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Time-like hypersurfaces of prescribed mean extrinsic curvature

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Time-like hypersurfaces of prescribed mean extrinsic curvature

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Abstract

The results on the initial boundary value problem for Einstein's vacuum field equation obtained in Friedrich and Nagy *Commun. Math. Phys.* **201** 619–655 rely on an unusual gauge. One of the defining gauge source functions represents the mean extrinsic curvature of the time-like leaves of a foliation that includes the boundary and covers a neighbourhood of it. The others steer the development of a frame field and coordinates on the leaves. In general their combined action is needed to control in the context of the reduced field equations the evolution of the leaves. In this article are derived the hyperbolic equations implicit in that gauge. It is shown that the latter are independent of the Einstein equations and well defined on arbitrary space-times. The analysis simplifies if boundary conditions with constant mean extrinsic curvature are stipulated. It simplifies further if the boundary is required to be totally geodesic.

Keywords: Einstein equations, initial boundary value problem, time-like hypersurfaces of prescribed mean extrinsic curvature

1. Introduction

In this article we consider a question that arises in the context of the initial boundary value problem for Einstein's vacuum field equation formulated in [7]. The setting is that of a smooth Lorentz metric g on a manifold M with boundary $S \cup T$ where S and T are smooth hypersurfaces of M which are space- and time-like respectively and intersect in the space-like surface $\Sigma = S \cap T$ that represents the common boundary of S and of T. The manifold T is assumed to be diffeomorphic to $\mathbb{R}_0^+ \times \Sigma$ with $\{0\} \times \Sigma$ identified with Σ. The assumptions on S are usually

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chosen according to the desired application. For definiteness we assume it to be compact with boundary Σ, though this is not needed for the following arguments. The set $M \backslash S$ is supposed to be in the future of S and on one side of T so that all past directed non-extendible time-like curves in M acquire an endpoint on $S \cup T$. As in [6] a smooth space-time (M, g) with these properties is referred to as an *ST-space-time*. The initial boundary value problem asks for the existence and uniqueness of solutions to Einstein's vacuum field equations $R_{\mu\nu} = 0$ that induce suitably prescribed Cauchy data on S and boundary data on T. Such a solution will be called an *ST-vacuum solution*.

The analysis of the standard Cauchy problem for Einstein's equations is typically based on coordinates x^ν that obey a *wave gauge* characterized by *coordinate gauge source functions* $F^\mu = F^\mu(x^\nu)$ [4]. In this gauge, often referred to as 'harmonic' if $F^\mu = 0$ and (somewhat absurdly) as 'generalized harmonic' if $F^\mu \neq 0$, Einstein's equations take the form of a system of wave equations of second order, the *reduced equations*. If g is a solution to this system for Cauchy data that satisfy the constraints and the gauge conditions on a space-like initial hyper-surface, the reduced equations imply with the Bianchi identity a *subsidiary system* which allows one to conclude that the coordinates satisfy the semi-linear system of wave equations

$$\Box_g x^\mu = F^\mu(x^\nu). \tag{1.1}$$

This ensures that g is in fact a solution to the Einstein equations. As pointed out in [4], the concept of a gauge source function gives access to a huge class of useful gauge conditions. If a solution admits coordinates that exist globally, they can be characterized, in principle, in terms of gauge source functions. Since they combine by (1.1) information on the coordinates as well as on the metric, the corresponding functions F^μ are usually not known *a priori*. A successful use of the concept thus requires clever choices of the functions $F^\mu(x^\nu)$ or suitable generalizations thereof. Interesting examples of such applications can be found in [14] in a purely analytical context and in [10, 12, 13] in numerical contexts.

The setting of [7] uses a formalism based on a frame e_k that satisfies $g(e_i, e_j) = g_{ij} = \text{diag}(1, -1, -1, -1)$. The additional gauge freedom introduced by it can in principle be dealt with by prescribing six *frame gauge source functions* $F_{jk} = F_{[jk]}(x^\mu)$ which can be given freely [4]. If these are suitably implemented, the reduced equations will propagate the frame so that it satisfies, similar to (1.1), the semi-linear wave equation

$$\Box_g e_j + g\left(\nabla_\mu e_j, \nabla^\mu e_k\right) g^{kl} e_l = F_j{}^k e_k. \tag{1.2}$$

There exist useful gauge conditions of a geometric nature, however, for which the functions F^μ and F_{jk} are not known. A simple example is given by the Gauss gauge, defined by a time function x^0 so that $e_0 = \text{grad}_g x^0$ is a time-like geodesic unit vector field, combined with an orthonormal frame field that comprises e_0 and is parallel propagated in the direction of e_0. Such a gauge, characterized by the conditions

$$e_0 = \partial_{x^0}, \qquad g(\nabla_{e_0} e_i, e_j) = 0, \tag{1.3}$$

allows us to extract from equations (2.12)–(2.14) a hyperbolic system of reduced equations. By (1.3) four frame coefficients and six connection coefficients are restricted to very special values. As in the case where one sets $F^\mu = 0$ and $F_{jk} = 0$, no gauge source functions seem to appear in the reduced equations. But being zero or one does not mean that they are not there.

The gauge defined by (1.3) differs from a gauge in terms of F^μ and F_{jk} in important ways. The gauge dependent quantities x^μ and e_k are subject to different propagation laws, which makes a difference in their applicability to specific problems. While the conditions related to F^μ and F_{jk} are independent of each other, the two conditions in (1.3) are not. They may be

read so that in a first step the second of these conditions is solved with initial data so that e_0 is orthogonal to the initial slice $\{x^0 = 0\}$ and then the coordinates are defined by the condition $\langle e_0, \mathrm{d}x^\mu \rangle = \delta^\mu{}_0$.

There exist combinations of the methods above or still quite different ways to remove the gauge freedom that allow us to prescribe certain functions freely in the reduced equations. Even if they are not related to the functions F^μ and F_{jk} or to frame and connection coefficients we will refer to such functions as *gauge source functions* because their presence in the reduced equations implicitly controls the evolution of the gauge.

The analysis of the initial boundary value problem in [7] rests on an unusual gauge and is characterized by unusual gauge source functions f and F^A, $A = 1, 2$. Here f corresponds to a function of the connection coefficients that controls in a neighbourhood of the boundary $T = T_0$ the evolution of a family of time-like hypersurfaces T_c, $0 \leqslant c < \epsilon$, which define a smooth space-time foliation that extends into the interior of the solution space-time. The two functions F^A, $A = 1, 2$, correspond to connection coefficients that control together with the geometric requirement (2.1) and its consequence (2.8) the evolution of a time-like vector field e_0 tangential to the hypersurfaces T_c. In the end the space-time coordinates x^μ turn out to be dragged along with e_0, so that the first of conditions (1.3) is satisfied, but that is not the way the coordinate x^3 is constructed. Some connection coefficients acquire symmetries by the requirement that the frame vector field e_3 is orthogonal to the hypersurfaces T_c. Given f and F^A the remaining connection coefficients are then controlled by the field equations.

With the fields f, F^A singled out as gauge source functions there can be extracted from the Einstein equations, in the representation used in [7], a hyperbolic system of reduced equations that allows one to formulate with suitably prescribed data a well-posed initial boundary value problem. The choice of these functions is motivated by the fact that in the given setting the Einstein equations do not supply evolution equations for f and F^A. Its final justification follows from the existence of a hyperbolic subsidiary system which implies that the reduced system preserves the constraints and gauge conditions and that the latter do what they have been chosen for.

This leads to a well-posedness result, local in time, for the initial boundary value problem for Einstein's field equations and one could leave it at that. In this article we would like to show, however, that there are explicit equations, analogous to (1.1), that reveal the relation between the gauge source functions f and F^A, the gauged structures, and the metric. Such equations will show that the desired gauge can be established under general assumptions and without any particular equation imposed on g. They give, in particular, additional confirmation that the gauge conditions do not impose restrictions on the solutions. Moreover, the knowledge of these equations could help us to find gauge source functions that extend the life time of the gauge and give control on the long term behaviour or other desired features of the solutions.

On the solution space-time the restriction of f to T_c represents the mean extrinsic curvature induced on this hypersurface and thus encodes an implicitly evolution law for T_c. This is in general not independent of the functions F^A. The mean extrinsic curvature χ of the boundary T constitutes in the setting of [7] a boundary datum which can be prescribed freely. Thus f should be smooth and coincide with χ on T but can be chosen rather arbitrary elsewhere. In the special case where the boundary data are given so that $\chi = \chi_* = \mathrm{const.}$ on T, one can choose $f = \chi_*$ on the foliation. As pointed out in [7], this leads to a considerable simplification. If χ is point-dependent, however, f must be point-dependent and the part of the gauge controlled by F^A comes into play. This problem is much more involved and has not been analysed so far.

The situation is reminiscent of the problem of constructing standard Cauchy data for Einstein's field equations with point-dependent mean extrinsic curvature. For a long time it was customary to construct Cauchy data with constant mean extrinsic curvature (CMC). The

main reason was that this led to technical simplifications but it had even been assumed occasionally that asymptotically flat solutions always admit such slices [2]. The analysis of the Einstein evolution equations, however, does not require such an assumption on the initial data. In 1982 Brill showed that there exist asymptotically flat solutions to Einstein's field equations which do not admit a Cauchy hypersurface with vanishing mean extrinsic curvature [2] and in 1988 Bartnik showed that there exist cosmological space-times which do not admit CMC Cauchy slices [1]. Neglecting data with point-dependent mean extrinsic curvature may thus exclude large classes of important space-times. Only recently have been obtained results on such data of some generality [9, 11].

Similarly, we shall miss out on large classes of space-times developing from initial and boundary data if the mean extrinsic curvature on the boundary is required to be constant. A point dependence of χ could induce the boundary T to shrink or bulge or oscillate. The nature and origin of the difficulties arising in the case of a point-dependent mean extrinsic curvature on the time-like boundary are, however, quite different from those arising in the construction of standard Cauchy data on space-like slices.

To explain the way the well posed initial boundary value problem is set up in [7], some considerations of [7] will be recalled in section 2. In section 3 are discussed various aspects of the special case of CMC $\chi = \chi_*$. This will in particular shed additional light on the result of Fournodavlos and Smulevici [3], who studied the case of boundaries that are totally geodesic. In the section 4 we finally consider the case of a point-dependent datum χ. It is shown in which sense the functions f and F^A are related to an implicit quasi-linear, symmetric hyperbolic system that fixes the gauge.

In the context of the initial boundary value problem long term evolution projects are beset with all the difficulties known from the standard Cauchy problem. There are issues, however, that are specific to this problem. If a solution (M, g) to Einstein's equation is given, a manifold of the form $T = \mathbb{R}_0^+ \times \Sigma$ can be smoothly embedded as a time-like hypersurface but it can also just be immersed so that the pull-back of g still defines a smooth Lorentzian metric on T but the image shows self-intersections. The ambient space-time then still induces on T smooth vacuum data of the type considered below. We can thus imagine situations where T is smoothly embedded close to its initial boundary Σ but then, as the space-time evolves, opposite sides of T start to move towards each other and threaten to touch and intersect. The point is that the data on T alone may not allow one to decide whether they are induced by an embedding or an immersion and even if they are induced by an embedding they may represent situations where discrete points of T are mapped to points in M lying quite close to each other. If one is ambitious enough to analyse such situations the gauge problem may need reconsiderations. In the following we shall not be ambitious and only consider the initial boundary value problem local in time.

2. The setting

In the following we consider four-dimensional ST-space-times (M, g) with boundary $S \cup T$ and edge $\Sigma = S \cap T$ as described in the introduction. Our goal is to construct solutions to Einstein's vacuum field equation $R_{\mu\nu}[g] = 0$ that arise from suitably prescribed initial data on S and boundary data on T. On the domain of dependence of the initial hypersurface S they are uniquely determined by the standard Cauchy problem for Einstein's vacuum field equation with Cauchy data on S. Seeking to construct local in time solutions to initial boundary value problems, a first step is to control the solutions in a neighbourhood of the edge Σ that simultaneously represents the boundary of S and of T. The choice of data on the boundary T depends very much on the chosen representation of the field equations. We follow here the discus-

sion of [7], which employs a frame formalism and uses the vacuum Bianchi equations for the conformal Weyl tensor.

2.1. Formalism and gauge conditions

On any smooth ST-space-time (M, g) can be chosen as follows an ST-*adapted gauge*, consisting of a smooth coordinate system x^μ, $\mu = 0, \ldots, 3$, and smooth local orthonormal frames e_k, $k = 0, \ldots, 3$ near T.

The function x^3 satisfies $x^3 = 0$ on T, $x^3 > 0$ elsewhere, so that the sets $T_c = \{x^3 = c = \text{const.}\}$ with $0 \leqslant c < c_*$ and $T_0 = T$ are time-like hypersurfaces diffeomorphic to T that smoothly foliate some neighbourhood W of T in M with $\mathrm{d}x^3 \neq 0$ on W.

The unit vector field e_0 on W is time-like, future directed, tangential to the hypersurfaces T_c, and orthogonal to the two-surfaces $S_c = S \cap T_c$ with $S_0 = \Sigma$. The space-like vector field e_3 on W represents the inward directed unit normals to the T_c.

The local vector fields e_A, $A = 1, 2$, define together with e_0 and e_3 a local orthonormal frame on some open subset of $S \cap W$, so that

$$g(e_i, e_j) = g_{ij} = \eta_{ij} = \text{diag}(1, -1, -1, -1).$$

The e_A are then tangential to the surfaces S_c. Denoting by D the Levi-Civita connection of the Lorentz-three-metric k induced by g on T_c we require the fields e_A to be D-Fermi transported in the direction of e_0 so that

$$g(e_0, e_0)D_{e_0}e_A + g(e_A, D_{e_0}e_0)e_0 - g(e_A, e_0)D_{e_0}e_0 = 0. \tag{2.1}$$

The fields e_A are then everywhere tangential to T_c.

The function x^0 defines a natural parameter on the integral curves of e_0 so that

$$\langle e_0, \mathrm{d}x^0 \rangle = 1 \quad \text{on } W, \qquad x^0 = 0 \quad \text{on } S \cap W. \tag{2.2}$$

Consider the vector field $X = (q^\#(\mathrm{d}x^3, \mathrm{d}x^3))^{-1} \, \text{grad}_q \, x^3$ tangential to S and orthogonal to the S_c, where q denotes the metric induced on S by g. An integral curve $\gamma(\sigma)$ of X with $\gamma(0) \in \Sigma$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(x^3(\gamma(\sigma))) = \left\langle \mathrm{d}x^3, \frac{\mathrm{d}}{\mathrm{d}\sigma}\gamma(\sigma) \right\rangle = \langle \mathrm{d}x^3, (q^\#(\mathrm{d}x^3, \mathrm{d}x^3))^{-1} \, \text{grad}_q \, x^3 \rangle = 1,$$

so that $\sigma = x^3(\gamma(\sigma))$, whence $\gamma(c) \in S_c$. The flow of X thus maps Σ diffeomorphically onto the S_c and we have a parametrization $W = \{(x^0, p, x^3)|x^0 \geqslant 0, p \in \Sigma, 0 \leqslant x^3 < c_*\}$. Choose local coordinates x^α, $\alpha = 1, 2$, on Σ, assume them to be dragged into the interior of S with the flow of X so that

$$q^\#(\mathrm{d}x^\alpha, \mathrm{d}x^3) = 0, \quad \alpha = 1, 2, \tag{2.3}$$

and then dragged along with the flow of e_0 so that

$$\langle e_0, \mathrm{d}x^\alpha \rangle = 0 \quad \text{on } W, \quad \alpha = 1, 2. \tag{2.4}$$

The x^α, $\alpha = 1, 2$, define local coordinates on the S_c and the x^α, $\alpha = 0, 1, 2$, define local coordinates on T_c for $0 \leqslant c < c_*$. For the frame coefficients satisfying $e_k = e^\mu{}_k \partial_\mu$ holds

$$e^\mu{}_0 = \delta^\mu{}_0, \quad e^3{}_A = 0, \quad e^3{}_3 > 0 \quad \text{on } T_c \quad \text{and} \quad e^0{}_A = 0, \quad A = 1, 2, \quad \text{on } S_c. \tag{2.5}$$

We will have to consider three types of projections. Since our frame is well adapted to the geometrical situation, corresponding projection formalisms can be avoided by distinguishing three groups of indices. They are given, with the values they take, by

$$a, c, d, e, f = 0, 1, 2; \qquad i, j, k, l, m, n = 0, 1, 2, 3; \qquad A, B, C, D = 1, 2.$$

For each group the summation rule is assumed. If ∇ denotes the connection defined by g, the connection coefficients $\Gamma_j{}^i{}_k$ in the frame e_k satisfy $\nabla_j e_k \equiv \nabla_{e_j} e_k = \Gamma_j{}^i{}_k e_i$ and $\Gamma_{jlk} = -\Gamma_{jkl}$ with $\Gamma_{jlk} = g_{li} \Gamma_j{}^i{}_k$. The second fundamental form induced on T_c in the frame e_a and the mean extrinsic curvature of the hypersurfaces T_c are given by

$$\chi_{ab} \equiv g(\nabla_{e_a} e_3, e_b) = -g(e_3, \nabla_{e_a} e_b) = \Gamma_a{}^3{}_b = \Gamma_{(a}{}^3{}_{b)},$$

$$\chi \equiv g^{ab} \chi_{ab} = g^{jk} \Gamma_j{}^3{}_k = \nabla_\mu e^\mu{}_3, \tag{2.6}$$

respectively. Because

$$D_a e_c \equiv D_{e_a} e_c = \Gamma_a{}^b{}_c e_b, \tag{2.7}$$

the $\Gamma_a{}^b{}_c$ define the inner connection D on T_c. The Fermi condition implies

$$\Gamma_0{}^A{}_B = 0, \tag{2.8}$$

$$D_{e_0} e_0 = F^A e_A, \qquad D_{e_0} e_A = -F_A e_0 \quad \text{with} \quad F^A = \Gamma_0{}^A{}_0, \qquad F_A = \eta_{AB} F^B. \tag{2.9}$$

The freedom of choosing the function x^3 and the time-like vector field e_0 we started with finds new expression in this formalism. With e_a as given above the functions $F^A = F^A(x^\mu)$ follow from these formulas. We can, however, also think of the functions F^A as being at our free disposal. If we solve equation (2.9) on T_c with arbitrarily prescribed functions $F^A = F^A(x^\alpha, c)$ and initial data satisfying $g(e_a, e_b) = \eta_{ab}$ on S_c and e_0 orthogonal to S_c, the solution will satisfy the relation $g(e_a, e_b) = \eta_{ab}$ on T_c. Since the field equations (2.12)–(2.14) do not provide propagation equations for the F^A this suggests to consider these functions in the reduced field equations as smooth gauge source function that can be freely prescribed. With the special choice

$$F^A = 0, \tag{2.10}$$

the field e_0 will be D-geodesic and the fields e_A parallel propagated. More general choices of F^A may allow one to avoid the development of caustics.

The freedom of choosing the leaves T_c of the foliation defined by the function x^3 is encoded in the mean extrinsic curvature $\chi = \chi(x^\alpha, c)$ induced on T_c. It will be discussed below in detail how T_c is determined by $\chi = \chi(x^\alpha, c)$ and the initial data on $S \cap W$. Since equations (2.12)–(2.14) do not provide a propagation equation for χ this function will be considered as a gauge source function that will be represented by a smooth function $f = f(x^\mu)$ in the reduced field equations. There is, however, a slight difference with the F^A. The function $\chi(x^\alpha, 0)$ will be used as a free boundary datum on T that determines the form of T. While the function f can be freely extended into the interior of the solution space-time, it must thus be given so that $f(x^\alpha, 0) = \chi(x^\alpha, 0)$ on T.

Due to the compactness of Σ, whence of $S_c = S \cap T_c$, the coordinates x^α, $\alpha = 1, 2$ and, in general, also the frame vector fields e_A are only defined locally on S_c. If they have to be redefined on some overlap region U in $W \cap S$ it is important to note that the corresponding

transformation is explicitly controlled along the integral curves of e_0 by the propagation laws imposed above and the transformation on U.

If the frame is subject to a transformation $e_A \to e_{A'} = s^A{}_{A'} e_A$ in some overlap patch $U \subset S \cap W$ with some point dependent transformation $s^A{}_{A'} \in SO(2)$ on S_c, we must require

$$F^A \to F^{A'} = s^{A'}{}_B F^B \quad \text{whence} \quad F_A \to F_{A'} = s^A{}_{A'} F_A \quad \text{with} \quad s^A{}_{A'} s^{A'}{}_B = \delta^A{}_B,$$

to preserve the first of equation (2.9) and the second equation then requires

$$D_{e_0} s^A{}_{A'} = 0,$$

which allows us to control the corresponding transformations

$$\eta_{AB} \to \eta_{A'B'} = \eta_{AB} s^A{}_{A'} s^B{}_{B'}, \qquad \beta_{AB} \to \beta_{A'B'} = \beta_{AB} s^A{}_{A'} s^B{}_{B'},$$

along the integral curves of e_0.

In the coordinates x^α, $\alpha = 1, 2$ described above f is assumed to be smooth with local representation $f = f(x^0, x^\alpha, x^3)$ on W. If the coordinates are subject on a subset of Σ to a transformation $x^\alpha \to x^{\beta'} = x^{\beta'}(x^\alpha)$ with inverse $x^\alpha = x^\alpha(x^{\beta'})$, this transformation transfers by the rules (2.3) and (2.4) into W, which allows us to control explicitly the new coordinate representation $f' = f'(x^0, x^{\alpha'}, x^3) = f(x^0, x^\beta(x^{\alpha'}), x^3)$. In a similar way transform the coordinate representations of the functions F^A on W and of the functions χ, η_{AB}, β_{AB}, α, and β on T discussed below.

2.2. The field equations

The basic unknowns in the representation of the field equations used in [7] are

$$e^\mu_k, \quad \Gamma_k{}^i{}_j, \quad C^i{}_{jkl}, \tag{2.11}$$

where $C^i{}_{jkl}$ is a tensor field with the algebraic properties of a conformal Weyl tensor which for a solution of the equations will in fact assume that meaning. The field equations are given by the *torsion free condition*

$$[e_i, e_j] = (\Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i) e_k, \tag{2.12}$$

where the square bracket denotes the commutator of the vector fields, the *curvature relation*

$$e_k(\Gamma_l{}^i{}_j) - e_l(\Gamma_k{}^i{}_j) + 2\Gamma_{[k}{}^i{}_{|m|}\Gamma_{l]}{}^m{}_j - 2\Gamma_m{}^i{}_j\Gamma_{[k}{}^m{}_{l]} = C^i{}_{jkl}, \tag{2.13}$$

where the left-hand side gives the curvature of the connection ∇ in terms of the connection coefficients and the frame, and the *vacuum Bianchi identity*

$$\nabla_i C^i{}_{jkl} = 0. \tag{2.14}$$

2.2.1. Splittings of the conformal Weyl tensor.
In the following we shall need two different decompositions of the conformal Weyl tensor. The decomposition used in [7] is defined by the time-like frame vector field $n = e_0$. *The N-electric* and *the N-magnetic* part of the conformal Weyl tensor are given by

$$E^n_{ik} = p_i{}^m p_k{}^n C_{mjnl} n^j n^l, \quad B^n_{ik} = p_i{}^m p_k{}^n \frac{1}{2} C_{mjpq} \epsilon^{pq}{}_{nl} n^j n^l,$$

respectively, where $\epsilon_{ijkl} = \epsilon_{[ijkl]}$ with $\epsilon_{0123} = 1$ and $p_{ij} = g_{ij} - n_i n_j$. These tensors are symmet-

ric, trace free, and *spatial* in the sense that $n^i E^n_{ik} = 0$ and $n^i B^n_{ik} = 0$. It holds

$$C_{ijkl} = 2 \left(q_{j[k}E^n_{l]i} - q_{i[k}E^n_{l]j} - n_{[k}B^n_{l]m}\epsilon^m{}_{ij} - n_{[i}B^n_{j]m}\epsilon^m{}_{kl} \right),$$

with $q_{ij} = g_{ij} - 2n_i n_j$ and $\epsilon_{jkl} = n^i \epsilon_{ijkl} = \epsilon_{0jkl}$, and also

$$C_{mnpq}n^m p^n{}_j p^p{}_k p^q{}_l = C_{0npq}p^n{}_j p^p{}_k p^q{}_l = -B^n_{jm}\epsilon^m{}_{kl},$$

$$C_{mnpq}p^m{}_i p^n{}_j p^p{}_k p^q{}_l = 2 \left(p_{j[k}E^n_{l]i} - p_{i[k}E^n_{l]j} \right).$$

The second decomposition is defined by the space-like frame vector field $N = e_3$. *The N-electric* and *the N-magnetic* part of the conformal Weyl tensor are given by

$$E^N_{ik} = k^m_i k^n_k C_{mjnl}N^j N^l, \qquad B^N_{ik} = k^m_i k^n_k \frac{1}{2} C_{mjpq}\epsilon^{pq}{}_{nl}N^j N^l.$$

These tensors are symmetric, trace free, satisfy $N^i E^N_{ik} = 0$, $N^i B^N_{ik} = 0$, and

$$C_{ijkl} = 2 \left(-l_{j[k}E^N_{l]i} + l_{i[k}E^N_{l]j} + N_{[k}B^N_{l]m}\bar{\epsilon}^m{}_{ij} + N_{[i}B^N_{j]m}\bar{\epsilon}^m{}_{kl} \right),$$

where $k_{ij} = g_{ij} + N_i N_j$, $l_{ij} = g_{ij} + 2N_i N_j$, $\bar{\epsilon}_{ijk} = \epsilon_{ijkl}N^l = \epsilon_{ijk3}$. It holds

$$C_{mnpq}N^m k^n{}_j k^p{}_k k^q{}_l = C_3 npq k^n{}_j k^p{}_k k^q{}_l = -B^N_{jm}\epsilon^m{}_{kl},$$

i.e.

$$C^3{}_{abc} = B^N_{ad}\epsilon^d{}_{bc3} \quad \text{or} \quad B^N_{ad} = \frac{1}{2}C_3 abc\epsilon^{bc}{}_{3d},$$

and

$$C_{mnpq}k^m{}_i k^n{}_j k^p{}_k k^q{}_l = 2 \left(-k_{j[k}E^N_{l]i} + k_{i[k}E^N_{l]j} \right).$$

The different parts are related by

$$\begin{aligned}
B^N_{00} &= C_{0312} = B^n_{33}, & B^N_{01} &= C_{0302} = E^n_{23}, & B^N_{02} &= C_{0310} = -E^n_{13}, \\
B^N_{10} &= C_{1312} = E^n_{32}, & B^N_{11} &= C_{1302} = -B^n_{22}, & B^N_{12} &= C_{1310} = B^n_{12}, \\
B^N_{20} &= C_{2312} = -E^n_{13}, & B^N_{21} &= C_{2302} = B^n_{12}, & B^N_{22} &= C_{3201} = -B^n_{11}, \\
E^N_{00} &= C_{0303} = E^n_{33}, & E^N_{01} &= C_{0313} = -B^n_{32}, & E^N_{02} &= C_{0323} = B^n_{31}, \\
E^N_{10} &= C_{1303} = -B^n_{32}, & E^N_{11} &= C_{1313} = -E^n_{22}, & E^N_{12} &= C_{1323} = E^n_{12}, \\
E^N_{20} &= C_{2303} = B^n_{31} & E^N_{21} &= C_{2313} = E^n_{21}, & E^N_{22} &= C_{2323} = -E^n_{11}.
\end{aligned}$$

2.2.2. The Gauss–Codazzi equations.
The tensor

$$k_{ij} = g_{ij} + N_i N_j = \eta_{ab}\delta^a{}_i \delta^b{}_j$$

represents the metric induced on the hypersurfaces T_c. We shall need the well known equations which relate the curvature tensor $R^i{}_{jkl}[g]$ of the metric g to fields living on the hypersurfaces T_c. Gauss' equation, which reads in our formalism

$$\bar{R}^a{}_{bcd}[k] = R^a{}_{bcd}[g] - \chi_c{}^a\chi_{db} + \chi_d{}^a\chi_{cb}, \tag{2.15}$$

relates it to the curvature tensor $\bar{R}^a{}_{bcd}[k]$ of k and the second fundamental form. Codazzi's equation

$$D_c\chi_{db} - D_d\chi_{cb} = R^3{}_{bcd}[g], \tag{2.16}$$

relates it to T_c-intrinsic derivatives of the second fundamental form. With the well known decomposition of the curvature tensor on three-dimensional spaces and the relations on the Weyl tensor above, Gauss' equation can be written on vacuum solutions

$$k_{b[d}R_{c]a}[k] + k_{a[c}R_{d]b}[k] + \frac{1}{3}R[k]k_{a[d}k_{c]b}$$

$$= 2\left(k_{b[d}E^N_{c]a} + k_{a[c}E^N_{d]b}\right) - \chi_{ca}\chi_{db} + \chi_{da}\chi_{cb},$$

where $R_{ab}[k]$ and $R[k]$ denote the Ricci tensor and the Ricci scalar of k. A contraction gives

$$\frac{1}{2}\left(R_{db}[k] + \frac{R[k]}{3}k_{db}\right) = E^N_{db} - \chi\chi_{bd} + \chi_d{}^c\chi_{bc}. \tag{2.17}$$

Codazzi's equation takes on a vacuum solution the form

$$D_c\chi_{db} - D_d\chi_{cb} = C^3{}_{bcd} = B^N{}_{be}\epsilon^e{}_{cd3}. \tag{2.18}$$

2.2.3. Reduced equations and boundary data. In the following a few remarks will be made on the reduced equations, the initial and boundary data, the corner conditions, and the subsidiary equations. For details (which are sometimes slightly rewritten here) the reader is referred to [7].

With the gauge conditions and the gauge source functions $F^A = F^A(x^\mu)$ and $f = f(x^\mu)$ equations (2.12)–(2.14) imply a symmetric hyperbolic system of *reduced equation* for the unknowns

$$e^\beta{}_a, \quad e^\mu{}_3, \quad \Gamma_A{}^B{}_0, \quad \Gamma_A{}^B{}_C, \quad \Gamma_3{}^A{}_B, \quad \Gamma_3{}^A{}_0, \quad \Gamma_3{}^3{}_A, \quad \Gamma_3{}^3{}_0,$$

$$\chi_{01}, \quad \chi_{02}, \quad \chi_{11}, \quad \chi_{12}, \quad \chi_{22}, \quad B^n_{ab}, \quad E^n_{ab}.$$

The field χ_{00}, which only occurs in non-differentiated form in these equations, is taken care of by writing $\chi_{00} = \chi_{11} + \chi_{22} + f$.

The *initial data* on S are given by *standard Cauchy data*, i.e. a solution to the vacuum constraints on space-like hypersurfaces, which extend smoothly to the boundary Σ of S.

To describe the *boundary conditions* we consider the trace free parts of the orthogonal projections of the N-electric and N-magnetic parts of the conformal Weyl tensor on T into the planes orthogonal to e_3 and e_0. The corresponding symmetric trace free tensors η_{AB} and β_{AB}, which by the relations given above can also be expressed in terms of the N-electric and N-magnetic parts of the conformal Weyl tensor, are given by

$$\beta_{11} = -\beta_{22} = -\frac{1}{2}(C_{3201} + C_{3102}) = \frac{1}{2}(B^n_{11} - B^n_{22}) = \frac{1}{2}(B^N_{11} - B^N_{22}), \tag{2.19}$$

$$\beta_{12} = \beta_{21} = C_{3101} = B^n_{12} = B^N_{12},$$

and

$$\eta_{11} = -\eta_{22} = \frac{1}{2}(C_{1010} - C_{2020}) = \frac{1}{2}(E_{11}^n - E_{22}^n) = \frac{1}{2}(E_{11}^N - E_{22}^N), \qquad (2.20)$$

$$\eta_{12} = \eta_{21} = E_{12} = C_{1020} = E_{12}^n = E_{12}^N.$$

In terms of the pseudo-orthonormal frame l, k, m satisfying (with e_3 inward pointing)

$$\sqrt{2}l = e_0 + e_3, \qquad \sqrt{2}k = e_0 - e_3, \qquad \sqrt{2}m = e_1 - ie_2,$$

and the Newman–Penrose notation for the curvature tensor, the relevant components of the conformal Weyl tensor are given by

$$\Psi_0 = C_{\mu\nu\sigma\pi}l^\mu m^\nu l^\sigma m^\pi = \eta_{11} + \beta_{12} + i(\beta_{11} - \eta_{12}),$$

$$\Psi_4 = C_{\mu\nu\sigma\pi}\bar{m}^\mu k^\nu \bar{m}^\sigma k^\pi = \eta_{11} - \beta_{12} + i(\beta_{11} + \eta_{12}).$$

The boundary conditions of [7] then take the form

$$f = \chi, \quad -\Psi_4 + \alpha\Psi_0 + \beta\bar{\Psi}_0 = q \quad \text{on } T, \qquad (2.21)$$

where α and β are complex-valued functions on T that satisfy

$$|\alpha| + |\beta| \leqslant 1, \qquad (2.22)$$

and q, the main datum besides χ on T, is a smooth complex-valued function on T that can be prescribed, consistent with the conditions discussed below, freely. Condition (2.22), which looks simpler than the corresponding condition given in [7], is obtained from the latter by diagonalizing the matrix B used there to express the restrictions on α and β.

As special examples, which will be of interest below, we note that the admissible choice $\alpha = 0$ and $\beta = 1$ results in the boundary condition

$$q = 2(\beta_{12} - i\beta_{11}) = 2B_{12}^n - i(B_{11}^n - B_{22}^n) = 2B_{12}^N - i(B_{11}^N - B_{22}^N) \quad \text{on } T. \qquad (2.23)$$

It only involves magnetic parts. The choice $\alpha = 0$, $\beta = -1$ gives the boundary condition

$$q = -2(\eta_{11} + i\eta_{12}) = E_{22}^n - E_{11}^n - 2iE_{12}^n = E_{22}^N - E_{11}^N - 2iE_{12}^N \quad \text{on } T, \quad (2.24)$$

which only involves electric parts of the conformal Weyl tensor.

To determine a smooth solution, the initial and the boundary data must satisfy a consistency condition, the so-called *corner condition* at the edge Σ. Because the reduced equations with given f and F^A are symmetric hyperbolic, the Cauchy data on S determine in our gauge a unique formal expansion type solution on $S = \{x^0 = 0\}$. For given functions α and β this expansion determines, in particular, a unique formal expansion of the fields on the left hand sides of (2.21) at Σ. The corner conditions require the data χ and q on the right hand sides to be prescribed consistent with these expansions at Σ.

For given Cauchy data it is always possible to find boundary data that satisfy this condition, which leaves the right hand sides of (2.21) essentially arbitrary away from Σ. If the boundary data are supposed to satisfy certain conditions, possibly suggested by some intended application, it requires an extra effort (and may not be possible) to construct Cauchy data, i.e. solutions to the constraint equations on S, that meet these requirements.

The main result of [7] says (in the notation introduced above):

Let be given smooth Cauchy data for Einstein's vacuum field equations on the compact three-manifold S with boundary Σ and smooth boundary data

$$\chi, q \quad \text{and functions} \quad \alpha, \beta \quad \text{on } T = \mathbb{R}^+ \times \Sigma, \tag{2.25}$$

satisfying (2.22). Choose on $M = \mathbb{R}^+ \times S$ smooth gauge source functions f with $f = \chi$ on T and F^A that satisfy together with the initial and boundary data the corner conditions at Σ defined by the reduced field equations. Then there exists for some $x_*^0 > 0$ a unique smooth solution g to Einstein's vacuum equations on the manifold $M' = [0, x_*^0[\times S \subset \mathbb{R}^+ \times S$ so that $S \equiv \{0\} \times S$ is space-like, $T' = [0, x_*^0[\times \Sigma \subset T$ is time-like, g induces the given Cauchy data on S, χ acquires on T' the meaning of the mean intrinsic curvature induced on T' by g and q coincides with the function of the conformal Weyl tensor of g on the left-hand side of the second equation of (2.21).

The first step to arrive at this result consists in setting up local initial boundary value problems for the reduced field equations and showing the existence of local solutions near given points of Σ. In a next step the local solutions are patched together to obtain a solution covering a neighbourhood of Σ. Because, as pointed out above, the transformations between local solutions can be explicitly controlled, there arises no problem. The solution near Σ is then patched together with the solution to the Cauchy problem for the reduced equations that is determined by the data on S.

This establishes the existence of a unique solution to the reduced equations on a manifold of the form $M' = [0, x_*^0[\times S$. In a final step a hyperbolic *subsidary system* is derived that supplies an argument that the solution to the reduced equations is in fact a solution to the Einstein equations.

Suppose (M, g) is a ST-vacuum solution. After choosing a gauge as above one can read off the data induced on S and T and the gauge source functions f and F^A near T (the corner conditions will, of course, be satisfied). Our result then shows that local in time the given ST-vacuum solution will be reconstructed uniquely by our method. Thus local in time all ST-vacuum solutions are covered by the existence result above.

3. Time-like hypersurfaces and mean extrinsic curvature

In this section we discuss the basic equation associated with the mean extrinsic curvature on time-like hypersurfaces. We consider then the case of CMC and derive the equations which are in this case implicit in the formulation of the initial boundary value problem of [7]. It follows a discussion of totally geodesic boundaries.

3.1. The basic equation

Let $(\mathcal{M}, g)$ be a four-dimensional space-time, $\mathcal{S}$ a space-like hypersurface which we assume for convenience to be a Cauchy hypersurface of $\mathcal{M}$, and $\mathcal{T}$ a time-like hypersurface which intersects $\mathcal{S}$ so that it cuts out from it a compact three-dimensional manifold S with compact space-like boundary $\Sigma = \mathcal{S} \cap \mathcal{T}$. The sets S and Σ may be thought of as identical with the ones labelled by the same symbols in the previous section. The set $\mathcal{T}$ does represent a space-time boundary but serves as a subsidiary hypersurface to establish a certain equation. It is fairly arbitrary, only when we arrive at condition (4.11) we will need to restrict it, together with the coordinates z^μ introduced below, further. The metric g is not required to satisfy any field equation. In the following the coordinate indices $\alpha, \beta, \gamma, \delta$ take values $0, 1, 2$ while the coordinate indices $\kappa, \lambda, \mu, \nu, \pi, \rho$ take values $0, 1, 2, 3$.

Let z^0, z^3 with $dz^0 \neq 0$, $dz^3 \neq 0$ denote smooth functions defined on a neighbourhood of Σ so that $\mathcal{S} = \{z^0 = 0\}$, $\mathcal{T} = \{z^3 = 0\}$, and $z^3 > 0$ on S on that neighbourhood. Set $S_c = \{z^0 = 0, z^3 = c = \text{const.}\}$ for $0 \leqslant c < c_*$ so that $S_0 = \Sigma$. By our assumptions we have $g^{00} = g^{\#}(dx^0, dx^0) > 0$ on $\mathcal{S}$, $g^{33} = g^{\#}(dx^3, dx^3) < 0$ on $\mathcal{T}$, and thus $g^{00} > 0$ and $g^{33} < 0$ on the S_c for suitably chosen $c_* > 0$. In the following this will always be assumed.

The functions z^0, z^3 are complemented by functions z^1, z^2 so that the z^μ define local coordinates, z^1, z^2, z^3 define local coordinates on S, z^1, z^2 define local coordinates on the S_c, and z^0, z^1, z^2 define local coordinates on $\mathcal{T}$. The vector field ∂_{z^0} is future directed.

To construct a time-like hypersurface T_c with $T_c \cap \mathcal{S} = S_c$ whose mean extrinsic curvature coincides with a given function χ we assume it to be essentially given as the graph of a function $\phi = \phi(z^\alpha)$ over $\mathcal{T}$ so that

$$T_c = \{\Phi(z^\mu) = c\} \quad \text{where} \quad \Phi(z^\mu) = z^3 - \phi(z^\alpha) \quad \text{with} \tag{3.1}$$
$$\phi|_{z^0=0} = 0 \quad \text{whence} \quad \phi_{,\alpha}|_{z^0=0} = \delta^0{}_\alpha \phi_{,0} \quad \text{on } S_c.$$

For T_c to be time-like we need

$$\nabla_\nu \Phi \nabla^\nu \Phi = g^{33} - 2g^{30}\phi_{,0} + g^{00}\phi_{,0}\phi_{,0} < 0 \quad \text{on } S_c. \tag{3.2}$$

This inequality is satisfied if

$$\frac{g^{03}}{g^{00}} - \sqrt{-\frac{g^{33}}{g^{00}} + \left(\frac{g^{03}}{g^{00}}\right)^2} < \phi_{,0} < \frac{g^{03}}{g^{00}} + \sqrt{-\frac{g^{33}}{g^{00}} + \left(\frac{g^{03}}{g^{00}}\right)^2}. \tag{3.3}$$

Since $g^{33} < 0$ and $g^{00} > 0$ on S_c the roots are real and the unit normal to T_c will be well defined near S_c and given by the restriction to T_c of the vector field

$$N^\mu = \nu\nabla^\mu\Phi = \nu(g^{\mu 3} - g^{\mu\alpha}\phi_{,\alpha}), \tag{3.4}$$

with

$$\nu = -(-\nabla_\nu\Phi\nabla^\nu\Phi)^{-\frac{1}{2}} = -(-g^{33} + 2g^{3\alpha}\phi_{,\alpha} - g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta})^{-\frac{1}{2}}.$$

The gradient fields in brackets on the right-hand side of (3.4) are orthogonal to $\{x^3 = c\}$ and $\mathcal{S}$ respectively and thus to the tangent spaces $T_p S_c$ of S_c. With $\phi_{,0}$ varying in the range (3.3) the vector N^μ exhausts all space-like directions in the two-plane orthogonal to the tangent space $T_p S_c$ of S_c. That plane contains the time-like vector

$$T^\mu = (g^{03} - \phi_{,0}g^{00})g^{\mu 3} - (g^{33} - \phi_{,0}g^{03})g^{\mu 0}, \tag{3.5}$$

which is orthogonal to N and $T_p S_c$ and unique up to a factor. It follows with (3.2)

$$g^{33} - \phi_{,0}g^{03} < \frac{1}{2}(g^{33} - g^{00}\phi_{,0}^2) < 0 \quad \text{on } S_c \text{ near } \Sigma. \tag{3.6}$$

The metric induced on T_c is given by

$$k_{\mu\nu} = g_{\mu\nu} + N_\mu N_\nu, \tag{3.7}$$

the induced second fundamental form is given by

$$\chi_{\mu\nu} = k_\mu{}^\kappa k_\nu{}^\lambda \nabla_\kappa N_\lambda = \nu k_\mu{}^\kappa k_\nu{}^\lambda \nabla_\kappa \nabla_\lambda \Phi, \tag{3.8}$$
$$= -\nu k_\mu{}^\kappa k_\nu{}^\lambda (\phi_{,\alpha\beta}\delta^\beta{}_\kappa \delta^\alpha{}_\lambda + \Gamma_\kappa{}^\rho{}_\lambda(\delta^3{}_\rho - \phi_{,\alpha}\delta^\alpha{}_\rho))$$

and the mean extrinsic curvature by

$$\chi = k^{\mu\nu}\chi_{\mu\nu} = k^{\mu\nu}\nabla_\mu N_\nu = \nabla_\mu N^\mu = \nu k^{\mu\nu}\nabla_\mu\nabla_\nu\Phi.$$

The equation which controls the evolution of ϕ and thus of T_c thus reads

$$-\nu k^{\alpha\beta}\phi_{,\alpha\beta} - \nu k^{\mu\nu}(\Gamma_\mu{}^3{}_\nu - \Gamma_\mu{}^\alpha{}_\nu\phi_{,\alpha}) = \chi, \tag{3.9}$$

where here and in the following the background fields $g_{\mu\nu}$ $\Gamma_\mu{}^\rho{}_\nu$ as well as ν, $k^{\mu\nu}$ etc have to be taken at the points $(z^\alpha, z^3) = (z^\alpha, \phi(z^\alpha) + c)$.

In the following we will consider the z^α also as coordinates on T_c. A vector field on T_c will then be of the form $X = X^\alpha\partial_{z^\alpha}$. Considered as a vector field in M is must be written $X' = X^\alpha\partial_{z^\alpha} + \phi_{,\alpha}X^\alpha\partial_{z^3}$. It holds then $g(X', Y') = \bar{k}(X, Y)$ with $\bar{k}$ the pull back of $k_{\mu\nu}\mathrm{d}z^\mu\mathrm{d}z^\nu$ to the hypersurface $T_c = \{(z^\mu) = (z^\alpha, \phi(z^\alpha) + c)\}$. It coincides with the pull back of $g_{\mu\nu}\mathrm{d}z^\mu\mathrm{d}z^\nu$ to T_c and is given by

$$\bar{k} = \bar{k}_{\alpha\beta}\,\mathrm{d}z^\alpha\,\mathrm{d}z^\beta = (g_{\alpha\beta} + 2g_{3(\alpha}\phi_{,\beta)} + g_{33}\phi_{,\alpha}\phi_{,\beta})\mathrm{d}z^\alpha\mathrm{d}z^\beta. \tag{3.10}$$

As long as T_c is time-like this metric is Lorentzian.

The principal part of equation (3.9) is governed by the symmetric tensor

$$k^{\alpha\beta} = \bar{k}^{\alpha\beta} \equiv g^{\alpha\beta} + \nu^2(g^{\alpha3} - g^{\alpha\gamma}\phi_{,\gamma})(g^{\beta3} - g^{\beta\delta}\phi_{,\delta}) \quad \text{on } T_c, \tag{3.11}$$

with

$$\nu^2 = (-g^{33} + 2g^{3\alpha}\phi_{,\alpha} - g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta})^{-1}.$$

It satisfies

$$\bar{k}_{\alpha\beta}\bar{k}^{\beta\gamma} = \delta^\beta{}_\alpha.$$

Suppose χ is is a given function of four variables. We write then $\chi = \chi(z^\alpha, \phi(z^\alpha) + c)$ on the right-hand side of (3.9). Equation (3.9) defines then a quasi-linear wave equation for ϕ. With a given, sufficiently small constant c, a given right-hand side χ as above, and initial data ϕ and $\phi_{,0}$ on S_c that satisfy (3.1) and (3.2) it determines a unique solution ϕ near S_c for which the hypersurface $T_c = \{z^3 = \phi(z^\alpha) + c\}$ is time-like. Because of the smooth dependence of the solutions on c, the initial data and χ, the hypersurfaces T_c represent the leaves of a smooth local foliation of $\mathcal{M}$ near Σ that defines a smooth function x^3 with $x^3 = c$ on T_c.

3.1.1. Boundaries of constant mean extrinsic curvature.
The coordinate dependence of the function χ has been specified above in a somewhat cursory way. In the context of the initial boundary value problem it requires in general further considerations. There is, however, a case, pointed out already in [7], where things simplify considerably. If the data (2.21) are given with

$$\chi = \chi_* = \text{constant} \quad \text{on } T, \tag{3.12}$$

we can set $f = \chi_*$ near T. Equation (3.9) simplifies and the construction of the hypersurfaces T_c and the coordinate x^3 completely decouples from setting up the frame vectors e_a and coordinates x^α and thus from the choice of the F^A. The reduced equations will ensure that their solution satisfies $\chi = \chi_*$ near T.

Simple though not uninteresting examples of such situations are given by the hypersurfaces $T = \{r = \text{const.} > 2m\}$ of the Schwarzschild solution in standard Schwarzschild coordinates. Their mean extrinsic curvature is given by

$$\chi = -\sqrt{1 - \frac{2m}{r}} \left(\frac{2r - 3m}{r(r - 2m)} \right). \tag{3.13}$$

This still leaves the freedom to prescribe as a datum the function q in (2.21) on T. Its deviation from the Schwarzschild values may be thought of as representing ingoing or outgoing gravitational radiation. The situation changes if we consider the Kerr solution in Boyer–Lindquist coordinates. The mean extrinsic curvature of the hypersurface $T = \{r = \text{const.} > 2m\}$ is then given by

$$\chi = -\frac{1}{2} \sqrt{\frac{r^2 - 2mr + a^2}{r^2 + a^2 \cos^2 \theta}} \left\{ \frac{2(r - m)}{r^2 - 2mr + a^2} + \frac{2r}{r^2 + a^2 \cos^2 \theta} \right\}.$$

3.1.2. Totally geodesic boundaries. An even more restricted case is considered by Fournodavlos and Smulevici [3], who study the existence of solutions with boundaries that are totally geodesic, emphasizing that the problem of geometric uniqueness discussed in [6] is absent here. We consider this situation here again because the present formalism sheds additional light onto the special nature of these issues in this case.

In general we are free to prescribe three functions on the boundary. On the face of it, the condition $\chi_{ab} = 0$, needed to make the boundary totally geodesics, thus looks much too strong. Precisely because all components of χ_{ab} are required to vanish, it turns out, however, that the problem can be reduced to the prescription of three functions. Suppose (M, g) is a *ST* vacuum solution with totally geodesic time-like boundary T so that $\chi_{ab} = 0$ on T. To set up a gauge as described in section 2, we can use equation (3.9) with $\chi = 0$ on the right-hand side to construct the function x^3, which has been handpicked before, so that $f = 0$ near T. For convenience we can then require (2.10) near T.

The Codazzi equation (2.18) implies with our assumption that

$$B^N{}_{ab} = 0 \quad \text{on } T, \tag{3.14}$$

and thus the, by (2.23) admissible, boundary conditions

$$\chi = 0, \quad \beta_{12} = B^N_{12} = 0, \qquad \beta_{11} = \frac{1}{2}(B^N_{11} - B^N_{22}) = 0 \quad \text{on } T. \tag{3.15}$$

Together with the Cauchy data induced by g on S these boundary conditions determine the solution uniquely near S. Because the solution is smooth, the corner conditions are satisfied, which guarantees that

$$\chi_{ab} = 0 \quad \text{on } \Sigma. \tag{3.16}$$

This shows that in our setting conditions (3.15) and (3.16) must necessarily be satisfied. It turns out that the boundary conditions (3.15) and initial data satisfying (3.16) are also sufficient for the construction of solutions with totally geodesic boundary. In fact, the reduced equations derived in [7] under general assumptions comprise the subsystem

$$D_0 \chi_{01} - D_1 \chi_{11} - D_2 \chi_{12} = D_1(f),$$
$$D_0 \chi_{02} - D_1 \chi_{12} - D_2 \chi_{22} = D_2(f),$$

$$D_0\chi_{11} - D_1\chi_{01} = -\beta_{12},$$
$$2D_0\chi_{12} - D_1\chi_{02} - D_2\chi_{01} = 2\beta_{11},$$
$$D_0\chi_{22} - D_2\chi_{02} = \beta_{12},$$

where $f = \chi_{00} - \chi_{11} - \chi_{22}$, $\beta_{11} = \frac{1}{2}(B_{11}^n - B_{22}^n) = \frac{1}{2}(B_{11}^N - B_{22}^N)$ and $\beta_{12} = B_{12}^n = B_{12}^N$. Because the vector fields e_a are tangential to T this subsystem defines a system intrinsic to the boundary T. It is symmetric hyperbolic. With the boundary condition (3.15) the right hand sides of the equations vanish on T. Observing the consistency condition (3.16), we can conclude, without explicit knowledge of the frame and the connection coefficients on T, that $\chi_{ab} = 0$ on T.

The argument shows that every ST-vacuum solution with totally geodesic boundary T can be obtained locally in time by solving the initial boundary value problem with boundary conditions that satisfy (3.15) and Cauchy data that imply (3.16). The main problem of characterizing all such solutions then reduces to the construction of Cauchy data for Einstein's vacuum field equations on three-manifolds S with boundary Σ for which the reduced equations determine a formal expansion with $\chi_{ab} = 0$, $B_{ab}^N = 0$ on Σ.

The particular choice of the gauge source functions made here is convenient but nowhere enters the argument. In fact, because of (3.14) conditions (3.15) will be satisfied in any frame. The obstructions to geometric uniqueness pointed out in [6] simply do not occur in this particular case. Anyway, the discussion of [6] does not ask so much for specific cases in which geometric uniqueness may hold but refers to the general problem with the complete freedom of prescribing three functions as boundary data.

In the case of anti-de Sitter type solutions that admit a smooth conformal boundary $\mathcal{J}$ at space-like and null infinity, the boundary is totally geodesic in a suitable conformal gauge as consequence of the field equations. Nevertheless, there is the freedom, in a sense similar to the second of conditions (2.21), to freely prescribe on $\mathcal{J}$ two functions derived from the conformal Weyl tensor. Moreover, with a condition similar to (2.23) the boundary conditions can be stated in a completely geometric way [5].

Since the Codazzi equation so much simplifies the above argument one may wonder whether something similar could be done by imposing conditions on the metric induced on T and using Gauss' equation (2.17). Because of the occurrence of the second fundamental form in that relation there appears to be no obvious way.

4. Prescribed mean extrinsic curvature

The case of point dependent mean intrinsic curvature is complicated because the function χ (or the gauge source function f) is not given in terms of some arbitrary coordinates like the z^α considered above. A relation like: $\chi = \chi(x^{\alpha'}, c)$ on T_c or $f = f(x^{\alpha'}, c)$ on T_c in section 2 is saying that we must think of χ or f as being given in the specific coordinates $x^{\alpha'}$ on T_c that are obtained by solving equations (2.2)–(2.9) on T_c. We write $x^{\alpha'}$ here to distinguish the index coming with x from the completely unrelated index of z^α. While the function f can be chosen freely away from the boundary $T = T_0$, the free data must be specified on T in some distinguished coordinate system. This coupling between the gauge on T and the way boundary data are prescribed, which is a specific feature of the initial boundary value problem, cannot be avoided unless the boundary conditions are completely stated in terms geometric structures (see the discussion in [6]). The functions $F^A = F^A(x^{\alpha'})$ will have to play a role when we specify χ or f.

Equations (2.2), (2.4) and (2.9), i.e.

$$D_{e_0} x^{\alpha'} = \delta^{\alpha'}{}_0, \quad D_{e_0} e_0 = F^A e_A, \quad D_{e_0} e_A = -F_A e_0, \tag{4.1}$$

are defined in terms of structures supposed to be induced by the ambient space-time on the hypersurface T_c which we want to determine by solving equation (3.9). The right-hand side of the latter must thus be arranged so as to correspond in the coordinates $x^{\alpha'}$ to the datum $\chi = \chi(x^{\alpha'}, c)$ (or $f = f(x^{\alpha'}, c)$) assumed as known.

This suggests to consider $F^A(x^{\alpha'}, c), f(x^{\alpha'}, c), \chi(x^{\alpha'}, c)$ as given functions and use (4.1) with $F^A = F^A(x^{\alpha'}(z^\beta, c), c)$ as a system of equations for $x^{\alpha'} = x^{\alpha'}(z^\beta, c)$ and the frame vectors e_a. The latter are thought to be given in the form $e_a^\alpha \partial_{z^\alpha}$ with $e_a^\alpha = e_a^\alpha(z^\beta)$ and can be expressed in terms of the coordinates $x^{\alpha'}$ once the coordinate transformation is available on T_c. The system (4.1) must be coupled to (3.9) with $\chi = \chi(x^{\alpha'}(z^\beta, c), c)$ or $f = f(x^{\alpha'}(z^\beta, c), c)$ on the right-hand side to determine ϕ and thus T_c. The dependence of the various functions on c will often be suppressed in the following because we mostly work with a fixed c.

This recipe does not work immediately. Two of equation (4.1) involve the Levi-Civita connection D of the metric $\bar{k}_{\alpha\beta}$ on T_c by (3.10), and thus the Christoffel symbols

$$\bar{\zeta}_\alpha{}_\gamma{}^\beta = \frac{1}{2}\bar{k}^{\beta\delta}(\bar{k}_{\delta\gamma,\alpha} + \bar{k}_{\alpha\delta,\gamma} - \bar{k}_{\alpha\gamma,\delta}) \quad \text{on } T_c. \tag{4.2}$$

They depend on the functions $g_{\mu\nu}(z^\alpha, \phi(z^\alpha) + c)$, their derivatives $g_{\mu\nu,\beta}(z^\alpha, \phi(z^\alpha) + c) + g_{\mu\nu,3}(z^\alpha, \phi(z^\alpha) + c)\phi_{,\beta}$, on the $\phi_{,\alpha}$, and in particular on the second derivatives $\phi_{,\alpha\beta}$. To control these functions we write equation (3.9) in the form

$$-\nu k^{\alpha\beta} \phi_{,\alpha\beta} - F = \chi(x^{\alpha'}(z^\beta)), \tag{4.3}$$

with

$$0 = D_\gamma(e^\alpha{}_0 D_\alpha x^{\beta'}) = e^\alpha{}_0 D_\gamma D_\alpha x^{\beta'} + D_\gamma e^\alpha{}_0 D_\alpha x^{\beta'}$$

$$= D_{e_0}(D_\gamma x^{\beta'}) + D_\gamma e^\alpha{}_0 (D_\alpha x^{\beta'}),$$

where the background function depend on $(z^\alpha, \phi(z^\alpha) + c)$. To obtain an equation that supplies the $\phi_{,\alpha\beta}$, we apply ∂_{z^γ} to the equation above and obtain

$$-\nu k^{\alpha\beta} (\phi_{,\gamma})_{,\alpha\beta} - G = \partial_{x^{\alpha'}} \chi \partial_{z^\gamma} x^{\alpha'} \tag{4.4}$$

with some function

$$G = G(\phi, \phi_{,\alpha}, \phi_{,\alpha\beta}, \text{background}),$$

where the background functions now involve $(z^\alpha, \phi(z^\alpha), \phi_{,\beta}(z^\alpha))$. For given smooth functions $x^{\alpha'}(z^\beta)$ the two equations above define a system of wave equations for ϕ and $\phi_{,\alpha}$. In addition to $x^{\alpha'}(z^\beta)$ the right-hand side of (4.4) requires, however, also control of

$$\partial_{z^\gamma} x^{\alpha'} = D_\gamma x^{\alpha'}. \tag{4.5}$$

The solvability of the coupled system (4.1), (4.3) and (4.4) thus depends on the possibility to

implement equations for this field. The first of equation (4.1) implies

$$0 = D_\gamma(e^\alpha{}_0 D_\alpha x^{\beta'}) = e^\alpha{}_0 D_\gamma D_\alpha x^{\beta'} + D_\gamma e^\alpha{}_0 D_\alpha x^{\beta'}$$

$$= D_{e_0}(D_\gamma x^{\beta'}) + D_\gamma e^\alpha{}_0 (D_\alpha x^{\beta'}).$$

To obtain equations for $D_\gamma e^\alpha{}_a$ we use

$$D_{e_0}(D_\gamma e^\beta{}_a) = e^\alpha{}_0 D_\gamma D_\alpha e^\beta{}_a + R^\beta{}_{\delta\alpha\gamma}[\bar{k}]e^\alpha{}_0 e^\delta{}_a$$

$$= D_\gamma(D_{e_0}e^\beta{}_a) - (D_\gamma e^\alpha{}_0)(D_\alpha e^\beta{}_a) + R^\beta{}_{\delta\alpha\gamma}[\bar{k}]e^\alpha{}_0 e^\delta{}_a.$$

For the first term on the right-hand side we get from equation (4.1)

$$D_\gamma(D_{e_0}e^\beta{}_0) = F^A{}_{,x^{\alpha'}}(D_\gamma x^{\alpha'})e^\beta{}_A + F^A(D_\gamma e^\beta{}_A),$$

$$D_\gamma(D_{e_0}e^\beta{}_A) = -F_{A,x^{\alpha'}}(D_\gamma x^{\alpha'})e^\beta{}_0 - F_A(D_\gamma e^\beta{}_0).$$

Thus together

$$D_{e_0}(D_\gamma x^{\beta'}) = -D_\gamma e^\alpha{}_0 (D_\alpha x^{\beta'}), \tag{4.6}$$

$$D_{e_0}(D_\gamma e^\beta{}_0) = F^A{}_{,x^{\alpha'}}(D_\gamma x^{\alpha'})e^\beta{}_A + F^A(D_\gamma e^\beta{}_A) \tag{4.7}$$

$$-(D_\gamma e^\alpha{}_0)(D_\alpha e^\beta{}_0) + R^\beta{}_{\delta\alpha\gamma}[\bar{k}]e^\alpha{}_0 e^\delta{}_0,$$

$$D_{e_0}(D_\gamma e^\beta{}_A) = -F_{A,x^{\alpha'}}(D_\gamma x^{\alpha'})e^\beta{}_0 - F_A(D_\gamma e^\beta{}_0), \tag{4.8}$$

$$-(D_\gamma e^\alpha{}_0)(D_\alpha e^\beta{}_A) + R^\beta{}_{\delta\alpha\gamma}[\bar{k}]e^\alpha{}_0 e^\delta{}_A.$$

These equations involve the curvature tensor of $\bar{k}$. Because the Christoffel symbols depend on $\phi_{,\alpha\beta}$ we can expect third derivatives of ϕ to enter the expression for the curvature tensor of $\bar{k}$. Because the background is known, however, we can use Gauss' equation

$$R^\beta{}_{\delta\alpha\gamma}[\bar{k}]e^\alpha{}_0 e^\delta{}_a = \tag{4.9}$$

$$\left(\sum_{\rho,\pi,\mu,\nu=0}^{3} R^\rho{}_{\pi\mu\nu}[g]k^\beta{}_\rho k^\pi{}_\delta k^\mu{}_\alpha k^\nu{}_\gamma - \chi^\beta{}_\alpha \chi_{\delta\gamma} + \chi^\beta{}_\gamma \chi_{\delta\alpha} \right) e^\alpha{}_0 e^\delta{}_a,$$

where the background field $R^\rho{}_{\pi\psi\nu}[g]$ is taken at $(z^\alpha, \phi(z^\alpha) + c)$ and (3.7) and (3.8) are used. The terms in large brackets only contain $\phi_{,\alpha}$ and $\phi_{,\alpha\beta}$.

   With (4.9) taken into account, the system (4.1), (4.3), (4.4) and (4.6)–(4.8) provides the desired closed system for the unknowns

$$x^{\alpha'}, \quad x^{\beta'}{}_{,\alpha}, \quad e^\beta{}_a, \quad D_\alpha e^\beta{}_a, \quad \phi, \quad \phi_{,\alpha}. \tag{4.10}$$

The initial data on S_c are determined, respectively chosen, as follows.
   As in (3.1) we assume

$$\phi|_{S_c} = 0 \quad \text{so that} \quad \phi_{,\alpha} = \delta^0{}_\alpha \phi_{,0} \quad \text{on } S_c.$$

The function $\phi_{,0}$ must be chosen on S_c so as to satisfy (3.3). Later on we will be led to consider the further condition (4.11).

The unique future directed unit vector field orthogonal to N and S_c is given by

$$e^{\mu}{}_0 = \frac{T^{\mu}}{\sqrt{(g_{\nu\rho}T^{\nu}T^{\rho})}} \quad \text{on } S_c,$$

where T^{μ} is the vector field (3.5) on S_c which will be tangential to T_c. Because $e^3{}_0 = \phi_{,0}e^0{}_0 = \phi_{,\alpha}e^{\alpha}{}_0$, $e^{\mu}{}_0$ is uniquely determined by

$$e^{\alpha}{}_0 = \frac{T^{\alpha}}{\sqrt{(g_{\nu\rho}T^{\nu}T^{\rho})}} = \frac{T^{\alpha}}{\sqrt{(\bar{k}_{\alpha\beta}T^{\alpha}T^{\beta})}}$$

$$= \nu\frac{(g^{03} - \phi_{,0}g^{00})g^{\alpha3} - (g^{33} - \phi_{,0}g^{03})g^{\alpha0}}{\sqrt{(g^{03})^2 - g^{33}g^{00}}} \quad \text{on } S_c.$$

It satisfies

$$e^0{}_0 = \nu\sqrt{(g^{03})^2 - g^{33}g^{00}} > 0.$$

The fields $e^{\alpha}{}_A$ are chosen tangential to S_c so that $e^0{}_A = 0$, whence $\bar{k}_{\alpha\beta}e^{\alpha}{}_0e^{\beta}{}_A = 0$. They are required to satisfy

$$\bar{k}_{\alpha\beta}e^{\alpha}{}_Ae^{\beta}{}_B = \bar{k}_{CD}e^C{}_Ae^D{}_B = g_{CD}e^C{}_Ae^D{}_B = -\delta_{AB}.$$

The forms dual to e_a are denoted by $\sigma^a = \sigma^a{}_{\alpha}\,dz^{\alpha}$ so that $\sigma^a{}_{\alpha}e^{\alpha}{}_b = \delta^a{}_b$. We assume

$$x^{0'} = 0, \quad \text{whence} \quad x^{0'}{}_{,A} = 0 \quad \text{on } S_c.$$

With the z^A defining local coordinates on S_c, we choose local coordinates $x^{A'} = x^{A'}(z^A)$ on S_c, which give $x^{A'}{}_{,A}$ on S_c with $\det(x^{A'}{}_{,A}) \neq 0$. Following (2.5) we need to require $\delta^{\alpha'}{}_0 = e^{\alpha'}{}_0 = e^{\alpha}{}_0x^{\alpha'}{}_{,\alpha}$ which implies

$$x^{0'}{}_{,0} = \frac{1}{e^0{}_0}, \qquad x^{A'}{}_{,0} = -\frac{1}{e^0{}_0}e^A{}_0x^{A'}{}_{,A}, \qquad e^{0'}{}_A = 0.$$

It remains to determine

$$D_{\alpha}e^{\beta}{}_b = \sigma^a{}_{\alpha}D_{e_a}e^{\beta}{}_b.$$

By (2.9) we must set, with the right hand sides given by the fields obtained so far,

$$D_{e_0}e^{\beta}{}_0 = F^Ae^{\beta}{}_A, \qquad D_{e_0}e^{\beta}{}_A = -F_Ae^{\beta}{}_0 \quad \text{on } S_c.$$

With the given information we can finally calculate

$$D_{e_A}e^{\beta}{}_0, \qquad D_{e_A}e^{\beta}{}_B \quad \text{on } S_c,$$

by using the Christoffel symbols (4.2). These involve the functions $\phi_{,\alpha\beta}$ on S_c. From the data given above we get

$$\phi_{,AB} = 0 \qquad \phi_{,0B} = (\phi_{,0})_{,B} \quad \text{on } S_c.$$

To obtain $\phi_{,00}$ we solve (3.9) on S_c for $k^{\alpha\beta}\phi_{,\alpha\beta}$ and observe that by (3.11)

$$k^{00} = \bar{k}^{00} \equiv g^{00} + \nu^2(g^{03} - g^{00}\phi_{,0})^2 > 0 \quad \text{on } S_c.$$

To discuss the solvability of our system we consider its principal part.

The first four of the unknowns (4.10) can be combined to an $\mathbb{R}^j$-valued unknown w that satisfies an equation of the form

$$e_0^\alpha w_{,\alpha} = \ldots,$$

while the remaining unknowns combine to an $\mathbb{R}^l$-valued unknown u that satisfies an equation of the form

$$\bar{k}^{\alpha\beta} u_{,\alpha\beta} = \ldots.$$

That e_0 is time-like with respect to $\bar{k}_{\alpha\beta}$ suggests that the coupled system can be written as a quasi-linear, symmetric hyperbolic system of first order [8].

In fact, in terms of the auxiliary unknowns $v_\alpha \equiv u_{,\alpha}$ the equation for u implies the system of first order

$$u_{,0} = v_0,$$
$$\bar{k}^{00}v_{0,0} + 2\bar{k}^{0A}v_{0,A} + \bar{k}^{A\,B}v_{A,B} = \ldots$$
$$-\bar{k}^{AB}v_{B,0} + \bar{k}^{AB}v_{0,B} = 0,$$

and the equations for w writes

$$e^0{}_0 w_{,0} + e^A{}_0 w_{,A} = \ldots$$

We have seen above that $e_0^0 > 0$ and $\bar{k}^{00} > 0$ on (whence near) S_c. If $\bar{k}^{AB}$ defines a negative definite symmetric bilinear form on (whence near) S_c, it follows that the combined system is quasi-linear, symmetric hyperbolic. It also implies then for its solutions the integrability condition $v_{B,0} = v_{0,B}$ and thus $(v_{A,B} - v_{B,A})_{,0} = v_{0,AB} - v_{0,BA} = 0$. With suitably given initial data the remaining integrability conditions follow. It remains to see under which conditions $\bar{k}^{AB}$ is negative definite.

Assume $a, b \in \mathbb{R}, y, z \in \mathbb{R}^k, k \geqslant 2$, and A, B are $k \times k$ matrices so that the following matrix equation holds with $k \times k$ unit matrix 1_k

$$\begin{pmatrix} a & {}^t y \\ y & A \end{pmatrix} \begin{pmatrix} b & {}^t z \\ z & B \end{pmatrix} = \begin{pmatrix} 1 & {}^t 0_k \\ 0_k & 1_k \end{pmatrix}.$$

If $b > 0$ and A is negative definite and symmetric, i.e. ${}^t u\, A\, u < 0$ for $u \neq 0$ and ${}^t A = A$, then B is negative definite and symmetric if and only if $a > 0$.

In fact, the matrix equation is equivalent to the relations

$$by + Az = 0, \qquad ab + {}^t yz = 1, \qquad y\,{}^t z + AB = 1_k, \qquad a\,{}^t z + {}^t yB = 0.$$

The second equation implies $a > 0$ if $y = 0$. Assume that $y \neq 0$. Being symmetric and negative definite, A has an inverse A^{-1}, which is also symmetric and negative definite. The first equation implies $z = -b\,A^{-1}y$, which gives with the second relation $1 = b(a - {}^t x\,A^{-1}y)$ and thus with our assumptions

$$a - {}^t yA^{-1}y > 0, \qquad b = \frac{1}{a - {}^t yA^{-1}y}, \qquad z = -\frac{1}{a - {}^t yA^{-1}y}A^{-1}y.$$

The third and fourth equations are then satisfied with the symmetric matrix

$$B = A^{-1} + \frac{1}{a - {}^t y A^{-1} y} (A^{-1} y)^t (A^{-1} y).$$

In terms of the positive definite matrix $W = -A^{-1}$ the condition ${}^t u \, B \, u < 0$ for $u \neq 0$ translates into

$$({}^t u W y)^2 < ({}^t y W y)({}^t u W u) + a({}^t u W u).$$

The Cauchy–Schwarz inequality with $u = y$ implies that $a > 0$.   □

Compare the matrix equation above with the relation $\bar{k}_{\alpha\beta}\bar{k}^{\beta\gamma} = \delta_\alpha^\gamma$. Since S is space-like and $\bar{k}_{AB} = g_{AB}$ on S_c, $\bar{k}_{AB}$ is negative definite. We saw above $\bar{k}^{00} > 0$ on S_c. Thus $\bar{k}^{AB}$ is negative definite if and only if $\bar{k}_{00} = g_{00} + 2g_{3(0}\phi_{,0)} + g_{33}\phi_{,0}\phi_{,0} > 0$ on S_c, or

$$\bar{k}_{00} = g_{\mu\nu} P^\mu P^\nu > 0 \quad \text{on } S_c \quad \text{with} \quad P^\mu = \delta_0^\mu + \phi_{,0}\delta_3^\mu. \tag{4.11}$$

It holds $N_\mu P^\mu = 0$. Thus P^μ is tangential to T_c on S_c, but without further assumptions it need not be time-like. By suitable choices of the hypersurface $\mathcal{T}$ and the coordinates z^μ, which were rather arbitrary so far, it can be arranged that P^μ is in fact time-like on, whence near S_c with $\phi_{,0}$ satisfying (3.3).

The Cauchy problem for the system (4.1), (4.3), (4.4) and (4.6)–(4.8) with initial data on open subsets of S_c as discussed above is then well posed. The corresponding solutions define pieces of the prospective hypersurface T_c. These local pieces can be patched together to obtain a part of the hypersurface T_c diffeomorphic to $[0, x_*^{0'}[\times S_c$, where $x_*^{0'} > 0$ and the unknown $x^{0'}$ takes values in $[0, x_*^{0'}[$.

Because these solutions depend smoothly on the initial data and, for sufficiently small $c_* > 0$, on $c \in [0, c_*[$, the hypersurfaces T_c define a foliation that is smooth in the sense that the function $x^{3'}$ obtained by setting $x^{3'} = c$ on T_c is smooth. Moreover, the coordinates $x^{\alpha'}$ and the vector fields e_0, e_A obtained on the patches can be glued together to give smooth coordinates and vector fields (expressed in terms of $x^{\alpha'}$) that satisfy equation (4.1) with the given functions $F^A = F^A(x^{\alpha'}, x^{3'})$ on the domain covered by the foliation. The mean extrinsic curvature of the T_c is given there by $\chi = \chi(x^{\alpha'}, x^{3'})$ (or $f = f(x^{\alpha'}, x^{3'})$).

5. Concluding remarks

It has been shown that the gauge based in section 2 on the choice of the function x^3 and the vector field e_0, both suitably adapted to the given time-like boundary T, can be completely reconstructed, together with the hypersurface T, on the basis of the given gauge source functions f and F^A. The construction imposes no conditions on the underlying space-time and, in particular, does not require the metric to satisfy any field equation. The system of differential equations required for this turns out to be fairly complicated and only quasi-linear. Obtaining information about the life time of a gauge is a notoriously difficult problem. The characterization in terms of f and F^A appears particularly difficult. A closer comparison with the way the gauge is discussed in section 2, which covers without complications a whole neighbourhood of the *given* hypersurface T, may give some insight into this.

Because the system considered in section 4 looks so difficult, it may be mentioned that fixing the gauge in terms of f and F^A introduces no additional complication into the reduced system extracted in [7] from equations (2.12) and (2.14). Part of the reason is that the curvature, which somewhat unexpectedly enters the system derived in section 4, is already an unknown in the

reduced system. In a similar way as equation (1.1) reduces to the relation $-g^{\nu\lambda}\Gamma_\nu{}^\mu{}_\lambda = F^\mu$ if it is expressed in terms of the coordinates x^μ that solve the equation, the equations of section 4 reduce to simpler expressions if they are expressed in terms of their solution so that $\mathcal{T}$ coincides with the hypersurface T_0 and $x^{3'} = z^3$ whence $\phi = 0$. The possibility to declare a particular set of functions as gauge source functions and the usefulness of this choice obviously depends on the chosen representation of the field equations.

Data availability statement

No new data were created or analysed in this study.

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