

E_9 EXCEPTIONAL FIELD THEORY
II. THE COMPLETE DYNAMICSGuillaume Bossard¹, Franz Ciceri², Gianluca Inverso^{3,4},
Axel Kleinschmidt^{2,5} and Henning Samtleben⁶¹*Centre de Physique Théorique, CNRS, Institut Polytechnique de Paris,
FR-91128 Palaiseau cedex, France*²*Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut)
Am Mühlenberg 1, DE-14476 Potsdam, Germany*³*Centre for Research in String Theory, School of Physics and Astronomy,
Queen Mary University of London, 327 Mile End Road, London, E1 4NS, United Kingdom*⁴*INFN, Sezione di Padova
Via Marzolo 8, 35131 Padova, Italy*⁵*International Solway Institutes
ULB-Campus Plaine CP231, BE-1050 Brussels, Belgium*⁶*Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS,
Laboratoire de Physique, FR-69342 Lyon, France*

We construct the first complete exceptional field theory that is based on an infinite-dimensional symmetry group. E_9 exceptional field theory provides a unified description of eleven-dimensional and type IIB supergravity covariant under the affine Kac–Moody symmetry of two-dimensional maximal supergravity. We present two equivalent formulations of the dynamics, which both rely on a pseudo-Lagrangian supplemented by a twisted self-duality equation. One formulation involves a minimal set of fields and gauge symmetries, which uniquely determine the entire dynamics. The other formulation extends \mathfrak{e}_9 by half of the Virasoro algebra and makes direct contact with the integrable structure of two-dimensional supergravity. Our results apply directly to other affine Kac–Moody groups, such as the Geroch group of general relativity.

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1 Introduction

This paper completes the work started in [1], constructing the exceptional field theory for the Kac–Moody group E_9 . Exceptional field theories (ExFT) are the duality covariant formulations of maximal supergravity built on the exceptional hidden symmetry groups E_n that arise in dimensional reduction of eleven-dimensional (or type IIB) supergravity [2–5]. E_n ExFT is based on a split of the coordinates of eleven-dimensional supergravity into D external and $n = 11 - D$ internal coordinates. The latter are embedded into an extended space-time with coordinates transforming in a representation of the exceptional group E_n . The original physical coordinates are recovered as the solution of an E_n covariant section constraint that constrains the coordinate dependence of all fields and gauge parameters. On the extended space-time, the original geometric diffeomorphisms and gauge symmetries are unified into generalised diffeomorphisms [6–18] which provide the central symmetry structure organising and defining the theories.

Exceptional field theories have been constructed for all finite-dimensional duality groups, i.e. for $n \leq 8$, and $D \geq 3$ external dimensions [19–26]. The resulting actions are modelled after the structure of D -dimensional maximal (gauged) supergravities, with all fields living on the full extended space-time (subject to the section constraint) and the non-abelian gauge structure of D -dimensional supergravity replaced by the infinite-dimensional algebraic structure of generalised diffeomorphisms. In particular, the scalar fields parametrise the coset space $E_n/K(E_n)$ and couple via a (gauged) sigma-model on this space, where $K(E_n)$ is the maximal compact subgroup of E_n .

For $n = 9$, the group E_9 is the infinite-dimensional affine group that appears as the global symmetry of maximal two-dimensional supergravity [27, 28], generalising the Geroch group in the reduction of four-dimensional Einstein gravity [29–32]. The affine Lie algebra \mathfrak{e}_9 of E_9 is by definition the sum of the centrally extended loop algebra $\hat{\mathfrak{e}}_8$ over \mathfrak{e}_8 and a one-dimensional derivation algebra.¹ The study of the internal sector of ExFT based on this affine algebra was initiated in [17] with the construction of generalised diffeomorphisms acting on fields whose internal coordinates Y^M transform in the basic lowest weight representation $R(\Lambda_0)_{-1}$ of E_9 , and a section constraint of the generic form

$$Y^{MN}{}_{PQ} \partial_M \otimes \partial_N = 0 \tag{1.1}$$

where $Y^{MN}{}_{PQ}$ is the E_9 invariant tensor (3.3). In E_9 ExFT, all fields live on the full extended space-time with two external coordinates x^μ and internal coordinates Y^M , subject to the section constraint (1.1).

¹We denote the derivation of \mathfrak{e}_9 by \mathfrak{d} and it is related to the Virasoro generator L_0 by $\mathfrak{d} = L_0 + h$ in a module of weight $h \in \mathbb{C}$.

In a first paper [1], we have constructed the internal part of the action of E_9 ExFT, usually referred to as the potential, as a scalar field Lagrangian invariant under generalised diffeomorphisms. More precisely, this is the truncation of ExFT obtained after dropping all dependence on the two external coordinates x^μ and truncating the gauge fields. The goal of this paper is to extend this construction to the full E_9 exceptional field theory.

With respect to its lower-rank cousins, E_9 ExFT comes with a number of additional technical challenges. First of all, the affine nature of the duality group requires all bosonic objects to appear in infinite-dimensional representations. This is reflected by the scalar fields parametrising the infinite-dimensional coset space

$$\frac{\widehat{E}_8 \times (\mathbb{R}_{L_0}^+ \times \mathbb{R}_{L_{-1}})}{K(E_9)}, \quad (1.2)$$

where \widehat{E}_8 denotes the centrally extended loop group over E_8 , and the factor $\mathbb{R}_{L_0}^+ \times \mathbb{R}_{L_{-1}}$ is obtained by exponentiating the Virasoro generators L_0 and L_{-1} .

Secondly, while the ExFT field content in generic dimensions is largely based on D -dimensional maximal supergravity as imposed by the tensor hierarchy [33–35], closure of the gauge algebra in general requires the introduction of additional p -forms of rank $p \geq D - 2$. These additional fields are covariantly constrained by algebraic conditions analogous to the section constraint (1.1). For sufficiently large D , the constrained fields may not be visible at the level of the Lagrangian, but starting from $D \leq 4$, they become an inevitable part of the dynamical equations of exceptional field theory. In particular, for $D = 2$, such constrained fields appear in all sectors of the theory. Indeed, the results of [1] show that the construction of an invariant scalar potential requires the coupling of an additional scalar field χ_M obeying the algebraic constraints

$$Y^{MN}{}_{PQ} \chi_M \otimes \partial_N = 0 = Y^{MN}{}_{PQ} \chi_M \otimes \chi_N, \quad (1.3)$$

analogous to (1.1). Similarly, the gauge fields

$$\mathbb{A}_\mu = (A_\mu^M, B_\mu^M{}_N), \quad (1.4)$$

combine vectors A_μ^M transforming in the basic lowest weight (coordinate) representation $R(\Lambda_0)_{-1}$ of E_9 with vector fields $B_\mu^M{}_N$ constrained in their last index analogously to (1.3). The set of vector fields (1.4) reflects the structure of gauge parameters of E_9 generalised diffeomorphisms [17].

Thirdly, as known from the lower-rank exceptional field theories, their construction in an even number of external dimensions D is typically hampered by the fact that in D -dimensional supergravity only a subgroup of the duality group E_n is realised off-shell while the full duality group is only realised on the equations of motion. The most straightforward construction of the ExFT dynamical field equations thus draws on a manifestly duality invariant pseudo-Lagrangian which must be supplemented by a set of first order (self-)duality equations, in the spirit of the so-called democratic formulation of supergravity theories [36]. In two-dimensional maximal supergravity, the Lagrangian only depends on the 128 propagating scalar fields parametrising $E_8/Spin(16)$, the dilaton ρ associated to L_0 , and the conformal factor σ of the external metric,

associated to the central charge K . The infinitely many fields parametrising the coset space (1.2) are defined on-shell by virtue of an infinite set of duality relations typically formulated in terms of a linear system [30–32, 28] and its expansion in terms of a spectral parameter. In order to reflect these structures, E_9 ExFT will be based on a manifestly duality invariant pseudo-Lagrangian which features all the scalar fields parametrising (1.2), together with duality equations relating them to the physical ones. However, the spectral parameter of the linear system introduced in [31, 32] depends on the $D = 2$ space-time coordinates, a property that is a priori incompatible with the definition of fields in E_9 lowest weight modules like A_μ^M in ExFT. This was not yet an issue in the construction of the ExFT scalar potential [1] which corresponds to a truncation of the theory in which all duality equations are consistently projected out. It will, however, be relevant in the construction of the full dynamics, and one of the main achievements of this paper will be to show that the ExFT duality equations reproduce the Breitenlohner–Maison linear system.

From the above discussion and the algebra of generalised diffeomorphisms, one might expect that the fields of E_9 ExFT comprise the coset scalar fields in (1.2), the unimodular metric $\tilde{g}_{\mu\nu}$, the constrained field χ_M and the vector fields (1.4). We will indeed define a minimal formulation of the theory featuring these fields (in the gauge in which the scalar fields lie in $E_9/K(E_9)$, i.e. with the L_{-1} scalar $\tilde{\rho} = 0$), as well as a single additional one-form field χ_μ . However, the relation of the E_9 ExFT duality equation to the Breitenlohner–Maison linear system cannot be unraveled in this minimal formulation. This can be resolved by adopting a formulation of the Breitenlohner–Maison linear system that circumvents the problem of the space-time dependent spectral parameter through the introduction of scalar fields in the negative Virasoro group [37, 38], enlarging the scalar target coset space (1.2) to

$$\frac{\widehat{E}_8 \rtimes \text{Vir}^-}{K(E_9)}, \quad (1.5)$$

where Vir^- is the group associated to the algebra $\mathfrak{vir}^- = \langle L_n | n \leq 0 \rangle$. We shall define a Virasoro-extended formulation of E_9 ExFT, in which the algebra of generalised diffeomorphisms is based on $\widehat{E}_8 \rtimes \text{Vir}^-$. The set of vector fields is enlarged accordingly, with one vector field $B_\mu^{(n)M}{}_N$ associated to each negative Virasoro generator L_{-n} . The truncation of the Virasoro-extended formulation of ExFT to $D = 2$ supergravity (via the trivial solution of the section constraint) will be shown to reproduce the Breitenlohner–Maison linear system in the formulation of [37, 38].

The scalar fields in (1.5) are parametrised by the coset representative

$$\mathcal{V} = \Gamma \hat{V} = \rho^{-L_0} e^{-\phi_1 L_{-1}} e^{-\phi_2 L_{-2}} \dots \mathring{V} e^{Y_{1A} T_{-1}^A} e^{Y_{2A} T_{-2}^A} \dots e^{-\sigma K} \quad (1.6)$$

upon introducing an additional infinite tower of scalar fields ϕ_n associated with the negative half of the Virasoro algebra. The fields ρ and ϕ_n combine into the Γ -factor in (1.6) whose presence replaces the space-time dependent spectral parameter of the standard linear system on (1.2) [31, 32]. The E_8 -valued matrix \mathring{V} in (1.6) carries the propagating scalar fields of $D = 2$ maximal supergravity while the negative mode generators T_{-n}^A of the loop algebra are associated with the infinite tower of dual potentials defined by the linear system.

The first-order duality relations encoding the ExFT scalar field dynamics are most compactly formulated in terms of the covariant currents

$$\mathfrak{J}_\mu = \mathcal{M}^{-1} \mathcal{D}_\mu \mathcal{M} = \mathfrak{J}_{\mu\alpha} T^\alpha = \sum_{m \in \mathbb{Z}} (\mathbb{J}_{\mu A}^m T_m^A + \mathbb{J}_{\mu m} L_m) + \mathbb{J}_{\mu \mathbb{K}} \mathbb{K}. \quad (1.7)$$

obtained from the generalised metric $\mathcal{M} = \mathcal{V}^\dagger \mathcal{V}$, and explicitly expanded in terms of the generators of \widehat{E}_8 and the full Virasoro algebra. As standard in ExFT, the covariant derivative \mathcal{D}_μ employs the vector fields (1.4) (and a tower of Virasoro descendants $B_\mu^{(n)M}{}_N$ of the constrained vectors) to gauge the action of Virasoro-extended E_9 generalised diffeomorphisms. Its explicit connection is defined in (4.6) in the main text. From (1.7), one may define a tower of shifted currents

$$\mathfrak{J}_\mu^{(m)} = \mathcal{S}_0(\Gamma^{-1} \mathcal{S}_m(\Gamma \mathfrak{J}_\mu \Gamma^{-1}) \Gamma) + \chi_{\mu,m}^\gamma \mathbb{K}, \quad (1.8)$$

where the operators \mathcal{S}_m shift the mode number of the \widehat{E}_8 loop generators and the Virasoro generators by m units and project out the component along the central charge \mathbb{K} , while Γ is the Vir^- part of the coset representative (1.6). Covariance of the shifted currents (1.8) under rigid $\widehat{E}_8 \rtimes \text{Vir}^-$ requires the introduction of a tower of new (external) one-forms $\chi_{\mu,m}^\gamma$ as their \mathbb{K} components.² This is in complete analogy to the appearance of the constrained scalar field χ_M (1.3) in the shifted current along internal derivatives which was introduced in [1] to define the ExFT scalar potential. As a key result of this paper, we obtain the ExFT scalar twisted self-duality equation in the form

$$\star \mathfrak{J} = \mathfrak{J}^{(1)}, \quad (1.9)$$

where \star denotes Hodge duality with respect to the unimodular two-dimensional metric in conformal gauge $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$. Upon dropping internal derivatives and gauge fields, the duality equation (1.9) reduces to a linear system for the field equations of $D = 2$ supergravity. The standard form of the linear system [31, 32] is recovered after integrating the vir^- part of the duality equation in order to obtain polynomial expressions for the additional scalar fields ϕ_m in (1.6) in terms of the fields ρ and $\tilde{\rho} \sim \phi_1$ parametrising the $\mathbb{R}_{L_0}^+ \rtimes \mathbb{R}_{L_{-1}}$ factor in (1.2).

Next, we construct in this paper a pseudo-Lagrangian which together with (1.9) determines the full dynamics of E_9 ExFT. According to the general structure of gauged supergravity in $D = 2$ dimensions [39], and more generally to the structure of two-dimensional gauged sigma models with WZW terms [40], the full Virasoro-extended E_9 ExFT pseudo-Lagrangian is expected to extend the potential term of [1] by a topological term coupling the currents of the coset space (1.5) to the gauge fields, as well as by a suitably gauged version of the linear system of duality equations. The pseudo-Lagrangian consists accordingly of the potential term of [1] and a topological term written as a top-form:

$$\begin{aligned} \rho^{-1} \mathcal{L}_{\text{top}} &= \mathcal{D} \chi_1^\gamma - \frac{1}{2} \omega^{\alpha\beta} \mathfrak{J}_\alpha \wedge \mathfrak{J}_\beta^{(1)} - \frac{4}{3} \sum_{n=2}^{\infty} (n^3 - n) \mathbb{P}_n \wedge (\mathbb{P}_{n+1} + \mathbb{P}_{n-1}) \\ &+ \sum_{n=1}^{\infty} \mathbb{P}_n \wedge (\chi_{1+n}^\gamma - \chi_{1-n}^\gamma) + \chi_{M,1}^\gamma \mathcal{F}^M + \widehat{\mathbb{F}}_1^\gamma + \mathcal{M} \widehat{\mathbb{F}}_1^\gamma. \end{aligned} \quad (1.10)$$

²The superscript γ is not an index, but rather a label related to the Γ conjugation in (1.8), see Section 4.

The various ingredients are defined and detailed in Section 4. In particular, the covariant derivative of $\chi_1^\gamma = \chi_{\mu,1}^\gamma dx^\mu$ is defined in (4.58). The second term in (1.10) is the gauged WZW-like term and carries the algebra cocycle $\omega^{\alpha\beta}$ corresponding to the central charge component of the commutator

$$\omega^{\alpha\beta} \mathfrak{J}_{\mu\alpha} \mathfrak{J}_{\nu\beta}^{(1)} = -[\mathfrak{J}_\mu, \mathfrak{J}_\nu^{(1)}]_{\mathbb{K}}. \quad (1.11)$$

The currents $\mathbf{P}_{\mu n} = \frac{1}{2}(\mathcal{D}_\mu \Gamma \Gamma^{-1})_{-n}$ for $n \geq 1$ are defined as the \mathfrak{vir}^- components of the Maurer–Cartan form from (1.6). The field $\chi_{M,1}^\gamma$ in (1.10) is related by the field redefinition (4.76) to the field χ_M appearing the potential in [1]. Finally, the last three terms in (1.10) are proportional to the non-abelian field strengths

$$\mathbb{F}_{\mu\nu} = (\mathcal{F}_{\mu\nu}^M, \mathcal{G}_{\mu\nu}^{(k)M}{}_N), \quad k \in \mathbb{N}^+, \quad (1.12)$$

of the gauge fields (1.4) (and their Virasoro-extensions). Their precise expressions are given in (4.42) and (4.45) below. Remarkably, the full ExFT pseudo-Lagrangian, after gauge-fixing $\phi_n \rightarrow 0$ for the \mathfrak{vir}^- scalar fields at $n \geq 2$, is simply given by the combination

$$\mathcal{L}_{\text{ext}} = \mathcal{L}_{\text{top}} - \star V, \quad (1.13)$$

of the topological term (1.10) with the potential V constructed in [1]. In their most explicit form, these two terms are given in (4.81) and (4.82) below. In particular, this pseudo-Lagrangian does not carry a traditional kinetic term for the scalar fields.

In order to make contact with the lower-rank exceptional field theories and to match the field equations with those of eleven-dimensional (and type IIB) supergravity, it turns out to be convenient to pass eventually from the Virasoro-extended to the minimal formulation of E_9 ExFT in which not only the \mathfrak{vir}^- scalar fields ϕ_n are gauge fixed to zero, but also the infinite tower of constrained and auxiliary one-forms $\{B_\mu^{(n)M}{}_N, \chi_{\mu,n}^\gamma\}$ featuring in (1.10) is integrated out in favour of the original gauge fields (1.4) and a single auxiliary one-form χ_μ . Although we shall prove that the minimal and Virasoro-extended formulations of the theory describe the same dynamics, these two formulations differ in structure and will be discussed separately. In particular, the current \mathcal{J} in the minimal formulation is valued in the Lie algebra of the group $\widehat{E}_8 \rtimes (\mathbb{R}_{L_0}^+ \times \mathbb{R}_{L_{-1}})$ which is not closed under Hermitian conjugation unlike the current (1.7). In this formulation, we relax the conformal gauge and consider an arbitrary unimodular metric $\tilde{g}_{\mu\nu}$, thereby allowing for the definition of external diffeomorphisms. To distinguish the currents in the two formulations we will use a different notation, with the minimal formulation current

$$\mathcal{J}_\mu = \mathcal{J}_{\mu\alpha} T^\alpha = \sum_{m \in \mathbb{Z}} J_{\mu A}^m T_m^A + J_{\mu 0} L_0 + J_{\mu -1} L_{-1} + J_{\mu \mathbb{K}} \mathbb{K}. \quad (1.14)$$

In particular, $J_{\mu \mathbb{K}}$ depends explicitly on the unimodular metric $\tilde{g}_{\mu\nu}$, see (5.34), and $J_{\mu -1} = B_\mu^M{}_M$. The duality equation takes the form of a twisted self-duality

$$\mathcal{J} = \rho^{-1} \star \mathcal{M}^{-1} (\mathcal{S}_1(\mathcal{J}) + \chi \mathbb{K})^\dagger \mathcal{M}, \quad (1.15)$$

associated to an E_9 invariant symmetric bilinear form. The resulting pseudo-Lagrangian, now written as a density, takes the form

$$\mathcal{L}_{\text{min}} = \mathcal{L}_1 + \mathcal{L}_2 + \frac{1}{4} \rho \varepsilon^{\mu\nu} \varepsilon^{\sigma\rho} \tilde{g}^{\kappa\lambda} \mathcal{D}_\mu \tilde{g}_{\sigma\kappa} \mathcal{D}_\nu \tilde{g}_{\rho\lambda} + \frac{\rho^{-1}}{4} \mathcal{M}^{MN} \partial_M \tilde{g}^{\mu\nu} \partial_N \tilde{g}_{\mu\nu} - V, \quad (1.16)$$

where \mathcal{L}_2 is a topological term similar in structure to (1.10), whereas \mathcal{L}_1 depends explicitly on the metric $\tilde{g}_{\mu\nu}$ and plays the role of a kinetic term. They are defined below in (5.45) and (5.46) and their variation with respect to $B_\mu{}^M{}_N$ is the contraction of the duality equation (1.15) with $\delta B_\mu{}^M{}_N$. The pseudo-Lagrangian (1.16) is a sum of terms separately invariant under internal diffeomorphisms, whose relative coefficients are determined by external diffeomorphism invariance, as is usually the case in ExFT. The invariance of the system of equations under external diffeomorphisms, including the duality equation (1.15), requires us to consider the additional equation

$$\mathcal{F}^M = \star\rho^{-1}T^{\alpha M}{}_N \mathcal{M}^{NP} \mathcal{J}_{P\alpha}^-, \quad (1.17)$$

in its entirety while it only appears as an Euler–Lagrange equation contracted with the constrained variation $\delta\chi_M$. The shifted internal current $\mathcal{J}_{M\alpha}^-$ is defined in (5.47) as in [1].

Upon partially solving the section constraint (1.1), most of the components of the duality equation (1.15) simply determine the non-vanishing components of the vector field $B_\mu{}^M{}_N$. All the solutions to the constraint (1.1) can be mapped under E_9 to a form compatible with the parabolic gauge (1.6), such that the fields only depend on the $D = 3$ external coordinates and the 248 internal coordinates of E_8 ExFT. Using this solution and integrating out the unconstrained components of $B_\mu{}^M{}_N$ one obtains that the pseudo-Lagrangian (1.16) becomes equivalent to the one of E_8 ExFT [22]. This proves that the dynamics of the theory reproduces the one of eleven-dimensional supergravity or type IIB supergravity depending on the choice of solution to the section constraint, confirming the dynamical content of the pseudo-Lagrangian (1.16).

The rest of this paper is organised as follows: In Section 2, we review the algebraic structures and how they appear in two-dimensional ungauged supergravity through a (Virasoro-extended) linear system. Section 3 describes the structure of E_9 generalised diffeomorphisms and the associated gauge fields, including their Virasoro-extensions. In Section 4, we construct the Virasoro-extended E_9 ExFT by first determining a gauge-invariant topological term. The Virasoro-extended topological term can be gauge-fixed and combined with the potential term. The minimal formulation is presented in Section 5 where the gauge-fixing is used to reduce the formulation of E_9 ExFT to a finite set of fields. In this formulation, we moreover study external diffeomorphisms and show consistency with E_8 ExFT. Because the two formulations of E_9 ExFT are different in structure, we have exposed them in a way that can be read independently, so that in particular Section 4 is not a prerequisite to Section 5 from 5.2 onwards. The equivalence of the two formulations is proved in Section 5.1. In Section 6, we discuss possible applications and generalisations of our results. Several appendices contain additional technical details.

2 Preliminaries on $D = 2$ supergravity and algebraic structures

In this section, we fix our notation for E_9 and its extension by Virasoro generators. We also review the so-called linear system [29–32, 28] of the bosonic part of $D = 2$ maximal supergravity and how the affine symmetry arises. In Section 2.3, we discuss a less well-known extension [37, 38] of the linear system that also features the Virasoro algebra.

2.1 E_9 , its Lie algebra and Virasoro extension

The split real affine Kac–Moody algebra \mathfrak{e}_9 has the basis T_m^A , \mathbf{K} and \mathbf{d} with non-trivial commutation relations

$$[T_m^A, T_n^B] = f^{AB}{}_C T_{m+n}^C + m \eta^{AB} \delta_{m,-n} \mathbf{K}, \quad [\mathbf{d}, T_m^A] = -m T_m^A. \quad (2.1)$$

The element \mathbf{K} is central in the Lie algebra and the indices $A, B, C = 1, \dots, 248$ parametrise the adjoint of the underlying exceptional \mathfrak{e}_8 Lie algebra where \mathfrak{e}_8 is realised as a subalgebra by only considering the elements T_0^A and has the Killing form η^{AB} . The mode number index $m \in \mathbb{Z}$ arises from the loop algebra construction of affine Lie algebras [41, 42]. The element \mathbf{d} is called the derivation. We shall also encounter the centrally extended loop algebra

$$\hat{\mathfrak{e}}_8 = \langle T_m^A, \mathbf{K} \rangle \quad (2.2)$$

that differs from \mathfrak{e}_9 by the omission of the derivation \mathbf{d} .

Lowest weight representations, denoted by $R(\Lambda)_h$, are determined by giving a weight Λ of $\hat{\mathfrak{e}}_8$ and a conformal weight h . The latter corresponds to the eigenvalue under \mathbf{d} while Λ summarises the eigenvalues under the Cartan subalgebra of $\hat{\mathfrak{e}}_8$, that consists of the Cartan subalgebra of \mathfrak{e}_8 and \mathbf{K} . The most relevant instance for us is the so-called basic representation $R(\Lambda_0)_h$ that is constructed in Fock space notation from an \mathfrak{e}_8 -invariant ground state $|0\rangle$

$$T_0^A |0\rangle = 0, \quad \mathbf{K} |0\rangle = |0\rangle, \quad \mathbf{d} |0\rangle = h |0\rangle. \quad (2.3)$$

The lowest weight condition means that $T_n^A |0\rangle = 0$ for $n > 0$ and the module is the unique irreducible quotient obtained by acting with T_n^A for $n < 0$ on the ground state.

The Kac–Moody group associated with \mathfrak{e}_9 will be denoted by E_9 and we refer to Appendix F for a discussion of some of the subtleties arising when defining this infinite-dimensional group. The subgroup that is generated by only T_m^A and \mathbf{K} is the centrally extended loop group and denoted by \hat{E}_8 in this paper.

The Sugawara construction [43, 42] provides an infinite set of additional Virasoro generators L_m ($m \in \mathbb{Z}$) acting on any lowest weight module $R(\Lambda)_h$. These generators satisfy the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c_{\text{vir}}}{12} m(m^2 - 1) \delta_{m,-n} \mathbf{K} \quad (2.4)$$

where the Virasoro central charge is determined by Λ . For the basic representation we have $c_{\text{vir}} = 8$. The commutation relations with the $\hat{\mathfrak{e}}_8$ generators in this representation are

$$[L_m, T_n^A] = -n T_{m+n}^A, \quad [L_m, \mathbf{K}] = 0, \quad (2.5)$$

and we see that the action of L_0 agrees with that of \mathbf{d} and they can be related in the lowest weight representation as $\mathbf{d} = L_0 + h$.

Denoting the Virasoro algebra by $\mathfrak{vir} = \langle L_m \mid m \in \mathbb{Z} \rangle$, we can form the extended algebra

$$\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}, \quad (2.6)$$

where the symbol \oplus indicates a semi-direct sum of Lie algebras since the Virasoro algebra acts on $\hat{\mathfrak{e}}_8$ according to (2.5). We shall denote the generators of $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ collectively as T^α , so that this index runs over $\hat{\mathfrak{e}}_8$ and \mathfrak{vir} . For any fixed $m \in \mathbb{Z}$ we define an $\hat{\mathfrak{e}}_8$ -invariant bilinear form by

$$\eta_{m\alpha\beta} T^\alpha \otimes T^\beta = \sum_{n \in \mathbb{Z}} \eta^{AB} T_n^A \otimes T_{m-n}^B - L_m \otimes \mathbf{K} - \mathbf{K} \otimes L_m. \quad (2.7)$$

For $m = 0$, this form coincides with the standard non-degenerate bilinear form \mathfrak{e}_9 when identifying L_0 with \mathfrak{d} . In this case we simply write $\eta_{\alpha\beta} \equiv \eta_{0\alpha\beta}$.

We shall also use the so-called shift operators \mathcal{S}_m for $m \in \mathbb{Z}$ that are defined on $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ by

$$\mathcal{S}_m(\mathbf{K}) = 0, \quad \mathcal{S}_m(L_n) = L_{m+n}, \quad \mathcal{S}_m(T_n^A) = T_{m+n}^A, \quad (2.8)$$

which implies that

$$\eta_{(n+m)\alpha\beta} T^\alpha \otimes T^\beta = \eta_{n\alpha\beta} T^\alpha \otimes \mathcal{S}_m(T^\beta) - L_{n+m} \otimes \mathbf{K}. \quad (2.9)$$

The shift operators are not invariant under $\hat{\mathfrak{e}}_8$ and their properties are discussed in more detail in [1] and in Appendix B. Therefore, when we define a shifted object we may want to introduce its \mathbf{K} completion, such that the completed expression transforms as an algebra-valued object. This will be explained in Section 4. Notice that the definition for \mathcal{S}_0 differs from that in [1]. There, we defined $\mathcal{S}_0(\mathbf{K}) = \mathbf{K}$ while in this paper we find it convenient to use $\mathcal{S}_0(\mathbf{K}) = 0$.

Finally, we define the Hermitian conjugates of the generators of $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ in the basic representation $R(\Lambda_0)_h$ by

$$L_n^\dagger = L_{-n}^\dagger, \quad \mathbf{K}^\dagger = \mathbf{K}, \quad T_n^{A\dagger} = \eta_{AB} T_{-n}^B. \quad (2.10)$$

Writing the Kac–Moody group as E_9 , this defines a unitary subgroup $K(E_9)$ consisting of the elements k that satisfy $kk^\dagger = k^\dagger k = 1$ when acting on $R(\Lambda_0)_h$. Under Hermitian conjugation the shift operators transform as

$$\mathcal{S}_m(X^\dagger) = (\mathcal{S}_{-m}(X))^\dagger, \quad X \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}. \quad (2.11)$$

2.2 $D = 2$ supergravity and the linear system

In this section we review how the infinite-dimensional algebras presented above appear as symmetries of two-dimensional supergravity [29–32, 28]. The entire discussion applies to two-dimensional gravity coupled to a dilaton and scalars in a non-linear sigma model on a coset space G/H based on a simple Lie group G . This includes the dimensional reduction to two dimensions of pure $D = 4$ General Relativity, with $G/H = SL(2)/SO(2)$, as well as reductions of many supergravity theories. For definiteness, we will take $G/H = E_8/Spin(16)$ as it appears in maximal supergravity, but all the results are easily generalised to the other cases. We thus begin with (ungauged) maximal supergravity in three dimensions, whose bosonic field content comprises a metric and scalar fields parametrising the symmetric space $E_8/Spin(16)$. Dimensionally reducing to two dimensions, the metric decomposes as³

$$ds_{\text{3D}}^2 = e^{2\sigma} \tilde{g}_{\mu\nu} dx^\mu dx^\nu + \rho^2 (d\varphi + w_\mu dx^\mu)^2 \quad (2.12)$$

³Notice that there is no notion of Einstein frame in two-dimensional gravity. The metric we present is thus just the dimensional split of the three-dimensional Einstein frame metric.

where $\tilde{g}_{\mu\nu}$ is the two-dimensional unimodular metric with $\det \tilde{g} = -1$, $e^{2\sigma}$ is the conformal factor, φ is the Kaluza–Klein coordinate, and all fields only depend on the two-dimensional space-time coordinates x^μ . The vector field w_μ is non-dynamical and we set it to zero in this section. The field ρ is a scalar in two dimensions and referred to as the dilaton. The $E_8/Spin(16)$ scalar fields are encoded in a coset representative $\mathring{V}(x)$ which transforms as

$$\mathring{V}(x) \rightarrow \mathring{h}(x)\mathring{V}(x)\mathring{g}, \quad \mathring{g} \in E_8, \quad \mathring{h}(x) \in Spin(16). \quad (2.13)$$

The equations of motion are phrased in terms of the coset and $Spin(16)$ components of the Maurer–Cartan form

$$d\mathring{V}\mathring{V}^{-1} = \mathring{P} + \mathring{Q}, \quad \mathring{P} = (\mathring{P})^\top, \quad \mathring{Q} = -(\mathring{Q})^\top. \quad (2.14)$$

The symbol $^\top$ denotes transposition in \mathfrak{e}_8 , i.e. the anti-involution that singles out the maximal compact subalgebra $\mathfrak{so}(16)$ and it agrees with restriction of the Hermitian conjugation defined in (2.10) to T_0^A . The scalar field ρ is free:

$$d \star d\rho = 0, \quad (2.15)$$

where \star denotes Hodge duality with respect to the unimodular metric $\tilde{g}_{\mu\nu}$.⁴ The equations of motion for the E_8 scalars are

$$d \star (\rho \mathring{V}^{-1} \mathring{P} \mathring{V}) = 0 \quad (2.16)$$

and finally the conformal factor is entirely specified by the Virasoro constraint which is most easily written in conformal gauge $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ and light-cone coordinates x^\pm such that $\eta_{+-} = \eta_{-+} = 1$ and $\eta_{\pm\pm} = 0$:

$$\partial_\pm \sigma \partial_\pm \rho - \frac{1}{2} \partial_\pm \partial_\pm \rho - \frac{1}{2} \rho \eta_{AB} P_\pm^A P_\pm^B = 0. \quad (2.17)$$

There is also a second-order equation for σ , which is implied by the other ones

$$\partial_+ \partial_- \sigma + \frac{1}{2} \eta_{AB} P_+^A P_-^B = 0. \quad (2.18)$$

Equation (2.15) implies that we can define a dual scalar field $\tilde{\rho}$ such that

$$d\tilde{\rho} = \star d\rho. \quad (2.19)$$

Combining this relation with (2.16) we can construct infinitely many scalar fields dual to the currents \mathring{P} . The first of these duality relations reads

$$dY_1 = 2\rho \mathring{V}^{-1} \star \mathring{P} \mathring{V}, \quad (2.20)$$

and the whole tower is best encoded into a linear system which we now describe in the form given in [32, 28].

⁴We use conventions such that $\star 1 = dx^0 \wedge dx^1$. For one-forms we have $(\star\omega)_\mu = \tilde{g}_{\mu\nu} \varepsilon^{\nu\rho} \omega_\rho$ with $\varepsilon^{01} = 1 = -\varepsilon_{01}$.

We introduce a (constant) spectral parameter $w \in \mathbb{C}$ and define an E_8 valued function $\hat{V}(w)$ with the requirement that it reduces to \mathring{V} as $w \rightarrow \infty$:⁵

$$\hat{V}(w) \in E_8, \quad \lim_{w \rightarrow \infty} \hat{V}(w) = \mathring{V}. \quad (2.21)$$

Then, equations (2.15) and (2.16) imply integrability of the linear system⁶

$$d\hat{V}(w)\hat{V}^{-1}(w) = \mathring{Q} - \frac{1 + \gamma^2}{1 - \gamma^2} \mathring{P} - \frac{2\gamma}{1 - \gamma^2} \star \mathring{P}. \quad (2.22)$$

In order to reproduce the ρ factor in (2.16), γ must be a space-time dependent function of ρ and $\tilde{\rho}$ and satisfy

$$\frac{d\gamma}{\gamma} = \frac{1 + \gamma^2}{1 - \gamma^2} \rho^{-1} d\rho + \frac{2\gamma}{1 - \gamma^2} \rho^{-1} d\tilde{\rho}, \quad (2.23)$$

which is solved by

$$\gamma_{\pm} = \frac{1}{\rho} \left(s(w) - \tilde{\rho} \pm \sqrt{(s(w) - \tilde{\rho})^2 - \rho} \right), \quad \gamma = \gamma_+ = \frac{1}{\gamma_-}. \quad (2.24)$$

Here, $s(w)$ is an integration constant through which γ depends on the spectral parameter. The simplest choice compatible with the asymptotics described below is

$$s(w) = w, \quad (2.25)$$

which we will use in the remainder of this section. More general choices are possible and we will make use of this fact later.⁷ Because of the square root, the function γ then defines a double covering of the w plane. The two sheets correspond to two solutions of (2.22) that are analytic continuations of each other and are captured by writing

$$\hat{V}(w) = V(\gamma(w)). \quad (2.26)$$

It is then straightforward to see that if $V(\gamma(w))$ satisfies (2.22), so does $V^{-\text{T}}(1/\gamma(w))$, where $V^{-\text{T}} = (V^{\text{T}})^{-1}$. We then have that the monodromy matrix

$$\mathfrak{M}(w) = V^{\text{T}}(1/\gamma(w))V(\gamma(w)) \quad (2.27)$$

is symmetric, single-valued in w and constant, i.e. independent of the space-time coordinates x^μ as a consequence of (2.22). The monodromy matrix entirely specifies a solution of the linear system. It can be acted upon by constant elements of the loop group over E_8 , namely E_8 -valued functions of the spectral parameter:

$$\mathfrak{M}(w) \rightarrow g^{\text{T}}(w)\mathfrak{M}(w)g(w), \quad g(w) \in E_8. \quad (2.28)$$

⁵Dependence on the space-time coordinates x^μ is understood.

⁶The negative signs in (2.22) may appear unusual compared to the expression in [32]. The explanation is that the γ parameter we use here corresponds to $1/t$ there.

⁷The meaning of the symbols s and w here is changed with respect to [32]. Also notice that [32] is in Euclidean signature, which causes some further sign flips with respect to our discussion here.

This defines the global \widehat{E}_8 symmetry of the model. For compatibility with (2.21), we require $V(\gamma(w))$ to be expandable as a series in γ and reduce to \mathring{V} as $\gamma \rightarrow \infty$. Then, the right \widehat{E}_8 action on $V(\gamma(w))$ must be compensated on the left by a local transformation that preserves its regularity properties:

$$V(\gamma(w), x) \rightarrow h(\gamma(w), x) V(\gamma(w), x) g(w), \quad h(\gamma(w), x) = h^{-T}(1/\gamma(w), x). \quad (2.29)$$

where we have denoted explicitly which objects are space-time dependent (and γ itself is space-time dependent through ρ and $\tilde{\rho}$). The compensating transformation belongs to *one* $K(E_9) = K(\widehat{E}_8)$ subgroup of \widehat{E}_8 and leaves the monodromy matrix invariant. It is defined in terms of a *field-dependent* anti-involution which acts by inversion of $\gamma(w)$ and differs from the Hermitian conjugation we introduced in (2.10) which, in the spectral parameter representation, acts by inversion of w . This distinction will become relevant in the next section. We see that $V(\gamma(w))$ is a coset representative for $\widehat{E}_8/K(E_9)$. The conformal factor e^σ also transforms under \widehat{E}_8 , compatibly with (2.17), in terms of a group cocycle that defines the central extension of \widehat{E}_8 .

An infinite tower of dual fields and the associated duality equations generalising (2.20) are obtained by expanding (2.22) around $w \rightarrow \infty$ with⁸

$$V(\gamma(w)) = \mathring{V} e^{Y_{1A} T_{-1}^A} e^{Y_{2A} T_{-2}^A} e^{Y_{3A} T_{-3}^A} \dots \quad (2.30)$$

Where we have defined

$$T_m^A = w^m T^A \quad (2.31)$$

with T^A a basis of generators for \mathfrak{e}_8 . We then see that the generators T_m^A with $m < 0$ correspond to shifts of the dual Y_m potentials that do not affect the physical fields in \mathring{V} . Positive loop level generators T_m^A with $m > 0$, instead, correspond to hidden symmetries that mix \mathring{V} with the dual potentials.

There are three more global symmetry generators manifest in the linear system: a rescaling of ρ combined with appropriate rescalings of the Y_m fields, captured by the action of the L_0 generator

$$L_0 = -w \frac{\partial}{\partial w}, \quad (2.32)$$

a shift of $\tilde{\rho}$ corresponding to the choice of integration constant in (2.19) combined with a redefinition of the Y_m fields captured by the Virasoro generator

$$L_{-1} = -\frac{\partial}{\partial w}, \quad (2.33)$$

and finally a constant shift of σ corresponding to the choice of integration constant in (2.17), corresponding to K . Notice that the generators L_0, L_{-1} do not commute with the loop algebra but normalise it.

⁸We provide more details on the definitions of Kac–Moody groups associated to a give Kac–Moody algebra in Appendix F.

2.3 Extended coset and twisted self-duality

In order to make contact with exceptional field theory, it is desirable to rewrite the linear system in a form that is manifestly covariant under all the global symmetries described above and independent of the $K(E_9)$ gauge. Such a form should also incorporate the duality equation between ρ and $\tilde{\rho}$ on the same footing as the ones for the Y_m fields. Furthermore, we will need to write our coset representatives and currents in a lowest weight representation of E_9 . The natural definition of the maximal unitary subgroup $K(E_9)$ will then be in terms of the Hermitian conjugation (2.10) — corresponding here to the inversion $w \rightarrow 1/w$ combined with E_8 transposition, rather than the field-dependent involution defined in (2.29). These issues are addressed by extending the group theoretical structures found so far and rephrasing the linear system as a twisted self-duality constraint on an enlarged set of dual fields.

This approach follows [37, 38] with several adaptations. It is based on the similarity between (2.23) and (2.22) and the association of ρ and $\tilde{\rho}$ with the Virasoro generators L_0 and L_{-1} . The basic idea is to regard $\gamma(w)$ as a diffeomorphism on the w plane. Infinitesimally, such changes of variables are generated by the Virasoro elements

$$L_m = -w^{m+1} \frac{\partial}{\partial w}. \quad (2.34)$$

There are some issues with such an interpretation, because the naive exponentiation of the Virasoro algebra does not form a group [44]. This is reflected for instance in the fact that (2.24) with $s = w$ defines a double covering of the w plane and hence while $\gamma^{-1}(w)$ is well defined, the double inverse is not unique. To circumvent these issues, we shall focus only on the behaviour of fields and group elements around $w \rightarrow +\infty$. We shall consider redefinitions of the spectral parameter that preserve the asymptotics at $+\infty$, namely we consider only transformations of the form

$$w \rightarrow f(w) = f_{-1}w + f_0 + f_1w^{-1} + f_2w^{-2} + \dots \quad (2.35)$$

with real coefficient and with $f_{-1} > 0$. It is not necessary for the power series to converge (i.e., we accept formal power series). Group multiplication is given by composition of two such transformations, and the coefficients of the inverse of (2.35) are finite expressions, uniquely determined order by order in terms of the f_i . There is a one-to-one correspondence between (2.35) and (formal) products of exponentials of the non-positive part of the Virasoro algebra:

$$F^{-1}w = f(w), \quad F^{-1} = \dots e^{-\frac{f_1}{f_{-1}}L_{-2}} e^{-\frac{f_0}{f_{-1}}L_{-1}} (f_{-1})^{-L_0}, \quad (2.36)$$

where each exponent is a rational function of finitely many f_i , whose explicit expression we will not need. We shall denote \mathfrak{vir}^- the algebra generated by L_m with $m \leq 0$ and by Vir^- the group we just described. The function (2.24) belongs to this set when expanded around $w \rightarrow +\infty$, so that we can write close to $w \rightarrow +\infty$

$$\frac{1}{\rho} \left(s(w) - \tilde{\rho} \pm \sqrt{(s(w) - \tilde{\rho})^2 - \rho} \right) = \frac{2s(w)}{\rho} - \frac{2\tilde{\rho}}{\rho} - \frac{\rho}{2s(w)} - \frac{\rho\tilde{\rho}}{2s(w)^2} + \dots \quad (2.37)$$

where we now also allow the integration constant $s(w)$ to take a more general form than just $s(w) = w$, compatibly with the regularity properties of $\hat{V}(w)$ stated above, so that

$$s(w) = s_{-1}w + s_0 + s_1w^{-1} + s_2w^{-2} + \dots \quad (2.38)$$

with $s_{-1} > 0$.⁹ Namely, $s(w)$ is itself a (constant) element of Vir^- .

The key point of this reformulation is to regard $\gamma(w)$ as an *arbitrary* space-time-dependent element of Vir^- , which we shall write as

$$\Gamma^{-1} = \dots e^{\phi_3 L_{-3}} e^{\phi_2 L_{-2}} e^{\phi_1 L_{-1}} \rho^{L_0}, \quad \gamma(w) = \Gamma^{-1} w = \frac{w}{\rho} - \frac{\phi_1}{\rho} - \frac{\phi_2}{\rho w} - \frac{\phi_3}{\rho w^2} + \dots \quad (2.39)$$

where $\phi_m(x)$, $m > 0$ are infinitely many scalar fields generalising the dual potential $\tilde{\rho}(x)$. The natural action of Vir^- elements on $\hat{V}(w)$ can be used, in particular, to rephrase (2.26) as

$$\hat{V}(w) = V(\gamma(w)) = \Gamma^{-1} V(w) \Gamma, \quad (2.40)$$

where expansion at $w \rightarrow +\infty$ is understood and we must now regard $\gamma(w)$ as the arbitrary series (2.39) rather than the expression that appears in the linear system. We therefore see that we may define an extended coset representative which includes the Vir^- group element above:

$$\mathcal{V} = \Gamma V(\gamma(w)) = V(w) \Gamma. \quad (2.41)$$

We can then combine this expression with the transformation property of $V(\gamma(w))$ in (2.29) to deduce how \mathcal{V} transforms under the loop group. One finds that the compensating transformation in (2.29) is brought to the left of \mathcal{V} after conjugation by Γ and therefore it depends on w directly, rather than through $\gamma(w)$:

$$\mathcal{V} = \Gamma V(\gamma(w)) \rightarrow h(w) \mathcal{V} g(w), \quad h(w) = h^{-\text{T}}(1/w) = (h^{-1}(w))^\dagger. \quad (2.42)$$

The compensating element $h(w)$ belongs again to a $K(E_9)$ subgroup of the loop group but now, crucially, it is defined by a field-independent involution that acts by inversion of w rather than γ . Indeed this is the spectral parameter representation of the Hermitian conjugation (2.10) that we were seeking. The extended coset representative also transforms under rigid Vir^- transformations such as (2.36):

$$\mathcal{V} = \Gamma V(\gamma(w)) \rightarrow \Gamma V(\gamma(w)) F = \Gamma F V(\gamma(f(w))). \quad (2.43)$$

In this case no compensating transformations are required. We conclude that we have found an extended set of fields parameterising the coset space

$$\frac{\hat{E}_8 \times \text{Vir}^-}{K(E_9)} \quad (2.44)$$

where $K(E_9)$ is defined in terms of the field independent Hermitian conjugation (2.10). This allows us to define the coset representative \mathcal{V} in arbitrary representations of \hat{E}_8 . At this point

⁹In the linear system, the choice of $s(w)$ amounts to a redefinition of the Y_m fields with $m \geq 2$.

we remind the reader that in the spectral parameter representation \mathbf{K} is represented trivially. The field associated to the central charge is σ and we will include it in \mathcal{V} later when we work in a faithful representation of the algebra.

We can now introduce the twisted self-duality constraint equivalent to (2.22) and (2.23). We introduce the Hermitian and anti-Hermitian parts of the Maurer–Cartan form

$$(\mathrm{d}\mathcal{V}\mathcal{V}^{-1})(w) = P(w) + Q(w), \quad P^\Gamma(1/w) = P(w), \quad Q^\Gamma(1/w) = -Q(w). \quad (2.45)$$

Notice that while $\mathrm{d}\mathcal{V}\mathcal{V}^{-1} \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$, P and Q take values also along the positive part of \mathfrak{vir} . The twisted self-duality constraint takes the form¹⁰

$$\star P(w) = \mathcal{S}_1(P(w)) = wP(w). \quad (2.46)$$

Under the symmetries of the extended coset space, $P(w)$ only transforms by conjugation with the local $K(E_9)$ transformation defined in (2.42). The shift operator does not commute with $K(E_9)$, but one can see that the commutator is again proportional to (2.46) and hence the twisted self-duality constraint is invariant. In Lorentzian signature $\star^2 P(w) = P(w)$. The right-hand side of (2.46) does not apparently square to P , but taking into account (2.11) and Hermiticity of P it can be easily shown to define a \mathbb{Z}_2 action. On the other hand, (2.46) also *implies* a cascade of duality relations

$$\star^{|m|} P(w) = \mathcal{S}_m(P(w)), \quad m \in \mathbb{Z}. \quad (2.47)$$

We now show that (2.46) is equivalent to the linear system in the triangular $K(E_9)$ gauge of (2.21) and (2.30). Let us first focus on the loop components of the extended Maurer–Cartan form. We define the components of $P(w)$:

$$P(w) = \sum_{m \in \mathbb{Z}} P_A^m T_m^A + \sum_{m \in \mathbb{Z}} P_m L_m, \quad (2.48)$$

where $P_A^0 T_0^A = \mathring{P}$ is the \mathfrak{e}_8 coset element and does not depend on w . Equation (2.47) implies

$$P_A^m = \star^{|m|} P_A^0, \quad (2.49)$$

which in turn gives

$$(\mathrm{d}\mathcal{V}\mathcal{V}^{-1})(w) \Big|_{\text{loop}} = \Gamma(\mathrm{d}\hat{V}\hat{V}^{-1})(w) \Gamma^{-1} = \mathring{Q} + \frac{1+w^{-2}}{1-w^{-2}} \mathring{P} + \frac{2w^{-1}}{1-w^{-2}} \star \mathring{P} \quad (2.50)$$

where it is understood that the denominators are expanded in a geometric series for $|w| > 1$. Because \mathring{P} and \mathring{Q} are w independent, we see that by conjugating this expression with Γ^{-1} , (2.22) is reproduced in a geometric expansion for $|\gamma(w)| > 1$. We now only have to show that the \mathfrak{vir}

¹⁰We stress that we have so far defined $P(w)$ in the spectral parameter representation so that $P(w)|_{\mathbf{K}} = 0$ by construction. The twisted self-duality constraint introduced here must be modified when written in a faithful representation of $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ to take into account the central charge sector. This will be done in Section 4.

components of (2.46) restrict the a priori arbitrary $\gamma(w)$ to solve (2.23). Following the same steps as above we find for the \mathfrak{vir} components of the Maurer–Cartan form

$$d\Gamma^{-1} = -\rho^{-1}d\rho\left(L_0 + 2\sum_{n=1}^{\infty}L_{-2n}\right) - \rho^{-1}\star d\rho\left(2\sum_{n=1}^{\infty}L_{-2n+1}\right) \quad (2.51)$$

$$= \rho^{-1}d\rho \frac{1+w^{-2}}{1-w^{-2}}w\frac{\partial}{\partial w} + \rho^{-1}\star d\rho \frac{2w^{-1}}{1-w^{-2}}w\frac{\partial}{\partial w}. \quad (2.52)$$

Observing that $d(\Gamma^{-1}w) = d\gamma(w) = -\Gamma^{-1}d\Gamma^{-1}w$, we see that it is sufficient to apply the expression above to w , multiply by Γ^{-1} from the left and substitute (2.19) to reproduce (2.23), which concludes our proof. This relation can also be reinterpreted as defining duality equations for all the ϕ_m fields. Expanding $d\Gamma^{-1}$ in the first few orders and using twisted self-duality we find

$$d\phi_1 = 2\star d\rho, \quad d\phi_2 = d(\rho^2), \quad d\phi_3 - \phi_1d\phi_2 = 2\rho^2\star d\rho, \quad \dots \quad (2.53)$$

We see that ϕ_1 is the same as $\tilde{\rho}$ up to a factor of 2 and in fact, contrary to the loop case where all dual potentials are non-locally related to each other, the whole tower of duality relations can be integrated to algebraic expressions in ρ and $\tilde{\rho}$ (or ϕ_1) exclusively, so that the only truly non-local relation is (2.19):

$$\phi_1 = 2\tilde{\rho} - 2s_0, \quad \phi_2 = \rho^2 - 2s_1, \quad \phi_3 = 2\rho^2(\tilde{\rho} - s_0) - 2s_2, \quad \dots \quad (2.54)$$

The integration constants s_i are those appearing in (2.38) and we have set $s_{-1} = 1/2$ using the rigid L_0 symmetry included in (2.43), as was already implied by the parameterisation (2.39). We stress that there is no solution of twisted self-duality such that the \mathfrak{vir}^- scalar fields vanish unless $d\rho = 0$.

We conclude this section by rewriting (2.46) in terms of the $K(E_9)$ invariant and $E_8 \rtimes \mathfrak{Vir}^-$ covariant current

$$J = 2\mathcal{V}^{-1}P\mathcal{V}. \quad (2.55)$$

Dressing (2.46) with \mathcal{V} , we find

$$\star J(w) = \gamma(w)J(w) = \mathcal{S}_1^\gamma(J(w)). \quad (2.56)$$

The operators \mathcal{S}_m^γ act as multiplication by $\gamma(w)^m$. Their definition in a lowest weight representation will be presented in Section 4 and Appendix B.

3 Gauge structure of E_9 exceptional field theory

In this section, we exhibit the E_9 ExFT gauge structures, beginning with a review of E_9 generalised diffeomorphisms and defining a Dorfman structure for E_9 . In Section 3.2 we then introduce a minimal set of vector fields that are needed to covariantise external derivatives, and we define their field strengths. We also define a natural set of transformation properties for these fields in terms of the Dorfman product. However, some modifications to these structures

will be required in order to construct the dynamical theory. As a necessary step to introduce the Virasoro-extended formulation of the theory, in Section 3.3 we extend the definition of generalised diffeomorphisms so that they gauge the larger algebra $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$. The vector field contents and their transformations are extended as well. In Section 5, when considering the minimal formulation of E_9 exceptional field theory, we will amend the (non-extended) gauge transformations of the vector fields and of their field strengths by a different choice of trivial parameters compared to this section, and also introduce extra terms dependent on the external metric $\tilde{g}_{\mu\nu}$.

3.1 Generalised diffeomorphisms and Dorfman structure

The internal space of E_9 exceptional field theory has coordinates Y^M transforming according to the $R(\Lambda_0)_{-1}$ and the generalised Lie derivative was introduced in [17]. It has a gauge parameter Λ^M that lies in the $R(\Lambda_0)_{-1}$ representation of E_9 and an ancillary parameter Σ^M_N lying in the representation $R(\Lambda_0)_0 \otimes \overline{R(\Lambda_0)_{-1}}$. Due to the Fock space structure of the $R(\Lambda_0)_h$ representation introduced in (2.3) we write the parameter Λ^M as a ket vector $|\Lambda\rangle$ and the ancillary parameter Σ^M_N as an operator Σ in this space that also changes the \mathfrak{d} weight. The two notations are related by expanding out the ket vector in a basis and writing $|\Lambda\rangle = \Lambda^M |e_M\rangle$. The coordinates Y^M arise in this way and the derivatives $\langle\partial| = \langle e^M | \partial_M$ transform in the conjugate representation $\overline{R(\Lambda_0)_{-1}}$.

The generalised Lie derivative acting on a vector $|V\rangle$ belonging to $R(\Lambda_0)_{-1}$ is given by

$$\mathcal{L}_{(\Lambda, \Sigma)} |V\rangle = \langle\partial_V | \Lambda\rangle |V\rangle - \eta_{\alpha\beta} \langle\partial_\Lambda | T^\alpha | \Lambda\rangle T^\beta |V\rangle - \langle\partial_\Lambda | \Lambda\rangle |V\rangle - \eta_{-1\alpha\beta} \text{Tr}(T^\alpha \Sigma) T^\beta |V\rangle. \quad (3.1)$$

Here, $\langle\partial|$ is the derivative with respect to the internal coordinates written as a bra vector and the subscript indicates which object the derivative is acting on. Thus, the first term in (3.1) is the transport term, the second the rotation term and the third a weight term. The final term realises the ancillary transformations. In index notation, (3.1) reads

$$\mathcal{L}_{(\Lambda, \Sigma)} V^M = \Lambda^N \partial_N V^M - \partial_N \Lambda^M V^N + Y^{MN}{}_{PQ} \partial_N \Lambda^P V^Q - \eta_{-1\alpha\beta} T^{\alpha P}{}_Q \Sigma^Q{}_P T^{\beta M}{}_N V^N, \quad (3.2)$$

where the Y tensor defining the section constraint in (1.1) is defined as

$$Y^{MN}{}_{PQ} = \delta_N^M \delta_Q^N - \delta_Q^M \delta_P^N - \eta_{\alpha\beta} T^{\alpha M}{}_Q T^{\beta N}{}_P. \quad (3.3)$$

More generally, the action of the generalised Lie derivative on a generic field Φ , with respect to the pair of parameters $\mathbb{A} = (|\Lambda\rangle, \Sigma)$, takes the form

$$\mathcal{L}_{\mathbb{A}} \Phi = \langle\partial_\Phi | \Lambda\rangle \Phi + [\mathbb{A}]_\alpha \delta^\alpha \Phi, \quad (3.4)$$

where we defined the linear combination $[\cdot]_\alpha$ for any pair of parameters

$$[\mathbb{A}]_\alpha \equiv \eta_{\alpha\beta} \langle\partial_\Lambda | T^\beta | \Lambda\rangle + \eta_{-1\alpha\beta} \text{Tr}(T^\beta \Sigma), \quad (3.5)$$

and where δ^α denotes an infinitesimal rigid $\mathfrak{e}_9 \oplus L_{-1}$ variation. We insist that this includes the variation with respect to \mathfrak{d} and not L_0 . For a vector $|V\rangle$ in the $R(\Lambda_0)_{-1}$ representation, as in

(3.1), the weight term that appears then follows from the infinitesimal scaling associated with the action $\mathbf{d}|V\rangle = (L_0 - 1)|V\rangle$.

Covariance of the generalised Lie derivative under rigid $\mathfrak{e}_9 \oplus L_{-1}$ transformations requires the combination $[\Lambda]_\alpha$ to be a projection onto the adjoint representation. This implies that the parameters $|\Lambda\rangle$ and Σ must transform separately as

$$\begin{aligned} X_\alpha \delta^\alpha |\Lambda\rangle &= -X_\alpha T^\alpha |\Lambda\rangle + X_0 |\Lambda\rangle, \\ X_\alpha \delta^\alpha \Sigma &= -X_\alpha [T^\alpha, \Sigma] - X_0 \Sigma + X_{-1} |\Lambda\rangle \langle \partial_\Lambda |, \end{aligned} \quad (3.6)$$

where here by definition

$$X_\alpha T^\alpha = X_0 L_0 + X_K K + \sum_{n \in \mathbb{Z}} X_n^A T_A^n + X_{-1} L_{-1}. \quad (3.7)$$

The last term in (3.6) indicates that the parameters $|\Lambda\rangle$ and Σ transform together in an indecomposable representation under L_{-1} . In the following, the ‘doubled notation’ $\mathbb{A} = (|\Lambda\rangle, \Sigma)$ will be consistently used for other indecomposable pairs of fields.

As is usual in ExFT, the algebra of generalised Lie derivatives closes only when an appropriate (strong) section constraint is fulfilled. The section constraint (1.1) is written explicitly in terms of \mathfrak{e}_9 generators [17]

$$\eta_{\alpha\beta} \langle \partial_1 | T^\alpha \otimes \langle \partial_2 | T^\beta + \langle \partial_1 | \otimes \langle \partial_2 | - \langle \partial_2 | \otimes \langle \partial_1 | = 0, \quad (3.8)$$

where $\langle \partial_1 |$ and $\langle \partial_2 |$ denote two partial derivatives acting on any objects in the theory. It moreover implies that

$$\begin{aligned} \eta_{-k\alpha\beta} \langle \partial_1 | T^\alpha \otimes \langle \partial_2 | T^\beta &= 0, \quad \text{for all } k \in \mathbb{N}^+, \\ \eta_{+1\alpha\beta} \left(\langle \partial_1 | T^\alpha \otimes \langle \partial_2 | T^\beta + \langle \partial_2 | T^\alpha \otimes \langle \partial_1 | T^\beta \right) &= 0. \end{aligned} \quad (3.9)$$

The ancillary parameter Σ is also section constrained [17], in the same way as happens for E_8 [22, 45]. This can be expressed by writing it out in bases as $\Sigma = \Sigma^M_N |e_M\rangle \langle e^N| = |\Sigma\rangle \langle \pi_\Sigma|$, where we have introduced a suggestive notation involving the bra vector $\langle \pi_\Sigma|$.¹¹ The constrained nature of Σ corresponds to replacing either $\langle \partial_1 |$ or $\langle \partial_2 |$ in (3.8) by it. The constraint on Σ also ensures that the trace operation in (3.1) is well-defined even though we are acting on an infinite-dimensional space. As an operator Σ has finite rank.

Any solution of the section constraint can be brought to the following form by an E_9 transformation:

$$\langle \partial | = \langle 0 | \partial_\varphi + \langle 0 | T_{+1}^A \partial_A, \quad (3.10)$$

where ∂_A must satisfy the section constraint of E_8 exceptional field theory [22].

The generalised Lie derivative closes according to

$$[\mathcal{L}_{\mathbb{A}_1}, \mathcal{L}_{\mathbb{A}_2}] = \mathcal{L}_{[\mathbb{A}_1, \mathbb{A}_2]_E}, \quad (3.11)$$

¹¹ Σ is not factorised, but the notation $|\Sigma\rangle \langle \pi_\Sigma|$ helps to describe the section constraint for Σ .

with $\mathbb{A}_1 = (|\Lambda_1\rangle, \Sigma_1)$ and $\mathbb{A}_2 = (|\Lambda_2\rangle, \Sigma_2)$, and where the exceptional Lie bracket is given by

$$[\mathbb{A}_1, \mathbb{A}_2]_E = (|\Lambda_{12}\rangle, \Sigma_{12}) \quad (3.12)$$

with [17]

$$\begin{aligned} |\Lambda_{12}\rangle &= \frac{1}{2} (\mathcal{L}_{(\Lambda_1,0)}|\Lambda_2\rangle - \mathcal{L}_{(\Lambda_2,0)}|\Lambda_1\rangle) , \\ \Sigma_{12} &= \mathcal{L}_{(\Lambda_1,0)}\Sigma_2 - \frac{1}{2}\eta_{-1\alpha\beta}\text{Tr}(T^\alpha\Sigma_1)T^\beta\Sigma_2 \\ &\quad - \frac{1}{4}\eta_{1\alpha\beta}\left(\langle\partial_{\Lambda_2}|T^\alpha|\Lambda_2\rangle T^\beta|\Lambda_1\rangle - \langle\partial_{\Lambda_2}|T^\alpha|\Lambda_1\rangle T^\beta|\Lambda_2\rangle\right)\langle\partial_{\Lambda_2}| - (1 \leftrightarrow 2) , \end{aligned} \quad (3.13)$$

The action of the generalised Lie derivative on a pair of parameters is given by

$$\begin{aligned} \mathcal{L}_{\mathbb{A}_1}\mathbb{A}_2 &= \left(\mathcal{L}_{\Lambda_1}|\Lambda_2\rangle , \mathcal{L}_{(\Lambda_1,0)}\Sigma_2 - \eta_{-1\alpha\beta}\text{Tr}(T^\alpha\Sigma_1)\left(T^\beta\Sigma_2 - \Sigma_2T^\beta\right) - \text{Tr}(\Sigma_1)|\Lambda_2\rangle\langle\partial_{\Lambda_2}| \right) \\ &= \left(\mathcal{L}_{\Lambda_1}|\Lambda_2\rangle , \langle\partial_{\Sigma_2}|\Lambda_1\rangle\Sigma_2 - \eta_{\alpha\beta}\langle\partial_{\Lambda_1}|T^\alpha|\Lambda_1\rangle T^\beta\Sigma_2 + \Sigma_2|\Lambda_1\rangle\langle\partial_{\Lambda_1}| \right. \\ &\quad \left. - \eta_{-1\alpha\beta}\text{Tr}(T^\alpha\Sigma_1)T^\beta\Sigma_2 - \text{Tr}(\Sigma_1)|\Lambda_2\rangle\langle\partial_{\Lambda_2}| \right) , \end{aligned} \quad (3.14)$$

where the term $\text{Tr}(\Sigma_1)|\Lambda_2\rangle\langle\partial_{\Lambda_2}|$ is due to the indecomposable structure of the pair \mathbb{A}_2 under L_{-1} transformations. In the second step of (3.14) we have used the section constraint to simplify a few terms.

In order to bring out the covariance properties of the gauge fields and field strengths in the next section, it is convenient to introduce a generalised Dorfman structure [46]. This is defined via the following (non-commutative and non-associative) product which involves exclusively pairs of parameters

$$\begin{aligned} \mathbb{A}_1 \circ \mathbb{A}_2 &= \left(\mathcal{L}_{\Lambda_1}\Lambda_2 , \mathcal{L}_{\Lambda_1}\Sigma_2 + \eta_{1\alpha\beta}\langle\partial_{\Lambda_1}|T^\alpha|\Lambda_1\rangle T^\beta|\Lambda_2\rangle\langle\partial_{\Lambda_1}| \right. \\ &\quad \left. + \eta_{\alpha\beta}\text{Tr}(T^\alpha\Sigma_1)T^\beta|\Lambda_2\rangle\langle\partial_{\Sigma_1}| - |\Lambda_2\rangle\langle\partial_{\Sigma_1}|\Sigma_1 \right) . \end{aligned} \quad (3.15)$$

A key property of the Dorfman product, which is not satisfied by the generalised Lie derivative, is that it obeys the Leibniz property

$$\mathbb{A}_1 \circ (\mathbb{A}_2 \circ \mathbb{A}_3) = (\mathbb{A}_1 \circ \mathbb{A}_2) \circ \mathbb{A}_3 + \mathbb{A}_2 \circ (\mathbb{A}_1 \circ \mathbb{A}_3) . \quad (3.16)$$

This relation can be proved by showing that the Dorfman product closes according to the anti-symmetric Dorfman bracket $[\mathbb{A}_1, \mathbb{A}_2]_D \equiv \frac{1}{2}(\mathbb{A}_1 \circ \mathbb{A}_2 - \mathbb{A}_2 \circ \mathbb{A}_1)$, and that the symmetric bracket $\{\mathbb{A}_1, \mathbb{A}_2\} \equiv \frac{1}{2}(\mathbb{A}_1 \circ \mathbb{A}_2 + \mathbb{A}_2 \circ \mathbb{A}_1)$ is trivial (with respect to \circ itself). These properties respectively correspond to the projections of (3.16) onto its antisymmetric and symmetric parts in \mathbb{A}_1 and \mathbb{A}_2 , and are discussed in more detail in Appendix A. Let us simply note here that, on the first entry of a pair of parameters, they imply that under the generalised Lie derivative

$$\mathcal{L}_{[\mathbb{A}_1, \mathbb{A}_2]_D} = \mathcal{L}_{[\mathbb{A}_1, \mathbb{A}_2]_E} , \quad \mathcal{L}_{\{\mathbb{A}_1, \mathbb{A}_2\}} = 0 , \quad (3.17)$$

where the E-bracket was defined in (3.13).

3.2 Covariant derivatives and field strengths

Using the Dorfman structure we can conveniently deduce the form and properties of the covariant field strengths associated to a pair of ‘vector fields’

$$\mathbb{A}_\mu = (A_\mu^M, B_\mu^M{}_N) = (|A_\mu\rangle, B_\mu). \quad (3.18)$$

The first component $|A_\mu\rangle$ is the usual vector field in two external dimensions, and it gauges the $|\Lambda\rangle$ -diffeomorphisms. It is thus valued in the basic representation of E_9 and it carries weight -1 under \mathfrak{d} . Its second component partner B_μ is associated to Σ -diffeomorphisms and is therefore constrained on its lower index and carries weight $+1$. This means the two components of \mathbb{A}_μ behave as their respective parameters under an infinitesimal $\mathfrak{e}_9 \oplus L_{-1}$ variation (3.6) and, in particular, also transform in an indecomposable representation under L_{-1} .

We define a covariant derivative as

$$\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{\mathbb{A}_\mu}. \quad (3.19)$$

As usual, its covariance under generalised diffeomorphisms is ensured if, acting on any vector V^M , one has

$$\delta_\Lambda \mathcal{D}_\mu V^M = \mathcal{L}_\Lambda \mathcal{D}_\mu V^M, \quad (3.20)$$

where $\Lambda = (|\Lambda\rangle, \Sigma)$. Using (3.11), we find that this can be achieved by the transformation

$$\begin{aligned} \delta_\Lambda \mathbb{A}_\mu &= \partial_\mu \Lambda - [\mathbb{A}_\mu, \Lambda]_E && + \text{trivial parameters} \\ &= \partial_\mu \Lambda - \mathbb{A}_\mu \circ \Lambda && + \text{trivial parameters} \\ &= \partial_\mu \Lambda + \Lambda \circ \mathbb{A}_\mu && + \text{trivial parameters,} \end{aligned} \quad (3.21)$$

in terms of the E-bracket (3.13) and the Dorfman product (3.15). We have given three equivalent ways of writing the transformation and they differ by trivial parameters according to (3.17). This is consistent since the transformation of \mathbb{A}_μ is only defined up to trivial parameters, as \mathbb{A}_μ only appears as the parameter of a generalised Lie derivative in (3.20).

In exceptional field theory, one typically chooses to work with the second transformation in (3.21) (without trivial parameters), which can be written in more compact form as

$$\delta_\Lambda \mathbb{A}_\mu = \mathbb{D}_\mu \Lambda, \quad (3.22)$$

in terms of the Dorfman covariant derivative

$$\mathbb{D}_\mu = \partial_\mu - \mathbb{A}_\mu \circ. \quad (3.23)$$

However, for the purpose of Section 5, we will consider in this paper the third version of the transformation in (3.21) (without trivial parameters). For the components of \mathbb{A}_μ we then get explicitly

$$\delta_\Lambda |A_\mu\rangle = \partial_\mu |\Lambda\rangle + \mathcal{L}_\Lambda |A_\mu\rangle, \quad (3.24)$$

and

$$\begin{aligned} \delta_{\wedge} B_{\mu} &= \partial_{\mu} \Sigma + \mathcal{L}_{\wedge} B_{\mu} \\ &+ \eta_{1\alpha\beta} \langle \partial_{\Lambda} | T^{\alpha} | \Lambda \rangle T^{\beta} | A_{\mu} \rangle \langle \partial_{\Lambda} | + \eta_{\alpha\beta} \text{Tr}(T^{\alpha} \Sigma) T^{\beta} | A_{\mu} \rangle \langle \partial_{\Sigma} | - | A_{\mu} \rangle \langle \partial_{\Sigma} | \Sigma, \end{aligned} \quad (3.25)$$

that we have written without displaying internal space indices. We also recall that the second term contains a term of the form $\text{Tr}(\Sigma) | A_{\mu} \rangle \langle \partial_A |$ which is due to the indecomposable structure of the pair of gauge fields \mathbb{A}_{μ} .

Using the covariant derivative (3.19) or its Dorfman version (3.23), we can introduce a pair of field strengths

$$\mathbb{F}_{\mu\nu} = (\mathcal{F}_{\mu\nu}^M, \mathcal{G}_{\mu\nu}^M) = (|\mathcal{F}_{\mu\nu}\rangle, \mathcal{G}_{\mu\nu}) \quad (3.26)$$

by

$$[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}] = -\mathcal{L}_{\mathbb{F}_{\mu\nu}}, \quad [\mathbb{D}_{\mu}, \mathbb{D}_{\nu}] = -\mathbb{F}_{\mu\nu} \circ, \quad (3.27)$$

where the former expression is implied by the latter. Note that these relations only define the field strengths up to trivial parameters. Under rigid $\mathfrak{e}_9 \oplus L_{-1}$ transformations the pair of field strengths behaves as the pairs of gauge parameters and gauge fields, namely is transforms in an indecomposable representation like in (3.6). As is customary in exceptional field theory, covariance of the field strengths under generalised diffeomorphisms requires us to introduce higher (external) forms which enter in their expressions as trivial parameters. Once again, for the purpose of Section 5, we can choose to write

$$\mathbb{F}_{\mu\nu} = 2\partial_{[\mu} \mathbb{A}_{\nu]} - [\mathbb{A}_{\mu}, \mathbb{A}_{\nu}]_E + \varpi \mathbf{C}_{\mu\nu}, \quad (3.28)$$

where $\mathbf{C}_{\mu\nu}$ denotes the set of two-forms defined in (A.24), while $\varpi \mathbf{C}_{\mu\nu}$ is the Dorfman doublet of associated trivial parameter using (A.11). The parts of the field strength components that are independent of the two-forms will be denoted by $|F_{\mu\nu}\rangle$ and $G_{\mu\nu}$. In form notation, their expressions follow from (3.13) and thus read¹²

$$\begin{aligned} |F\rangle &= d|A\rangle - \frac{1}{2} \langle \partial_A | A' \rangle |A\rangle + \frac{1}{2} \eta_{\alpha\beta} \langle \partial_A | T^{\alpha} | A \rangle T^{\beta} | A' \rangle + \frac{1}{2} \langle \partial_A | A \rangle |A'\rangle, \\ G &= dB - \langle \partial_B | A \rangle B + \eta_{\alpha\beta} \langle \partial_A | T^{\alpha} | A \rangle T^{\beta} B + B | A \rangle \langle \partial_A | + \frac{1}{2} \eta_{-1\alpha\beta} \text{Tr}(T^{\alpha} B) T^{\beta} B \\ &- \frac{1}{4} \eta_{1\alpha\beta} \langle \partial_A | T^{\alpha} | A \rangle T^{\beta} | A' \rangle \langle \partial_A | - \frac{1}{4} \eta_{1\alpha\beta} \langle \partial_A | T^{\alpha} | A' \rangle T^{\beta} | A \rangle \langle \partial_A |, \end{aligned} \quad (3.29)$$

The explicit dependence on two-forms in (3.28) as well as their transformations under generalised diffeomorphisms are given in detail in Appendix A, and are tuned to cancel the non-covariant variations of (3.29). This ultimately ensures that the field strengths transform covariantly as

$$\delta_{\wedge} \mathbb{F} = \wedge \circ \mathbb{F}. \quad (3.30)$$

¹²We use the notation $|A'\rangle$ to distinguish the vector field that is not derived by the bra $\langle \partial_A |$, so that $\langle \partial_A | \otimes |A\rangle \otimes |A'\rangle$ means in components $\partial_M A^N A^P$.

We stress that these are not the final transformation properties for the field strengths of E_9 exceptional field theory, for several reasons. In the extended formulation of Section 4, an extended \circ product is necessary to capture the gauge transformations of a larger set of vector fields. This is explained in the next subsection. In the minimal formulation of Section 5, the transformation properties of both the B field and its field strength \mathcal{G} are modified by extra terms involving the external unimodular metric $\tilde{g}_{\mu\nu}$. Furthermore, both formulations of E_9 exceptional field theory imply an Euler–Lagrange equation for $|\mathcal{F}\rangle$ that appears with a projection. This projected equation is gauge-invariant by construction. If one wants to extend the duality equation to be unprojected and gauge-invariant, the gauge transformation of \mathbb{F} must be modified by trivial parameters. This applies to both formulations, and we shall see explicitly in the minimal formulation in Section 5 that external diffeomorphism invariance requires the inclusion of the unprojected duality equation.

We finish this section by presenting a set of Bianchi identities. Using (3.27) to formally evaluate the action of three antisymmetrised Dorfman covariant derivative on a generic pair $\mathbb{V} = (|V\rangle, V)$ leads to the relation $\mathbb{D}(\mathbb{F} \circ \mathbb{V}) = \mathbb{F} \circ \mathbb{D}\mathbb{V}$, which means that

$$(\mathbb{D}\mathbb{F}) \circ \mathbb{V} = 0. \quad (3.31)$$

On the first component of \mathbb{V} this relation reduces to $\mathcal{L}_{\mathbb{D}\mathbb{F}}|V\rangle = 0$, and implies the following two identities

$$\langle \partial | \mathcal{D}\mathcal{F} \rangle = 0, \quad [\mathbb{D}\mathbb{F}]_\alpha = 0, \quad (3.32)$$

where the projection in the second equation was defined in (3.5).

3.3 Extended gauge structure

The extension of the group theoretical structure discussed in Section 2.3 in the context of two-dimensional supergravity suggests the existence of an extended gauge structure for exceptional field theory. In this section, we show that one can indeed consistently enlarge the algebra that is gauged by generalised diffeomorphisms to $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$. Many of the identities and results of the previous section still hold in this extended setting and we will allow ourselves to be more schematic.

We start by considering an extension of the generalised Lie derivative (3.4) in which the rotation term now involves an infinitesimal $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$ variation. The set of gauge parameters is thus also enlarged, and now consists of

$$\mathbb{A} = (|\Lambda\rangle, \Sigma^{(k)}) \quad k \in \mathbb{N}^+. \quad (3.33)$$

On a vector $|V\rangle$ that lies in $R(\Lambda_0)_{-1}$, we have

$$\begin{aligned} \mathcal{L}_{\mathbb{A}}|V\rangle &= \langle \partial_V | \Lambda \rangle |V\rangle + [\mathbb{A}]_\alpha \delta^\alpha |V\rangle \\ &= \langle \partial_V | \Lambda \rangle |V\rangle - \eta_{\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle T^\beta |V\rangle - \langle \partial_\Lambda | \Lambda \rangle |V\rangle - \sum_{k=1}^{\infty} \eta_{-k} \alpha\beta \text{Tr}(T^\alpha \Sigma^{(k)}) T^\beta |V\rangle, \end{aligned} \quad (3.34)$$

where the parameter $\Sigma^{(1)}$ corresponds to Σ in the previous section, and where we have defined $[\Lambda]_\alpha$ as the natural extension of the projection (3.5) onto $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$

$$[\Lambda]_\alpha \equiv \eta_{\alpha\beta} \langle \partial_\Lambda | T^\beta | \Lambda \rangle + \sum_{k=1}^{\infty} \eta_{-k\alpha\beta} \text{Tr}(T^\beta \Sigma^{(k)}). \quad (3.35)$$

We recall that the weight term in (3.34) is due to the fact that δ^α includes a variation with respect to \mathfrak{d} and not L_0 . Acting on a generic field, we use (3.4) together with (3.35).

In order for the projection $[\Lambda]_\alpha$ to transform in the adjoint, the set of gauge parameters must transform under rigid $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$ variations as

$$\begin{aligned} X_\alpha \delta^\alpha | \Lambda \rangle &= -X_\alpha T^\alpha | \Lambda \rangle + X_0 | \Lambda \rangle, \\ X_\alpha \delta^\alpha \Sigma^{(k)} &= -X_\alpha [T^\alpha, \Sigma^{(k)}] + \sum_{0 \leq p < k} (2p - k) X_{-p} \Sigma^{(k-p)} + k X_{-k} | \Lambda \rangle \langle \partial_\Lambda |, \end{aligned} \quad (3.36)$$

for all $k \in \mathbb{N}^+$, where

$$X_\alpha T^\alpha = \sum_{p=0}^{\infty} X_{-p} L_{-p} + X_K K + \sum_{n \in \mathbb{Z}} X_n^A T_A^n. \quad (3.37)$$

The parameters $\Sigma^{(k)}$ thus carry weight k under \mathfrak{d} . Note that, similarly to the unextended case, the gauge parameters $\Lambda = (|\Lambda\rangle, \Sigma^{(k)})$ transform in an indecomposable representation under L_{-p} for all $p > 0$. Let us also point out that a further extension of generalised diffeomorphisms to include gaugings of the full Virasoro algebra would not be compatible with the section constraints. The set of constraints (3.8) and (3.9) are indeed invariant under \mathfrak{vir}^- transformations, but not under any of the transformations generated L_k with $k > 0$, for instance an infinitesimal L_1 transformation applied to (3.8) maps $\eta_{\alpha\beta}$ to $\eta_{1\alpha\beta}$, but the symmetrisation in (3.9) is not reproduced.

The generalised Lie derivative (3.34) still closes as in (3.11), but according to an extended E-bracket $[\Lambda_1, \Lambda_2]_E = (|\Lambda_{12}\rangle, \Sigma_{12}^{(k)})$ which reads explicitly

$$\begin{aligned} |\Lambda_{12}\rangle &= \frac{1}{2} (\mathcal{L}_{(\Lambda_1,0)} |\Lambda_2\rangle - \mathcal{L}_{(\Lambda_2,0)} |\Lambda_1\rangle), \\ \Sigma_{12}^{(k)} &= \mathcal{L}_{(\Lambda_1,0)} \Sigma_2^{(k)} - \frac{1}{2} \sum_{p=1}^{\infty} \eta_{-p\alpha\beta} \text{Tr}(T^\alpha \Sigma_1^{(p)}) T^\beta \Sigma_2^{(k)} - \frac{1}{2} \sum_{0 < p < k} (2p - k) \text{Tr}(\Sigma_1^{(p)}) \Sigma_2^{(k-p)} \\ &\quad - \frac{1}{4} \delta_1^k \eta_{1\alpha\beta} \left(\langle \partial_{\Lambda_2} | T^\alpha | \Lambda_2 \rangle T^\beta | \Lambda_1 \rangle - \langle \partial_{\Lambda_2} | T^\alpha | \Lambda_1 \rangle T^\beta | \Lambda_2 \rangle \right) \langle \partial_{\Lambda_2} | - (1 \leftrightarrow 2). \end{aligned} \quad (3.38)$$

The proof of closure only relies on the use of the section constraints (3.8) and (3.9) which were already necessary for consistency of the unextended gauge algebra, and is presented in Appendix A. Note that $\Sigma_{12}^{(k)}$ is still correctly on section.

The gauge parameters $\Sigma^{(k)}$ are highly degenerate. The only new gauge transformations introduced by $\Sigma^{(k)}$ with $k \geq 2$ are those generated by L_{-k} with $k > 1$, associated to $\text{Tr}(\Sigma^{(k)})$ and acting as shifts on the dual potentials ϕ_k introduced in Section 2.3. The $\hat{\mathfrak{e}}_8$ transformations

generated by $\Sigma^{(k)}$ can all be reabsorbed into $\Sigma^{(1)}$ up to trivial parameters. To see this, we take the Σ components of (3.35) and apply (2.9) to write

$$\sum_{k=1}^{\infty} \eta_{-k} \alpha\beta \text{Tr}(T^\alpha \Sigma^{(k)}) T^\beta = \sum_{k=1}^{\infty} \left(\eta_{\alpha\beta} \text{Tr}(\Sigma^{(k)} T^\alpha) \mathcal{S}_{-k}(T^\beta) - \text{Tr}(\Sigma^{(k)} L_{-k}) \mathbf{K} \right). \quad (3.39)$$

By rigid E_9 covariance we can assume that the $\Sigma^{(k)}$ solve the section constraint as in (3.10). Substituting into the above expression and focussing on the gauging of loop generators, one finds that $\Sigma^{(1)}$ gauges some \mathfrak{e}_8 generators T_0^A , a larger set of T_{-1}^A generators, and all T_{-n}^A for $n \geq 2$. Higher $\Sigma^{(k)}$ then gauge the same algebra shifted by $1 - k$ along the negative loop levels. Hence, each $\Sigma^{(k)}$ for $k \geq 2$ gauges a loop subalgebra of the one gauged by $\Sigma^{(1)}$. The only new generators gauged by $\Sigma^{(k)}$ for $k \geq 2$ are then L_{-k} , associated with $\text{Tr}(\Sigma^{(k)})$. Rigid E_9 covariance of $[\mathbb{A}]_\alpha$ then guarantees that this conclusion holds for any other solution of the section constraint.

In order to simplify the definition and transformation properties of the vector fields and their field strengths, we introduce a \circ product naturally extending (3.15)

$$\begin{aligned} \mathbb{A}_1 \circ \mathbb{A}_2 = & \left(\mathcal{L}_{\mathbb{A}_1} \mathbb{A}_2, \mathcal{L}_{\mathbb{A}_1} \Sigma_2^{(k)} + \delta_1^k \eta_{1\alpha\beta} \langle \partial_{\Lambda_1} | T^\alpha | \Lambda_1 \rangle T^\beta | \Lambda_2 \rangle \langle \partial_{\Lambda_1} | + \eta_{\alpha\beta} \text{Tr}(T^\alpha \Sigma_1^{(k)}) T^\beta | \Lambda_2 \rangle \langle \partial_{\Sigma_1} | \right. \\ & \left. - | \Lambda_2 \rangle \langle \partial_{\Sigma_1} | \Sigma_1^{(k)} - (k-1) \left(\text{Tr}(\Sigma_1^{(k)}) | \Lambda_2 \rangle \langle \partial_{\Sigma_1} | - \Sigma_1^{(k)} \langle \partial_{\Sigma_1} | \Lambda_2 \rangle \right) \right). \end{aligned} \quad (3.40)$$

This expression satisfies (3.17), meaning that it is equivalent to the extended E-bracket (3.38) up to the addition of trivial parameters. However, this extended \circ product does not satisfy the Leibniz identity (3.16). More precisely, it fails to satisfy it by a trivial parameter. We can nevertheless use it to define the transformation properties of the vector fields and their field strengths in direct analogy with Section 3.2. We then introduce additional constrained gauge fields $B_\mu^{(k)}$, and their field strengths $\mathcal{G}_{\mu\nu}^{(k)}$, associated to the new generalised diffeomorphisms generated by L_{-k} . Our notations are thus naturally extended as follows

$$\mathbb{A}_\mu = (|A_\mu\rangle, B_\mu^{(k)}), \quad \mathbb{F}_{\mu\nu} = (|\mathcal{F}_{\mu\nu}\rangle, \mathcal{G}_{\mu\nu}^{(k)}), \quad (3.41)$$

with $k \in \mathbb{N}^+$. Under rigid $\hat{\mathfrak{e}}_9 \oplus \mathfrak{vir}^-$ variations these sets of gauge fields and field strengths form indecomposable pairs and transform as the set of parameters $\mathbb{A} = (|\Lambda\rangle, \Sigma^{(k)})$ in (3.36). The extended gauge connection still transforms according to (3.21) and the field strengths are defined according to the first expression in (3.27) which we reproduce here:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\mathcal{L}_{\mathbb{F}_{\mu\nu}}. \quad (3.42)$$

The second expression in (3.27) now only holds up to a trivial parameter. The extended field strengths contain a trivial combination of two-form fields as discussed in the previous section, but now including the novel extended trivial parameters described in Appendix A.4. Then, combining (3.11) with (3.42) we find that the transformation property (3.30) still holds in the extended setting, with any trivial parameter that may arise on the right-hand side absorbed into the two-form transformation.¹³ Finally, the Bianchi identities (3.32) still hold in the extended

¹³As stressed after (3.30) and discussed in Section 5, invariance under external diffeomorphisms requires to introduce an unprojected duality equation for $|\mathcal{F}\rangle$ which in turn requires to amend (3.30) by a trivial parameter. We will not discuss external diffeomorphisms in the extended formulation, so (3.30) will suffice in this setting.

setting, as they can be also deduced by formally evaluating the action of three \mathcal{D} differentials on a covariant field.

4 Virasoro-extended E_9 exceptional field theory

We are now equipped to introduce the field content of E_9 exceptional field theory and build a pseudo-Lagrangian to determine its dynamics. The approach we follow in this section is based on the formalism introduced in Section 2.3. In particular, the twisted self-duality constraint will be the natural covariantisation of (2.46) and (2.56). This makes the connection to two-dimensional supergravity and to the linear system straightforward, and has the further advantage that scalar field currents covariant under generalised diffeomorphisms can be naturally defined in terms of a gauge connection \mathbb{A} transforming according to (3.21) and (3.40). We will however only develop this formalism in the conformal gauge $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$. We refer to this approach as ‘extended’ because of the presence of the \mathfrak{vir}^- scalar fields ϕ_n , their associated symmetries and gauge fields $B^{(n)}$. The ‘minimal’ approach to E_9 exceptional field theory and its dynamics is introduced in Section 5.

In this section we always leave wedge products as understood, writing for instance $\mathfrak{J}\mathfrak{J}$ in place of $\mathfrak{J} \wedge \mathfrak{J}$.

4.1 Covariant currents and twisted self-duality

We begin by introducing the $(\widehat{E}_8 \rtimes \mathfrak{Vir}^-)/K(E_9)$ coset representative

$$\mathcal{V} = \Gamma \hat{V} = \rho^{-L_0} e^{-\phi_1 L_{-1}} e^{-\phi_2 L_{-2}} \dots \overset{\circ}{V} e^{Y_{1A} T_{-1}^A} e^{Y_{2A} T_{-2}^A} \dots e^{-\sigma K} \quad (4.1)$$

where we work now in a faithful representation of the algebra so that $K \neq 0$. We will regard $e^{-\sigma K}$ as contained in \hat{V} .

These expressions involve products of infinite products. However, the coefficients of any generator T_{-m}^A or L_{-m} in the Maurer–Cartan form are finite expressions that can be computed by truncating the infinite products to the first m factors. We then define the generalised metric (of weight 0)¹⁴

$$\mathcal{M} = \mathcal{V}^\dagger \mathcal{V}, \quad (4.2)$$

that transforms under $g \in \widehat{E}_8 \rtimes \mathfrak{Vir}^-$ as

$$\mathcal{M} \rightarrow g^\dagger \mathcal{M} g. \quad (4.3)$$

¹⁴Because Hermitian conjugation does not map \mathfrak{Vir}^- nor the completed loop group to themselves, expressions like \mathcal{M} need qualification. One way is to first identify the $K(E_9)$ singlet element in $\overline{R(\Lambda_0)_0} \otimes_{\text{sym}} \overline{R(\Lambda_0)_0}$, which we denote as $\underline{\Delta}_{MN}$ (the underlined indices transform under the local $K(E_9)$ symmetry), and then regard the generalised metric as a field dependent element of $\overline{R(\Lambda_0)_0} \otimes_{\text{sym}} \overline{R(\Lambda_0)_0}$, which we write as $\mathcal{M}_{MN} = \underline{\Delta}_{PQ} \mathcal{V}_M^P \mathcal{V}_N^Q$. However, it will turn out to be more convenient to treat \mathcal{M} as a group element. All manipulations we encounter are then justified by switching to the Unendlichein formulation in terms of the Hermitian and anti-Hermitian projections \mathcal{P} and \mathcal{Q} of the Maurer–Cartan form, see (4.26) and Section 4.4. We provide more details on the definitions of the group E_9 and its representations in Appendix F.

We will always take \mathcal{V} and \mathcal{M} in the $R(\Lambda_0)_0$ representation and write explicitly the ρ factors when they act on representations of non-zero conformal weight h .

Taking an internal derivative of \mathcal{M} we define the internal current

$$\mathcal{J}_{M\alpha}T^\alpha = \mathcal{M}^{-1}\partial_M\mathcal{M}, \quad (4.4)$$

which satisfies the section constraint along the index M . Translating this index to bracket notation we shall write $\mathcal{J}_{M\alpha}T^\alpha\langle e^M| = \langle \mathcal{J}_\alpha| \otimes T^\alpha$ with $\langle e^M|$ a basis for $\overline{R(\Lambda_0)_{-1}}$. Under generalised diffeomorphisms \mathcal{M} transforms covariantly, namely $\delta_\Lambda\mathcal{M} = \mathcal{L}_\Lambda\mathcal{M}$ with

$$\begin{aligned} \mathcal{M}^{-1}\mathcal{L}_\Lambda\mathcal{M} &= \langle \mathcal{J}_\alpha|\Lambda\rangle T^\alpha + \left[\eta_{\alpha\beta}\langle\partial_\Lambda|T^\beta|\Lambda\rangle + \sum_{r=1}^{\infty}\eta_{-r\alpha\beta}\text{Tr}[\Sigma^{(r)}T^\beta] \right] \left(T^\alpha + \mathcal{M}^{-1}(T^\alpha)^\dagger\mathcal{M} \right), \\ &= \langle \mathcal{J}_\alpha|\Lambda\rangle T^\alpha + [\Lambda]_\alpha \left(T^\alpha + \mathcal{M}^{-1}(T^\alpha)^\dagger\mathcal{M} \right), \end{aligned} \quad (4.5)$$

Notice that compared to [1], we now include the action of infinitely many Virasoro generators in both the gauge parameters (through $\text{Tr}(\Sigma^{(m)})$) and in the conjugation by \mathcal{M} . In the second line we have used the shortcut expression $[\Lambda]_\alpha$ introduced in (3.35). We can then define the external current that is covariant under rigid $\widehat{E}_8 \times \text{Vir}^-$ transformations as well as generalised diffeomorphisms

$$\mathfrak{J} = \mathcal{M}^{-1}\mathcal{D}\mathcal{M} = \mathcal{M}^{-1}d\mathcal{M} - \langle \mathcal{J}_\alpha|\Lambda\rangle T^\alpha + [\Lambda]_\alpha \left(T^\alpha + \mathcal{M}^{-1}(T^\alpha)^\dagger\mathcal{M} \right). \quad (4.6)$$

We expand the algebra components of \mathfrak{J} as follows:

$$\mathfrak{J} = \mathfrak{J}_\alpha T^\alpha = \sum_{m \in \mathbb{Z}} (J_A^m T_m^A + J_m L_m) + J_K K. \quad (4.7)$$

The twisted self-duality constraint of exceptional field theory will essentially be the covariantisation of the two-dimensional expression (2.56), but we now need to take into account the K component of the current which was trivially represented in Section 2.3. The shift operators \mathcal{S}_m and \mathcal{S}_m^γ introduced in (2.8) and at the end of Section 2.3 can be interpreted as multiplication by w^m and $\gamma(w)^m$ respectively, and we can thus relate them by Γ conjugation using (2.39):

$$\mathcal{S}_m^\gamma(X) = \mathcal{S}_0(\Gamma^{-1}\mathcal{S}_m(\Gamma X\Gamma^{-1})\Gamma) \quad (4.8)$$

for any $X \in \widehat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$. Any \mathcal{S}_m^γ can be expanded as a series in \mathcal{S}_k operators, with $k \leq m$ and ρ and ϕ_m dependent coefficients. For instance, (2.39) is reproduced as

$$\mathcal{S}_1^\gamma = \frac{1}{\rho} \left[\mathcal{S}_1 - \phi_1 \mathcal{S}_0 - \phi_2 \mathcal{S}_{-1} - \phi_3 \mathcal{S}_{-2} - \left(\phi_4 + \frac{1}{2}\phi_2^2 \right) \mathcal{S}_{-3} + \dots \right]. \quad (4.9)$$

The \mathcal{S}_0 operator on the right-hand side of (4.8) is only necessary to remove any spurious K component that might be generated by Γ conjugation because of the \mathfrak{vir} central charge. Several properties of these Vir^- field dependent shift operators \mathcal{S}_m^γ are described in Appendix B. In particular, notice that because of the factors of ρ generated by Γ conjugation all \mathcal{S}_m^γ operators carry weight 0, whereas \mathcal{S}_m carry weight m .

Consider now the shift $\mathcal{S}_m^\gamma(\mathfrak{J})$ of the covariant current. The transformation properties of \mathfrak{J} and Γ imply that under a rigid transformation $X \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$

$$\delta_X \mathcal{S}_m^\gamma(\mathfrak{J}) = -[X, \mathcal{S}_m^\gamma(\mathfrak{J})] - \omega^{\alpha\beta} X_\alpha (\mathcal{S}_m^\gamma(\mathfrak{J}))_\beta \mathbf{K}, \quad (4.10)$$

where we are varying both \mathfrak{J} and the Γ implicit in the definition of the shift operator. The expression $\omega^{\alpha\beta}$ is the algebra cocycle corresponding to (minus) the central charge component of the $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ commutator: given $X, Z \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ with components

$$X = X_\alpha T^\alpha = \sum_{m \in \mathbb{Z}} (X_A^m T_m^A + X_m L_m) + X_{\mathbf{K}} \mathbf{K}, \quad (4.11)$$

and similarly for Z , we define

$$\omega^{\alpha\beta} X_\alpha Z_\beta = -[X, Z]|_{\mathbf{K}} = -\eta^{AB} \sum_{n \in \mathbb{Z}} n X_A^n Z_B^{-n} - \frac{c_{\mathfrak{vir}}}{12} \sum_{n \in \mathbb{Z}} (n^3 - n) X_n Z_{-n}. \quad (4.12)$$

To compensate for this cocycle term in (4.10) we follow the same strategy as in [1] and define covariant shifted currents

$$\mathfrak{J}^{(m)} = \mathcal{S}_m^\gamma(\mathfrak{J}) + \chi_m^\gamma \mathbf{K}, \quad (4.13)$$

where the new one-form fields χ_m^γ satisfy the indecomposable transformation property

$$\delta_X \chi_m^\gamma = \omega^{\alpha\beta} X_\alpha \mathfrak{J}_\beta^{(m)}, \quad (4.14)$$

where we have used $\omega^{\alpha\mathbf{K}} = 0$ to write $\mathfrak{J}^{(m)}$ in place of $\mathcal{S}_m^\gamma(\mathfrak{J})$. This transformation property guarantees that the shifted current transforms as an algebra element (of $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$) under $X \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$:

$$\delta_X \mathfrak{J}^{(m)} = -[X, \mathfrak{J}^{(m)}]. \quad (4.15)$$

More generally, for any $Z \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ transforming as an algebra element under $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$, we may define a \mathbf{K} completion \widehat{Z}_m^γ of $\mathcal{S}_m^\gamma(Z)$ such that

$$Z^{(m)} = \mathcal{S}_m^\gamma(Z) + \widehat{Z}_m^\gamma \mathbf{K}, \quad \delta_X Z^{(m)} = -[X, Z^{(m)}]. \quad (4.16)$$

When the \mathbf{K} completion of an object can be expressed entirely in terms of other fields of the theory, we will use the notation introduced above. If the \mathbf{K} completion is an independent field, we use a new symbol for it, as we did for \mathfrak{J} and χ_m^γ .

We define the χ_m^γ fields to be covariant of weight 0 under generalised diffeomorphisms. Namely, their transformation descends from (4.14):

$$\delta_\Lambda \chi_m^\gamma = \langle \partial_\chi | \Lambda \rangle \chi_m^\gamma + \omega^{\alpha\beta} [\Lambda]_\alpha \mathfrak{J}_\beta^{(m)}. \quad (4.17)$$

This implies in turn that the $\mathfrak{J}^{(m)}$ are covariant, their transformation under generalised diffeomorphisms follows from (3.4) and (4.15):

$$\delta_\Lambda \mathfrak{J}^{(m)} = \langle \partial_{\mathfrak{J}} | \Lambda \rangle \mathfrak{J}^{(m)} - [\Lambda]_\alpha [T^\alpha, \mathfrak{J}^{(m)}]. \quad (4.18)$$

Notice that (4.12) depends implicitly on the representation in which we write \mathfrak{J} , through the **vir** central charge c_{vir} . This means that we must specify in which representation we write \mathfrak{J} in order to define the χ_m^γ fields. We have defined \mathfrak{J} in the $R(\Lambda_0)_0$ representation and hence $c_{\text{vir}} = 8$ as this is the natural choice for E_9 exceptional field theory, because its internal derivatives, gauge parameters and generalised metric all sit in (tensor products of) $R(\Lambda_0)_h$ representations and their duals. Our construction applies to loop groups over any other simple Lie group, and in each case c_{vir} will take a different value. Hence it will be more convenient to leave c_{vir} unspecified in (4.12) to keep track of where it appears.

The current satisfies $\mathfrak{J} = \mathcal{M}^{-1} \mathfrak{J}^\dagger \mathcal{M}$. This allows us to relate the action of opposite shifts by bringing \mathcal{M} through a shift operator

$$\mathcal{S}_m^\gamma(\mathfrak{J}) = \mathcal{M}^{-1} (\mathcal{S}_{-m}^\gamma(\mathfrak{J}))^\dagger \mathcal{M} + \omega^\alpha(\mathcal{M}) \mathfrak{J}_\alpha^{(-m)} \mathbf{K}, \quad (4.19)$$

where we have introduced the (group) cocycle

$$\omega^\alpha(\mathcal{M}) = \mathcal{M}^{-1} \mathcal{S}_0(T^\alpha)^\dagger \mathcal{M} - \mathcal{S}_0(\mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M}) = \mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M}|_{\mathbf{K}} - \delta_{\mathbf{K}}^\alpha. \quad (4.20)$$

Several properties of such cocycles are collected in Appendix B. We then see that we can identify χ^γ fields associated to opposite shifts up to a cocycle, by imposing

$$\mathcal{M}^{-1} (\mathfrak{J}^{(m)})^\dagger \mathcal{M} = \mathfrak{J}^{(-m)} \quad (4.21)$$

so that

$$\chi_{-m}^\gamma = \chi_m^\gamma + \omega^\alpha(\mathcal{M}) \mathfrak{J}_\alpha^{(m)} = \chi_m^\gamma - \omega^\alpha(\mathcal{M}) \mathfrak{J}_\alpha^{(-m)}. \quad (4.22)$$

Finally, we obviously want $\mathfrak{J}^{(0)} = \mathfrak{J}$ and hence identify $\chi_0^\gamma = \mathbf{J}_\mathbf{K}$.

With the definitions and transformation properties above we can state the covariant twisted self-duality equation of exceptional field theory

$$\star \mathfrak{J} = \mathfrak{J}^{(1)}, \quad (4.23)$$

which reduces to (2.56) along the loop and **vir** components for $\langle \partial | = 0$ and $B^{(m)} = 0$. The **K** component of this equation reads

$$\star \mathbf{J}_\mathbf{K} = \chi_1^\gamma, \quad (4.24)$$

which is not a duality equation for σ but rather shows that χ_1^γ is an auxiliary field that does not carry any on-shell degrees of freedom. While $\star^2 \mathfrak{J} = \mathfrak{J}$, the shift operator on the right-hand side does not square to \mathcal{S}_0 .¹⁵ This means that by iterating (4.23) along the loop and **vir** components we recover a cascade of duality relations analogous to (2.47). In order to complete the **K** sector of such relations we then introduce the appropriate χ_m^γ fields for every shift, obtaining

$$\star^{|m|} \mathfrak{J}^{(n)} = \mathfrak{J}^{(m+n)}, \quad m, n \in \mathbb{Z}. \quad (4.25)$$

We stress that the loop and **vir** components of these relations are simply linear combinations of the components of (4.23) and are therefore redundant. On the other hand, the **K** component

¹⁵We can also rewrite (4.23) in the equivalent form $\star \mathfrak{J} = \mathcal{M}^{-1} (\mathfrak{J}^{(1)})^\dagger \mathcal{M}$ so that the operator on the right-hand side is involutive.

reads $\star^{|m|}\chi_n^\gamma = \chi_{m+n}^\gamma$ which relates all χ^γ fields to each other and to J_K . The relations (4.23) and (4.25) are manifestly invariant under rigid $\widehat{E}_8 \rtimes \text{Vir}^-$ and generalised diffeomorphisms, and compatible with (4.22).

We can also introduce the covariant versions of the Hermitian and anti-Hermitian projections of the Maurer–Cartan one-form

$$D\mathcal{V}\mathcal{V}^{-1} = \mathcal{P} + \mathcal{Q}, \quad \mathcal{P}^\dagger = \mathcal{P}, \quad \mathcal{Q}^\dagger = -\mathcal{Q}. \quad (4.26)$$

The relation to the currents \mathfrak{J} is conjugation by \mathcal{V} :

$$\mathfrak{J} = 2\mathcal{V}^{-1}\mathcal{P}\mathcal{V}, \quad (4.27)$$

and we can require a similar relation for the shifted currents which, using (B.10), leads to the definitions

$$\mathcal{P}^{(m)} = \mathcal{S}_m(\mathcal{P}) + \tilde{\chi}_m K, \quad \tilde{\chi}_m = \tilde{\chi}_{-m} = \frac{1}{2}\chi_m^\gamma - \frac{1}{2}\omega^\alpha(\mathcal{V})\mathfrak{J}_\alpha^{(m)} \quad (4.28)$$

so that $\mathfrak{J}^{(m)} = 2\mathcal{V}^{-1}\mathcal{P}^{(m)}\mathcal{V}$. The one-forms $\tilde{\chi}_m$ complete the K component of $\mathcal{S}_m(\mathcal{P})$ so that $\mathcal{P}^{(m)}$ transforms with a commutator under $K(\mathfrak{e}_9)$, analogously to the K completions of \mathcal{S}_m^γ introduced in (4.16).

With these definitions we can rewrite twisted self-duality (4.23) and (4.25) as

$$\star\mathcal{P} = \mathcal{P}^{(1)}, \quad \star^{|m|}\mathcal{P}^{(n)} = \mathcal{P}^{(m+n)}. \quad (4.29)$$

We shall expand \mathcal{P} and \mathcal{Q} similarly to (4.7):

$$\mathcal{P} = \sum_{m \in \mathbb{Z}} (\mathbb{P}_A^m T_m^A + \mathbb{P}_m L_m) + \mathbb{P}_K, \quad \mathcal{Q} = \sum_{m \in \mathbb{Z}} (\mathbb{Q}_A^m T_m^A + \mathbb{Q}_m L_m). \quad (4.30)$$

In terms of these components, twisted self-duality implies in particular

$$\mathbb{P}_A^m = \star^{|m|}\mathbb{P}_A^0, \quad \mathbb{P}_m = \star^{|m|}\mathbb{P}_0. \quad (4.31)$$

4.2 Shifted Maurer–Cartan equations

We will now construct a topological pseudo-Lagrangian for vir^- extended E_9 exceptional field theory. The Euler–Lagrange equations obtained by varying this pseudo-Lagrangian must be supplemented with the twisted self-duality constraint (4.23) (or (4.29)) to reproduce the equations of motion. In particular, in analogy with lower-rank exceptional field theories, we will require that the Euler–Lagrange equations for the χ_m^γ fields as well as the $B^{(m)}$ fields vanish upon using twisted self-duality, reflecting the fact that these fields do not encode any physical degrees of freedom. We will work only in conformal gauge with the two-dimensional unimodular metric fixed to the Minkowski metric: $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$. From the point of view of Kaluza–Klein decomposition of a higher-dimensional metric, it is always possible to partially gauge fix higher-dimensional diffeomorphisms so that a 2×2 non-degenerate block on the diagonal of the higher dimensional metric is reduced to $\eta_{\mu\nu}$ (or $\delta_{\mu\nu}$). In other words, even when $\tilde{g}_{\mu\nu}$ depends on the Y^M coordinates

subject to the section constraint, reflecting that our theory is in fact a higher-dimensional supergravity, it is still possible to switch to conformal gauge without loss of generality. As a result, however, the equations of motion obtained from the pseudo-Lagrangian and twisted self-duality must be supplemented with the covariantised version of the Virasoro constraint (2.17). We will come back to this in Section 4.6. We will restore $\tilde{g}_{\mu\nu}$ in Section 5.

The two-dimensional current (2.55) satisfies the Maurer–Cartan equation $dJ + \frac{1}{2}[J, J] = 0$, where the wedge product is understood and the commutators are implicitly graded with the rank of the p -forms, namely for A and B any $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ valued forms, $[A, B] = A_\alpha \wedge B_\beta f^{\alpha\beta}{}_\gamma T^\gamma$. For the covariant current \mathfrak{J} this expression becomes

$$\mathcal{D}\mathfrak{J} + \frac{1}{2}[\mathfrak{J}, \mathfrak{J}] + \langle \mathcal{J}_\alpha | \mathcal{F} \rangle T^\alpha + [\mathbb{F}]_\alpha (T^\alpha + \mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M}) = 0, \quad (4.32)$$

which is deduced from (4.6) and the first equation in (3.27). We will apply this identity to a shifted current $\mathfrak{J}^{(m)}$ in order to construct a candidate topological term for exceptional field theory. The idea is that while all components of \mathfrak{J} satisfy the above integrability condition, the central charge component of $\mathfrak{J}^{(m)}$ is χ_m^γ which is a fundamental field. Thus, we will be able to write a set of two-forms based on $\mathfrak{J}^{(m)}$ and its derivatives that transform by conjugation under rigid $\hat{E}_8 \rtimes \text{Vir}^-$ and such that, because of the identity (4.32), only their \mathbb{K} components are non-vanishing. As a consequence, these expressions are automatically invariant under rigid $\hat{E}_8 \rtimes \text{Vir}^-$. We will then find that only one such expression is compatible with twisted self-duality, covariant under generalised diffeomorphisms and independent of the two-form fields appearing in the covariant field strengths \mathbb{F} . That will be the main part of our pseudo-Lagrangian.

We begin by taking a covariant derivative of $\mathfrak{J}^{(m)}$:

$$\begin{aligned} \mathcal{D}\mathfrak{J}^{(m)} &= \mathcal{D}\mathcal{S}_m^\gamma(\mathfrak{J}) + \mathcal{D}\chi_m^\gamma \mathbb{K} \\ &= \mathcal{S}_0 \left(\mathcal{D}(\Gamma^{-1} \mathcal{S}_m(\Gamma \mathfrak{J} \Gamma^{-1}) \Gamma) \right) + \mathcal{D}\chi_m^\gamma \mathbb{K} \\ &= \mathcal{S}_0 \left(\Gamma^{-1} (\mathcal{S}_m([\mathcal{D}\Gamma \Gamma^{-1}, \Gamma \mathfrak{J} \Gamma^{-1}]) - [\mathcal{D}\Gamma \Gamma^{-1}, \mathcal{S}_m(\Gamma \mathfrak{J} \Gamma^{-1})]) \Gamma \right) + \mathcal{S}_m^\gamma(\mathcal{D}\mathfrak{J}) + \mathcal{D}\chi_m^\gamma \mathbb{K} \\ &= m \sum_{q=0}^{\infty} (\mathcal{D}\Gamma \Gamma^{-1})_{-q} \mathcal{S}_{m-q}^\gamma(\mathfrak{J}) + \mathcal{S}_m^\gamma(\mathcal{D}\mathfrak{J}) + \mathcal{D}\chi_m^\gamma \mathbb{K}. \end{aligned} \quad (4.33)$$

In the last line $(\mathcal{D}\Gamma \Gamma^{-1})_{-q}$ is the L_{-q} component of the Maurer–Cartan form and we used (B.2). We now apply (4.32) to the second term and use (B.12) to arrive at the identity

$$\begin{aligned} \mathcal{D}\mathfrak{J}^{(m)} + \frac{1}{2}[\mathfrak{J}, \mathfrak{J}^{(m)}] + \frac{m}{2} \sum_{q=1}^{\infty} (\mathcal{D}\Gamma \Gamma^{-1})_{-q} (\mathcal{S}_{m+q}^\gamma(\mathfrak{J}) - \mathcal{S}_{m-q}^\gamma(\mathfrak{J})) + \langle \mathcal{J}_\alpha | \mathcal{F} \rangle \mathcal{S}_m^\gamma(T^\alpha) \\ + [\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(T^\alpha + \mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M}) + \left(\frac{1}{2} \omega^{\alpha\beta} \mathfrak{J}_\alpha \mathfrak{J}_\beta^{(m)} - \mathcal{D}\chi_m^\gamma \right) \mathbb{K} = 0. \end{aligned} \quad (4.34)$$

This expression contains several shift operators which we now want to complete by adding terms proportional to \mathbb{K} , as described in (4.16), so that the completed expression transforms as an algebra-valued object under rigid $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$. For each shift of \mathfrak{J} we will of course add

the associated χ^γ one-form. The shift of the internal current $\langle \mathcal{J}_\alpha |$ is completed in analogy with the external one and with [1] by introducing an *internal* scalar field $\langle \chi_m^\gamma |$ in the $\overline{R(\Lambda_0)}_{-1}$ representation, defining

$$\langle \mathcal{J}_\alpha^{(m)} | \otimes T^\alpha = \langle \mathcal{J}_\alpha | \otimes \mathcal{S}_m^\gamma(T^\alpha) + \langle \chi_m^\gamma | \otimes \mathbf{K}, \quad (4.35)$$

and imposing the rigid transformation property

$$\delta_X \langle \chi_m^\gamma | = \langle \chi_m^\gamma | X - X_0 \langle \chi_m^\gamma | + \omega^{\alpha\beta} X_\alpha \langle \mathcal{J}_\beta^{(m)} |, \quad X \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-. \quad (4.36)$$

The internal index of the internal current (represented by $\langle |$) comes from an internal derivative and satisfies the section constraint (3.8), therefore $\langle \chi_m^\gamma |$ must satisfy the section constraint, too. Eventually we will find that only one such scalar appears in the pseudo-Lagrangian and that it is related to the $\langle \chi |$ field introduced in [1].

Finally, we need to construct the \mathbf{K} completion of the expression $[[\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(T^\alpha + \mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M})]$ as in (4.16). For reference let us write down

$$[[\mathbb{F}]_\alpha = \eta_{\alpha\beta} \langle \partial_{\mathcal{F}} | T^\beta | \mathcal{F} \rangle + \sum_{n=1}^{\infty} \eta_{-n\alpha\beta} \text{Tr}(\mathcal{G}^{(n)} T^\beta). \quad (4.37)$$

This expression transforms covariantly under $\hat{E}_8 \times \text{Vir}^-$. In particular, the rigid indecomposable transformation (3.36) of $(\mathcal{F}, \mathcal{G}^{(m)})$ guarantees Vir^- covariance. When we apply the shift operator, Vir^- covariance of $[[\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(T^\alpha)]$ still holds because \mathcal{S}_m^γ commutes with Vir^- thanks to the transformation of Γ in (4.8), and because $[[\mathbb{F}]_\alpha]$ does not take values along the positive \mathfrak{vir} generators, so that no central charge term is generated. The same holds true when we apply \mathcal{M} conjugation before the shift operator. Hence, we simply have

$$\delta_{L_{-k}} \left([[\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(T^\alpha + \mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M})] \right) = -[[\mathbb{F}]_\alpha [L_{-k}, \mathcal{S}_m^\gamma(T^\alpha + \mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M})]]. \quad (4.38)$$

In order to look at the loop transformation, we first focus on $[[\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(T^\alpha)]$ and use the fact that \mathcal{S}_m^γ can be written as a series of constant \mathcal{S}_k operators such as (4.9), and the coefficients are \hat{E}_8 invariant:

$$\mathcal{S}_m^\gamma = \sum_{k \leq m} a_m^k(\rho, \phi) \mathcal{S}_k \quad (4.39)$$

Thus, we look at a single \mathcal{S}_k operator acting on $[[\mathbb{F}]_\alpha]$ and perform a finite $g \in \hat{E}_8$ transformation. Using (B.14) we find

$$[[\mathbb{F}]_\alpha \mathcal{S}_k(g^{-1} T^\alpha g)] = [[\mathbb{F}]_\alpha (g^{-1} \mathcal{S}_k(T^\alpha) g - \omega_{-k}^\alpha(g^{-1}) \mathbf{K})] \quad (4.40)$$

Opening up the cocycle term using (B.7), (B.15) and (2.9), it reads

$$\begin{aligned} -[[\mathbb{F}]_\alpha \omega_{-k}^\alpha(g^{-1})] &= -\omega^\alpha(g^{-1}) \left(\eta_{k\alpha\beta} \langle \partial_{\mathcal{F}} | T^\beta | \mathcal{F} \rangle + \sum_{n=1}^{\infty} \eta_{k-n\alpha\beta} \text{Tr}(\mathcal{G}^{(n)} T^\beta) \right) \\ &= \langle \partial_{\mathcal{F}} | (g L_k g^{-1} - L_k) | \mathcal{F} \rangle + \sum_{n=1}^{\infty} \text{Tr}(\mathcal{G}^{(n)} (g L_{k-n} g^{-1} - L_{k-n})) \end{aligned} \quad (4.41)$$

Where we used (B.16) in the second line. Recalling that this is just the transformation of one term in the series of shifts defining \mathcal{S}_m^γ and that of course $L_{k-n} = \mathcal{S}_k(L_{-n})$, we are led to the definition

$$\widehat{\mathbb{F}}_m^\gamma = -\langle \partial_{\mathcal{F}} | \mathcal{S}_m^\gamma(L_0) | \mathcal{F} \rangle - \sum_{n=1}^{\infty} \text{Tr} \left(\mathcal{G}^{(n)} \mathcal{S}_m^\gamma(L_{-n}) \right), \quad (4.42)$$

such that in the loop variation of the combination

$$[\mathbb{F}]_\alpha^{(m)} T^\alpha = [\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(T^\alpha) + \widehat{\mathbb{F}}_m^\gamma \mathbb{K} \quad (4.43)$$

the cocycle terms coming from (4.40) cancel out. Furthermore, one can see that due to the indecomposable Vir^- transformation of the field strengths, (4.42) is Vir^- invariant. This guarantees that (4.43) transforms as an algebra element. In order to define the \mathbb{K} completion of the term $[\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(\mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M})$, we observe that the \mathcal{M} conjugate of (4.43) also transforms as an algebra element. Sending $m \rightarrow -m$ and writing $\mathcal{M} = \Gamma^\dagger \widehat{M} \Gamma$ with $\widehat{M}^\dagger = \widehat{M} \in \widehat{E}_8$, conjugating (4.43) with \mathcal{M} and applying Hermitian conjugation gives us

$$\begin{aligned} & [\mathbb{F}]_\alpha \mathcal{M}^{-1}(\mathcal{S}_{-m}^\gamma(T^\alpha))^\dagger \mathcal{M} + \widehat{\mathbb{F}}_{-m}^\gamma \mathbb{K} \\ &= [\mathbb{F}]_\alpha \left(\mathcal{M} \mathcal{S}_0 \left(\Gamma^{-1} \mathcal{S}_{-m}(\Gamma T^\alpha \Gamma^{-1}) \Gamma \right) \mathcal{M}^{-1} \right)^\dagger + \widehat{\mathbb{F}}_{-m}^\gamma \mathbb{K} \\ &= [\mathbb{F}]_\alpha \left(\mathcal{S}_0 \left(\Gamma^\dagger \widehat{M} \mathcal{S}_{-m}(\Gamma T^\alpha \Gamma^{-1}) \widehat{M}^{-1} \Gamma^{-\dagger} \right) \right)^\dagger + \left(\widehat{\mathbb{F}}_{-m}^\gamma + \omega^\alpha(\mathcal{M}) [\mathbb{F}]_\alpha^{(-m)} \right) \mathbb{K} \\ &= [\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(\mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M}) + \left(\widehat{\mathbb{F}}_{-m}^\gamma + \omega^\alpha(\mathcal{M}) [\mathbb{F}]_\alpha^{(-m)} \right) \mathbb{K} \\ &= [\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(\mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M}) + {}^{\mathcal{M}}\widehat{\mathbb{F}}_m^\gamma \mathbb{K}, \end{aligned} \quad (4.44)$$

where in the third line we brought \mathcal{M} through \mathcal{S}_0 generating a cocycle term (written in terms of (4.43) using $\omega^{\mathbb{K}}(\mathcal{M}) = 0$), and in the fourth line we brought \widehat{M} through \mathcal{S}_{-m} , which does not generate cocycles because of the overall \mathcal{S}_0 projection. We then propagated Hermitian conjugation through \mathcal{S}_0 . The \mathbb{K} component of the third and fourth line is the expression we are looking for. In the last line, for later convenience, we defined

$${}^{\mathcal{M}}\widehat{\mathbb{F}}_m^\gamma = \widehat{\mathbb{F}}_{-m}^\gamma + \omega^\alpha(\mathcal{M}) [\mathbb{F}]_\alpha^{(-m)}. \quad (4.45)$$

We can now complete each shift operator in (4.34) with the appropriate central charge term arriving at the identity

$$\begin{aligned} & \mathcal{D}\mathfrak{J}^{(m)} + \frac{1}{2}[\mathfrak{J}, \mathfrak{J}^{(m)}] + m \sum_{n=1}^{\infty} \text{P}_n(\mathfrak{J}^{(m+n)} - \mathfrak{J}^{(m-n)}) \\ & \quad + \langle \mathcal{J}^{(m)} | \mathcal{F} \rangle + [\mathbb{F}]_\alpha^{(m)} T^\alpha + [\mathbb{F}]_\alpha \mathcal{S}_m^\gamma(\mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M}) + {}^{\mathcal{M}}\widehat{\mathbb{F}}_m^\gamma \mathbb{K} \\ &= \left(\mathcal{D}\chi_m^\gamma - \frac{1}{2}\omega^{\alpha\beta} \mathfrak{J}_\alpha \mathfrak{J}_\beta^{(m)} + m \sum_{n=1}^{\infty} \text{P}_n(\chi_{m+n}^\gamma - \chi_{m-n}^\gamma) + \langle \chi_m^\gamma | \mathcal{F} \rangle + \widehat{\mathbb{F}}_m^\gamma + {}^{\mathcal{M}}\widehat{\mathbb{F}}_m^\gamma \right) \mathbb{K} \\ &=: \mathfrak{X}^{(m)} \mathbb{K}. \end{aligned} \quad (4.46)$$

In the first line we have used that the \mathfrak{vir}^- components of the Maurer–Cartan form are proportional to the \mathfrak{vir} components of \mathcal{P} : $P_{\pm n} = \frac{1}{2}(\mathcal{D}\Gamma\Gamma^{-1})_{-n}$ for $n \neq 0$. The internal shifted current $\langle \mathcal{J}^{(m)} |$ is defined in (4.35). Because the left-hand side of this equation transforms by conjugation under rigid $\widehat{E}_8 \rtimes \text{Vir}^-$, so does the right-hand side, which is therefore invariant.

4.3 Gauge invariant topological term

We will now show that only $\mathfrak{X}^{(1)}$ can be used to construct a pseudo-Lagrangian. First, we notice that by \mathcal{M} conjugation $\mathfrak{X}^{(m)} = \mathfrak{X}^{(-m)}$ and hence we focus on $m > 0$ ($\mathfrak{X}^{(0)} = 0$ identically). Then, we stress that all $\mathfrak{X}^{(m)}$ have weight 0 under the generalised Lie derivative:

$$\mathcal{L}_\Lambda \mathfrak{X}^{(m)} = \langle \partial_{\mathfrak{X}} | \Lambda \rangle \mathfrak{X}^{(m)}. \quad (4.47)$$

Even assuming a covariant transformation property for $\mathfrak{X}^{(m)}$ under generalised diffeomorphisms (so that their variation equals the generalised Lie derivative, which we will prove for $\mathfrak{X}^{(1)}$ below), we need objects of weight 1 to be able to integrate out the transport term above. Our candidate pseudo-Lagrangians are therefore of the form $\rho \mathfrak{X}^{(m)}$. Knowing this, we must check that the Euler–Lagrange equations for the χ^γ forms and the B fields vanish upon imposing twisted self-duality. The B -field variation is more involved and will in fact require an extra correction to the action, but the χ^γ variation is straightforward. Using (4.22) to relate χ^γ forms associated to opposite shifts and then varying with respect to χ_m^γ , $m > 0$, we find

$$\delta(\rho \mathfrak{X}^{(m)}) \propto P_0 \delta \chi_m^\gamma + m \left(\sum_{p>m} P_{p-m} \delta \chi_p^\gamma - \sum_{p=1}^{m-1} P_{m-p} \delta \chi_p^\gamma - \sum_{p>0} P_{m+p} \delta \chi_p^\gamma \right) \quad (4.48)$$

and one sees for instance that the $\delta \chi_1^\gamma$ term is proportional to $P_{m-1} + P_{m+1}$ for any $m \geq 2$, which does not vanish under (4.29). Instead, for $m = 1$ this expression simplifies to

$$\sum_{p=1}^{\infty} (P_{p-1} - P_{p+1}) \delta \chi_p^\gamma, \quad (4.49)$$

which vanishes upon imposing twisted self-duality. This indicates that the correct topological term should be based on $\mathfrak{X}^{(1)}$.¹⁶

We now prove gauge invariance of $\rho \mathfrak{X}^{(1)}$ (up to a total internal derivative). The first three terms in (4.46) are manifestly covariant under generalised diffeomorphisms. Let us focus on $\widehat{\mathbb{F}}_1^\gamma$ and compute its non-covariant variation

$$\Delta_\Lambda = \delta_\Lambda - \mathcal{L}_\Lambda. \quad (4.50)$$

¹⁶Rewriting the Euler–Lagrange equations (4.49) in terms of $\mathcal{D}\Gamma\Gamma^{-1}$ one finds an expression analogous to (2.51), with covariantised derivatives and $\mathcal{D}\tilde{\rho}$ in place of $\star\mathcal{D}\rho$ (and $\phi_1 \equiv 2\tilde{\rho}$). This shows that the one-forms χ_m^γ are Lagrange multipliers imposing that $\gamma(w)$, as defined in (2.39), reduces to (2.37) and hence essentially to the function used in the Breitenlohner–Maison linear system. Lagrangian mechanics does not generally allow integrating out one field using the Euler–Lagrange equations of another, hence we cannot naively substitute the relations descending from (4.49) directly into the pseudo-Lagrangian.

Because the field strengths transform with the extended \circ product (3.40) as in (3.30), one has $\Delta_\Lambda \mathbb{F} = \Lambda \circ \mathbb{F} - \mathcal{L}_\Lambda \mathbb{F}$. Substituting into (4.37), (4.42) and (4.45), after some work we find

$$\Delta_\Lambda [\mathbb{F}]_\alpha = -[\Lambda]_\alpha \langle \partial_\Lambda | \mathcal{F} \rangle, \quad (4.51)$$

$$\Delta_\Lambda \widehat{\mathbb{F}}_1^\gamma = -\widehat{\Lambda}_1^\gamma \langle \partial_\Lambda | \mathcal{F} \rangle + \rho^{-1} \langle \partial_\Sigma | \Sigma^{(1)} | \mathcal{F} \rangle - \rho^{-1} \text{Tr}(\Sigma^{(1)}) \langle \partial_\Sigma | \mathcal{F} \rangle, \quad (4.52)$$

$$\Delta_\Lambda {}^{\mathcal{M}}\widehat{\mathbb{F}}_1^\gamma = -{}^{\mathcal{M}}\widehat{\Lambda}_1^\gamma \langle \partial_\Lambda | \mathcal{F} \rangle \quad (4.53)$$

where $\widehat{\Lambda}_1^\gamma$ and ${}^{\mathcal{M}}\widehat{\Lambda}_1^\gamma$ are defined as in (4.42) and (4.45) but with $(|\Lambda\rangle, \Sigma^{(k)})$ in place of $(|\mathcal{F}\rangle, \mathcal{G}^{(k)})$. We prove these results in Appendix C.1. Notice that in the last two lines the partial derivatives act on $\Lambda = (|\Lambda\rangle, \Sigma^{(k)})$ contained within $\widehat{\Lambda}_1^\gamma$ and ${}^{\mathcal{M}}\widehat{\Lambda}_1^\gamma$, but not on the scalar fields. We arrive at the result

$$\Delta_\Lambda \mathfrak{X}^{(1)} = \left(\Delta_\Lambda \langle \chi_1^\gamma | - (\widehat{\Lambda}_1^\gamma + {}^{\mathcal{M}}\widehat{\Lambda}_1^\gamma) \langle \partial_\Lambda | + \rho^{-1} \langle \partial_\Sigma | \Sigma^{(1)} - \rho^{-1} \text{Tr}(\Sigma^{(1)}) \langle \partial_\Sigma | \right) | \mathcal{F} \rangle. \quad (4.54)$$

Notice that the entire expression in the parenthesis satisfies the section constraint. This means that we can define the generalised diffeomorphism transformation of $\langle \chi_1^\gamma |$ to cancel the above variation:

$$\Delta_\Lambda \langle \chi_1^\gamma | = (\widehat{\Lambda}_1^\gamma + {}^{\mathcal{M}}\widehat{\Lambda}_1^\gamma) \langle \partial_\Lambda | - \rho^{-1} \langle \partial_\Sigma | \Sigma^{(1)} + \rho^{-1} \text{Tr}(\Sigma^{(1)}) \langle \partial_\Sigma |. \quad (4.55)$$

With this definition, we have that $\rho \mathfrak{X}^{(1)}$ is $\widehat{E}_8 \rtimes \text{Vir}^-$ invariant and generalised diffeomorphism invariant up to a total internal derivative. If we follow the same approach for $\rho \mathfrak{X}^{(m)}$, $m > 1$, we find that $\Delta_\Lambda \widehat{\mathbb{F}}_m^\gamma$ contains terms that cannot be reabsorbed into the variation of $\langle \chi_m^\gamma |$, and therefore $\rho \mathfrak{X}^{(m)}$ is not invariant under generalised diffeomorphisms. This is also described in Appendix C.1. We again conclude that only $\rho \mathfrak{X}^{(1)}$ is a suitable candidate for our topological term.

One last cross-check on the suitability of $\rho \mathfrak{X}^{(1)}$ as a candidate pseudo-Lagrangian is that the two-forms appearing in the covariant field strengths (3.28) must not contribute. These terms can potentially appear in the $\widehat{\mathbb{F}}_{\pm 1}^\gamma$ contributions to $\mathfrak{X}^{(1)}$, but a direct computation shows that they vanish up to a total internal derivative. To see this, we start by recalling the definition (4.42) and use (4.39) to write $\mathcal{S}_m^\gamma(L_n)$ as a series of \mathfrak{vir} generators L_k with $k \leq m+n$. Then, we see that $\widehat{\mathbb{F}}_{-1}^\gamma$ (which appears through (4.45)) only involves contractions with negative \mathfrak{vir} generators and hence no trivial parameters contribute (such traces appear in the \mathbf{K} component of $[\mathbb{F}]_\alpha$ and therefore anything that contributes to them is by definition not trivial). A similar argument shows that the two-forms included in $\mathcal{G}^{(k)}$, $k \geq 2$ do not contribute to $\widehat{\mathbb{F}}_1^\gamma$, and that contributions from the $|\mathcal{F}\rangle$ dependent term drop out as total derivatives. One is then left with evaluating the two-form contributions of $\text{Tr}(\mathcal{G}^{(1)} L_0)$. One immediately sees that the two-forms associated to the trivial parameters (A.5a) to (A.5c) appear only through a total derivative and hence are discarded. The parameters (A.5d) to (A.5f) do not involve any derivative and direct calculation shows their contribution vanishes. We exemplify this for (A.5d). Using the symmetries of the U parameter, its contribution to $\widehat{\mathbb{F}}_1^\gamma$ is proportional to

$$\eta_{1\alpha\beta} \left(\langle \pi_1 | [L_0, T^\alpha] | U_1 \rangle \langle \pi_2 | T^\beta | U_2 \rangle + \langle \pi_1 | T^\alpha | U_1 \rangle \langle \pi_2 | [L_0, T^\beta] | U_2 \rangle \right) = -\eta_{1\alpha\beta} \langle \pi_1 | T^\alpha | U_1 \rangle \langle \pi_2 | T^\beta | U_2 \rangle \quad (4.56)$$

which vanishes by the section constraint since U is symmetric in $\langle \pi_1 |$ and $\langle \pi_2 |$. The general result can be deduced from equations (5.18), (A.13) and (A.14). Again, it is straightforward to see that two-form independence does not hold for $\rho \mathfrak{X}^{(m)}$ for any $m \geq 2$.

The expression $\rho \mathfrak{X}^{(1)}$ is not the final topological term. The **vir** components of \mathcal{P} defined in (4.27) are invariant under rigid $\widehat{E}_8 \rtimes \text{Vir}^-$ and transform as scalars under generalised diffeomorphisms, hence any bilinear in these components (times ρ) defines an invariant action. This apparent ambiguity is fixed by computing the Euler–Lagrange equations of ρ and the B fields, which we will do in Section 4.6. For now, we just claim that one such contribution is necessary to reproduce the correct equations of motion, giving

$$\mathcal{L}_{\text{top}} = \rho \mathfrak{X}^{(1)} - \rho \frac{c_{\text{vir}}}{6} \sum_{n=2}^{\infty} (n^3 - n) \mathbf{P}_n (\mathbf{P}_{n+1} + \mathbf{P}_{n-1}),$$

so that explicitly

$$\begin{aligned} \rho^{-1} \mathcal{L}_{\text{top}} &= \mathcal{D}\chi_1^\gamma - \frac{1}{2} \omega^{\alpha\beta} \mathfrak{J}_\alpha \mathfrak{J}_\beta^{(1)} - \frac{c_{\text{vir}}}{6} \sum_{n=2}^{\infty} (n^3 - n) \mathbf{P}_n (\mathbf{P}_{n+1} + \mathbf{P}_{n-1}) \\ &\quad + \sum_{n=1}^{\infty} \mathbf{P}_n (\chi_{1+n}^\gamma - \chi_{1-n}^\gamma) + \langle \chi_1^\gamma | \mathcal{F} \rangle + \widehat{\mathbb{F}}_1^\gamma + \mathcal{M} \widehat{\mathbb{F}}_1^\gamma. \end{aligned} \quad (4.57)$$

There are several c_{vir} dependent couplings in this pseudo-Lagrangian. Beyond the explicit term we have just added in the first line, the $\omega^{\alpha\beta}$ cocycle contains a **vir** component, and a similar cocycle is contained within $\mathcal{D}\chi_1^\gamma$:

$$\mathcal{D}\chi_1^\gamma = (d - \langle \partial_\chi | A \rangle) \chi_1^\gamma + \omega^{\alpha\beta} [A]_\alpha \mathfrak{J}_\beta^{(1)}. \quad (4.58)$$

This reflects the fact that the definition of χ_m^γ depends on the choice of representation in which we write \mathfrak{J} . We will see in the next section that all c_{vir} dependent couplings cancel out when we rewrite the pseudo-Lagrangian in terms of $\tilde{\chi}_m$ defined in (4.28), whose transformation properties are c_{vir} independent.

4.4 Unendlichbein formulation

We can reformulate (4.57) in terms of the Hermitian projection \mathcal{P} of the Maurer–Cartan form and the $\tilde{\chi}_m$ forms introduced in (4.28). In order to do so, we first provide more details on how several objects we need are defined and how they transform. The coset representative (4.1) transforms as follows under infinitesimal rigid $\widehat{E}_8 \rtimes \text{Vir}^-$ transformations

$$\delta^\alpha \mathcal{V} = \mathcal{V} T^\alpha + h^\alpha \mathcal{V}, \quad h^\alpha \in K(\mathfrak{e}_9), \quad (4.59)$$

where h^α is a local compensating transformation. The Maurer–Cartan form and its projections (4.26) are then expanded as¹⁷

$$\mathcal{D}\mathcal{V}\mathcal{V}^{-1} = d\mathcal{V}\mathcal{V}^{-1} - (\langle \partial_{\mathcal{V}} | A \rangle \mathcal{V}) \mathcal{V}^{-1} - [A]_\alpha (\mathcal{V} T^\alpha \mathcal{V}^{-1} + h^\alpha), \quad (4.60)$$

$$\mathcal{P} = \frac{1}{2} (d\mathcal{V}\mathcal{V}^{-1} + \text{h.c.}) - \langle \mathcal{P}_\alpha | A \rangle T^\alpha - [A]_\alpha \frac{1}{2} (\mathcal{V} T^\alpha \mathcal{V}^{-1} + \text{h.c.}), \quad (4.61)$$

$$\mathcal{Q} = \frac{1}{2} (d\mathcal{V}\mathcal{V}^{-1} - \text{h.c.}) - \langle \mathcal{Q}_\alpha | A \rangle T^\alpha - [A]_\alpha \frac{1}{2} (\mathcal{V} T^\alpha \mathcal{V}^{-1} - \text{h.c.}) - [A]_\alpha h^\alpha, \quad (4.62)$$

where $\langle \mathcal{P}_\alpha |$ and $\langle \mathcal{Q}_\alpha |$ are the projections of the internal Maurer–Cartan form $(\partial_M \mathcal{V}) \mathcal{V}^{-1} \langle e^M |$ which appears in the first line contracted with $|A\rangle$. Taking the differential of the above expressions we find the gauged Maurer–Cartan equations¹⁸

$$\mathcal{D}\mathcal{P} - [\mathcal{Q}, \mathcal{P}] = -\langle \mathcal{P}_\alpha | \mathcal{F} \rangle T^\alpha - [\mathbb{F}]_\alpha \frac{1}{2} (\mathcal{V} T^\alpha \mathcal{V}^{-1} + \text{h.c.}), \quad (4.63)$$

$$\mathcal{D}\mathcal{Q} - \frac{1}{2} [\mathcal{Q}, \mathcal{Q}] - \frac{1}{2} [\mathcal{P}, \mathcal{P}] = -\langle \mathcal{Q}_\alpha | \mathcal{F} \rangle T^\alpha - [\mathbb{F}]_\alpha \frac{1}{2} (\mathcal{V} T^\alpha \mathcal{V}^{-1} - \text{h.c.}) - [\mathbb{F}]_\alpha h^\alpha. \quad (4.64)$$

With this information we can in principle repeat the computation of the shifted Maurer–Cartan equation of the previous section in terms of \mathcal{P} rather than \mathfrak{J} . More simply, we can take the left-hand side of (4.46) (with $m = 1$), conjugate the expression with \mathcal{V} and use (4.27) and (4.28) to find

$$\begin{aligned} \mathfrak{X}^{(1)} &= 2\mathcal{D}\tilde{\chi}_1 + 2\omega^{\alpha\beta} \mathcal{Q}_\alpha \mathcal{P}_\beta^{(1)} + 2 \sum_{n=1}^{\infty} \mathbb{P}_n (\tilde{\chi}_{n+1} - \tilde{\chi}_{n-1}) \\ &\quad + 2\langle \tilde{\chi}_1 | \mathcal{F} \rangle + \widehat{\mathbb{F}}_1^\gamma + \widehat{\mathbb{F}}_{-1}^\gamma + \omega^\alpha (\mathcal{V}) ([\mathbb{F}]_\alpha^{(1)} + [\mathbb{F}]_\alpha^{(-1)}), \end{aligned} \quad (4.65)$$

where $\langle \tilde{\chi}_1 |$ is the internal equivalent of the one-form $\tilde{\chi}_1$ and completes the K component of $\langle \mathcal{P}_\alpha | \otimes \mathcal{S}_1(T^\alpha)$. It is related to $\langle \chi_1^\gamma |$ just like its one-form siblings in (4.28). The last three terms in the second line are the K completions of $[\mathbb{F}]_\alpha \mathcal{S}_{\pm 1}(\mathcal{V} T^\alpha \mathcal{V}^{-1})$. It is important to notice that in this expression the h^α compensator cancels out. To see this, we expand the covariant derivative of $\tilde{\chi}_1$:

$$\mathcal{D}\tilde{\chi}_1 = d\tilde{\chi}_1 - \langle \partial_{\tilde{\chi}} | A \rangle \tilde{\chi}_1 - [A]_\alpha (h^\alpha)_\beta \omega^{\beta\delta} \mathcal{P}_\delta^{(1)} \quad (4.66)$$

where $h^\alpha = (h^\alpha)_\beta T^\beta$. This reflects the fact that $\tilde{\chi}$ transforms under $K(E_9)$ such that

$$(h^\alpha)_\beta \delta_{K(\mathfrak{e}_9)}^\beta \mathcal{P}^{(1)} = [h^\alpha, \mathcal{P}^{(1)}]. \quad (4.67)$$

¹⁷Notice that the local compensating transformation h^α satisfies $(h^\alpha)^\dagger = -h^\alpha$ and does not take values along \mathfrak{vir} . On the other hand, \mathcal{Q} also takes values in the anti-Hermitian part of \mathfrak{vir} .

¹⁸Notice that the rigid $\mathfrak{e}_8 \oplus \mathfrak{vir}^-$ variation of h^α cannot be entirely specified without reference to the specific gauge choice of \mathcal{V} and associated Killing vectors. Only the antisymmetrised part admits a covariant algebraic expression $2\delta^{[\alpha} h^{\beta]} = f^{\alpha\beta\gamma} h^\gamma + [h^\alpha, h^\beta]$. This can be used for instance to compute $\delta^\alpha (\mathcal{D}\mathcal{V}\mathcal{V}^{-1}) = \mathcal{D}h^\alpha + [h^\alpha, \mathcal{D}\mathcal{V}\mathcal{V}^{-1}]$ as expected, and hence the second covariant differential acting on $\mathcal{D}\mathcal{V}\mathcal{V}^{-1}$ is well-defined. On the other hand, to derive (4.64) we simply use $\mathcal{D}^2 = -\mathcal{L}_{\mathbb{F}}$ so that no $\mathcal{D}h^\alpha$ term appears and we do not need knowledge of $\delta^\alpha h^\beta$. This is of course standard in computing gauged Maurer–Cartan equations on coset spaces.

The cocycle term in $\mathcal{D}\tilde{\chi}_1$ then cancels out with the h^α contained in the \mathcal{Q} connection in the second term. In fact, the entire first line of (4.65) can be identified with the generalised diffeomorphism and local $K(E_9)$ covariant derivative of $\tilde{\chi}_1$. To see this, we can rewrite the last term of the first line as $-2\sum_{n\in\mathbb{Z}}\mathbf{Q}_n\tilde{\chi}_{1+n}$, using the relation $\mathbf{Q}_n = -\text{sgn}(n)\mathbf{P}_n$, which is valid because the Maurer–Cartan form does not take values along the positive levels of the \mathfrak{vir} algebra.¹⁹

We anticipated in (4.57) that in order to write the full topological Lagrangian we need to subtract a $c_{\mathfrak{vir}}$ dependent term quadratic in the \mathfrak{vir} components of \mathcal{P} . We can now give a better description of such a subtraction. The algebra cocycle in the first line of (4.65) contains a Virasoro component according to the definition (4.12), and using again that $\mathbf{Q}_n = -\text{sgn}(n)\mathbf{P}_n$ we can rewrite it as a term quadratic in \mathcal{P} :

$$-\frac{c_{\mathfrak{vir}}}{12}\sum_{n\in\mathbb{Z}}(n^3-n)\mathbf{Q}_n\mathbf{P}_{n+1}=\frac{c_{\mathfrak{vir}}}{12}\sum_{n=2}^{\infty}(n^3-n)\mathbf{P}_n(\mathbf{P}_{n+1}+\mathbf{P}_{n-1}). \quad (4.68)$$

As stated in (4.57) and proved below in Section 4.6 this is exactly the term that must be removed from the pseudo-Lagrangian in order to reproduce the correct equations of motion. We can therefore rewrite (4.57) as follows

$$\begin{aligned} \rho^{-1}\mathcal{L}_{\text{top}} &= 2\mathcal{D}\tilde{\chi}_1 - 2\eta^{AB}\sum_{n\in\mathbb{Z}}n\mathbf{Q}_A^n\mathbf{P}_B^{-n-1} + 2\sum_{n=1}^{\infty}\mathbf{P}_n(\tilde{\chi}_{n+1}-\tilde{\chi}_{n-1}) \\ &\quad + 2\langle\tilde{\chi}_1|\mathcal{F}\rangle + \widehat{\mathbb{F}}_1^\gamma + \widehat{\mathbb{F}}_{-1}^\gamma + \omega^\alpha(\mathcal{V})\left([\mathbb{F}]_\alpha^{(1)} + [\mathbb{F}]_\alpha^{(-1)}\right). \end{aligned} \quad (4.69)$$

Notice that in this expression there are no $c_{\mathfrak{vir}}$ dependent couplings. The group cocycle in the second line does not generate $c_{\mathfrak{vir}}$ dependent terms because the Virasoro components of $[\mathbb{F}]_\alpha$ only run along \mathfrak{vir}^- . Hence, the pseudo-Lagrangian is independent of the representation in which \mathcal{P} is defined. This reflects the fact that the $\tilde{\chi}_m$ only transform under local $K(E_9)$ transformations and so their transformation does not include any \mathfrak{vir} cocycle. This must be contrasted with the $c_{\mathfrak{vir}}$ dependent transformation of χ_m^γ . In fact, the whole first line of (4.69) can be regarded as the generalised diffeomorphisms and $K(E_9)$ covariant derivative of $\tilde{\chi}_1$ at $c_{\mathfrak{vir}}=0$, a fact that will greatly simplify the computation of the scalar field equations of motion. The reason $c_{\mathfrak{vir}}$ does not cancel out from (4.57) is that the relation (4.28) between $\tilde{\chi}_1$ and χ_1^γ is based on a group cocycle that is itself $c_{\mathfrak{vir}}$ dependent. Such a dependence also disappears if we use generalised diffeomorphisms to gauge fix $\phi_n \rightarrow 0$ for $n \geq 2$. This is proved explicitly in Appendix C.2.

4.5 \mathfrak{Vir}^- gauge fixing and full pseudo-Lagrangian

The coset representative (4.1) contains exponentials of all the negative Virasoro generators. By contrast, the E_9 exceptional field theory scalar potential we constructed in [1] only includes exponentials of L_0 and L_{-1} . We will not attempt here to generalise the scalar potential of [1] to

¹⁹Such sum over \mathbf{Q}_n appears in the $K(E_9)$ covariant derivative of $\tilde{\chi}_1$ because constant shift operators \mathcal{S}_m do not commute with \mathfrak{vir} and hence the K completions of $\mathcal{S}_m(\mathcal{P})$ must transform equivalently to w^m under the Virasoro components of \mathcal{Q} .

include the other Virasoro fields. Instead, we will use the gauge freedom associated to $\text{Tr}(B^{(n)})$ to set

$$\phi_n \rightarrow 0, \quad n \geq 2. \quad (4.70)$$

We have stressed in Section 2.3 that (4.70) does not solve the two-dimensional twisted self-duality constraint. In exceptional field theory, this is remedied by the Stückelberg coupling of ϕ_n to $\text{Tr}(B^{(n)})$ through the covariant derivative. This means that we can impose (4.70) at the price of keeping $\text{Tr}(B^{(n)}) \neq 0$ in order to respect the \mathfrak{vir} components of (4.23), as each and all \mathfrak{vir} components of \mathfrak{J} must be dual to $\mathcal{D}\rho$ and hence generally non-zero. This has an important consequence. The gauge-fixed residual generalised diffeomorphisms are generated by $|\Lambda\rangle$ and $\Sigma^{(1)}$ (because traceless $\Sigma^{(n)}$, $n \geq 2$ parameters can be reabsorbed into $\Sigma^{(1)}$ up to trivial parameters), taking the form described in Section 3.1 and familiar from the previous papers [17, 1]. However, the $B^{(n)}$ fields and their field strengths $\mathcal{G}^{(n)}$ are non-vanishing for any $n > 0$.

We choose to keep $\phi_1 \neq 0$. In fact, we shall substitute $\phi_1 \rightarrow \tilde{\rho}$ in the rest of this section, arriving at the coset representative

$$\mathcal{V} = \rho^{-L_0} e^{-\tilde{\rho}L_{-1}} \overset{\circ}{V} e^{Y_{1A}T_{-1}^A} e^{Y_{2A}T_{-2}^A} \dots e^{-\sigma\mathbf{K}}, \quad (4.71)$$

which matches the one we used in [1]. The duality equation between ρ and $\tilde{\rho}$ as it descends from (4.23) reads, expanding the covariant derivatives,

$$2 \star(\text{d} - \langle \partial_\rho | A \rangle - \langle \partial_A | A \rangle)\rho = (\text{d} - \langle \partial_{\tilde{\rho}} | A \rangle - \langle \partial_A | A \rangle)\tilde{\rho} - \text{Tr}(B^{(1)}). \quad (4.72)$$

The factor of 2 on the right-hand side does not match the two-dimensional relation (2.19) if we drop internal derivatives and B fields, reflecting the fact that this is really the covariantisation of the duality relation between ρ and ϕ_1 given in (2.53), but again this is remedied here by $\text{Tr}(B^{(1)})$ which trivialises the relation between ρ and $\tilde{\rho}$ and hence also the distinction between $\tilde{\rho}$ and ϕ_1 .

The gauge fixing reduces $\gamma(w)$ to the finite expression

$$\gamma(w) \rightarrow \frac{w - \tilde{\rho}}{\rho}. \quad (4.73)$$

Comparing with the expression (2.24) appearing in the two-dimensional linear system, we see that the gauge-fixed function is missing the square root. Reproducing the square root in terms of exponentials of Virasoro generators is what had made it necessary to introduce the ϕ_n fields and their duality relations, but again this is now compensated for by the presence of the B fields. We can use (4.73) to expand \mathcal{S}_m^γ in terms of the constant \mathcal{S}_m as binomial series. In particular,

$$\mathcal{S}_1^\gamma \rightarrow \rho^{-1} (\mathcal{S}_1 - \tilde{\rho} \mathcal{S}_0), \quad \mathcal{S}_{-1}^\gamma \rightarrow \rho \sum_{n=0}^{\infty} \tilde{\rho}^n \mathcal{S}_{-1-n}. \quad (4.74)$$

Notice that the series on the right-hand side appears in the scalar potential we found in [1], acting on the internal current $\langle \mathcal{J}_\alpha |$. We thus see that \mathcal{S}_{-1}^γ is the \mathfrak{vir}^- generalisation of that expression (up to an overall factor of ρ).

In order to match the expression of the scalar potential and its field content we also need to relate $\langle \chi_1^\gamma |$ to the $\langle \chi |$ field appearing in [1]. Up to a factor of ρ (which is just due to different weight assignments), the latter gives the \mathbb{K} completion of the (gauge fixed) \mathcal{S}_{-1}^γ shift of the internal current, written in terms of the series (4.74). Indeed we defined there

$$\langle \mathcal{J}_\alpha^- | \otimes T^\alpha = \langle \mathcal{J}_\alpha | \otimes \sum_{n=0}^{\infty} \tilde{\rho}^n \mathcal{S}_{-1-n}(T^\alpha) + \langle \chi | \otimes \mathbb{K}, \quad (4.75)$$

which equals $\rho^{-1} \langle \mathcal{J}_\alpha^{(-1)} |$ and transforms as an algebra valued object under rigid $\hat{\mathfrak{e}}_8 \oplus \langle L_{-1} \rangle$. To match this relation we perform the field redefinition

$$\langle \chi_1^\gamma | = \rho \langle \chi | - 2 \langle P_1 | - \rho \omega^\alpha(\mathcal{M}) \langle \mathcal{J}_\alpha^- |. \quad (4.76)$$

where $\langle P_1 | = -\frac{1}{2} \rho^{-1} \langle \partial | \tilde{\rho}$. The presence of $\langle P_1 |$ has no effect on rigid $\hat{\mathfrak{e}}_8 \oplus \langle L_{-1} \rangle$ covariance, instead it is motivated by matching the transformation properties of $\langle \chi |$ under generalised diffeomorphisms as found in [1]. The non-covariant variation of $\langle \chi_1^\gamma |$ is given in (4.55). Those of $\langle P_1 |$ and $\langle \mathcal{J} |$ were derived in [1] and we reproduce them here:

$$\Delta_\Lambda \langle \mathcal{J}_\alpha | \otimes T^\alpha = [\Lambda]_\alpha (T^\alpha + \mathcal{M}^{-1} (T^\alpha)^\dagger \mathcal{M}) \langle \partial_\Lambda |, \quad (4.77)$$

$$\Delta_\Lambda \langle P_1 | = -\frac{\tilde{\rho}}{2\rho} \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda | - \frac{1}{2\rho} \text{Tr}(\Sigma^{(1)}) \langle \partial_\Sigma |. \quad (4.78)$$

Combining these relations with $\Delta_\Lambda \omega^\alpha(\mathcal{M}) = 0$ and using relations and definitions analogous to (4.42), (4.44) and (4.45) for \mathbb{A} , we then find

$$\Delta_\Lambda \langle \chi | = \frac{1}{\rho} (\widehat{\mathbb{A}}_{-1}^\gamma + {}^{\mathcal{M}} \widehat{\mathbb{A}}_{-1}^\gamma) \langle \partial_\Lambda | - \frac{1}{\rho^2} \langle \partial_\Sigma | \Sigma^{(1)} - \frac{\tilde{\rho}}{\rho^2} \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda |. \quad (4.79)$$

We show in Appendix C.2 that this transformation property matches the one we derived in [1].

The final pseudo-Lagrangian density, in form notation, gauge-fixed to (4.70), reads

$$\mathcal{L}_{\text{ext}} = \mathcal{L}_{\text{top}} - \star V \quad (4.80)$$

with the topological term expanded as follows²⁰

$$\begin{aligned} \rho^{-1} \mathcal{L}_{\text{top}} &= d\chi_1^\gamma - \langle \partial_\chi | A \rangle \chi_1^\gamma + \frac{1}{2\rho} \eta^{AB} \sum_{n \in \mathbb{Z}} n (J_A^n + 2[\mathbb{A}]_A^n) (J_B^{-n-1} - \tilde{\rho} J_B^{-n}) \\ &\quad - \frac{1}{2} \rho^{-1} \mathcal{D} \tilde{\rho} (\chi_2^\gamma - J_\mathbb{K}) + \frac{1}{2} \sum_{n=2}^{\infty} \rho^{-n} \sum_{k=0}^{n-2} \binom{n-2}{k} (-\tilde{\rho})^k \text{Tr}(B^{(n-k)}) (\chi_{1+n}^\gamma - \chi_{1-n}^\gamma) \\ &\quad + \langle \rho \chi - 2P_1 - \rho \omega^\alpha(\mathcal{M}) \mathcal{J}_\alpha^- | \mathcal{F} \rangle + \widehat{\mathbb{F}}_1^\gamma + {}^{\mathcal{M}} \widehat{\mathbb{F}}_1^\gamma, \end{aligned} \quad (4.81)$$

where J_A^n are the loop components of the covariant current (4.7), $[\mathbb{A}]_A^m$ are the loop components of $[\mathbb{A}]_\alpha$ defined analogously to (3.35), $\mathcal{D} \tilde{\rho} = d\tilde{\rho} - \langle \partial | (\tilde{\rho} | A) \rangle - \text{Tr}(B^{(1)})$, the cocycle $\omega^\alpha(\mathcal{M})$ is

²⁰Alternatively, one can use the more compact Unendlichbein formulation (4.69), together with equation (4.116) of [1]. In this case, notice that $\langle \tilde{\chi} |$ as defined there equals $2 \langle \tilde{\chi}_1 | + 2 \langle P_1 |$ as defined here.

defined in (4.20), the internal shifted current $\langle \mathcal{J}_\alpha^- |$ is defined in (4.75), and finally $\widehat{\mathbb{F}}_1^\gamma$ and ${}^{\mathcal{M}}\widehat{\mathbb{F}}_1^\gamma$ are defined in (4.42) and (4.45). The c_{vir} dependent couplings in (4.57) cancel out with the gauge fixing as argued at the end of the previous section and proved explicitly in Appendix C.2. The scalar potential V comes from [1]:

$$\begin{aligned}
V &= \frac{1}{4\rho} \sum_{n \in \mathbb{Z}} \sum_{k=0}^{\infty} \tilde{\rho}^k \eta^{AB} \langle J_A^{k-n} | \mathcal{M}^{-1} | J_B^n \rangle - \frac{1}{\rho} \langle P_0 | \mathcal{M}^{-1} | J_K \rangle \\
&\quad - \frac{2}{\rho} \langle P_1 | \mathcal{M}^{-1} | \rho \chi - P_1 \rangle - \omega^\alpha(\mathcal{M}) \langle P_1 | \mathcal{M}^{-1} | \mathcal{J}_\alpha^- \rangle \\
&\quad - \frac{1}{2\rho} \langle \mathcal{J}_\alpha | T^\beta \mathcal{M}^{-1} T^{\alpha\dagger} | \mathcal{J}_\beta \rangle + \frac{\rho}{2} \langle \mathcal{J}_\alpha^- | T^\beta \mathcal{M}^{-1} T^{\alpha\dagger} | \mathcal{J}_\beta^- \rangle + \frac{1}{\rho} \langle P_0 | T^\alpha \mathcal{M}^{-1} | \mathcal{J}_\alpha \rangle. \tag{4.82}
\end{aligned}$$

We used $|\mathcal{J}_\alpha\rangle = (\langle \mathcal{J}_\alpha |)^\dagger$ (and similarly for other objects) to simplify the notation, thus regarding $|\mathcal{J}_\alpha\rangle$ as an element of $R(\Lambda_0)_{+1}$. The expression $\omega^\alpha(\mathcal{M}) \langle \mathcal{J}_\alpha^- |$ read instead $\rho^{-2} \Omega^\alpha(\mathcal{M}) \langle \mathcal{J}_\alpha |$ in [1]. We define $\Omega^\alpha(\mathcal{M})$ and prove equivalence of the two expressions in Appendix B.

Note that the topological term and the potential are both invariant under (internal) generalised diffeomorphisms. The relative coefficient between the two in (4.80) is fixed by requiring that, when partially solving the section constraint as in (3.10), the field equations for the various scalar fields reproduce those of E_8 exceptional field theory. This can for instance be verified by using the field equation (4.99) obtained by varying with respect to $\langle \chi |$. We expect that the coefficient could be alternatively fixed by imposing invariance under conformal external diffeomorphisms. In the minimal formulation of Section 5, where the conformal gauge is relaxed, the relative coefficient between the topological term and the potential will be fixed (and confirmed) explicitly by requiring invariance under external diffeomorphisms.

4.6 Equations of motion

We shall now take a look at the equations of motion of Virasoro-extended E_9 exceptional field theory. These are obtained as the Euler–Lagrange equations of the pseudo-Lagrangian constructed in the previous sections, combined with the twisted self-duality condition (4.23) (or equivalently (4.29)) and with the covariantisation of the Virasoro constraint (2.17)

$$\mathcal{D}_\pm \sigma \mathcal{D}_\pm \rho - \frac{1}{2} \mathcal{D}_\pm \mathcal{D}_\pm \rho - \frac{1}{2} \rho \eta^{AB} (\mathbf{P}_\pm)_A^0 (\mathbf{P}_\pm)_B^0 = 0, \tag{4.83}$$

written here in light-cone coordinates. Up to twisted self-duality it is invariant under rigid $\widehat{E}_8 \times \text{Vir}^-$ transformations and, by covariance, under generalised diffeomorphisms. This constraint must be imposed because we only wrote the covariant action in conformal gauge. We will relax conformal gauge and see the Virasoro constraint arise as an Euler–Lagrange equation in Section 5. The equivalence of this Euler–Lagrange equation with (4.83) is shown in Appendix D. In Appendix C.4 we will exemplify how the equations of motion derived here reproduce those of E_8 exceptional field theory, by matching the equations of motion for the E_8 scalar fields. A complete matching at the level of the action will be given in the minimal formalism of Section 5. In computing the equations of motion we will not explicitly vary the scalar potential, which was

already shown to match E_8 exceptional field theory in [1], and instead focus on the variation of the topological term. These explicit computations are also meant to clarify some details of the Vir^- extended formalism, such as the need for the regularisation of certain series induced by the cocycle term in the action and motivating the addition of the extra invariant term in (4.57) compared to $\rho\mathfrak{X}^{(1)}$.

The trivial variations

We have already addressed the Euler–Lagrange equations for the one-forms χ_m^γ (or equivalently $\tilde{\chi}_m$) in (4.49), which shows that these fields appear as Lagrange multipliers for a subset of the relations imposed by twisted self-duality, and hence their equations of motion are trivialised once the latter is imposed. Let us now look at the field variations with respect to the $B^{(k)}$ fields, which should also trivialise upon imposing twisted self-duality. Without loss of generality we will also gauge fix

$$\phi_n \rightarrow 0, \quad \forall n \geq 1, \quad (4.84)$$

just like we did in the previous section, but now also setting $\tilde{\rho} \rightarrow 0$. The scalar potential of course does not contribute to the variation and we will use (4.69) as a starting point. We therefore consider the variation

$$\delta A = (0, \delta B^{(k)}). \quad (4.85)$$

(We will consider the variation with respect to $|A\rangle$ below, starting from (4.123).)

We have already stressed that the $K(E_9)$ compensator h^α appearing in (4.60)–(4.62) and other expressions cancels out in the topological term. To simplify the exposition, in this section we will then use the definitions (4.60)–(4.62) and (4.66), but removing by hand the compensator

$$h^\alpha \rightarrow 0, \quad (4.86)$$

since it is guaranteed to cancel out anyway. With this in mind, we write the B field variation of the Maurer–Cartan form, of \mathcal{P} and of \mathcal{Q} as²¹

$$\delta(\mathcal{D}\mathcal{V}\mathcal{V}^{-1}) = -[\delta A]_\alpha \mathcal{V}T^\alpha \mathcal{V}^{-1} = -2[\underline{\delta A}]_\alpha T^\alpha, \quad (4.87)$$

$$\delta\mathcal{P} = -[\underline{\delta A}]_\alpha (T^\alpha + \text{h.c.}), \quad (4.88)$$

$$\delta\mathcal{Q} = -[\underline{\delta A}]_\alpha (T^\alpha - \text{h.c.}), \quad (4.89)$$

where we have used (4.86) and we introduced the underlined notation to denote dressing by \mathcal{V} :

$$[\underline{A}]_\alpha T^\alpha = \frac{1}{2} [A]_\alpha \mathcal{V}T^\alpha \mathcal{V}^{-1}. \quad (4.90)$$

To further simplify the exposition, we shall use rigid E_9 invariance to assume that the solution of the section constraint is of the form (3.10). This guarantees that $[A]_\alpha$ takes values in \mathfrak{vir}^- and the non-positive loop levels only, the ones whose exponentials appear in our choice of

²¹Because we are ignoring the compensator, $\mathcal{D}\mathcal{V}\mathcal{V}^{-1}$ is formally no longer valued in a parabolic subalgebra of $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$ but gains positive loop components proportional to $[A]_\alpha$. This is reflected in the following computations.

coset representative (4.1), except for an $[\mathbb{A}]_A^{+1}$ component. In particular, the $B^{(k)}$ fields do not contribute to the \mathbb{K} component, namely

$$\mathrm{Tr}(B^{(k)}L_{-k}) = 0. \quad (4.91)$$

Furthermore, the $_A$ index of $[\mathbb{A}]_A^{+1}$ satisfies the E_8 section constraint and reads explicitly

$$[\mathbb{A}]_C^{+1} = \partial_C \langle 0|A \rangle = \partial_C w, \quad (4.92)$$

where in a reduction from three to two dimensions w corresponds to the Kaluza–Klein vector, see (2.12).

Looking at (4.69), the variation of the last term of the first line clearly vanishes upon imposing (4.29). The only other contribution from the first line comes from the loop cocycle:

$$-2\rho\eta^{AB}\delta\left(\sum_{n\in\mathbb{Z}}n\mathbb{Q}_A^n\mathbb{P}_B^{-n-1}\right) = 4\rho\eta^{AB}[\delta\mathbb{A}]_A^0(\star\mathbb{P}_B^0 + [\mathbb{A}]^{+1B}) + 4\rho\eta^{AB}[\delta\mathbb{A}]_A^{-2}[\mathbb{A}]_B^{+1}, \quad (4.93)$$

where we have repeatedly used twisted self-duality and also used the fact that $[\delta\mathbb{A}]_A^0$ satisfies the E_8 section (because of the section constraint on $\delta B^{(k)}$) to remove a contraction with a term $[\mathbb{A}]_A^{+1}$ (the coset dressing reduces to conjugation by \mathring{V} which cancels out in the contraction). The $[\mathbb{A}]_B^{+1}$ components appear when converting $\mathbb{Q}_A^{\pm 1}$ to $\mathbb{P}_A^{\pm 1}$ in order to use twisted self-duality on the latter. Explicitly,

$$\mathbb{Q}_A^1 = -\mathbb{P}_A^1 - 2[\mathbb{A}]_A^1, \quad \mathbb{Q}_A^{-1} = -\mathbb{P}_A^{-1} + 2[\mathbb{A}]^{1A}, \quad \mathbb{Q}_A^n = -\mathrm{sgn}(n)\mathbb{P}_A^n, \quad |n| \geq 2, \quad (4.94)$$

where the position of the ε_8 index in the middle equation is due to the ε_8 transposition $[\mathbb{A}]^{1A} = \eta^{AB}[\mathbb{A}]_B^1$ and we could write more explicitly $\mathbb{Q}_A^{-1} = -\mathbb{P}_A^{-1} + 2\delta_{AB}\eta^{BC}[\mathbb{A}]_C^1$. We will use this notation throughout this section and in Appendix C.

We now look at the variation of the second line of (4.69). The $\langle \tilde{\chi}_1|\mathcal{F} \rangle$ term does not contain any B fields so we can ignore it. The other terms can be rewritten in a more compact form thanks to the gauge fixing (4.84), which allows us to reinterpret the $\omega^\alpha(\mathcal{V})$ terms as arising from dressing a \mathfrak{vir} generator with the coset representative, as discussed in Appendix B. This gives the convenient expression

$$\begin{aligned} & \rho\widehat{\mathbb{F}}_1^\gamma + \rho\widehat{\mathbb{F}}_{-1}^\gamma + \rho\omega^\alpha(\mathcal{V})\left([\mathbb{F}]_\alpha^{(1)} + [\mathbb{F}]_\alpha^{(-1)}\right) \\ & \stackrel{(4.84)}{=} -\rho\langle\partial_{\mathcal{F}}|\mathcal{V}^{-1}(L_1 + L_{-1})\mathcal{V}|\mathcal{F}\rangle - \sum_{k=1}^{\infty}\rho^{1-k}\mathrm{Tr}\left(\mathcal{G}^{(k)}\mathcal{V}^{-1}(L_{1-k} + L_{-1-k})\mathcal{V}\right). \end{aligned} \quad (4.95)$$

In order to vary this expression we use closure of the generalised Lie derivative, (3.17), and (3.42) to deduce the natural relation²²

$$\delta\mathbb{F} = \mathbb{D}\delta\mathbb{A} + \text{trivial parameters}. \quad (4.96)$$

²²We still use the definition (3.23) for \mathbb{D} even if the extended \circ product does not satisfy the Leibniz identity. The trivial parameters on the right-hand side of (4.96) would vanish in the non-extended case.

In our case the right-hand side reduces to $(0, \mathcal{D}\delta B^{(k)})$ using (4.85) and noticing that the difference between the extended \circ product and the generalised Lie derivative vanishes when the first component of a doubled object is zero. The trivial parameters can be ignored as we know they at most contribute by a total derivative (equivalently, they can be set to vanish by combining the B field variations with an ad-hoc two-form gauge transformation). Using now the fact that the right-hand side of the expression $\mathcal{V}(\mathcal{D}\delta B^{(k)})\mathcal{V}^{-1}$ still satisfies the section constraint in the form (3.10), hence traces with negative Virasoro generators vanish, we find that the B field variation of (4.95) reduces to

$$-\mathrm{Tr}\left(\mathcal{V}(\mathcal{D}\delta B^{(1)})\mathcal{V}^{-1}L_0\right) = \mathrm{Tr}\left(\mathcal{V}\delta B^{(1)}\mathcal{V}^{-1}[L_0, \mathcal{D}\mathcal{V}\mathcal{V}^{-1}]\right) \quad (4.97)$$

up to total (internal and external) derivatives. We are again using (4.86) to simplify the notation. Only the loop components of the Maurer–Cartan form contribute. Furthermore, because $\mathcal{V} \in E_9$ by (4.84) and $\eta_{-1\alpha\beta}$ is E_9 invariant up to a weight term, one can easily combine (3.35), (4.85) and (4.90) to find

$$2\rho[\underline{\delta A}]_A^n = \eta_{AB}\mathrm{Tr}\left(\mathcal{V}\delta B^{(1)}\mathcal{V}^{-1}T_{-n-1}^B\right), \quad n \geq -2, \quad (4.98)$$

where we are restricting to $n \geq -2$ so that higher $\delta B^{(k)}$ contributions can be removed by trivial parameters (for smaller n , the \mathcal{V} conjugation implies a contribution from $\mathrm{Tr}(\delta B^{(2)})$ and higher, which are not trivial). Opening up the commutator one then quickly deduces that (4.97) exactly cancels out (4.93). The B field Euler–Lagrange equations vanish upon imposing twisted self-duality as required.

It is now straightforward to check that the B field variation of the extra term we included in (4.57) (compared to $\rho\mathfrak{X}^{(1)}$) does not vanish, and hence its inclusion in the pseudo-Lagrangian is essential for the above result to hold.

The scalar field variations

In varying the constrained scalar field $\langle\chi|$ (equivalently $\langle\chi_1^y|$ or $\langle\tilde{\chi}_1|$) we include explicitly the contribution from the scalar potential. From (4.81) and (4.82) we find the equation

$$\boxed{\langle\delta\chi|:} \quad \langle\delta\chi| \left(|\mathcal{F}\rangle + 2\star\rho^{-1}\mathcal{M}^{-1}|P_1\rangle - \star T^\alpha\mathcal{M}^{-1}|\mathcal{J}_\alpha^-\rangle \right) = 0. \quad (4.99)$$

This result does not rely on any specific solution of the section constraint, but is based on the gauge-fixing (4.70) in order to display the variation of the scalar potential explicitly. Recall that $\langle\delta\chi|$ satisfies the section constraint, hence the equation above only arises from the pseudo-Lagrangian when contracted with a constrained object.

From now on we no longer rely on a gauge-fixing for the Virasoro scalar fields and we do not commit to a specific solution of the section constraint. We also do not vary the scalar potential explicitly. Nevertheless, all computations displayed below hold if we impose the gauge-fixing (4.70), and therefore the complete equations of motion can be derived by varying (4.82) and combining it with the results in this section.

The variation of the topological term with respect to σ is also straightforward, because it only appears in (4.69) through $\tilde{\chi}_0 = \mathcal{P}_K = -\mathcal{D}\sigma$. Up to twisted self-duality we then get the equation of motion

$$\boxed{\delta\sigma:} \quad \mathcal{D} \star \mathcal{D}\rho - \frac{1}{2} \star \frac{\delta V}{\delta\sigma} = 0. \quad (4.100)$$

The extended Virasoro fields ϕ_n are pure gauge, hence their equations of motion are redundant and we do not need to compute them. We then focus on the variation of the scalars ρ , $\mathring{V} \in E_8$ and Y_n^A , $n \geq 1$. We can encode their variation in terms of an Hermitian algebra element π :

$$\pi = \pi^\dagger = \delta\mathcal{V}\mathcal{V}^{-1} \in \mathfrak{e}_9. \quad (4.101)$$

This means that the variation above is a combination of a field variation and a $K(E_9)$ transformation:

$$\delta\mathcal{V} = \delta_{\text{fields}}\mathcal{V} + \zeta\mathcal{V}, \quad \zeta \in K(\mathfrak{e}_9), \quad \zeta_A^n = \text{sgn}(n)\pi_A^n. \quad (4.102)$$

Recall now that at the end of Section 4.4 we noticed that the first line of (4.69) corresponds to the generalised diffeomorphism and $K(E_9)$ covariant derivative of $\tilde{\chi}_1$, with \mathcal{Q} as composite connection, except for the substitution $c_{\text{vir}} \rightarrow 0$ which removes a piece of the cocycle (4.68). It is then convenient to include the composite connection \mathcal{Q} in \mathcal{D} :

$$\widehat{\mathcal{D}}\Phi = \mathcal{D}\Phi + \mathcal{Q}_\alpha \delta_{K(\hat{\mathfrak{e}}_8 \oplus \text{vir})}^\alpha \Phi, \quad (4.103)$$

where the second term denotes the transformation of a field Φ under the local $K(\mathfrak{e}_9)$, extended by the anti-Hermitian part of vir , where \mathcal{Q} takes values.²³ With this definition, for instance, the first of (4.64) reads $\widehat{\mathcal{D}}\mathcal{P} = 0$ and the variation of \mathcal{P} and \mathcal{Q} then reads

$$\delta\mathcal{P} = \widehat{\mathcal{D}}\pi = \mathcal{D}\pi - [\mathcal{Q}, \pi], \quad \delta\mathcal{Q} = [\pi, \mathcal{P}], \quad (4.104)$$

with $\mathcal{D}\pi = d\pi - \langle \partial\pi|A \rangle - [A]_\alpha [h^\alpha, \pi]$. Notice that as usual the compensator h^α cancels out between the covariant derivative and the \mathcal{Q} commutator.

It turns out to be convenient to combine the scalar field variations with a special choice of variation for the $\tilde{\chi}_m$ forms. Since the latter contribute trivially to the equations of motion once twisted self-duality is imposed, this choice will not affect our results. We then introduce a spurious object $\tilde{\pi}_m$ meant to complete the K component of $\mathcal{S}_m(\pi)$, just like $\tilde{\chi}_m$ does for $\mathcal{S}_m(\mathcal{P})$:

$$\pi^{(m)} = \mathcal{S}_m(\pi) + \tilde{\pi}_m K. \quad (4.105)$$

We then choose $\delta\tilde{\chi}_m = \widehat{\mathcal{D}}\tilde{\pi}_m$ so that the variations of the shifted \mathcal{P} take the simple form

$$\delta\mathcal{P}^{(m)} = \widehat{\mathcal{D}}\pi^{(m)} = \mathcal{D}\pi^{(m)} + \mathcal{Q}_\alpha \delta_{K(\hat{\mathfrak{e}}_8 \oplus \text{vir})}^\alpha \pi^{(m)}, \quad (4.106)$$

²³The presence of $K(\text{vir})$ can appear strange since only vir^- are global symmetries of the theory and they do not require compensating local transformations. This is due to the fact that the ungauged theory is in fact also invariant under infinitesimal positive vir transformations and the decomposition of the Maurer–Cartan form into \mathcal{P} and \mathcal{Q} reflects this. Most objects we use have therefore natural transformation properties under $K(\text{vir})$ obtained by extension of their $K(\mathfrak{e}_9)$ transformation and standard commutation relations, which simplifies many manipulations in the following.

where $\delta_K^\alpha(\hat{\epsilon}_s \oplus \mathfrak{vir})\pi^{(m)}$ includes a sum over Virasoro shifts just like the last term of the first line in (4.65). Explicitly,

$$\mathcal{Q}_\alpha \delta_K^\alpha(\hat{\epsilon}_s \oplus \mathfrak{vir})\pi^{(m)} = -[\mathcal{Q}, \pi^{(m)}] + m \sum_{k \in \mathbb{Z}} \mathcal{Q}_k \pi^{(m+k)}. \quad (4.107)$$

After several manipulations, we will set $\tilde{\pi}_m \rightarrow 0$ without loss of generality.

With this setup, in order to compute the variation of the first line in (4.69), we can equivalently compute

$$\delta(2\rho \widehat{\mathcal{D}}\tilde{\chi}_1) = \delta(2\rho \widehat{\mathcal{D}}\mathcal{P}^{(1)})|_{\mathbb{K}} \quad (4.108)$$

and set $c_{\mathfrak{vir}} \rightarrow 0$ in the cocycle. We then expand to find

$$\delta(2\rho \widehat{\mathcal{D}}\mathcal{P}^{(1)}) = 2\delta\rho \widehat{\mathcal{D}}\mathcal{P}^{(1)} - 2\rho [\delta\mathcal{Q}, \mathcal{P}^{(1)}] + 2\rho \widehat{\mathcal{D}}\delta\mathcal{P}^{(1)}, \quad (4.109)$$

where we have already used twisted self-duality to remove a series over the \mathfrak{vir} components of \mathcal{Q} . Let us look at the middle term. Projecting onto \mathbb{K} and using (4.104) we find that it vanishes. For instance, using twisted self-duality the loop components of π contribute as follows (recalling that we are keeping wedge products as understood)

$$f^{ABC} \sum_{n,q} (-1)^n n \pi_A^q \star^{q+1} P_B^0 P_C^0 - \eta^{AB} \sum_{n,q} (-1)^n n q \pi_A^{-q} \star^{q+1} P_0 P_B^0 = 0. \quad (4.110)$$

A similar computation applies to the L_0 component of π and since we set $c_{\mathfrak{vir}} \rightarrow 0$, that is all we need.

Let us then look at the last term in (4.109), which is proportional to $\widehat{\mathcal{D}}^2\pi^{(1)}$. The square of the covariant differential has the general form

$$\widehat{\mathcal{D}}^2 = -\mathcal{L}_{\mathbb{F}} + \left(\mathcal{D}\mathcal{Q} - \frac{1}{2}[\mathcal{Q}, \mathcal{Q}] \right)_\alpha \delta_K^\alpha(\hat{\epsilon}_s \oplus \mathfrak{vir}), \quad (4.111)$$

and we recognise the last term as the covariant field strength of the composite \mathcal{Q} connection. We can use the Maurer–Cartan equations (4.64) to further manipulate this expression. In order to do so, notice that twisted self-duality implies $[\mathcal{P}, \mathcal{P}] = 0$. The \mathbb{K} component is straightforward. For the rest, we use $\mathcal{S}_0(\mathcal{P}) = \mathcal{S}_m(\mathcal{S}_{-m}(\mathcal{P}))$ for any $m \neq 0$ so that

$$\mathcal{S}_0([\mathcal{P}, \mathcal{P}]) = \mathcal{S}_{-m}([\mathcal{P}, \mathcal{S}_m(\mathcal{P})]) + m \sum_{p \in \mathbb{Z}} P_p \mathcal{S}_p(\mathcal{P}). \quad (4.112)$$

Let us now take m odd. By twisted self-duality, the first term of the right-hand side vanishes. The second term flips sign under $m \rightarrow -m$ while the left-hand side does not, hence they both vanish. We can then substitute into (4.111) the left-hand side of the Maurer–Cartan equation (4.64) for the curvature of \mathcal{Q} , and apply it to $\pi^{(1)}$. We find (setting $h^\alpha \rightarrow 0$ as usual as it cancels out anyway)

$$\begin{aligned} \widehat{\mathcal{D}}^2\pi^{(1)} &= -\langle \partial_\pi | \mathcal{F} \rangle \pi^{(1)} + \left(\frac{1}{2} \langle \mathcal{Q}_\alpha | \mathcal{F} \rangle + \underline{[\mathbb{F}]}_\alpha \right) [T^\alpha - T^{\alpha\dagger}, \pi^{(1)}] \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{1}{2} \langle \mathcal{Q}_n | \mathcal{F} \rangle + \underline{[\mathbb{F}]}_n \right) (\pi^{(1+n)} - \pi^{(1-n)}). \end{aligned} \quad (4.113)$$

Projecting this expression on \mathbb{K} and setting to zero the central charge component $\tilde{\pi}_1$ of $\pi^{(1)}$, we find that only the middle term contributes and arrive at the result

$$2\rho \widehat{\mathcal{D}} \delta \mathcal{P}^{(1)}|_{\mathbb{K}} = 2\rho \eta^{AB} \sum_{n \in \mathbb{Z}} n \left(\langle Q_A^n | \mathcal{F} \rangle + \underline{[\mathbb{F}]}_A^n - \underline{[\mathbb{F}]}^{-nA} \right) \pi_B^{-n-1}. \quad (4.114)$$

Notice that the variation of ρ does not contribute to this expression.

Let us now look at the variation of the second line of (4.69). Using (B.4) one finds that the ρ variation of $\omega^\alpha(\mathcal{V})$ vanishes. Furthermore, from the definition (B.11) one easily sees that both $\widehat{\mathbb{F}}_m^\gamma$ and $[\mathbb{F}]_\alpha^{(m)}$ depend on ρ only through an overall ρ^{-m} factor. Therefore, the ρ variation of the second line of (4.69) reads

$$2\delta\rho \left(\langle \tilde{\chi}_1 | \mathcal{F} \rangle + \widehat{\mathbb{F}}_{-1}^\gamma + \omega^\alpha(\mathcal{V}) [\mathbb{F}]_\alpha^{(-1)} \right). \quad (4.115)$$

Looking now at the \widehat{E}_8 scalars variation, we must take into account that $\langle \tilde{\chi}_1 |$ transforms under the $K(\mathfrak{e}_9)$ transformation ζ we introduced in (4.102) to make π Hermitian, so that

$$\delta \langle \tilde{\chi}_1 | \mathcal{F} \rangle = \eta^{AB} \sum_{n \in \mathbb{Z}} |n| \pi_A^n \langle P_B^{-n-1} | \mathcal{F} \rangle, \quad (4.116)$$

while the variation of $\omega^\alpha(\mathcal{V})$ gives

$$\delta \omega^\alpha(\mathcal{V}) = [\pi, \mathcal{V} T^\alpha \mathcal{V}^{-1}]|_{\mathbb{K}}. \quad (4.117)$$

We can now put these results together. The ρ variation of the topological term is the sum of the first term in (4.109) and of (4.115). Applying twisted self-duality, we find the equation of motion

$$\boxed{\delta\rho:} \quad \widehat{\mathcal{D}} \star \mathcal{P}_{\mathbb{K}} + \langle \tilde{\chi}_1 | \mathcal{F} \rangle + \widehat{\mathbb{F}}_{-1}^\gamma + \omega^\alpha(\mathcal{V}) [\mathbb{F}]_\alpha^{(-1)} = \frac{1}{2} \star \frac{\delta V}{\delta \rho}, \quad (4.118)$$

where we need to keep in mind that $c_{\text{vir}} \rightarrow 0$ in the cocycle contained in the first term. We can in fact further apply twisted self-duality to it, in order to make contact with the second order equation for σ for two-dimensional supergravity (2.18). To do so, we shall temporarily assume a solution of section of the form (3.10) and write

$$\begin{aligned} \widehat{\mathcal{D}} \star \mathcal{P}_{\mathbb{K}} &= \mathcal{D} \star \mathcal{P}_{\mathbb{K}} - \eta^{AB} \sum_{n \in \mathbb{Z}} n Q_A^n \star P_B^{-n} \\ &= -\mathcal{D} \star \mathcal{D} \sigma + \eta^{AB} \sum_{n \in \mathbb{Z}} |n| P_A^n \star P_B^{-n} + 4\eta^{AB} \underline{[\mathbb{A}]}_A^{+1} \star P_B^{-1} \\ &= -\mathcal{D} \star \mathcal{D} \sigma + 2\eta^{AB} P_A^0 \star P_B^0 \left(\sum_{n=1}^{\infty} (-1)^n n \right) + 4\eta^{AB} \underline{[\mathbb{A}]}_A^{+1} P_B^0 \\ &= -\mathcal{D} \star \mathcal{D} \sigma - \frac{1}{2} \eta^{AB} P_A^0 \star P_B^0 + 4\eta^{AB} \underline{[\mathbb{A}]}_A^{+1} P_B^0, \quad (\text{with } h^\alpha \rightarrow 0) \end{aligned} \quad (4.119)$$

In the second line we have converted Q_A^n to P_A^n and we stressed that we set $h^\alpha \rightarrow 0$ everywhere as usual, because it would cancel out between \mathcal{Q} and the covariant derivative anyway. In the third line we have used twisted self-duality and in the last step we have regularised the divergent sum

as $\sum_{n=1}^{\infty} (-1)^n n \rightarrow -\frac{1}{4}$. This is achieved for instance by taking the $z \rightarrow 1$ limit of a geometric series $\sum_{n=1}^{\infty} (-z)^n p(n)$ which converges for $|z| < 1$ and $p(n)$ any polynomial in n . We motivate this choice of regularisation in Appendices C.3 and C.4. We then see that if we reduce to two-dimensional supergravity by setting $\langle \partial | = 0$ and $B^{(k)} = 0$, this equation of motion reduces to (2.18). Notice how this result hinges on the removal of the c_{bit} dependent terms of the cocycle in $\widehat{\mathcal{D}} \star \mathcal{P}_K$. Had we not removed the term (4.68) from the pseudo-Lagrangian, it would have contributed here through a term proportional to $(\mathcal{D}\rho)^2$ (with a regularisation similar to the one above).

Let us now look at the variation of the loop scalar fields. Using the Maurer–Cartan equation (4.64) for \mathcal{P} combined with twisted self-duality (4.29), we find the convenient identity

$$\widehat{\mathcal{D}} \star^{|n|} \mathbf{P}_A^0 = -\langle P_A^{n+1} | \mathcal{F} \rangle - \underline{[\mathbb{F}]_A^{n+1}} - \underline{[\mathbb{F}]_A^{-n-1} A}, \quad n \in \mathbb{Z}, \quad (4.120)$$

which implies in particular that all the even components of the right-hand side are equal to each other, and the same holds for the odd ones. Combining this expression with (4.114), (4.116) and (4.117), the latter contracted with $2\rho([\mathbb{F}]_A^{(1)} + [\mathbb{F}]_A^{(-1)})$, writing $\langle Q_A^n | = -\text{sgn}(n)\langle P_A^n |$ and using that π is Hermitian we then find the equation of motion

$$\boxed{\delta\mathring{V}, \delta\mathbf{Y}_n:} \quad 2\rho \pi^{0A} (\widehat{\mathcal{D}} \star \mathbf{P}_A^0 + 2\underline{[\mathbb{F}]_A^{+1}}) - 4\rho \sum_{k=1}^{\infty} \pi^{-kA} (\underline{[\mathbb{F}]_A^{k-1}} - \underline{[\mathbb{F}]_A^{k+1}}) = \star \delta V. \quad (4.121)$$

The π_A^0 component of this equation still contains infinitely many dual fields within the covariant derivative. Using twisted self-duality to express it only in terms of physical fields, we are led to a regularisation analogous to the one used for (4.119). In analogy to that computation we temporarily choose a section solution of the form (3.10) to simplify the vector field dependence and write explicitly

$$\begin{aligned} \widehat{\mathcal{D}} \star \mathbf{P}_A^0 &= \mathcal{D} \star \mathbf{P}_A^0 - f^{CD}{}_A \sum_{n \in \mathbb{Z}} \mathbf{Q}_C^n \star \mathbf{P}_D^{-n} + \sum_{n \in \mathbb{Z}} n (\mathbf{Q}_{-n} \star \mathbf{P}_A^n - \mathbf{Q}_A^n \star \mathbf{P}_{-n}) \\ &= \mathcal{D} \star \mathbf{P}_A^0 - f^{CD}{}_A \mathbf{Q}_C^0 \star \mathbf{P}_D^0 - 2 \sum_{n=1}^{\infty} n (\mathbf{Q}_A^n \star \mathbf{P}_{-n} + \mathbf{Q}_n \star \mathbf{P}_A^{-n}) \\ &\quad + 2f^{CD}{}_A (\underline{[\mathbb{A}]_C^1} - \underline{[\mathbb{A}]^{1C}}) \mathbf{P}_D^0 - 2\mathbf{P}_0 (\underline{[\mathbb{A}]_C^1} + \underline{[\mathbb{A}]^{1C}}) \\ &= \mathcal{D} \star \mathbf{P}_A^0 - f^{CD}{}_A \mathbf{Q}_C^0 \star \mathbf{P}_D^0 + 2\mathbf{P}_0 \star \mathbf{P}_A^0 \sum_{n=1}^{\infty} (-1)^n n \\ &\quad + 2f^{CD}{}_A (\underline{[\mathbb{A}]_C^1} - \underline{[\mathbb{A}]^{1C}}) \mathbf{P}_D^0 - 2\mathbf{P}_0 (\underline{[\mathbb{A}]_C^1} + \underline{[\mathbb{A}]^{1C}}) \\ &= \rho^{-1} \mathcal{D} (\rho \star \mathbf{P}_A^0) - f^{CD}{}_A \mathbf{Q}_C^0 \star \mathbf{P}_D^0 + 2f^{CD}{}_A (\underline{[\mathbb{A}]_C^1} - \underline{[\mathbb{A}]^{1C}}) \mathbf{P}_D^0 - 2\mathbf{P}_0 (\underline{[\mathbb{A}]_A^1} + \underline{[\mathbb{A}]^{1A}}). \end{aligned} \quad (4.122)$$

We check in Appendix C.4 that the π_A^0 equation of motion correctly reproduces the equation of motion of the scalar fields in E_8 exceptional field theory. As a further cross-check, we can reduce to two-dimensional supergravity by setting $\langle \partial | \rightarrow 0$ and check that the equations of motion derived so far are then compatible with setting the $B^{(k)}$ fields to 0 as follows. The vector fields $|A\rangle$ and $\langle \chi_1^\gamma |$ decouple when $\langle \partial | = 0$, and the field equation for ρ reduces to $d \star d\rho = 0$.

Extending then (4.120) also along the \mathfrak{vir} components and using the Maurer–Cartan equations as well as the π_A^0 equation above, we find that all \mathfrak{vir}^- components of $[\mathbb{F}]_\alpha$, as well as the coset projection of $[\mathbb{F}]_A^n$ must vanish. Plugging these results into the rest of equation (4.121), we then have that all non-trivial components of the $B^{(k)}$ field strengths must vanish, hence without loss of generality we can set $B^{(k)} \rightarrow 0$ as claimed. Together with the fact that (4.23) correctly reduces to two-dimensional twisted self-duality when $\langle \partial | = 0 = B^{(k)}$, this guarantees that two-dimensional supergravity is reproduced from Virasoro-extended E_9 ExFT.

The $|A\rangle$ vector field variation

The starting point is again the pseudo-Lagrangian (4.69), which we will vary with respect to the vector field $|A\rangle$. We recall that we work in the ungauged-fixed setting where $\phi_m \neq 0$ for all $m > 0$. We also re-use the notation introduced in (4.85), but this time with

$$\delta\mathbb{A} = (|\delta A\rangle, 0). \quad (4.123)$$

The variations of \mathcal{P} and \mathcal{Q} follow directly from their expressions (4.61) and (4.62). They read

$$\delta\mathcal{P} = -\langle \mathcal{P} | \delta A \rangle - \underline{[\delta\mathbb{A}]}_\alpha (T^\alpha + \text{h.c.}), \quad (4.124)$$

$$\delta\mathcal{Q} = -\langle \mathcal{Q} | \delta A \rangle - \underline{[\delta\mathbb{A}]}_\alpha (T^\alpha - \text{h.c.}), \quad (\text{with } h^\alpha \rightarrow 0), \quad (4.125)$$

where $\underline{[\delta\mathbb{A}]} = \underline{[\delta\mathbb{A}]}_\alpha T^\alpha$ takes values in $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$. As explained previously, we consistently ignore the h^α contributions as they ultimately cancel out in the pseudo-Lagrangian.

Let us start by focusing on the variation of the first term of the first and second line in (4.69). With (4.66) and (4.96), we find

$$2\rho \delta (\mathcal{D}\tilde{\chi}_1 + \langle \tilde{\chi}_1 | \mathcal{F} \rangle) = 2\rho (\langle \partial_{\tilde{\chi}} | \tilde{\chi}_1 - \mathcal{D}\langle \tilde{\chi}_1 | + \mathbb{P}_0 \langle \tilde{\chi}_1 |) | \delta A \rangle, \quad (4.126)$$

up to internal and external total derivatives. For the cocycle term of the first line, one computes

$$\begin{aligned} -2\rho \eta^{AB} \delta \left(\sum_{n \in \mathbb{Z}} n Q_A^n P_B^{-n-1} \right) &= 2\rho \eta^{AB} \sum_{n \in \mathbb{Z}} n \left(Q_A^n \langle P_B^{-n-1} | - P_A^{-n-1} \langle Q_B^n | \right) \\ &\quad + 4\rho \eta^{AB} \sum_{n \in \mathbb{Z}} (n+1) (Q + P)_A^n \underline{[\delta\mathbb{A}]}_B^{-n-1}, \end{aligned} \quad (4.127)$$

where we used twisted self-duality in writing the last line. For the third term of the first line in (4.69), the variation of \mathbb{P}_n does not contribute since $(\tilde{\chi}_{n+1} - \tilde{\chi}_{n-1})$ vanishes by twisted self-duality. The only contribution comes from the variation of $\tilde{\chi}_0 = \mathbb{P}_K$, and we then simply have

$$2\rho \delta \left(\sum_{n=1}^{\infty} \mathbb{P}_n (\tilde{\chi}_{n+1} - \tilde{\chi}_{n-1}) \right) = 2\rho \mathbb{P}_1 (\langle P_K | \delta A \rangle + 2 \underline{[\delta\mathbb{A}]}_K). \quad (4.128)$$

The variation of the remaining terms in the second line of the pseudo-Lagrangian (4.69) requires more care. To derive the variation of $\widehat{\mathbb{F}}_1^\gamma$ and $\widehat{\mathbb{F}}_{-1}^\gamma$, which are defined in (4.42), we use

the following property which holds exactly for $n < 1$ and up to a total derivative for $n = 1$:

$$\begin{aligned}\delta \widehat{\mathbb{F}}_n^\gamma &= (\widehat{\mathbb{D}\delta\mathbb{A}})_n^\gamma = (d - \delta_{\mathbb{A}})\widehat{\delta\mathbb{A}}_n^\gamma - n \sum_{q=0}^{\infty} (\mathcal{D}\Gamma\Gamma^{-1})_{-q} \widehat{\delta\mathbb{A}}_{n-q}^\gamma \\ &= \mathcal{D}\widehat{\delta\mathbb{A}}_n^\gamma - \Delta_{\mathbb{A}}\widehat{\delta\mathbb{A}}_n^\gamma - n \sum_{q=0}^{\infty} (\mathcal{D}\Gamma\Gamma^{-1})_{-q} \widehat{\delta\mathbb{A}}_{n-q}^\gamma, \quad \forall n \leq 1,\end{aligned}\tag{4.129}$$

where we used (4.96) in the first step. Note that the double object $\delta\mathbb{A}$ transforms with the extended \circ product under a gauge variation, and is therefore treated on the same footing as \mathbb{F} . This allows, in the second step, to extract the differential operator $(d - \delta_{\mathbb{A}})$ out the combination $\widehat{\delta\mathbb{A}}_n^\gamma$. In the process we however generate the last term of the first line, which subtracts the action of the gauge variation on the Virasoro scalars contained in $\widehat{\delta\mathbb{A}}_n^\gamma$. This is described in Appendix C.1. The expression of the latter takes the same form as in (4.42). The non-covariant part of its gauge variation with respect to \mathbb{A} , defined in (4.50) and denoted by $\Delta_{\mathbb{A}}$, can be deduced from (C.14). We thus find

$$\begin{aligned}\delta(\rho \widehat{\mathbb{F}}_{-1}^\gamma) &= \rho \mathcal{D}\widehat{\delta\mathbb{A}}_{-1}^\gamma + \rho \sum_{q=0}^{\infty} (\mathcal{D}\Gamma\Gamma^{-1})_{-q} \widehat{\delta\mathbb{A}}_{-1-q}^\gamma + \rho \widehat{\mathbb{A}}_{-1}^\gamma \langle \partial_{\mathbb{A}} | \delta A \rangle, \\ \delta(\rho \widehat{\mathbb{F}}_1^\gamma) &= \rho \mathcal{D}\widehat{\delta\mathbb{A}}_1^\gamma - \rho \sum_{q=0}^{\infty} (\mathcal{D}\Gamma\Gamma^{-1})_{-q} \widehat{\delta\mathbb{A}}_{1-q}^\gamma + \rho \widehat{\mathbb{A}}_1^\gamma \langle \partial_{\mathbb{A}} | \delta A \rangle \\ &\quad - (\langle \partial_B | B^{(1)} - \text{Tr}(B^{(1)}) \langle \partial_B | | \delta A \rangle),\end{aligned}\tag{4.130}$$

$$\tag{4.131}$$

where $\widehat{\mathbb{A}}_n^\gamma$ is again of the same form as (4.42). The second equation holds only up to a total derivative. To treat the last terms in the pseudo-Lagrangian (4.69) which involve the $\omega^\alpha(\mathcal{V})$ cocycle, let us first consider $[\mathbb{F}]_\alpha^{(n)}$ defined in (4.43). Up to a \mathbb{K} component, the latter can be written as $\mathcal{S}_n^\gamma([\mathbb{F}])_\alpha$, where in order to simplify the notation in the next few steps, we will write $[\mathbb{F}] = [\mathbb{F}]_\alpha T^\alpha$ and similarly for other objects such as $[\mathbb{A}]$, their derivatives and their variation. Taking into account that $\omega^K(\mathcal{V}) = 0$, we then have for $n \leq 1$,

$$\begin{aligned}\delta(\omega^\alpha(\mathcal{V})[\mathbb{F}]_\alpha^{(n)}) &= \omega^\alpha(\mathcal{V})\mathcal{S}_n^\gamma([\mathbb{D}\delta\mathbb{A}])_\alpha \\ &= \omega^\alpha(\mathcal{V})\left(\mathcal{D}\mathcal{S}_n^\gamma([\delta\mathbb{A}])_\alpha - \Delta_{\mathbb{A}}\mathcal{S}_n^\gamma([\delta\mathbb{A}])_\alpha - n \sum_{q=0}^{\infty} (\mathcal{D}\Gamma\Gamma^{-1})_{-q} \mathcal{S}_{n-q}^\gamma([\delta\mathbb{A}])_\alpha\right).\end{aligned}\tag{4.132}$$

We used the same arguments as in (4.129) in order to extract the differential operator out of the shift operator. At this point, observe that according to (B.10), we can write

$$\omega^\alpha(\mathcal{V})\mathcal{S}_n^\gamma([\delta\mathbb{A}])_\alpha = \mathcal{V}\mathcal{S}_n^\gamma([\delta\mathbb{A}])\mathcal{V}^{-1}|_{\mathbb{K}} = \mathcal{V}[\delta\mathbb{A}]^{(n)}\mathcal{V}^{-1}|_{\mathbb{K}} - \widehat{\delta\mathbb{A}}_n^\gamma,\tag{4.133}$$

where $[\delta\mathbb{A}]^{(n)}$ is, in analogy with (4.43), the \mathbb{K} -completed version of $\mathcal{S}_n^\gamma([\delta\mathbb{A}])$ and therefore transforms with a commutator under rigid $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$. Bearing this in mind we get, after an

integration by parts,

$$\begin{aligned}
\delta\left(\rho\omega^\alpha(\mathcal{V})[\mathbb{F}]_\alpha^{(n)}\right) &= \mathcal{D}\left(\rho\mathcal{V}[\delta\mathbb{A}]^{(n)}\mathcal{V}\right)|_{\mathbb{K}} - \rho\mathcal{D}\widehat{\delta\mathbb{A}}_n^\gamma + \rho\mathbb{P}_0\left(\widehat{\delta\mathbb{A}}_n^\gamma + \omega^\alpha(\mathcal{V})\mathcal{S}_n^\gamma([\delta\mathbb{A}])\right) \\
&\quad - n\rho\sum_{q=0}^{\infty}(\mathcal{D}\Gamma\Gamma^{-1})_{-q}\omega^\alpha(\mathcal{V})\mathcal{S}_{n-q}^\gamma([\delta\mathbb{A}])_\alpha - \rho\omega^\alpha(\mathcal{V})\Delta_\mathbb{A}\mathcal{S}_n^\gamma([\delta\mathbb{A}])_\alpha \\
&\quad - \rho[(\mathcal{P} + \mathcal{Q}), \mathcal{V}\mathcal{S}_n^\gamma([\delta\mathbb{A}])\mathcal{V}^{-1}]|_{\mathbb{K}}, \tag{4.134}
\end{aligned}$$

where the non-covariant variation of $[\delta\mathbb{A}]_\alpha$ takes the same form as in (4.51). The first term can be dropped as it reduces to total internal and external derivatives. Combining this result with (4.130) and (4.131) we find, after rearranging the sums of the $(\mathcal{D}\Gamma\Gamma^{-1})$ terms and using twisted self-duality,

$$\begin{aligned}
&\delta\left(\rho\widehat{\mathbb{F}}_1^\gamma + \rho\widehat{\mathbb{F}}_{-1}^\gamma + \rho\omega^\alpha(\mathcal{V})\left([\mathbb{F}]_\alpha^{(1)} + [\mathbb{F}]_\alpha^{(-1)}\right)\right) \\
&= \rho\sum_{n=1,-1}\left(\widehat{\mathbb{A}}_n^\gamma + \omega^\alpha(\mathcal{V})\mathcal{S}_n^\gamma([\mathbb{A}])_\alpha\right)\langle\partial_\mathbb{A}|\delta\mathbb{A}\rangle - (\langle\partial_B|B^{(1)} - \text{Tr}(B^{(1)})\langle\partial_B|\rangle|\delta\mathbb{A}\rangle \\
&\quad - 2\rho\mathbb{P}_1\left(\widehat{\delta\mathbb{A}}_0^\gamma + \omega^\alpha(\mathcal{V})\mathcal{S}_0^\gamma([\delta\mathbb{A}])_\alpha\right) - 2\rho\sum_{n=1,-1}[(\mathcal{P} + \mathcal{Q}), \mathcal{S}_n([\delta\mathbb{A}])]|_{\mathbb{K}}. \tag{4.135}
\end{aligned}$$

Using (4.133) and $\widehat{\delta\mathbb{A}}_0^\gamma = [\delta\mathbb{A}]_{\mathbb{K}}$, the parenthesis in the last line can be written as $2[\delta\mathbb{A}]_{\mathbb{K}}$ and will therefore cancel against the last term in (4.128). Note also that, in writing the last term of (4.135), we have pulled the conjugation by \mathcal{V} inside of the the shift operator and used the notation (4.90). The loop contribution of this commutator will cancel against the last line of (4.127), while its Virasoro contribution should vanish by itself since the pseudo-Lagrangian we vary does not depend on c_{vir} . This is indeed the case because both the Maurer–Cartan form $(\mathcal{P} + \mathcal{Q})$ and $[\delta\mathbb{A}]$ only take values in $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$.

Adding up all the variations, we then obtain the following final form for the field equation

$$\begin{aligned}
\boxed{|\delta\mathbb{A}\rangle}: \quad &\langle\partial_{\tilde{\chi}}|\tilde{\chi}_1 - \mathcal{D}\langle\tilde{\chi}_1| + \mathbb{P}_0\langle\tilde{\chi}_1| + \mathbb{P}_1\langle P_{\mathbb{K}}| + \eta^{AB}\sum_{n\in\mathbb{Z}}n\left(\mathbb{Q}_A^n\langle P_B^{-n-1}| - \mathbb{P}_A^{-n-1}\langle\mathbb{Q}_B^n|\right) \\
&+ \frac{1}{2}\sum_{n=1,-1}\left(\widehat{\mathbb{A}}_n^\gamma + \omega^\alpha(\mathcal{V})[\mathbb{A}]_\alpha^{(n)}\right)\langle\partial_\mathbb{A}|\rangle - \frac{1}{2\rho}\langle\partial_B|B^{(1)} + \frac{1}{2\rho}\text{Tr}(B^{(1)})\langle\partial_B|\rangle = 0, \tag{4.136}
\end{aligned}$$

where we divided by an overall factor of 2ρ and removed the projection on $|\delta\mathbb{A}\rangle$. Note that the terms in the bracket of the second line correspond to the \mathbb{K} completion of $[\mathbb{A}]_\alpha\mathcal{S}_n(\mathcal{V}T^\alpha\mathcal{V}^{-1})$, analogously to how $\tilde{\chi}_m$ completes $\mathcal{S}_m(\mathcal{P})$ in (4.28). We also emphasise that $\langle\partial_\mathbb{A}|\rangle$ only acts on the vector fields $|A\rangle$, $B^{(k)}$ and not on the scalar fields.

While it is not manifest, one can verify that the complete set of field equations is invariant under rigid E_9 . The equation (4.136) is separately invariant and one way of checking this is by using the invariance of the \mathbb{K} -completion of a shifted Maurer–Cartan equation, such as (4.34), but now with one internal and one external derivative.

5 Minimal E_9 exceptional field theory

In this section, we shall show that there is a formulation of E_9 exceptional field theory with a finite set of fields, i.e. $\mathcal{M} \in E_9, |A\rangle, B, \chi, \langle\chi|$, even though they are in infinite-dimensional representations of E_9 . As explained in Section 4.5, one can gauge-fix all the Virasoro fields ϕ_n to zero. The price to pay is that the truncation to two-dimensional supergravity, with fields not depending on the internal coordinates, must necessarily involve non-zero constrained fields $B^{(k)}$. This obscures the relation to the linear system in two dimensions. But it has the advantage that this allows us to eliminate the infinite set of constrained one-forms $B^{(k)}$ and one-forms χ_k^γ in favour of a single constrained one-form $B \sim B_\mu^M{}_N$ and a single one-form $\chi \sim \chi_\mu$ (that still transforms indecomposably with the current) as we shall demonstrate in this section. Moreover, the twisted self-duality equation for the scalar fields can be written using a E_9 invariant bilinear form, and the pseudo-Lagrangian takes a form that is more similar to lower-rank ExFTs. In particular, we relax the conformal gauge and define external diffeomorphisms in this minimal formulation. We will exhibit that the complete pseudo-Lagrangian is determined by internal and external diffeomorphism invariance, similar to E_n ExFT for $n \leq 8$. For the appropriate choice of solution to the section constraint, we will finally show that the Euler–Lagrange equations of the pseudo-Lagrangian (combined with the duality equation), reproduce the known field equations of E_8 exceptional field theory. All expressions will be finite and we shall not need to resort to formal geometric series summation as in the Virasoro-extended formulation.

5.1 Integrating out the auxiliary fields

At vanishing Virasoro fields ϕ_n , i.e. for $\Gamma = \rho^{-L_0}$, one can eliminate the constrained fields $B^{(k)}$ and the scalar fields χ_k^γ in favour of a single constrained field B and a single scalar field χ . Notice that we also gauge-fix $\phi_1 = \tilde{\rho} = 0$, unlike in Section 4.5. In this case, the coset representative \mathcal{V} in (4.1) and the generalised metric \mathcal{M} belong to E_9 . In order to obtain the minimal duality equation from (4.25), it will be convenient to first rewrite (4.23) as

$$\tilde{\mathfrak{J}} = \rho^{-1} \star (\mathcal{M}^{-1} \mathcal{S}_1(\tilde{\mathfrak{J}})^\dagger \mathcal{M} + \rho \chi_1^\gamma \mathbf{K}) . \quad (5.1)$$

To analyse the duality equation (5.1) we decompose the current (4.6) as

$$\tilde{\mathfrak{J}}_\alpha = \mathcal{J}_\alpha^b - \sum_{k=1}^{\infty} \left(\eta_{-k\alpha\beta} \text{Tr}[T^\beta B^{(k)}] + \rho^{-2k} \eta_{k\alpha\beta} \text{Tr}[\mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M} B^{(k)}] \right) , \quad (5.2)$$

where

$$\mathcal{J}^b = \mathcal{M}^{-1} \mathcal{D}^b \mathcal{M} = \mathcal{M}^{-1} (\text{d} - \langle \partial_{\mathcal{M}} | A \rangle) \mathcal{M} - \eta_{\alpha\beta} \langle \partial_A | (T^\alpha + \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M}) | A \rangle T^\beta , \quad (5.3)$$

is expressed using a ‘bare’ covariant derivative

$$\mathcal{D}^b = \text{d} - \mathcal{L}_{(A,0)} \quad (5.4)$$

that does not contain the $B^{(k)}$ terms.

It will be convenient to similarly redefine the fields χ_k^γ in terms of bare fields χ_k^b by separating out the $B^{(k)}$ components according to

$$\chi_k^\gamma = \rho^{-k} \chi_k^b + \rho^{-k} \sum_{q=0}^{\infty} \text{Tr} \left[(L_{-1+k-q} + \rho^{-2-2q} \mathcal{M}^{-1} L_{-k-1-q} \mathcal{M}) B^{(1+q)} \right]. \quad (5.5)$$

The field χ_k^b is the K-completion of $\mathcal{S}_k(\mathcal{J}^b)$ and the terms involving B play a similar role for the B -terms in (5.2). We also recall from the text under (4.25) that for $k \geq 1$

$$\chi_{1+k}^\gamma = \star^k \chi_1^\gamma. \quad (5.6)$$

The components of (5.1) along the Virasoro generators give that

$$\text{Tr}[B^{(1+k)}] = \rho^k \star^k \text{Tr}[B^{(1)}], \quad (5.7)$$

relating the traces of all higher $B^{(k)}$ to that of $B^{(1)}$. Therefore we can rewrite the current \mathfrak{J} as

$$\begin{aligned} \mathfrak{J} &= \mathcal{J}^b - \eta_{-1\alpha\beta} \text{Tr} \left[\sum_{k=0}^{\infty} \mathcal{S}_{-k}(T^\alpha) B^{(1+k)} \right] (T^\beta + \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M}) \\ &\quad + \sum_{k=0}^{\infty} \rho^k \star^k \text{Tr}[B^{(1)}] (L_{-1-k} + \mathcal{M}^{-1} L_{1+k} \mathcal{M}) \end{aligned} \quad (5.8)$$

in which the constrained fields $B^{(1+k)}$ only appear through

$$\text{Tr} \left[\sum_{k=0}^{\infty} \mathcal{S}_{-k}(T^\alpha) B^{(1+k)} \right] = \text{Tr} \left[T^\alpha \sum_{k=0}^{\infty} \widehat{\mathcal{S}}_{-k}(B^{(1+k)}) \right]. \quad (5.9)$$

The operation $\widehat{\mathcal{S}}_{-k}(B^{(1+k)})$ is defined implicitly such that the relation (A.41) holds for any $T^\alpha \in \widehat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$ as explained in Appendix A.4. This implies that one can choose a particular gauge for the one-form gauge transformations defined in Appendix A.4 such that (5.7) extends to the whole constrained fields and fixes for all $k \geq 0$

$$B^{(1+k)} = \rho^k \star^k B. \quad (5.10)$$

Note that, although $B^{(1)} = B$ according to this equation, B does not transform in the same way as $B^{(1)}$ under internal diffeomorphisms so they should not be understood as being the same constrained fields. This is because the fields $B^{(k)}$ only appear through the combination $\sum_{k=0}^{\infty} \widehat{\mathcal{S}}_{-k}(B^{(1+k)})$ above, and the identification (5.10) only holds for this combination traced with a generator $T^\alpha \in \mathfrak{e}_9 \oplus \langle L_{-1} \rangle$. To define the gauge transformation of the field B one identifies the gauge transformation of this combination

$$\delta_\Lambda \sum_{k=0}^{\infty} \widehat{\mathcal{S}}_{-k}(B^{(1+k)}) = \mathcal{L}_{(\Lambda,0)} \sum_{k=0}^{\infty} \widehat{\mathcal{S}}_{-k}(B^{(1+k)}) - \eta_{1\alpha\beta} \langle \partial_\Lambda | T^\alpha | A \rangle T^\beta | \Lambda \rangle \langle \partial_\Lambda |, \quad (5.11)$$

and defines the gauge transformation of the constrained field B consistently with it, i.e. such that

$$\sum_{k=0}^{\infty} \rho^k \star^k \widehat{\mathcal{S}}_{-k}(\delta_{\Lambda} B) = \sum_{k=0}^{\infty} \rho^k \star^k \widehat{\mathcal{S}}_{-k}(\mathcal{L}_{(\Lambda,0)} B) - \eta_{1\alpha\beta} \langle \partial_{\Lambda} | T^{\alpha} | A \rangle T^{\beta} | \Lambda \rangle \langle \partial_{\Lambda} | + \dots \quad (5.12)$$

where the ellipses correspond to terms that vanish upon tracing with T^{α} . Inverting the geometric series, one obtains the gauge transformation of the constrained field B up to a trivial parameter

$$\begin{aligned} \delta_{\Lambda} B &= \mathcal{L}_{(\Lambda,0)} B + (1 - \rho \star \widehat{\mathcal{S}}_{-1}) \left(\eta_{1\alpha\beta} \langle \partial_{\Lambda} | T^{\alpha} | \Lambda \rangle T^{\beta} | A \rangle \langle \partial_{\Lambda} | \right) + \dots \\ &= \mathcal{L}_{(\Lambda,0)} B + \eta_{1\alpha\beta} \langle \partial_{\Lambda} | T^{\alpha} | \Lambda \rangle T^{\beta} | A \rangle \langle \partial_{\Lambda} | \\ &\quad - \rho \star \left(\eta_{\alpha\beta} \langle \partial_{\Lambda} | T^{\alpha} | \Lambda \rangle T^{\beta} | A \rangle + \langle \partial_{\Lambda} | \Lambda \rangle | A \rangle - \langle \partial_{\Lambda} | A \rangle | \Lambda \rangle \right) \langle \partial_{\Lambda} | . \end{aligned} \quad (5.13)$$

Note that although we inverted the formal geometric series to obtain this formula from the Virasoro-extended gauge transformation, it is well defined and E_9 -covariant. The final line is the definition of the gauge transformation of B .

Writing $\chi^{\flat} \equiv \chi_1^{\flat}$ for short and substituting (5.10) on both sides of the duality equation (5.1), one obtains

$$\begin{aligned} \mathcal{J}^{\flat} - \sum_{k=0}^{\infty} \star^k \left(\rho^k \eta_{-1-k\alpha\beta} \text{Tr}[T^{\alpha} B] + \rho^{-2-k} \eta_{1+k\alpha\beta} \text{Tr}[\mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} B] \right) T^{\beta} \\ = \rho^{-1} \star \left(\mathcal{M}^{-1} \mathcal{S}_1(\mathcal{J}^{\flat})^{\dagger} \mathcal{M} + \chi^{\flat} \mathbf{K} \right) - \sum_{k=1}^{\infty} \star^k \left(\rho^{-k} \eta_{k-1\alpha\beta} \text{Tr}[\mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} B] + \rho^k \eta_{-1-k\alpha\beta} \text{Tr}[T^{\alpha} B] \right) T^{\beta}. \end{aligned} \quad (5.14)$$

One finds that most of the terms cancel and the remaining equation is

$$\mathcal{J}^{\flat} - \eta_{-1\alpha\beta} \text{Tr}[T^{\alpha} B] T^{\beta} = \rho^{-1} \star \left(\mathcal{M}^{-1} \mathcal{S}_1(\mathcal{J}^{\flat})^{\dagger} \mathcal{M} - \eta_{\alpha\beta} \text{Tr}[T^{\alpha} B] \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M} + \chi^{\flat} \mathbf{K} \right). \quad (5.15)$$

This equation can be written as

$$\mathcal{J} = \rho^{-1} \star \left(\mathcal{M}^{-1} \mathcal{S}_1(\mathcal{J})^{\dagger} \mathcal{M} + \chi \mathbf{K} \right), \quad (5.16)$$

for the current \mathcal{J} and the field χ defined as

$$\mathcal{J} = \mathcal{J}^{\flat} - \eta_{-1\alpha\beta} \text{Tr}[T^{\alpha} B] T^{\beta}, \quad \chi = \chi^{\flat} + \text{Tr}[L_0 B], \quad (5.17)$$

distributing the B dependence. This provides an alternative formulation of the theory that we refer to as the minimal formulation and discuss in detail in the remainder of this section.

Because $\mathcal{J} \neq \mathcal{M}^{-1} \mathcal{J}^{\dagger} \mathcal{M}$, this duality equation does not imply an infinite chain of relations between \mathcal{J} and its shifts $\mathcal{S}_n(\mathcal{J})$ as in equation (4.25) of the Virasoro-extended formulation. Rather, (4.25) translates in the minimal formulation into finite linear combinations of the duality equation (5.16) and its Hermitian conjugate where some of the \mathcal{J}^{\flat} components cancel out. We will see that (5.16) is invariant under internal diffeomorphisms with the gauge transformation (5.13).

Let us now consider similar consequences for the the pseudo-Lagrangian. The potential term is not modified so we only need to discuss the topological term at vanishing Virasoro fields $\phi_n = 0$ for $n \geq 1$. We will then rearrange the terms in (4.81) in the case $\mathcal{M} \in E_9$, i.e. for $\tilde{\rho} = 0$. By the definition (4.42), we have

$$\begin{aligned} \rho \widehat{\mathbb{F}}_1^\gamma &= -\langle \partial_{\mathcal{F}} | L_1 | \mathcal{F} \rangle - \sum_{k=0}^{\infty} \text{Tr} [L_{-k} \mathcal{G}^{(1+k)}] \\ &= -\langle \partial_F | L_1 | F \rangle - \sum_{k=0}^{\infty} \text{Tr} [L_{-k} G^{(1+k)}] \\ &\quad + \langle \partial | \left(\frac{1}{2} \eta_{1\alpha\beta} T^\alpha | C_1 \rangle \langle \pi_C | T^\beta | C_2 \rangle + | C_1^+ \rangle \text{Tr} [C_2^+] - C_2^+ | C_1 \rangle \right), \end{aligned} \quad (5.18)$$

where the two-forms (which enter the field strengths as discussed in Appendix A.2) appear through a total derivative that we will neglect in the following. For $\mathcal{M} \in E_9$, one also computes

$$\begin{aligned} &\mathcal{M}^{-1} (\widehat{\mathbb{F}}_{-1}^\gamma \mathbb{K} + [\mathbb{F}]_\alpha \mathcal{S}_{-1}^\gamma (T^\alpha)^\dagger) \mathcal{M} \\ &= \rho \left(\eta_{-1\alpha\beta} \langle \partial_{\mathcal{F}} | T^\alpha | \mathcal{F} \rangle + \sum_{k=1}^{\infty} \eta_{-1-k\alpha\beta} \text{Tr} [T^\alpha \mathcal{G}^{(k)}] \right) \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M} \\ &= \rho^{-1} \left(\eta_{1\alpha\beta} \langle \partial_{\mathcal{F}} | \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} | \mathcal{F} \rangle + \sum_{k=1}^{\infty} \rho^{-2k} \eta_{1+k\alpha\beta} \text{Tr} [\mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} \mathcal{G}^{(k)}] \right) T^\beta, \end{aligned} \quad (5.19)$$

and because ${}^{\mathcal{M}}\widehat{\mathbb{F}}_1^\gamma$ is defined as the \mathbb{K} component of this expression according to (4.44), one obtains

$$\rho {}^{\mathcal{M}}\widehat{\mathbb{F}}_1^\gamma = -\langle \partial_F | \mathcal{M}^{-1} L_{-1} \mathcal{M} | F \rangle - \sum_{k=1}^{\infty} \rho^{-2k} \text{Tr} [\mathcal{M}^{-1} L_{-1-k} \mathcal{M} G^{(k)}], \quad (5.20)$$

where the fact that $\widehat{\mathbb{F}}_{-1}^\gamma$ does not depend on the two-forms was used to replace $|\mathcal{F}\rangle$ and $\mathcal{G}^{(k)}$ by $|F\rangle$ and $G^{(k)}$, respectively.

Moreover, one has $\rho^{-1} \mathcal{D}\tilde{\rho} = -\text{Tr}[B^{(1)}]$ at $\tilde{\rho} = 0$ and one can therefore rearrange the following terms in (4.81) according to

$$\begin{aligned} &\rho d\chi_1^\gamma - \rho \langle \partial_\chi | A \rangle \chi_1^\gamma + \eta^{AB} \sum_{n \in \mathbb{Z}} n [\mathbb{A}]_A^n (J_B^{-n-1} - \tilde{\rho} J_B^{-n}) \\ &= \mathcal{D}^b(\rho \chi_1^\gamma) - \sum_{n \in \mathbb{Z}} (n+1) \sum_{k=0}^{\infty} \text{Tr} [T_{n-k}^A B^{(1+k)}] J_A^n + \frac{1}{2} J_0 \rho \chi_1^\gamma, \end{aligned} \quad (5.21)$$

where we used the bare covariant derivative introduced in (5.4). For $\mathcal{M} \in E_9$ it will be convenient to introduce the shifted cocycles $\omega_{-k}^\alpha(\mathcal{M})$ defined in (B.14), that satisfies

$$\mathcal{M}^{-1} L_n \mathcal{M} = \rho^{-2n} L_n - \rho^{2k-2n} \eta_{m-k\alpha\beta} \omega_{-k}^\alpha(\mathcal{M}) T^\beta, \quad (5.22)$$

such that $\omega_0^\alpha(\mathcal{M}) = \omega(\mathcal{M})$. Using (B.15), the expression (4.22) for χ_{-k}^γ can be simplified to

$$\chi_{-k}^\gamma = \chi_k^\gamma + \rho^k \omega_{-k}^\alpha(\mathcal{M}) \mathfrak{J}_\alpha. \quad (5.23)$$

Combining all these terms, one finally obtains the following form of the topological term (4.81) at $\tilde{\rho} = 0$, up to the total derivative in the two-form fields in (5.18) that we do not include for brevity,

$$\begin{aligned} \mathcal{L}_{\text{top}} = & \mathcal{D}^b(\rho\chi_1^\gamma) - \sum_{n \in \mathbb{Z}} (n+1) \sum_{k=0}^{\infty} \text{Tr}[T_{n-k}^A B^{(1+k)}] \mathcal{J}_A^n + \frac{1}{2} \mathcal{J}_K \text{Tr} B^{(1)} + \frac{1}{2} \eta^{AB} \sum_n n \mathcal{J}_A^n \mathcal{J}_B^{-1-n} \\ & + \frac{1}{2} \mathcal{J}_0 \rho \chi_1^\gamma + \frac{1}{2} \sum_{k=1}^{\infty} \rho^{1-k} \text{Tr} B^{(k)} \chi_{1+k}^\gamma + \frac{1}{2} \sum_{k=0}^{\infty} (\rho^{-1-k} \chi_{1+k}^\gamma + \omega_{-1-k}^\alpha(\mathcal{M}) \mathfrak{J}_\alpha) \text{Tr} B^{(2+k)} \\ & + \rho \langle \chi_1^\gamma | F \rangle - \langle \partial_F | (L_1 + \mathcal{M}^{-1} L_{-1} \mathcal{M} | F \rangle - \sum_{k=0}^{\infty} \text{Tr} [(L_{-k} + \rho^{-2-2k} \mathcal{M}^{-1} L_{-2-k} \mathcal{M}) G^{(1+k)}] . \end{aligned} \quad (5.24)$$

With the split (5.2) and (5.5), the above topological term can be written as

$$\begin{aligned} \mathcal{L}_{\text{top}} = & \mathcal{D}^b \chi_1^b + \frac{1}{2} \eta^{AB} \sum_n n \mathcal{J}_A^b \mathcal{J}_B^{-1-n} + \frac{1}{2} \eta_{1\alpha\beta} \langle \partial_A | T^\alpha | A \rangle \langle \partial_A | (L_0 + \rho^{-2} \mathcal{M}^{-1} L_{-2} \mathcal{M}) T^\beta | A' \rangle \\ & + \rho \langle \chi_1^\gamma | F \rangle - \langle \partial_F | (L_1 + \mathcal{M}^{-1} L_{-1} \mathcal{M} | F \rangle \\ & + \frac{1}{2} \mathcal{J}_0^b \chi_1^b + \frac{1}{2} \text{Tr}[\mathcal{J}^b B^{(1)}] + \frac{1}{2} \text{Tr}[(\mathcal{S}_{-1}(\mathcal{J}^b) + \rho^{-2} \chi_1^b + \omega_{-1}^\alpha(\mathcal{M}) \mathcal{J}_\alpha^b) B^{(2)}] \\ & + \frac{1}{2} \sum_{k \geq 1} \text{Tr}[(\mathcal{S}_{-1-k}(\mathcal{J}^b) + \rho^{-2-2k} \chi_{1+k}^b + \omega_{-1-k}^\alpha(\mathcal{M}) \mathcal{J}_\alpha^b) (B^{(k+2)} - \rho^2 B^{(k)})] \\ & + \frac{1}{2} \sum_{k \geq 1} \sum_{q \geq 1} \rho^{-2k} \eta_{1+k-q\alpha\beta} \text{Tr}[T^\alpha B^{(q)}] \text{Tr}[\mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M} B^{(k)}] , \end{aligned} \quad (5.25)$$

where we used the explicit expression of $G^{(k)}$

$$\begin{aligned} G^{(k)} = & \mathcal{D}^b B^{(k)} - \frac{1}{2} \delta_1^k \eta_{1\alpha\beta} \langle \partial_A | T^\alpha | A \rangle T^\beta | A' \rangle \langle \partial_A | + \frac{1}{2} \sum_{q=1}^{\infty} \eta_{-q\alpha\beta} \text{Tr}[T^\alpha B^{(q)}] T^\beta B^{(k)} \\ & + \frac{1}{2} \sum_{r=1}^k (2r-1-k) \text{Tr}[B^{(r)}] B^{(1+k-r)} \end{aligned} \quad (5.26)$$

and the fact that the covariant derivative of χ_1^γ compensates the one of these field strengths using

$$\begin{aligned} & \mathcal{D}^b(\rho\chi_1^\gamma) - \sum_{k=0}^{\infty} \text{Tr} [(L_{-k} + \rho^{-2-2k} \mathcal{M}^{-1} L_{-2-k} \mathcal{M}) \mathcal{D}^b B^{(1+k)}] \\ & = \mathcal{D}^b \chi_1^b + \sum_{k=1}^{\infty} \text{Tr} [\mathcal{D}^b (\rho^{-2k} \mathcal{M}^{-1} L_{-1-k} \mathcal{M}) B^{(k)}] . \end{aligned} \quad (5.27)$$

The Euler–Lagrange equations of motion of the fields $B^{(k)}$ and χ_k^γ do not imply the duality equation (5.1), so one cannot integrate them out in the usual sense. In fact, integrating them out naïvely gives rise to inconsistencies due to formal indefinite sums that appear in the topological term once one substitutes the solutions to their Euler–Lagrange equations. However, it turns out

to be consistent to set them to the values in (5.6) and (5.10). In general, one cannot substitute a solution to a duality equation (like (5.6) and (5.10)) into a pseudo-Lagrangian. In the present case, the Euler–Lagrange equations of motions for the fields \mathcal{M} , $|A\rangle$ and $\langle\chi|$ are automatically preserved, but the Euler–Lagrange equations for $B^{(1)}$ and χ_1^γ are not. We shall see nonetheless that the latter are consistent with the duality equation. Indeed, after the substitution of (5.6) and (5.10) in the pseudo-Lagrangian, the B field and χ field Euler–Lagrange equations become projected components of the duality equation (5.16), which is itself by construction consistent with the original duality equation (5.1). Therefore we obtain that the two pseudo-Lagrangians together with their respective duality equations define the same set of equations. Note that the use of the pseudo-Lagrangian was always to determine the Euler–Lagrange equations of motions for the fields \mathcal{M} , $|A\rangle$ and $\langle\chi|$ only, whereas the Euler–Lagrange equations of the constrained fields $B^{(k)}$ and χ_k^γ were redundant with the duality equations. So it may not be that surprising that this manipulation turns out to be consistent.

Because (5.6) and (5.10) involve the Hodge star operator, their substitution in the topological term (5.25) gives the sum $\mathcal{L}_1 + \mathcal{L}_2$ of a kinetic term

$$\mathcal{L}_1 = \frac{1}{2}\rho \operatorname{Tr}[(\mathcal{S}_{-1}(\mathcal{J}^b) + \rho^{-2}\chi^b + \omega_{-1}^\alpha(\mathcal{M})\mathcal{J}_\alpha^b) \star B] - \frac{1}{4}\rho^{-1}\eta_{\alpha\beta}\operatorname{Tr}[T^\alpha B] \star \operatorname{Tr}[\mathcal{M}^{-1}T^{\beta\dagger}\mathcal{M}B], \quad (5.28)$$

and a topological term

$$\begin{aligned} \mathcal{L}_2 = & \mathcal{D}^b\chi^b + \frac{1}{2}\eta^{AB}\sum_n nJ_A^b J_B^{b-1-n} + \frac{1}{2}\eta_{1\alpha\beta}\langle\partial_A|T^\alpha|A\rangle\langle\partial_A|(L_0 + \rho^{-2}\mathcal{M}^{-1}L_{-2}\mathcal{M})T^\beta|A'\rangle \\ & + \rho\langle\chi_1^\gamma|F\rangle - \langle\partial_F|(L_1 + \mathcal{M}^{-1}L_{-1}\mathcal{M}|F\rangle + \frac{1}{2}\mathcal{J}_0^b\chi^b + \frac{1}{2}\operatorname{Tr}[\mathcal{J}^b B]. \end{aligned} \quad (5.29)$$

To obtain this result we have used in particular

$$\begin{aligned} & \frac{1}{2}\sum_{k=1}^{\infty}\sum_{q=1}^{\infty}\rho^{-2k}\eta_{1+k-q\alpha\beta}\operatorname{Tr}[T^\alpha B^{(q)}]\operatorname{Tr}[\mathcal{M}^{-1}T^{\beta\dagger}\mathcal{M}B^{(k)}] \\ = & \frac{1}{4}\sum_{k=1}^{\infty}\sum_{q=1}^{\infty}(\rho^{-2k}\eta_{1+k-q\alpha\beta} - \rho^{2-2k}\eta_{k-1-q\alpha\beta})\operatorname{Tr}[T^\alpha B^{(q)}]\operatorname{Tr}[\mathcal{M}^{-1}T^{\beta\dagger}\mathcal{M}B^{(k)}] \\ = & -\frac{1}{4}\rho^{-2}\eta_{\alpha\beta}\operatorname{Tr}[T^\alpha B^{(1)}]\operatorname{Tr}[\mathcal{M}^{-1}T^{\beta\dagger}\mathcal{M}B^{(2)}] \\ & -\frac{1}{4}\sum_{k=1}^{\infty}\eta_{-k\alpha\beta}\left(\operatorname{Tr}[T^\alpha B^{(k)}]\operatorname{Tr}[\mathcal{M}^{-1}T^{\beta\dagger}\mathcal{M}B^{(1)}] + \rho^{-2}\operatorname{Tr}[T^\alpha B^{(k+1)}]\operatorname{Tr}[\mathcal{M}^{-1}T^{\beta\dagger}\mathcal{M}B^{(2)}]\right) \\ & +\frac{1}{4}\sum_{k=1}^{\infty}\sum_{q=1}^{\infty}\rho^{-2k}\eta_{1+k-q\alpha\beta}\operatorname{Tr}[T^\alpha B^{(q)}]\operatorname{Tr}[\mathcal{M}^{-1}T^{\beta\dagger}\mathcal{M}(B^{(k)} - \rho^{-2}B^{(k+2)})]. \end{aligned} \quad (5.30)$$

This last equation involves indefinite formal sums that we have regularised using geometric series regularisation as in (4.119) such that both the second and the last line cancel after the substitution (5.10).

Now one can easily verify that the Euler–Lagrange equations of B and χ^b derived from the pseudo-Lagrangian $\mathcal{L}_1 + \mathcal{L}_2$ in (5.28) and (5.29), are proportional to the duality equation (5.16)

$$\delta(\mathcal{L}_1 + \mathcal{L}_2) = \frac{1}{2}\operatorname{Tr}\left[\delta B\left(\rho^{-1}\star\mathcal{M}^{-1}(\mathcal{S}_1(\mathcal{J}) + \chi\mathcal{K})^\dagger\mathcal{M} - \mathcal{J}\right)\right] + \frac{1}{2}\delta\chi^b(\rho^{-1}\star\operatorname{Tr}[B] - J_0). \quad (5.31)$$

This concludes the consistency of the substitution of (5.6) and (5.10) in (5.25). Note that although the relation between the topological term (5.25) and $\mathcal{L}_1 + \mathcal{L}_2$ requires a regularisation, the two terms (5.28) and (5.29) are well defined and E_9 invariant. We shall use them to define the pseudo-Lagrangian in the next section. In this minimal formulation we will see that the Euler–Lagrange equations are finite and do not require any regularisation.

5.2 Internal diffeomorphisms invariance and pseudo-Lagrangian

We introduce now the unimodular metric $\tilde{g}_{\mu\nu}$ such that $\det \tilde{g} = -1$ and the two-dimensional metric is $g_{\mu\nu} = e^{2\sigma} \tilde{g}_{\mu\nu}$. Because this formulation of the theory is not manifestly covariant under Σ gauge transformations, it will be useful to use the bare covariant derivative that was defined in (5.4) as $\mathcal{D}^b = d - \mathcal{L}_{(A,0)}$ instead of $\mathcal{D} = d - \mathcal{L}_{(A,B)}$. For a non-constant unimodular metric $\tilde{g}_{\mu\nu}$, the definition of the current \mathcal{J}^b has to be generalised from (5.3) to

$$\mathcal{J}_{\mu\alpha}^b T^\alpha = \mathcal{M}^{-1} \mathcal{D}_\mu^b \mathcal{M} + \tilde{g}^{\nu\sigma} (\partial_\nu - \langle \partial_{\tilde{g}} | A_\nu \rangle) \tilde{g}_{\mu\sigma} \mathbf{K}. \quad (5.32)$$

This additional central component is necessary for the covariance of the duality equation under external diffeomorphisms as we shall see later. It can also be understood by identifying this central component written in terms of the two-dimensional metric $g_{\mu\nu}$

$$\mathcal{J}_{\mu\alpha}^b T^\alpha = \tilde{\mathcal{M}}^{-1} \mathcal{D}_\mu^b \tilde{\mathcal{M}} + g^{\nu\sigma} (\mathcal{D}_\nu^b g_{\mu\sigma} - \mathcal{D}_\mu^b g_{\nu\sigma}) \mathbf{K}, \quad (5.33)$$

as the gravitational flux (see e.g. [47, Eq. 4.13a]), where $\tilde{\mathcal{M}}$ does not include the conformal factor σ . According to (5.17), the current $\mathcal{J} = \mathcal{J}_\alpha T^\alpha$ involves only the \mathcal{M} -independent B term as

$$\mathcal{J}_{\mu\alpha} T^\alpha = \mathcal{M}^{-1} \mathcal{D}_\mu^b \mathcal{M} - \eta_{-1\alpha\beta} \text{Tr}[T^\beta B_\mu] T^\alpha + \tilde{g}^{\nu\sigma} (\partial_\nu - \langle \partial_{\tilde{g}} | A_\nu \rangle) \tilde{g}_{\mu\sigma} \mathbf{K}, \quad (5.34)$$

and therefore \mathcal{J} is not equal to $\mathcal{M}^{-1} \mathcal{J}^\dagger \mathcal{M}$. It does not transform covariantly under generalised diffeomorphisms either, but the duality equation (5.16), that we reproduce here for convenience

$$\mathcal{J}_{\mu\alpha} T^\alpha = \rho^{-1} \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\nu} \mathcal{M}^{-1} (\mathcal{S}_1(\mathcal{J}_{\nu\alpha} T^\alpha) + \chi_\nu \mathbf{K})^\dagger \mathcal{M}, \quad (5.35)$$

is invariant under internal diffeomorphisms, as we prove below. The Hodge star operator is written out with respect to the unimodular metric $\tilde{g}_{\mu\nu}$ with the convention $\varepsilon_{\mu\nu} = \tilde{g}_{\mu\sigma} \tilde{g}_{\nu\rho} \varepsilon^{\sigma\rho}$ and $\varepsilon^{01} = -\varepsilon_{01} = 1$.

We define the gauge transformations of parameter $|\Lambda\rangle$

$$\begin{aligned} \delta_\Lambda \tilde{g}_{\mu\nu} &= \langle \partial \tilde{g}_{\mu\nu} | \Lambda \rangle, \\ \delta_\Lambda \mathcal{M} &= \langle \partial_{\mathcal{M}} | \Lambda \rangle \mathcal{M} + \eta_{\alpha\beta} \langle \partial_\Lambda | T^\beta | \Lambda \rangle (T^{\alpha\dagger} \mathcal{M} + \mathcal{M} T^\alpha), \\ \delta_\Lambda | A_\mu \rangle &= \partial_\mu | \Lambda \rangle + \langle \partial_A | \Lambda \rangle | A_\mu \rangle - \eta_{\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle T^\beta | A_\mu \rangle - \langle \partial_\Lambda | \Lambda \rangle | A_\mu \rangle, \end{aligned} \quad (5.36)$$

and

$$\begin{aligned}
\delta_\Lambda B_\mu &= \langle \partial_B | \Lambda \rangle B_\mu - \eta_{\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle [T^\beta, B_\mu] + \langle \partial_\Lambda | \Lambda \rangle B_\mu + \eta_{1\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle T^\beta | A_\mu \rangle \langle \partial_\Lambda | \\
&\quad - \rho \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\nu} (\eta_{\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle T^\beta | A_\nu \rangle + \langle \partial_\Lambda | \Lambda \rangle | A_\nu \rangle - \langle \partial_\Lambda | A_\nu \rangle | \Lambda \rangle) \langle \partial_\Lambda | \\
\delta_\Lambda \chi_\mu &= \langle \partial_\chi | \Lambda \rangle \chi_\mu + \langle \partial_\Lambda | \Lambda \rangle \chi_\mu + \sum_n n \langle \partial_\Lambda | T_n^A | \Lambda \rangle J_{\mu A}^{n-1} - \eta_{\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle \langle \partial_\Lambda | \mathcal{M}^{-1} L_{-1} \mathcal{M} T^\beta | A_\mu \rangle \\
&\quad - \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda | \mathcal{M}^{-1} L_{-1} \mathcal{M} | A_\mu \rangle + \langle \partial_\Lambda | A_\mu \rangle \langle \partial_\Lambda | \mathcal{M}^{-1} L_{-1} \mathcal{M} | \Lambda \rangle \\
&\quad - \rho \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\nu} (\eta_{\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle \langle \partial_\Lambda | L_0 T^\beta | A_\nu \rangle + \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda | L_0 | A_\nu \rangle - \langle \partial_\Lambda | A_\nu \rangle \langle \partial_\Lambda | L_0 | \Lambda \rangle) .
\end{aligned} \tag{5.37}$$

Note that the gauge transformation of the gauge field is not $\mathcal{D}_\mu | \Lambda \rangle$ as usual in exceptional field theory, but it differs by a trivial diffeomorphism as shown in (3.21). In particular, the parts of the gauge fields transformations which are independent of the Hodge dual match with (3.24) and (3.25). The Hodge dual terms in the variations of B and χ follow directly from (5.13). They can also be derived from the gauge transformations in [47], in which the Hodge dual terms and the gauge field transformation (5.36) appear naturally.

To prove that the duality equation (5.35) is invariant under internal diffeomorphisms one computes the gauge transformation of the current

$$\begin{aligned}
\delta_\Lambda \mathcal{J} &= \langle \partial_{\mathcal{J}} | \Lambda \rangle \mathcal{J} + \eta_{\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle [\mathcal{J}, T^\beta] \\
&\quad + \eta_{\alpha\beta} (\eta_{\gamma\delta} \langle \partial_\Lambda | T^\gamma | \Lambda \rangle \langle \partial_\Lambda | T^\alpha T^\delta | A \rangle + \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda | T^\alpha | A \rangle - \langle \partial_\Lambda | A \rangle \langle \partial_\Lambda | T^\alpha | \Lambda \rangle) \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M} \\
&\quad + \eta_{-1\alpha\beta} \rho \star (\eta_{\gamma\delta} \langle \partial_\Lambda | T^\gamma | \Lambda \rangle \langle \partial_\Lambda | T^\alpha T^\delta | A \rangle + \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda | T^\alpha | A \rangle - \langle \partial_\Lambda | A \rangle \langle \partial_\Lambda | T^\alpha | \Lambda \rangle) T^\beta ,
\end{aligned} \tag{5.38}$$

and we identify the first line with its covariant transformation, while the two last lines are found to project out in the duality equation, up to a central element that is compensated by the gauge transformation of χ .

The duality equation (5.35) is a twisted self-duality equation that can be obtained from a truncation of the E_{11} twisted self-duality equation defined in [47]. Twisted self-duality in $D = 2p$ dimensions can normally be written as an equality between a p -form field strength dressed with the scalar matrix $\mathcal{M} \in G$ and its Hodge star contracted with a G -invariant bilinear form. For E_9 , the one-form field strength combines the current \mathcal{J} together with the one-form χ in the module $\langle L_1 \rangle^* \oplus \mathfrak{e}_9 \oplus \langle L_{-1} \rangle$, where $\langle L_1 \rangle^* \oplus \mathfrak{e}_9$ is the module conjugate to $\mathfrak{e}_9 \oplus \langle L_1 \rangle$ that describes the indecomposable representation of the field χ with the current \mathcal{J} . The symmetric E_9 -invariant bilinear form is

$$\rho \langle (\chi_\mu, \mathcal{J}_\mu), (\chi_\nu, \mathcal{J}_\nu) \rangle = \sum_{n \in \mathbb{Z}} \eta^{AB} J_{\mu A}^n J_{\nu B}^{-n-1} - J_{\mu-1} J_{\nu\kappa} - J_{\mu\kappa} J_{\nu-1} - J_{\mu 0} \chi_\nu - \chi_\mu J_{\nu 0} . \tag{5.39}$$

This bilinear form, together with the action of E_9 on the module $\langle L_1 \rangle^* \oplus \mathfrak{e}_9 \oplus \langle L_{-1} \rangle$, define the twisted self-duality equation in $\langle L_1 \rangle^* \oplus \mathfrak{e}_9 \oplus \langle L_{-1} \rangle$.

$$(\chi, \mathcal{J}) = (\mathcal{I}_1[\chi], \mathcal{I}_1[\mathcal{J}]) , \tag{5.40}$$

with the involution²⁴

$$\mathcal{I}_1[\mathcal{J}] = \rho^{-1} \star \mathcal{M}^{-1}(\mathcal{S}_1(\mathcal{J}) + \chi \mathbf{K})^\dagger \mathcal{M}, \quad \mathcal{I}_1[\chi] = \rho \star J_{\mathbf{K}} - \rho^2 \omega_{-1}^\alpha(\mathcal{M}) \mathcal{I}_1[\mathcal{J}]_\alpha. \quad (5.41)$$

The equation $\chi = \mathcal{I}_1[\chi]$ follows from the central component of (5.35)

$$J_{\mathbf{K}} = \rho^{-1} \star \chi + \rho \omega_{-1}^\alpha(\mathcal{M}) \star \mathcal{J}_\alpha. \quad (5.42)$$

One checks indeed that

$$\begin{aligned} \mathcal{I}_1[\mathcal{I}_1[\mathcal{J}]] &= \rho^{-1} \mathcal{M}^{-1} \star \left(\mathcal{S}_1(\rho^{-1} \mathcal{M}^{-1} \star (\mathcal{S}_1(\mathcal{J}) + \chi \mathbf{K})^\dagger \mathcal{M}) + \chi \mathbf{K} \right)^\dagger \mathcal{M} + \rho^{-1} \star \mathcal{I}_1[\chi] \mathbf{K} \\ &= \mathcal{J}, \end{aligned} \quad (5.43)$$

and $\mathcal{I}_1[\mathcal{I}_1[\chi]] = \chi$. Although (5.40) is manifestly involutive, it is more convenient to write it as an equation in the Lie algebra $\mathfrak{e}_9 \oplus \langle L_{-1} \rangle$ as (5.35).

The pseudo-Lagrangian of the theory for a non-constant unimodular metric $\tilde{g}_{\mu\nu}$ is defined as

$$\mathcal{L}_{\min} = \mathcal{L}_1 + \mathcal{L}_2 + \frac{1}{4} \rho \varepsilon^{\mu\nu} \varepsilon^{\sigma\rho} \tilde{g}^{\kappa\lambda} \mathcal{D}_\mu \tilde{g}_{\sigma\kappa} \mathcal{D}_\nu \tilde{g}_{\rho\lambda} + \frac{\rho^{-1}}{4} \langle \partial \tilde{g}^{\mu\nu} | \mathcal{M}^{-1} | \partial \tilde{g}_{\mu\nu} \rangle - V, \quad (5.44)$$

where the kinetic term

$$\mathcal{L}_1 = \frac{1}{2} \rho^{-1} \tilde{g}^{\mu\nu} \left(\text{Tr}[(\mathcal{M}^{-1} \mathcal{S}_1(\mathcal{J}_\mu)^\dagger \mathcal{M} + \chi_\mu) B_\nu] + \frac{1}{2} \eta_{\alpha\beta} \text{Tr}[T^\alpha B_\mu] \text{Tr}[\mathcal{M}^{-1} (T^\beta)^\dagger \mathcal{M} B_\nu] \right) \quad (5.45)$$

and the topological term²⁵

$$\begin{aligned} \mathcal{L}_2 dx^0 \wedge dx^1 &= \mathcal{D}\chi + \frac{1}{2} J_0 \chi + \frac{1}{2} J_{-1} J_{\mathbf{K}} + \frac{1}{2} \eta^{AB} \sum_n n J_A^n J_B^{-1-n} + \rho^2 (\chi - \omega_{-1}^\alpha(\mathcal{M}) \mathcal{J}_\alpha | F) \\ &\quad - \langle \partial_F | (L_1 + \mathcal{M}^{-1} L_{-1} \mathcal{M}) | F \rangle - \text{Tr}[L_0 G] + \frac{1}{2} \rho^{-2} \eta_{1\alpha\beta} \langle \partial_A | T^\alpha | A \rangle \langle \partial_A | \mathcal{M}^{-1} L_{-2} \mathcal{M} T^\beta | A' \rangle, \end{aligned} \quad (5.46)$$

are the direct generalisations of (5.28) and (5.29) with a unimodular metric $\tilde{g}_{\mu\nu}$. Note that we wrote explicitly $\langle \chi_1^\gamma | = \rho \langle \chi - \omega_{-1}^\alpha(\mathcal{M}) \mathcal{J}_\alpha |$ in terms of the field $\langle \chi |$ that appears in the potential term V [1]. We recall that the (internal) $\langle \chi |$ transforms as the central component of

$$\langle \mathcal{J}_\alpha^- | \otimes T^\alpha = \langle \mathcal{J}_\alpha | \otimes \mathcal{S}_{-1}(T^\alpha) + \langle \chi | \otimes \mathbf{K}, \quad (5.47)$$

whereas the (external) vector field χ instead transforms as the central component of $\mathcal{S}_1(\mathcal{J}) + \chi \mathbf{K}$. In addition to the dependence in the general unimodular metric $\tilde{g}_{\mu\nu}$, (5.45) and (5.46) differ from (5.28) and (5.29) in that we have included (most of) the dependence in the field B_μ in the current (5.35), the covariant derivative $\mathcal{D}\chi$ and the field strength G , using similar steps as in passing from (5.24) to (5.25).

²⁴The conjugation by \mathcal{M} uses the $K(E_9)$ invariant bilinear form defined at $\mathcal{M} = \mathbb{1}$ as

$$((\chi_\mu, \mathcal{J}_\mu), (\chi_\nu, \mathcal{J}_\nu)) = \sum_{n \in \mathbb{Z}} \delta^{AB} J_\mu^n J_\nu^n - J_{\mu 0} J_{\nu \mathbf{K}} - J_{\mu \mathbf{K}} J_{\nu 0} - J_{\mu -1} \chi_\nu - \chi_\mu J_{\nu -1}.$$

²⁵In this expression we extracted the top-form factor on the left, with the convention that $AB = \varepsilon^{\mu\nu} A_\mu B_\nu dx^0 \wedge dx^1$ and $A \star B = \tilde{g}^{\mu\nu} A_\mu B_\nu dx^0 \wedge dx^1$ for one-forms in the two-dimensional external space.

In (5.44), the first two terms $\mathcal{L}_1 + \mathcal{L}_2$ together, the two terms involving the derivative of the unimodular metric and the potential V are individually invariant under internal diffeomorphisms up to a total derivative. We shall see that their relative coefficients are determined by external diffeomorphisms.

In order to study the invariances of the proposed pseudo-Lagrangian (5.44), we recall a few definitions. The field strength F and G are defined as in (3.29), while the covariant derivative

$$\mathcal{D}\chi = d\chi - \langle \partial_\chi | A \rangle \chi - \langle \partial_A | A \rangle \chi - \sum_n n \langle \partial_A | T_n^A | A \rangle J_A^{n-1} - \sum_n (n+1) \text{Tr}[T_n^A B] J_A^n - \text{Tr}[B] J_K, \quad (5.48)$$

follows from the indecomposable representation of the field χ under $\mathfrak{e}_9 \oplus \langle L_{-1} \rangle$.²⁶ One derives the Bianchi identity for \mathcal{J}

$$\begin{aligned} & \mathcal{D}\mathcal{J} + \mathcal{J}^2 + \langle \mathcal{J} | F \rangle + \eta_{\alpha\beta} \langle \partial_F | (T^\alpha + \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M}) | F \rangle T^\beta + \eta_{-1\alpha\beta} \text{Tr}[T^\alpha G] T^\beta \\ &= \frac{1}{2} \eta_{-1\alpha\beta} \eta_{1\gamma\delta} \langle \partial_A | T^\gamma | A \rangle \langle \partial_A | T^\alpha T^\delta | A' \rangle \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M}. \end{aligned} \quad (5.49)$$

The non-covariance of the right-hand-side follows from the property that \mathcal{J} does not include the term in $-\eta_{-1\alpha\beta} \text{Tr}[T^\alpha B] \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M}$ in its definition. One can also derive the rigid E_9 invariant topological term using the same construction as in (4.46). One can indeed write a Bianchi identity for the shifted current that transforms under \mathfrak{e}_9 by a commutator with the \mathfrak{e}_9 generators T^α . This Bianchi identity is satisfied up to a central element that is equal to the topological term \mathcal{L}_2

$$\begin{aligned} & \mathcal{D}(\mathcal{S}_1(\mathcal{J}) + \chi K) + \frac{1}{2} J_0(\mathcal{S}_1(\mathcal{J}) + \chi K) + \frac{1}{2} J_{-1} \mathcal{J} + \frac{1}{2} \mathcal{J}(\mathcal{S}_1(\mathcal{J}) + \chi K) + \frac{1}{2} (\mathcal{S}_1(\mathcal{J}) + \chi K) \mathcal{J} \\ &+ \langle \mathcal{J}_\alpha | \mathcal{S}_1(T^\alpha) | F \rangle + \langle \rho^2(\chi - \omega_{-1}(\mathcal{M}) \mathcal{J}_\alpha) | F \rangle + \eta_{1\alpha\beta} \langle \partial_F | (T^\alpha + \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M}) | F \rangle T^\beta \\ &+ \eta_{\alpha\beta} \text{Tr}[T^\alpha G] T^\beta - \frac{1}{2} \rho^{-2} \eta_{2\alpha\beta} \eta_{1\gamma\delta} \langle \partial_A | T^\gamma | A \rangle \langle \partial_A | \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} T^\delta | A' \rangle T^\beta = \mathcal{L}_2 K, \end{aligned} \quad (5.50)$$

which implies that \mathcal{L}_2 is rigid E_9 invariant. The term \mathcal{L}_1 is manifestly invariant under rigid E_9 and can be determined such that the Euler–Lagrange equation for B is a projection of the duality equation.

Let us now show that $\mathcal{L}_1 + \mathcal{L}_2$ transforms as a density under internal generalised diffeomorphisms. The non-covariant variation of \mathcal{L}_1 gives

$$\begin{aligned} \Delta_\Lambda \mathcal{L}_1 &= \frac{1}{2} \rho \tilde{g}^{\mu\nu} (\eta_{\gamma\delta} \langle \partial_\Lambda | T^\gamma | \Lambda \rangle \langle \partial_\Lambda | T^\alpha T^\delta | A_\mu \rangle + \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda | T^\alpha | A_\mu \rangle \\ &\quad - \langle \partial_\Lambda | A_\mu \rangle \langle \partial_\Lambda | T^\alpha | \Lambda \rangle) (\mathcal{J}_{\nu\alpha} + \mathcal{I}_1[\mathcal{J}_{\nu\alpha}]) \end{aligned} \quad (5.51)$$

To vary \mathcal{L}_2 we must first derive the transformation of the field strengths. To avoid a cumbersome computation one first observes that the gauge transformation of G is mostly determined from the gauge transformation of the field strength F by the Dorfman product (3.15) according to (3.30). Therefore writing $\delta_\Lambda | F \rangle$ as $\mathcal{L}_{(\Lambda,0)} | F \rangle$ plus an explicit trivial parameter allows to determine the gauge transformation of G , up to doubly constrained trivial parameters (A.5d)–(A.5f) that only affects G and the non-covariant piece of the B field gauge transformation (i.e. the second line

²⁶Note that $\mathcal{D}\chi$ differs from $\mathcal{D}\chi_1^\gamma$ in Section 4, because χ_1^γ is invariant under L_0 and L_{-1} , while χ transforms respectively into itself and J_K under these two generators.

in (5.37)) that must be computed separately. But because the terms in the doubly constrained parameters leave invariant $\text{Tr}[L_0G]$ in the pseudo-Lagrangian (see Appendix A.1), we do not need to compute them. One computes in this way

$$\begin{aligned}\delta_\Lambda|F\rangle &= \mathcal{L}_{(\Lambda,0)}|F\rangle + \frac{1}{2}(\mathcal{L}_{d|\Lambda}|A\rangle - \mathcal{L}_{|A}d|\Lambda\rangle) + \frac{1}{2}\eta_{-1\alpha\beta}\eta_{1\gamma\delta}\langle\partial_\Lambda|T^\gamma|A\rangle\langle\partial_\Lambda|T^\alpha T^\delta|\Lambda\rangle T^\beta|A\rangle \quad (5.52) \\ \delta_\Lambda G &= \mathcal{L}_{(\Lambda,0)}G + \eta_{1\alpha\beta}\langle\partial_\Lambda|T^\alpha|\Lambda\rangle T^\beta|F\rangle\langle\partial_\Lambda| \\ &\quad + \frac{1}{2}\eta_{1\alpha\beta}\langle\partial_\Lambda|T^\alpha|A\rangle T^\beta d|\Lambda\rangle\langle\partial_\Lambda| - \frac{1}{2}\eta_{1\alpha\beta}\langle\partial_A|T^\alpha|A\rangle T^\beta d|\Lambda\rangle\langle\partial_A| \\ &\quad - \frac{1}{2}\eta_{1\gamma\delta}\langle\partial_\Lambda|T^\gamma|A\rangle(\eta_{\alpha\beta}T^\alpha|A\rangle\langle\partial_\Lambda|T^\beta T^\delta|\Lambda\rangle\langle\partial| - |A\rangle\langle\partial|T^\delta|\Lambda\rangle\langle\partial_\Lambda| + T^\delta|\Lambda\rangle\langle\partial|A\rangle\langle\partial_\Lambda|) \\ &\quad + \mathcal{D}\left(\rho \star (\eta_{\alpha\beta}\langle\partial_\Lambda|T^\alpha|A\rangle T^\beta|\Lambda\rangle + \langle\partial_\Lambda|A\rangle|\Lambda\rangle - \langle\partial_\Lambda|\Lambda\rangle|A\rangle)\langle\partial_\Lambda|\right) + \dots, \quad (5.53)\end{aligned}$$

where the first line in $\delta_\Lambda G$ directly follows from the Dorfman product (3.15), the second and third lines keep track on the trivial parameter with one constrained index in the transformation of $|F\rangle$, the last line is computed directly and the ellipses stand for a trivial parameter with two constrained indices.

With this we can now study the gauge transformation of the topological term (5.46). When varying $\langle\partial_F|L_1|F\rangle + \text{Tr}[L_0G]$, one obtains that all the contributions cancel up to a total derivative, except for the one obtained by commuting $\langle\partial_F|$ with $\mathcal{L}_{|A}$ on $\delta_\Lambda|F\rangle$ and the last term (involving the Hodge star) in the variation of $\delta_\Lambda G$. To understand this cancellation, one notes that $\langle\partial_F|L_1|F\rangle + \text{Tr}[L_0G]$ is invariant under all one-form gauge transformations up to a total internal derivative (A.14), and therefore, the trivial parameter components of the variations (5.52) and (5.53) cancel up to a total derivative in the variation of $\langle\partial_F|L_1|F\rangle + \text{Tr}[L_0G]$. Eventually one obtains

$$\begin{aligned}&\Delta_\Lambda(-2\langle\partial_F|L_1|F\rangle - 2\text{Tr}[L_0G]) \\ &= 2\eta_{\alpha\beta}\langle\partial_\Lambda|T^\alpha|\Lambda\rangle\langle\partial_\Lambda|L_1T^\beta|F\rangle + 2\langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|L_1|F\rangle + 2\eta_{1\alpha\beta}\langle\partial_\Lambda|T^\alpha|F\rangle\langle\partial_\Lambda|L_0T^\beta|\Lambda\rangle \\ &\quad - 2\mathcal{D}\left[\star\rho(\langle\partial_\Lambda|T_\alpha|\Lambda\rangle\langle\partial_\Lambda|L_0T^\alpha|A\rangle + \langle\partial_\Lambda|A\rangle\langle\partial_\Lambda|L_0|A\rangle - \langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|L_0|A\rangle)\right] \\ &= 2\langle\partial_\Lambda|L_1|\Lambda\rangle\langle\partial_\Lambda|F\rangle \\ &\quad - 2(d - \langle\partial|A\rangle)\left[\star\rho(\langle\partial_\Lambda|T_\alpha|\Lambda\rangle\langle\partial_\Lambda|L_0T^\alpha|A\rangle + \langle\partial_\Lambda|A\rangle\langle\partial_\Lambda|L_0|A\rangle - \langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|L_0|A\rangle)\right] \\ &\quad - 2\rho(\langle\partial_A|[L_0, T_\alpha]|A\rangle + \eta_{-1\alpha\gamma}\text{Tr}[[L_0, T^\gamma]B]) \\ &\quad \star(\langle\partial_\Lambda|T_\beta|\Lambda\rangle\langle\partial_\Lambda|T^\alpha T^\beta|A\rangle + \langle\partial_\Lambda|A\rangle\langle\partial_\Lambda|T^\alpha|A\rangle - \langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|T^\alpha|A\rangle), \quad (5.54)\end{aligned}$$

where the second line is a total derivative. We also have after some algebra

$$\begin{aligned}&\Delta_\Lambda(-2\langle\partial_F|\mathcal{M}^{-1}L_{-1}\mathcal{M}|F\rangle + \rho^{-2}\eta_{1\alpha\beta}\langle\partial_{A_2}|T^\alpha|A_1\rangle\langle\partial_{A_2}|\mathcal{M}^{-1}L_{-2}\mathcal{M}T^\beta|A_2\rangle) \quad (5.55) \\ &= 2\langle\partial_\Lambda|T_\alpha|\Lambda\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|F\rangle + 2\langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|F\rangle + 2\langle\partial_\Lambda|A\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|d\Lambda\rangle \\ &\quad + 2\langle\partial_\Lambda|d\Lambda\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|A\rangle + 2\langle\partial_\Lambda|T_\alpha|A\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|d\Lambda\rangle \\ &\quad + \langle\partial_\Lambda|T_\alpha|\Lambda\rangle(\langle\partial_\Lambda|T_\beta T^\alpha|A_1\rangle\langle\partial_{A_2}|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\beta|A_2\rangle - \langle\partial_{A_1}|T_\beta T^\alpha|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|A_2\rangle) \\ &\quad - \langle\partial_\Lambda|T^\alpha|\Lambda\rangle(\langle\partial_\Lambda|A_1\rangle\langle 2\partial_\Lambda + \partial_{A_1} + 2\partial_{A_2}|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\beta|A_2\rangle + \langle\partial_{A_2}|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|A_2\rangle) \\ &\quad + \langle\partial_{A_2}|\Lambda\rangle\langle\partial_\Lambda|T_\alpha|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|A_2\rangle + \langle\partial_\Lambda|\Lambda\rangle\langle\partial_{A_1}|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|A_2\rangle \\ &\quad + \langle\partial_{A_2}|\Lambda\rangle\langle\partial_\Lambda|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|A_2\rangle - 2\langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|A_1\rangle\langle\partial_\Lambda + \partial_{A_1} + \partial_{A_2}|\mathcal{M}^{-1}L_{-1}\mathcal{M}|A_2\rangle\end{aligned}$$

where we have written $T_\alpha = \eta_{\alpha\beta} T^\beta$ for the \mathfrak{e}_9 generators and one should not forget that the transport term in the Lie derivative of $|A\rangle$ does contribute to the non-covariant variation of the second term when two derivatives act on the parameter $|\Lambda\rangle$. One also computes

$$\Delta_\Lambda(2\langle\rho^2(\chi - \omega_{-1}^\alpha(\mathcal{M})J_\alpha)|F\rangle) = -2\langle\partial_\Lambda|(L_1 + \mathcal{M}^{-1}L_{-1}\mathcal{M})|\Lambda\rangle\langle\partial_\Lambda|F\rangle, \quad (5.56)$$

and finally that

$$\begin{aligned} & \Delta_\Lambda(2D\chi + J_0\chi + J_{-1}J_K + \eta^{AB} \sum_n n J_A^n J_B^{1-n}) \\ &= \langle\partial_\Lambda|T_\alpha|A\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}\mathcal{S}_1(J)^\dagger\mathcal{M}T^\alpha|\Lambda\rangle + \langle\partial_\Lambda|A\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}\mathcal{S}_1(J)^\dagger\mathcal{M}|\Lambda\rangle \\ &+ \langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}\mathcal{S}_1(J)^\dagger\mathcal{M}|A\rangle + 2\langle\partial_\Lambda|T_\alpha|F\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|\Lambda\rangle + 2\langle\partial_\Lambda|F\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|\Lambda\rangle \\ &- 2\langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|F\rangle - 2\langle\partial_\Lambda|T_\alpha|A\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|\Lambda\rangle \\ &- 2\langle\partial_\Lambda|A\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|\Lambda\rangle + 2\langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|A\rangle \\ &- \langle\partial_\Lambda|T_\alpha|\Lambda\rangle(\langle\partial_\Lambda|T_\beta T^\alpha|A_1\rangle\langle\partial_{A_2}|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\beta|A_2\rangle - \langle\partial_{A_1}|T_\beta T^\alpha|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\beta|A_2\rangle) \\ &+ \langle\partial_\Lambda|T_\alpha|\Lambda\rangle(\langle\partial_\Lambda|A_1\rangle\langle 2\partial_\Lambda + \partial_{A_1} + 2\partial_{A_2}|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|A_2\rangle + \langle\partial_{A_2}|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|A_2\rangle) \\ &- \langle\partial_{A_2}|\Lambda\rangle\langle\partial_\Lambda|T_\alpha|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\alpha|A_2\rangle - \langle\partial_\Lambda|\Lambda\rangle\langle\partial_{A_1}|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|A_2\rangle \\ &- \langle\partial_{A_2}|\Lambda\rangle\langle\partial_\Lambda|A_1\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}L_{-1}\mathcal{M}|A_2\rangle + 2\langle\partial_\Lambda|\Lambda\rangle\langle\partial_\Lambda|A_1\rangle\langle\partial_\Lambda + \partial_{A_1} + \partial_{A_2}|\mathcal{M}^{-1}L_{-1}\mathcal{M}|A_2\rangle. \end{aligned} \quad (5.57)$$

Combining all terms one gets eventually

$$\Delta_\Lambda(\mathcal{L}_1 + \mathcal{L}_2) = 0, \quad (5.58)$$

such that $\mathcal{L}_1 + \mathcal{L}_2$ is indeed invariant under internal diffeomorphisms. Moreover, one checks that in $\mathcal{L}_1 + \mathcal{L}_2$ is invariant under the Σ and 1-form gauge transformations defined in Appendix A.5.

The potential V was shown to be invariant under internal diffeomorphisms in [1], and the two additional terms in $\tilde{g}_{\mu\nu}$ are manifestly invariant under internal diffeomorphisms up to a total derivative. Therefore the pseudo-Lagrangian (5.44) consists of three pieces that are separately invariant under internal diffeomorphisms.

We shall see in the next section, that external diffeomorphism invariance also requires to introduce one additional duality equation in the theory. Just like the B field Euler–Lagrange equation, obtained by varying the pseudo-Lagrangian with respect B , gives a projection of the duality equation (5.35), the Euler–Lagrange equation for the constrained field $\langle\chi|$ gives a projection of the duality equation²⁷

$$|\mathcal{F}_{\mu\nu}\rangle = -\rho^{-1}\varepsilon_{\mu\nu}T^\alpha\mathcal{M}^{-1}|\mathcal{J}_\alpha^-\rangle, \quad (5.59)$$

where

$$|\mathcal{F}\rangle = |F\rangle + \langle\partial_C|T_\alpha|C_{(1)}T^\alpha|C_2\rangle + \langle\pi_C|T_\alpha|C_{[1]}T^\alpha|C_2\rangle + 2\langle\pi_C|C_{[1]}\rangle|C_2\rangle + \eta_{-1\alpha\beta}\text{Tr}[T^\alpha C_2^+ T^\beta]|C_1^+\rangle, \quad (5.60)$$

²⁷This equation follows from the duality equations in [47] upon branching to E_9 .

where we use the same notation as in (A.5a), (A.5b) and (A.5c) and the shifted internal current $\langle \mathcal{J}_\alpha^- |$ is defined in (5.47). These two-forms can be written in components

$$|C_{(1)} \otimes |C_{2)} \rangle \Leftrightarrow C^{(MN)}, \quad |C_{[1]} \otimes |C_{2]} \rangle \otimes \langle \pi_C | \Leftrightarrow C^{[MN]}_P, \quad |C_1^+ \rangle \otimes C_2^+ \Leftrightarrow C^{+M;N}_P, \quad (5.61)$$

where $C^{(MN)}$ is symmetric, $C^{[MN]}_P$ antisymmetric in MN , and $C^{[MN]}_P$ and $C^{+M;N}_P$ are both constrained on their index P . Note in particular that $|C_1 \rangle \otimes |C_2 \rangle$ represents a unique field that does not factorise, so $\langle \partial_C | \otimes |C_1 \rangle \otimes |C_2 \rangle \Leftrightarrow \partial_P C^{MN}$ is a total derivative. As trivial parameters, they are defined such that they drop out when $|\mathcal{F}\rangle$ is contracted with a constrained bra, as in the equation of motion of $\langle \chi |$ for example. One checks using (5.52) that this duality equation (5.59) is indeed invariant under internal diffeomorphisms provided one defines the (inhomogeneous) transformation of the two-form potentials as²⁸

$$\begin{aligned} \tilde{\Delta}_\Lambda |C_1 \rangle \otimes |C_{2\mu\nu} \rangle &= \frac{1}{2} (\partial_{[\mu} |\Lambda \rangle \otimes |A_{\nu]} \rangle - |A_{[\mu} \rangle \otimes \partial_{\nu]} |\Lambda \rangle), \\ \tilde{\Delta}_\Lambda |C_{[1]} \rangle \otimes |C_{2]\mu\nu} \rangle \otimes \langle \pi_C | &= \frac{1}{2} (\partial_{[\mu} |\Lambda \rangle \otimes |A_{\nu]} \rangle + |A_{[\mu} \rangle \otimes \partial_{\nu]} |\Lambda \rangle) \otimes \langle \partial_\Lambda - \partial_A |, \\ \tilde{\Delta}_\Lambda |C_{1\mu\nu}^+ \rangle \otimes C_2^+ &= -\rho^{-1} \varepsilon_{\mu\nu} \mathcal{M}^{-1} |\partial_\Lambda \rangle \otimes |\Lambda \rangle \langle \partial_\Lambda | - \eta_{1\alpha\beta} \langle \partial_\Lambda | T^\alpha |A_{[\mu} \rangle |A_{\nu]} \rangle \otimes T^\beta |\Lambda \rangle \langle \partial_\Lambda |. \end{aligned} \quad (5.62)$$

Note that the two-form indices can be placed on any C_i since they are representing the same field according to our notation (5.61). The first term in the non-covariant variation of $|C_{1\mu\nu}^+ \rangle \otimes C_2^+$ does not follow from the Dorfman structure and does not appear in (A.30), but is defined to compensate the non-covariant transformation of the shifted current [1, Eq. (4.30)]

$$\begin{aligned} \Delta_\Lambda T^\alpha \mathcal{M}^{-1} |\mathcal{J}_\alpha^- \rangle &= \langle \partial_\Lambda | T^\alpha |\Lambda \rangle (\eta_{-1\alpha\beta} T^\beta \mathcal{M}^{-1} |\partial_\Lambda \rangle + \rho^{-2} \eta_{1\alpha\beta} \mathcal{M}^{-1} T^{\beta\dagger} |\partial_\Lambda \rangle) \\ &= \eta_{-1\alpha\beta} \langle \partial_\Lambda | T^\alpha |\Lambda \rangle T^\beta \mathcal{M}^{-1} |\partial_\Lambda \rangle, \end{aligned} \quad (5.63)$$

where we used that the second term vanishes according to the section constraint. Therefore the field strength does not transform covariantly as in (3.30), but includes moreover the same non-covariant variation as the internal current.

We shall see in the next subsection that we need to include (at least a projection of) this duality equation in order to obtain a system of equations invariant under external diffeomorphisms.

5.3 External diffeomorphisms invariance

Similarly as in [22], we define the external diffeomorphisms as

$$\begin{aligned} \delta_\xi \mathcal{M} &= \xi^\mu \mathcal{D}_\mu^\flat \mathcal{M} - \mathcal{D}_\mu \xi^\mu \mathcal{M}, \\ \delta_\xi \tilde{g}_{\mu\nu} &= \xi^\sigma \mathcal{D}_\sigma \tilde{g}_{\mu\nu} + 2\mathcal{D}_{(\mu} \xi^\sigma \tilde{g}_{\nu)\sigma} - \tilde{g}_{\mu\nu} \mathcal{D}_\sigma \xi^\sigma, \\ \delta_\xi |A_\mu \rangle &= \rho^{-1} \varepsilon_{\mu\nu} \xi^\nu T^\alpha \mathcal{M}^{-1} |\mathcal{J}_\alpha^- \rangle + \rho^{-2} \tilde{g}_{\mu\nu} \mathcal{M}^{-1} |\partial \xi^\nu \rangle, \end{aligned} \quad (5.64)$$

where

$$\mathcal{D}_\mu \xi^\nu = \partial_\mu \xi^\nu - \langle \partial \xi^\nu | A_\mu \rangle. \quad (5.65)$$

²⁸Here the inhomogeneous transformation $\tilde{\Delta}_\Lambda$ is defined in (A.29). It is the \mathbb{C} -independent component of $\delta_\Lambda \mathbb{C}$.

Note that we do not use the full covariant derivative $\mathcal{D}_\mu \mathcal{M}$ because Σ gauge invariance is not manifest and we have gauge fixed the L_{-1} gauge symmetry by setting $\tilde{\rho} = 0$. Writing the duality equation (5.59) as $|\mathcal{E}_{\mu\nu}\rangle = 0$ with

$$|\mathcal{E}_{\mu\nu}\rangle \equiv |\mathcal{F}_{\mu\nu}\rangle + \rho^{-1} \varepsilon_{\mu\nu} T^\alpha \mathcal{M}^{-1} |\mathcal{J}_\alpha^-\rangle, \quad (5.66)$$

one finds that this transformation of the vector potential indeed corresponds to a covariant diffeomorphism modulo the duality equation (5.59), i.e.

$$\delta_\xi |A_\mu\rangle = \xi^\nu |\mathcal{F}_{\nu\mu}\rangle + \rho^{-2} \tilde{g}_{\mu\nu} \mathcal{M}^{-1} |\partial\xi^\nu\rangle + \xi^\nu |\mathcal{E}_{\mu\nu}\rangle. \quad (5.67)$$

This is the same argument that is used to define [22, Eq. (3.40)].

We will determine the transformation of the fields χ_μ and B_μ such that the twisted self-duality equation (5.35)

$$\mathcal{E}_\mu \equiv \mathcal{J}_\mu - \rho^{-1} \mathcal{M}^{-1} \star (\mathcal{S}_1(\mathcal{J}_\mu) + \chi_\mu \mathbf{K})^\dagger \mathcal{M} \quad (5.68)$$

transforms into itself and other equations of motion under external generalised diffeomorphisms. We will show below that these transformations can be defined such that

$$\delta_\xi \mathcal{E}_\mu = \xi^\nu \mathcal{D}_\nu^\flat \mathcal{E}_\mu + \mathcal{D}_\mu \xi^\nu \mathcal{E}_\nu + \xi^\nu \left(-\mathcal{M}^{-1} \mathcal{L}_{(\mathcal{E}_{\mu\nu}, 0)} \mathcal{M} + \rho^{-1} \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\rho} \mathcal{M}^{-1} \mathcal{S}_1(\mathcal{M}^{-1} \mathcal{L}_{(\mathcal{E}_{\rho\nu}, 0)} \mathcal{M})^\dagger \mathcal{M} \right), \quad (5.69)$$

so that we find that external diffeomorphism invariance requires to consider

$$\eta_{\alpha\beta} \langle \partial_\mathcal{E} |T^\beta| \mathcal{E}_{\mu\nu}\rangle = 0 \quad (5.70)$$

as an equation of motion. This equation is more constraining than the Euler–Lagrange equation $\langle \delta\chi | \mathcal{E}_{\mu\nu}\rangle = 0$.

One may be used to the property that exceptional field theories can usually be defined without being forced to introduce the duality equations for the non-propagating higher form fields, as is the case e.g. in [20–22]. For E_9 one may expect that the Euler–Lagrange equation for the coset scalar fields would imply an equation of the form

$$\eta_{\alpha\beta} \langle \partial_\mathcal{E} |T^\beta| \mathcal{E}_{\mu\nu}\rangle + \eta_{-1\alpha\beta} \text{Tr}[T^\beta \mathcal{E}_{\mu\nu}] \stackrel{?}{=} 0 \quad (5.71)$$

for a $\mathcal{E}_{\mu\nu}$ involving the two-form field strength $\mathcal{G}_{\mu\nu}$ such that this equation would not depend explicitly in the two-form potential \mathbb{C} . However, the scalar fields Euler–Lagrange equations only give a projection of the \mathcal{S}_1 shift of such an equation to $\mathfrak{e}_9 \ominus K(\mathfrak{e}_9)$, which does not imply that it holds.

To compute the variation of (5.68) under external diffeomorphisms, it will be convenient to define the non-covariant component of an external diffeomorphism as

$$\delta_\xi = \mathcal{L}_\xi + \Delta_\xi \quad (5.72)$$

such that

$$\mathcal{L}_\xi \mathcal{J}_\mu = \xi^\nu \mathcal{D}_\nu^\flat \mathcal{J}_\mu + \mathcal{D}_\mu \xi^\nu \mathcal{J}_\nu, \quad (5.73)$$

and similarly for any vector field. Using this definition, one computes that

$$\begin{aligned} \Delta_\xi \mathcal{J}_\mu^\flat &= \xi^\nu \eta_{-1\alpha\beta} \eta_{1\gamma\delta} \langle \partial_A | T^\gamma | A'_{[\mu} \rangle \langle \partial_A | T^\alpha T^\delta | A_{\nu]} \rangle (T^\beta + \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M}) \\ &\quad - \xi^\nu (\langle \mathcal{J}_\alpha | E_{\mu\nu} \rangle + \eta_{\alpha\beta} \langle \partial_E | (T^\beta + \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M}) | E_{\mu\nu} \rangle) T^\alpha - \xi^\nu \tilde{g}^{\sigma\rho} \langle \partial \tilde{g}_{\mu\sigma} | E_{\rho\nu} \rangle \mathbb{K} \\ &\quad - \rho^{-2} \tilde{g}_{\mu\nu} \langle \partial \xi^\nu | \mathcal{M}^{-1} | \mathcal{J}_\alpha \rangle T^\alpha - \rho^{-1} \varepsilon_{\mu\nu} \langle \partial \xi^\nu | T_\alpha T^\beta \mathcal{M}^{-1} | \mathcal{J}_\beta^- \rangle (T^\alpha + \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M}) \\ &\quad - \rho^{-2} \langle \partial \tilde{g}_{\mu\nu} | \mathcal{M}^{-1} | \partial \xi^\nu \rangle \mathbb{K} + \left(\mathcal{D}_\nu \mathcal{D}_\mu \xi^\nu - 2 \mathcal{D}_\mu \mathcal{D}_\nu \xi^\nu + \tilde{g}_{\mu\rho} \tilde{g}^{\nu\sigma} \mathcal{D}_\nu \mathcal{D}_\sigma \xi^\rho \right) \mathbb{K} \\ &\quad - \rho^{-2} \eta_{\alpha\beta} \tilde{g}_{\mu\nu} \langle \partial | (T^\beta \mathcal{M}^{-1} + \mathcal{M}^{-1} T^{\beta\dagger}) | \partial \xi^\nu \rangle T^\alpha - \rho^{-2} \tilde{g}_{\mu\nu} \langle \partial \xi^\nu | [T_\alpha, T^\beta] \mathcal{M}^{-1} | \mathcal{J}_\beta \rangle T^\alpha, \end{aligned} \quad (5.74)$$

where $\langle \partial |$ in the last line indicates that this term is a total internal derivative and

$$|E_{\mu\nu}\rangle = |F_{\mu\nu}\rangle + \rho^{-1} \varepsilon_{\mu\nu} T^\alpha \mathcal{M}^{-1} | \mathcal{J}_\alpha^- \rangle, \quad (5.75)$$

is the component of equation (5.59) that does not include the two-form gauge fields. The first term $\langle \mathcal{J}_\alpha | E_{\mu\nu} \rangle = \langle \mathcal{J}_\alpha | \mathcal{E}_{\mu\nu} \rangle$, but to get the two-form dependence in the second term, we need to define $\Delta_\xi B$ and $\Delta_\xi \chi$ to depend explicitly on the two-forms. One computes that (5.69) is satisfied if one defines the variations

$$\begin{aligned} \Delta_\xi B_\mu &= \eta_{1\alpha\beta} \xi^\nu \langle \partial_A | T^\alpha | A'_{[\mu} \rangle T^\beta | A_{\nu]} \rangle \langle \partial_A | \\ &\quad - \rho \tilde{g}_{\mu\rho} \varepsilon^{\rho\sigma} \xi^\nu (\eta_{\alpha\beta} \langle \partial_A | T^\alpha | A'_{[\sigma} \rangle T^\beta | A_{\nu]} \rangle + \langle \partial_A | A'_{[\rho} | A_{\nu]} \rangle + \langle \partial_A | A_{[\rho} | A'_{\nu]} \rangle) \langle \partial_A | \\ &\quad + \rho^{-1} \varepsilon_{\mu\nu} \left(2 \xi^\nu \mathcal{M}^{-1} | \partial_\xi \rangle \langle \partial_\xi | + \tilde{g}^{\nu\sigma} \mathcal{M}^{-1} (| \partial \xi^\rho \rangle \langle \partial \tilde{g}_{\rho\sigma} | + | \partial \tilde{g}_{\rho\sigma} \rangle \langle \partial \xi^\rho |) \right. \\ &\quad \left. - T^\alpha \mathcal{M}^{-1} (| \partial \xi^\nu \rangle \langle \mathcal{J}_\alpha | + | \mathcal{J}_\alpha \rangle \langle \partial \xi^\nu |) + \mathcal{M}^{-1} | \partial \xi^\nu \rangle \langle J_0 | \right) \\ &\quad + \tilde{g}_{\mu\nu} \left(\mathcal{M}^{-1} T^{\alpha\dagger} | \partial \xi^\nu \rangle \langle \mathcal{J}_\alpha^- | + T^\alpha \mathcal{M}^{-1} | \mathcal{J}_\alpha^- \rangle \langle \partial \xi^\nu | \right) + \xi^\nu \delta_{C_{\sigma\nu}} B_\mu, \\ \Delta_\xi \chi_\mu &= -\xi^\nu \left(\eta_{\alpha\beta} \langle \partial_A | T^\alpha | A'_{[\mu} \rangle \langle \partial_A | \mathcal{M}^{-1} L_{-1} \mathcal{M} T^\beta | A_{\nu]} \rangle + \langle \partial_A | A'_{[\mu} \rangle \langle \partial_A | \mathcal{M}^{-1} L_{-1} \mathcal{M} | A_{\nu]} \rangle \right. \\ &\quad \left. + \langle \partial_A | A_{[\mu} \rangle \langle \partial_A | \mathcal{M}^{-1} L_{-1} \mathcal{M} | A'_{\nu]} \rangle \right) \\ &\quad - \xi^\nu \tilde{g}_{\mu\rho} \varepsilon^{\rho\sigma} (\eta_{\alpha\beta} \langle \partial_A | T^\alpha | A'_{[\sigma} \rangle \langle \partial_A | L_0 T^\beta | A_{\nu]} \rangle + \langle \partial_A | A'_{[\sigma} \rangle \langle \partial_A | L_0 | A_{\nu]} \rangle + \langle \partial_A | A_{[\sigma} \rangle \langle \partial_A | L_0 | A'_{\nu]} \rangle) \\ &\quad - \rho \varepsilon_{\mu\rho} \xi^\nu \langle \partial \tilde{g}^{\rho\sigma} | E_{\sigma\nu} \rangle + \rho \varepsilon_{\mu\rho} \tilde{g}^{\sigma\rho} (\mathcal{D}_\nu \mathcal{D}_\sigma \xi^\nu - 2 \mathcal{D}_\sigma \mathcal{D}_\nu \xi^\nu) + \rho \varepsilon_{\mu\nu} \tilde{g}^{\sigma\rho} \mathcal{D}_\sigma \mathcal{D}_\rho \xi^\nu \\ &\quad + \rho^{-2} \tilde{g}_{\mu\nu} \left(\langle \partial | (L_1 \mathcal{M}^{-1} + \mathcal{M}^{-1} L_{-1}) | \partial \xi^\nu \rangle + \langle \partial \xi^\nu | [L_1, T^\alpha] \mathcal{M}^{-1} | \mathcal{J}_\alpha \rangle \right. \\ &\quad \left. + \langle \partial \xi^\nu | (L_0 T^\alpha \mathcal{M}^{-1} + T^\alpha \mathcal{M}^{-1} L_0) | \mathcal{J}_\alpha^- \rangle + \rho^2 \langle \partial \xi^\nu | \mathcal{M}^{-1} \mathcal{S}_1 T^{\alpha\dagger} | \mathcal{J}_\alpha \rangle \right) \\ &\quad + \rho^{-1} \tilde{g}_{\mu\rho} \varepsilon^{\rho\sigma} \langle \partial \tilde{g}_{\sigma\nu} | (L_0 \mathcal{M}^{-1} + \mathcal{M}^{-1} L_0 - \mathcal{M}^{-1}) | \partial \xi^\nu \rangle \\ &\quad + \rho^{-1} \varepsilon_{\mu\nu} \left(\langle \partial_\xi + \partial_{\mathcal{M}} | (\mathcal{M}^{-1} L_0 + L_0 \mathcal{M}^{-1}) | \partial \xi^\nu \rangle + \langle \partial \xi^\nu | \mathcal{M}^{-1} L_0 | J_0 \rangle + \langle \partial \xi^\nu | T^\alpha \mathcal{M}^{-1} | \mathcal{J}_\alpha \rangle \right. \\ &\quad \left. + \langle \partial \xi^\nu | (L_1 + \mathcal{M}^{-1} L_{-1} \mathcal{M}) T^\alpha \mathcal{M}^{-1} | \mathcal{J}_\alpha^- \rangle \right) + \xi^\nu \delta_{C_{\sigma\nu}} \chi_\mu, \end{aligned} \quad (5.76)$$

where $\xi^\nu \delta_{C_{\sigma\nu}}$ is the one-form gauge transformation of parameter $C_{\sigma\nu}$ where σ is the index of the vector parameter whereas ν is contracted with ξ^ν , i.e.²⁹

$$\begin{aligned}
\xi^\nu \delta_{C_{\sigma\nu}} B &= -\frac{1}{2} \iota_\xi \eta_{1\alpha\beta} T^\alpha |C_{[1]} \rangle \langle \pi_C | T^\beta | C_{2]} \rangle \langle \partial_C | + \langle \partial_C | T^\beta | C_{2]} \rangle \langle \pi_C | \rangle \\
&\quad + \iota_\xi \eta_{\alpha\beta} \text{Tr}[T^\alpha C_2^+ | T^\beta | C_1^+ \rangle \langle \partial_{C^+} | - \iota_\xi | C_1^+ \rangle \langle \partial_{C^+} | C_2^+ + \iota_\xi \langle \partial_{C^+} | C_1^+ \rangle C_2^+ \\
&\quad + \rho \star \iota_\xi \left(\eta_{\alpha\beta} T^\alpha |C_{[1]} \rangle \langle \pi_C | T^\beta | C_{2]} \rangle + 2 |C_{[1]} \rangle \langle \pi_C | C_{2]} \rangle \right) \langle \partial_C | - \rho \star \iota_\xi \eta_{-1\alpha\beta} \text{Tr}[T^\alpha C_2^+ | T^\beta | C_1^+ \rangle \langle \partial_{C^+} | , \\
\xi^\nu \delta_{C_{\sigma\nu}} \chi &= -\iota_\xi \eta_{\alpha\beta} \langle \pi_C | T^\alpha | C_{[1]} \rangle \langle \partial_C | \mathcal{M}^{-1} L_{-1} \mathcal{M} T^\beta | C_{2]} \rangle - 2 \iota_\xi \langle \pi_C | C_{[1]} \rangle \langle \partial_C | \mathcal{M}^{-1} L_{-1} \mathcal{M} | C_{2]} \rangle \\
&\quad - \rho \star \iota_\xi \left(\eta_{\alpha\beta} \langle \pi_C | T^\alpha | C_{[1]} \rangle \langle \partial_C | L_0 T^\beta | C_{2]} \rangle + 2 \langle \pi_C | C_{[1]} \rangle \langle \partial_C | L_0 | C_{2]} \rangle \right) \\
&\quad - \eta_{-1\alpha\beta} \iota_\xi \text{Tr}[T^\alpha C_2^+ | \langle \partial_{C^+} | \mathcal{M}^{-1} L_{-1} \mathcal{M} T^\beta | C_1^+ \rangle - \rho \star \iota_\xi \eta_{-1\alpha\beta} \text{Tr}[T^\alpha C_2^+ | \langle \partial_{C^+} | L_0 T^\beta | C_1^+ \rangle .
\end{aligned} \tag{5.77}$$

Because we consider $|\mathcal{E}_{\mu\nu}\rangle = 0$ as an equation of motion, invariance of the system also requires to compute its own transformation under external diffeomorphisms. One computes that $|\mathcal{E}_{\mu\nu}\rangle$ varies into

$$\delta_\xi |\mathcal{E}_{\mu\nu}\rangle = -\rho^{-2} \varepsilon_{\mu\nu} \tilde{g}_{\sigma\lambda} \varepsilon^{\lambda\rho} \mathcal{E}_\rho \mathcal{M}^{-1} |\partial \xi^\sigma\rangle , \tag{5.78}$$

with the definitions

$$\begin{aligned}
\delta_\xi \langle \chi | &= -\langle \partial \xi^\mu | \left(\rho^{-1} \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\nu} J_{\nu 0} + \rho^{-2} (\chi_\mu + \text{Tr}[\mathcal{M}^{-1} L_0 \mathcal{M} B_\mu] + B_\mu - \text{Tr}[B_\mu]) + \omega_{-1}^\alpha(\mathcal{M}) \mathcal{J}_{\mu\alpha} \right) \\
&\quad - \sum_n n \xi^\mu \langle \partial_A | T_{n+1}^A | A_\mu \rangle \langle J_A^n | + \xi^\mu \langle \partial_A | (L_{-1} + \mathcal{M}^{-1} L_1 \mathcal{M}) | A_\mu \rangle \langle \partial_A | ,
\end{aligned} \tag{5.79}$$

and

$$\begin{aligned}
\Delta_\xi |C\rangle \otimes |C_{\mu\nu}\rangle &= \frac{1}{2} (\delta_\xi |A_{[\mu}\rangle \otimes |A_{\nu]}\rangle - |A_{[\mu}\rangle \otimes \delta_\xi |A_{\nu]}\rangle) , \\
\Delta_\xi |C_{[1]} \rangle \otimes |C_{2] \mu\nu}\rangle \otimes \langle \pi_C | &= \frac{1}{2} (\delta_\xi |A_{[\mu}\rangle \otimes |A_{\nu]}\rangle + |A_{[\mu}\rangle \otimes \delta_\xi |A_{\nu]}\rangle) \otimes \langle \partial_{\delta_\xi A} - \partial_A | , \\
\Delta_\xi |C_{1\mu\nu}^+ \rangle \otimes C_2^+ &= \rho^{-1} \varepsilon_{\mu\nu} \mathcal{M}^{-1} \left(|\partial_A\rangle \otimes \xi^\sigma |A_\sigma\rangle \langle \partial_A | - \rho^{-1} \tilde{g}_{\rho\sigma} \varepsilon^{\sigma\nu} |\partial \xi^\rho\rangle \otimes B_\nu \right) .
\end{aligned} \tag{5.80}$$

The duality equations (5.35) and (5.59) therefore transform into themselves under external diffeomorphisms. The minimal set of duality equations transforming into each others under external diffeomorphisms is

$$\mathcal{E}_{\mu\alpha} = 0 , \quad \eta_{\alpha\beta} \langle \partial \mathcal{E} | T^\beta | \mathcal{E}_{\mu\nu}\rangle = 0 . \tag{5.81}$$

One may be worried that the external diffeomorphisms variations of the ancillary fields B and χ depend on the two-form potentials while the pseudo-Lagrangian does not. The resolution is that the pseudo-Lagrangian is only invariant under external diffeomorphisms up to bilinear terms in the duality equations, including equation (5.59) that depends on the two-form potentials. To see this we observe that varying B and χ only in the pseudo-Lagrangian one obtains that the dependence in the two-form potentials can be absorbed into field strengths terms, i.e. that

$$\text{Tr}[\mathcal{E} \delta_{\iota_\xi C} B] + \mathcal{E}_0 \delta_{\iota_\xi C} \chi^\flat + \langle \partial_{\mathcal{F}} | \left(\mathcal{S}_1(\mathcal{E}) - \mathcal{E}_0 L_1 - \mathcal{E}_0 \mathcal{M}^{-1} L_{-1} \mathcal{M} - \mathcal{E} \rho \star \right) \iota_\xi | \mathcal{F} \rangle \tag{5.82}$$

²⁹where $\iota_\xi dx^\mu = \xi^\mu$.

does not depend on the two-form potential and so we expect the non-covariant variation of the pseudo-Lagrangian to give

$$\Delta_\xi \mathcal{L} = -\frac{1}{2} \langle \partial_{|\mathcal{E}} | \left(\mathcal{S}_1(\mathcal{E}) - \mathcal{E}_0 L_1 - \mathcal{E}_0 \mathcal{M}^{-1} L_{-1} \mathcal{M} - \mathcal{E} \rho \star \right) \iota_\xi | \mathcal{E} \rangle + d\mathcal{B}_\xi - \langle \partial | \mathcal{B}_\xi \rangle \quad (5.83)$$

for some specific boundary terms \mathcal{B}_ξ and $|\mathcal{B}_\xi\rangle$. We stress here that it is enough for the pseudo-Lagrangian to be invariant under a symmetry up to terms quadratic in the duality equations for its Euler–Lagrange equations to transform into themselves and the duality equations under that symmetry. On the contrary, it would not be sufficient for the pseudo-Lagrangian to be invariant up to terms linear in the duality equations. A symmetry of the equations of motion only require that they transform into themselves under the symmetry variations. If the Euler–Lagrange equations of a pseudo-Lagrangian transform under the symmetry into duality equations, then one finds that the pseudo-Lagrangian is only invariant under that symmetry up to terms quadratic in the duality equations. This is discussed in detail in Appendix E.

As a consistency check, one can verify that the relation between the quadratic terms in the duality equation in (5.83) are consistently related to the terms proportional to the duality equations in the external diffeomorphism variation of the B fields according to the general discussion of Appendix E. This can easily be verified, because the equation of motion $\mathcal{E}_B = 0$ of the constrained field B is simply the projection of (5.68) upon tracing with δB , so its variation under external diffeomorphisms follows directly from (5.69) as³⁰

$$\begin{aligned} \text{Tr}[\delta B \delta_\xi \mathcal{E}_B] &= -\frac{1}{2} \text{Tr}[\delta B \delta_\xi \mathcal{E}] \quad (5.84) \\ &= -\frac{1}{2} \text{Tr}[\delta B \mathcal{L}_\xi \mathcal{E}] - \frac{1}{2} \text{Tr}[T^\alpha \delta B] \iota_\xi \langle \mathcal{J}_\alpha | \mathcal{E} \rangle + \frac{1}{2} \text{Tr}[\mathcal{M}^{-1} \mathcal{S}_1 T^{\alpha\dagger} \mathcal{M} \delta B] \rho^{-1} \star \iota_\xi \langle \mathcal{J}_\alpha | \mathcal{E} \rangle \\ &\quad - \frac{1}{2} \langle \partial_{|\mathcal{E}} | \delta B \iota_\xi | \mathcal{E} \rangle + \frac{1}{2} \text{Tr}[\delta B] \iota_\xi \langle \partial_{|\mathcal{E}} | \mathcal{E} \rangle - \frac{1}{2} \text{Tr}[\mathcal{M}^{-1} L_{-1} \mathcal{M} \delta B] \rho^{-1} \star \iota_\xi \langle \partial_{|\mathcal{E}} | \mathcal{E} \rangle \\ &\quad + \frac{1}{2} \eta_{\alpha\beta} \text{Tr}[\mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M} \delta B] \langle \partial_{|\mathcal{E}} | \left(\rho^{-1} \star (\mathcal{S}_1(T^\alpha) - \delta_0^\alpha (L_1 + \mathcal{M}^{-1} L_{-1} \mathcal{M})) - T^\alpha \right) \iota_\xi | \mathcal{E} \rangle \end{aligned}$$

where the two first lines are proportional to Euler–Lagrange equations whereas the last line is proportional to the duality equation (5.59). Assuming (5.83) we can vary the right-hand side with respect to B to obtain

$$\begin{aligned} \delta \Delta_\xi \mathcal{L} &= \frac{1}{2} \langle \partial_{|\mathcal{E}} | \delta B \iota_\xi | \mathcal{E} \rangle - \frac{1}{2} \text{Tr}[\delta B] \iota_\xi \langle \partial_{|\mathcal{E}} | \mathcal{E} \rangle + \frac{1}{2} \text{Tr}[L_0 \delta B] \iota_\xi \langle \partial_{|\mathcal{E}} | \mathcal{E} \rangle \quad (5.85) \\ &\quad + \frac{1}{2} \eta_{\alpha\beta} \text{Tr}[\mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M} \delta B] \langle \partial_{|\mathcal{E}} | \left(\rho^{-1} \star (\mathcal{S}_1(T^\alpha) - \delta_0^\alpha (L_1 + \mathcal{M}^{-1} L_{-1} \mathcal{M})) - T^\alpha \right) \iota_\xi | \mathcal{E} \rangle , \end{aligned}$$

where again the first line is proportional to Euler–Lagrange equations whereas the second matches precisely the term above in $\text{Tr}[\delta B \delta_\xi \mathcal{E}_B]$ proportional to the duality equation (5.59), consistently with (E.5) in Appendix E.

We have not fully derived the invariance of the pseudo-Lagrangian up to quadratic terms in the duality equations. We shall only check that external diffeomorphism invariance fixes all free coefficients in the pseudo-Lagrangian.

We check this in steps. Let us start with external diffeomorphisms that do not depend on the internal coordinates and with $|A\rangle = 0$ for simplicity. We denote the corresponding transformation by $\delta_\xi^{(0)}$. To check the invariance of the pseudo-Lagrangian under such diffeomorphisms,

³⁰We vary δB at $\delta \chi^b = 0$, not $\delta \chi = 0$.

we only need to consider the terms involving $J_{\mu\kappa}$ and χ in $\mathcal{L}_1 + \mathcal{L}_2$, as well as the third term in (5.44). The terms linear in the field $\text{Tr}[B_\mu]$ transform together covariantly by construction since they are proportional to the duality equation (5.42). The term that remains is $D\chi + \frac{1}{2}J_0\chi$ in \mathcal{L}_2 . One computes that

$$\delta_\xi^{(0)}(\mathcal{L}_1 + \mathcal{L}_2) = \frac{1}{2}\partial_\mu(\xi^\mu(\mathcal{L}_1 + \mathcal{L}_2)) + \rho \varepsilon^{\mu\nu}(\partial_\mu - \langle \vec{\partial} | A_\mu \rangle)(\varepsilon^{\sigma\lambda} \tilde{g}_{\lambda\rho} \partial_\nu \partial_\sigma \xi^\rho). \quad (5.86)$$

To compute the variation of the third term we found useful to use

$$\frac{1}{4}\varepsilon^{\mu\nu} \varepsilon^{\sigma\rho} \tilde{g}^{\kappa\lambda} \partial_\mu \tilde{g}_{\sigma\kappa} \partial_\nu \tilde{g}_{\rho\lambda} = \partial_\mu(\varepsilon^{\mu\nu} \tilde{g}_{01} \tilde{g}_{11}^{-1} \partial_\nu \tilde{g}_{11}), \quad (5.87)$$

and vary instead

$$\begin{aligned} \delta_\xi^{(0)} \tilde{g}_{11} &= \xi^\mu \partial_\mu \tilde{g}_{11} + \tilde{g}_{11}(\partial_1 \xi^1 - \partial_0 \xi^0) + 2\tilde{g}_{01} \partial_1 \xi^0 \\ \delta_\xi^{(0)} \frac{\tilde{g}_{01}}{\tilde{g}_{11}} &= \xi^\mu \partial_\mu \frac{\tilde{g}_{01}}{\tilde{g}_{11}} + \partial_0 \xi^1 + \frac{\tilde{g}_{01}}{\tilde{g}_{11}}(\partial_0 \xi^0 - \partial_1 \xi^1) + \frac{\tilde{g}_{00}}{\tilde{g}_{11}} \partial_1 \xi^0 \end{aligned} \quad (5.88)$$

from which one computes that

$$\begin{aligned} \delta_\xi^{(0)} \left(\frac{\tilde{g}_{01}}{\tilde{g}_{11}} \partial_\mu \tilde{g}_{11} \right) &= \xi^\nu \partial_\nu \left(\frac{\tilde{g}_{01}}{\tilde{g}_{11}} \partial_\mu \tilde{g}_{11} \right) + \xi^\nu \partial_\mu \xi^\nu \left(\frac{\tilde{g}_{01}}{\tilde{g}_{11}} \partial_\nu \tilde{g}_{11} \right) - \varepsilon^{\nu\sigma} \tilde{g}_{\sigma\rho} \partial_\mu \partial_\nu \xi^\rho \\ &\quad + \partial_\mu \left(\tilde{g}_{11} \partial_0 \xi^1 + \left(\tilde{g}_{00} + \frac{2}{\tilde{g}_{11}} \right) \partial_1 \xi^0 \right), \end{aligned} \quad (5.89)$$

where the total derivative in the second line will drop out in the variation of the pseudo-Lagrangian. Using this formula one obtains

$$\delta_\xi^{(0)} \left(\frac{1}{4} \rho \varepsilon^{\mu\nu} \varepsilon^{\sigma\rho} \tilde{g}^{\kappa\lambda} \partial_\mu \tilde{g}_{\sigma\kappa} \partial_\nu \tilde{g}_{\rho\lambda} \right) = \partial_\vartheta \left(\xi^\vartheta \frac{1}{4} \rho \varepsilon^{\mu\nu} \varepsilon^{\sigma\rho} \tilde{g}^{\kappa\lambda} \partial_\mu \tilde{g}_{\sigma\kappa} \partial_\nu \tilde{g}_{\rho\lambda} \right) - \rho \varepsilon^{\mu\nu} \partial_\mu(\varepsilon^{\sigma\lambda} \tilde{g}_{\lambda\rho} \partial_\nu \partial_\sigma \xi^\rho). \quad (5.90)$$

Thus, the pseudo-Lagrangian is invariant under purely external diffeomorphisms, i.e. at $|A\rangle = 0$ for $\xi^\mu(x)$.

Now if we take the second order derivative terms in the unimodular metric and the vector field $\xi^\mu(x, y)$, we obtain from $\mathcal{L}_1 + \mathcal{L}_2$

$$\begin{aligned} \delta_\xi(\mathcal{L}_1 + \mathcal{L}_2) &= \dots + \rho^{-1} \left(\langle \partial_\xi | \mathcal{M}^{-1} | \partial \xi^\sigma \rangle \tilde{g}^{\mu\nu} \mathcal{D}_\mu \tilde{g}_{\nu\sigma} - \langle \partial \tilde{g}_{\mu\sigma} | \mathcal{M}^{-1} | \partial \xi^\sigma \rangle \mathcal{D}_\nu \tilde{g}^{\mu\nu} \right) \\ &= \dots + \langle \partial | (\rho^{-1} \mathcal{M}^{-1} | \partial \xi^\sigma \rangle \tilde{g}^{\mu\nu} \mathcal{D}_\mu \tilde{g}_{\nu\sigma}) - \partial_\mu (\rho^{-1} \langle \partial \tilde{g}_{\nu\sigma} | \mathcal{M}^{-1} | \partial \xi^\sigma \rangle \tilde{g}^{\mu\nu}) \\ &\quad + \rho^{-1} \left(\langle \partial g_{\nu\sigma} | \mathcal{M}^{-1} \mathcal{D}_\mu | \partial \xi^\sigma \rangle \tilde{g}^{\mu\nu} - \frac{1}{2} \langle \partial \tilde{g}_{\mu\nu} | \mathcal{M}^{-1} | \partial \xi^\sigma \rangle \mathcal{D}_\sigma \tilde{g}^{\mu\nu} \right) \end{aligned} \quad (5.91)$$

which cancels the non-covariant variation of $\frac{\rho^{-1}}{4} \langle \partial \tilde{g}^{\mu\nu} | \mathcal{M}^{-1} | \partial \tilde{g}_{\mu\nu} \rangle$.

The variation of the potential gives the terms

$$\begin{aligned} \Delta_\xi V &= -\frac{1}{2} \rho^{-1} \langle \partial \xi^\mu | (T^\alpha \mathcal{J}_\mu^b + \mathcal{J}_\mu^b T^\alpha) \mathcal{M}^{-1} | \mathcal{J}_\alpha \rangle \\ &\quad + \frac{1}{2} \rho^{-1} \langle \partial \xi^\mu | (T^\alpha \mathcal{S}_1(\mathcal{J}_\mu^b) + \mathcal{S}_1(\mathcal{J}_\mu^b) T^\alpha) \mathcal{M}^{-1} | \mathcal{J}_\alpha^- \rangle + \dots \end{aligned} \quad (5.92)$$

which are compensated by the terms

$$\begin{aligned}\Delta_\xi \mathcal{L}_1 &= \frac{1}{2}\rho^{-1}\langle\partial\xi^\mu|(T^\alpha\mathcal{S}_1(\mathcal{J}_\mu^b) + \mathcal{M}^{-1}\mathcal{S}_1(\mathcal{J}_\mu^b)^\dagger\mathcal{M}T^\alpha)\mathcal{M}^{-1}|\mathcal{J}_\alpha^-\rangle + \dots \\ \Delta_\xi \mathcal{L}_2 &= -\frac{1}{2}\rho^{-1}\langle\partial\xi^\mu|(T^\alpha\mathcal{J}_\mu^b + \mathcal{J}_\mu^b T^\alpha)\mathcal{M}^{-1}|\mathcal{J}_\alpha\rangle \\ &\quad + \frac{1}{2}\rho^{-1}\langle\partial\xi^\mu|(\mathcal{S}_1(\mathcal{J}_\mu^b) - \mathcal{M}^{-1}\mathcal{S}_1(\mathcal{J}_\mu^b)^\dagger\mathcal{M})T^\alpha\mathcal{M}^{-1}|\mathcal{J}_\alpha^-\rangle + \dots\end{aligned}\tag{5.93}$$

in the variation of \mathcal{L}_1 and \mathcal{L}_2 , fixing in this way their relative coefficients in \mathcal{L}_{\min} .

We conclude therefore that the invariance of the pseudo-Lagrangian (5.44) under external diffeomorphisms, up to total derivatives and terms quadratic in the duality equations, fixes the relative coefficients of all the terms in (5.44). In this section we have proved that the duality equations (5.35) and (5.59) transforms into each others under external diffeomorphisms, and verified several consistency checks for the invariance of the Euler–Lagrange equations. The complete invariance under external diffeomorphisms will be confirmed in the next section, by showing that upon partially solving the section constrained one obtains the E_8 exceptional field theory [22]. Because the latter is invariant under three-dimensional external diffeomorphisms, we conclude that the E_9 exceptional field theory pseudo-Lagrangian (5.44) must indeed satisfy (5.83).

5.4 Embedding of E_8 exceptional field theory

The most convenient way to prove that the E_9 exceptional field theory does describe supergravity solution is to show that it reproduces E_8 exceptional field theory upon choosing the partial section solution

$$\langle\partial| = \langle 0|(\partial_\varphi + T_1^A\partial_A),\tag{5.94}$$

where ∂_A is the internal E_8 derivative that still satisfies itself the E_8 exceptional field theory section condition [11, 22]

$$\partial_A \otimes \partial_B + \partial_B \otimes \partial_A + f_{AE}{}^C f^{DE}{}_B \partial_C \otimes \partial_D = 0.\tag{5.95}$$

In (5.94) and below we use φ to denote the circle coordinate in the Kaluza–Klein ansatz, which is an internal coordinate (5.94) in E_9 exceptional field theory and an external S^1 coordinate in E_8 exceptional field theory with three-dimensional external space-time involving a local S^1 fibration, see (2.12).

Semi-flat current and Kaluza–Klein ansatz

We use the E_8 parabolic gauge

$$\mathcal{V} = (\rho^{-L_0} e^{-\sigma\mathring{V}}) U\tag{5.96}$$

with $\mathring{V} \in E_8$ and $U = \prod_{k=1}^{\infty} \exp(Y_{kA} T_{-k}^A)$ the negative mode component in E_9 . Then one defines the semi-flat current [47]

$$U^{-1} \tilde{\mathcal{J}}^b U = \mathcal{J}^b, \quad U^{-1} \tilde{\mathcal{J}} U = \mathcal{J}, \quad \chi = \tilde{\chi} - \omega_{-1}^\alpha(U) \mathcal{J}_\alpha = \tilde{\chi} + \omega_{-1}^\alpha(U^{-1}) \tilde{\mathcal{J}}_\alpha.\tag{5.97}$$

This gives rise to well-defined finite expressions, because the action of U^{-1} on a constrained bra is finite

$$\langle \partial | U^{-1} = \langle 0 | (\partial_\varphi - \eta^{AB} Y_{1A} \partial_B + T_1^A \partial_A) . \quad (5.98)$$

The duality equation (5.16) then reduces to

$$\tilde{\mathcal{J}} = \rho^{-1} \star (\mathcal{M}_0^{-1} S_1 (\tilde{\mathcal{J}})^\dagger \mathcal{M}_0 + \tilde{\chi}) \quad (5.99)$$

where we define $\mathcal{M}_0 = e^{-2\sigma} \rho^{-2L_0} M \in \mathbb{R}_+ \times \mathbb{R}_+ \times E_8$, the mode 0 component of \mathcal{M} (i.e. commuting with L_0), not to be confused with M that denotes the E_8 matrix (written M_{AB} in the adjoint representation of E_8). Since

$$U \cdot \mathcal{M}^{-1} d\mathcal{M} \cdot U^{-1} = \mathcal{M}_0^{-1} d\mathcal{M}_0 + dU \cdot U^{-1} + \mathcal{M}_0^{-1} (dU \cdot U^{-1})^\dagger \mathcal{M}_0 \quad (5.100)$$

we have similarly

$$\begin{aligned} \tilde{J}_\alpha T^\alpha &= \mathcal{M}_0^{-1} (d - \langle \partial_{\mathcal{M}_0} | A) \mathcal{M}_0 + (d - \langle \partial_U | A) U \cdot U^{-1} + \mathcal{M}_0^{-1} ((d - \langle \partial_U | A) U \cdot U^{-1})^\dagger \mathcal{M}_0 \\ &\quad - \eta_{\alpha\beta} \langle \partial_A | U^{-1} T^\beta U | A \rangle (T^\alpha + \mathcal{M}_0^{-1} T^{\alpha\dagger} \mathcal{M}_0) - \eta_{-1\alpha\beta} \text{Tr}[T^\beta U B U^{-1}] T^\alpha . \end{aligned} \quad (5.101)$$

The semi-flat current components are then finite by construction and we have in particular the components along L_0 and \mathbb{K}

$$\begin{aligned} \tilde{J}_0 &= -2\rho^{-1} (d\rho - \langle \partial_A + \partial_\rho | A \rangle \rho) = -2\rho^{-1} \mathcal{D}\rho , \\ \tilde{J}_\mathbb{K} &= -2(d\sigma - \langle \partial_\sigma | A \rangle) + 2\langle \partial_A | U^{-1} L_0 U | A \rangle + \tilde{g}^{\nu\sigma} (\partial_\nu \tilde{g}_{\mu\sigma} - \langle \partial \tilde{g}_{\mu\sigma} | A_\nu \rangle) dx^\mu \\ &= -2\mathcal{D}\sigma + \tilde{g}^{\nu\sigma} \mathcal{D}_\nu \tilde{g}_{\mu\sigma} dx^\mu . \end{aligned} \quad (5.102)$$

Because the pseudo-Lagrangian involves infinitely many dual fields, we will need to eliminate them in order to reproduce the E_8 exceptional field theory Lagrangian. For this we shall add terms to the pseudo-Lagrangian quadratic in the duality equations, which by construction do not modify the equations of motion since their variation vanishes upon using the duality equation. The duality equation along the loop algebra generators from (5.16) are

$$\tilde{J}_A^{-n} = \rho^{2n-1} H^B{}_A \star \tilde{J}_B^{n-1} , \quad (5.103)$$

where we introduced the E_8 matrix $H^B{}_A = \eta^{BC} M_{CA}$ that defines the conjugation $H^A{}_B T_0^B = M^{-1} (T_0^A)^\dagger M$ and

$$\tilde{J}_A^{b-n} = \rho^{2n} H^B{}_A \tilde{J}_B^{bn} . \quad (5.104)$$

We also need the explicit form of the ancillary and constrained field B in the E_8 solution (5.94) of the E_9 section constraint. As a constrained derivative decomposes as (5.94), one can write the general ansatz for B

$$B = |b\rangle \langle 0| + |b_A\rangle \langle 0| T_1^A , \quad (5.105)$$

with $|b_A\rangle$ constrained on its A index according to the E_8 section constraint (5.95). Writing out the various ways in which B occurs in the pseudo-Lagrangian, this gives the components

$$\begin{aligned}
\mathrm{Tr}[B] &= \langle 0|b\rangle + \langle 0|T_1^A|b_A\rangle \equiv b_{\mathbb{K}} \\
\mathrm{Tr}[L_0 B] &= \langle 0|T_1^A|b_A\rangle \equiv C^A{}_A \\
\mathrm{Tr}[L_{-k} B] &= 0 \quad \forall k \geq 1 \\
\mathrm{Tr}[T_{-1-k}^A B] &= 0 \quad \forall k \geq 1 \\
\eta_{AB} \mathrm{Tr}[T_{-1}^B B] &= \langle 0|b_A\rangle \equiv -B_A \\
\eta_{AB} \mathrm{Tr}[T_0^B B] &= f_{AB}{}^C \langle 0|T_1^B|b_C\rangle \equiv f_{AB}{}^C C^B{}_C \\
\eta_{AB} \mathrm{Tr}[T_k^B B] &= \eta_{AB} \langle 0|T_k^B|b\rangle + \eta_{AB} \langle 0|T_1^C T_k^B|b_C\rangle \equiv \mathcal{B}_A^{-1-k} \quad \forall k \geq 1
\end{aligned} \tag{5.106}$$

where B_A and $C^B{}_A$ are constrained on their A indices while $b_{\mathbb{K}}$ and \mathcal{B}_A^{-1-k} are arbitrary. We shall see that B_A is the vector field in the Kaluza–Klein ansatz of the constrained field in three dimensions, while $C^B{}_A$ is the component of a two-form potential in three dimensions. For the vector field $|A\rangle$ we use the similar ansatz

$$|A_\mu\rangle = (w_\mu + \eta_{AB} A_\mu^A T_{-1}^B + \eta_{AC} \eta_{BD} C_\mu^{A;B} T_{-1}^C T_{-1}^D + \dots) |0\rangle, \tag{5.107}$$

where w_μ is the Kaluza–Klein vector of the three-dimensional external metric (2.12). The vector A_μ^A in (5.107) is the Kaluza–Klein component of the three-dimensional vector (in the adjoint of E_8), and $C^{A;B}$, valued in $\mathbf{3875} \oplus \mathbf{1} \oplus \mathbf{248}$ with

$$\langle 0|T_1^A T_1^B|A\rangle = 2C^{(A;B)} + f_{CE}{}^B f^{AE}{}_D C^{C;D}, \tag{5.108}$$

comes from the two-forms in three dimensions [22].³¹ To describe the Kaluza–Klein ansatz for the vectors we note from [1] that $Y_1^A = A_\varphi^A$, while the constrained field

$$\langle \chi| = \rho^{-1} \langle \tilde{\chi}| + \omega_1^\alpha (U^{-1}) \langle \tilde{J}_\alpha| = \rho^{-1} \langle 0|(\tilde{\chi}_\varphi + \rho^{-1} B_{\varphi A}) + \omega_1^\alpha (U^{-1}) \langle \tilde{J}_\alpha|. \tag{5.109}$$

The Kaluza–Klein ansatz is defined such that the three-dimensional vector fields take the form

$$A^{3\mathrm{D}A} = A_\varphi^A (d\varphi + w) + A_\mu^A dx^\mu, \quad B_A^{3\mathrm{D}} = B_{\varphi A} (d\varphi + w) + B_{\mu A} dx^\mu + \rho \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\nu} \partial_A w_\nu dx^\mu. \tag{5.110}$$

The additional term in the Kaluza–Klein Ansatz of the constrained vector $B_A^{3\mathrm{D}}$ is consistent with the E_8 section constraint, and can be ascribed to the non-covariant transformation of the field under external diffeomorphisms [22]. The three-dimensional covariant derivatives

$$D^{3\mathrm{D}} = dx^\mu D_\mu + d\varphi D_\varphi = d^{3\mathrm{D}} - \mathcal{L}_{(A^{3\mathrm{D}A}, B_A^{3\mathrm{D}})}, \tag{5.111}$$

of the metric components are defined in form notations as

$$\begin{aligned}
D^{3\mathrm{D}} \sigma &= d^{3\mathrm{D}} \sigma - A^{3\mathrm{D}A} \partial_A \sigma - \partial_A A^{3\mathrm{D}A}, & D^{3\mathrm{D}} \rho &= d^{3\mathrm{D}} \rho - \partial_A (A^{3\mathrm{D}A} \rho), \\
D^{3\mathrm{D}} \tilde{g}_{\mu\nu} &= d^{3\mathrm{D}} \tilde{g}_{\mu\nu} - A^{3\mathrm{D}A} \partial_A \tilde{g}_{\mu\nu}, & D^{3\mathrm{D}} w_\mu &= d^{3\mathrm{D}} w_\mu - A^{3\mathrm{D}A} \partial_A w_\mu,
\end{aligned} \tag{5.112}$$

³¹The antisymmetric $\mathbf{248}$ component $C^{[A;B]} = -\frac{1}{60} f^{AB}{}_E f_{CD}{}^E C^{C;D}$ was not considered in [22] because it could be absorbed in a redefinition of constrained two-form $C^B{}_A$ in the E_8 theory.

and the E_9 covariant derivative decomposes as

$$\mathcal{D}_\mu = D_\mu - w_\mu D_\varphi \quad (5.113)$$

on σ , $\tilde{g}_{\mu\nu}$ and w_μ , whereas there is an additional term for ρ

$$\mathcal{D}_\mu \rho = D_\mu \rho - w_\mu D_\varphi \rho - \rho D_\varphi w_\mu . \quad (5.114)$$

Since the field B appears in the pseudo-Lagrangian through $\text{Tr}[T^\alpha U B U^{-1}]$, we shall also introduce the shorthand notation $\eta_{AB} \text{Tr}[T_k^B U B U^{-1}] = \tilde{\mathcal{B}}_A^{-1-k}$ for the loop components with $k \geq 1$. For example, $\tilde{\mathcal{B}}_A^{-2}$ expands as

$$\tilde{\mathcal{B}}_A^{-2} = \mathcal{B}_A^{-2} + f_{AB}{}^E f_{EC}{}^D A_\varphi^B C^C{}_D + \eta_{AB} A_\varphi^B b_K - \frac{1}{2} f_{AB}{}^E f_{EC}{}^D A_\varphi^B A_\varphi^C B_D - f_{AB}{}^C Y_2^B B_C . \quad (5.115)$$

We also have for the (mode 0) E_8 component

$$\eta_{AB} \text{Tr}[T_0^B U B U^{-1}] = f_{AB}{}^C \tilde{C}^B{}_C = f_{AB}{}^C (C^B{}_C - A_\varphi^B B_C) . \quad (5.116)$$

The duality equations (5.103) for $n \geq 2$ give

$$\tilde{\mathcal{B}}_A^{-n} = \tilde{J}_A^{b-n} - \rho^{2n-1} H^B{}_A \star \tilde{J}_B^{n-1} , \quad (5.117)$$

and since the fields $\tilde{\mathcal{B}}_A^{-n}$ are arbitrary, one can integrate them out in the pseudo-Lagrangian by setting them to this value.

Expansion of the pseudo-Lagrangian

We are now ready to analyse the pseudo-Lagrangian (5.44). We will first show that $\mathcal{L}_1 + \mathcal{L}_2$ can be written as a finite set of terms plus an infinite sum of terms quadratic in the duality equations (5.103). For this purpose one computes that \mathcal{L}_1 decomposes as

$$\begin{aligned} \mathcal{L}_1 &= \frac{\rho^{-1}}{2} \tilde{g}^{\mu\nu} \left(\text{Tr}[\tilde{B}_\mu \mathcal{M}_0^{-1} (\mathcal{S}_1(\tilde{J}_\nu^b) + \tilde{\chi}_\nu K)^\dagger \mathcal{M}_0] - \frac{1}{2} \eta_{\alpha\beta} \text{Tr}[T^\alpha \tilde{B}_\mu] \text{Tr}[\mathcal{M}_0^{-1} T^{\beta\dagger} \mathcal{M}_0 \tilde{B}_\nu] \right) \quad (5.118) \\ &= \frac{\rho}{2} M^{AB} \tilde{g}^{\mu\nu} \left(-\tilde{J}_{\mu A}^b B_{\nu B} + \rho^{-2} \tilde{J}_{\mu A}^{b-1} f_{BC}{}^D \tilde{C}_\nu^C{}_D + \sum_{n=2}^{\infty} \rho^{-2n} \tilde{J}_{\mu A}^{b-n} \tilde{\mathcal{B}}_{\nu B}^{-n} \right) + \frac{\rho^{-1}}{2} \tilde{g}^{\mu\nu} (\tilde{\chi}_\mu^b + \tilde{C}_{\mu A}^A) \tilde{b}_{\nu K} \\ &\quad - \frac{\rho}{4} M^{AB} \tilde{g}^{\mu\nu} \left(B_{\mu A} B_{\nu B} + \rho^{-2} f_{AC}{}^D \tilde{C}_\mu^C{}_D f_{BE}{}^F \tilde{C}_\nu^E{}_F + \sum_{n=2}^{\infty} \rho^{-2n} \tilde{\mathcal{B}}_{\mu A}^{-n} \tilde{\mathcal{B}}_{\nu B}^{-n} \right) \\ &= -\frac{\rho}{4} M^{AB} \tilde{g}^{\mu\nu} \sum_{n=2}^{\infty} \rho^{-2n} (\tilde{J}_{\mu A}^{-n} - \rho^{2n-1} H^C{}_A (\star \tilde{J})_{\mu C}^{n-1}) (\tilde{J}_{\nu B}^{-n} - \rho^{2n-1} H^D{}_B (\star \tilde{J})_{\nu D}^{n-1}) \\ &\quad + \frac{\rho^{-1}}{2} \tilde{g}^{\mu\nu} (\tilde{\chi}_\mu - \rho (\star \tilde{J})_{\mu K}) (b_{\nu K} - \rho (\star \tilde{J})_{\nu 0}) \\ &\quad - \frac{\rho^{-1}}{4} M^{AB} \tilde{g}^{\mu\nu} (\tilde{J}_{\mu A}^{-1} - \rho H^C{}_A (\star \tilde{J})_{\mu C}^0) (\tilde{J}_{\nu B}^{-1} - \rho H^D{}_B (\star \tilde{J})_{\nu D}^0) \\ &\quad - \frac{\rho}{4} \eta_{AB} \tilde{g}^{\mu\nu} (\eta^{AC} \tilde{J}_{\mu C}^0 + M^{AC} B_{\mu C}) (\eta^{BD} \tilde{J}_{\nu D}^0 + M^{BD} B_{\nu D}) + \rho \tilde{g}^{\mu\nu} \tilde{J}_{\mu K} \tilde{J}_{\nu 0} \\ &\quad + \frac{1}{2} \varepsilon^{\mu\nu} \eta^{AB} \sum_{n=0}^{\infty} \tilde{J}_{\mu A}^n \tilde{J}_{\nu B}^{-1-n} - \frac{1}{2} \varepsilon^{\mu\nu} \tilde{J}_{\mu K} \tilde{J}_{\nu -1} - \frac{1}{2} \varepsilon^{\mu\nu} \tilde{J}_{\mu 0} \tilde{\chi}_\nu , \end{aligned}$$

where we have used in particular that

$$\tilde{g}^{\mu\nu}(\star\tilde{J})_{\mu C}^0(\star\tilde{J})_{\nu D}^0 = -\tilde{g}^{\mu\nu}\tilde{J}_{\mu C}^0\tilde{J}_{\nu D}^0, \quad \eta^{AB}B_{\mu A}B_{\nu B} = 0. \quad (5.119)$$

For the topological term \mathcal{L}_2 , we note in the first place that a rigid E_9 transformation g (not to be confused with the metric $g_{\mu\nu}$) in the centrally extended loop group \hat{E}_8 , i.e. with $\rho(g) = 1$, gives

$$\mathcal{D}\chi \rightarrow \mathcal{D}\chi + \omega_{-1}^\alpha(g^{-1})(\mathcal{D}\mathcal{J})_\alpha. \quad (5.120)$$

One computes that

$$\begin{aligned} \mathcal{D}\chi &= d\tilde{\chi} - \langle \partial | (|A\rangle\tilde{\chi}) - \omega_{-1}^\alpha(U)(\mathcal{D}\mathcal{J})_\alpha + \eta^{AB} \sum_{n=1}^{\infty} n \tilde{J}_A^{-n} \tilde{J}_B^{n-1} - \text{Tr}[UBU^{-1}]\tilde{J}_K \\ &\quad + \langle \partial_A | U^{-1}T_{-1}^A U | A \rangle \tilde{J}_A^{-2} + \langle \partial_A | U^{-1}\mathcal{M}_0^{-1}(T_1^A)^\dagger \mathcal{M}_0 U | A \rangle \tilde{J}_A^0, \end{aligned} \quad (5.121)$$

where we used

$$d\omega_{-1}^\alpha(U) = R(U^{-1})^\alpha_\beta \delta\omega_{-1}^\beta(dU \cdot U^{-1}), \quad \omega_{-1}^\beta(\exp(X)) = \delta\omega_{-1}^\beta(X) + \mathcal{O}(X^2), \quad (5.122)$$

and the property that $\langle \partial_A | U^{-1}T_{-n}^A = 0$ for $n \geq 2$ in the E_8 partial section solution.

The first line of (5.121) gives together with the other terms of \mathcal{L}_2

$$\begin{aligned} &- 2\omega_{-1}^\alpha(U)(\mathcal{D}\mathcal{J})_\alpha + 2\eta^{AB} \sum_{n=1}^{\infty} n \tilde{J}_A^{-n} \tilde{J}_B^{n-1} - 2\text{Tr}[UBU^{-1}]\tilde{J}_K \\ &+ J_0\chi + J_{-1}J_K + \eta^{AB} \sum_n n J_A^n J_B^{1-n} + 2\langle \rho^2(\chi - \omega_{-1}^\alpha(M)\mathcal{J}_\alpha) | F \rangle \\ &- 2\langle \partial_F | (L_1 + \mathcal{M}^{-1}L_{-1}\mathcal{M}) | F \rangle - 2\text{Tr}[L_0G] + \rho^{-2}\eta_{1\alpha\beta}\langle \partial_A | T^\alpha | A' \rangle \langle \partial_A | \mathcal{M}^{-1}L_{-2}\mathcal{M}T^\beta | A \rangle \\ &= \tilde{J}_0\tilde{\chi} - \text{Tr}[\tilde{B}]\tilde{J}_K - \eta^{AB} \sum_{n=0}^{\infty} \tilde{J}_A^n \tilde{J}_B^{1-n} + 2\rho\langle \tilde{\chi} | F \rangle - 2\text{Tr}[L_0UGU^{-1}] \\ &- 2\langle \partial_F | U^{-1}(L_1 + \mathcal{M}_0^{-1}L_{-1}\mathcal{M}_0)U | F \rangle \end{aligned} \quad (5.123)$$

$$+ \rho^{-2}\eta_{1\alpha\beta}\langle \partial_A | U^{-1}T^\alpha U | A' \rangle \langle \partial_A | U^{-1}\mathcal{M}_0^{-1}L_{-2}\mathcal{M}_0T^\beta U | A \rangle, \quad (5.124)$$

where we used the rigid E_9 -invariance of \mathcal{L}_2 . Note that the last line and the second term of the next-to-last vanish when using the partial solution to the section constraint (5.94) associated to E_8 . We therefore have in total

$$\begin{aligned} \mathcal{L}_2 &= d\tilde{\chi} - \langle \partial | (U^{-1}|\tilde{A}\rangle\tilde{\chi}) + \frac{1}{2}\tilde{J}_0\tilde{\chi} - \frac{1}{2}\text{Tr}[B]\tilde{J}_K - \frac{1}{2}\eta^{AB} \sum_{n=0}^{\infty} \tilde{J}_A^n \tilde{J}_B^{1-n} + \rho\langle \tilde{\chi} | F \rangle - \text{Tr}[L_0UGU^{-1}] \\ &- \langle \partial_F | U^{-1}L_1U | F \rangle + \langle \partial_A | U^{-1}T_{-1}^A U | A \rangle \tilde{J}_A^{-2} + \langle \partial_A | U^{-1}\mathcal{M}_0^{-1}(T_1^A)^\dagger \mathcal{M}_0 U | A \rangle \tilde{J}_A^0. \end{aligned} \quad (5.125)$$

Combining this with \mathcal{L}_1 from (5.118), we have

$$\begin{aligned} \mathcal{L}_1 + \mathcal{L}_2 &= \mathcal{Z} - \frac{\rho}{4}\eta_{AB}\tilde{g}^{\mu\nu}(\eta^{AC}\tilde{J}_{\mu C}^0 + M^{AC}B_{\mu C})(\eta^{BD}\tilde{J}_{\nu D}^0 + M^{BD}B_{\nu D}) + \frac{1}{2}\rho\tilde{g}^{\mu\nu}\tilde{J}_{\mu K}\tilde{J}_{\nu 0} \\ &+ \rho\langle \tilde{\chi} | F \rangle - \text{Tr}[L_0UGU^{-1}] + \langle \partial_U | U^{-1}L_1U | F \rangle + \langle \partial_A | U^{-1}T_{-1}^A U | A \rangle \tilde{J}_A^{-2} \\ &+ \langle \partial_A | U^{-1}\mathcal{M}_0^{-1}(T_1^A)^\dagger \mathcal{M}_0 U | A \rangle \tilde{J}_A^0, \end{aligned} \quad (5.126)$$

where

$$\begin{aligned}
\mathcal{Z} = & -\frac{\rho}{4}M^{AB}\sum_{n=2}^{\infty}\rho^{-2n}(\tilde{J}_A^{-n}-\rho^{2n-1}H^C{}_A\star\tilde{J}_C^{n-1})\star(\tilde{J}_B^{-n}-\rho^{2n-1}H^D{}_B\star\tilde{J}_D^{n-1}) \\
& +\frac{1}{2}\rho^{-1}(\tilde{\chi}-\rho\star\tilde{J}_K)\star(b_K-\rho\star\tilde{J}_0)+d\tilde{\chi}-\langle\partial|(U^{-1}|\tilde{A})\tilde{\chi}+U^{-1}L_1U|F)\rangle \\
& -\frac{\rho^{-1}}{4}M^{AB}(\tilde{J}_A^{-1}-\rho H^C{}_A\star\tilde{J}_C^0)\star(\tilde{J}_B^{-1}-\rho H^D{}_B\star\tilde{J}_D^0)
\end{aligned} \tag{5.127}$$

does not contribute to the equations of motion, as it is composed out of bilinear terms in the components of the duality equations (5.103) and total derivatives.

The before to last term in the pseudo-Lagrangian (5.44) becomes

$$\frac{\rho^{-1}}{4}\langle\partial\tilde{g}^{\mu\nu}|\mathcal{M}^{-1}|\partial\tilde{g}_{\mu\nu}\rangle=\frac{\rho^{-1}}{4}e^{2\sigma}D_\varphi\tilde{g}^{\mu\nu}D_\varphi\tilde{g}_{\mu\nu}+\frac{1}{4}\rho e^{2\sigma}M^{AB}\partial_A\tilde{g}^{\mu\nu}\partial_B\tilde{g}_{\mu\nu}. \tag{5.128}$$

We recall from [1] that the E_9 potential decomposes as

$$\begin{aligned}
-V = & -eV^{3D}-\frac{1}{4}\rho e^{2\sigma}M^{AB}\partial_A\tilde{g}^{\mu\nu}\partial_B\tilde{g}_{\mu\nu}+\frac{\rho^3}{2}M^{AB}\tilde{g}^{\mu\nu}\partial_Aw_\mu\partial_Bw_\nu \\
& -\frac{1}{4}\rho^{-1}e^{2\sigma}\eta_{AB}j_\varphi^Aj_\varphi^B-2\rho^{-1}e^{2\sigma}D_\varphi\sigma D_\varphi\sigma \\
& -\frac{1}{2}\rho^{-3}e^{2\sigma}(\rho\tilde{\chi}_\varphi-A_\varphi^AB_{\varphi A}+\partial_A Y_2^A+\frac{1}{2}f_{AB}{}^CA_\varphi^A\partial_CA_\varphi^B)^2
\end{aligned} \tag{5.129}$$

where the derivative of the three-dimensional metric in the first line does not appear in the E_9 potential,³² but is part of the E_8 potential in [22], so must be subtracted. We also recall that

$$\langle\tilde{\chi}|=\langle 0|(\tilde{\chi}_\varphi+\rho^{-1}B_{\varphi A}T_1^A). \tag{5.130}$$

To analyse the pseudo-Lagrangian we recombine the various terms coming from the five components of (5.44) into the new five components

$$\mathcal{L}_{\min}=\mathcal{Z}+\mathcal{L}_{\text{kin}}^{3D}+\mathcal{L}_{\text{EH}}^{3D}+\mathcal{L}_{\text{CS}}^{3D}+\mathcal{L}_{\text{KK}}^{3D}-eV^{3D}, \tag{5.131}$$

with

$$\begin{aligned}
\mathcal{L}_{\text{kin}}^{3D} = & -\frac{\rho}{4}\eta_{AB}\tilde{g}^{\mu\nu}(\eta^{AC}\tilde{J}_{\mu C}^0+M^{AC}B_{\mu C})(\eta^{BD}\tilde{J}_{\nu D}^0+M^{BD}B_{\nu D})-\frac{1}{4}\rho^{-1}e^{2\sigma}\eta_{AB}j_\varphi^Aj_\varphi^B \\
& +\langle\partial_A|U^{-1}\mathcal{M}_0^{-1}(T_1^A)^\dagger\mathcal{M}_0U|A\rangle\tilde{J}_A^0+\frac{\rho^3}{2}M^{AB}\tilde{g}^{\mu\nu}\partial_Aw_\mu\partial_Bw_\nu,
\end{aligned} \tag{5.132a}$$

$$\begin{aligned}
\mathcal{L}_{\text{EH}}^{3D} = & \frac{1}{2}\rho\tilde{g}^{\mu\nu}\tilde{J}_{\mu K}\tilde{J}_{\nu 0}-2\rho^{-1}e^{2\sigma}D_\varphi\sigma D_\varphi\sigma-\frac{1}{4}\rho^3\sqrt{-g}g^{\mu\sigma}g^{\nu\rho}f_{\mu\nu}f_{\sigma\rho}-\rho\tilde{g}^{\mu\nu}\partial_Aw_\mu F_{\varphi\nu}^{3DA} \\
& +\frac{1}{4}\rho\varepsilon^{\mu\nu}\varepsilon^{\sigma\rho}\tilde{g}^{\kappa\lambda}\mathcal{D}_\mu\tilde{g}_{\sigma\kappa}\mathcal{D}_\nu\tilde{g}_{\rho\lambda}+\frac{\rho^{-1}}{4}e^{2\sigma}D_\varphi\tilde{g}^{\mu\nu}D_\varphi\tilde{g}_{\mu\nu},
\end{aligned} \tag{5.132b}$$

$$\begin{aligned}
\mathcal{L}_{\text{CS}}^{3D} = & \rho\langle\tilde{\chi}|F\rangle-\text{Tr}[L_0UGU^{-1}]+\langle\partial_U|U^{-1}L_1U|F\rangle+\langle\partial_A|U^{-1}T_{-1}^AU|A\rangle\tilde{J}_A^{-2} \\
& -(\rho\tilde{\chi}_\varphi-A_\varphi^AB_{\varphi A}+\partial_A Y_2^A+\frac{1}{2}f_{AB}{}^CA_\varphi^A\partial_CA_\varphi^B)f+\rho\tilde{g}^{\mu\nu}\partial_Aw_\mu F_{\varphi\nu}^{3DA},
\end{aligned} \tag{5.132c}$$

$$\begin{aligned}
\mathcal{L}_{\text{KK}}^{3D} = & -\frac{1}{2}\rho^{-3}e^{2\sigma}(\rho\tilde{\chi}_\varphi-A_\varphi^AB_{\varphi A}+\partial_A Y_2^A+\frac{1}{2}f_{AB}{}^CA_\varphi^A\partial_CA_\varphi^B)^2 \\
& +(\rho\tilde{\chi}_\varphi-A_\varphi^AB_{\varphi A}+\partial_A Y_2^A+\frac{1}{2}f_{AB}{}^CA_\varphi^A\partial_CA_\varphi^B)f_{01}+\frac{1}{4}\rho^3\sqrt{-g}g^{\mu\sigma}g^{\nu\rho}f_{\mu\nu}f_{\sigma\rho}.
\end{aligned} \tag{5.132d}$$

³²Note that [1, Eq. (5.34)] is written in the conformal gauge $ds_{3D}^2=e^{2\sigma}\eta_{\mu\nu}dx^\mu dx^\nu+\rho^2d\varphi^2$ and the internal derivatives of $\tilde{g}_{\mu\nu}$ and w_μ do not appear in the E_9 potential as defined in [1].

Here, we have added by hand the last lines in $\mathcal{L}_{\text{CS}}^{\text{3D}}$ and $\mathcal{L}_{\text{KK}}^{\text{3D}}$ and subtracted the same terms from the first line in $\mathcal{L}_{\text{EH}}^{\text{3D}}$. We have also introduced the Kaluza–Klein field strength

$$f = dw - A^A \partial_A w - w \partial_\varphi w = \langle 0|F \rangle = dx^\mu D_\mu w - w D_\varphi w . \quad (5.133)$$

Let us start with $\mathcal{L}_{\text{kin}}^{\text{3D}}$. One checks that the combination

$$\begin{aligned} \eta^{AC} \tilde{J}_{\mu C}^0 + M^{AC} B_{\mu C} &= j_\mu^A - \rho \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\nu} (\eta^{AB} + M^{AB}) \partial_B w_\nu \\ &= j_\mu^{\text{3DA}} - w_\mu j_\varphi^A - \rho \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\nu} (\eta^{AB} + M^{AB}) \partial_B w_\nu . \end{aligned} \quad (5.134)$$

where

$$j^{\text{3DA}} = j_\varphi^A (d\varphi + w) + j_\mu^A dx^\mu , \quad (5.135)$$

is the covariant E_8 current [22]. One computes moreover that

$$\langle \partial_A | U^{-1} \mathcal{M}_0^{-1} (T_1^A)^\dagger \mathcal{M}_0 U | A \rangle = \rho^2 M^{AB} \partial_B w \quad (5.136)$$

so that (5.132a) gives indeed the scalar field kinetic term of the E_8 exceptional field theory

$$\mathcal{L}_{\text{kin}}^{\text{3D}} = -\frac{\rho}{4} \eta_{AB} \tilde{g}^{\mu\nu} (j_\mu^{\text{3DA}} - w_\mu j_\varphi^A) (j_\nu^{\text{3DB}} - w_\nu j_\varphi^B) - \frac{1}{4} \rho^{-1} e^{2\sigma} \eta_{AB} j_\varphi^A j_\varphi^B . \quad (5.137)$$

For the Einstein–Hilbert term we first use that

$$[D_\varphi, \mathcal{D}_\mu] w_\nu = -F_{\varphi\mu}^{\text{3DA}} \partial_A w_\nu - D_\varphi w_\mu D_\varphi w_\nu , \quad (5.138)$$

and then compute that³³

$$\begin{aligned} \mathcal{L}_{\text{EH}}^{\text{3D}} &= \rho \left(-\frac{1}{4} \tilde{g}^{\mu\nu} \tilde{g}^{\sigma\rho} \tilde{g}^{\kappa\lambda} \mathcal{D}_\mu \tilde{g}_{\sigma\kappa} \mathcal{D}_\nu \tilde{g}_{\rho\lambda} + \frac{1}{2} \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma} \tilde{g}^{\kappa\lambda} \mathcal{D}_\mu \tilde{g}_{\sigma\kappa} \mathcal{D}_\nu \tilde{g}_{\rho\lambda} \right) \\ &\quad + \tilde{g}^{\mu\nu} \mathcal{D}_\mu \rho (2\mathcal{D}_\nu \sigma - \tilde{g}^{\sigma\rho} \mathcal{D}_\sigma \tilde{g}_{\nu\rho}) + \rho \tilde{g}^{\mu\nu} [D_\varphi, \mathcal{D}_\mu] w_\nu + \rho \tilde{g}^{\mu\nu} D_\varphi w_\mu D_\varphi w_\nu \\ &\quad - 2\rho^{-1} e^{2\sigma} D_\varphi \sigma D_\varphi \sigma - \frac{1}{4} \rho^3 \sqrt{-g} g^{\mu\sigma} g^{\nu\rho} f_{\mu\nu} f_{\sigma\rho} + \frac{\rho^{-1} e^{2\sigma}}{4} D_\varphi \tilde{g}^{\mu\nu} D_\varphi \tilde{g}_{\mu\nu} \\ &= D_\varphi (\rho \tilde{g}^{\mu\nu} \mathcal{D}_\mu w_\nu) - \mathcal{D}_\mu (\rho \tilde{g}^{\mu\nu} D_\varphi w_\nu) \\ &\quad + \rho \left(-\frac{1}{4} \tilde{g}^{\mu\nu} \tilde{g}^{\sigma\rho} \tilde{g}^{\kappa\lambda} \mathcal{D}_\mu \tilde{g}_{\sigma\kappa} \mathcal{D}_\nu \tilde{g}_{\rho\lambda} + \frac{1}{2} \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma} \tilde{g}^{\kappa\lambda} \mathcal{D}_\mu \tilde{g}_{\sigma\kappa} \mathcal{D}_\nu \tilde{g}_{\rho\lambda} \right) - \frac{1}{4} \rho^3 \sqrt{-g} g^{\mu\sigma} g^{\nu\rho} f_{\mu\nu} f_{\sigma\rho} \\ &\quad + (D_\mu - w_\mu D_\varphi) \rho (2\tilde{g}^{\mu\nu} \mathcal{D}_\nu \sigma + \mathcal{D}_\nu \tilde{g}^{\mu\nu}) - 2\rho \tilde{g}^{\mu\nu} D_\varphi w_\mu \mathcal{D}_\nu \sigma - \rho \mathcal{D}_\mu w_\nu D_\varphi \tilde{g}^{\mu\nu} \\ &\quad + \tilde{g}^{\mu\nu} (D_\varphi w_\mu (D_\nu - w_\nu D_\varphi) \rho - \mathcal{D}_\mu w_\nu D_\varphi \rho) - 2\rho^{-1} e^{2\sigma} D_\varphi \sigma D_\varphi \sigma + \frac{\rho^{-1} e^{2\sigma}}{4} D_\varphi \tilde{g}^{\mu\nu} D_\varphi \tilde{g}_{\mu\nu} \\ &= \sqrt{-g^{\text{3D}}} \widehat{R}^{\text{3D}} - \mathcal{D}_\mu \left(\rho \tilde{g}^{\sigma\rho} \tilde{g}^{\mu\nu} (\mathcal{D}_\sigma \tilde{g}_{\rho\nu} - \mathcal{D}_\nu \tilde{g}_{\sigma\rho}) \right) , \end{aligned} \quad (5.139)$$

³³We used here the following formula for the Einstein–Hilbert term in D dimensions

$$\begin{aligned} \sqrt{-g} \widehat{R} &= \sqrt{-g} \left(-\frac{1}{4} g^{\mu\nu} g^{\sigma\rho} g^{\kappa\lambda} D_\mu g_{\sigma\kappa} D_\nu g_{\rho\lambda} + \frac{1}{2} g^{\mu\sigma} g^{\nu\rho} g^{\kappa\lambda} D_\mu g_{\rho\kappa} D_\nu g_{\sigma\lambda} + \frac{1}{4} g^{\mu\nu} g^{\sigma\rho} g^{\kappa\lambda} D_\mu g_{\sigma\rho} D_\nu g_{\kappa\lambda} \right. \\ &\quad \left. - \frac{1}{2} g^{\mu\sigma} g^{\nu\rho} g^{\kappa\lambda} D_\mu g_{\kappa\lambda} D_\nu g_{\sigma\rho} \right) + D_\mu \left(\sqrt{-g} g^{\sigma\rho} g^{\mu\nu} (D_\sigma g_{\rho\nu} - D_\nu g_{\sigma\rho}) \right) \end{aligned}$$

with $D_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - A_\sigma^A \partial_A g_{\mu\nu} - \frac{2}{D-2} g_{\mu\nu} \partial_A A_\sigma^A$.

where on all individual terms the E_9 covariant derivative \mathcal{D}_μ gives $D_\mu - w_\mu D_\varphi$, except when it acts as a total covariant derivative, in which case it is a true total derivative.³⁴

The identification of the Chern–Simons term requires more work and we shall first simplify some of the expressions appearing in (5.132c). Using (5.98) and $\langle 0|L_1 = 0$ one gets

$$\begin{aligned}\langle \partial|U^{-1}L_1U &= \langle 0|T_2^A U \partial_A = \langle 0|(T_2^A + f_{BC}{}^A Y_1^B T_1^C + 2Y_2^A) \partial_A, \\ \langle \partial|U^{-1}T_{-1}^A U &= \eta^{AB} \langle 0| \partial_B,\end{aligned}\tag{5.140}$$

so that

$$\langle \partial_U|U^{-1}L_1U|F\rangle + \langle \partial_A|U^{-1}T_{-1}^A U|A\rangle \tilde{J}_A^{-2} = 2\partial_A Y_2^A f + f_{BC}{}^A \partial_A A_\varphi^B \langle 0|T_1^C|F\rangle + \eta^{AB} \partial_A w \tilde{J}_B^{-2}.\tag{5.141}$$

One computes moreover

$$\begin{aligned}\langle 0|T_1^A|F\rangle &= dA^A - \frac{1}{2}A^B \partial_B A^A - \frac{1}{2}f^{EC}{}_D f_{EB}{}^A \partial_C A^D A^B - \frac{1}{2}\partial_B A^B A^A \\ &\quad - w(\partial_\varphi A^A + \partial_B C^{(A;B)} + \frac{1}{2}f_{CE}{}^A f^{BE}{}_D \partial_B C^{C;D}) + \partial_B w(C^{(A;B)} + \frac{1}{2}f_{CE}{}^B f^{AE}{}_D C^{C;D}), \\ \eta_{AB} \text{Tr}[T_{-1}^B G] &= -dB_A + A^B \partial_B B_A + \partial_B A^B B_A - B_B \partial_A A^B + \frac{1}{2}f_{CD}{}^B A^C \partial_A \partial_B A^D \\ &\quad + w(\partial_\varphi B_A - \frac{1}{2}f_{CD}{}^B \partial_A \partial_B C^{C;D}) + (b_\kappa - C^B{}_B) \partial_A w + \partial_B w C^B{}_A + \frac{1}{2}f_{CD}{}^B C^{C;D} \partial_A \partial_B w, \\ \text{Tr}[L_0 G] &= dC^A{}_A - \partial_A(A^A C^B{}_B) - \partial_\varphi(w C^A{}_A) - \partial_\varphi A^A B_A - \frac{1}{2}f_{BC}{}^A A^B \partial_\varphi \partial_A A^C \\ &\quad + \eta^{AB} \mathcal{B}_A^{-2} \partial_B w + \frac{1}{2}w f_{CD}{}^A \partial_\varphi \partial_A C^{C;D} + \frac{1}{2}\partial_\varphi \partial_A w f_{CD}{}^A C^{C;D},\end{aligned}\tag{5.142}$$

where we recognise the components of the three-dimensional field strengths [22]

$$\begin{aligned}F^{3DA} &= dA^{3DA} - \frac{1}{2}A^{3DB} \partial_B A^{3DA} - \frac{1}{2}f^{EC}{}_D f_{EB}{}^A \partial_C A^{3DD} A^{3DB} - \frac{1}{2}\partial_B A^{3DB} A^{3DA}, \\ G_A^{3D} &= dB_A^{3D} - A^{3DB} \partial_B B_A^{3D} - \partial_B A^{3DB} B_A^{3D} + B_B^{3D} \partial_A A^{3DB} - \frac{1}{2}f_{CD}{}^B A^{3DC} \partial_A \partial_B A^{3DD}.\end{aligned}\tag{5.143}$$

In particular, we have

$$\begin{aligned}\langle 0|U^{-1}T_1^A U|F\rangle &= F^A - w \partial_B (C^{(A;B)} + \frac{1}{2}f_{CE}{}^A f^{BE}{}_D C^{C;D}) \\ &\quad + \partial_B w (C^{(A;B)} + A_\varphi^A A^B) + \frac{1}{2}f_{CE}{}^B f^{AE}{}_D (C^{C;D} + A_\varphi^A A^B),\end{aligned}\tag{5.144}$$

where F^A is the Kaluza–Klein component of the three-dimensional field strength

$$F^{3DA} = \eta^{AB} \tilde{J}_B^{-1} (d\varphi + w) + F^A.\tag{5.145}$$

We also need the expression

$$\begin{aligned}&\eta^{AB} \partial_A w \tilde{J}_B^{-2} \\ &= \eta^{AB} \partial_A w \tilde{J}_B^{-2} - \eta^{AB} \partial_A w \mathcal{B}_B^{-2} + A_\varphi^A ((b_\kappa - C^B{}_B) \partial_A w + \partial_B w C^B{}_A) + A_\varphi^A A_\varphi^B B_A \partial_B w\end{aligned}\tag{5.146}$$

³⁴For total derivatives $\mathcal{D}_\mu(\rho X^\mu) = \partial_\mu(\rho X^\mu) - \partial_\varphi(w_\mu \rho X^\mu) - \partial_A(A_\mu^A \rho X^\mu)$ using (5.114).

where the bare current expands as

$$\begin{aligned}
\eta^{AB} \tilde{J}_B^{b-2} = & (d - w\partial_\varphi - A^B \partial_B - 2\partial_B A^B - 2\partial_\varphi w) Y_2^A - f_{CE}^A f^{BE}{}_D (Y_2^C \partial_B A^D + A_\varphi^C Y_2^D \partial_B w) \\
& + \frac{1}{2} f_{BC}^A A_\varphi^B (d - w\partial_\varphi - A^D \partial_D) A_\varphi^C - \frac{1}{2} f^{FG}{}_D f_{CF}^A f_{EG}^B A_\varphi^C A_\varphi^D \partial_B A^E \\
& - f_{BC}^A A_\varphi^B \partial_\varphi A^C - \frac{1}{6} f^{FG}{}_C f_{DF}^A f_{EG}^B A_\varphi^C A_\varphi^D A_\varphi^E \partial_B w \\
& - f_{BC}^A \partial_\varphi C^{B;C} - 2f_{CD}^A A_\varphi^C \partial_B C^{(D;B)} - f^{FG}{}_D f_{CF}^A f_{EG}^B A_\varphi^C \partial_B C^{D;E} \\
& - \langle 0|T_1^B T_2^A \partial_B|A \rangle + f^{AB}{}_C Y_3^C \partial_B w .
\end{aligned} \tag{5.147}$$

The last term with the degree 3 fields simplify to a total derivative when contracted with $\partial_A w$ as

$$- \partial_A w \langle 0|T_1^B T_2^A \partial_B|A \rangle = -\partial_B (\partial_A w \langle 0|T_1^B T_2^A|A \rangle) \tag{5.148}$$

using that the L_0 degree 3 component of the basic representation does not contain a generic symmetric representation.

We can now recombine all these terms in (5.132c). We first observe that the dependence in \mathcal{B}_A^{-2} and b_κ cancels between $-\text{Tr}[L_0 U G U^{-1}]$ and $\langle \partial_A|U^{-1} T_{-1}^A U|A \rangle \tilde{J}_A^{-2}$. With more work one obtains that the dependence in the three-dimensional two-form components also combine to a total derivative. We compute (where A^{3DA} , F^{3DA} and B_A^{3D} are the forms in two dimensions)

$$\begin{aligned}
& \mathcal{L}_{\text{CS}}^{3D} \tag{5.149} \\
= & (B_{\varphi A} + f_{BC}^A \partial_A A_\varphi^B) F^{3DC} + \partial_\varphi A^{3DA} B_A^{3D} + \frac{1}{2} f_{BC}^A A^{3DB} \partial_\varphi \partial_A A^{3DC} \\
& + A_\varphi^A \left(d B_A^{3D} - A^{3DB} \partial_B B_A^{3D} - \partial_B A^{3DB} B_A^{3D} + B_B^{3D} \partial_A A^{3DB} - \frac{1}{2} f_{CD}^B A^{3DC} \partial_A \partial_B A^{3DD} \right) \\
& - \frac{1}{2} f^{FG} (E f_{C|F}^A f_{D)G}^B \partial_A \left(\frac{1}{3} A_\varphi^C A_\varphi^D A_\varphi^E w \partial_B w + w A_\varphi^C \partial_B (A_\varphi^D A^E) \right) \\
& + \frac{1}{2} f_{CD}^A \left(-\partial_{[A} (w A_\varphi^C \partial_B] A_\varphi^D A^B) + \partial_{(A} (A_\varphi^B A_\varphi^C \partial_B) (w A^D) + \partial_B (A_\varphi^B \partial_A A_\varphi^C w A^D) \right) \\
& - d C^A{}_A + \partial_A (A^A C^B{}_B) + \partial_\varphi (w C^A{}_A) - \frac{1}{2} \partial_A (w f_{CD}^A \partial_\varphi C^{C;D}) - \frac{1}{2} \partial_\varphi (\partial_A w f_{CD}^A C^{C;D}) \\
& - \frac{1}{2} (f^{FG}{}_D f_{CF}^{[A} f_{EG}^{B]}) + 2f_{CD}^{[A} \delta_E^{B]} \partial_A (2w A_\varphi^C \partial_B C^{(D;E)} + w \partial_B A_\varphi^C C^{(D;E)}) \\
& + \frac{1}{2} f_{CD}^A \partial_A (A_\varphi^B) \partial_B (w C^{C;D}) + (2-1) \partial_A Y_2^A (dw - A^A \partial_A w - w \partial_\varphi w) \\
& + \partial_A w \left((d - w\partial_\varphi - A^B \partial_B - 2\partial_B A^B - 2\partial_\varphi w) Y_2^A - f_{CE}^A f^{BE}{}_D (Y_2^C \partial_B A^D + A_\varphi^C Y_2^D \partial_B w) \right) \\
= & B_{\varphi A} F^{3DA} + B_A^{3D} (d A_\varphi^A - \partial_\varphi A^{3DA} + A_\varphi^B \partial_B A^{3DA} - A^{3DB} \partial_B A_\varphi^A) \\
& - \frac{1}{2} f_{BC}^A (d A^{3DB} \partial_A A_\varphi^C - d A_\varphi^B \partial_A A^{3DC} + \partial_\varphi A^{3DB} \partial_A A^{3DC}) \\
& - \frac{1}{3} f_{CD}^A (A_\varphi^B A^{3DC} \partial_A \partial_B A^{3DD} - A^{3DB} A_\varphi^C \partial_A \partial_B A^{3DD} + A^{3DB} A^{3DC} \partial_A \partial_B A_\varphi^D) \\
& - \frac{1}{6} f^{FG}{}_C f_{FD}^A f_{GE}^B (A_\varphi^C \partial_A A^{3DD} \partial_B A^{3DE} - 2A^{3DC} \partial_A A_\varphi^D \partial_B A^{3DE}) + \partial_A(\dots) + d(\dots) ,
\end{aligned}$$

where we have used repeatedly the section constrained and identity [22, Eq. (A.1)] to get for example

$$\begin{aligned}
& \frac{1}{2} f^{FG}{}_C f_{FD}{}^A f_{GE}{}^B A^C \partial_A A_\varphi^D \partial_B A^E \\
&= -\frac{1}{6} f^{FG}{}_C f_{FD}{}^A f_{GE}{}^B (A_\varphi^C \partial_A A^D \partial_B A^E - 2A^C \partial_A A_\varphi^D \partial_B A^E) \\
& \quad + \frac{1}{6} f_{CD}{}^A (A_\varphi^B A^C \partial_A \partial_B A^D + 2A^B A_\varphi^C \partial_A \partial_B A^D + A^B A^C \partial_A \partial_B A_\varphi^D) \\
& \quad + \frac{1}{6} f_{CD}{}^{(A} \partial_A (A^C \partial_B (A^D A^B)) - \partial_B A^C A_\varphi^D A^B)
\end{aligned} \tag{5.150}$$

and to cancel the terms symmetric the derivatives indices involving $C^{(D;E)}$. We recognise therefore $\mathcal{L}_{\text{CS}}^{3\text{D}}$ as the Chern–Simons term obtained in [22] up to total derivative terms.³⁵

Finally, we compute for the Kaluza–Klein term (5.132d) that

$$\mathcal{L}_{\text{KK}}^{3\text{D}} = -\frac{1}{2} \rho^{-3} e^{2\sigma} (\rho \tilde{\chi}_\varphi - A_\varphi^A B_{\varphi A} + \partial_A Y_2^A + \frac{1}{2} f_{AB}{}^C A_\varphi^A \partial_C A_\varphi^B - \rho^3 e^{-2\sigma} f_{01})^2. \tag{5.151}$$

This last term does not appear in [22], but one can integrate out the auxiliary field $\tilde{\chi}_\varphi$ to eliminate it. By construction this term is quadratic in the contraction with $\langle 0|$ of the duality equation (5.59)

$$U|\mathcal{F}_{\mu\nu}\rangle = -\rho^{-1} \varepsilon_{\mu\nu} T^\alpha \mathcal{M}_0^{-1} U^{-1\dagger} |\tilde{\mathcal{J}}_\alpha^-\rangle. \tag{5.152}$$

We have therefore identified

$$\mathcal{L}^{3\text{D}} = \mathcal{L}_{\text{kin}}^{3\text{D}} + \mathcal{L}_{\text{EH}}^{3\text{D}} + \mathcal{L}_{\text{CS}}^{3\text{D}} - eV^{3\text{D}} \tag{5.153}$$

as the E_8 exceptional field theory Lagrangian of [22] up to total derivative terms. The E_9 exceptional field theory pseudo-Lagrangian (5.44) on the partial section (5.94) decomposes according to (5.131) as the E_8 exceptional field theory Lagrangian plus an infinite sum of terms quadratic in the components of the duality equations (5.35) and (5.59).

Additional duality equations

The E_8 decomposition of the duality equation (5.35) does not only give the E_8 Euler–Lagrange equations, but also an infinite set of duality equations. The duality equations (5.103) for $n \geq 2$ determine the auxiliary fields $\tilde{\mathcal{B}}_A^{-n}$, and do not give any non-trivial new equation. The $n = 1$ component of (5.103) gives however the non-trivial equation

$$\mathcal{F}_{\mu\varphi}^{3\text{D}A} = \rho \tilde{g}_{\mu\sigma} \varepsilon^{\sigma\nu} \eta^{AB} (j_{\nu B}^{3\text{D}} - B_{\nu B}^{3\text{D}} - w_\nu (j_{\varphi B} - B_{\varphi B})) \tag{5.154}$$

with

$$\begin{aligned}
\mathcal{F}_{\mu\varphi}^{3\text{D}A} &= F_{\mu\varphi}^{3\text{D}A} - \partial_B \left(2C_\mu^{(A;B)} + f_{CE}{}^A f^{BE}{}_D C_\mu^{C;D} + A_\mu^A A_\varphi^B + \frac{1}{2} f_{CE}{}^{(A} f^{B)E}{}_D A_\mu^C A_\varphi^D \right) \\
& \quad + f^{AB}{}_C (C_{\mu B}^C - \frac{1}{2} w_\mu f_{EF}{}^C A_\varphi^E \partial_B A_\varphi^F) \\
&= F_{\mu\varphi}^{3\text{D}A} + \partial_B \left(2C_{\mu\varphi}^{3\text{D}(A;B)} + f_{CE}{}^A f^{BE}{}_D C_{\mu\varphi}^{3\text{D}C;D} \right) + f^{AB}{}_C C_{\mu\varphi}^{3\text{D}C}{}_B.
\end{aligned} \tag{5.155}$$

³⁵And up to a sign misprint in the third term in [22].

One recognises therefore a component of the three-dimensional duality equation

$$\mathcal{F}^{3D A} + \eta^{AB} \star_{3D} (j_B^{3D} - B_B^{3D}) = 0 . \quad (5.156)$$

This equation is consistent with the three-dimensional equation Euler–Lagrange equation

$$\delta B_A (\mathcal{F}^{3D A} + \eta^{AB} \star_{3D} j_B^{3D}) = 0 . \quad (5.157)$$

since the last term in B_B vanishes using the section constraint.

Note that in the equivalent equation in the Virasoro-extended formulation (4.23), the component J_A^{-1} of the current \mathfrak{J} includes an additional term in $\eta_{AB} \text{Tr}[T_{-1}^B B^{(2)}]$ with respect to J_A^{-1} in (5.34). The component of $B^{(2)}$ can be used to eliminate the term in B_B^{3D} in (5.156). This term can also be reabsorbed in a redefinition of $C_{\mu\varphi}^{3DC} B$ in the minimal formulation of the theory, but only if one breaks explicitly the covariance of the Ansatz to e.g. $GL(1) \times E_7$. So we may define instead of (5.156), the three-dimensional duality equation

$$\mathcal{F}^{3D A} + \eta^{AB} \star_{3D} j_B^{3D} = 0 . \quad (5.158)$$

Similarly, one can compute the component of the duality equation (5.59) coming from $\langle 0|T_1^A U|\mathcal{E}\rangle = 0$ to get

$$\mathcal{F}_{\mu\nu}^{3D A} + 2w_{[\mu} \mathcal{F}_{\nu]\varphi} = -\rho^{-1} e^{2\sigma} \eta^{AB} (j_{\varphi B} - B_{\varphi B}) \quad (5.159)$$

with

$$\begin{aligned} \mathcal{F}_{\mu\nu}^{3D A} &= F_{\mu\nu}^{3D A} + \partial_B \left(2C_{\mu\nu}^{(A;B)} + f_{CE}{}^A f^{BE}{}_D C_{\mu\nu}^{C;D} - w_{[\mu} (2C_{\nu]}^{(A;B)} + f_{CE}{}^A f^{BE}{}_D C_{\nu]}^{C;D}) \right. \\ &\quad \left. + w_{[\mu} (A_{\nu]}^A A_{\varphi}^B) + \frac{1}{2} f_{CE}{}^A f^{BE}{}_D A_{\nu]}^C A_{\varphi}^D \right) \\ &\quad + f^{AB}{}_C (C_{\mu\nu B}^C + \frac{1}{4} f_{EF}{}^C A_{\varphi}^E A_{[\mu}^F \partial_B w_{\nu]}) + 2w_{[\mu} C_{\nu] B}^C + \rho^{-1} e^{2\sigma} \varepsilon_{\mu\nu} \partial_B A_{\varphi}^C \\ &= F_{\mu\nu}^{3D A} + \partial_B \left(2C_{\mu\nu}^{3D(A;B)} + f_{CE}{}^A f^{BE}{}_D C_{\mu\nu}^{3D C;D} \right) + f^{AB}{}_C C_{\mu\nu}^{3D C}{}_B , \end{aligned} \quad (5.160)$$

where $C_{\mu\nu}^{A;B}$ and $C_{\mu\nu A}^B$ come from the E_9 two-forms. Since external and internal diffeomorphisms in E_9 both contain the E_8 external diffeomorphisms, it is to be expected that these two equations transform into each other, so that we need to consider (5.59). This is in fact the only non-trivial component of equation (5.70), because all other components can be solved trivially by fixing an unconstrained two-form.

The duality equation (5.59) includes one more duality equation that can be interpreted as the three-dimensional three-form field strength equation. It gives

$$\langle 0|T_1^A T_1^B U|\mathcal{E}_{\mu\nu}\rangle = (\delta_C^A \delta_D^B + \frac{1}{2} f_{CE}{}^A f^{BE}{}_D) (-\mathcal{G}_{\varphi\mu\nu}^{C;D} + 2\rho \varepsilon_{\mu\nu} M^{CF} J_{F;D}) , \quad (5.161)$$

for a three-form field strength $\mathcal{G}^{A;B}$ in E_8 exceptional field theory. The antisymmetric component $\langle 0|T_2^A|\mathcal{F}_{\mu\nu}\rangle$ includes an unconstrained two-form $C_{\varphi\mu\nu}^A$, so this equation can be solved by fixing $C_{\varphi\mu\nu}^A$. The only non-trivial component of (5.161) is therefore in the $\mathbf{3875} \oplus \mathbf{1}$ of E_8 , as expected from the tensor hierarchy. The field strength $\langle 0|T_1^A T_1^B T_1^C|\mathcal{F}_{\mu\nu}\rangle$ also includes an unconstrained

two-form in the same E_8 representation, and similarly does $\langle 0|T_1^{A_1}T_1^{A_2}\dots T_1^{A_n}|\mathcal{F}_{\mu\nu}\rangle$ for any number n of generators T_1^A greater than three, so all the other components of (5.59) can be solved by fixing the corresponding two-form components.

We conclude therefore that the duality equations (5.35) and (5.59) on section give all the duality equations expected from the E_8 tensor hierarchy, and all other components are tautological equations that can be solved by fixing unconstrained auxiliary fields in function of the other fields.

6 Conclusion

In this paper, we have constructed the complete dynamics of E_9 exceptional field theory in two different formulations.

The Virasoro-extended formulation in Section 4 is based on the $\widehat{E}_8 \rtimes \text{Vir}^-$ symmetry, where the Virasoro generators L_m for $m \leq 0$ originate in the extended linear system for $D = 2$ supergravity [37, 38] discussed in Section 2.3. Indeed, this formulation of E_9 exceptional field theory allows us to naturally reproduce $D = 2$ supergravity and its linear system by setting to zero the internal derivative $\langle \partial |$ as well as all constrained fields $B^{(k)}$ and $\langle \chi_1^\gamma |$. In this formulation, a proper gauge connection for generalised diffeomorphisms can be defined, allowing for instance the construction of covariant external currents for the scalar fields, and the vector fields and their field strengths transform in a manner familiar from lower-rank ExFTs. This also makes the construction of the topological term more intuitive. The field content of the Virasoro-extended formulation includes the field $\tilde{\rho} \sim \phi_1$ dual to the $D = 2$ dilaton ρ . This field plays an important role when using Weyl coordinates for studying $D = 2$ solutions, and the gauging of its shift symmetry is central to the construction of $D = 2$ gauged supergravities in [39]. The formulation that we have presented is only phrased in the conformal gauge $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ of the $D = 2$ external metric. This gauge can be imposed for any higher-dimensional solution that one may want to describe in ExFT, but it would nonetheless be interesting to generalise the Virasoro-extended formulation to arbitrary external metrics. Even keeping conformal gauge, one should be able to define conformal external diffeomorphisms and we expect that they will fix all relative coefficients in the pseudo-Lagrangian. Furthermore, we have only presented a manifestly $\widehat{E}_8 \rtimes \text{Vir}^-$ invariant formulation of the topological term, while the expression for the scalar potential we have constructed in previous work [1] is based on internal currents gauge-fixed to $\phi_n = 0$ for $n \geq 2$. Because these fields are pure gauge, we do not lose any information on the dynamics by this gauge choice, but it would also be interesting to construct the manifest Vir^- extension of the scalar potential. Since we know the transformation properties of all fields under $\widehat{\mathfrak{e}}_8 \oplus \mathfrak{vir}^-$ extended generalised diffeomorphisms, one can achieve this goal by appropriately acting with local L_{-n} transformations on the known expression for the potential. We have derived all the equations of motion implied by our pseudo-Lagrangian. One may verify that these are compatible with the E_8 ExFT equations and we have shown this explicitly for the scalar equation in Appendix C.4.

The second, minimal, formulation of E_9 exceptional field theory was given in Section 5. It involves finitely many fields (in infinite-dimensional representations of E_9) and can be obtained

from the Virasoro-extended formulation by gauge-fixing all the negative Virasoro fields to zero, including the field $\tilde{\rho}$ associated to L_{-1} . The pseudo-Lagrangian takes a slightly more conventional form, as the sum of a kinetic, a topological and a potential term and is defined for an arbitrary unimodular metric $\tilde{g}_{\mu\nu}$. We have defined the external diffeomorphisms and have shown that they leave the duality equations (5.35) and (5.59) invariant, and fix all free coefficients in the pseudo-Lagrangian in a way familiar from E_n ExFT for $n \leq 8$. We have verified that E_8 exceptional field theory, including the Einstein equation, follows correctly from the minimal formulation. In particular one obtains the E_8 exceptional field theory Lagrangian from the minimal pseudo-Lagrangian up to terms quadratic in the duality equations. All the expressions are finite in the minimal formulation, and no infinite series regularisation is involved. It is noteworthy that the field content and the duality equations in the minimal formulation are consistent with E_{11} exceptional field theory [48, 47], and could in principle be derived from it upon branching E_{11} under its $SL(2) \times E_9$ subgroup.³⁶ Even though the field $\tilde{\rho}$ is gauged-fixed to zero in the minimal E_9 ExFT of Section 5, it seems plausible that an intermediate formulation including $\tilde{\rho}$ exists. First investigations show that one can write the duality equation at $\tilde{\rho} \neq 0$ provided one introduces one additional constrained vector field $B^{(2)}$ in such a ‘next-to-minimal’ formulation. The field $\tilde{\rho}$ will be crucial for supersymmetrising E_9 ExFT, see [28, 54, 55] for work on supersymmetry in ungauged $D = 2$ supergravity and fermionic representations of $K(\mathfrak{e}_9)$.

Both the minimal and the Virasoro-extended formulation are based on pseudo-Lagrangians that are supplemented by a set of first-order self-duality equations for the scalar fields. In order to render the model accessible to canonical tools, it may be useful to further extend this framework into a genuine Lagrangian formulation presumably upon sacrificing manifest two-dimensional Lorentz invariance, along the lines of [56], see also [57, 58] for earlier work on chiral scalars.

It should be stressed that the formalisms developed in this paper can be directly applied to construct extended field theories based on duality groups \hat{G} that are the affine extensions of any finite-dimensional, simple Lie group G . This is simply achieved by exchanging E_8 in this paper by G , since we have not used detailed information about the structure constants $f^{AB}{}_C$ in (2.1) anywhere. The fact that the structure of generalised diffeomorphisms and the section constraint is the same for any simple Lie group G was proved in [17, Sec. 6]. For appropriate choices of G , our results define ‘‘half-maximal’’ exceptional field theories along the lines of [59, 46], as well as extended field theories based on any symmetric space in three space-time dimensions that lifts at least to four dimensions, irrespective of supersymmetry [60, 61]. For instance, taking $G = SL(2)$ we expect that the matching to E_8 ExFT discussed in Section 5 would instead provide a matching to the $SL(2)$ -covariant theory of [16]. Our results then provide a description of $D = 4$ general relativity formally covariant under the Geroch group.

There are a number of potential applications of E_9 exceptional field theory. As with other ExFTs, one use of E_9 exceptional field theory is to study in more detail gauged supergravity

³⁶The E_{11} ExFT duality equations were given in [48, 47], and a pseudo-Lagrangian, sharing some features with the E_9 minimal formulation, will be presented in [49]. A concrete conjecture for E_{11} and an associated extended space-time in the context of maximal supergravity was first made in [50, 51] and further developed in [52, 53].

in $D = 2$. Gaugings of E_9 have been investigated in [39], but the general scalar potential is not known. It will be very interesting to construct this in an E_9 covariant form from E_9 ExFT using a generalised Scherk–Schwarz reduction similar to [62–64]. This would lead for instance to theories with AdS_2 vacua, generalising [65], and that could be of interest in the $\text{AdS}_2/\text{CFT}_1$ correspondence. The gaugings defined in [39] always involve, beyond some subalgebra of $\hat{\mathfrak{e}}_8$, the L_{-1} generator,³⁷ and possible non-Lagrangian gaugings involving the L_0 generator were discussed in [17]. The Virasoro-extended formulation of E_9 exceptional field theory suggests the existence of $D = 2$ supergravities that gauge more general subalgebras of \mathfrak{vir}^- involving arbitrarily negative Virasoro generators. It would certainly be interesting to study this possibility via a generalised Scherk–Schwarz ansatz. It would moreover be interesting to study generalised Scherk–Schwarz reductions with mild violations of the section constraint to describe for example massive type IIA consistent truncations. This would require to check the invariance under generalised diffeomorphisms in the presence of such a mild violation, or to define a deformed version of E_9 ExFT as in [66] (see also [67]).

Supergravity in $D = 2$ is the natural habitat of exotic branes that are characterised by having co-dimension at most two and thus sufficiently many isometries to be describable in two dimensions [68, 69]. The truly exotic branes (with tension scaling like $g_s^{-\alpha}$ for $\alpha > 2$ in terms of the string coupling) are related by discrete duality transformations to (smeared versions of) the more conventional D- and NS-branes [70, 71]. E_9 ExFT can then provide a framework for studying uplifts of these exotic objects and a unified description of their duality orbits [72, 73].

Another interesting avenue of research might be to explore the fate of the known integrable structure of $D = 2$ ungauged maximal supergravity [28, 74] within ExFT. The integrable structure arises when the trivial solution $\langle \partial | = 0$ to the section constraint is chosen, which corresponds to the toroidal reduction. Whether or not there are any remnants of this integrable structure for more general backgrounds with a non-trivial dependence in the internal coordinates is an open problem, and one may hope that E_9 ExFT could shed some light on this question.

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A On trivial parameters and generalised diffeomorphisms

In this appendix, we collect some details on trivial parameters for the E_9 generalised Lie derivative (3.1) and its Virasoro extension (3.34). These are by definition the non-vanishing pairs $\Lambda = (|\Lambda\rangle, \Sigma)$, respectively the non-vanishing $\Lambda = (|\Lambda\rangle, \Sigma^{(k)})$, that satisfy

$$\mathcal{L}_\Lambda |V\rangle = 0 \quad \text{for any } |V\rangle. \quad (\text{A.1})$$

³⁷This is denoted L_{+1} in [39].

The existence of such trivial parameters is possible due to the section constraint (3.8) as usual in exceptional geometry. We first define the (non-extended) trivial parameters in Appendix A.1, show the closure of the algebra of generalised diffeomorphisms (3.21) on the one-form gauge fields and describe the two-form gauge fields and their gauge transformations in Appendix A.2. Closure of the Virasoro-extended generalised diffeomorphisms (3.34) is explicitly checked in Appendix (A.3). Finally, in Appendix A.4, we discuss the trivial parameters of these extended diffeomorphisms, which include the previous ones and an infinite set of additional trivial parameters.

A.1 Trivial parameters

The commutator of two generalised Lie derivatives gives a generalised Lie derivative according to (3.11), but the commutator of two generalised diffeomorphisms acting on the Dorfman pair of gauge fields \mathbb{A} produces an additional one-form gauge transformation $\delta_{\mathbf{R}}\mathbb{A}$, which is a trivial parameter for the Lie derivative, i.e.

$$\mathcal{L}_{\delta_{\mathbf{R}}\mathbb{A}}|V\rangle = 0 \tag{A.2}$$

for any $|V\rangle$. It is standard in gauged supergravity that the gauge algebra only closes on the one-form gauge field up to a one-form gauge transformation of the two-form gauge field in the tensor hierarchy [33]. The E_n gauged supergravity two-forms are valued in the symmetric tensor product of the one-form gauge field representation that we denote $R(\Lambda_n)$ (in Bourbaki labelling for $n \leq 8$), in the orthogonal complement of the generic irreducible representation $R(2\Lambda_n)$. Continuing this structure to E_9 , this gives a two-form gauge field $C_{\mu\nu}^{MN}$ valued in³⁸

$$|C_{[1]} \rangle \otimes |C_{[2]} \rangle \in R(\Lambda_0)_{-1} \vee R(\Lambda_0)_{-1} \ominus R(2\Lambda_0)_{-2} , \tag{A.3}$$

and the corresponding one-form gauge parameter R_{μ}^{MN} . In exceptional field theory in dimension $D \leq 4$, the gauge invariance of the theory requires also the inclusion of an additional two-form field with one constrained index in $\overline{R(\Lambda_n)}$, and the others in the antisymmetric tensor product $\wedge^{4-D} R(\Lambda_n)$ [21, 22]. For E_9 , this gives a two-form gauge field $C_{\mu\nu M}^{PQ}$ valued in

$$|C_{[1]} \rangle \otimes |C_{[2]} \rangle \langle \pi_C | \in R(\Lambda_0)_{-1} \wedge R(\Lambda_0)_{-1} \otimes \overline{R(\Lambda_0)_{-1}} , \tag{A.4}$$

and the corresponding one-form gauge parameter $R_{\mu}^{PQ}{}_M$. In E_9 exceptional field theory, there is moreover an additional one-form gauge parameter with one constrained index and one-form gauge parameters with two constrained indices. We expect that D -forms with two constrained indices should similarly appear in the field strength of the constrained $(D-1)$ -form in lower rank exceptional field theories.

³⁸This also follows from a level decomposition of E_{11} (T. Nutma, unpublished).

One computes that the following Dorfman pairs of parameters $\delta_{\mathbf{R}}\mathbb{A} = (\delta_{\mathbf{R}}|A\rangle, \delta_{\mathbf{R}}B)$ are trivial when acting on any $|V\rangle$ according to (A.2)

$$\left(\eta_{\alpha\beta} \langle \partial_R | T^\alpha | R_{(1)} T^\beta | R_{(2)} \rangle, 0 \right), \quad (\text{A.5a})$$

$$\left(\eta_{\alpha\beta} \langle \pi_R | T^\alpha | R_{[1]} T^\beta | R_{[2]} \rangle + 2 \langle \pi_R | R_{[1]} | R_{[2]} \rangle, \right. \\ \left. \frac{1}{2} \eta_{1\alpha\beta} T^\alpha | R_{[1]} \left[\langle \pi_R | T^\beta | R_{[2]} \rangle \langle \partial_R | + \langle \partial_R | T^\beta | R_{[2]} \rangle \langle \pi_R | \right] \right), \quad (\text{A.5b})$$

$$\left(\eta_{-1\alpha\beta} \text{Tr}[T^\alpha R_2^+ T^\beta | R_1^+], \right. \\ \left. - \eta_{\alpha\beta} \text{Tr}[T^\alpha R_2^+ T^\beta | R_1^+] \langle \partial_{R^+} | + | R_1^+ \rangle \langle \partial_{R^+} | R_2^+ - \langle \partial_{R^+} | R_1^+ \rangle R_2^+ \right), \quad (\text{A.5c})$$

$$\left(0, \frac{1}{2} \eta_{1\alpha\beta} \langle \pi_{U_1} | T^\beta | U_{(1)} T^\alpha | U_{(2)} \rangle \langle \pi_{U_2} | + \frac{1}{2} \eta_{1\alpha\beta} \langle \pi_{U_2} | T^\beta | U_{(1)} T^\alpha | U_{(2)} \rangle \langle \pi_{U_1} | \right), \quad (\text{A.5d})$$

$$\left(0, \eta_{\alpha\beta} \text{Tr}[T^\alpha X_{(1)} T^\beta X_{(2)}] + \text{Tr}[X_{(1)} X_{(2)} - X_{(1)} X_{(2)}] \right), \quad (\text{A.5e})$$

$$\left(0, \eta_{-1\alpha\beta} \text{Tr}[T^\alpha W_{(1)} T^\beta W_{(2)}] \right). \quad (\text{A.5f})$$

The notation here for one-form parameters is such that

$$|R_{(1)}\rangle \otimes |R_{(2)}\rangle \iff R^{PQ} = R^{(PQ)}, \quad (\text{A.6a})$$

$$|R_{[1]}\rangle \otimes |R_{[2]}\rangle \langle \pi_R | \iff R^{PQ}_N = R^{[PQ]}_N \quad \text{constrained in } N, \quad (\text{A.6b})$$

$$|R_1^+\rangle \otimes R_2^+ \iff R_+^{P;Q}_N \quad \text{constrained in } N, \quad (\text{A.6c})$$

$$|U_1\rangle \langle \pi_{U_1} | \otimes |U_2\rangle \langle \pi_{U_2} | \iff U^P_M{}^Q_N = U^{(P}_{(M}{}^{Q)}_N) \quad \text{constrained in both } M \text{ and } N, \quad (\text{A.6d})$$

$$X_1 \otimes X_2 \iff X^P_M{}^Q_N = X^Q_N{}^P_M \quad \text{constrained in both } M \text{ and } N, \quad (\text{A.6e})$$

$$W_1 \otimes W_2 \iff W^P_M{}^Q_N = W^Q_N{}^P_M \quad \text{constrained in both } M \text{ and } N \quad (\text{A.6f})$$

and in (A.5e) the notation $X_1 X_2$ is the product of operators, which corresponds to the trace $X^M_P{}^P_N$. The notation $|R_{(1)}\rangle \otimes |R_{(2)}\rangle$ denotes a single one-form, so that $|\partial_R\rangle$ acts on all of it, and similarly for $|R_{[1]}\rangle \otimes |R_{[2]}\rangle \langle \pi_R |$ and $|R_1^+\rangle \otimes R_2^+$. The semi-colon for $R_+^{P;Q}_N$ is used to separate the two tensor factors.

Because the parameter (A.5a) vanishes when $|R_{(1)}\rangle \otimes |R_{(2)}\rangle \in R(2\Lambda_0)_{-2}$, we can indeed interpret it as the one-form gauge parameter expected from the tensor hierarchy, consistently with (A.3). For simplicity we never write the projection to the orthogonal complement of $R(2\Lambda_0)_{-2}$, but this component will always be projected out in the relevant expressions.

In order to show that these parameters are trivial one has to use the section constraints as well as the invariance of the bilinear forms $\eta_{m\alpha\beta}$ under the action of $\hat{\mathbf{e}}_8$ and

$$\eta_{n\alpha\beta} [L_m, T^\alpha] \otimes T^\beta + \eta_{n\alpha\beta} T^\alpha \otimes [L_m, T^\beta] = (m-n) \eta_{m+m\alpha\beta} T^\alpha \otimes T^\beta \\ - \frac{c_{\text{vir}}}{6} m(m^2-1) \delta_{m,-n} \mathbf{K} \otimes \mathbf{K}. \quad (\text{A.7})$$

To illustrate this, we look at the example where $\delta_{\mathbf{R}}\mathbb{A}$ is the second parameter (A.5b). The transport and weight terms can be shown to vanish easily using (3.8), so that we are left with

$$\begin{aligned}
\mathcal{L}_{\delta_{\mathbf{R}}\mathbb{A}}|V\rangle &= -\eta_{\alpha\beta}\eta_{\gamma\delta}\langle\pi_R|T^\gamma|R_{[1]}\rangle\langle\partial_R|T^\alpha T^\delta|R_{[2]}\rangle T^\beta|V\rangle - 2\eta_{\alpha\beta}\langle\pi_R|R_{[1]}\rangle\langle\partial_R|T^\alpha|R_{[2]}\rangle T^\beta|V\rangle \\
&\quad + \frac{1}{2}\eta_{-1\alpha\beta}\eta_{1\gamma\delta}\left(\langle\partial_R|T^\gamma|R_{[1]}\rangle\langle\pi_R|T^\alpha T^\delta|R_{[2]}\rangle + \langle\pi_R|T^\gamma|R_{[1]}\rangle\langle\partial_R|T^\alpha T^\delta|R_{[2]}\rangle\right) T^\beta|V\rangle \\
&= -\frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\left(\langle\pi_R|T^\gamma|R_{[1]}\rangle\langle\partial_R|[T^\alpha, T^\delta]|R_{[2]}\rangle - \langle\pi_R|[T^\alpha, T^\gamma]|R_{[1]}\rangle\langle\partial_R|T^\delta|R_{[2]}\rangle\right) T^\beta|V\rangle \\
&\quad - \eta_{\alpha\beta}\left(\langle\pi_R|R_{[1]}\rangle\langle\partial_R|T^\alpha|R_{[2]}\rangle + \langle\partial_R|R_{[1]}\rangle\langle\pi_R|T^\alpha|R_{[2]}\rangle\right) T^\beta|V\rangle \\
&\quad + \frac{1}{2}\eta_{-1\alpha\beta}\eta_{1\gamma\delta}\left(\langle\partial_R|T^\gamma|R_{[1]}\rangle\langle\pi_R|[T^\alpha, T^\delta]|R_{[2]}\rangle + \langle\pi_R|T^\gamma|R_{[1]}\rangle\langle\partial_R|[T^\alpha, T^\delta]|R_{[2]}\rangle\right) T^\beta|V\rangle \\
&= 0, \tag{A.8}
\end{aligned}$$

where in the second step we have used (3.8) and (3.9) to write commutators as well as invariance of $\eta_{\gamma\delta}$ to split the first term into two. In the last step we have used the following identity [17, Eq. (2.26)] for any two operators X and Y

$$\begin{aligned}
&\eta_{m\alpha\beta}\eta_{m-n\gamma\delta}\text{Tr}[T^\gamma X]\text{Tr}[[T^\alpha, T^\delta]Y]T^\beta - \eta_{p\alpha\beta}\eta_{m-p\gamma\delta}\text{Tr}[T^\gamma X]\text{Tr}[[T^\alpha, T^\delta]Y]T^\beta \\
&= (n-p)\eta_{m\alpha\beta}\left(\text{Tr}[T^\alpha X]\text{Tr}[Y] - \text{Tr}[X]\text{Tr}[T^\alpha Y]\right)T^\beta - (n-p)\eta_{m\alpha\beta}\text{Tr}[T^\alpha X]\text{Tr}[T^\beta Y]\mathbf{K} \\
&\quad + \frac{c_{\text{vir}}}{12}(n(n^2-1) - p(p^2-1))\delta_{m,0}\text{Tr}[X]\text{Tr}[Y]\mathbf{K}, \tag{A.9}
\end{aligned}$$

with $m = n = 0$ and $p = -1$.

We will now show that these trivial parameters are all the ones we need to close the algebra of generalised diffeomorphisms and define the field strength. In particular we can use these trivial parameters to verify (3.17). For this purpose we define the sextuplet of one-form gauge parameters \mathbf{R}_μ as

$$\begin{aligned}
\mathbf{R} = \left(|R_{(1)}\rangle \otimes |R_{(2)}\rangle, |R_{[1]}\rangle \otimes |R_{[2]}\rangle \langle\pi_R|, |R_1^+\rangle \otimes R_2^+, \right. \\
\left. |U_{(1)}\rangle \langle\pi_{U_1}| \otimes |U_{(2)}\rangle \langle\pi_{U_2}|, X_{(1)} \otimes X_{(2)}, W_{(1)} \otimes W_{(2)} \right). \tag{A.10}
\end{aligned}$$

The trivial parameters (A.5) can be written with the linear map ϖ that maps a sextuplet of parameters \mathbf{R} to a Dorfman doublet as follows

$$\begin{aligned}
\varpi\mathbf{R} &= \left(\eta_{\alpha\beta}\langle\partial_R|T^\alpha|R_{(1)}\rangle T^\beta|R_{(2)}\rangle + \eta_{\alpha\beta}\langle\pi_R|T^\alpha|R_{[1]}\rangle T^\beta|R_{[2]}\rangle + 2\langle\pi_R|R_{[1]}\rangle|R_{[2]}\rangle \right. \\
&\quad \left. + \eta_{-1\alpha\beta}\text{Tr}[T^\alpha R_2^+]T^\beta|R_1^+\rangle, \right. \\
&\quad \left. \frac{1}{2}\eta_{1\alpha\beta}T^\alpha|R_{[1]}\rangle \left[\langle\pi_R|T^\beta|R_{[2]}\rangle \langle\partial_R| + \langle\partial_R|T^\beta|R_{[2]}\rangle \langle\pi_R| \right] \right. \\
&\quad \left. - \eta_{\alpha\beta}\text{Tr}[T^\alpha R_2^+]T^\beta|R_1^+\rangle \langle\partial_{R^+}| + |R_1^+\rangle \langle\partial_{R^+}|R_2^+ - \langle\partial_{R^+}|R_1^+\rangle R_2^+ \right. \\
&\quad \left. + \frac{1}{2}\eta_{1\alpha\beta}\langle\pi_{U_1}|T^\beta|U_{(1)}\rangle T^\alpha|U_{(2)}\rangle \langle\pi_{U_2}| + \frac{1}{2}\eta_{1\alpha\beta}\langle\pi_{U_2}|T^\beta|U_{(1)}\rangle T^\alpha|U_{(2)}\rangle \langle\pi_{U_1}| \right. \\
&\quad \left. + \eta_{\alpha\beta}\text{Tr}[T^\alpha X_{(1)}]T^\beta X_{(2)} + \text{Tr}[X_{(1)}X_{(2)}] - X_{(1)}X_{(2)} + \eta_{-1\alpha\beta}\text{Tr}[T^\alpha W_{(1)}]T^\beta W_{(2)} \right). \tag{A.11}
\end{aligned}$$

By definition we have that $\mathcal{L}_{\varpi\mathbf{R}}$ vanishes on any field, and one checks moreover that

$$\varpi\mathbf{R} \circ \mathbb{A} = 0 \tag{A.12}$$

for any parameter \mathbb{A} . One also computes that

$$\begin{aligned}
& \left(\langle \partial_A | L_n | A \rangle + \text{Tr}[L_{n-1} B] \right) \Big|_{\mathbb{A}=\varpi \mathbf{R}} \\
&= \frac{n}{2} \eta_{m\alpha\beta} \langle \partial_R | T^\alpha | R_{(1)} \rangle \langle \partial_R | T^\beta | R_{(2)} \rangle + \frac{n}{2} \eta_{m\alpha\beta} \langle \pi_R | T^\alpha | R_{[1]} \rangle \langle \partial_R | T^\beta | R_{(2)} \rangle \\
&\quad + \eta_{m-1\alpha\beta} \langle \partial_{R^+} | T^\alpha | R_1^+ \rangle \text{Tr}[T^\beta R_2^+] + \frac{n-2}{2} \eta_{m\alpha\beta} \langle \pi_1 | T^\alpha | U_{(1)} \rangle \langle \pi_2 | T^\beta | U_{(2)} \rangle \\
&\quad + \frac{n-1}{2} \eta_{m-1\alpha\beta} \text{Tr}[T^\alpha X_{(1)}] \text{Tr}[T^\beta X_{(2)}] + \frac{n}{2} \eta_{m-2\alpha\beta} \text{Tr}[T^\alpha W_{(1)}] \text{Tr}[T^\beta W_{(2)}] , \tag{A.13}
\end{aligned}$$

which does not vanish for $n \geq 1$. For $n = 1$ this term simplifies to

$$\left(\langle \partial_A | L_1 | A \rangle + \text{Tr}[L_0 B] \right) \Big|_{\mathbb{A}=\varpi \mathbf{R}} = \frac{1}{2} \eta_{1\alpha\beta} \langle \pi_R | T^\alpha | R_{[1]} \rangle \langle \partial_R | T^\beta | R_{(2)} \rangle + \eta_{\alpha\beta} \langle \partial_{R^+} | T^\alpha | R_1^+ \rangle \text{Tr}[T^\beta R_2^+] \tag{A.14}$$

and is a total derivative. For $n \geq 2$ it is not a total derivative. This implies for example that $\widehat{\mathbb{F}}_m^Y$ in (4.46) depends explicitly on the two-form fields for $m \geq 2$, so that only $\mathfrak{X}^{(1)}$ defines an appropriate topological term.

Since the map (A.11) acts only on the internal indices, it applies regardless of whether \mathbf{R} are space-time p -forms or scalars. To prove (3.17) we now define the bilinear map \mathfrak{u} that gives for any two Dorfman doublets \mathbb{A}_1 and \mathbb{A}_2 the sextuplet of parameters

$$\begin{aligned}
\mathfrak{u}(\mathbb{A}_1, \mathbb{A}_2) = & \left(-\frac{1}{2} |\Lambda_{(1)} \rangle \otimes |\Lambda_{(2)} \rangle , -\frac{1}{2} |\Lambda_{[1]} \rangle \otimes |\Lambda_{(2)} \rangle \langle \partial_{\Lambda_1} - \partial_{\Lambda_2} | , -|\Lambda_{(2)} \rangle \otimes \Sigma_1 , \right. \\
& \left. |\Lambda_{(1)} \rangle \langle \partial_{\Lambda_1} | \otimes |\Lambda_{(2)} \rangle \langle \partial_{\Lambda_1} | , -\Sigma_1 \otimes |\Lambda_{(2)} \rangle \langle \partial_{\Lambda_2} | - |\Lambda_{(2)} \rangle \langle \partial_{\Lambda_2} | \otimes \Sigma_1 , -\Sigma_{(1} \otimes \Sigma_{2)} \right) . \tag{A.15}
\end{aligned}$$

With this definition one checks that

$$\mathbb{A}_1 \circ \mathbb{A}_2 - [\mathbb{A}_1, \mathbb{A}_2]_E = \varpi \mathfrak{u}(\mathbb{A}_1, \mathbb{A}_2) , \tag{A.16}$$

which proves (3.17) by taking respectively the antisymmetric and the symmetric components in \mathbb{A}_1 and \mathbb{A}_2 . More generally, one can show

$$(\mathbb{A}_{(1} \circ \mathbb{A}_{2)}) \circ \mathbb{A}_3 = 0 , \tag{A.17}$$

$$2 \mathbb{A}_{[1} \circ (\mathbb{A}_{2]} \circ \mathbb{A}_3) = [\mathbb{A}_1, \mathbb{A}_2]_D \circ \mathbb{A}_3 , \tag{A.18}$$

where the first equation follows in fact directly from (A.12) and (A.16). These relations correspond respectively to the triviality of the symmetric bracket $\{\mathbb{A}_1, \mathbb{A}_2\} = \mathbb{A}_{(1} \circ \mathbb{A}_{2)}$ and to the closure of the Dorfman product according to the antisymmetric bracket $[\mathbb{A}_1, \mathbb{A}_2]_D = \mathbb{A}_{[1} \circ \mathbb{A}_{2]}$. As argued in Section 3.1, they together imply the Leibniz identity (3.16).

Using the Leibniz identity and (A.16), one then computes that the gauge transformation

$$\delta_{\mathbb{A}} \mathbb{A} = d\mathbb{A} + \mathbb{A} \circ \mathbb{A} \tag{A.19}$$

closes as

$$\delta_{\mathbb{A}_2} \delta_{\mathbb{A}_1} \mathbb{A} - \delta_{\mathbb{A}_1} \delta_{\mathbb{A}_2} \mathbb{A} = \delta_{[\mathbb{A}_1, \mathbb{A}_2]_E} \mathbb{A} + \varpi (\mathfrak{u}(\mathbb{A}_1, d\mathbb{A}_2) - \mathfrak{u}(\mathbb{A}_2, d\mathbb{A}_1)) . \tag{A.20}$$

In summary, we write the one-form gauge transformation of the gauge fields as

$$\delta_{\mathbf{R}}\mathbb{A} = \varpi\mathbf{R} , \quad (\text{A.21})$$

such that

$$\delta_{\mathbb{A}_2}\delta_{\mathbb{A}_1}\mathbb{A} - \delta_{\mathbb{A}_1}\delta_{\mathbb{A}_2}\mathbb{A} = \delta_{[\mathbb{A}_1, \mathbb{A}_2]_E}\mathbb{A} + \delta_{\mathbf{R}_{12}}\mathbb{A} , \quad (\text{A.22})$$

with

$$\mathbf{R}_{12} = \mathfrak{v}(\mathbb{A}_1, d\mathbb{A}_2) - \mathfrak{v}(\mathbb{A}_2, d\mathbb{A}_1) . \quad (\text{A.23})$$

Note that the Leibniz property is only satisfied up to a trivial parameter in the Virasoro-extended formulation introduced in Section 3.3. This is nonetheless sufficient for the algebra of generalised diffeomorphisms to close on the gauge field \mathbb{A} . In this case the same construction applies, but \mathbf{R}_{12} is modified by the corresponding violation of the Leibniz identity.

A.2 Covariant field strengths and two-forms

The E_9 field strength (3.28) depends on a sextuplet of two-form fields \mathbf{C} defined as in (A.10) as

$$\mathbf{C} = \left(|C_{(1)}\rangle \otimes |C_{(2)}\rangle , |C_{[1} \otimes |C_{2]}\rangle \langle \pi_C | , |C_1^+\rangle \otimes |C_2^+\rangle , \right. \\ \left. |C_{(1)}^-\rangle \langle \pi_1 | \otimes |C_{(2)}^-\rangle \langle \pi_2 | , C_{(1} \otimes C_{2)} , C_{(1}^+ \otimes C_{2)}^+ \right) , \quad (\text{A.24})$$

with the same symmetries as for \mathbf{R} in (A.6). The two-form C^{MN} is the one expected in the tensor hierarchy according to (A.3). The two-forms C^{MN}_P and $C^{M;N}_+$ carry one constrained index, and are similar to the two-forms introduced in [21, 22] that extend the tensor hierarchy in exceptional field theory. The two-forms $C^M_{-P}{}^N{}_Q$, $C^M_{P}{}^N{}_Q$ and $C^M_{+P}{}^N{}_Q$ carry two constrained indices P and Q , and appear in the field strength \mathcal{G} of the one-form B , but not in $|\mathcal{F}\rangle$.

Using (A.16), one can write the field strength (3.28) as

$$\mathbb{F} = d\mathbb{A} - \frac{1}{2}[\mathbb{A}, \mathbb{A}]_E + \varpi\mathbf{C} = d\mathbb{A} - \frac{1}{2}\mathbb{A} \circ \mathbb{A} + \varpi\left(\mathbf{C} + \frac{1}{2}\mathfrak{v}(\mathbb{A}, \mathbb{A})\right) , \quad (\text{A.25})$$

which is convenient to prove (3.30). Using Leibniz (3.16) and (A.16) one computes the gauge transformation of the field strength

$$\delta_{\mathbb{A}}\mathbb{F} = \mathbb{A} \circ \mathbb{F} \\ + \varpi\left(\delta_{\mathbb{A}}\mathbf{C} + \mathfrak{v}(d\mathbb{A}, \mathbb{A}) + \frac{1}{2}\mathfrak{v}(\mathbb{A} \circ \mathbb{A}, \mathbb{A}) + \frac{1}{2}\mathfrak{v}(\mathbb{A}, \mathbb{A} \circ \mathbb{A}) - \mathfrak{v}(\mathbb{A}, \varpi(\mathbf{C} + \frac{1}{2}\mathfrak{v}(\mathbb{A}, \mathbb{A}))) - \mathfrak{v}(\varpi(\mathbf{C} + \frac{1}{2}\mathfrak{v}(\mathbb{A}, \mathbb{A})), \mathbb{A})\right) . \quad (\text{A.26})$$

Therefore there exists a gauge transformation of \mathbf{C} such that (3.30) is satisfied. This gauge transformation is only defined up to an element in the kernel of ϖ . It is convenient to chose a particular element in the kernel to simplify the gauge transformation, so we define

$$\delta_{\mathbb{A}}\mathbf{C} = \mathfrak{v}(\mathbb{A}, \varpi(\mathbf{C} + \frac{1}{2}\mathfrak{v}(\mathbb{A}, \mathbb{A}))) + \mathfrak{v}(\varpi(\mathbf{C} + \frac{1}{2}\mathfrak{v}(\mathbb{A}, \mathbb{A})), \mathbb{A}) \\ - \mathfrak{v}(d\mathbb{A}, \mathbb{A}) - \frac{1}{2}\mathfrak{v}(\mathbb{A} \circ \mathbb{A}, \mathbb{A}) - \frac{1}{2}\mathfrak{v}(\mathbb{A}, \mathbb{A} \circ \mathbb{A}) - \frac{1}{2}\kappa(\mathbb{A}, \mathbb{A}, \mathbb{A}) , \quad (\text{A.27})$$

where $\varpi\kappa(\mathbb{A}, \mathbb{A}, \mathbb{A}) = 0$ and we choose

$$\begin{aligned}
\kappa(\mathbb{A}, \mathbb{A}, \mathbb{A}) = & \left(|\Lambda\rangle \otimes \eta_{-1\alpha\beta} \text{Tr}[T^\alpha B] T^\beta |A\rangle + \eta_{-1\alpha\beta} \text{Tr}[T^\alpha B] T^\beta |A\rangle \otimes |\Lambda\rangle , \right. \\
& \left(|\Lambda\rangle \otimes \eta_{-1\alpha\beta} \text{Tr}[T^\alpha B] T^\beta |A\rangle - \eta_{-1\alpha\beta} \text{Tr}[T^\alpha B] T^\beta |A\rangle \otimes |\Lambda\rangle \right) \langle \partial_\Lambda - \partial_A - \partial_B | , \\
& -|\Lambda\rangle \otimes \left(\text{Tr}[T_\alpha B] T^\alpha |A\rangle \langle \partial_B | + |A\rangle \langle \partial_B | B - \langle \partial_B | A \rangle B \right. \\
& \quad \left. + \langle \partial_A | T_\alpha | A \rangle T^\alpha B + B |A\rangle \langle \partial_A | - \text{Tr}[B] |A\rangle \langle \partial_A | \right) \\
& -\mathcal{L}_\mathbb{A}(|A\rangle \otimes B) + \eta_{-1\alpha\beta} \text{Tr}[T^\alpha B] T^\beta |A\rangle \otimes \Sigma - \text{Tr}[\Sigma] |A'\rangle \otimes |A\rangle \langle \partial_A | \\
& -\frac{1}{2} |\Lambda\rangle \otimes \eta_{1\alpha\beta} \left(\langle \partial_A | T^\alpha | A \rangle T^\beta |A'\rangle \langle \partial_A | - \langle \partial_A | T^\alpha | A' \rangle T^\beta |A\rangle \langle \partial_A | \right) , \\
& \left. \dots , \dots , \dots \right) , \tag{A.28}
\end{aligned}$$

where we have not computed the expression of the three doubly constrained parameters because they are not needed anywhere.

The homogeneous part in \mathbf{C} is not the Lie derivative of \mathbf{C} , but mixes the various components of \mathbf{C} . We define the inhomogeneous variation of \mathbf{C}

$$\tilde{\Delta}_\mathbb{A} \mathbf{C} = \delta_\mathbb{A} \mathbf{C} - \mathfrak{i}(\mathbb{A}, \varpi \mathbf{C}) - \mathfrak{i}(\varpi \mathbf{C}, \mathbb{A}) , \tag{A.29}$$

that defines the piece depending explicitly on the vector fields \mathbb{A} . One finds

$$\begin{aligned}
\tilde{\Delta}_\mathbb{A} |C_{(1)}\rangle \otimes |C_{(2)}\rangle &= \frac{1}{4} (d|\Lambda\rangle \otimes |A\rangle) - |A\rangle \otimes d|\Lambda\rangle , \tag{A.30} \\
\tilde{\Delta}_\mathbb{A} |C_{[1]}\rangle \otimes |C_{[2]}\rangle \otimes \langle \pi_C | &= \frac{1}{4} (d|\Lambda\rangle \otimes |A\rangle + |A\rangle \otimes d|\Lambda\rangle) \otimes \langle \partial_\Lambda - \partial_A | , \\
\tilde{\Delta}_\mathbb{A} |C_1^+\rangle \otimes C_2^+ &= -\frac{1}{2} \eta_{1\alpha\beta} \langle \partial_\Lambda | T^\alpha | A \rangle |A\rangle \otimes T^\beta |A\rangle \langle \partial_\Lambda | \\
&\quad - |A\rangle \otimes d\Sigma - \frac{1}{2} \eta_{\alpha\beta} \text{Tr}[T^\alpha \Sigma] |A\rangle \otimes T^\beta |A\rangle \langle \partial_\Sigma | \\
&\quad + \frac{1}{2} |A\rangle \otimes |A\rangle \langle \partial_\Sigma | \Sigma + \frac{1}{2} \text{Tr}[\Sigma] |A'\rangle \otimes |A\rangle \langle \partial_A | .
\end{aligned}$$

Note nonetheless that this is *not* the gauge transformation that is relevant in minimal E_9 exceptional field theory, because the invariance of the duality equation (5.59) requires $|\mathcal{F}\rangle$ to transform as the internal current, and the gauge transformation of $|C_1^+\rangle \otimes C_2^+$ in exceptional field theory includes an additional term (5.62).³⁹

A.3 Closure of extended generalised diffeomorphisms

The Virasoro-extended generalised diffeomorphisms (3.34) close according to the extended E-bracket (3.38). In order to show this, we only need to consider the commutators $[\mathcal{L}_{(\Lambda_1, 0)}, \mathcal{L}_{(0, \Sigma_2^{(k)})}]$ and $[\mathcal{L}_{(0, \Sigma_1^{(p)})}, \mathcal{L}_{(0, \Sigma_2^{(k)})}]$, where $\Sigma_1^{(p)}$ and $\Sigma_2^{(k)}$ denote two complete sets of extended gauge parameters. Closure of the $|\Lambda\rangle$ -diffeomorphisms onto themselves follows directly from the closure of the unextended generalised diffeomorphisms according to (3.13).

³⁹We only introduced this equation in the minimal formulation, but invariance under conformal diffeomorphisms would require to include this equation in the Virasoro-extended formulation as well.

Evaluating the first commutator on a generalised vector $|V\rangle$ in $R(\Lambda_0)_0$, we find using (3.36) and the section constraints,

$$\begin{aligned}
& [\mathcal{L}_{(\Lambda_1,0)}, \mathcal{L}_{(0,\Sigma_2^{(k)})}]|V\rangle \\
&= - \sum_{k=1}^{\infty} \eta_{-k} \alpha\beta \text{Tr} \left(T^\alpha (\langle \partial_\Sigma | \Lambda_1 \rangle \Sigma_2^{(k)} - \eta_{\gamma\delta} \langle \partial_\Lambda | T^\gamma | \Lambda_1 \rangle [T^\delta, \Sigma_2^{(k)}] + k \langle \partial_\Lambda | \Lambda_1 \rangle \Sigma_2^{(k)}) \right) T^\beta |V\rangle \\
&= \mathcal{L}_{(0, \mathcal{L}_{(\Lambda_1,0)} \Sigma_2^{(k)})} |V\rangle. \tag{A.31}
\end{aligned}$$

This confirms the first term in the expression (3.38) of Σ_{12} . Let us now consider the second commutator

$$\begin{aligned}
& [\mathcal{L}_{(0,\Sigma_1^{(p)}), \mathcal{L}_{(0,\Sigma_2^{(k)})}]|V\rangle \\
&= \sum_{k,q=1}^{\infty} \eta_{-k} \alpha\beta \eta_{-q} \gamma\delta \text{Tr}(\Sigma_1^{(k)} T^\alpha) \text{Tr}(\Sigma_2^{(q)} T^\gamma) [T^\beta, T^\delta] |V\rangle \\
&= \sum_{k,q=1}^{\infty} \left(- \eta_{-k} \alpha\beta \eta_{-q} \gamma\delta \text{Tr}(\Sigma_1^{(k)} T^\alpha) \text{Tr}(\Sigma_2^{(q)} T^\delta T^\beta) + (k-q) \eta_{-k-q} \alpha\gamma \text{Tr}(\Sigma_1^{(k)}) \text{Tr}(\Sigma_2^{(q)} T^\alpha) \right) T^\gamma |V\rangle \\
&= - \sum_{q=1}^{\infty} \eta_{-q} \gamma\delta \text{Tr}(\tilde{\Sigma}_{12}^{(q)} T^\delta) T^\gamma |V\rangle = \mathcal{L}_{(0, \tilde{\Sigma}_{12}^{(q)})} |V\rangle, \tag{A.32}
\end{aligned}$$

where in the second line we used (A.7), and with

$$\tilde{\Sigma}_{12}^{(q)} = - \sum_{k=1}^{\infty} \eta_{-k} \alpha\beta \text{Tr}(\Sigma_1^{(k)} T^\alpha) T^\beta \Sigma_2^{(q)} - \sum_{0 < k < q} (2k-q) \text{Tr}(\Sigma_1^{(k)}) \Sigma_2^{(q-k)}. \tag{A.33}$$

In writing the above expression, we used the antisymmetry of the commutator in $\Sigma_1^{(p)}$ and $\Sigma_2^{(k)}$. This reproduces correctly the $\Sigma\Sigma$ terms of $\Sigma_{12}^{(k)}$ in (3.38).

A.4 Extended trivial parameters

The Virasoro-extended generalised diffeomorphisms (3.34) include by construction the trivial parameters $(\delta_{\mathbf{R}}|A), \delta_{\mathbf{R}}B^{(1)})$ defined in (A.5), as well as an infinite sequence of parameters, with $\delta_{\mathbf{R}}|A) = 0$,

$$\begin{aligned}
\delta_{\mathbf{R}}B^{(1)} &= \sum_{k=2}^{\infty} \left(\eta_{2-k} \alpha\beta \langle \pi_{(1)} | T^\alpha | U_2^{(k)} \rangle T^\beta | U_1^{(k)} \rangle \langle \pi_{(2)} | + \eta_{1-k} \alpha\beta \text{Tr}[T^\alpha X_2^{(k)}] T^\beta X_1^{(k)} \right. \\
&\quad \left. + \eta_{-k} \alpha\beta \text{Tr}[T^\alpha W_2^{(k)}] T^\beta W_1^{(k)} \right) + \langle \pi_{(1)} | U_2^{(2)} \rangle | U_1^{(2)} \rangle \langle \pi_{(2)} | - \langle \pi_{(1)} | U_1^{(2)} \rangle | U_2^{(2)} \rangle \langle \pi_{(2)} |, \tag{A.34}
\end{aligned}$$

and for $k \geq 2$

$$\begin{aligned}
\delta_{\mathbf{R}}B^{(k)} &= \eta_{1-\alpha\beta} \langle \pi_{(1)} | T^\alpha | U_1^{(k)} \rangle T^\beta | U_2^{(k)} \rangle \langle \pi_{(2)} | + k \langle \pi_{(1)} | U_2^{(k+1)} \rangle | U_1^{(k+1)} \rangle \langle \pi_{(2)} | - k \langle \pi_{(1)} | U_1^{(k+1)} \rangle | U_2^{(k+1)} \rangle \langle \pi_{(2)} | \\
&\quad + \text{Tr}[T^\alpha X_1^{(k)}] T^\alpha X_2^{(k)} - (k-2) \text{Tr}[X_1^{(k)}] X_2^{(k)} + (k-1) \text{Tr}[X_2^{(k)}] X_1^{(k)} - X_2^{(k)} X_1^{(k)} \\
&\quad + (k-2) \text{Tr}[W_2^{(k-1)}] W_1^{(k-1)} - (k-2) \text{Tr}[W_1^{(k-1)}] W_2^{(k-1)} + \eta_{-1-\alpha\beta} \text{Tr}[T^\alpha W_1^{(k)}] T^\beta W_2^{(k)}. \tag{A.35}
\end{aligned}$$

As in (A.6), all bra covectors are constrained, but no additional symmetry is assumed

$$\begin{aligned}
|U_1^{(k)}\rangle\langle\pi_{(1)}| \otimes |U_2^{(k)}\rangle\langle\pi_{(2)}| &\iff U^{(k)P}{}_M{}^Q{}_N = U^{(k)P}{}_{(M}{}^Q{}_{N)} \quad \text{constrained in both } M \text{ and } N, \\
X_1^{(k)} \otimes X_2^{(k)} &\iff X^{(k)P}{}_M{}^Q{}_N \quad \text{constrained in both } M \text{ and } N, \\
W_1^{(k)} \otimes W_2^{(k)} &\iff W^{(k)P}{}_M{}^Q{}_N \quad \text{constrained in both } M \text{ and } N.
\end{aligned} \tag{A.36}$$

These constrained parameters are very redundant, and it is in principle sufficient to only consider the parameters $W_1^{(k)} \otimes W_2^{(k)}$ and $X_1^{(2)} \otimes X_2^{(2)}$ to obtain a complete basis of trivial parameters. One can in particular eliminate the traceless component of all $\delta_{\mathbf{R}}B^{(k)}$ with $k \geq 3$ using trivial parameters $W_1^{(k-1)} \otimes W_2^{(k-1)} = |0\rangle\langle 0| \otimes W^{(k-1)}$. By construction the traces $\text{Tr}[\delta_{\mathbf{R}}B^{(k)}]$ cannot be eliminated, since they appear independently multiplying the Virasoro generator L_{-1-k} in the Lie derivative (3.34). For any $k \geq 3$ one can indeed always find $W^{(n)}$ such that

$$\delta_{\mathbf{R}}B^{(k)} = -(k-2)W^{(k-1)} + (k-2)\text{Tr}[W^{(k-1)}]|0\rangle\langle 0| - L_{-1}W^{(k)} + \text{Tr}[\Sigma^{(k)}]|0\rangle\langle 0|. \tag{A.37}$$

One can also eliminate the traceless component of $\delta_{\mathbf{R}}B^{(2)}$ using an appropriate trivial parameter $X_1^{(2)} \otimes X_2^{(2)}$, but not with an E_8 invariant ansatz as above for $k \geq 3$. If one uses the E_8 solution to the section constraint (3.10), there is a gauge parameter

$$X_1^{(2)} \otimes X_2^{(2)} = |0\rangle\langle 0| \otimes X^{(2)} + X^C{}_{B;A}T_{-1C}|0\rangle\langle 0|T_1^A \otimes |0\rangle\langle 0|T_1^B \tag{A.38}$$

with $X^C{}_{B;A}$ satisfying (5.95) for its indices A and B , such that⁴⁰

$$\delta_{\mathbf{R}}B^{(2)} = -L_0X^{(2)} - X^{(2)}|0\rangle\langle 0| + \text{Tr}[X^{(2)}]|0\rangle\langle 0| - X^B{}_{B;A}|0\rangle\langle 0|T_1^A - L_{-1}W^{(2)} + \text{Tr}[\Sigma^{(2)}]|0\rangle\langle 0|. \tag{A.39}$$

This proves, as noted in Section 3.3, that any loop transformation generated by $\Sigma^{(k)}$ can be reabsorbed into $\Sigma^{(1)}$. To write this systematically, we use (2.9) to rewrite (3.35) as

$$[\mathbb{A}]_\alpha = \eta_{\alpha\beta}\langle\partial_\Lambda|T^\beta|\Lambda\rangle + \sum_{k=1}^{\infty} \left(\eta_{-1\alpha\beta}\text{Tr}[\Sigma^{(k)}\mathcal{S}_{1-k}(T^\beta)] - \delta_\alpha^{L-k}\text{Tr}(\Sigma^{(k)}) \right). \tag{A.40}$$

The first term only contributes to the gauging of $\hat{\mathbf{e}}_8$ generators. The fact that any loop transformation generated by $\Sigma^{(k)}$ can be reabsorbed into $\Sigma^{(1)}$ then means that we can implicitly define shift operators $\hat{\mathcal{S}}_{-k}$ acting on any Σ parameter rather than on algebra-valued objects, with

$$\text{Tr}\left(\hat{\mathcal{S}}_{-k}(\Sigma)T^\alpha\right) = \text{Tr}\left(\Sigma\mathcal{S}_{-k}(T^\alpha)\right), \quad \forall k \in \mathbb{N}, \quad T^\alpha \in \hat{\mathbf{e}}_8 \oplus \mathfrak{vir}^-. \tag{A.41}$$

For instance, (3.35) may be rewritten as

$$[\mathbb{A}]_\alpha = \eta_{\alpha\beta}\langle\partial_\Lambda|T^\beta|\Lambda\rangle + \sum_{k=1}^{\infty} \left(\eta_{-1\alpha\beta}\text{Tr}\left[\hat{\mathcal{S}}_{1-k}(\Sigma^{(k)})T^\beta\right] - \delta_\alpha^{L-k}\text{Tr}(\Sigma^{(k)}) \right). \tag{A.42}$$

⁴⁰The only traceless $\delta_{\mathbf{R}}B^{(2)}$ that cannot be obtained from $X^{(2)}$ are such that $L_0[\delta_{\mathbf{R}}B^{(2)}] = 0$ and $\delta_{\mathbf{R}}B^{(2)}|0\rangle = 0$, which gives $\delta_{\mathbf{R}}B^{(2)} = \delta_{\mathbf{R}}B_A^{(2)}|0\rangle\langle 0|T_1^A$. To prove that all constrained vector $\delta_{\mathbf{R}}B_A^{(2)}$ can be written as $X^B{}_{B;A}$ for a doubly constrained tensor $X^C{}_{B;A}$, one can use the E_7 solution to the constraint (5.95).

The operators $\widehat{\mathcal{S}}_{-k}$ do not admit an explicit E_9 covariant expression. They are defined only up to the addition of trivial parameters of the types U , X and W in (A.5d)–(A.5f) and (A.36) and we have used this freedom to make (A.41) valid also for $T^\alpha = L_0$.

Using these formal shift operators, a basis for the extra trivial parameters appearing in the Virasoro-extended formalism is implicitly given by traceless $\delta_{\mathbf{R}}B^{(k)}$ parameters satisfying for fixed $p \geq 2$

$$\mathrm{Tr}(\delta_{\mathbf{R}}B^{(p)}) = 0, \quad \delta_{\mathbf{R}}B^{(1)} = -\widehat{\mathcal{S}}_{1-p}(\delta_{\mathbf{R}}B^{(p)}), \quad \delta_{\mathbf{R}}B^{(n)} = 0 \quad \text{for } n \notin \{p, 1\}. \quad (\text{A.43})$$

A.5 One-form gauge transformations in the minimal formulation

In the minimal formulation the gauge transformations of the field B are not (3.25) and $\varpi_{\mathbf{R}}$, but must instead be modified for the duality equation (5.35) to be invariant. The Σ gauge transformations of the fields that leave (5.35) invariant are

$$\begin{aligned} \delta_{(0,\Sigma)}\tilde{g}_{\mu\nu} &= 0, \\ \delta_{(0,\Sigma)}\mathcal{M} &= \eta_{-1\alpha\beta}\mathrm{Tr}[T^\beta\Sigma](T^{\alpha\dagger}\mathcal{M} + \mathcal{M}T^\alpha), \\ \delta_{(0,\Sigma)}|A\rangle &= -\eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha\Sigma]T^\beta|A\rangle \\ \delta_{(0,\Sigma)}B &= \mathrm{d}\Sigma + \eta_{\alpha\beta}\mathrm{Tr}[T^\alpha\Sigma]T^\beta|A\rangle\langle\partial_\Sigma| - |A\rangle\langle\partial_\Sigma|\Sigma + \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha\Sigma]T^\beta B \\ &\quad - \rho \star \widehat{\mathcal{S}}_{-1}(\mathrm{d}\Sigma + \eta_{\alpha\beta}\mathrm{Tr}[T^\alpha\Sigma]T^\beta|A\rangle\langle\partial_\Sigma| - |A\rangle\langle\partial_\Sigma|\Sigma) \\ \delta_{(0,\Sigma)}\chi &= \sum_n (n+1)\mathrm{Tr}[T_n^A\Sigma]J_A^n - 2\rho \star \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha\Sigma]\langle\partial_\Sigma|T^\beta|A\rangle \\ &\quad - \mathrm{Tr}[(\rho^{-1}L_{-1} + \rho^{-2}\mathcal{M}^{-1}L_{-2}\mathcal{M})(\mathrm{d}\Sigma + \eta_{\alpha\beta}\mathrm{Tr}[T^\alpha\Sigma]T^\beta|A\rangle\langle\partial_\Sigma| - |A\rangle\langle\partial_\Sigma|\Sigma)], \end{aligned} \quad (\text{A.44})$$

where $\widehat{\mathcal{S}}_{-1}$ is defined implicitly from (A.41). The ambiguity in the definition of $\widehat{\mathcal{S}}_{-1}$ can be absorbed in a redefinition of the one-form gauge parameters U , X and W . The one-form gauge transformations of $|A\rangle$, B and χ are defined as

$$\begin{aligned} \delta_{\mathbf{R}}|A\rangle &= \langle\partial_R|T_\alpha|R_{(1)}T^\alpha|R_{(2)}\rangle + \langle\pi_R|T_\alpha|R_{(1)}T^\alpha|R_{(2)}\rangle + 2\langle\pi_R|R_{(1)}|R_{(2)}\rangle + \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha R_2^+]T^\beta|R_{(1)}\rangle \\ \delta_{\mathbf{R}}B &= \frac{1}{2}\eta_{1\alpha\beta}T^\alpha|R_{(1)}\rangle \left[\langle\pi_R|T^\beta|R_{(2)}\rangle\langle\partial_R| + \langle\partial_R|T^\beta|R_{(2)}\rangle\langle\pi_R| \right] \\ &\quad - \eta_{\alpha\beta}\mathrm{Tr}[T^\alpha R_2^+]T^\beta|R_{(1)}^+\rangle\langle\partial_{R^+}| + |R_{(1)}^+\rangle\langle\partial_{R^+}|R_{(2)}^+ - \langle\partial_{R^+}|R_{(1)}^+\rangle R_{(2)}^+ \\ &\quad - \rho \star (T_\alpha|R_{(1)}\rangle\langle\pi_R|T^\alpha|R_{(2)}\rangle + 2|R_{(1)}\rangle\langle\pi_R|R_{(2)}\rangle)\langle\partial_R| + \rho \star \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha R_2^+]T^\beta|R_{(1)}^+\rangle\langle\partial_{R^+}| \\ &\quad + \frac{1}{2}\eta_{1\alpha\beta}\langle\pi_{U_1}|T^\beta|U_{(1)}T^\alpha|U_{(2)}\rangle\langle\pi_{U_2}| + \frac{1}{2}\eta_{1\alpha\beta}\langle\pi_{U_2}|T^\beta|U_{(1)}T^\alpha|U_{(2)}\rangle\langle\pi_{U_1}| \\ &\quad + \eta_{\alpha\beta}\mathrm{Tr}[T^\alpha X_{(1)}]T^\beta X_{(2)} + \mathrm{Tr}[X_{(1)}X_{(2)}] - X_{(1)}X_{(2)} + \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha W_{(1)}]T^\beta W_{(2)} \\ \delta_{\mathbf{R}}\chi &= \eta_{\alpha\beta}\langle\pi_R|T^\alpha|R_{(1)}\rangle\langle\partial_R|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\beta|R_{(2)}\rangle + 2\langle\pi_R|R_{(1)}\rangle\langle\partial_R|\mathcal{M}^{-1}L_{-1}\mathcal{M}|R_{(2)}\rangle \\ &\quad + \rho \star \left(\eta_{\alpha\beta}\langle\pi_R|T^\alpha|R_{(1)}\rangle\langle\partial_\Lambda|L_0T^\beta|R_{(2)}\rangle + 2\langle\pi_R|R_{(1)}\rangle\langle\partial_R|L_0|R_{(2)}\rangle \right) \\ &\quad + \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha R_2^+]\langle\partial_{R^+}|\mathcal{M}^{-1}L_{-1}\mathcal{M}T^\beta|R_{(1)}\rangle + \rho \star \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha R_2^+]\langle\partial_{R^+}|L_0T^\beta|R_{(1)}\rangle, \end{aligned} \quad (\text{A.45})$$

where the only modification of $\delta_{\mathbf{R}}B$ with respect to $\varpi_{\mathbf{R}}$ are the terms involving \star . These additional terms are defined such that the duality equation (5.35) is invariant under these one-form gauge transformations. One moreover checks that the pseudo-Lagrangian (5.44) is invariant under Σ and \mathbf{R} gauge transformations, up to a total derivative.

B Shift operators and cocycles

We summarise here some details on the shift operators and cocycles encountered in the text and on their transformation properties. First, we stress that our definition of \mathcal{S}_0 in this paper projects out \mathbf{K} , contrary to \mathcal{S}_0 in [1] which acts as the identity:

$$\mathcal{S}_0(\mathbf{K})^{\text{[here]}} = 0, \quad \mathcal{S}_0(\mathbf{K})^{\text{[1]}} = \mathbf{K}, \quad \mathcal{S}_k^{\text{[here]}} = \mathcal{S}_k^{\text{[1]}} \quad \forall k \neq 0. \quad (\text{B.1})$$

For generic algebra elements $X, Z \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$, the shift operator \mathcal{S}_m defined in (2.8) has the commutation property

$$\mathcal{S}_m([X, Z]) = [X, \mathcal{S}_m(Z)] + m \sum_{n \in \mathbb{Z}} X_{L_n} \mathcal{S}_{m+n}(Z) + \omega^{\alpha\beta} X_\alpha (\mathcal{S}_m(Z))_\beta \mathbf{K}, \quad (\text{B.2})$$

where

$$\omega^{\alpha\beta} X_\alpha Z_\beta = -[X, Z]|_{\mathbf{K}} = -\eta^{AB} \sum_{n \in \mathbb{Z}} n X_A^n Z_B^{-n} - \frac{c_{\text{vir}}}{12} \sum_{n \in \mathbb{Z}} (n^3 - n) X_n Z_{-n} \quad (\text{B.3})$$

is the Lie algebra cocycle in any highest/lowest weight representation of $\hat{\mathfrak{e}}_8$. Notice that $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$. For a finite $\hat{E}_8 \rtimes \text{Vir}^-$ element g we define

$$\omega^\alpha(g) \mathbf{K} = g \mathcal{S}_0(T^\alpha) g^{-1} - \mathcal{S}_0(g T^\alpha g^{-1}), \quad g \in \hat{E}_8 \rtimes \text{Vir}^-, \quad (\text{B.4})$$

which we may also write as $\omega^\alpha(g) = g T^\alpha g^{-1}|_{\mathbf{K}} - \delta_{\mathbf{K}}^\alpha$. This identity can also be used to define a cocycle for the generalised metric (4.2):

$$\omega^\alpha(\mathcal{M}) \mathbf{K} = \mathcal{M} \mathcal{S}_0(T^\alpha) \mathcal{M}^{-1} - \mathcal{S}_0(\mathcal{M} T^\alpha \mathcal{M}^{-1}). \quad (\text{B.5})$$

Using $\mathbf{K}^\dagger = \mathbf{K}$, one can easily show

$$\omega^\alpha(\mathcal{M})(\mathcal{M}^{-1} X^\dagger \mathcal{M})_\alpha = -\omega^\alpha(\mathcal{M}) X_\alpha, \quad (\text{B.6})$$

which implies for instance $\omega^\alpha(\mathcal{M}) \mathfrak{J}_\alpha^{(m)} = -\omega^\alpha(\mathcal{M}) \mathfrak{J}_\alpha^{(-m)}$ and $\omega^\alpha(\mathcal{M}) \mathfrak{J}_\alpha = 0$. It is also useful to note that

$$\omega^{\alpha\mathbf{K}} = 0 = \omega^{\mathbf{K}}(\cdot). \quad (\text{B.7})$$

Any finite $\hat{E}_8 \rtimes \text{Vir}^-$ element can be decomposed as follows:

$$g = F \ell, \quad F \in \text{Vir}^-, \quad \ell \in \hat{E}_8. \quad (\text{B.8})$$

Recall that \mathcal{S}_m in the spectral parameter representation acts as multiplication by w^m . Then, taking into account $F^{-1}w = f(w)$ as in (2.36), it is natural to define a shift operator \mathcal{S}_m^f that acts as multiplication by $f(w)^m$. This is obtained by conjugating \mathcal{S}_m with F , then projecting out any \mathbf{K} component generated by \mathfrak{vir} cocycles by means of \mathcal{S}_0 :

$$\mathcal{S}_m^f(X) = \mathcal{S}_0(F^{-1} \mathcal{S}_m(F X F^{-1}) F) = \mathcal{S}_0(g^{-1} \mathcal{S}_m(g X g^{-1}) g). \quad (\text{B.9})$$

Then, we can write the finite transformation properties of the standard shift operator as

$$g^{-1} \mathcal{S}_m(g X g^{-1}) g = \mathcal{S}_m^f(X) - \omega^\alpha(g) \left(\mathcal{S}_m^f(X) \right)_\alpha \mathbf{K}. \quad (\text{B.10})$$

The ρ and ϕ_m dependent shift operator \mathcal{S}_m^γ is defined as in (B.9):

$$\mathcal{S}_m^\gamma(X) = \mathcal{S}_0(\Gamma^{-1}\mathcal{S}_m(\Gamma X \Gamma^{-1})\Gamma) = \mathcal{S}_0(\mathcal{V}^{-1}\mathcal{S}_m(\mathcal{V} X \mathcal{V}^{-1})\mathcal{V}), \quad (\text{B.11})$$

which reproduces (4.8). We can commute \mathcal{S}_m^γ with an $\hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$ element to get

$$\mathcal{S}_m^\gamma([X, Z]) = [X, \mathcal{S}_m^\gamma(Z)] + m \sum_{n \in \mathbb{Z}} (\Gamma X \Gamma^{-1})_{L_n} \mathcal{S}_{m+n}^\gamma(Z) + \omega^{\alpha\beta} X_\alpha (\mathcal{S}_m^\gamma(Z))_\beta \mathbf{K}. \quad (\text{B.12})$$

This is used for instance in computing the shifted Maurer–Cartan equation (4.34).

There are several other useful definitions and properties of the cocycles introduced in [1] that are only valid for $g \in E_9$. In this case the decomposition (B.8) is rewritten as

$$g = \rho(g)^{-2L_0} \ell, \quad (\text{B.13})$$

where $\rho(g)$ is a constant element of the monoparametric subgroup generated by L_0 and is not to be confused with the scalar field ρ . We can generalise (B.4) to \mathcal{S}_k and define shifted cocycles

$$\omega_{-k}^\alpha(g)\mathbf{K} = \rho(g)^{-2k} g \mathcal{S}_k(T^\alpha) g^{-1} - \mathcal{S}_k(g T^\alpha g^{-1}), \quad g \in E_9, \quad (\text{B.14})$$

and we write $\omega_0^\alpha(g) = \omega^\alpha(g)$ for simplicity. The shift can be moved from the cocycle to the object it contracts:

$$\omega_{-k}^\alpha(g) X_\alpha = \rho(g)^{-2k} \omega^\alpha(g) (\mathcal{S}_k(X))_\alpha, \quad X \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}, \quad g \in E_9. \quad (\text{B.15})$$

This shows that expressions like $\omega^\alpha(g) (\mathcal{S}_k^\gamma(X))_\alpha$ can be expanded as series of shifted cocycles if $g \in E_9$. These shifted cocycles also appear in the conjugation of a Virasoro generator by a loop transformation:

$$g^{-1} L_k g = \rho(g)^{-2k} L_k - \omega_{-k}^\alpha(g) \eta_{\alpha\beta} T^\beta = \rho(g)^{-2k} (L_k - \omega^\alpha(g) \eta_{k\alpha\beta} T^\beta), \quad g \in E_9. \quad (\text{B.16})$$

In the second equality we moved the shift from the cocycle to $\eta_{\alpha\beta}$ and used (2.9). All the definitions and properties of the shifted cocycles also apply when we substitute $g \rightarrow \mathcal{M}$ (and $\rho(\mathcal{M}) = \rho$) provided we gauge-fix $\phi_m = 0$ (including $\phi_1 = \tilde{\rho} = 0$) so that (formally) $\mathcal{M} \in E_9$.

In [1] we also defined a generalisation of $\omega_1^\alpha(\mathcal{M})$ to $\tilde{\rho} \neq 0$, using the fact that in this case $\mathcal{M} \in \hat{E}_8 \rtimes SL(2)$ and the $SL(2)$ component acts on w as fractional linear transformations:⁴¹

$$\Omega^\alpha(\mathcal{M})\mathbf{K} = \mathcal{M}^{-1} \left(\rho^2 \sum_{n=0}^{\infty} \tilde{\rho}^n \mathcal{S}_{1+n}(T^{\alpha\dagger}) + \tilde{\rho} \mathcal{S}_0(T^{\alpha\dagger}) \right) \mathcal{M} - \mathcal{S}_1(\mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M}). \quad (\text{B.17})$$

We will now prove that

$$\Omega^\alpha(\mathcal{M}) \langle \mathcal{J}_\alpha | = \rho^2 \omega^\alpha(\mathcal{M}) \langle \mathcal{J}_\alpha^- |. \quad (\text{B.18})$$

⁴¹Recall that we are using $\mathcal{S}_0(\mathbf{K}) = 0$ as in the rest of this paper, which differs from [1] where we were defining \mathcal{S}_0 to be the identity. This slightly affects how we are writing $\Omega^\alpha(\mathcal{M})$ here.

To show this, we use (B.5) on the \mathcal{S}_0 term so that we can use the first of (4.74), and bring Hermitian conjugation out of the shift operators in the $\tilde{\rho}$ series so that we can also use the second of (4.74). This gives us

$$\mathcal{S}_1^\gamma(\mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M}) + \frac{\tilde{\rho}}{\rho} \omega^\alpha(\mathcal{M}) \mathbb{K} = \mathcal{M}^{-1} \left(\mathcal{S}_{-1}^\gamma(T^\alpha) \right)^\dagger \mathcal{M} - \frac{1}{\rho} \Omega^\alpha(\mathcal{M}) \mathbb{K}. \quad (\text{B.19})$$

Contracting with $\langle \mathcal{J}_\alpha |$ and using $\langle \mathcal{J}_\alpha | \otimes \mathcal{M}^{-1}(T^\alpha)^\dagger \mathcal{M} = \langle \mathcal{J}_\alpha | \otimes T^\alpha$, we find that the cocycle on the left-hand side does not contribute. On the other hand, we already know from the computation in (4.44) that

$$\mathcal{S}_1^\gamma(\mathcal{M}^{-1} X^\dagger \mathcal{M}) = \mathcal{M}^{-1} \left(\mathcal{S}_{-1}^\gamma(X) \right)^\dagger \mathcal{M} - \omega^\alpha(\mathcal{M}) (\mathcal{S}_{-1}^\gamma(X))_\alpha \mathbb{K}, \quad (\text{B.20})$$

for any $X \in \hat{\mathfrak{e}}_8 \oplus \mathfrak{vir}$. Substituting $X_\alpha \rightarrow \langle \mathcal{J}_\alpha |$, we find that the \mathbb{K} components of the last two expressions must coincide and using the definition (4.75) we conclude.

C Details on the Virasoro-extended formalism

This appendix contains additional details for some of the calculations and aspects of the Virasoro-extended formulation of E_9 ExFT presented in Section 4.

C.1 Field strength variations

The variation of the field strengths under generalised diffeomorphisms was stated in (4.51)–(4.53) that we prove here, beginning with the first equation. The field strength $\mathbb{F} = (\mathcal{F}, \mathcal{G}^{(k)})$ transforms with the extended \circ product (3.40) under generalised diffeomorphisms. This means that their non-covariant variation (4.50) reads $\Delta_\Lambda \mathbb{F} = \mathbb{A} \circ \mathbb{F} - \mathcal{L}_\Lambda \mathbb{F}$ and in particular $\Delta_\Lambda | \mathcal{F} \rangle = 0$. In order to substitute into (4.37), we then compute

$$\Delta_\Lambda \langle \partial_{\mathcal{F}} | T^\lambda | \mathcal{F} \rangle = - \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda | T^\lambda | \mathcal{F} \rangle - [\mathbb{A}]_\alpha \langle \partial_{\Lambda, \Sigma} | T^\lambda T^\alpha | \mathcal{F} \rangle, \quad (\text{C.1})$$

$$\begin{aligned} \Delta_\Lambda \text{Tr}(\mathcal{G}^{(k)} T^\lambda) &= \delta_1^k \eta_{1\alpha\beta} \langle \partial_\Lambda | [T^\lambda, T^\alpha] | \mathcal{F} \rangle \langle \partial_\Lambda | T^\beta | \Lambda \rangle + \eta_{\alpha\beta} \text{Tr}(\Sigma^{(k)} T^\alpha) \langle \partial_\Sigma | [T^\lambda, T^\beta] | \mathcal{F} \rangle \\ &\quad - k \text{Tr}(\Sigma^{(k)}) \langle \partial_\Sigma | T^\lambda | \mathcal{F} \rangle + (k-1) \text{Tr}(\Sigma^{(k)} T^\lambda) \langle \partial_\Sigma | \mathcal{F} \rangle, \end{aligned} \quad (\text{C.2})$$

where in the second equation we used the section constraint (3.8) to introduce the commutators in the first line. Substituting into (4.37), the first equation is contracted with $\eta_{\delta\lambda} T^\delta$ and the second one with $\eta_{-k\delta\lambda} T^\delta$ and summed over k . It is then useful to rewrite such expressions in terms of the (rescaled) level 2 coset generators acting on triple tensor products [17]

$$\overset{12}{C}_m = -\eta_{m\alpha\beta} T^\alpha \otimes T^\beta \otimes \mathbb{K}, \quad \overset{13}{C}_m = -\eta_{m\alpha\delta} T^\alpha \otimes \mathbb{K} \otimes T^\delta, \quad \overset{23}{C}_m = -\eta_{m\beta\delta} \mathbb{K} \otimes T^\beta \otimes T^\delta, \quad (\text{C.3})$$

which satisfy several useful relations given in equations (2.24)–(2.26) of [17]. We can then write, using the notation $\Sigma^{(k)} = |\Sigma^{(k)} \rangle \langle \pi|$ even if $\Sigma^{(k)}$ is not a tensor product

$$\begin{aligned} \eta_{-k\delta\lambda} \Delta_\Lambda \text{Tr}(\mathcal{G}^{(k)} T^\lambda) \otimes T^\delta &= \delta_1^k \langle \partial_\Lambda | \otimes \langle \partial_\Lambda | \left[\overset{23}{C}_{-1}, \overset{12}{C}_1 \right] | \Lambda \rangle \otimes | \mathcal{F} \rangle + \langle \pi | \otimes \langle \partial_\Sigma | \left[\overset{23}{C}_{-k}, \overset{12}{C}_0 \right] | \Sigma^{(k)} \rangle \otimes | \mathcal{F} \rangle \\ &\quad + \langle \pi | \otimes \langle \partial_\Sigma | \left(k \overset{23}{C}_{-k} + (1-k) \overset{13}{C}_{-k} \right) | \Sigma^{(k)} \rangle \otimes | \mathcal{F} \rangle \end{aligned} \quad (\text{C.4})$$

and we now use equations (2.24) and (2.26) of [17] (which can be proved using (A.7)) to write

$$[\overset{23}{C}_{-1}, \overset{12}{C}_1] = \overset{13}{C}_0 - \overset{12}{C}_0 - \overset{23}{C}_0 + [\overset{23}{C}_0, \overset{12}{C}_0] \quad (\text{C.5})$$

$$[\overset{23}{C}_{-k}, \overset{12}{C}_0] = -k \overset{13}{C}_0 - k \overset{12}{C}_0 + k \overset{23}{C}_0 + [\overset{23}{C}_0, \overset{12}{C}_{-k}]. \quad (\text{C.6})$$

It is then straightforward to arrive at (4.51) by expanding back the coset generators, repeatedly using the section constraints and adding up all variations into (4.37).

Let us now look at the non-covariant variation of $\widehat{\mathbb{F}}_m^{\gamma}$ for generic $m \in \mathbb{Z}$. We will use the fact that \mathcal{S}_m^{γ} can be expanded in a ρ and ϕ_n dependent series of constant \mathcal{S}_n operators with $n \leq m$. Thus, we compute the non-covariant variation of each term in such a series (using $\mathcal{S}_n(L_p) = L_{p+n}$)

$$\widehat{\mathbb{F}}_n = -\langle \partial_{\mathcal{F}} | L_n | \mathcal{F} \rangle - \sum_{k=1}^{\infty} \text{Tr}(\mathcal{G}^{(k)} L_{n-k}), \quad (\text{C.7})$$

and add them up at the end. We can still use (C.1) and (C.2) by setting $T^{\lambda} \rightarrow L_n$ for the former and $T^{\lambda} \rightarrow L_{n-k}$ for the latter. Looking at the first term of the latter for $k=1$, we can first focus at the commutator term when T^{β} is along the loop components. We use the identity

$$[L_{n-1}, T_p^A] = [L_n, \mathcal{S}_{-1}(T_p^A)] - \mathcal{S}_{n-1}(T_p^A) \quad (\text{C.8})$$

which, combined with (2.9), gives us

$$\begin{aligned} & \eta_{1\alpha\beta} \langle \partial_{\Lambda} | T^{\beta} | \Lambda \rangle \langle \partial_{\Lambda} | [L_{n-1}, T^{\alpha}] | \mathcal{F} \rangle \\ &= \eta_{\alpha\beta} \langle \partial_{\Lambda} | T^{\alpha} | \Lambda \rangle \langle \partial_{\Lambda} | [L_n, T^{\beta}] | \mathcal{F} \rangle - \eta_{m\alpha\beta} \langle \partial_{\Lambda} | T^{\alpha} | \Lambda \rangle \langle \partial_{\Lambda} | T^{\beta} | \mathcal{F} \rangle \\ &+ \langle \partial_{\Lambda} | \Lambda \rangle \langle \partial_{\Lambda} | L_n | \mathcal{F} \rangle - \langle \partial_{\Lambda} | L_n | \Lambda \rangle \langle \partial_{\Lambda} | \mathcal{F} \rangle \end{aligned} \quad (\text{C.9})$$

where we added and subtracted the $\mathbb{K} \otimes \mathbf{vir}$ components of each bilinear form to make the identity hold along all components. This is the only contribution of order $(\langle \partial_{\Lambda} |)^2$ coming from $\Delta_{\Lambda} \mathcal{G}^{(k)}$, as the only other ones with two derivatives acting on $|\Lambda\rangle$ come from (C.1). Then, putting together all components of order $(\langle \partial_{\Lambda} |)^2$ we find

$$\Delta_{\Lambda} \widehat{\mathbb{F}}_n = \langle \partial_{\Lambda} | \mathcal{S}_n(L_0) | \Lambda \rangle \langle \partial_{\Lambda} | \mathcal{F} \rangle + \eta_{m\alpha\beta} \langle \partial_{\Lambda} | T^{\alpha} | \Lambda \rangle \langle \partial_{\Lambda} | T^{\beta} | \mathcal{F} \rangle + \mathcal{O}(\langle \partial_{\Sigma} |) \quad (\text{C.10})$$

where in the first term we have highlighted the shift operator coming from writing \mathcal{S}_m^{γ} as a series. With similar steps this expression is completed with the Σ dependent terms

$$\mathcal{O}(\langle \partial_{\Sigma} |) = \sum_{k=1}^{\infty} \text{Tr}(\Sigma^{(k)} \mathcal{S}_n(L_{-k})) \langle \partial_{\Sigma} | \mathcal{F} \rangle + n \sum_{k=1}^{\infty} \eta_{n-k\alpha\beta} \text{Tr}(\Sigma^{(k)} T^{\alpha}) \langle \partial_{\Sigma} | T^{\beta} | \mathcal{F} \rangle. \quad (\text{C.11})$$

We have again highlighted the shift operator \mathcal{S}_n . The second terms of the last two expressions vanish by the section constraints (3.9) if $n < 1$. When $n = 1$, only terms proportional to $\Sigma^{(1)}$ contribute, and using the section constraint we simplify them to

$$\eta_{0\alpha\beta} \text{Tr}(\Sigma^{(1)} T^{\alpha}) \langle \partial_{\Sigma} | T^{\beta} | \mathcal{F} \rangle = \langle \partial_{\Sigma} | \Sigma^{(1)} | \mathcal{F} \rangle - \text{Tr}(\Sigma^{(1)}) \langle \partial_{\Sigma} | \mathcal{F} \rangle. \quad (\text{C.12})$$

Recalling that we are really interested in the non-covariant variation of $\widehat{\mathbb{F}}_m^\gamma$, which is a series in the objects we just varied, for $n \leq m$, we find

$$\Delta_\Lambda \widehat{\mathbb{F}}_m^\gamma = \langle \partial_\Lambda | \mathcal{S}_m^\gamma(L_0) | \Lambda \rangle \langle \partial_\Lambda | \mathcal{F} \rangle + \sum_{k=1}^{\infty} \text{Tr}(\Sigma^{(k)} \mathcal{S}_m^\gamma(L_{-k})) \langle \partial_\Sigma | \mathcal{F} \rangle, \quad m < 1, \quad (\text{C.13})$$

and we recognise the definition (4.42) for $\widehat{\mathbb{A}}_m^\gamma$. When $m = 1$, from (4.9) we see that (C.12) contributes with an overall ρ^{-1} factor. In total we have

$$\Delta_\Lambda \widehat{\mathbb{F}}_m^\gamma = -\widehat{\mathbb{A}}_m^\gamma \langle \partial_{\Lambda, \Sigma} | \mathcal{F} \rangle + \delta_1^m \rho^{-1} (\langle \partial_\Sigma | \Sigma^{(1)} - \text{Tr}(\Sigma^{(1)}) \langle \partial_\Sigma | \rangle | \mathcal{F} \rangle), \quad m \leq 1. \quad (\text{C.14})$$

This reproduces (4.52) in particular. Combining this result with (4.51) and (4.45), (4.53) is also readily found. Finally, notice that for $m > 1$ (C.14) is modified by the second terms in (C.10) and (C.11) which contribute in forms that cannot be simplified by the section constraints. As a result, the expression contracting $|\mathcal{F}\rangle$ in $\Delta_\Lambda \widehat{\mathbb{F}}_m^\gamma$ is not anymore directly subject to the section constraint. This means that such non-covariant variations appear in the transformation properties of $\mathfrak{X}^{(m)}$ for $m > 1$ and that they cannot be reabsorbed by (re)defining the variation of the term $\langle \chi_m^\gamma | \mathcal{F} \rangle$.

Let us now prove (4.129). The idea is that $\mathbb{D}\delta\mathbb{A} = (d - \delta_\mathbb{A})\delta\mathbb{A}$ where $\delta_\mathbb{A}$ is the transformation under generalised diffeomorphisms of $\delta\mathbb{A}$, with \mathbb{A} as parameter. Using the definition (4.42) of the operator $\widehat{(\)}_m^\gamma$ we then write

$$\widehat{(\mathbb{D}\delta\mathbb{A})}_m^\gamma = -\langle \partial_\mathbb{A} + \partial_{\delta\mathbb{A}} | \mathcal{S}_m^\gamma(L_0) | (d - \delta_\mathbb{A})\delta\mathbb{A} \rangle - \sum_{k=1}^{\infty} \text{Tr} [((d - \delta_\mathbb{A})\delta B^{(k)}) \mathcal{S}_m^\gamma(L_{-k})]. \quad (\text{C.15})$$

We have included here the variation of the $B^{(k)}$ fields to be more general, as it does not complicate the proof. The operator $d - \delta_\mathbb{A}$ commutes with the partial derivatives in the first term so that, in order to bring it out of $\widehat{(\)}_m^\gamma$, we just need to add and remove its action on the shift operators. Using (4.8) and covariance under generalised diffeomorphisms of the coset representative we are led to compute

$$-\langle \partial_{\delta\mathbb{A}} | (\mathcal{D}\mathcal{S}_m^\gamma(L_0)) | \delta\mathbb{A} \rangle - \sum_{k=1}^{\infty} \text{Tr} [\delta B^{(k)} \mathcal{D}\mathcal{S}_m^\gamma(L_{-k})], \quad (\text{C.16})$$

which must be subtracted from $(d - \delta_\mathbb{A})\widehat{\delta\mathbb{A}}_m^\gamma$. Each derivative of the shift operators reads (with $k = 0$ for the first term)

$$\begin{aligned} & \mathcal{D}\mathcal{S}_0(\Gamma^{-1}\mathcal{S}_m(\Gamma L_{-k}\Gamma^{-1})\Gamma) \\ &= \mathcal{S}_0(\Gamma^{-1}\mathcal{S}_m([\mathcal{D}\Gamma\Gamma^{-1}, \Gamma L_{-k}\Gamma^{-1}])\Gamma) - \mathcal{S}_0(\Gamma^{-1}[\mathcal{D}\Gamma\Gamma^{-1}, \mathcal{S}_m(\Gamma L_{-k}\Gamma^{-1})]\Gamma) \\ &= m \sum_{p=0}^{\infty} (\mathcal{D}\Gamma\Gamma^{-1})_{-p} \mathcal{S}_{m-p}^\gamma(L_{-k}), \end{aligned} \quad (\text{C.17})$$

where we used (B.2) in the last step. Plugging this back into (C.16) we reproduce (4.129).

C.2 Some details on \mathfrak{vir}^- gauge fixing

The gauge-fixing of \mathfrak{vir}^- in the Virasoro-extended formulation was considered in Section 4.5. We here give some additional details.

$c_{\mathfrak{vir}}$ dependent couplings

We first reconsider the $c_{\mathfrak{vir}}$ dependent terms in (4.57) and show explicitly that after we gauge-fix $\phi_m \rightarrow 0$, $m \geq 2$ they cancel out. We begin with the term coming from $\omega^{\alpha\beta} \mathfrak{J}_\alpha \mathfrak{J}_\beta^{(1)}$. This is (minus) the contribution of the \mathfrak{vir} components of \mathfrak{J} to the central charge term in the commutator $[\mathfrak{J}, \mathfrak{J}^{(1)}]$. The latter can be rewritten as

$$-\frac{1}{2}[\mathfrak{J}, \mathfrak{J}^{(1)}]_{\mathbb{K}} = -2[\mathcal{P}, \mathcal{P}^{(1)}]_{\mathbb{K}} - 2\omega^\alpha (\mathcal{V}^{-1}) f^{\beta\delta}{}_\alpha \mathcal{P}_\beta \mathcal{P}_\delta^{(1)}. \quad (\text{C.18})$$

We are interested in the terms quadratic in the \mathfrak{vir} components of \mathcal{P} , in which case the cocycle term does not contribute because conjugation by ρ^{L_0} and $e^{\tilde{\rho}L_{-1}}$ does not generate central terms. The former term then reads

$$-2[\mathcal{P}, \mathcal{P}^{(1)}]_{\mathbb{K}} = -\frac{c_{\mathfrak{vir}}}{6} \sum_{n \in \mathbb{Z}} (n^3 - n) \mathbb{P}_n \mathbb{P}_{n+1} + \dots, \quad (\text{C.19})$$

where the dots denote terms dependent on the loop components of \mathcal{P} . Using (4.61) and the fact that \mathbb{P}_n only contributes for $|n| \geq 2$, we then arrive at

$$\begin{aligned} -\frac{1}{2}[\mathfrak{J}, \mathfrak{J}^{(1)}]_{\mathbb{K}} &= -\frac{c_{\mathfrak{vir}}}{24} (n^3 - n) \mathbb{J}_n \mathbb{J}_{-n-1} + \dots \\ &= -\frac{c_{\mathfrak{vir}}}{12} \sum_{m=0}^{\infty} [\mathbb{A}]_{-m} [\Gamma L_{-m} \Gamma^{-1} + \text{h.c.}, \mathcal{P}^{(1)}]_{\mathbb{K}} + \dots \end{aligned} \quad (\text{C.20})$$

Adding to this expression also the term proportional to $[\mathbb{A}]_n \mathbb{J}_{-n-1}$ coming from the cocycle in (4.58), we arrive at the identity

$$\frac{c_{\mathfrak{vir}}}{24} \sum_{n \in \mathbb{Z}} (n^3 - n) (\mathbb{J}_n + 2[\mathbb{A}]_n) \mathbb{J}_{-n-1} = -\frac{c_{\mathfrak{vir}}}{12} \sum_{m=0}^{\infty} [\mathbb{A}]_{-m} [\Gamma L_{-m} \Gamma^{-1} - \text{h.c.}, \mathcal{P}^{(1)}]_{\mathbb{K}}. \quad (\text{C.21})$$

Using (4.62) it is then straightforward to show that this expression matches the left-hand side of (4.68) and hence cancels out with the last term in the first line of (4.57), removing all $c_{\mathfrak{vir}}$ dependent couplings from the pseudo-Lagrangian.

Matching $\langle \chi |$ transformation

We now proceed to proving that (4.79) agrees with the transformation of $\langle \chi |$ in [1]. The $\hat{\epsilon}_8$ invariant bilinear forms (2.7) are not invariant under generic \mathfrak{vir} transformations. This means in particular that in an expression like

$$\eta_{m\alpha\beta} \mathcal{M}^{-1} (T^\alpha)^\dagger \mathcal{M} \otimes T^\beta \quad (\text{C.22})$$

we cannot simply bring the \mathcal{M} conjugation through to the other factor. Formally writing

$$\mathcal{M} = \Gamma^\dagger \Gamma g_{\mathcal{M}}, \quad g_{\mathcal{M}} \in \widehat{E}_8, \quad (\text{C.23})$$

we see that the problem lies in $\Gamma^\dagger \Gamma$ which involves exponentials of all \mathfrak{vir} generators. When we gauge fix $\phi_m \rightarrow 0$, $m \geq 2$, however, only exponentials of L_0 , L_{-1} and L_1 are left, and parameterise a well-defined $SL(2)$ element

$$\Gamma^\dagger \Gamma \rightarrow m \in SL(2), \quad (\text{C.24})$$

which acts on w by fractional linear transformations. In this case it is possible to bring the \mathcal{M} conjugation through in (C.25) and this was done in the appendix of [1]. We summarise here some results:

$$\eta_{-1\alpha\beta} \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} \otimes T^\beta = \frac{1}{\rho^2} (\eta_{1\alpha\beta} - 2\tilde{\rho} \eta_{\alpha\beta} + \tilde{\rho}^2 \eta_{-1\alpha\beta}) T^\alpha \otimes \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M}, \quad (\text{C.25})$$

$$\eta_{\alpha\beta} \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} \otimes T^\beta = \frac{1}{\rho^2} (\tilde{\rho} \eta_{1\alpha\beta} + (\rho^2 - 2\tilde{\rho}^2) \eta_{\alpha\beta} + (\tilde{\rho}^3 - \rho^2 \tilde{\rho}) \eta_{-1\alpha\beta}) T^\alpha \otimes \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M},$$

$$\eta_{1\alpha\beta} \mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} \otimes T^\beta = \frac{1}{\rho^2} (\tilde{\rho}^2 \eta_{1\alpha\beta} + 2(\rho^2 \tilde{\rho} - \tilde{\rho}^3) \eta_{\alpha\beta} + (\rho^2 - \tilde{\rho}^2)^2 \eta_{-1\alpha\beta}) T^\alpha \otimes \mathcal{M}^{-1} T^{\beta\dagger} \mathcal{M}.$$

Let us then prove that (4.79) matches the non-covariant variation of $\langle \chi |$ derived in [1], when gauge fixed to $\phi_m = 0$, $m \geq 2$ and $\phi_1 = \tilde{\rho}$. To do so, we must expand $\widehat{\mathcal{L}}_{-1}^\gamma$ and ${}^{\mathcal{M}}\widehat{\mathcal{L}}_{-1}^\gamma$. The former is expanded as in equation (4.42). The latter is defined as in (4.45), with $\mathbb{F} \rightarrow \mathbb{A}$:

$${}^{\mathcal{M}}\widehat{\mathcal{L}}_{-1}^\gamma = \widehat{\mathcal{L}}_1^\gamma + \omega^\alpha(\mathcal{M}) \left([\mathbb{A}]_\beta \mathcal{S}_1^\gamma(T^\beta) \right)_\alpha. \quad (\text{C.26})$$

Again, the expansion of the first term on the right-hand side follows (4.42). Let us now look at the cocycle term. We expand the shift operator as in (4.74) and write $[\mathbb{A}]_\beta$ explicitly using (3.5) (recall that $\Sigma^{(m)} = 0$ for $m \geq 2$ because we are gauge-fixed):

$$\omega^\alpha(\mathcal{M}) \left([\mathbb{A}]_\beta \mathcal{S}_1^\gamma(T^\beta) \right)_\alpha = \omega^\alpha(\mathcal{M}) \frac{1}{\rho} \left((\eta_{1\alpha\beta} - \tilde{\rho} \eta_{\alpha\beta}) \langle \partial_\Lambda | T^\beta | \Lambda \rangle + (\eta_{\alpha\beta} - \tilde{\rho} \eta_{-1\alpha\beta}) \text{Tr}(\Sigma^{(1)} T^\beta) \right), \quad (\text{C.27})$$

where we also used (2.9) and $\omega^K(\mathcal{M}) = 0$. We must now expand the contraction of the cocycle with the invariant bilinears. First, we open the cocycle and write

$$\begin{aligned} \omega^\alpha(\mathcal{M}) \eta_{k\alpha\beta} \mathbb{K} \otimes T^\beta &= \eta_{k\alpha\beta} \left(\mathcal{M}^{-1} \mathcal{S}_0(T^{\alpha\dagger}) \mathcal{M} - \mathcal{S}_0(\mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M}) \right) \otimes T^\beta \\ &= \eta_{k\alpha\beta} \left(\mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M} - \mathcal{S}_0(\mathcal{M}^{-1} T^{\alpha\dagger} \mathcal{M}) \right) \otimes T^\beta + \mathbb{K} \otimes L_k \end{aligned} \quad (\text{C.28})$$

Using then (C.25) on this expression for $k = -1, 0, 1$, we see that the first term becomes a sum of terms of the form $\eta_{k\mathbb{K}\beta}$ for $k = -1, 0, 1$. The last term instead stays as it is. Adding everything up we then find

$$\begin{aligned} \omega^\alpha(\mathcal{M}) \left([\mathbb{A}]_\beta \mathcal{S}_1^\gamma(T^\beta) \right)_\alpha &= \langle \partial_\Lambda | \mathcal{S}_1^\gamma(L_0) | \Lambda \rangle - \frac{1}{\rho} \langle \partial_\Lambda | \mathcal{M}^{-1} (\tilde{\rho} L_0 + (\rho^2 - \tilde{\rho}^2) L_1) \mathcal{M} | \Lambda \rangle \\ &\quad + \text{Tr} \left(\Sigma^{(1)} \mathcal{S}_1^\gamma(L_{-1}) \right) - \frac{1}{\rho} \text{Tr} \left(\Sigma^{(1)} \mathcal{M}^{-1} (L_0 - \tilde{\rho} L_1) \mathcal{M} \right). \end{aligned} \quad (\text{C.29})$$

Combining this expression with $\widehat{\Lambda}_1^\gamma$ and $\widehat{\Lambda}_{-1}^\gamma$, expanding the shift operators and plugging everything into (4.79) we arrive at

$$\begin{aligned} \Delta_\Lambda \langle \chi | = & - \langle \partial_\Lambda | \left(\sum_{n=0}^{\infty} \tilde{\rho}^n L_{-1-n} + \mathcal{M}^{-1} \left(\frac{\tilde{\rho}}{\rho^2} L_0 + \left(1 - \frac{\tilde{\rho}^2}{\rho^2} \right) L_1 \right) \mathcal{M} \right) | \Lambda \rangle \langle \partial_\Lambda | - \frac{\tilde{\rho}}{\rho^2} \langle \partial_\Lambda | \Lambda \rangle \langle \partial_\Lambda | \\ & - \text{Tr} \left[\Sigma^{(1)} \left(\sum_{n=0}^{\infty} \tilde{\rho}^n L_{-2-n} + \mathcal{M}^{-1} \left(\frac{1}{\rho^2} L_0 - \frac{\tilde{\rho}}{\rho^2} L_1 \right) \mathcal{M} \right) \right] \langle \partial_\Sigma | - \frac{1}{\rho^2} \langle \partial_\Sigma | \Sigma^{(1)}, \end{aligned} \quad (\text{C.30})$$

which reproduces equations (4.67) and (4.92) of [1].

C.3 Series regularisation and integrability conditions

In computing the equations of motion in the extended formalism of Section 4.6, we have found that for some of them, using twisted self-duality to eliminate all dual fields leads to divergent sums that need to be regularised. This necessity is essentially due to the fact that commutators of \mathcal{P} and \mathcal{Q} cannot be reduced to a finite expression using only the E_8 commutation relations, especially when \mathcal{S}_m operators are involved, because \mathcal{P} and \mathcal{Q} are infinitely extended over both the positive and negative loop and \mathfrak{vir} levels. We give more details on this in Appendix F. To motivate the regularisation procedure we adopt in this paper, we can look at the integrability condition for twisted self-duality (2.46) in purely two-dimensional (super)gravity, which we proved in Section 2.3, but formulated now only in terms of P and Q as defined in (2.45) and their Maurer–Cartan equation. The matching of integrability conditions of (2.46) we give in Section 2.3 does not involve any regularisation because it is based on the Maurer–Cartan form $d\mathcal{V}\mathcal{V}^{-1}$ which takes values only along non-positive loop and \mathfrak{vir} levels. Then, the only commutators one encounters are those needed to apply the Maurer–Cartan equation to the linear system (2.22) (more precisely, its w expansion with generic $\gamma(w)$) and they are all finite. Writing the integrability conditions in terms of P and Q instead, we encounter commutators that are not finite *per se*, and we will see that in order to reproduce the result based on $d\mathcal{V}\mathcal{V}^{-1}$ the regularisations employed in Section 4 become necessary.

The Maurer–Cartan equation for P reads $dP - [Q, P] = 0$. Applying an \mathcal{S}_1 operator to this expression, using (B.2) and then (2.47), we arrive at

$$\mathcal{S}_0(d \star P - [Q, \star P]) = 0. \quad (\text{C.31})$$

The \mathfrak{e}_8 component of this equation was already computed, in ExFT, in Section 4.6. Let us check that up to a regularisation, (C.31) reproduces (2.15), (2.16) and nothing else. We will need that in the triangular gauge (2.30) and by twisted self-duality

$$P_A^m = \star^m P_A^0, \quad Q_A^m = -\text{sgn}(m) \star^m P_A^0, \quad m \neq 0, \quad (\text{C.32})$$

$$P_m = \star^m (d\Gamma\Gamma^{-1})_0, \quad Q_m = -\text{sgn}(m) \star^m (d\Gamma\Gamma^{-1})_0, \quad Q_0 = 0. \quad (\text{C.33})$$

Let us take the L_{-p} , $p \geq 0$ component of this equation which, using twisted self-duality, reads

$$\begin{aligned}
& 2d\star^{p+1}(d\Gamma\Gamma^{-1})_0 + [(d\Gamma\Gamma^{-1})^\dagger, \star(d\Gamma\Gamma^{-1})]|_{L_{-p}} \tag{C.34} \\
&= 2d\star^{p+1}(d\Gamma\Gamma^{-1})_0 + \sum_{n \geq 0} (2m+p) (d\Gamma\Gamma^{-1})_{-m} \star (d\Gamma\Gamma^{-1})_{-p-m} \\
&= 2d\star^{p+1}(d\Gamma\Gamma^{-1})_0 + (d\Gamma\Gamma^{-1})_0 \star^{p+1} (d\Gamma\Gamma^{-1})_0 \left(2p + 8 \sum_{m=1}^{\infty} (-1)^m m + 4p \sum_{m=1}^{\infty} (-1)^m \right) \\
&= 2d\star^{p+1}(d\Gamma\Gamma^{-1})_0 - 2(d\Gamma\Gamma^{-1})_0 \star^{p+1} (d\Gamma\Gamma^{-1})_0 \\
&= -\rho^{-1}d \star d\rho \text{ if } p \text{ even, } 0 \text{ otherwise,}
\end{aligned}$$

which recovers the free equation of motion for ρ . We have resummed the divergent series $\sum_{m=1}^{\infty} (-1)^m m \rightarrow -1/4$ and $\sum_{m=1}^{\infty} (-1)^m \rightarrow -1/2$ using the $z \rightarrow 1$ limit of a geometric series $\sum_{n=1}^{\infty} (-z)^n p(n)$ for $p(n)$ a polynomial in n . Now for the loop part,

$$\begin{aligned}
& d\star P|_{T_p^C} - [Q, \star P]|_{T_p^C} \tag{C.35} \\
&= d\star^{p+1} P_C^0 - f^{AB}{}_C \sum_{n \in \mathbb{Z}} Q_A^n \star P_B^{p-n} + \sum_{n \in \mathbb{Z}} n (Q_{p-n} \star P_C^n - Q_C^n \star P_{p-n}).
\end{aligned}$$

We redefine $n \rightarrow p - n$ in the second-to-last term. Using that, by twisted self-duality, Q_A^n and Q_n are n -odd, while the P are n -even, we find that several terms cancel out and we are left with

$$\begin{aligned}
&= d\star^{p+1} P_C^0 - f^{AB}{}_C Q_A^0 \star^{p+1} P_B^0 - 2 \sum_{n=1}^{\infty} n (Q_C^n \star P_{p-n} + Q_n \star P_C^{p-n}) \tag{C.36} \\
&= d\star^{p+1} P_C^0 - f^{AB}{}_C Q_A^0 \star^{p+1} P_B^0 + 4 \frac{1+(-1)^p}{2} (d\Gamma\Gamma^{-1})_0 \star^{p+1} P_C^0 \sum_{n=1}^{\infty} (-1)^n n \\
&= \begin{cases} dP_C^0 - f^{AB}{}_C Q_A^0 P_B^0 = 0 & p \text{ odd,} \\ \rho^{-1}d(\rho \star P_C^0) - f^{AB}{}_C Q_A^0 \star P_B^0 & p \text{ even,} \end{cases}
\end{aligned}$$

which reproduces (2.16) given $P_A^0 = \mathring{P}_A$ and $Q_A^0 = \mathring{Q}_A$, which is valid because of the triangular $K(E_9)$ gauge. We have again regularised $\sum_{n=1}^{\infty} (-1)^n n \rightarrow -1/4$.

C.4 Matching of scalar equations of motion to E_8 ExFT

As a further motivation for the regularisations employed in Section 4.6 and studied in the previous section, we provide here the matching of the \mathring{V} scalar field equations of motion derived there to those of E_8 exceptional field theory. A full matching of the E_9 theory to the E_8 one has been given at the level of the pseudo-Lagrangians in the minimal formalism of Section 5.

We begin with the π^{0A} component of equation (4.121), looking only at the left-hand side since the scalar potential has already been matched to E_8 exceptional field theory in [1]. We will rewrite the relevant term as

$$\rho(\widehat{D} \star P_A^0 + 2[\underline{\mathbb{F}}_A^{+1}]) \mathring{V} T^A \mathring{V}^{-1}, \tag{C.37}$$

where we have conjugated the expression by the $E_8/Spin(16)$ coset representative \mathring{V} and rescaled by a factor of 2. This means that this expression should be traced with $2\pi_B^0(\mathring{V}T^B\mathring{V}^{-1}) = M^{-1}\delta M$, which is the natural expression for the variation of the E_8 generalised metric M_{AB} .

We will now choose the solution of the section constraint (3.10). With this, we can expand

$$\underline{[A]}_A^1 T^A = \frac{1}{2}\rho(\partial_A w)\mathring{V}T^A\mathring{V}^{-1}, \quad (\text{C.38})$$

$$\underline{[A]}_A^0 T^A = \frac{1}{2}(f_{AB}{}^C\partial_C A^B - B_A + f_{AB}{}^C Y_1^B\partial_C w)\mathring{V}T^A\mathring{V}^{-1}, \quad (\text{C.39})$$

where $T^A \in E_8$ and $w = \langle 0|A \rangle$ (not to be confused with the spectral parameter). In analogy with (5.107) and (5.106), we have defined $A^C = \langle 0|T_{+1}^C|A \rangle$ and $-B_A = \text{Tr}(B^{(1)}T_{-1A})$.

Looking at the first two terms in the expansion (4.122) of $\widehat{\mathcal{D}} \star \mathbf{P}_A^0$, we write

$$\begin{aligned} & (\mathcal{D}(\rho \star \mathbf{P}_A^0) - \rho f^{CD}{}_A \mathbf{Q}_C^0 \star \mathbf{P}_D^0)\mathring{V}^{-1}T^A\mathring{V} \\ &= d(\rho\mathring{V}^{-1} \star \mathbf{P}_A^0 T^A \mathring{V}) - \langle \partial | (|A\rangle \rho \mathring{V}^{-1} \star \mathbf{P}_A^0 T^A \mathring{V}) \\ & \quad + \rho (f_{AB}{}^C \partial_C A^B - B_A + f_{AB}{}^C Y_1^B \partial_C w) [T^A, \mathring{V}^{-1} \star \mathbf{P}_F^0 T^F \mathring{V}]. \end{aligned} \quad (\text{C.40})$$

We now implement the field redefinitions

$$\begin{aligned} A^C &= A^{3D}{}^C - w A_\varphi^C, \\ B_C &= B^{3D}{}_C - w B_{\varphi C} - \rho \star \partial_C w \end{aligned} \quad (\text{C.41})$$

where $A_\mu^{3D}{}^C$ and $B_\mu^{3D}{}_C$ are the x^0 and x^1 components of the E_8 vector fields, while A_φ^C and $B_{\varphi C}$ are their components along the third space-time direction denoted by φ . We do not need a suffix 3D for the latter, but we identify them with

$$A_\varphi^C = Y_1^C, \quad B_{\varphi C} = 2\rho \eta_{CD} \langle \tilde{\chi}_1 | T_{-1}^D | 0 \rangle. \quad (\text{C.42})$$

This was already established in [1]. Using (4.61), we then have that

$$2\mathring{V}^{-1}\mathbf{P}_A^0 T^A \mathring{V} = j^{3D} - w j_\varphi - \rho \star \partial_A w (T^A + M^{-1}T^A M) \quad (\text{C.43})$$

where $j^{3D} = j_\mu^{3D} dx^\mu$ is \mathfrak{e}_8 valued and

$$j_{\widehat{\mu}}^{3D} = (j_\mu^{3D}, j_\varphi) = M^{-1}D_{\widehat{\mu}}M \quad (\text{C.44})$$

is the E_8 scalar field current and $D_{\widehat{\mu}}$ the E_8 ExFT covariant derivative:

$$M^{-1}D_{\widehat{\mu}}M = M^{-1}\partial_{\widehat{\mu}}M - A_{\widehat{\mu}}^{3D}{}^C M^{-1}\partial_C M + (f_{AB}{}^C \partial_C A_{\widehat{\mu}}^{3D}{}^B - B_{\widehat{\mu}A}^{3D})(T^A + M^{-1}(T^A)^\top M). \quad (\text{C.45})$$

We then have that (C.40) becomes, after the field redefinitions,

$$\begin{aligned} & \frac{1}{2}(D - wD_\varphi - (D_\varphi w))(\rho \star j^{3D} - \rho \star w j_\varphi - \rho^2 \star \partial_A w (T^A + M^{-1}T^A M)) \\ & - \frac{1}{2}\rho^2 \partial_A w [T^A, j^{3D} - w j_\varphi - \rho^2 \star \partial_B w (T^B + M^{-1}T^B M)]. \end{aligned} \quad (\text{C.46})$$

The $\partial_A w \star \partial_B w [T^A, T^B]$ term vanishes by the section constraint.

We must now add to the above expression the $[\underline{A}]_A^1$ dependent terms of (4.122) as well as the last one in (C.37). For the latter we compute, applying (C.41),

$$2[\underline{F}]_A^1 \dot{V}^{-1} T^A \dot{V} = \rho \partial_A (\langle 0 | \mathcal{F} \rangle) T^A = \rho (D - w D_\varphi) (\partial_A w) T^A + \rho D_\varphi w \partial_A w T^A \quad (\text{C.47})$$

where w transforms as a scalar (weight 0) under E_8 generalised diffeomorphisms. For the last two terms of (4.122) we also need $\mathcal{D}\rho$ after the redefinitions (C.41):

$$\mathcal{D}\rho = (D - w D_\varphi - (D_\varphi w)) \rho \quad (\text{C.48})$$

so that

$$\begin{aligned} & 2\rho f^{CD} {}_A([\underline{A}]_C^1 - [\underline{A}]^{1C}) \mathbf{P}_D^0 \dot{V}^{-1} T^A \dot{V} + 2\mathcal{D}\rho [\underline{A}]_A^1 \dot{V}^{-1} (T^A + T_A) \dot{V} \\ &= \frac{1}{2} \rho^2 \partial_A w [T^A - M^{-1} T_A M, j^{3D} - w j_\varphi - \rho \star \partial_B w (T^B + M^{-1} T_B M)] \\ & \quad + \rho (D - w D_\varphi - (D_\varphi w)) \rho \partial_A w (T^A + M^{-1} T_A M) \\ & \simeq \rho^2 \partial_A w [T^A, j^{3D} - w j_\varphi - \rho \star \partial_B w (T^B + M^{-1} T_B M)] \\ & \quad + \rho (D - w D_\varphi - (D_\varphi w)) \rho \partial_A w (T^A + M^{-1} T_A M), \end{aligned} \quad (\text{C.49})$$

where in the last step we are using that the whole expression appears traced with $M^{-1} \delta M$. Again, the $\partial_A w \star \partial_B w [T^A, T^B]$ term vanishes by the section constraint. Putting everything together we find

$$\frac{1}{2} (M^{-1} \delta M)^A [(D - w D_\varphi - (D_\varphi w)) (\rho \star j_A^{3D} - \rho \star w j_{\varphi A}) + \rho^3 f_{AB}{}^C M^{BD} \partial_C w \star \partial_D w] + \dots = 0 \quad (\text{C.50})$$

where the dots correspond to the contribution from the E_9 scalar potential.

Comparing with the E_8 ExFT equations of motion, we have that the variation of the kinetic term reads (up to total derivatives, denoting by e the determinant of the 3d vielbein)

$$\begin{aligned} \delta \left(-\frac{1}{4} e g^{\widehat{\mu}\widehat{\nu}} \eta^{AB} j_{\widehat{\mu}A}^{3D} j_{\widehat{\nu}B}^{3D} \right) &= \frac{1}{2} (M^{-1} \delta M)^A D_{\widehat{\mu}} (e g^{\widehat{\mu}\widehat{\nu}} j_{\widehat{\mu}A}^{3D}) \\ &= \frac{1}{2} (M^{-1} \delta M)^A [(D - w D_\varphi - (D_\varphi w)) (\rho \star j_A^{3D} - \rho \star w j_{\varphi A}) + D_\varphi (\rho^{-1} e^{2\sigma}) j_{\varphi A}], \end{aligned} \quad (\text{C.51})$$

where the last term is purely internal (only internal derivatives and no 2d vector fields) and hence is reproduced by the E_9 scalar potential contributions in the ‘...’ above. Finally, the E_8 scalar potential contains one term dependent on the Kaluza–Klein vector w_μ :

$$-e V_{E_8} = \dots + \frac{1}{4} e M^{AB} \partial_A g^{\widehat{\mu}\widehat{\nu}} \partial_B g_{\widehat{\mu}\widehat{\nu}} = \dots - \frac{1}{2} \rho^3 M^{AB} \partial_A w_\mu \partial_B w^\mu. \quad (\text{C.52})$$

Again the dots represent purely internal terms. Variation of this term reproduces exactly the last term in (C.50), proving the claim.

D The E_9 Virasoro constraint

In order to match the Virasoro constraint (4.83) in the Virasoro-extended formulation of the theory with the Euler–Lagrange equation for $\tilde{g}_{\mu\nu}$ in its minimal formulation, one must assume that the section constraint is solved consistently with the E_8 parabolic gauge (5.96) such that B takes the form (5.105). This is always possible since all solutions to the section constraint are E_9 conjugate to this one. Although matching these two equations in this way requires to break E_9 symmetry, both equations are E_9 invariant and they therefore must be equivalent.

The terms in the pseudo-Lagrangian (5.44) that contribute to the Euler–Lagrange equation for $\tilde{g}_{\mu\nu}$ are

$$\mathcal{L}_1 + \frac{1}{2}\varepsilon^{\mu\nu}J_{\mu-1}J_{\nu\kappa} + \frac{1}{2}\rho\varepsilon^{\mu\nu}\varepsilon^{\sigma\rho}\tilde{g}^{\kappa\lambda}\mathcal{D}_\mu\tilde{g}_{\sigma\kappa}\mathcal{D}_\nu\tilde{g}_{\rho\lambda} + \frac{\rho^{-1}}{4}\langle\partial\tilde{g}^{\mu\nu}|\mathcal{M}^{-1}|\partial\tilde{g}_{\mu\nu}\rangle. \quad (\text{D.1})$$

By construction, the variation of \mathcal{L}_1 involves infinitely many terms, with

$$\delta\mathcal{L}_1 = \frac{1}{2}\rho^{-1}\delta\tilde{g}^{\mu\nu}\left(\text{Tr}\left[(\mathcal{M}^{-1}\mathcal{S}_1(\mathcal{J}_\mu^b)^\dagger\mathcal{M} + \chi_\mu^b)B_\nu\right] - \frac{1}{2}\eta_{\alpha\beta}\text{Tr}[T^\alpha B_\mu]\text{Tr}[\mathcal{M}^{-1}T^{\beta\dagger}\mathcal{M}B_\nu]\right). \quad (\text{D.2})$$

But if we assume (5.105) and therefore (5.106), one can use the semi-flat formulation introduced in Section 5.4 with the same steps to show that the variation of the pseudo-Lagrangian (5.44) with respect to $\tilde{g}_{\mu\nu}$ reduces to the variation of

$$\begin{aligned} \mathcal{Z} - \frac{\rho}{4}\eta_{AB}\tilde{g}^{\mu\nu}(\eta^{AC}\tilde{J}_{\mu C}^0 + M^{AC}B_{\mu C})(\eta^{BD}\tilde{J}_{\nu D}^0 + M^{BD}B_{\nu D}) + 2\tilde{g}^{\mu\nu}\mathcal{D}_\mu\rho\mathcal{D}_\nu\sigma + \mathcal{D}_\mu\rho\mathcal{D}_\nu\tilde{g}^{\mu\nu} \\ + \frac{1}{2}\rho\varepsilon^{\mu\nu}\varepsilon^{\sigma\rho}\tilde{g}^{\kappa\lambda}\mathcal{D}_\mu\tilde{g}_{\sigma\kappa}\mathcal{D}_\nu\tilde{g}_{\rho\lambda} + \frac{\rho^{-1}}{4}\langle\partial\tilde{g}^{\mu\nu}|\mathcal{M}^{-1}|\partial\tilde{g}_{\mu\nu}\rangle, \end{aligned} \quad (\text{D.3})$$

where \mathcal{Z} is the term (5.127) that vanishes upon using the duality equation (5.35). One can therefore ignore \mathcal{Z} in deriving the Euler–Lagrange equation for $\tilde{g}^{\mu\nu}$ to get a manifestly finite result.

Using the same solution to the section constraint for the fields $B^{(k)}$ in the extended formulation and setting all the Virasoro fields to zero, one obtains that

$$2\mathbb{P}_{\mu A}^0\mathring{V}^{-1}T_0^A\mathring{V} = (\tilde{J}_{\mu A}^0 + \eta_{AC}M^{BC}B_{\mu B})T_0^A, \quad (\text{D.4})$$

where $B_A = -\text{Tr}(B^{(1)}T_{-1A})$. Substituting this result in the Euler–Lagrange equation (D.3) for $\tilde{g}_{\mu\nu}$ evaluated at $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ gives precisely (4.83) at vanishing Virasoro fields.

E On symmetries of pseudo-Lagrangians

In this appendix, we consider in some generality the definition of symmetries of a pseudo-Lagrangian. This discussion is relevant to the invariance of the theory under external diffeomorphisms as discussed in Section 5.3. Denote the fields of a theory collectively as ϕ^I , with a pseudo-action $S = \int \mathcal{L}$, its associated Euler–Lagrange equations $\mathcal{E}_I = \frac{\delta S}{\delta\phi^I} = 0$, and assume a separate set of duality equations $\mathcal{E}_A = 0$.

Having a symmetry with parameter ξ means that the equations of motion transform into each other according to

$$\delta_\xi \mathcal{E}_I = A_I^J(\xi, \phi) \mathcal{E}_J + B_I^B(\xi, \phi) \mathcal{E}_B, \quad \delta_\xi \mathcal{E}_A = C_A^J(\xi, \phi) \mathcal{E}_J + D_A^B(\xi, \phi) \mathcal{E}_B. \quad (\text{E.1})$$

This is certainly the case if both the duality equations and the pseudo-action are invariant under that symmetry, in which case $B_I^B = C_A^J = 0$. But we want to define a minimal requirement. To this purpose consider

$$\begin{aligned} \frac{\delta}{\delta \phi^I} \delta_\xi S &= \frac{\delta}{\delta \phi^I} \int \delta_\xi \phi^J \frac{\delta S}{\delta \phi^J} \\ &= \int \delta_\xi \phi^J \frac{\delta}{\delta \phi^J} \frac{\delta S}{\delta \phi^I} + \int \frac{\delta \delta_\xi \phi^J}{\delta \phi^I} \frac{\delta S}{\delta \phi^J} \\ &= \delta_\xi \mathcal{E}_I + \int \frac{\delta \delta_\xi \phi^J}{\delta \phi^I} \mathcal{E}_J. \end{aligned} \quad (\text{E.2})$$

One re-obtains that the Euler–Lagrange equations transform into themselves if $\delta_\xi S = 0$. However, having a symmetry as defined by (E.1), only requires the weaker condition that $\frac{\delta}{\delta \phi^I} \delta_\xi S$ must be proportional to the equations of motion \mathcal{E}_I and the duality equations \mathcal{E}_A . For this to be the case it is sufficient for $\delta_\xi S$ to be quadratic in the duality equations

$$\delta_\xi S = \int \left(\alpha^{IJ}(\xi, \phi) \mathcal{E}_I \mathcal{E}_J + \beta^{IB}(\xi, \phi) \mathcal{E}_I \mathcal{E}_B + \gamma^{AB}(\xi, \phi) \mathcal{E}_A \mathcal{E}_B \right). \quad (\text{E.3})$$

One can always redefine the symmetry such that the two first terms vanish, with

$$\delta'_\xi \phi^I = \delta_\xi \phi^I - \alpha^{IJ}(\xi, \phi) \mathcal{E}_J - \beta^{IB}(\xi, \phi) \mathcal{E}_B \quad (\text{E.4})$$

but the last term cannot be eliminated in general. One has then

$$B_I^B(\xi, \phi) = \int \left(\frac{\delta \gamma^{AB}(\xi, \phi)}{\delta \phi^I} \mathcal{E}_A + 2\gamma^{AB}(\xi, \phi) \frac{\delta}{\delta \phi^I} \mathcal{E}_A \right) \quad (\text{E.5})$$

and we get that $\gamma^{AB} \neq 0$ if $B_I^B(\xi, \phi)$ does not vanish. So if the Euler–Lagrange equations do not transform into each other under a symmetry, but also mix with the duality equations, then the pseudo-Lagrangian is only invariant up to terms quadratic in the duality equations. However, it is not sufficient that the action is invariant up to terms linear in the duality equations.

F On Kac–Moody groups

For a given Kac–Moody Lie algebra there are different definitions of an associated Kac–Moody group and representations, see [75, 76]. In the case of centrally extended loop groups, the two standard notions of a minimal group and maximal (or completed) group can be described as follows.

The minimal group, denoted \widehat{E}_8^m consists of maps from the complex plane into the group E_8 , where the maps are restricted to Laurent polynomials around infinity (or the origin). In terms of the spectral $w \in \mathbb{C}$, one starts with elements $g^m(w)$ that can be written as

$$g^m(w) = \sum_{\ell=\ell_1}^{\ell_2} w^{-\ell} g_\ell \quad (\text{F.1})$$

with $\ell_1 \leq \ell_2$ (finite integers) and such that for each value of w one has $g^m(w) \in E_8$ (for a chosen matrix representation). This space of maps forms a group (under pointwise multiplication) and can be centrally extended [77]. The resulting group is \widehat{E}_8^m .

The completed group \widehat{E}_8^{c-} replaces Laurent polynomials by formal Laurent series around infinity. In terms of an expansion in powers of w , this means that we are allowing arbitrarily negative powers of w

$$g^{c-}(w) = \sum_{\ell=\ell_1}^{\infty} w^{-\ell} g_\ell. \quad (\text{F.2})$$

The $-$ indicates both that the powers are arbitrarily negative and that this corresponds to a completion in the positive Borel direction, which is consistent with our choice of letting $T_m^A \cong w^m T^A$ correspond to negative (positive) roots for $m > 0$ ($m < 0$), respectively. We also require $g^{c-}(w) \in E_8$ as a group element in the field of formal Laurent series in w . As the conditions for being in the group are algebraic conditions on the matrix entries (such as $\det(g^{c-}(w)) = 1$), these are also expressible in terms of formal Laurent series and can be imposed without requiring the series to converge. Since formal Laurent series form a field, elements of this type form a group and can again be centrally extended.⁴² The logarithmic derivative $dg^{c-}(w) (g^{c-}(w))^{-1}$ is an element of the completed Lie algebra $\widehat{\mathfrak{e}}_8^{c-}$ that consists of \mathfrak{e}_8 -valued Laurent series around infinity with central extension. From the point of view of the root space, we allow for an infinite linear combination of negative root generators in $\widehat{\mathfrak{e}}_8^{c-}$ but only a finite linear combination of positive root generators. It possible to define a Lie bracket on $\widehat{\mathfrak{e}}_8^{c-}$.

The explicit coset representative that is given in (2.30) clearly belongs to this completed group and thus we take for the scalar fields the group \widehat{E}_8^{c-} . Elements of this group can act on highest weight modules $\overline{R(\Lambda)_h}$ since the exponentials of positive highest generators all are finite sums (rather than series) due to the existence of a highest weight. When we try to act on a lowest weight modules $R(\Lambda)_h$ the exponentials do not terminate but the computation for any given weight space is a finite sum so that one could consider infinite linear combinations in the space $R(\Lambda)_h$, which is sometimes called the completed module [76] and which we denote by $R(\Lambda)_h^{c-}$. The (algebraic) dual of highest weight module $\overline{R(\Lambda)_h}$ is a completed lowest weight module $R(\Lambda)_h^{c-}$ and the pairing is invariant under \widehat{E}_8^{c-} .

The Chevalley involution maps the (negatively) completed Lie algebra $\widehat{\mathfrak{e}}_8^{c-}$ to the (positively) completed Lie algebra $\widehat{\mathfrak{e}}_8^{c+}$ since it interchanges negative and positive roots. If $\widehat{\mathfrak{e}}_8^{c-}$ are Laurent

⁴²We note that, when using the Geroch group for generating solutions of the Einstein equations [78], an intermediate version of the loop group is used that is given by meromorphic functions on (covers of) Riemann surfaces.

series around infinity, then $\widehat{\mathfrak{e}}_8^{c+}$ are Laurent series around the origin. Therefore, the projections⁴³

$$\begin{aligned} P(w) &= \frac{1}{2} \left[dV^{c-}(w) (V^{c-}(w))^{-1} + \left(dV^{c-}(w) (V^{c-}(w))^{-1} \right)^\dagger \right], \\ Q(w) &= \frac{1}{2} \left[dV^{c+}(w) (V^{c+}(w))^{-1} - \left(dV^{c-}(w) (V^{c-}(w))^{-1} \right)^\dagger \right], \end{aligned} \quad (\text{F.3})$$

of the Maurer–Cartan form lie in the doubly completed space $\widehat{\mathfrak{e}}_8^{c+-} := \widehat{\mathfrak{e}}_8^{c+} + \widehat{\mathfrak{e}}_8^{c-}$ for $V^{c-}(w) \in \widehat{E}_8^{c-}$. There is no Lie bracket on $\widehat{\mathfrak{e}}_8^{c+-}$ that can be defined by a finite number of operations from the one on \mathfrak{e}_8 since this would require multiplying Laurent series around infinity with Laurent series around zero which is not a well-defined operation. The completed group \widehat{E}_8^{c-} does not act on $\widehat{\mathfrak{e}}_8^{c+-}$ for the same reason, only \widehat{E}_8^m does. Therefore, strictly speaking, the current $J(w) = 2(V^{c-}(w))^{-1} P(w) V^{c-}(w)$ is ill-defined. There is a well-defined action of $K(\widehat{E}_8^m) = K(E_9)$ on the components P and Q of the Maurer–Cartan form. This is sufficient for defining the action of generalised diffeomorphisms since the derivative is a constrained object. Besides using the Unendlichbein approach to avoid the problem of the ill-defined current, one can consider E_9 exceptional field theory in a level decomposition with respect to a finite-dimensional Levi subgroup, such as E_8 . This semi-flat formulation was used in Section 5.4 and yields a current $\tilde{\mathcal{J}}$ that is in $\widehat{\mathfrak{e}}_8^{c+-}$ since the conjugation of the symmetrised Maurer–Cartan form is reduced to elements of the Levi subgroup and all other objects in the theory are conjugated by the remaining unipotent elements of \widehat{E}_8^{c-} appropriately.

Because the coset representative is an element of \widehat{E}_8^{c-} ,⁴⁴ the representations it acts on must be of the right type: either completed lowest weight modules (such as $|A\rangle$) or minimal highest modules (such as $\langle\partial|$). For instance, typical expressions we encounter are $\langle\partial|V^{-1} \dots V^{-1\dagger}|A\rangle$ with $V^{-1\dagger} \in \widehat{E}_8^{c+}$ that can act on completed lowest weight modules (while V^{-1} has finite action on $\langle\partial|$).

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⁴³Notice that these are not the same objects as defined in Section 2.3 as for simplicity we are not including conjugation by Γ in this appendix.

⁴⁴In (2.36), we have also written an infinite exponential of negative Virasoro generators which are therefore in the completed $\text{Vir}^- = \text{Vir}^{-c}$ group as defined there. The associated algebra can act consistently on the completed Lie algebra $\widehat{\mathfrak{e}}_8^{c-}$, so that (2.40) is well defined.

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