

# $T\bar{T}$ deformation on multi-quantum mechanics and regenesis

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## Abstract

We study the  $T\bar{T}$  deformation on a multi-quantum mechanical systems. By introducing the dynamical coordinate transformation, we obtain the deformed theory as well as the solution. We further study the thermo-field-double state under the  $T\bar{T}$  deformation on these systems, including conformal quantum mechanical system, the Sachdev-Ye-Kitaev model, and the model satisfying Eigenstate Thermalization Hypothesis. We find common regenesis phenomena where the signal injected into one local system can regenerate from the other local system. From the bulk picture, we study the deformation on Jackiw-Teitelboim gravity governed by Schwarzian action and find that the regenesis phenomena here are not related to the causal structure of semi-classical wormhole.

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## 1. INTRODUCTION

The  $T\bar{T}$  deformation of field theory has attracted much research interest in recent years both from viewpoint of field theory and in the context of holographic duality. The  $T\bar{T}$  deformation of 2D rotation and translation invariant field theory is defined by [1–3]. The deformation is triggered by the irrelevant and double-trace operator  $T\bar{T} = -\det(T_{\mu\nu})$ . Although the  $T\bar{T}$  deformation flows towards UV, it has numerous intriguing properties. A remarkable property is integrability [2, 4, 5]. If the un-deformed theory is integrable, there exists a set of infinite commuting conserved charges or KdV charges in the deformed theory. If the theory is maximally chaotic, the deformed theory holds the maximal chaos [6, 7], which agrees with the fact that  $T\bar{T}$  deformation is irrelevant.

The  $T\bar{T}$  deformation of the  $(0+1)$ -dimensional quantum mechanical (QM) system is studied in [8, 9]. When the QM system is taken as the Sachdev-Ye-Kitaev (SYK) model [10–15], the deformed SYK model exhibits the maximal chaotic behavior as the undeformed model. One can also refer to [16] for one-dimension deformation of boson gas.

Our objective here is to study the  $T\bar{T}$  deformation in multi-QM systems. This is a wide class of integrable deformations of the QM which can be regarded as a transformation of the Hamiltonian  $H \rightarrow f(H)$ . Since the  $T\bar{T}$  deformation is generally non-local, as shown in

[3], the  $T\bar{T}$  deformation on multi-QM systems effectively couple the local system with each other and generate a non-local phenomenon. In this paper, we study the causal correlation caused by the  $T\bar{T}$  deformation on the bi-QM system, where the two local QM systems, labeled by  $L$  and  $R$ , share the Hamiltonian in the same form.

We will focus on a particular entangled state on bi-QM system, thermo-field-double (TFD) state, where the local system is in a thermal state and the local entropy is caused by entanglement. When the QM system enjoys holographic duality, the geometric correspondence of the TFD state is an eternal black hole [17, 18].

When the two QM systems are coupled with each other and their interactions match the entanglement structure of the TFD state, a phenomenon similar to the quantum teleportation appears, where the signal injected into one QM system can regenerate from the other QM system [19]. The teleportation of the quantum state is constructed in the SYK model [20] and in 2D CFT [21]. We call this phenomenon regeneration.

The geometric correspondence via holography is the “traversable wormhole” [22–25]. A signal injected into the external black hole from one boundary at a proper time can transverse the Einstein-Rosen bridge and reach the other boundary. The traversability of the wormhole is closely associated with the violation of averaged null energy condition (ANEC). The ANEC conjectures states that the integral of null energy on null ray must be non-negative in any UV complete QFT. The ANEC has been proven in many special cases [26–28]. The ANE can measure the change of causal structure when we perturb solution of vacuum Einstein equation by matter stress tensor. When ANE is negative, the null ray in unperturbed metric becomes time-like in perturbed metric. In classical general relativity, existence of traversable wormhole implies negative ANE. To construct a traversable wormhole, the authors in [22, 23] add the double-trace deformations  $O_L O_R$  between the two sides of the black hole. Under this deformation, the ANEC is violated and the Einstein-Rosen bridge of the eternal black hole becomes traversable.

However, not all the regeneration phenomena have geometric correspondence in semi-classical approximation [19, 23]. In the interference region, where the signal is injected earlier than the scrambling time, the back-reaction to the wormhole destroys the correlation between  $O_L$  and  $O_R$  and contributes a non-zero phase on the correlator carrying the signal. The signal regenerates from the other side at the time which is reversed to the time of the injection. Such kind of regeneration phenomenon is called “quantum traversable wormhole” [19].

The above double-trace deformation on the bi-QM system is relevant and able to change the ground state [24, 25]. This paper will consider the  $T\bar{T}$  deformation on the bi-QM systems, which is the double-trace deformation with stress tensors. Because it’s a non-local and irrelevant deformation, we expect to find the regeneration phenomena contributed by ultraviolet (UV) channels. Similarly, we will focus on TFD states. In the usual constructions of a traversable wormhole, the non-local deformation should match the entanglement structure of the TFD state, namely, the  $O_L$  and  $O_R$  constructing the deformation should be correlated initially. However, the  $T\bar{T}$  deformation is unique and is not related to the entanglement structure. So we will expect a relatively weak but general regeneration phenomenon.

The organization of this paper is as follows. In section 2, we give a general framework of the  $T\bar{T}$  deformation on a single or multi QM system. In section 3, we study the first order  $T\bar{T}$  deformation on bi-QM system in TFD states. Taking conformal QM, the SYK model, and the system satisfying Eigenstate Thermalization Hypothesis (ETH) as examples, we show general regeneration phenomena where a signal can pass from one QM system to the other QM system. In section 4, we study the  $T\bar{T}$  deformation on the wormhole in Schwarzian

theory where the result is in agreement with the analysis from the bi-QM system. We end in section 5 with summarise and prospects.

## 2. $T\bar{T}$ DEFORMATION ON $(0+1)$ -DIMENSIONAL SYSTEMS

This section, we give some general approaches to the study of the  $T\bar{T}$  deformation on multi-QM system, which will be utilized in the following sections.

### 2.1. Solution of $T\bar{T}$ deformed Hamiltonian

Consider a pair of canonical variables  $\{q, p\}$  and the original Hamiltonian  $H_0(q, p)$ . Given a solution of the Hamiltonian equation

$$q(t) = \tilde{q}(q_0, p_0, t), \quad p(t) = \tilde{p}(q_0, p_0, t), \quad (2.1)$$

whose initial condition is  $q_0 = q(0)$  and  $p_0 = p(0)$ , we consider a new Hamiltonian

$$H = f(H_0), \quad (2.2)$$

whose form may be taken as the  $T\bar{T}$  deformation proposed by [8, 9]. The new Hamiltonian equations are

$$q' = f'(H_0) \frac{\partial H_0}{\partial p}, \quad p' = -f'(H_0) \frac{\partial H_0}{\partial x}. \quad (2.3)$$

One can find the solution of the deformed theory with the same initial condition

$$q(t) = \tilde{q}(q_0, p_0, T), \quad p(t) = \tilde{p}(q_0, p_0, T), \quad T = f'(H_0(q_0, p_0)) t, \quad (2.4)$$

where we call  $T$  the dynamical coordinate.

The above process of finding new solution can be generalized to the theory  $H_0(\vec{q}, \vec{p})$  with multi pairs of canonical variables  $\{\vec{q} = (q_1, q_2, \dots, q_n), \vec{p} = (p_1, p_2, \dots, p_n)\}$ . The new solution is

$$q_s(t) = \tilde{q}_s(\vec{q}_0, \vec{p}_0, T), \quad p_s(t) = \tilde{p}_s(\vec{q}_0, \vec{p}_0, T), \quad T = f'(H_0(\vec{q}_0, \vec{p}_0)) t, \quad s = 1, 2, \dots, n, \quad (2.5)$$

which satisfies the initial condition  $\vec{q}_0 = (q_1(0), q_2(0), \dots, q_n(0))$  and  $\vec{p}_0 = (p_1(0), p_2(0), \dots, p_n(0))$ .

### 2.2. $T\bar{T}$ Deformation and dynamical coordinate

The  $T\bar{T}$  deformation can be formulated as a dynamical change of coordinates. In this section, we would like to explore the  $T\bar{T}$  deformation in the  $(0+1)$ -dimensional version of [29–37]. One can couple an action  $S_0$  to a  $(0+1)$ -dimensional "gravity":

$$S[e_\mu, v^\mu, \phi] = S_{\text{grav}}[e_\mu, v^\mu] + S_0[e_\mu, \phi] \quad (2.6)$$

$$S_{\text{grav}}[e_\mu, v^\mu] = \frac{1}{\lambda} \int dt e_t B(e_t v^t), \quad (2.7)$$

where the 1-form  $e_\mu$  is the dynamical tetrad, the unknown function  $B$  is determined later, and the vector  $v^\mu$  is a fixed co-tetrad corresponding to the metric on which the deformed theory lives. One can take  $v^t = 1$  and then

$$v^T = \frac{dT}{dt}, \quad e_T = 1, \quad e_t = \frac{dT}{dt}. \quad (2.8)$$

Taking scalar theory as an example, we consider the  $S_0$  as follow

$$S_0 = \int dt e_t \left( \frac{1}{2(e_t)^2} \partial_t \phi \partial_t \phi - V(\phi) \right), \quad (2.9)$$

or one can use the first order formalism

$$S_0 = \int dt e_t \left( \frac{1}{e_t} p \partial_t \phi - H_0(\phi, p) \right), \quad (2.10)$$

where the  $p$  is the canonical momentum in the phase space and  $H_0$  is the corresponding Hamiltonian of the undeformed theory. The equation of motion of  $e_t$  gives

$$e_t v^t B'(e_t v^t) + B(e_t v^t) - \lambda H_0 = 0. \quad (2.11)$$

In  $T$  coordinate, from (2.8), it becomes

$$\frac{dT}{dt} B' \left( \frac{dT}{dt} \right) + B \left( \frac{dT}{dt} \right) - \lambda H_0 = 0. \quad (2.12)$$

Using  $dT = f'(H_0) dt$ , one can obtain a relation between  $f$  and  $B$

$$f'(H_0) B'(f'(H_0)) + B(f'(H_0)) - \lambda H_0 = 0, \quad (2.13)$$

whose solution is

$$B(f'(H)) = \lambda H - \frac{\lambda f(H)}{f'(H)} + \frac{C}{f'(H)}, \quad (2.14)$$

where  $C$  is a constant.

In  $t$  coordinate, the solution of (2.11) with the  $B$  in the form (2.14) is  $e_t = f'(H_0)$ . By integrating out  $e_t$  in the action, the resulting action is

$$S = \int dt (p \partial_t \phi - f(H_0)), \quad (2.15)$$

where the constant term  $C/\lambda$  has been dropped.

For  $T\bar{T}$  deformation [9], we have

$$f(H) = \frac{1 - \sqrt{1 - 8H\lambda}}{4\lambda}, \quad (2.16)$$

and then

$$B(x) = \frac{(x-1)^2}{8x^2}. \quad (2.17)$$

If  $S_0$  takes the form given by (2.9), the deformed action after integrating out  $e_t$  is given by

$$S = \int dt \left( \frac{\sqrt{4\partial_t \phi \partial_t \phi \lambda + 1} \sqrt{1 - 8\lambda V(\phi)} - 1}{4\lambda} \right). \quad (2.18)$$

One can start from the single 1D Liouville action to obtain the deformed action, which is the same as the deformed one given in [9].

### 2.3. $T\bar{T}$ deformation on multi-fields

For the theory of multi scalars  $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_n)$  in (0+1) dimension, one can obtain the  $T\bar{T}$  deformed action as follows. Consider the original Lagrange

$$\mathcal{L}_0 = \frac{1}{2} \sum_s \phi'_s \phi'_s - V(\vec{\phi}). \quad (2.19)$$

The Hamiltonian is

$$H_0 = \frac{1}{2} \sum_s p_s p_s + V(\vec{\phi}), \quad (2.20)$$

where  $p_s$  is the canonical momentum corresponding to  $\phi_s$ . Consider the  $T\bar{T}$  deformation

$$H_\lambda = \frac{1 - \sqrt{1 - 8\lambda H_0}}{4\lambda} \quad (2.21)$$

The deformed Lagrange is

$$\mathcal{L}_\lambda = \frac{\sqrt{(1 + 4\lambda \sum_s \phi'_s \phi'_s)(1 - 8\lambda V(\vec{\phi}))}}{4\lambda} \quad (2.22)$$

It satisfies the flow equation

$$\frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = \frac{-T_\lambda^2}{1/2 - 2\lambda T_\lambda}, \quad (2.23)$$

where the deformed energy-momentum tensor is

$$T_\lambda = \sum_s \phi'_s \frac{\partial \mathcal{L}_\lambda}{\partial \phi'_s} - \mathcal{L}_\lambda. \quad (2.24)$$

## 3. CAUSAL CORRELATION CAUSED BY THE $T\bar{T}$ DEFORMATION

### 3.1. The first order $T\bar{T}$ deformation on bi-QM system

Consider a quantum mechanical system (QM) with Hilbert space  $\mathcal{H}$  and Hamiltonian  $H$ . Let  $D = \dim \mathcal{H}$ . Denote the spectrum density of  $H$  as  $\rho(E)$ . The summation of energy spectrum can be written as  $\sum_E = D \int dE \rho(E)$ .

Consider two-copy of quantum mechanical systems  $\text{QM}_L$  and  $\text{QM}_R$  with Hilbert space  $\mathcal{H} \otimes \mathcal{H}$  and Hamiltonian  $H_0 = H_L + H_R$  where  $H_L = H \otimes 1$  and  $H_R = 1 \otimes H$ .

We consider the global Hamiltonian

$$H_\lambda = f(H_0), \quad (3.1)$$

which is the  $T\bar{T}$  deformation (2.16) at first order

$$f(H) = H + 2\lambda H^2. \quad (3.2)$$

The  $T\bar{T}$  term couples  $\text{QM}_L$  and  $\text{QM}_R$  with each other. Since the  $T\bar{T}$  deformation is irrelevant, the ground state of the deformed theory remains unchanged in general. We introduce the  $T\bar{T}$  deformation on states with two strategies.

### 3.2. $T\bar{T}$ quenched TFD state

#### 3.2.1. State

We prepare the non-normalized TFD state without any  $T\bar{T}$  deformation

$$|\Psi\rangle = \sum_E e^{-\beta E/2} |E\rangle_L |E\rangle_R \quad (3.3)$$

whose reduce density matrix on each side are

$$\rho = \sum_E e^{-\beta E} |E\rangle \langle E|. \quad (3.4)$$

Their normalization are

$$|\tilde{\Psi}\rangle = |\Psi\rangle / \sqrt{Z(\beta)}, \quad \tilde{\rho} = \rho/Z(\beta), \quad Z(\beta) = \sum_E e^{-\beta E}. \quad (3.5)$$

Consider TFD state  $|\Psi\rangle$  at  $t = 0$  and evolve it with the deformed Hamiltonian  $H_\lambda$ , namely

$$|\Psi(t)\rangle = e^{-itf(H_0)} |\Psi\rangle. \quad (3.6)$$

Notice that the reduced density matrices remain unchanged

$$\rho(t) = \rho. \quad (3.7)$$

So the Renyi entropies between  $\text{QM}_L$  and  $\text{QM}_R$  are independent of time.

#### 3.2.2. Correlation

Consider a local and Hermitian operator  $O$  acting on  $\mathcal{H}$ . Its copies on each QM system are

$$O_L = O \otimes 1, \quad O_R = 1 \otimes O^T, \quad (3.8)$$

where the transpose is taken on the energy basis of  $H_0$ .

To study the causal correlation between two QM systems under the  $T\bar{T}$  quench, we calculate the retarded correlator

$$G_{LR}^R(t_1, t_2) = -i\Theta(t_-) \langle \tilde{\Psi} | [O_L(it_1), O_R(it_2)] | \tilde{\Psi} \rangle = 2\Theta(t_-) \text{Im} \langle \tilde{\Psi} | O_L(it_1) O_R(it_2) | \tilde{\Psi} \rangle, \quad (3.9)$$

where  $t_\pm = t_1 \pm t_2$  and  $O(\tau) = e^{\tau H_{LR}} O e^{-\tau H_{LR}}$ . It is the linear response of the protocol, sending a signal from  $\text{QM}_R$  at time  $t_2$  and measuring  $\text{QM}_L$  at time  $t_1$ . Let's first calculate the correlator with  $\tau_1 > \tau_2$  on energy basis

$$\langle \Psi | O_L(\tau_1) O_R(\tau_2) | \Psi \rangle \quad (3.10)$$

$$= \sum_{E_1 E_2} O_{12} O_{21} \exp \left\{ -\frac{\beta}{2} E_1 - \frac{\beta}{2} E_2 + \tau_1 f(2E_1) - \tau_2 f(2E_2) - \tau_{12} f(E_1 + E_2) \right\} \quad (3.11)$$

$$= \sum_{E_1 E_2} O_{12} O_{21} \exp \left\{ -\frac{\beta}{2} E_+ + 2\lambda E_-^2 \tau_- + E_- \tau_+ (1 + 4\lambda E_+) \right\} \quad (3.12)$$

$$= \sum_{E_1 E_2} O_{12} O_{21} \exp \left\{ -\frac{\beta}{2} E_+ \right\} \int_{-i\infty}^{i\infty} d\beta' K(-2\lambda\tau_-, (1 + 4\lambda E_+) \tau_+ + \beta') \exp \{-\beta' E_-\}, \quad (3.13)$$

where  $O_{ij} = \langle E_i | O | E_j \rangle$ ,  $E_{\pm} = E_1 \pm E_2$ ,  $\tau_{\pm} = \tau_1 \pm \tau_2$ , and the kernel

$$K(\alpha, \beta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE e^{-\alpha E^2 + \beta E} = \frac{-i}{2\sqrt{\pi\alpha}} \exp \frac{\beta^2}{4\alpha}. \quad (3.14)$$

To analytically calculate the above transformation, we consider the weakly-coupled limit  $|\lambda| \ll 1/E_{\beta}$ , where  $E_{\beta} = \langle \tilde{\Psi} | H_s | \tilde{\Psi} \rangle$  is the energy at inverse temperature  $\beta$ . So we can do the approximation  $|\lambda|E_{\pm} \ll 1$ , so that

$$G_{LR}(t_1, t_2) = \langle \tilde{\Psi} | O_L(it_1) O_R(it_2) | \tilde{\Psi} \rangle \approx i \int_{-\infty}^{\infty} du K(-2i\lambda t_-, it_+ + iu) G_W(u; \beta), \quad (3.15)$$

where the half-circle Wightman correlator is  $G_W(u; \beta) = \text{Tr}[e^{-(\beta/2+iu)H} O e^{-(\beta/2-iu)H} O] / Z(\beta)$ . The approximation is exact when  $t_+ = 0$ . Applying complex conjugate to (3.15), we find

$$\lambda \leftrightarrow -\lambda, \quad G_{LR}(t_1, t_2) \leftrightarrow G_{LR}(t_1, t_2)^* \quad (3.16)$$

at the weakly-coupled limit. So we consider positive  $\lambda$ .

Furthermore, at weakly-coupled limit  $|\lambda| \ll 1/E_{\beta}$ , we can use the saddle point approximation  $u = -t_+ + \delta u$ , so that

$$G_{LR}(t_1, t_2) \approx \int_{-\infty}^{\infty} d\delta u \sqrt{\frac{i}{8\pi\lambda t_-}} \exp \frac{-i\delta u^2}{8\lambda t_-} \left( G_W(-t_+; \beta) + \frac{1}{2} \delta u^2 G_W''(-t_+; \beta) \right) \quad (3.17)$$

$$= G_W(-t_+; \beta) - 2i\lambda t_- G_W''(-t_+; \beta). \quad (3.18)$$

So the retarded correlator is approximately

$$G_{LR}^R(t_1, t_2) \approx -4\lambda t_- \Theta(t_-) G_W''(-t_+; \beta), \quad (3.19)$$

which is just the result from the first-order perturbation on  $\lambda$ , since

$$[O_L(\tau_1), O_R(\tau_2)] = 4\lambda \tau_- \dot{O}_L^{(0)}(\tau_1) \dot{O}_R^{(0)}(\tau_2) + \mathcal{O}[\lambda^2], \quad (3.20)$$

where  $O^{(0)}(\tau) = e^{\tau(H_L+H_R)} O e^{-\tau(H_L+H_R)}$ . Because of the entanglement structure,  $G_W''(-t_+; \beta)$  is maximized at  $t_+ = 0$ . So the signal comes out from  $\text{QM}_L$  around the time  $t_1 = -t_2$ .

A similar regeneration phenomenon appears if we apply an instantaneous  $T\bar{T}$  quench on the TFD state

$$H_{\lambda}(t) = H_L + H_R + 2\lambda(H_L + H_R)^2 \delta(t) \quad (3.21)$$

The retarded correlator at the first-order perturbation on  $\lambda$  is

$$G_{LR}^R(t_1, t_2) = -i\Theta(t_-) \left\langle \tilde{\Psi} \left| \left[ e^{i2\lambda(H_L+H_R)^2} O_L^{(0)}(it_1) e^{-i2\lambda(H_L+H_R)^2}, O_R^{(0)}(it_2) \right] \right| \tilde{\Psi} \right\rangle \quad (3.22)$$

$$\approx 4\lambda \Theta(t_-) \left\langle \tilde{\Psi} \left| \dot{O}_L^{(0)}(it_1) \dot{O}_R^{(0)}(it_2) \right| \tilde{\Psi} \right\rangle \quad (3.23)$$

$$= -4\lambda \Theta(t_-) G_W''(t_+; \beta) \quad (3.24)$$

When  $t_1 = -t_2 = t > 0$ ,  $G_{LR}^R(t, -t) \approx -4\lambda G''(0; \beta)$  does not depend on  $t$  at all. Signal can pass from  $\text{QM}_R$  to  $\text{QM}_L$  instantly.



Both kinds of  $T\bar{T}$  quench leads to non-vanishing retarded correlators. The entanglement structure of TFD state leads to the quantum correlation between the operator  $O_L$  and  $O_R$ . Since the operators also perturb the energy correlation, under the  $T\bar{T}$  deformation the quantum correlation becomes the causal correlation. We can imagine the process that we send a signal into  $\text{QM}_R$  at a time and measure it on  $\text{QM}_L$  at the reversed time with the highest intensity. It is similar to the traversal phenomenon under non-local double-trace deformation in the interference region [19, 20]. Since this phenomenon is mainly related to the two-point function on canonical ensemble, it is not associated with chaos nor scrambling [12, 38, 39].

### 3.3. $T\bar{T}$ deformed TFD state

#### 3.3.1. State

Alternatively, we can prepare a new TFD state with the  $T\bar{T}$  deformed Hamiltonian

$$|\Psi_\lambda\rangle = \sum_E e^{-\beta f(E)/2} |E\rangle_L |E\rangle_R \quad (3.25)$$

$$\rho_\lambda = \sum_E e^{-\beta f(E)} |E\rangle \langle E| \quad (3.26)$$

and let it evolve with the deformed Hamiltonian

$$|\Psi_\lambda(t)\rangle = e^{-itf(H_0)} |\Psi_\lambda\rangle \quad (3.27)$$

$$\rho_\lambda(t) = \rho_\lambda. \quad (3.28)$$

where  $\rho_\lambda$  is the reduced density matrix on each QM system. The state can be normalized as  $|\tilde{\Psi}_\lambda\rangle = |\Psi_\lambda\rangle / \sqrt{Z_\lambda(\beta)}$  where the deformed partition function is  $Z_\lambda(\beta) = \text{Tr}[e^{-\beta f(H)}]$ . The Renyi entropies are time-independent. Since  $f'(E) > 1$  for  $\lambda > 0$ , the deformation enhances the imbalance of energy distribution  $e^{-\beta f(E)}$ , namely, low energy states have higher probabilities. So, at the same temperature, the entanglement in the  $T\bar{T}$  deformed TFD state is generally lower than that in the  $T\bar{T}$  quenched TFD state.

#### 3.3.2. Correlation

The correlator on the  $T\bar{T}$  deformed TFD state is

$$\langle \Psi_\lambda | O_L(\tau_1) O_R(\tau_2) | \Psi_\lambda \rangle \quad (3.29)$$

$$= \sum_{E_1 E_2} O_{12} O_{21} \exp \left\{ -\frac{\beta}{2} f(E_1) - \frac{\beta}{2} f(E_2) + \tau_1 f(2E_1) - \tau_2 f(2E_2) - \tau_{12} f(E_1 + E_2) \right\} \quad (3.30)$$

$$= \sum_{E_1 E_2} O_{12} O_{21} \exp \left\{ -\frac{\beta}{2} E_+(1 + \lambda E_+) + 2\lambda E_-^2 \left( \tau_- - \frac{\beta}{4} \right) + E_- \tau_+(1 + 4\lambda E_+) \right\} \quad (3.31)$$

$$= \sum_{E_1 E_2} O_{12} O_{21} \exp \left\{ -\frac{\beta}{2} E_+(1 + \lambda E_+) \right\} \int_{-i\infty}^{i\infty} d\beta' K \left( -2\lambda \left( \tau_- - \frac{\beta}{4} \right), (1 + 4\lambda E_+) \tau_+ + \beta' \right) \exp \{ -\beta' E_- \}, \quad (3.32)$$

Similarly, at the weakly-coupled limit  $|\lambda| \ll 1/E_\beta = 1/\langle \tilde{\Psi}_\lambda | H_s | \tilde{\Psi}_\lambda \rangle$ , we do the approximation  $|\lambda|E_+ \ll 1$ , such that

$$G_{LR}(t_1, t_2)_\lambda = \langle \tilde{\Psi}_\lambda | O_L(it_1)O_R(it_2) | \tilde{\Psi}_\lambda \rangle \approx i \int_{-\infty}^{\infty} du K \left( -2\lambda \left( it_- - \frac{\beta}{4} \right), it_+ + iu \right) G_W(u; \beta), \quad (3.33)$$

which can be obtained by replacement  $it_- \rightarrow it_- - \frac{\beta}{4}$  in the  $G_{LR}(t_1, t_2)$  in (3.15). From (3.20), at the first order perturbation on  $\lambda$ , the retarded correlator  $G_{LR}^R(t_1, t_2)_\lambda$  on  $|\tilde{\Psi}_\lambda(t)\rangle$  is the same as  $G_{LR}^R(t_1, t_2)$ .

### 3.4. Applications

#### 3.4.1. Conformal QM

We can apply the above formula to conformal QM. For a primary operator  $O$  with dimension  $\Delta$ , whose Wightman correlator is

$$G_W(t; \beta) = \left( \frac{\pi}{\beta} \operatorname{sech} \frac{\pi t}{\beta} \right)^{2\Delta}. \quad (3.34)$$

From (3.15), the correlator on the  $T\bar{T}$  quenched TFD state is

$$G_{LR}(t_1, t_2) = \left( \frac{\pi}{\beta} \right)^{2\Delta} \sqrt{\frac{i}{8\pi x}} \int_{-\infty}^{\infty} du \operatorname{sech}^{2\Delta}(\pi u) \exp \frac{(u + t_+/\beta)^2}{i8x}, \quad x = \frac{\lambda t_-}{\beta \beta}, \quad (3.35)$$

as shown in Figs. 1, 2 and 3. In Figs. 1, the peak appears near the time scale  $\beta^2/\lambda$ , which indicates the best regeneration. The correlator on the  $T\bar{T}$  deformed TFD state is

$$G_{LR}(t_1, t_2)_\lambda = \left( \frac{\pi}{\beta} \right)^{2\Delta} \sqrt{\frac{i}{8\pi x}} \int_{-\infty}^{\infty} du \operatorname{sech}^{2\Delta}(\pi u) \exp \frac{(u + t_+/\beta)^2}{i8x}, \quad x = \frac{\lambda}{\beta} \left( \frac{t_-}{\beta} + \frac{i}{4} \right), \quad (3.36)$$

whose behavior is close to the case of  $T\bar{T}$  quenched TFD state, except that the correlation is slightly suppressed due to the loss of entanglement.

#### 3.4.2. The SYK model

Consider the SYK model as the QM, whose local Hamiltonian is [12–15]

$$H = \frac{i^{\frac{q}{2}}}{q!} \sum_{j_1, \dots, j_q} J_{j_1, \dots, j_q} \psi^{j_1} \dots \psi^{j_q}, \quad \overline{J_{j_1, \dots, j_q}^2} = \frac{2^{q-1}(q-1)! \mathcal{J}^2}{qN^{q-1}}. \quad (3.37)$$

The SYK model exhibits the behavior of free fermions in the ultraviolet (UV) and conformal symmetry in the infrared (IR). The two-point function interpolating between UV and IR

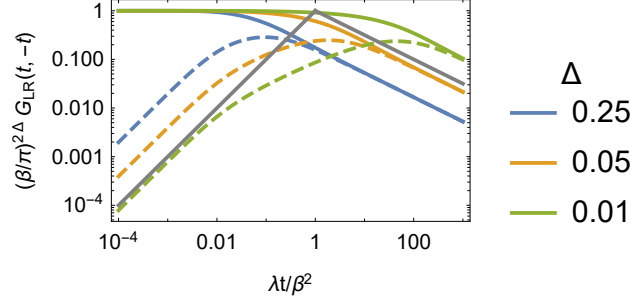


FIG. 1.  $G_{LR}(t, -t)$  for conformal QM, whose real (imaginary) part is denoted as solid (dashed) line. The gray lines denotes power law  $t$  and  $t^{-1/2}$ .

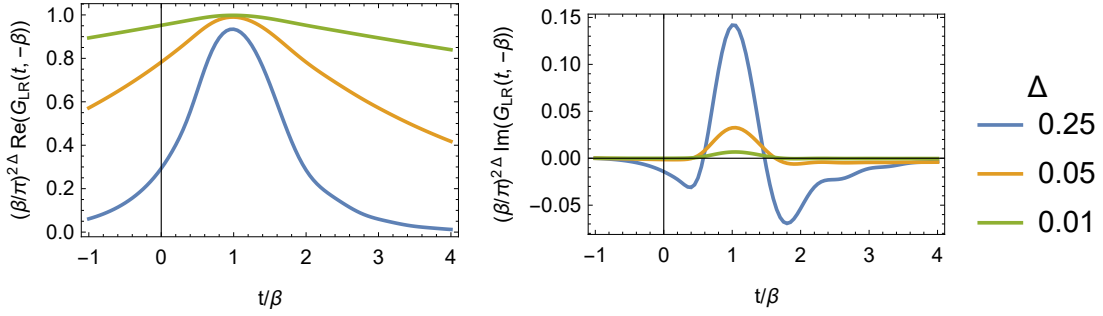


FIG. 2.  $G_{LR}(t, -\beta)$  for conformal QM, where  $\lambda/\beta = 0.01$ .

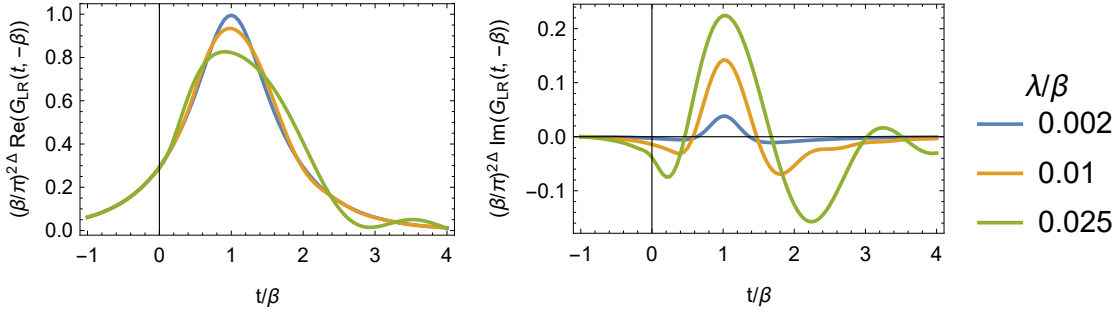


FIG. 3.  $G_{LR}(t, -\beta)$  for conformal QM, where  $\Delta = 0.25$ .

can be solved at the large- $q$  limit. The Wightman function is [14]

$$G_W(t; \beta) = \frac{1}{2} \left[ \left( \frac{\cos \frac{\pi v}{2}}{\cosh \frac{\pi v t}{\beta}} \right)^2 \right]^{1/q}, \quad \pi v = \beta \mathcal{J} \cos \frac{\pi v}{2}. \quad (3.38)$$

For the  $T\bar{T}$  quenched TFD state, the correlator at weakly coupled limit  $|\lambda| \ll 1/E_\beta \sim \beta \mathcal{J}^2/N$  is similar to the conformal result

$$G_{LR}(t_1, t_2) = \left( \frac{\pi v}{\beta \mathcal{J}} \right)^{2/q} \frac{1}{2} \sqrt{\frac{i}{8\pi x}} \int_{-\infty}^{\infty} du \operatorname{sech}^{2/q}(\pi v u) \exp \frac{(u + t_+/ \beta)^2}{i8x}, \quad x = \frac{\lambda t_-}{\beta \beta}, \quad (3.39)$$

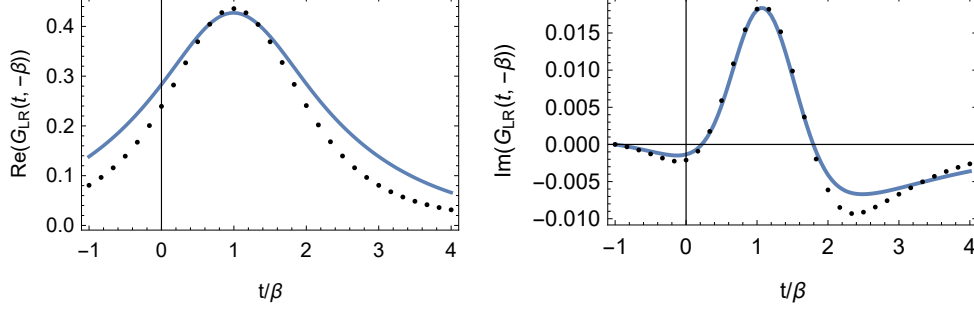


FIG. 4.  $G_{LR}(t, -\beta)$  for the SYK model. Dots denote the result from exact diagonalization. Curves denote the result from (3.39). Parameters are  $q = 4, N = 20, \mathcal{J} = 1, \beta = 2, \lambda = 0.02$ .

which is close to the result from exact diagonalization in Fig. 4. For the  $T\bar{T}$  deformed TFD state, the correlator is

$$G_{LR}(t_1, t_2)_\lambda = \left(\frac{\pi v}{\beta \mathcal{J}}\right)^{2/q} \frac{1}{2} \sqrt{\frac{i}{8\pi x}} \int_{-\infty}^{\infty} du \operatorname{sech}^{2/q}(\pi v u) \exp\left(\frac{(u + t_+/\beta)^2}{i8x}\right), \quad x = \frac{\lambda}{\beta} \left(\frac{t_-}{\beta} + \frac{i}{4}\right). \quad (3.40)$$

### 3.4.3. The system satisfying the ETH

We can apply the above formula to the system satisfying the ETH. Consider a Hermitian operator  $O$  satisfying [40–43]

$$O_{ab} \approx \frac{1}{\sqrt{D}} F(E_+, E_-) R_{ab}, \quad a \neq b \quad (3.41)$$

$$\langle R_{ab} \rangle = 0, \quad \langle R_{ab} R_{cd}^* \rangle = \delta_{ac} \delta_{bd}, \quad E_{\pm} = E_a \pm E_b, \quad (3.42)$$

where  $R_{ab}$  is random matrix. We further assume that the operator  $O$  has bandwidth  $\Gamma$

$$F(E_+, E_-) \sim A(E_+/2) e^{-|E_-|/\Gamma}. \quad (3.43)$$

The off-diagonal part of the correlator of the operator on the  $T\bar{T}$  quenched TFD state is

$$\langle \Psi | O_L(\tau_1) O_R(\tau_2) | \Psi \rangle - \sum_a O_{aa}^2 \exp\{-\beta E_a\} \quad (3.44)$$

$$= \frac{1}{D} \sum_{a \neq b} \langle R_{ab} R_{ba} \rangle A\left(\frac{E_+}{2}\right)^2 \exp\left\{-\frac{\beta}{2} E_+ + 2\lambda E_-^2 \tau_- + E_- \tau_+ (1 + 4\lambda E_+) - \frac{2}{\Gamma} |E_-|\right\} \quad (3.45)$$

$$\approx \frac{1}{D} \sum_{a \neq b} A\left(\frac{E_+}{2}\right)^2 \exp\left\{-\frac{\beta}{2} E_+ + 2\lambda E_-^2 \tau_- + E_- \tau_+ - \frac{2}{\Gamma} |E_-|\right\} \quad (3.46)$$

where we have used the weakly-coupled limit  $|\lambda| \ll 1/E_\beta$ . For large  $D$ , the energy band  $\Lambda = E_{\max} - E_{\min}$  is much larger than the bandwidth  $\Gamma$ . So we can calculate the integral in the approximation of flat spectrum difference

$$\sum_{a \neq b} \approx D^2 \int_0^\Lambda dE_a dE_b \rho(E_a) \rho(E_b) \approx D^2 \int_0^{2\Lambda} dE_+ \rho(E_+/2) \int_{-\infty}^{+\infty} dE_-. \quad (3.47)$$

Then the off-diagonal part is simplified as

$$\begin{aligned}
& D \int_0^{2\Lambda} dE_+ A \left( \frac{E_+}{2} \right)^2 \rho \left( \frac{E_+}{2} \right) \exp \left\{ -\frac{\beta}{2} E_+ \right\} \int_{-\infty}^{+\infty} dE_- \exp \left\{ 2\lambda E_-^2 \tau_- + E_- \tau_+ - \frac{2}{\Gamma} |E_-| \right\} \\
& = 2 \left[ D \int_0^\Lambda dE A(E)^2 \rho(E) e^{-\beta E} \right] \sqrt{\frac{\pi}{-8\lambda\tau_-}} \left[ g \left( \frac{2/\Gamma - \tau_+}{\sqrt{-8\lambda\tau_-}} \right) + g \left( \frac{2/\Gamma + \tau_+}{\sqrt{-8\lambda\tau_-}} \right) \right], \quad (3.48)
\end{aligned}$$

where

$$g(z) = e^{z^2} \operatorname{erfc}(z) = e^{z^2} \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^z dt e^{-t^2} \right). \quad (3.49)$$

Let  $A^2 = Z[\beta]^{-1} D \int_0^\Lambda dE A(E)^2 \rho(E) e^{-\beta E}$ . The retarded correlator is

$$G_{LR}^R(t_1, t_2) \approx 4A^2 \operatorname{Im} \left\{ \sqrt{\frac{\pi}{-i8\lambda t_-}} \left[ g \left( \frac{2/\Gamma - it_+}{\sqrt{-8i\lambda t_-}} \right) + g \left( \frac{2/\Gamma + it_+}{\sqrt{-8i\lambda t_-}} \right) \right] \right\}. \quad (3.50)$$

Asymptotically,

$$G_{LR}^R(t, -t) \rightarrow 2A^2 \begin{cases} 32\Gamma^3 \lambda t_- \frac{4-3\Gamma^2 t_+^2}{(4+\Gamma^2 t_+^2)^3}, & t_- \rightarrow 0 \\ \sqrt{\frac{\pi}{\lambda t_-}}, & t_- \rightarrow \infty \end{cases}, \quad (3.51)$$

whose power law behavior is the same as the conformal result in Fig. 1.

Replacing  $it_- \rightarrow it_- - \frac{\beta}{4}$ , we obtain the retarded correlator on the  $T\bar{T}$  deformed TFD state

$$G_{LR}^R(t_1, t_2)_\lambda \approx 4A^2 \operatorname{Im} \left\{ \sqrt{\frac{\pi}{2\lambda(\beta - 4it_-)}} \left[ g \left( \frac{2/\Gamma - it_+}{\sqrt{2\lambda(\beta - 4it_-)}} \right) + g \left( \frac{2/\Gamma + it_+}{\sqrt{2\lambda(\beta - 4it_-)}} \right) \right] \right\}. \quad (3.52)$$

#### 4. $T\bar{T}$ DEFORMATION ON SCHWARZIAN THEORY

In this section, we consider the  $T\bar{T}$  deformation on the eternal black hole in Jackiw-Teitelboim (JT) gravity [44, 45]

$$I = \frac{1}{16\pi G} \left[ \int d^2x \sqrt{-g} \Phi (R + 2) + 2 \int_b dx \sqrt{-h} \Phi_b K \right] \quad (4.1)$$

with the boundary condition

$$ds_h^2 = -dt^2/\epsilon^2, \quad \Phi_b = \Phi_r/\epsilon, \quad (4.2)$$

where  $b$  denotes the boundary,  $h$  is the induced metric on the boundary,  $\Phi_b$  is the value of the dilaton  $\Phi$  on the boundary, and  $\epsilon$  is the UV cutoff. Integrating out the dilaton  $\Phi$ , we have  $R + 2 = 0$ , whose solution is the  $\text{AdS}_2$  space. The  $\text{AdS}_2$  space in global coordinate and Rindler coordinate are

$$ds^2 = \frac{-dv^2 + d\sigma^2}{\sin^2 \sigma} = -\sinh^2 \rho d\varphi^2 + d\rho^2. \quad (4.3)$$

Consider two boundary  $L$  and  $R$  with reparametrizations  $(\varphi_L(t), \rho_L(t))$  and  $(\varphi_R(t), \rho_R(t))$ . To satisfy the boundary condition, the reparametrizations are expanded as

$$\sinh \rho_L(t) = -\frac{1}{\epsilon \varphi'_L(t)} - \frac{\epsilon \varphi''_L(t)^2}{2\varphi'_L(t)^3} + \mathcal{O}(\epsilon^2), \quad (4.4)$$

$$\sinh \rho_R(t) = \frac{1}{\epsilon \varphi'_R(t)} + \frac{\epsilon \varphi''_R(t)^2}{2\varphi'_R(t)^3} + \mathcal{O}(\epsilon^2). \quad (4.5)$$

The action is reduced to the two-sited Schwarzian theory

$$I = \frac{1}{8\pi G} \int_{L,R} dx \sqrt{-h} \Phi_b(K-1) \quad (4.6)$$

$$= -C \int dt \left[ \text{Sch} \left( -\coth \frac{\varphi_L}{2}, u \right) + \text{Sch} \left( \tanh \frac{\varphi_R}{2}, u \right) \right] + \mathcal{O}[\epsilon^2] \quad (4.7)$$

$$= \frac{C}{2} \int dt \left[ \frac{\varphi''_L(t)^2}{\varphi'_L(t)^2} + \varphi'_L(t)^2 + \frac{\varphi''_R(t)^2}{\varphi'_R(t)^2} + \varphi'_R(t)^2 \right] + \text{surface term}, \quad C = \frac{\Phi_r}{8\pi G} \quad (4.8)$$

Similar to the argument in [24], the action has  $\text{SL}(2)$  gauge symmetry and the gauge charges vanish. So the solution can be transformed into the  $LR$ -symmetric form  $\varphi_L(u) = \varphi_R(t) = \varphi(t)$ . Following [8], we will consider the  $T\bar{T}$  deformation. We use Ostrogradsky formalism to write down the canonical variables [46]

$$\begin{aligned} q_1 &= \varphi, & q_2 &= \varphi', \\ p_1 &= \frac{\partial L}{\partial \varphi'} - \partial_t \frac{\partial L}{\partial \varphi''} = C \left[ \frac{\varphi''^2}{\varphi'^3} + \varphi' - \frac{\varphi^{(3)}}{\varphi'^2} \right], & p_2 &= \frac{\partial L}{\partial \varphi''} = C \frac{\varphi''}{\varphi'^2}. \end{aligned} \quad (4.9)$$

The Hamiltonian in Ostrogradsky formalism is

$$H_0 = H_L + H_R = p_1 q_2 + \frac{1}{4C} p_2^2 q_2^2 - C q_2^2. \quad (4.10)$$

The solution of TFD state at inverse temperature  $\beta$  is

$$\varphi(t) = \varphi_\beta(t) = \frac{2\pi t}{\beta}, \quad (4.11)$$

and all the canonical variables are determined by (4.9). With (4.4), the solution of the reparametrization describes two boundary trajectories on the constant radius in the Rindler patch of  $\text{AdS}_2$  space, whose geometry is a wormhole connecting two boundaries from higher-dimensional perspective.

We firstly consider a general deformation

$$H_\lambda = f(H_0). \quad (4.12)$$

Under the deformation, the canonical relations are determined by the deformed Hamiltonian equation

$$q'_i = \frac{\partial H_\lambda}{\partial p'_i}, \quad p'_i = -\frac{\partial H_\lambda}{\partial q'_i}. \quad (4.13)$$

Given a solution of the Hamiltonian equation of  $H_0$ , such as the  $\varphi_\beta(t)$  in (4.11), we can find the solution of  $H_\lambda$  from (2.5). Introduce the dynamical time

$$T = kt, \quad k = f'(H_0[\varphi_\beta(t)]), \quad (4.14)$$

where  $H_0[\varphi_\beta(t)] = 4\pi^2 C/\beta^2$  is the value of  $H_0$  on the canonical variables determined by the solution in (4.11) with the canonical relation in (4.9). The solution of the Hamiltonian equation of  $H_\lambda$  is

$$\varphi(t) = \varphi_\beta(kt) = \varphi_{\beta/k}(t) = 2\pi kt/\beta, \quad (4.15)$$

$$q_1 = 2\pi kt/\beta, \quad q_2 = 2\pi/\beta, \quad p_1 = 2\pi C/\beta, \quad p_2 = 0. \quad (4.16)$$

The energy is  $H_\lambda[\varphi_\beta(kt)] = f(4\pi^2 C/\beta^2)$ .

Now, choose the  $T\bar{T}$  deformation

$$H_\lambda = f(H_0) = \frac{1 - \sqrt{1 - 8H_0\lambda}}{4\lambda}. \quad (4.17)$$

Then  $k = 1/\sqrt{1 - 32\pi^2\lambda C/\beta^2}$  and  $H_\lambda[\varphi_\beta(kt)] = \left(1 - \sqrt{1 - 32\pi^2\lambda C/\beta^2}\right)/4\lambda$ . So the weakly-coupled limit in Sec. 3 means  $|\lambda| \ll \beta^2/C$  and  $k \approx 1$  here.

The solution (4.15) has two interpretations, which separately correspond to the two strategies in Sec. 3. First, recall that the local state  $\rho(t)$  in (3.7) is a thermal state of the undeformed Hamiltonian  $H_0$ . So the solution  $\varphi(t) = \varphi_{\beta/k}(t)$  is interpreted as the  $T\bar{T}$  quenched TFD state  $|\Psi(t)\rangle$  at inverse temperature  $\beta/k$  in (3.6). Second, recall that the local state  $\rho_\lambda(t)$  in (3.28) is a thermal state of the deformed Hamiltonian  $f(H_0)$ , whose dynamical time is  $kt$  as well. So the solution  $\varphi(t) = \varphi_\beta(kt)$  is interpreted as the  $T\bar{T}$  deformed TFD state  $|\Psi_\lambda(t)\rangle$  at inverse temperature  $\beta$  in (3.27).

The two boundaries of the  $T\bar{T}$  quenched/deformed TFD state are space-like separated. So the causal correlation found in Sec. 3 is not associated with the causal structure of a semi-classical wormhole. It is similar to the ‘‘quantum traversable wormhole’’ in Ref. [19]. Without the  $T\bar{T}$  deformation, the vanishing of the retarded correlator is the result of the perfect cancellation between the two propagators  $\langle O_L^{(0)} O_R^{(0)} \rangle$  and  $\langle O_R^{(0)} O_L^{(0)} \rangle$ , which are dual to the process of a virtual particle traveling from  $R$  to  $L$  and from  $L$  to  $R$  in the bulk respectively. With the  $T\bar{T}$  deformation, the virtual particle can release two gravitons and annihilate with each other on the boundaries, via the  $H_L H_R$  term in the  $T\bar{T}$  deformation, as shown in Fig. 5. The propagators acquire different factors, resulting in the propagation of real particles.

More precisely, we can directly calculate retarded correlator  $G_{LR}^R(t_1, t_2)$  at first order of  $\lambda$  by using the Schwarzian action. Take the  $G_{LR}^R(t_1, t_2)$  in (3.23) as an example, where the  $T\bar{T}$  quench is applied instantaneously. Considering the reparametrization mode  $\varphi(t) = t + \varepsilon(t)$ , we expand the dynamical part of the Euclidean Schwarzian action, the correlator and, the non-local  $T\bar{T}$  term with respect to  $\varepsilon(t)$ ,

$$I_\varepsilon = \frac{C}{2} \int d\tau (\varepsilon''(\tau)^2 - \varepsilon'(\tau)^2 + \mathcal{O}[\varepsilon^3]), \quad (4.18)$$

$$\langle O(\tau_1) O(\tau_2) \rangle = \left[ \frac{(1 + \varepsilon'(\tau_1))(1 + \varepsilon'(\tau_2))}{4} \csc^2 \frac{\tau_1 - \tau_2 + \varepsilon(\tau_1) - \varepsilon(\tau_2)}{2} \right]^\Delta, \quad (4.19)$$

$$-4\lambda H_L H_R = \mathcal{O}[\varepsilon] - 4\lambda C^2 (\varepsilon^{(3)}(0) + \varepsilon'(0)) (\varepsilon^{(3)}(\pi) + \varepsilon'(\pi)) + \mathcal{O}[\varepsilon^3], \quad \beta = 2\pi. \quad (4.20)$$

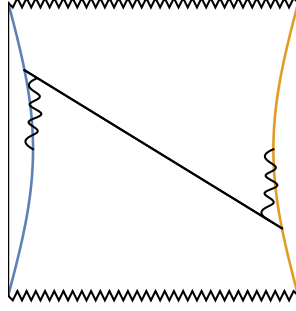


FIG. 5. The Witten diagram in retarded correlator  $G_{LR}^R$  at first the order of  $\lambda$  in the global coordinate of  $\text{AdS}_2$ , where the solid line is the propagator of matter field and the wavy lines are the propagator of boundary graviton (reparametrization model). The boundaries are space-like separated.

The quadratic term in  $I_\varepsilon$  gives the propagator of reparametrization model  $\langle \varepsilon(\tau)\varepsilon \rangle = -(\pi - |\tau|)(\pi + |\tau| + 2 \sin |\tau|)/(4\pi C)$  [45]. The brackets  $\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$  in the commutator  $-4\lambda \langle [[H_L H_R, O_L(t_1)], O(t_2)] \rangle$  are factorized into  $\langle \varepsilon\varepsilon \rangle \langle \varepsilon\varepsilon \rangle$  as shown in Fig. 5. The  $\mathcal{O}[\varepsilon^3]$  term in  $I_\varepsilon$  and the  $\mathcal{O}[\varepsilon]$  term in  $-4\lambda H_L H_R$  will not contribute to the commutator. The final result is just (3.24) at tree level.

Legendre transforming the deformed Hamiltonian and letting  $q_1 = \varphi$ ,  $q_2 = e^\phi$ , we obtain the deformed Lagrange

$$\mathcal{L}_\lambda = \frac{C e^\phi (\varphi'^2 + \phi'^2)}{\varphi'} + \frac{(e^\phi - \varphi')^2}{8\lambda \varphi' e^\phi}. \quad (4.21)$$

Solving  $\varphi$  and substituting it in the Lagrange, we have

$$\mathcal{L}_\lambda = \frac{\sqrt{(1 + 8C\lambda\phi'^2)(1 + 8C\lambda e^{2\phi})} - 1}{4\lambda}, \quad (4.22)$$

which agrees with the  $T\bar{T}$  deformation of the Liouville QM  $\mathcal{L} = \frac{C}{2}(\phi_L'^2 + \phi_R'^2 + e^{2\phi_L} + e^{2\phi_R})$ , given in (2.22), with  $\phi_L = \phi_R = \phi$ .

To keep the correction of finite cutoff  $\epsilon$ , we substitute (4.4) into the action (4.6) and obtain the Lagrange

$$\mathcal{L}_\epsilon = C \sum_{s=L,R} \frac{1}{2} \left( \varphi_s'^2 + \frac{\varphi_s''^2}{2\varphi_s'^2} \right) + \frac{\epsilon^2}{8} \left( -\frac{57\varphi_s''^4}{\varphi_s'^4} - \varphi_s'^4 + \frac{8\varphi_s''^2}{\varphi_s'^2} - 6\varphi_s''^2 \right) + \mathcal{O}[\epsilon^3] \quad (4.23)$$

If we substitute the solution (4.11) in the above Lagrange, we find  $\mathcal{L}_\epsilon = 4\pi^2 C \beta^{-2} - 16\pi^4 C^2 \lambda / \beta^{-4}$ , where we have used  $\lambda = 2\pi\epsilon^2 G / \Phi_R$  according to [9, 47]. However, if we substitute the deformed solution (4.15) into (4.22), we find  $\mathcal{L}_\lambda = 4\pi^2 C \beta^{-2} - 32\pi^4 C^2 \lambda / \beta^{-4}$ , which is different from  $\mathcal{L}_\epsilon$ . So the effect of the non-local  $T\bar{T}$  deformation considered in this paper is not simply equivalent to moving the boundaries into the bulk [48]. It couples the two boundaries and leads to non-local dynamics.

## 5. SUMMARY AND PROSPECT

In this paper, we consider the  $T\bar{T}$  deformation on multi-QM system. Given a solution of the original theory, we can find a solution of the deformed theory, which is related to



the original solution by rescaling the time. Motivated by the rescaling, we introduce the dynamical tetrad acting as 1-dimensional gravity. By integrating out the dynamical tetrad, we can obtain the deformed action. The  $T\bar{T}$  deformation on multi-scalars theory follows a similar form.

The  $T\bar{T}$  deformation on bi-QM system effectively couples the two local systems with each other. We further consider the TFD states on the bi-QM system. The signal injected into one system at a time can come out from the other system at the reverse time. The time of best regeneration scales as  $\beta^2/\lambda$  in conformal QM. In the SYK model, our analytical result at the large- $q$  limit is close to the result from exact diagonalization. For the theory satisfying ETH, we find that the regeneration mainly depends on the bandwidth of the operator carrying the signal in the energy basis.

Finally, we study such  $T\bar{T}$  deformation on two-sided Schwarzian action which describes the leading non-conformal dynamics of the eternal black hole in JT gravity. We obtain the deformed Lagrange and find that the deformed solution is an external black hole with rescaled time, whose two boundaries are space-like separated. It shows that the regeneration found in bi-QM system is not associated with the causal structure of a semi-classical wormhole.

Here we mainly study the regeneration phenomenon of the TFD state under  $T\bar{T}$  deformation. Since the  $T\bar{T}$  coupling is directly related to energy, the energy transport also deserves investigation in the future [49]. Our study of the regeneration phenomenon under the  $T\bar{T}$  deformation gives a new perspective of the information process and the causal structure of  $T\bar{T}$  deformed field theories. We also expect that the regeneration phenomenon under  $T\bar{T}$  deformation is common in highly-entangled states since the  $T\bar{T}$  deformation needs not to match the entanglement structure. It is natural to extend the  $T\bar{T}$  deformation on the  $CFT_2$  with multiple fields and check the regeneration of the deformed TFD states [6, 7, 50]. In terms of [51], one can choose proper two-sided  $T\bar{T}$  coupling to reconstruct the bulk geometry of the deformed TFD state and compare the correlators from gravity and from field theory.

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