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Spin Hall effects in General Relativity

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Abstract

The propagation of test fields, such as electromagnetic, Dirac, or linearized gravity, on a fixed spacetime manifold is often studied by using the geometrical optics approximation. In the limit of infinitely high frequencies, the geometrical optics approximation provides a conceptual transition between the test field and an effective point-particle description. The corresponding point-particles, or wave rays, coincide with the geodesics of the underlying spacetime. For most astrophysical applications of interest, such as the observation of celestial bodies, gravitational lensing, or the observation of cosmic rays, the geometrical optics approximation and the effective point-particle description represent a satisfactory theoretical model. However, the geometrical optics approximation gradually breaks down as test fields of finite frequency are considered.

In this thesis, we consider the propagation of test fields on spacetime, beyond the leading-order geometrical optics approximation. By performing a covariant Wentzel-Kramers-Brillouin analysis for test fields, we show how higher-order corrections to the geometrical optics approximation can be obtained. The higher-order corrections are related to the dynamics of the spin internal degree of freedom of the considered test field. We obtain an effective point-particle description, which contains spin-dependent corrections to the geodesic motion obtained using geometrical optics. This represents a covariant generalization of the well-known spin Hall effect, usually encountered in condensed matter physics and in optics. Our analysis is applied to electromagnetic and massive Dirac test fields, but it can easily be extended to other fields, such as linearized gravity. In the electromagnetic case, we present several examples where the gravitational spin Hall effect of light plays an important role. These include the propagation of polarized light rays in black hole spacetimes and cosmological spacetimes, as well as polarization-dependent effects on the shape of black hole shadows. Furthermore, we show that our effective point-particle equations for polarized light rays reproduce well-known results, such as the spin Hall effect of light in an inhomogeneous medium, and the relativistic Hall effect of polarized electromagnetic wave packets encountered in Minkowski spacetime.

List of publications

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Contents

1. Introduction

Contents

Hall effects are well know in many areas of physics and represent the basis of many applications of practical interest. The most basic example is the ordinary Hall effect, discovered by Edwin Herbert Hall in 1879 [100]. This effect is observed for charged particles travelling in a conductor in the x direction, when a magnetic field is applied in the transverse y direction. Then, due to the Lorentz force, the charged particles are deflected in the z direction. The magnitude of the deflection is related to the magnitude of the applied magnetic field and the absolute value of the charge, while the direction of the deflection is given by the orientation of the applied magnetic field along the y direction and the sign of the charge.

The ordinary Hall effect, despite its simplicity, successfully captures the main features of Hall effects in general. That is, a particle (or localized wave packet) with some internal degree of freedom and travelling in the x, under the influence of some external agent, acting in the y direction, gets deflected in the z direction, and the orientation and magnitude of the deflection depend on the internal degree of freedom. In the case of the ordinary Hall effect, the internal degree of freedom is represented by the charge of the particle. Thus, Hall effects can be viewed as a consequence of the coupling between the external and internal degrees of freedom of particles or wave packets.

In this thesis, our attention is directed towards spin Hall effects, which represent a subclass of Hall effects where the relevant internal degree of freedom is the spin of a particle or wave packet. The main mechanism behind spin Hall effects, representing the coupling of external and internal degrees of freedom, is called the spin-orbit interaction [158, 71, 37]. The spin represents the internal degree of freedom, and the orbital part represents external degrees of freedom, such as position and velocity.

Examples of spin Hall effects are commonly encountered in many areas of physics, such as condensed matter physics or optics. The spin Hall effect of electrons can be observed for electrons travelling in certain materials exhibiting spin-orbit coupling. In this case, electrons with opposite spin get deflected in opposite directions, transverse to their direction of propagation [158, 71]. Similarly, in optics one observes the spin Hall effect of light. This effect is present for polarized light propagating in a medium with inhomogeneous refractive index. In this case, the polarization represents the spin internal degree of freedom. Light rays of opposite circular polarization are deflected in opposite directions, perpendicular to their direction of propagation and to the gradient of the refractive index [37]. These effects have been intensively studied for the past 40 years, leading to solid theoretical foundations, as well as many experimental observations. Beyond their primary role in fundamental physics, spin Hall effects are playing an important role for many applications in metrology [179], spintronics [174, 108], photonics [118, 178], optical communications [111] and image processing [180].

In optics, the spin Hall effect of light is usually derived by considering the WKB approximation of Maxwell's equations in a medium with inhomogeneous refractive index [33, 31, 148]. At the lowest order in the WKB approximation, one obtains the well-known geometrical optics ray equations, and there is no coupling between the external (position and velocity of the ray) and internal (polarization) degrees of freedom. However, by taking into account terms of higher order in the WKB approximation, one obtains effective ray equations containing polarization-dependent correction terms, representing the spin Hall effect of light. In this case, the evolution of the polarization involves the Berry connection, and the polarization-dependent terms in the effective ray equations are represented by the corresponding Berry curvature. This approach emphasizes the fact that the mathematical description of physical phenomena in terms of fields is fundamental, while any point-particle description should be viewed just as an approximation of the underlying fields. The WKB approximation facilitates the transition from the field equations to the effective point-particle description, and the spin Hall effect of light is represented by polarization-dependent corrections arising in the effective point-particle description. In this case, the effective point-particle dynamics is expected to approximate the dynamics of a localized wave packet of the electromagnetic field (e.g., a polarized laser beam). This is actually the case, as confirmed by several experiments [103, 36].

Given the important role of spin Hall effects in condensed matter physics and optics, it is natural to explore the possibility of similar effects arising in general relativity. In particular, we are interested in studying the dynamics of particles or wave packets with spin internal degree of freedom, on a fixed spacetime background representing a solution of the Einstein field equations. In this context, any spin-dependent propagation of a particle or wave packet, which arises solely due to the presence of the gravitational field, is referred to as a gravitational spin Hall effect [128].

In general relativity, the motion of free-falling test particles without any internal structure is represented by the geodesics of the underlying spacetime. For a broad range of astrophysical applications, such as the motion of planets or the propagation of light rays, the geodesic motion provides a good approximation. However, test particles can also have internal structure (such as spin, electric charge, Yang-Mills charge, etc.), which can mutually interact with the external degrees of freedom. There are several approaches within general relativity that study the dynamics of particles or wave packets with internal degrees of freedom [128]. For example, the well-known Mathisson-Papapetrou-Dixon equations are meant to describe massive test particles with spin [59]. Various extensions of these equations for the massless case have also been considered [160, 152]. Another approach is the use of WKB-type approximations for field equations such as Maxwell's equations [85, 84, 154] or the Dirac equation [9, 147]. This can lead to spin-dependent extensions of the usual geometrical optics results.

However, the previously mentioned models do not provide a satisfactory description of gravitational spin Hall effects. Each model suffers from at least one serious drawback, such as lack of covariance, limited applicability to a small class of spacetimes, or lack of proper physical interpretation. Furthermore, there are also contradictory claims between some models, as discussed in [128]. The main goal of this thesis is to provide a covariant derivation of the gravitational spin Hall effect, which does not suffer from the same drawbacks as the previous models, and which should settle previously encountered inconsistencies.

The gravitational spin Hall effect of light is derived by using the WKB approximation for the vacuum Maxwell equations on a fixed spacetime background. The derivation is covariant and the considered spacetime is arbitrary. We obtain an effective point-particle description, consisting of a set of ray equations and a transport equation for the polarization. The ray equations contain polarization-dependent corrections to the null geodesics, representing the gravitational spin Hall effect of light. We present these results in close analogy with the description of the spin Hall effect of light in optics - the transport equation for the polarization is expressed in terms of the Berry connection, and the correction terms in the ray equations are expressed in terms of the Berry curvature.

Considering the WKB approximation for the Dirac equation on a fixed spacetime background, we derive the gravitational spin Hall effect of electrons. Formally, the derivation follows the same steps as in the electromagnetic case, and the resulting point-particle dynamics is also governed by the corresponding Berry connection and Berry curvature.

The present derivation of the gravitational spin Hall effect opens the way for several applications. It is shown that in certain black hole spacetimes (such as Schwarzschild or Kerr) the gravitational spin Hall effect of light results in a polarization dependent scattering of light rays. Even though the effect is small, the presence of a finite scattering angle suggests that the effect could be measurable sufficiently far away from the black hole. Furthermore, it is shown how the gravitational spin Hall effect of light can affect the observation of black hole shadows. While there is no effect of Schwarzschild black holes, the shadows of Kerr black holes are modified in a polarization-dependent way.

There are several natural directions in which the results of this thesis can be extended, some of which are currently being investigated by the author. First of all, the procedure presented here for the derivation of the gravitational spin Hall effect could easily be applied for studying the propagation of gravitational waves on a fixed spacetime background, within the framework of linearized gravity. This should lead to a derivation of the spin Hall effect of gravitational waves. Second, instead of the WKB approximation, one could use the more general Gaussian beam approximation [143] (see also Ref. [130]). This has the advantage of providing a much clearer transition between wave packet dynamics and point-particle dynamics, and it is expected that the shape of the wave packet could also affect its propagation, as suggested in Ref. [30]. Lastly, the effective ray equations describing the gravitational spin Hall effect should be studied in more detail. One important aspect that should be addressed is the existence of conserved quantities for the effective ray equations, as well as any possible relations with conservation laws associated with the underlying field equations. This is partially answered in section 2.4.3, where it is shown that in certain spacetimes, such as Kerr, one can use the Lagrangian formulation of the effective ray equations to obtain conserved quantities. However, this approach can only provide conserved quantities associated with Killing vector fields, and it is not clear if the effective ray equations also admit conserved quantities associated to Killing tensor fields (such as a generalized version of Carter's constant for spinning particles).

1.1. Overview of the thesis

We continue this introductory chapter with a brief overview of spin Hall effects in general, based on [128]. This is meant to introduce the basic concepts used in the description of spin Hall effects, such as the Berry phase, and to provide an overview of the currently available descriptions of spin Hall effects in general relativity. In Chapter 2 we present the main results of this thesis, based on [129]. The gravitational spin Hall effect of light is derived by means of a covariant WKB analysis of Maxwell's equations on a fixed arbitrary spacetime background. The treatment is in close analogy with well-known derivations of the spin Hall effect in optics. The spin-orbit interaction is represented through the Berry phase, and the correction terms in the modified ray equations are represented in terms of the Berry curvature. The modified ray equations are examined in several particular cases, and it is shown how several known results are recovered. A *Mathematica* code has been developed for numerically integrating the modified ray equations on a given spacetime background. The code can be found in Appendix A.7, and can easily be used to explore further examples. In Chapter 3, we perform a semiclassical analysis of the massive Dirac equation on a fixed arbitrary spacetime background, along the same lines as in the previous chapter. We obtain the dynamics of the internal spin degree of freedom in terms of the Berry connection, and we propose a derivation of modified ray equations, describing the spin Hall effect of electrons on a curved spacetime background.

1.2. Spin Hall effects in materials

In condensed matter physics, the spin Hall effect (SHE) of electrons was first predicted in 1971 [72, 73], and describes the appearance of a spin current, transverse to the electric charge current propagating in a material. The effect was first observed by Bakun et al. in 1984 [11] as the inverse spin Hall effect, and only later on, in 2004, was the direct spin Hall effect observed in semiconductors [110]. The source of this effect is the relativistic spin-orbit coupling between a particle's spin and its center of mass motion inside a potential. Detailed reviews about the spin Hall effect of electrons can be found in [158, 71].

A similar effect, called the spin Hall effect of light, is present in the case of electromagnetic waves propagating inside an inhomogeneous optical medium. In this case, the spin-orbit coupling comes from the interaction of the polarization degree of freedom with the gradient of the refractive index of the medium, resulting in a transverse shift of the wave packet motion, in a direction perpendicular to the gradient of the refractive index. The first known examples related to the spin Hall effect of light are the Goos-Hänchen effect [91], originally reported in 1947, and the Imbert-–Fedorov effect [79, 104], reported in 1955. These effects involve polarization-dependent transverse shifts of light beams undergoing refraction or total internal reflection. A recent review of these effects can be found in [32]. Later on, polarization-dependent propagation of light inside an inhomogeneous optical medium was reported under the name "optical Magnus effect" [63, 114], in analogy with the Magnus effect experienced by spinning objects moving through a fluid. This was followed by the work of Onoda et al. [131] (who introduced the term "Hall effect of light"), Bliokh et al. [33, 34, 36] and Duval et al. [67, 68, 65]. The first experimental observation of the spin Hall effect of light came in 2008 [103, 36]. Reviews about the current state of the research can be found in [37, 116].

Focusing on the spin Hall effect of light in inhomogeneous media, we briefly review the main mechanism behind the effect, and we present the corresponding equations of motion. We discuss the different types of angular momentum that electromagnetic waves can carry, and how spin-orbit interactions of light result from the conservation of the total angular momentum. Next, the notion of the Berry phase is introduced, and its relation to the spin Hall effect of light is explained. Finally, the equations of motion of polarized light rays in an inhomogeneous medium are introduced.

1.2.1. Angular momentum of light

It is well known that electromagnetic waves can carry angular momentum [106]. Following classical Maxwell's theory, the angular momentum density is given by the cross product of position vector $\mathbf{x}$ with the Poynting vector $\mathbf{E} \times \mathbf{B}$. The total angular momentum of the electromagnetic field is the space integral of this quantity [106]:

$$\mathbf{J} = \epsilon_0 \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \, dx^3, \tag{1.1}$$

where ϵ_0 is the vacuum permittivity. Furthermore, the total angular momentum can be split into two parts:

$$\mathbf{J} = \mathbf{S} + \mathbf{L} = \epsilon_0 \int (\mathbf{E} \times \mathbf{A})\, dx^3 + \epsilon_0 \sum_{i=1}^{3} \int E_i \left(\mathbf{x} \times \nabla\right) A_i dx^3. \tag{1.2}$$

In general, the above split is gauge dependent, due to the presence of the vector potential $\mathbf{A}$. However, this issue can be resolved by imposing the Coulomb gauge

$$A^\alpha = (0, \mathbf{A}), \qquad \nabla \cdot \mathbf{A} = 0. \tag{1.3}$$

In this case, the above split of the total angular momentum coincides with the gauge invariant definition for spin and orbital angular momentum [39]. The first term, $\mathbf{S}$, represents the spin angular momentum, and can be associated with the polarization of the electromagnetic wave. The second term, $\mathbf{L}$, represents the orbital angular momentum and was mostly ignored until the early 1990s, when it was shown that Laguerre-Gaussian light beams carry well defined spin and orbital angular momentum [3]. Detailed reviews about how the angular momentum of light shaped the last 25 years of developments in the science of light, covering both theoretical and experimental ground, can be found in [8, 17, 35].

When considering the propagation of light in inhomogeneous optical media, it is convenient to adopt the paraxial beam approximation. This means that the considered electromagnetic wave packet does not spread significantly during its propagation, so it can effectively be described by a ray trajectory. Within this approximation, considering a beam with mean wave vector $\mathbf{P}$ (and $P = |\mathbf{P}|$), the total angular momentum of light can be split into three distinct components [37, 35]:

- Spin angular momentum (SAM): this corresponds to the first term in Eq. (1.2), and it is related to the polarization of electromagnetic waves. The SAM per photon can take values $\sigma = \pm\hbar$, and in flat spacetime it is aligned with the direction of propagation of the beam:

$$\mathbf{S} = \sigma \frac{\mathbf{P}}{P}. \tag{1.4}$$

- Intrinsic orbital angular momentum (IOAM): this is characteristic for electromagnetic beams with helical wavefronts, such as Laguerre-Gaussian [3], Bessel [171] or exponential beams [26]. Beams with IOAM are generally described by a topological charge ℓ, which represents the twisting degree of the wavefronts. The IOAM per photon can take any integer value $\ell = 0, \pm\hbar, \pm 2\hbar, ...$, and in flat spacetime it is aligned with the direction of propagation of the beam:

$$\mathbf{L_{int}} = \ell \frac{\mathbf{P}}{P}. \quad (1.5)$$

- Extrinsic orbital angular momentum (EOAM): this is in direct analogy with the mechanical angular momentum for massive particles, and it is present for beams propagating at a distance from the origin of the coordinate system (the origin might correspond to some special point of an applied external potential). The EOAM is given by the cross product of the centroid of the propagating beam, $\mathbf{R}$, and its momentum, $\mathbf{P}$:

$$\mathbf{L_{ext}} = \mathbf{R} \times \mathbf{P}. \quad (1.6)$$

The second term in Eq. (1.2) is the sum of the IOAM and EOAM. Thus, the total angular momentum of paraxial light beams can be written as:

$$\mathbf{J} = \mathbf{S} + \mathbf{L} = \mathbf{S} + \mathbf{L_{int}} + \mathbf{L_{ext}}. \quad (1.7)$$

The conservation of the total angular momentum will induce the spin-orbit interactions of light, resulting in the spin Hall effect of light and other related effects [37]. For example, if we consider a system where only SAM and EOAM are present, the conservation of the total angular momentum will induce the spin Hall effect of light. Another possible example is a system with IOAM and EOAM, where the conservation of the total angular momentum will result in a similar effect, called the orbital Hall effect [30, 37]. In particular, IOAM plays a special role since the topological charge ℓ can take any integer value, thus one can in principle prepare beams that carry significant amounts of angular momentum. Optical beams with IOAM up to $10^4\hbar$ per photon have been reported [81].

Furthermore, the discussion presented here is not limited to electromagnetic waves. The same splitting of the total angular momentum can be considered for any other spin-field, and the conservation of the total angular momentum will give raise to the corresponding spin-orbit interactions. In particular, it is worth emphasizing that electrons carrying IOAM are attracting a lot of attention [168, 123, 113, 16, 28], and gravitational waves carrying IOAM have also been theoretically studied in [27, 112, 29, 12].

1.2.2. Berry phase

The Berry phase plays a central role in the description of SHEs, both in Optics [131, 37, 31], and in Condensed Matter Physics [19, 127, 157, 175]. For example, by considering relativistic wave equations, such as the Dirac equation or Maxwell's equations, the evolution of the spin degree of freedom will be influenced by the

Berry phase, while the spin-orbit coupling will imprint the effect of the Berry phase on the corresponding point-particle equations of motion, resulting in a SHE.

As originally described by Michael Berry [20], the adiabatic evolution of a quantum system changes the wavefunction by an additional phase factor, referred to as Berry phase or geometrical phase. The quantum system is considered to remain in some nth eigenstate of the Hamiltonian $\hat{H}(\mathbf{R})$:

$$\hat{H}(\mathbf{R}) \left|\Psi_n(\mathbf{R})\right\rangle = E_n(\mathbf{R}) \left|\Psi_n(\mathbf{R})\right\rangle , \tag{1.8}$$

where $\mathbf{R} = \mathbf{R}(t)$ represents the set of parameters varying adiabatically. The adiabatic evolution of the parameters is considered in the sense of Kato [109], and it will define a parallel transport of the wavefunction along the path in parameter space [51]. A well-known example of such a system is a spin-$\frac{1}{2}$ particle in a slowly changing magnetic field $\mathbf{B}(t)$ [51]. In this case, the set of parameters $\mathbf{R}(t)$ is identified with the magnetic field $\mathbf{B}(t)$, and for magnetic fields of constant magnitude the parameter space will have S^2 topology.

When the parameters $\mathbf{R}$ vary along a closed loop C in parameter space, such that $\mathbf{R}(0) = \mathbf{R}(T)$, the wavefunction acquires an additional Berry phase $\gamma_n(C)$:

$$\left|\Psi_n(\mathbf{R}(T))\right\rangle = e^{i\gamma_n(C)} e^{-\frac{i}{\hbar}\int_0^T E_n(\mathbf{R}(t))dt} \left|\Psi_n(\mathbf{R}(0))\right\rangle , \tag{1.9}$$

$$\gamma_n(C) = i \oint_C \left\langle\Psi_n(\mathbf{R})\right| \nabla_{\mathbf{R}} \left|\Psi_n(\mathbf{R})\right\rangle \cdot d\mathbf{R} = \oint_C \mathbf{A}_{\mathbf{R}} \cdot d\mathbf{R}. \tag{1.10}$$

The Berry phase can be expressed in terms of the Berry vector potential, $\mathbf{A}_{\mathbf{R}}$, also called the Berry connection. Furthermore, if we consider an arbitrary hypersurface in parameter space, such that $\partial\Sigma = C$, and by using Stokes' theorem, we can rewrite the Berry phase as:

$$\gamma_n(C) = \int_\Sigma \nabla \times \mathbf{A}_{\mathbf{R}} \cdot d\mathbf{S} = \int_\Sigma \mathbf{F}_{\mathbf{R}} \cdot d\mathbf{S}. \tag{1.11}$$

In the above expression $\mathbf{F}_{\mathbf{R}}$ is called the Berry curvature, since it describes the geometrical properties of the parameter space. In analogy with classical electrodynamics, we can think of $\mathbf{A}_{\mathbf{R}}$ as a "magnetic" vector potential, and of $\mathbf{F}_{\mathbf{R}}$ as the corresponding "magnetic" field in the parameter space. Then, one can regard the Berry phase $\gamma_n(C)$ as the flux of $\mathbf{F}_{\mathbf{R}}$ through the surface Σ [51].

Shortly after Berry's original paper, an elegant mathematical formulation was introduced by Barry Simon, who represented the geometrical phase factor by the holonomy of a connection on a Hermitian line bundle [155]. Later on, generalizations of the Berry phase were introduced by Wilczek and Zee for systems with degenerate spectra [173], and by Aharonov and Anandan for systems undergoing general cyclic evolution, that is not necessarily adiabatic [2, 4]. Extensions for noncyclic evolution exist as well [151, 126, 137].

From the definition of the Berry phase presented above, one might conclude that this is a purely Quantum Mechanical effect, and it should not be present at the level of classical theories. However, as it can be seen from [89, 5], the Berry phase naturally occurs in classical field theories as well.

Generally, the study of SHEs involves the propagation of localized wave packets inside some inhomogeneous medium. Nevertheless, it is instructive to look at the following basic example. If we consider electromagnetic waves described by classical Maxwell's equations, we can easily see how the Berry phase arises naturally, without considering any Quantum Mechanical effects [25, 45, 88]. The intrinsic topological structure of Maxwell's equations in vacuum is revealed as soon as one performs a plane wave expansion for electromagnetic waves. Using this description, electromagnetic waves are characterized by a wave vector $\mathbf{k}$ and a complex polarization vector $\mathbf{e}(\mathbf{k})$, together with the transversality condition $\mathbf{k} \cdot \mathbf{e}(\mathbf{k}) = 0$. Furthermore, the space of possible wave vectors is constrained by the dispersion relation (also called on-shell condition) $|\mathbf{k}|^2 = \omega^2(\mathbf{k})$, which implies that the $\mathbf{k}$-space will have S^2 topology [45]. The polarization vectors $\mathbf{e}(\mathbf{k})$ form a 2-dimensional complex vector space, and due to the transversality condition they will lie in a tangent plane to the spherical space of $\mathbf{k}$ vectors.

By identifying the parameter space from the standard treatment of the Berry phase with the $\mathbf{k}$-space of electromagnetic waves, one can see how the Berry phase arises at the classical level [98, 99]. Considering an electromagnetic wave that follows a closed loop in $\mathbf{k}$-space, the polarization vector $\mathbf{e}(\mathbf{k})$ will be parallel transported around this loop, and, due to the curvature of the $\mathbf{k}$-space, it will get rotated by a geometrical phase factor proportional to the solid angle enclosed by the loop [51] (a visual example of this process is also presented in [88]). This rotation of the polarization vector was already known in 1938, when it was investigated by Rytov [150], followed by the work of Vladimirskii [170] (for this reason, the effect is generally referred to as Rytov or Rytov-Vladimirskii rotation). The effect was experimentally observed for the first time in 1984 by Ross [145], followed by the work of Chiao, Tomita and Wu [48, 165].

Even though it will not be considered in the present review, a similar effect, called the Pancharatnam phase, will also arise if the polarization state space is identified as the parameter space and the adiabatic evolution of the polarization vector is considered [135, 21]. This effect was also observed experimentally [22].

However, regarding curved spacetime, there are few theoretical studies discussing the Berry phase, and no experimental results. A first study of the Berry phase for waves propagating in a weak gravitational field was presented in [44], and further developed by several authors [10, 52, 6, 43, 42, 80, 134].

1.2.3. Spin Hall effect equations of motion

The spin Hall effect of light in inhomogeneous optical media can be viewed as a consequence of the spin-orbit coupling between SAM and EOAM, resulting in the helicity dependence of the ray trajectories. In terms of the Berry phase, the spin Hall effect of light can be described by considering **k**-space as parameter space. Then the Berry curvature of **k**-space will act as a "Lorentz force" on "charged" particles, where the "charge" will be represented by the helicity of photons. Thus, the spin Hall effect of light can be viewed as a consequence of Berry curvature in momentum space [131].

The point-particle equations of motion describing the spin Hall effect of light have been obtained by different authors, using different methods. These include postulating an effective ray Lagrangian or Hamiltonian [131], using geometrical optics with a modified eikonal ansatz for Maxwell's equations [33, 34], or considering a mechanical model for photons, as inspired by the description of spinning particles in General Relativity [67].

Considering an inhomogeneous medium with a refractive index $n(X)$, the equations of motion of polarized light rays, describing the spin Hall effect of light are [148]

$$\dot{\mathbf{P}} = \frac{cP}{n^2}\nabla n, \qquad \dot{\mathbf{X}} = \frac{c\mathbf{P}}{nP} + s\frac{\dot{\mathbf{P}} \times \mathbf{P}}{P^3}, \tag{1.12}$$

where $s = \pm 1$, depending on the state of circular polarization of the light ray.

The last term in the $\dot{X}$ equation is the correction term that determines the spin Hall effect of light and can be interpreted as a Lorentz force produced by the Berry curvature in momentum space, with the photon helicity acting as a charge [37]. As will be discussed in 2, a similar Berry curvature term appears in the description of the gravitational spin Hall effect of light, along with other terms related to the curvature of spacetime.

In the limit of very short wavelengths, the spin Hall effect of light is suppressed, and we recover the classical equations of motion for photons in a medium with arbitrary refractive index n. The spin Hall effect of light becomes more visible as one increases the wavelength, but one should keep in mind that these equations were derived under the assumption that the wavelength is much smaller than the length scale over which the medium properties vary significantly.

The theoretical predictions of Eq. (1.12) were first verified in 2008 by Hosten and Kwait [103]. Their experiment used the technique of quantum weak measurements to amplify the small transverse shift coming from the spin Hall effect of light. This was followed a few months later by the experiment of Bliokh, Niv, Kleiner and Hasman [36]. In this case, the authors managed to amplify the spin Hall effect of light by multiple reflections inside a glass cylinder. Afterwards, the effect was detected by several other groups, using different experimental methods [97, 115, 119, 159]. A more detailed account of the experimental results can be found in [37, 158].

1.3. Spin Hall effects in general relativity

Considering the dynamics of a localized wave packets or a spinning particle, by the gravitational spin Hall effect, we mean any spin-dependent correction of this dynamics, in comparison to the dynamics of a scalar field or geodesic motion. This should extend to general relativity the spin Hall effects known from condensed matter physics and optics. The role of the inhomogeneous medium is now played by spacetime itself, and the spin-orbit coupling is a consequence of the interaction between the spin degree of freedom and the curvature of spacetime. This effect is expected to be present for all spin-fields (some examples are the Dirac field, electromagnetic waves and linear gravitational waves) propagating in a nontrivial fixed spacetime.

One motivation for studying the gravitational spin Hall effect comes from the fact that electromagnetic waves propagating in a curved spacetime are formally described by the same set of equations as electromagnetic waves propagating inside some optical medium, in flat spacetime. The properties of the optical medium can be related to the components of the metric tensor describing the curved spacetime [142, 23, 24]. This type of analogy was first recognized by Eddington, who suggested that the gravitational light bending around the Sun could also be obtained if we consider an appropriate distribution of a refractive material [74]. This was later developed by Gordon [92], and Plebanski [142]. For a more recent discussion see [23, 24].

Following Plebanski [142], a spacetime described by the metric tensor $g_{\mu\nu}$ can be viewed as an effective medium with perfect impedance matching, described by a tensorial permittivity ϵ_{ij}, a tensorial permeability μ_{ij}, and a magnetoelectric coupling vector α_i (here, Latin indices run from 1 to 3):

$$\epsilon^{ij} = \mu^{ij} = -\sqrt{-\det g}\frac{g^{ij}}{g_{00}}, \qquad \alpha_i = -\frac{g_{0i}}{g_{00}}. \tag{1.13}$$

This correspondence is an example of what is called analogue gravity [14], where certain properties of a curved spacetime are reproduced in laboratories using other physical systems. However, this analogy has its limitations and should be used with care. The main limitation is that it breaks covariance, and simply writing the metric using different coordinates can result in analog materials with completely different properties [78].

The experimental observation of the spin Hall effect of light in inhomogeneous optical media, together with the correspondence to curved spacetime, suggests that this effect should also play a role for waves propagating in curved spacetimes, in which context it is usually neglected. It is conceivable that the gravitational spin Hall effect might have experimentally observable consequences, for example, in the form of corrections to gravitational lensing.

Various approaches have been proposed in the literature in order to describe the gravitational spin Hall effect. A detailed review of these theoretical models can be found in Ref. [128]. Below we briefly mention these models, together with their main features and limitations.

1.3.1. Spinning particles in general relativity

The equation for the worldline of a massive spinning test body in the context of the pole-dipole approximation was first derived by Mathisson [122] and Papapetrou [136] by integrating the conservation law of the energy momentum tensor $\nabla_\nu T^{\mu\nu}$ for a multipole expansion of the energy momentum tensor $T^{\mu\nu}$. A covariant derivation was given by Tulczyjew [167] and Dixon [57]. The latter contains multipole expansions to any order; see also Ref. [156]. There are many alternative derivations in the literature [156, 144, 169, 160, 13]. A Hamiltonian formulation for the Mathisson-Papapetrou-Dixon equations can be found in Ref. [13], and the systematic presentation of the Hamiltonian for different orders in spin can be found in Ref. [169]. A particularly transparent derivation can be found in Ref. [161], and a slightly more general derivation in Ref. [59]. A more mathematical derivation, including a full discussion of the symplectic structure of the phase space of the dynamic variables, can be found in Ref. [160], albeit only available in French. For the definition of multipole moments see Ref. [58].

The Mathisson-Papapetrou-Dixon equations are given by:

$$\dot{p}^\mu = -\frac{1}{2} R^\mu{}_{\nu\kappa\lambda} u^\nu S^{\kappa\lambda}, \tag{1.14}$$

$$\dot{S}^{\alpha\beta} = p^{[\alpha} u^{\beta]}, \tag{1.15}$$

where u^μ denotes the four-velocity of the particle (the timelike unit tangent vector of the worldline $u^\mu u_\mu = -1$), p^μ is the total momentum of the particle, and $S^{\mu\nu}$ is the totally antisymmetric spin tensor. The system (1.14)-(1.15) has 10 equations for 13 unknowns (3 for u^μ, 4 for p^μ and 6 for $S^{\mu\nu}$) and is therefore underdetermined. In particular, we are missing an equation that determines u^μ. This is usually fixed with so called spin supplementary conditions. The most commonly used spin supplementary conditions are the following ones:

- Tulczyjew-Dixon: $S^{\mu\nu} p_\nu = 0$
- Pirani: $S^{\mu\nu} u_\nu = 0$ [141]
- Corinaldesi-Papapetrou: $(\partial_t)_\mu S^{\mu\alpha} = 0$, for stationary spacetimes.

Note that the worldlines obtained from different spin supplementary conditions do not coincide. They are usually interpreted as different gauge choices for the "center of mass" of the extended bodies [53]. According to Dixon [59], the Tulczyjew-Dixon spin supplementary condition, $S^{\mu\nu} p_\nu = 0$, is the only spin supplementary condition that fixes a unique world line for an extended body, without requiring the introduction of any other additional structure. For a review on the effect of the different spin supplementary conditions and their physical interpretation, see [54, 55]. For the Tulczyjew-Dixon spin supplementary condition, $m = p^\mu u_\mu$ can be interpreted as the mass, which is constant along the worldline. For the Pirani spin supplementary condition, the mass is given by $p^\mu p_\mu = \tilde{m}$, which is again conserved along the worldline. For both spin supplementary conditions, the magnitude of the spin, $s^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}$, is constant along the worldlines. It was shown in [53] that various choices are in fact physically equivalent, provided that higher-order quadrupole terms are ignored. Therefore, choosing a spin supplementary condition comes down to practicality and personal preferences.

The fact that the Mathisson-Papapetrou-Dixon equations can be adapted for massless particles was first mentioned by Souriau [160], and then worked out in detail by Saturnini [152] (both references only available in French). They start with the Mathisson-Papapetrou-Dixon equations (1.14) and (1.15), assume the Tulczyjew-Dixon spin supplementary condition, $S^{\mu\nu} p_\nu = 0$, $S^{\mu\nu} \neq 0$, and the momentum to be null, $p^\mu p_\mu = 0$. They obtain the following set of equations, to which we will refer to as the Souriau-Saturnini equations:

$$u^\mu = p^\mu + \frac{2}{R_{\alpha\beta\lambda\nu}S^{\alpha\beta}S^{\lambda\nu}} S^{\alpha\mu} R_{\alpha\beta\lambda\nu} S^{\lambda\nu} p^\beta, \tag{1.16}$$

$$\dot{p}^\mu = s \frac{\sqrt{-g}\epsilon^{\alpha\beta\rho\sigma} R_{\alpha\beta\lambda\nu} S^{\lambda\nu} R_{\rho\sigma\gamma\delta} S^{\gamma\delta}}{8 R_{\alpha\beta\lambda\nu} S^{\alpha\beta} S^{\lambda\nu}} p^\mu, \tag{1.17}$$

$$\dot{S}^{\mu\nu} = p^{[\mu} u^{\nu]}, \tag{1.18}$$

where g is the metric determinant.

In Ref. [152], Saturnini showed that for a certain choice of initial condition for the spin, a radially ingoing null geodesic would satisfy Eqs. (1.16)-(1.18) and the observation of redshift would not change for massless particles with spin. Using numerical simulations, he also observed that, for certain initial conditions in Schwarzschild spacetimes, the Eqs. (1.16)-(1.18) with different helicities produce trajectories that are symmetric with respect to the plane of a reference null geodesic with zero spin. However, he deemed the effect to be too small to be observable.

In Ref. [68, 65], Duval et al. present a derivation of the spin Hall effect of light by using a symplectic framework, similar to the one introduced by Souriau and Saturnini in the context of spinning particles in curved spacetime.

In Ref. [66], Duval and Schücker studied the Souriau-Saturnini equations in a Robertson Walker spacetime. By numerically integrating Eqs. (1.16)-(1.18) with a non-zero orthogonal component in the spin vector, they obtained spacelike spiral trajectories that wind around a reference null geodesic, or equivalently, a reference trajectory for a spinning massless particle with zero orthogonal spin component in the spin vector. They argue that, for "reasonable cosmologies, redshifts, and atomic periods", the physical distance between the spiral and the null geodesic is of the order of the wavelength, even though according to their analysis it is in principle unbounded.

In Ref. [70], Duval, Marsot, and Schücker extended the analysis to Schwarzschild spacetimes. For the numerical simulations, they assumed initial conditions at the perihelion, the point of closest approach to the star on the trajectory. From their perturbative analysis, they recover two deflection angles, one between the trajectory and the geodesic plane, given by:

$$\beta \sim -\left(1 - \frac{2GM}{r_0}\right) \frac{\chi\lambda_0}{2\pi r_0}, \tag{1.19}$$

and one between the geodesic plane and the momentum carried by the spinning photon:

$$\gamma \sim s \frac{GM\lambda_0}{2\pi r_0^2}. \tag{1.20}$$

Here, $s = \pm 1$ is the photon helicity. This second deflection angle is proportional to the one derived in [95]. It is reassuring that the deflection angle comes out similarly with two completely different methods.

The Mathisson-Papapetrou-Dixon equations and the Souriau-Saturnini equations represent a covariant description of spinning particles on arbitrary spacetime backgrounds. However, at least in the massless case, it is not clear to what extent are these equations approximating the behavior of localized electromagnetic wave packets in curved spacetime.

1.3.2. Relativistic quantum mechanical approach

The first connection between the motion of spinning particles in curved spacetime and the spin Hall effect was introduced by Bérard and Mohrbach in 2006 [19]. The authors studied the adiabatic evolution of a Dirac particle by using the Foldy-Wouthuysen transformation [82] and presented a generalization of this method for arbitrary spin-fields by using the Bargmann-Wigner equations [15], and a generalized version of the Foldy-Wouthuysen transformation [46, 107]. In this way, the position operator of spinning particles acquires an anomalous contribution, related to a non-Abelian Berry connection [19]. Based on this method, Gosselin, Bérard, and Mohrbach studied the gravitational spin Hall effect of electrons [94] and photons [95] in a static gravitational field.

Restricting attention to the case of photons discussed in Ref. [95], the authors describe electromagnetic waves using the Bargmann-Wigner equations of a massless spin-1 field. In general, the Bargmann-Wigner equations describe massive or massless free spin-j fields, and consist of $2j$ coupled partial differential equations, each equation having a similar structure as a Dirac equation [15, 96]. Considering the case of a spin-1 field in the curved spacetime described by the metric $g_{\mu\nu}$, the Bargmann-Wigner equations take the following form:

$$(-i\hbar\gamma^\mu \nabla_\mu + m)_{\alpha_1 \alpha_1'} \Psi_{\alpha_1' \alpha_2} = 0, \tag{1.21}$$

$$(-i\hbar\gamma^\mu \nabla_\mu + m)_{\alpha_2 \alpha_2'} \Psi_{\alpha_1 \alpha_2'} = 0, \tag{1.22}$$

where the field $\Psi_{\alpha_1\alpha_2}$ is a completely symmetric 4-spinor of rank 2, the primed indices are contracted, the gamma matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and ∇_μ is the covariant derivative for spinor fields. When setting $m = 0$, it can be shown that these equations are equivalent to the homogeneous Maxwell's equations [96].

In order to obtain the equations of motion describing the gravitational spin Hall effect of photons in a static gravitational field, Gosselin et al. [95] used a generalized Foldy-Wouthuysen transformation, together with their semiclassical diagonalization procedure described in [19, 93]. Even though their results describe a general static spacetime with torsion [95], here we will restrict our attention to the particular case of a Schwarzschild background, with the metric expressed in isotropic coordinates as is Eq. (2.211). In this case, the following equations of motion, describing the gravitational spin Hall effect of photons, were obtained by Gosselin et al. [95]:

$$\dot{\mathbf{P}} = -cP\nabla F, \qquad \dot{\mathbf{X}} = c\frac{\mathbf{P}}{P}F + s\frac{\dot{\mathbf{P}} \times \mathbf{P}}{P^3}, \tag{1.23}$$

where $F = \sqrt{-\frac{g_{tt}}{g_{xx}}}$ contains the metric components, $s = \pm\hbar$ is the photon helicity, $P = h/\lambda$ is the magnitude of the photon momentum, and the vector notation is $\mathbf{P} = (P_x, P_y, P_z)$, $\mathbf{X} = (X, Y, Z)$. The gravitational spin Hall effect is given by the second term in the equation for $\dot{\mathbf{X}}$. Clearly, this is a helicity dependent correction, which vanishes when we neglect the helicity of the photon. In this case, the equations of motion reduce to the usual null geodesics, and describe ordinary light bending around a Schwarzschild black hole. Also, the gravitational spin Hall effect correction term is proportional to the wavelength λ of the photon, since $\dot{\mathbf{P}} \propto P \propto \lambda^{-1}$, $\mathbf{P} \propto \lambda^{-1}$, and $P^3 \propto \lambda^{-3}$. Thus, the gravitational spin Hall effect vanishes in the limit of very short wavelengths or infinitely high frequencies.

An alternative derivation of Eqs. (1.23) can be obtained by treating the Schwarzschild spacetime as an effective inhomogeneous medium. By using the equivalence between Maxwell's equations in curved spacetime and inside a material in flat spacetime, as discussed in Eq. (1.13), an effective refractive index can be attributed to the Schwarzschild spacetime, $n = 1/F$, and the same methods as for the spin Hall effect of light in inhomogeneous optical media can be applied. For example, Eqs. (1.23) can be easily obtained by inserting $n = 1/F$ into Eqs. (1.12).

One of the main disadvantages of the method used in [95] is that the use of the Bargmann-Wigner equations blurs the connection with Maxwell's equations, while the Foldy-Wouthuysen transformation, and the semiclassical diagonalization procedures unnecessarily introduce Planck's constant, giving the general impression that the gravitational spin Hall effect is of quantum mechanical origin (this is clearly not the case, since Planck's constant cancels in the second term in the equation for $\dot{\mathbf{X}}$). Another drawback of the approach of Gosselin et al. is that their treatment is limited to static spacetimes, and it is not clear how the method should be extended to more general spacetimes.

1.3.3. Geometrical optics approach

The geometrical optics approximation is widely used in general relativity for various wave equations. The starting point for this approach is a WKB ansatz

$$\begin{aligned}\Psi(x) &= A[x, \nabla S(x), \epsilon] e^{iS(x)/\epsilon},\\ A[x, \nabla S(x), \epsilon] &= \sum_{n=0} \epsilon^n A_n[x, \nabla S(x)].\end{aligned} \tag{1.24}$$

where $S(x)$ is a rapidly oscillating real phase function, A is a slowly varying amplitude (this is a scalar, vector, tensor or spinor, depending on the nature of the considered field), and a small expansion parameter ϵ. Note that, in contrast to most of the physics literature on the subject, we are explicitly allowing the amplitude A to depend on the phase gradient ∇S. This is justified by the mathematical formulation of the WKB approximation [18, 76], where ∇S determines a Lagrangian submanifold of T^*M, and the amplitude A is defined on this Lagrangian submanifold, by a transport equation. The WKB ansatz is inserted into the considered field equation, and the result is analyzed at each order in ϵ.

The standard treatment for the propagation of electromagnetic waves in general relativity is achieved by investigating Maxwell's equations in curved spacetime. Null geodesics can be obtained from Maxwell's equations by considering the lowest order geometrical optics approximation [124, 139, 140]. However, at this level of approximation, there is no influence of the polarization degree of freedom on the ray trajectories. To obtain a theoretical description of the gravitational spin Hall effect, higher-order terms should be considered in the geometrical optics approximation.

Starting with Maxwell's equations in curved spacetime, and by considering certain corrections to the standard geometrical optics approximation, several authors obtained polarization-dependent trajectories for light rays in a curved spacetime [85, 86, 177, 61, 62, 154, 84] (see also [101] for a more general discussion). However, some of the predictions presented in these papers are in contradiction with the results discussed in sections 1.3.1 and 1.3.2. For example, polarization-dependent trajectories were predicted in [85, 177] on a Kerr spacetime. However, according to the authors, this effect disappears in the limit of a Schwarzschild spacetimes, in contrast to what we discussed in the previous sections. Another problem of the approach presented in [85, 177] is that it only works for stationary spacetimes, and it is not clear how to extend it beyond this regime.

A similar procedure was applied in [177] to study the propagation of gravitational waves, with similar results as discussed above. The only difference comes from the fact that gravitational waves are described by a massless spin-2 field, so we have helicity $s = \pm 2$. These claims are in contradiction with the results of Yamamoto [176], which predicted a spin Hall effect for gravitational waves propagating in Schwarzschild spacetimes.

The geometrical optics approach has also been used for studying the gravitational spin Hall effect of massive Dirac fields [147, 9]. It has been shown by these authors that the resulting ray equations coincide with the linearized Mathisson-Papapetrou-Dixon equations.

2. Gravitational spin Hall effect of light

Contents

This chapter contains the main results of the present thesis - a covariant derivation of the gravitational spin Hall effect of light, based on a WKB analysis of the vacuum Maxwell's equations. The presentation closely follows Ref. [129], with some extra examples and details added.

We start by introducing the vacuum Maxwell's equations on a fixed background. Then, the WKB approximation is used to obtain an approximation of the Maxwell field action. At the lowest order in the WKB expansion parameter ϵ, the corresponding Euler-Lagrange equations reproduce the well-known results of geometrical optics, while the first-order correction terms describe polarization effects, crucial in the derivation of the gravitational spin Hall effect. Using these Euler-Lagrange equations, we derive an effective dispersion relation, containing polarization-dependent correction terms. The transition between the effective dispersion relation and the effective ray equations describing the gravitational spin Hall effect of light is realized by considering the effective dispersion relation as a Hamilton-Jacobi equation and applying the method of characteristics. Then, the effective ray equations are analyzed, and several examples are presented.

2.1. Maxwell's equations

We consider electromagnetic waves in vacuum as test fields on a Lorentzian manifold $(M, g_{\mu\nu})$. These can be described by the electromagnetic tensor $\mathscr{F}_{\alpha\beta}$, which is a skew-symmetric real 2-form, satisfying the vacuum Maxwell's equations [124, Sec. 22.4]

$$\nabla^\alpha \mathscr{F}_{\alpha\beta} = 0, \qquad \nabla_{[\alpha} \mathscr{F}_{\beta\gamma]} = 0. \tag{2.1}$$

Electromagnetic waves can also be described by using the electromagnetic four-potential $\mathscr{A}_\alpha$, which is a real 1-form. Then, the electromagnetic tensor $\mathscr{F}_{\alpha\beta}$ is obtained as

$$\mathscr{F}_{\alpha\beta} = 2\nabla_{[\alpha} \mathscr{A}_{\beta]}, \tag{2.2}$$

and Eq. (2.1) becomes [124, Sec. 22.4]

$$\hat{D}_\alpha{}^\beta \mathscr{A}_\beta = 0, \qquad \hat{D}_\alpha{}^\beta = \nabla^\beta \nabla_\alpha - \delta^\beta_\alpha \nabla^\mu \nabla_\mu. \tag{2.3}$$

This equation can be obtained as the Euler-Lagrange equation from the following action:

$$J = \frac{1}{4} \int_M \mathrm{d}^4x \, \sqrt{g} \, \mathscr{F}_{\alpha\beta} \mathscr{F}^{\alpha\beta} = \frac{1}{2} \int_M \mathrm{d}^4x \, \sqrt{g} \, \mathscr{A}^\alpha \hat{D}_\alpha{}^\beta \mathscr{A}_\beta, \tag{2.4}$$

where the last equality is obtained using integration by parts.

Note that the physically relevant quantity is always the electromagnetic tensor $\mathscr{F}_{\alpha\beta}$, while the electromagnetic four-potential $\mathscr{A}_\alpha$ can, in general, contain pure gauge terms. In the following, we shall adopt the Lorenz gauge for the electromagnetic electromagnetic four-potential $\mathscr{A}_\alpha$:

$$\nabla_\alpha \mathscr{A}^\alpha = 0. \tag{2.5}$$

2.2. WKB approximation

In general, finding exact solutions for the vacuum Maxwell's equations (2.3) is not possible, and certain approximations need to be considered. Since we are interested in describing the propagation of electromagnetic waves in vacuum, a natural choice is the WKB approximation, which we introduce below.

2.2.1. WKB ansatz

We assume that the electromagnetic four-potential $\mathscr{A}_\alpha$ admits a WKB expansion of the form

$$\begin{aligned} \mathscr{A}_\alpha(x) &= \mathrm{Re}\left[A_\alpha(x, k(x), \epsilon) e^{iS(x)/\epsilon}\right], \\ A_\alpha(x, k(x), \epsilon) &= A_{0\alpha}(x, k(x)) + \epsilon A_{1\alpha}(x, k(x)) + \mathscr{O}(\epsilon^2), \end{aligned} \tag{2.6}$$

where S is a real scalar function, A_α is a complex amplitude, and ϵ is a small expansion parameter. The gradient of S is denoted as

$$k_\mu(x) = \nabla_\mu S(x). \tag{2.7}$$

In general, higher-order terms can be kept in the expansion of the amplitude A_α in Eq. (2.6). However, as we shall see in the following sections, for the purpose of describing spin Hall effects we can ignore terms of $\mathscr{O}(\epsilon^2)$.

Note that we are allowing the complex amplitude A_α to depend on the phase gradient $k_\alpha(x)$, in order to emphasize that A_α is defined on the Lagrangian submanifold $x \mapsto (x, k(x)) \in T^*M$. Such a dependency is often ignored in the physics literature, but it is generally present in mathematically rigorous treatments of the WKB approximations, such as Refs. [18, Sec. 3.3] or [76]. In particular, the dependency of A in k appears naturally in the geometrical optics equation (2.39), and we observe in Sec. 2.3.4 that the polarization vector and the polarization basis naturally depend on k, which is why the k-dependence was introduced in Eq. (2.6).

The small expansion parameter ϵ indicates that the phase of the vector potential rapidly oscillates and its variations are much faster than those corresponding to the complex amplitude $A_\alpha(x, k, \epsilon)$. We consider a timelike observer traveling along the worldline $\lambda \mapsto y^\alpha(\lambda)$ with proper time λ. Then, we can see the relation between ϵ and the wave frequency ω measured by this observer:

$$\omega = -\frac{t^\alpha k_\alpha}{\epsilon}. \tag{2.8}$$

Here, $t^\alpha = \mathrm{d}y^\alpha/\mathrm{d}\lambda$ is the velocity vector field of the observer. The phase function S and ϵ are dimensionless quantities. Working with geometrized units, such that $c = G = 1$ [172, Appendix F], the velocity t^α is dimensionless, and k_α has the dimension of inverse length. Hence, ω has the dimension of the inverse length, as expected for frequency. Then, the observer sees the frequency going to infinity as ϵ goes to 0.

We can illustrate the validity condition of the geometrical optics approximation for a Schwarzschild black hole, with Schwarzschild radius r_s. For a source of light that is falling into the black hole, the gravitational redshift formula implies that the frequency ω_∞ measured by an observer at infinity in the rest frame of the central object is smaller than the frequency measured by an observer at a finite distance from the black hole. Then, a criterion for the high-frequency limit to hold is

$$\epsilon = (\omega_\infty r_s)^{-1} \ll 1. \tag{2.9}$$

Note that we could have taken any observer at a finite distance from the black hole as a criterion. The choice of the observer at infinity provides the simplest expression.

2.2.2. WKB analysis of the Lorenz gauge

Maxwell's equations in the form (2.3) do not have a well-posed Cauchy problem. In particular, they admit pure gauge solutions. This problem is usually eliminated by introducing a gauge condition. Here we shall focus on the Lorenz gauge condition

$$\nabla_\alpha \mathscr{A}^\alpha = 0. \tag{2.10}$$

We reproduce here, in the context of a WKB analysis, the classical argument regarding the gauge fixing for Maxwell's equations (see, for instance [49, Lemma

10.2]). Using the identity

$$\hat{D}_\alpha{}^\beta \mathscr{A}_\beta - \nabla_\alpha \nabla^\beta \mathscr{A}_\beta = -\nabla^\beta \nabla_\beta \mathscr{A}_\alpha + R_{\alpha\beta} \mathscr{A}^\beta, \tag{2.11}$$

one observes that, if Maxwell's equations (2.3) and the Lorenz gauge (2.10) are satisfied, then the wave equation

$$-\nabla^\beta \nabla_\beta \mathscr{A}_\alpha + R_{\alpha\beta} \mathscr{A}^\beta = 0 \tag{2.12}$$

holds. Conversely, by solving Eq. (2.12), with Cauchy data satisfying the constraint and gauge conditions, one obtains a solution to Maxwell's equations in the Lorenz gauge.

Note that we consider here approximate solutions to Maxwell's equations

$$\hat{D}_\alpha{}^\beta \mathscr{A}_\beta = \mathscr{O}(\epsilon^0). \tag{2.13}$$

Hence, it is sufficient to consider that the Lorenz gauge is satisfied at the appropriate order:

$$\nabla_\alpha \mathscr{A}^\alpha = \mathscr{O}(\epsilon^1). \tag{2.14}$$

We reproduce the standard argument recovering Maxwell's equation in the Lorenz gauge from the wave Eq. (2.12), taking into account that we are considering only approximate solutions. Assume that the wave equation holds:

$$-\nabla^\beta \nabla_\beta \mathscr{A}_\alpha + R_{\alpha\beta} \mathscr{A}^\beta = \mathscr{O}(\epsilon^0). \tag{2.15}$$

Upon inserting the WKB ansatz, this is equivalent to

$$\begin{aligned} k^\beta k_\beta A_{0\alpha} &= 0, \\ i k^\beta k_\beta A_{1\alpha} + A_{0\alpha} \nabla^\beta k_\beta + 2 k^\beta \nabla_\beta A_{0\alpha} &= 0. \end{aligned} \tag{2.16}$$

Furthermore, assume that the initial data for the wave equation (2.15) satisfy

$$\begin{aligned} k_\alpha A_0{}^\alpha &= 0, \\ \nabla_\alpha A_0{}^\alpha + i k_\alpha A_1{}^\alpha &= 0. \end{aligned} \tag{2.17}$$

Equation (2.15) implies that

$$\nabla^\beta \nabla_\beta \left(\nabla_\alpha \mathscr{A}^\alpha \right) = \mathscr{O}(\epsilon^{-1}). \tag{2.18}$$

The initial data (2.17) for Eq. (2.15) imply that, initially,

$$\nabla_\alpha \mathscr{A}^\alpha = \mathscr{O}(\epsilon^1). \tag{2.19}$$

Observe that the condition

$$T^\beta \nabla_\beta \left(\nabla_\alpha \mathscr{A}^\alpha \right) = \mathscr{O}(\epsilon^0) \tag{2.20}$$

is automatically satisfied, where T^β is a unit future-oriented normal vector to the hypersurface on which initial data are prescribed. Hence, the equation satisfied by the Lorenz gauge source function (2.18) admits initial data as in Eqs. (2.19) and (2.20) vanishing at the appropriate order in ϵ [at $\mathscr{O}(\epsilon^1)$ and $\mathscr{O}(\epsilon^0)$, respectively]. This implies that Maxwell's equations

$$\hat{D}_\alpha{}^\beta \mathscr{A}_\beta = \mathscr{O}(\epsilon^0), \tag{2.21}$$

which can be expanded as

$$\begin{aligned} k^\beta A_{0[\beta} k_{\alpha]} &= 0, \\ 2k^\beta \nabla_\beta A_{0\alpha} - \left(\nabla_\beta A_0{}^\beta + i k_\beta A_1{}^\beta \right) k_\alpha - k^\beta \nabla_\alpha A_{0\beta} & \\ - A_0{}^\beta \nabla_\beta k_\alpha + A_{0\alpha} \nabla_\beta k^\beta + i k^\beta k_\beta A_{1\alpha} &= 0, \end{aligned} \tag{2.22}$$

are satisfied in the Lorenz gauge

$$\nabla_\alpha \mathscr{A}^\alpha = \mathscr{O}(\epsilon^1) \Leftrightarrow \left\{ \begin{aligned} k_\alpha A_0{}^\alpha &= 0 \\ \nabla_\alpha A_0{}^\alpha + i k_\alpha A_1{}^\alpha &= 0 \end{aligned} \right. . \tag{2.23}$$

2.2.3. Assumptions on the initial conditions

We end this section by summarizing the initial conditions that we shall use in the WKB ansatz for Maxwell's equations.

1. The Lorenz gauge (2.23) is satisfied initially. This condition is used to obtain a well-defined solution to the equations of motion, as discussed in Sec. 2.2.2.
2. The initial phase gradient k_α is a future-oriented null covector. As will be seen, the condition that k_α is null is a compatibility condition that follows from the Euler-Lagrange equations and the Lorenz gauge condition (2.23) at the lowest order in ϵ; cf. dispersion relation (2.31) below.
3. Initially, the beam has circular polarization; cf. Eq. (2.70). In Sec. 2.3.4 we show that the initial state of circular polarization is conserved. In Sec. 2.4.2 this assumption ensures a consistent transition transition between the effective dispersion relation and the effective ray equations. Heuristically speaking, due to the spin Hall effect, a localized wave packet that initially has linear polarization can split into two localized wave packets of opposite circular polarization. While this does not represent a problem at the level of

Maxwell's equations (which are partial differential equations), the same behavior cannot be captured by the effective ray equations (which are ordinary differential equations) obtained in Sec. 2.4.2.

2.3. Higher-order geometrical optics

2.3.1. WKB approximation of the field action

We compute the WKB approximation for our field theory by inserting the WKB ansatz (2.6) in the field action (2.4):

$$\begin{aligned} J &= \int_M \mathrm{d}^4x \, \sqrt{g} \, \mathrm{Re}\left(A^\alpha e^{iS/\epsilon}\right) \hat{D}_\alpha{}^\beta \mathrm{Re}\left(A_\beta e^{iS/\epsilon}\right) \\ &= \frac{1}{4} \int_M \mathrm{d}^4x \, \sqrt{g} \left[A^{*\alpha} e^{-iS/\epsilon} \hat{D}_\alpha{}^\beta \left(A_\beta e^{iS/\epsilon}\right) + \text{c.c.}\right] \\ &\quad + \frac{1}{4} \int_M \mathrm{d}^4x \, \sqrt{g} \left[A^\alpha e^{iS/\epsilon} \hat{D}_\alpha{}^\beta \left(A_\beta e^{iS/\epsilon}\right) + \text{c.c.}\right]. \end{aligned} \tag{2.24}$$

If S has a nonvanishing gradient, then $e^{iS/\epsilon}$ is rapidly oscillating. In this case, for f sufficiently regular, the method of stationary phase [64, Sec. 2.3] implies

$$\int_M \mathrm{d}^4x \, \sqrt{g} \, e^{\pm i2S(x)/\epsilon} f(x) = \mathcal{O}(\epsilon^2). \tag{2.25}$$

Upon expanding the derivative terms in Eq. (2.24), and keeping only terms of the lowest two orders in ϵ, we obtain the following WKB approximation of the field action [for convenience, we are shifting the powers of ϵ, such that the lowest-order term is of $\mathcal{O}(\epsilon^0)$]:

$$-\epsilon^2 J = \int_M \mathrm{d}^4x \, \sqrt{g} \Big[D_\alpha{}^\beta A^{*\alpha} A_\beta - \frac{i\epsilon}{2} \overset{v}{\nabla}{}^\mu D_\alpha{}^\beta \left(A^{*\alpha} \nabla_\mu A_\beta - A_\beta \nabla_\mu A^{*\alpha}\right)\Big] + \mathcal{O}(\epsilon^2), \tag{2.26}$$

where

$$\begin{aligned} D_\alpha{}^\beta &= \frac{1}{2} k_\mu k^\mu \delta_\alpha^\beta - \frac{1}{2} k_\alpha k^\beta, \\ \overset{v}{\nabla}{}^\mu D_\alpha{}^\beta &= k^\mu \delta_\alpha^\beta - \frac{1}{2} \delta_\alpha^\mu k^\beta - \frac{1}{2} g^{\mu\beta} k_\alpha. \end{aligned} \tag{2.27}$$

Here, $D_\alpha{}^\beta$ represents the symbol [87] of the operator $\hat{D}_\alpha{}^\beta$, evaluated at the phase space point $(x, p) = (x, k)$, and we are using the notation $\overset{v}{\nabla}{}^\mu D_\alpha{}^\beta$ for the vertical derivative (Appendix A.1) of $D_\alpha{}^\beta$, evaluated at the phase space point $(x, p) = (x, k)$.

The action depends on the following fields: $S(x)$, $\nabla_\mu S(x)$, $A_\alpha(x, \nabla S)$, $\nabla_\mu [A_\alpha(x, \nabla S)]$, $A^{*\alpha}(x, \nabla S)$, $\nabla_\mu [A^{*\alpha}(x, \nabla S)]$. Following the calculations in Appendix A.2, the Euler-Lagrange equations are

$$D_\alpha{}^\beta A_\beta - i\epsilon \left(\overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right) \nabla_\mu A_\beta - \frac{i\epsilon}{2} \left(\nabla_\mu \overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right) A_\beta = \mathcal{O}(\epsilon^2), \tag{2.28}$$

$$D_\alpha{}^\beta A^{*\alpha} + i\epsilon \left(\overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right) \nabla_\mu A^{*\alpha} + \frac{i\epsilon}{2} \left(\nabla_\mu \overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right) A^{*\alpha} = \mathcal{O}(\epsilon^2), \tag{2.29}$$

$$\nabla_\mu \left[\left(\overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right) A^{*\alpha} A_\beta - \frac{i\epsilon}{2} \left(\overset{v}{\nabla}{}^\mu \overset{v}{\nabla}{}^\nu D_\alpha{}^\beta\right) \left(A^{*\alpha} \nabla_\nu A_\beta - A_\beta \nabla_\nu A^{*\alpha}\right) \right] = \mathcal{O}(\epsilon^2). \tag{2.30}$$

In the above equations, the symbol $D_\alpha{}^\beta$ and its vertical derivatives are all evaluated at the phase space point (x, k). Note that the same set of equations can be obtained in a more traditional way, by inserting the WKB ansatz (2.6) directly into the field equation (2.3), or by following the approach presented in Ref. [60]. More generally, a detailed discussion about the variational formulation of the WKB approximation can be found in Ref. [166].

2.3.2. Zeroth-order geometrical optics

Starting with Eqs. (2.28)-(2.30), and keeping only terms of $\mathcal{O}(\epsilon^0)$, we obtain

$$D_\alpha{}^\beta A_{0\beta} = 0, \tag{2.31}$$

$$D_\alpha{}^\beta A_0{}^{*\alpha} = 0, \tag{2.32}$$

$$\nabla_\mu \left[\left(\overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right) A_0{}^{*\alpha} A_{0\beta} \right] = 0. \tag{2.33}$$

Equation (2.31) can also be written as

$$D_\alpha{}^\beta A_{0\beta} = \frac{1}{2} \left(k_\mu k^\mu \delta_\alpha^\beta - k_\alpha k^\beta\right) A_{0\beta} = 0. \tag{2.34}$$

The matrix $D_\alpha{}^\beta$ admits two eigenvalues when k_α is not a null covector. The first eigenvalue is $\frac{1}{2} k_\mu k^\mu$ with eigenspace consisting of covectors perpendicular to k_α. The second eigenvalue is 0 with eigenvector k_α. When k_α is null, the matrix $D_\alpha{}^\beta$

is nilpotent. It admits a unique eigenvalue 0 whose eigenspace is the orthogonal to k_α, which contains the covector k_α.

The Lorenz gauge condition (2.23) implies that $A_{0\alpha}$ is orthogonal to k_α. Hence, a necessary condition for Eq. (2.31) to admit a nontrivial solution is that k_α is a null covector. It is also possible to deduce that k_α is a null covector without using the gauge condition. For completeness, we present this argument below.

Equation (2.34) admits nontrivial solutions if and only if $A_{0\beta}$ is an eigenvector of ${D_\alpha}^\beta$ with zero eigenvalue. Two cases should be discussed: k_α is a null covector, or k_α is not a null covector.

Assume first that k_α is not a null covector, $k^\mu k_\mu \neq 0$. Then, Eq. (2.34) leads to

$$A_{0\alpha} = \frac{k^\beta A_{0\beta}}{k_\mu k^\mu} k_\alpha. \tag{2.35}$$

This entails that

$$A_{0[\alpha} k_{\beta]} = 0 \quad \text{or} \quad \mathscr{F}_{\alpha\beta} = \nabla_{[\alpha} \mathscr{A}_{\beta]} = \mathscr{O}(\epsilon^0). \tag{2.36}$$

In other words, when k_α is not a null covector, the corresponding solution is, at the lowest order in ϵ, a pure gauge solution. Since the corresponding electromagnetic field vanishes, we do not consider this case further.

If k_α is null, $k^\mu k_\mu = 0$, Eq. (2.34) implies

$$k^\beta A_{0\beta} = 0. \tag{2.37}$$

This is consistent with the Lorenz gauge condition (2.23) at the lowest order in ϵ. A similar argument can be applied for the complex-conjugate Eq. (2.32), from which we obtain $k_\alpha {A_0}^{*\alpha} = 0$.

Using Eqs. (2.31)-(2.33), we obtain the well-known system of equations governing the geometrical optics approximation at the lowest order in ϵ:

$$k_\mu k^\mu = 0, \tag{2.38}$$

$$k^\alpha A_{0\alpha} = k_\alpha {A_0}^{*\alpha} = 0, \tag{2.39}$$

$$\nabla_\mu \left(k^\mu \mathscr{I}_0 \right) = 0, \tag{2.40}$$

where $\mathscr{I}_0 = {A_0}^{*\alpha} A_{0\alpha}$ is the lowest-order intensity (more precisely, $\mathscr{I}_0$ is proportional to the wave action density [166]). Equation (2.40) is obtained from Eq. (2.33) by using the orthogonality condition (2.39). Using Eq. (2.7), we have

$$\nabla_\mu k_\alpha = \nabla_\alpha k_\mu, \tag{2.41}$$

and we can use Eq. (2.38) to derive the geodesic equation for k_μ:

$$k^\nu \nabla_\nu k_\mu = 0. \tag{2.42}$$

2.3.3. First-order geometrical optics

Here, we examine Eqs. (2.28) and (2.29) at order ϵ^1 only:

$$D_\alpha{}^\beta A_{1\beta} - i\left(\overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right)\nabla_\mu A_{0\beta} - \frac{i}{2}\left(\nabla_\mu \overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right) A_{0\beta} = 0, \tag{2.43}$$

$$D_\alpha{}^\beta A_1{}^{*\alpha} + i\left(\overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right)\nabla_\mu A_0{}^{*\alpha} + \frac{i}{2}\left(\nabla_\mu \overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right) A_0{}^{*\alpha} = 0. \tag{2.44}$$

Using Eq. (2.27), we can also rewrite Eq. (2.43) as follows:

$$\begin{aligned} k^\mu \nabla_\mu A_{0\alpha} - \frac{1}{2}k_\alpha \nabla_\mu A_0{}^\mu - \frac{1}{2}k_\beta \nabla_\alpha A_0{}^\beta - \frac{i}{2}k_\alpha k^\beta A_{1\beta} \\ + \frac{1}{2}A_{0\alpha}\nabla_\mu k^\mu - \frac{1}{4}A_0{}^\beta \nabla_\beta k_\alpha - \frac{1}{4}A_0{}^\beta \nabla_\alpha k_\beta = 0. \end{aligned} \tag{2.45}$$

Using Eq. (2.41), we can rewrite the last two terms as

$$-\frac{1}{4}A_0{}^\beta \nabla_\beta k_\alpha - \frac{1}{4}A_0{}^\beta \nabla_\alpha k_\beta = -\frac{1}{2}A_0{}^\beta \nabla_\alpha k_\beta. \tag{2.46}$$

Using Eq. (2.39), we also have

$$0 = \nabla_\alpha\left(k_\beta A_0{}^\beta\right) = k_\beta \nabla_\alpha A_0{}^\beta + A_0{}^\beta \nabla_\alpha k_\beta. \tag{2.47}$$

Then, Eq. (2.45) becomes

$$k^\mu \nabla_\mu A_{0\alpha} + \frac{1}{2}A_{0\alpha}\nabla_\mu k^\mu - \frac{1}{2}k_\alpha\left(\nabla_\mu A_0{}^\mu + i k_\mu A_1{}^\mu\right) = 0. \tag{2.48}$$

The last term can be eliminated by using the Lorenz gauge (2.23). The same steps can be applied to the complex-conjugate Eq. (2.44):

$$\begin{aligned} k^\mu \nabla_\mu A_{0\alpha} + \frac{1}{2}A_{0\alpha}\nabla_\mu k^\mu = 0, \\ k^\mu \nabla_\mu A_0{}^{*\beta} + \frac{1}{2}A_0{}^{*\beta}\nabla_\mu k^\mu = 0. \end{aligned} \tag{2.49}$$

Furthermore, using the lowest-order intensity $\mathscr{I}_0$, we can write the amplitude in the following way:

$$A_{0\alpha} = \sqrt{\mathscr{I}_0}\, a_{0\alpha}, \qquad A_0{}^{*\alpha} = \sqrt{\mathscr{I}_0}\, a_0{}^{*\alpha}, \tag{2.50}$$

where $a_{0\alpha}$ is a unit complex covector (i.e., $a_0{}^{*\alpha} a_{0\alpha} = 1$) describing the polarization. Then, from Eq. (2.49), together with Eq. (2.40), we obtain

$$k^\mu \nabla_\mu a_{0\alpha} = k^\mu \nabla_\mu a_0{}^{*\alpha} = 0. \tag{2.51}$$

The parallel propagation of the complex covector $a_{0\alpha}$ along the integral curve of k^μ is another well-known result of the geometrical optics approximation.

2.3.4. The polarization vector in a null tetrad

We observed that the polarization vector satisfies the orthogonality condition

$$k^\alpha a_{0\alpha} = 0. \tag{2.52}$$

Consider a null tetrad [138, Sec. 3] $\{k_\alpha, n_\alpha, m_\alpha, \bar{m}_\alpha\}$ satisfying

$$\begin{gathered} m_\alpha \bar{m}^\alpha = 1, \qquad k_\alpha n^\alpha = -1, \\ k_\alpha k^\alpha = n_\alpha n^\alpha = m_\alpha m^\alpha = \bar{m}_\alpha \bar{m}^\alpha = 0, \\ k_\alpha m^\alpha = k_\alpha \bar{m}^\alpha = n_\alpha m^\alpha = n_\alpha \bar{m}^\alpha = 0. \end{gathered} \tag{2.53}$$

Note that we use the metric signature opposite to that used in Ref. [138, Sec. 3]. The covectors $n_\alpha, m_\alpha, \bar{m}_\alpha$ are not assumed to be parallel-propagated along the geodesic generated by k^α. It is only k_α that is parallel-propagated along the geodesic generated by k^α, in accordance with Eq. (2.42). Since the null tetrad is adapted to the covector k_α, the orthogonality conditions (2.53) imply that m_α and $\bar{m}_\alpha$ are functions of k_α. The polarization covector $a_{0\alpha}$ is orthogonal to k_α, so we can decompose it as

$$a_{0\alpha}(x,k) = z_1(x) m_\alpha(x,k) + z_2(x) \bar{m}_\alpha(x,k) + z_3(x) k_\alpha(x), \tag{2.54}$$

where z_1, z_2, and z_3 are complex scalar functions. Since $a_{0\alpha}$ is a unit complex covector, the scalar functions z_1 and z_2 are constrained by

$$z_1^* z_1 + z_2^* z_2 = 1. \tag{2.55}$$

It is important to note that the decomposition (2.54), and more specifically, the choice of m_α, requires choosing a null covector n_α. Fixing n_α is equivalent to choosing a unit timelike covector field t_α that can represent a family of timelike observers. We can always take n_α as

$$t_\alpha = \frac{1}{2\epsilon\omega} k_\alpha + \epsilon\omega n_\alpha. \tag{2.56}$$

Once n_α (or t_α) is fixed, the remaining SO(2) gauge freedom in the choice of m_α is described by the spin rotation

$$k_\alpha \mapsto k_\alpha, \quad n_\alpha \mapsto n_\alpha, \quad m_\alpha \mapsto e^{i\phi(x)} m_\alpha, \tag{2.57}$$

for $\phi(x) \in \mathbb{R}$. Polarization measurements will always depend on the choice of m_α and $\bar{m}_\alpha$. However, as shown in Sec. 2.4.2, the modified ray equations describing the gravitational spin Hall effect of light do not depend on the particular choice of m_α and $\bar{m}_\alpha$. Thus, we can work with any smooth choice of m_α and $\bar{m}_\alpha$ that satisfy Eq. (2.53).

Using Eqs. (2.54) and (2.42), the parallel-transport equation for the polarization vector becomes

$$\begin{aligned} 0 &= k^\mu \nabla_\mu a_{0\alpha} \\ &= z_1 k^\mu \nabla_\mu m_\alpha + z_2 k^\mu \nabla_\mu \bar{m}_\alpha + m_\alpha k^\mu \nabla_\mu z_1 + \bar{m}_\alpha k^\mu \nabla_\mu z_2 + k_\alpha k^\mu \nabla_\mu z_3. \end{aligned} \tag{2.58}$$

Contracting the above equation with $\bar{m}^\alpha$, m^α, and n^α, we obtain

$$\begin{aligned} k^\mu \nabla_\mu z_1 &= -z_1 \bar{m}^\alpha k^\mu \nabla_\mu m_\alpha, \\ k^\mu \nabla_\mu z_2 &= -z_2 m^\alpha k^\mu \nabla_\mu \bar{m}_\alpha, \\ k^\mu \nabla_\mu z_3 &= -(z_1 m^\alpha + z_2 \bar{m}^\alpha) k^\mu \nabla_\mu n_\alpha. \end{aligned} \tag{2.59}$$

Recall that in the above equations, the covectors m_α and $\bar{m}_\alpha$ are functions of x and $k(x)$. The covariant derivatives are applied as follows:

$$\begin{aligned} k^\mu \nabla_\mu m_\alpha &= k^\mu \nabla_\mu \left[m_\alpha(x,k) \right] \\ &= k^\mu \left(\overset{h}{\nabla}_\mu m_\alpha \right)(x,k) + k^\mu \left(\nabla_\mu k_\nu \right) \left(\overset{v}{\nabla}{}^\nu m_\alpha \right)(x,k) \\ &= k^\mu \overset{h}{\nabla}_\mu m_\alpha, \end{aligned} \tag{2.60}$$

where $\overset{h}{\nabla}_\mu$ is the horizontal derivative (Appendix A.1). It is convenient to introduce the two-dimensional unit complex vector

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{2.61}$$

which is analogous to the Jones vector in optics [83, 31, 148, 149]. We also use the Hermitian transpose $z^\dagger$, defined as follows:

$$z^\dagger = \begin{pmatrix} z_1^* & z_2^* \end{pmatrix}. \tag{2.62}$$

Then, the equations for z_1 and z_2 can be written in a more compact form:

$$k^\mu \nabla_\mu z = i k^\mu B_\mu \sigma_3 z, \tag{2.63}$$

where σ_3 is the third Pauli matrix,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.64}$$

and B_μ is the real 1-form extending to general relativity the Berry connection used in optics [31, 148]:

$$B_\mu(x,k) = \frac{i}{2}\left(\bar{m}^\alpha \overset{h}{\nabla}_\mu m_\alpha - m_\alpha \overset{h}{\nabla}_\mu \bar{m}^\alpha\right) = i\bar{m}^\alpha \overset{h}{\nabla}_\mu m_\alpha. \tag{2.65}$$

Furthermore, if we restrict z to an affinely parametrized null geodesic $\tau \mapsto x^\mu(\tau)$, with $\dot{x}^\mu = k^\mu$, we can write

$$\dot{z} = ik^\mu B_\mu \sigma_3 z, \tag{2.66}$$

where $\dot{z} = \dot{x}^\mu \nabla_\mu z$. Integrating along the worldline, we obtain

$$z(\tau) = \begin{pmatrix} e^{i\gamma(\tau)} & 0 \\ 0 & e^{-i\gamma(\tau)} \end{pmatrix} z(0), \tag{2.67}$$

where γ represents the Berry phase [31, 148],

$$\gamma(\tau_1) = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau k^\mu B_\mu. \tag{2.68}$$

Using either Eq. (2.59) or Eq. (2.66), we see that the evolution of z_1 and z_2 is decoupled in the circular polarization basis, and the following quantities are conserved along k^μ:

$$\begin{aligned} 1 &= z_1^* z_1 + z_2^* z_2 = z^\dagger z, \\ s &= z_1^* z_1 - z_2^* z_2 = z^\dagger \sigma_3 z. \end{aligned} \tag{2.69}$$

Based on our assumptions on the initial conditions (Sec. 2.2.3), we only consider beams which are circularly polarized, i.e. one of the conditions

$$z(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{or} \qquad z(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70}$$

holds. Thus, we have $s = \pm 1$, depending on the choice of the initial polarization state.

The results described in this section are similar to the description of the polarization of electromagnetic waves traveling in a medium with an inhomogeneous index of refraction [31].

2.3.5. Extended geometrical optics

Now, we take Eqs. (2.28)-(2.30), but without splitting them order by order in ϵ. Our aim is to derive an effective Hamilton-Jacobi system that would give us $\mathcal{O}(\epsilon)$ corrections to the ray equations.

Effective dispersion relation

By contracting Eq. (2.28) with $A^{*\alpha}$ and Eq. (2.29) with A_β, and adding them together, we obtain the following equation:

$$D_\alpha{}^\beta A^{*\alpha} A_\beta - \frac{i\epsilon}{2}\left(\overset{v}{\nabla}{}^\mu D_\alpha{}^\beta\right)\left(A^{*\alpha}\nabla_\mu A_\beta - A_\beta \nabla_\mu A^{*\alpha}\right) = \mathcal{O}(\epsilon^2). \tag{2.71}$$

Using Eqs. (2.27) and (2.39), we can rewrite the above equation as follows:

$$\begin{aligned} &\frac{1}{2} k_\mu k^\mu (A_0{}^{*\alpha} A_{0\alpha} + \epsilon A_0{}^{*\alpha} A_{1\alpha} + \epsilon A_1{}^{*\alpha} A_{0\alpha}) \\ &\quad - \frac{i\epsilon}{2} k^\mu \left(A_0{}^{*\alpha} \nabla_\mu A_{0\alpha} - A_{0\alpha} \nabla_\mu A_0{}^{*\alpha}\right) \\ &\quad + \frac{i\epsilon}{4} k_\alpha \left(A_0{}^{*\mu} \nabla_\mu A_0{}^\alpha - A_0{}^\mu \nabla_\mu A_0{}^{*\alpha}\right) = \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.72}$$

Using Eq. (2.39), we obtain

$$0 = A_0{}^{*\mu} \nabla_\mu \left(k_\alpha A_0{}^\alpha\right) = k_\alpha A_0{}^{*\mu} \nabla_\mu A_0{}^\alpha + A_0{}^{*\mu} A_0{}^\alpha \nabla_\mu k_\alpha, \tag{2.73}$$

so we can write

$$\frac{i\epsilon}{4} k_\alpha \left(A_0{}^{*\mu} \nabla_\mu A_0{}^\alpha - A_0{}^\mu \nabla_\mu A_0{}^{*\alpha}\right) = -\frac{i\epsilon}{2} \nabla_\mu k_\alpha A_0{}^{*[\mu} A_0{}^{\alpha]} = 0, \tag{2.74}$$

where the last equality is due to Eq. (2.41). Then, Eq. (2.71) becomes

$$\begin{aligned} &\frac{1}{2} k_\mu k^\mu (A_0{}^{*\alpha} A_{0\alpha} + \epsilon A_0{}^{*\alpha} A_{1\alpha} + \epsilon A_1{}^{*\alpha} A_{0\alpha}) \\ &\quad - \frac{i\epsilon}{2} k^\mu \left(A_0{}^{*\alpha} \nabla_\mu A_{0\alpha} - A_{0\alpha} \nabla_\mu A_0{}^{*\alpha}\right) = \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.75}$$

Let us introduce the $\mathcal{O}(\epsilon^1)$ intensity

$$\mathcal{I} = \mathcal{A}^{*\alpha} \mathcal{A}_\alpha = A_0{}^{*\alpha} A_{0\alpha} + \epsilon A_0{}^{*\alpha} A_{1\alpha} + \epsilon A_1{}^{*\alpha} A_{0\alpha} + \mathcal{O}(\epsilon^2). \tag{2.76}$$

Then, we can rewrite the amplitude as

$$A_\alpha = \sqrt{\mathcal{I}} a_\alpha = \sqrt{\mathcal{I}} \left(a_{0\alpha} + \epsilon a_{1\alpha}\right) + \mathcal{O}(\epsilon^2), \tag{2.77}$$

where a_α is a unit complex covector. Then, from Eq. (2.75) we obtain

$$\frac{1}{2} k_\mu k^\mu - \frac{i\epsilon}{2} k^\mu \left(a_0{}^{*\alpha} \nabla_\mu a_{0\alpha} - a_{0\alpha} \nabla_\mu a_0{}^{*\alpha}\right) = \mathcal{O}(\epsilon^2). \tag{2.78}$$

This can be viewed as an effective dispersion relation, containing $\mathscr{O}(\epsilon)$ corrections to the geometrical optics equation (2.38). Finally, let us introduce

$$K_\mu = k_\mu - \frac{i\epsilon}{2}\left({a_0}^{*\alpha}\nabla_\mu a_{0\alpha} - a_{0\alpha}\nabla_\mu {a_0}^{*\alpha}\right) \tag{2.79}$$

and rewrite the effective dispersion relation as

$$\frac{1}{2}K_\mu K^\mu = \mathscr{O}(\epsilon^2). \tag{2.80}$$

It is worth noting that this equation can also be obtained directly from the effective field action (2.26), specifically by varying the latter with respect to $\mathscr{I}$.

Effective transport equation

Using Eqs. (2.27), (2.38), and (2.39), the effective transport equation (2.30) becomes

$$\begin{aligned}\nabla_\mu\Bigg[&k^\mu\left({A_0}^{*\alpha}A_{0\alpha} + \epsilon {A_0}^{*\alpha}A_{1\alpha} + \epsilon {A_1}^{*\alpha}A_{0\alpha}\right) - \frac{i\epsilon}{2}g^{\mu\nu}\left({A_0}^{*\alpha}\nabla_\nu A_{0\alpha} - A_{0\alpha}\nabla_\nu {A_0}^{*\alpha}\right)\\ &+\frac{i\epsilon}{4}\left({A_0}^{*\alpha}\nabla_\alpha {A_0}^\mu - {A_0}^\mu\nabla_\alpha {A_0}^{*\alpha}\right) + \frac{i\epsilon}{4}\left({A_0}^{*\mu}\nabla_\alpha {A_0}^\alpha - {A_0}^\alpha\nabla_\alpha {A_0}^{*\mu}\right)\\ &-\frac{\epsilon}{2}k_\alpha\left({A_0}^{*\mu}{A_1}^\alpha + {A_1}^{*\alpha}{A_0}^\mu\right)\Bigg] = \mathscr{O}(\epsilon^2).\end{aligned} \tag{2.81}$$

We can perform the following replacements in the above equation:

$$\begin{aligned}{A_0}^{*\alpha}\nabla_\alpha {A_0}^\mu &= \nabla_\alpha\left({A_0}^{*\alpha}{A_0}^\mu\right) - \nabla_\alpha {A_0}^{*\alpha}{A_0}^\mu,\\ \nabla_\alpha {A_0}^{*\mu}{A_0}^\alpha &= \nabla_\alpha\left({A_0}^{*\mu}{A_0}^\alpha\right) - {A_0}^{*\mu}\nabla_\alpha {A_0}^\alpha.\end{aligned} \tag{2.82}$$

After rearranging terms, the effective transport equation becomes

$$\begin{aligned}\nabla_\mu\Bigg[&k^\mu\left({A_0}^{*\alpha}A_{0\alpha} + \epsilon {A_0}^{*\alpha}A_{1\alpha} + \epsilon {A_1}^{*\alpha}A_{0\alpha}\right) - \frac{i\epsilon}{2}g^{\mu\nu}\left({A_0}^{*\alpha}\nabla_\nu A_{0\alpha} - A_{0\alpha}\nabla_\nu {A_0}^{*\alpha}\right)\\ &-\frac{i\epsilon}{2}{A_0}^\mu\left(\nabla_\alpha {A_0}^{*\alpha} - ik_\alpha {A_1}^{*\alpha}\right) + \frac{i\epsilon}{2}{A_0}^{*\mu}\left(\nabla_\alpha {A_0}^\alpha + ik_\alpha {A_1}^\alpha\right)\\ &+\frac{i\epsilon}{4}\nabla_\alpha\left({A_0}^{*[\alpha}{A_0}^{\mu]}\right)\Bigg] = \mathscr{O}(\epsilon^2).\end{aligned} \tag{2.83}$$

The last term above vanishes due to the symmetry of the Ricci tensor:

$$\begin{aligned}\nabla_\mu \nabla_\alpha \left({A_0}^{*[\alpha} {A_0}^{\mu]} \right) &= \nabla_{[\mu} \nabla_{\alpha]} \left({A_0}^{*\alpha} {A_0}^{\mu} \right) \\ &= \left({R_{\alpha\nu\mu}}^{\nu} - {R_{\mu\nu\alpha}}^{\nu} \right) {A_0}^{*\alpha} {A_0}^{\mu} \\ &= \left(R_{\alpha\mu} - R_{\mu\alpha} \right) {A_0}^{*\alpha} {A_0}^{\mu} \\ &= 0.\end{aligned} \tag{2.84}$$

Furthermore, after using the Lorenz gauge condition (2.23), we are left with the following form of the effective transport equation:

$$\begin{aligned}\nabla_\mu \Bigg[& k^\mu ({A_0}^{*\alpha} A_{0\alpha} + \epsilon {A_0}^{*\alpha} A_{1\alpha} + \epsilon {A_1}^{*\alpha} A_{0\alpha}) \\ & - \frac{i\epsilon}{2} g^{\mu\nu} \left({A_0}^{*\alpha} \nabla_\nu A_{0\alpha} - A_{0\alpha} \nabla_\nu {A_0}^{*\alpha} \right) \Bigg] = \mathscr{O}(\epsilon^2).\end{aligned} \tag{2.85}$$

Introducing the intensity $\mathscr{I}$ and the vector K^μ, we obtain

$$\nabla_\mu \left\{ \mathscr{I} \left[k^\mu - \frac{i\epsilon}{2} g^{\mu\nu} \left({a_0}^{*\alpha} \nabla_\nu a_{0\alpha} - a_{0\alpha} \nabla_\nu {a_0}^{*\alpha} \right) \right] \right\} = \nabla_\mu \left(\mathscr{I} K^\mu \right) = \mathscr{O}(\epsilon^2). \tag{2.86}$$

This is an effective transport equation for the intensity $\mathscr{I}$, which includes $\mathscr{O}(\epsilon)$ corrections to the geometrical optics Eq. (2.40). As discussed in Ref. [166], the direction of K^μ coincides with the direction of the wave action flux.

2.4. Effective ray equations

2.4.1. Hamilton-Jacobi system at the leading order

The lowest-order geometrical optics equations (2.38) and (2.40) can be viewed as a system of coupled partial differential equations:

$$\frac{1}{2} g^{\mu\nu} k_\mu k_\nu = 0, \tag{2.87}$$

$$\nabla_\mu \left(\mathscr{I}_0 k^\mu \right) = 0, \tag{2.88}$$

where $k_\mu = \nabla_\mu S$. Equation (2.87) is a Hamilton-Jacobi equation for the phase function S, and Eq. (2.88) is a transport equation for the intensity $\mathscr{I}_0$ [125]. The

Hamilton-Jacobi equation can be solved using the method of characteristics. This is done by defining a Hamiltonian function on T^*M, such that

$$H(x, \nabla S) = \frac{1}{2} g^{\mu\nu} k_\mu k_\nu = 0. \tag{2.89}$$

It is obvious that in this case, the Hamiltonian is

$$H(x, p) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \tag{2.90}$$

Note that in contrast to the dispersion relation (2.89), the Hamiltonian (2.90) is a function on the whole phase space T^*M, with p_μ being an arbitrary covector. Hamilton's equations take the following form:

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = g^{\mu\nu} p_\nu, \tag{2.91}$$

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2} \partial_\mu g^{\alpha\beta} p_\alpha p_\beta. \tag{2.92}$$

Given a solution $\{x^\mu(\tau), p_\mu(\tau)\}$ for Hamilton's equations, we obtain a solution of the Hamilton-Jacobi Eq. (2.89) by taking [90, p. 433]:

$$S(x^\mu(\tau_1), p_\mu(\tau_1)) = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \left[\dot{x}^\mu p_\mu - H(x, p)\right] + \text{const}. \tag{2.93}$$

Note that the above equation represents an action, with the corresponding Lagrangian related to the Hamiltonian (2.90) by a Legendre transformation [1, Ex. 3.6.10]. The Euler-Lagrange equation is equivalent to the geodesic equation [1, Th. 3.7.1] and with Hamilton's equations (2.91) and (2.92). Once the Hamilton-Jacobi equation is solved, the transport Eq. (2.88) can also be solved, at least in principle [125]. However, our main interest is in the ray equations governed by the Hamiltonian (2.90). The corresponding Hamilton's equations (2.91) and (2.92) describe null geodesics. These equations can easily be rewritten as

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0, \tag{2.94}$$

or in the explicitly covariant form:

$$p^\nu \nabla_\nu p^\mu = \dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0. \tag{2.95}$$

2.4.2. Effective Hamilton-Jacobi system

The effective dispersion relation (2.80), together with the effective transport equation (2.86) introduce $\mathcal{O}(\epsilon^1)$ corrections over the system discussed above:

$$\frac{1}{2} g^{\mu\nu} k_\mu k_\nu - \frac{i\epsilon}{2} k^\mu \left({a_0}^{*\alpha} \nabla_\mu a_{0\alpha} - a_{0\alpha} \nabla_\mu {a_0}^{*\alpha}\right) = \mathcal{O}(\epsilon^2), \tag{2.96}$$

$$\nabla_\mu \left\{ \mathscr{I} \left[k^\mu - \frac{i\epsilon}{2} g^{\mu\nu} \left({a_0}^{*\alpha} \nabla_\nu a_{0\alpha} - a_{0\alpha} \nabla_\nu {a_0}^{*\alpha} \right) \right] \right\} = \mathscr{O}(\epsilon^2). \tag{2.97}$$

Using Eq. (2.54), the effective dispersion relation becomes

$$\frac{1}{2} g^{\mu\nu} k_\mu k_\nu - \frac{i\epsilon}{2} k^\mu \left(z^\dagger \partial_\mu z - \partial_\mu z^\dagger z \right) - \epsilon s k^\mu B_\mu = \mathscr{O}(\epsilon^2), \tag{2.98}$$

where $B_\mu = B_\mu(x, k)$ is the Berry connection introduced in Eq. (2.65), and $s = \pm 1$, depending on the initial polarization. Using Eq. (2.67), together with the assumption on the initial polarization, we can write:

$$-\frac{i\epsilon}{2} k^\mu \left(z^\dagger \partial_\mu z - \partial_\mu z^\dagger z \right) = \epsilon s k^\mu \partial_\mu \gamma. \tag{2.99}$$

Since the value of s is fixed by the initial conditions, the only unknowns are the phase function S and the Berry phase γ. We can write an effective Hamilton-Jacobi equation for the total phase $\tilde{S} = S + \epsilon s \gamma$:

$$\begin{aligned} H\left(x, \nabla \tilde{S}\right) &= \frac{1}{2} g^{\mu\nu} k_\mu k_\nu + \epsilon s k^\mu \partial_\mu \gamma - \epsilon s k^\mu B_\mu + \mathscr{O}(\epsilon^2) \\ &= \frac{1}{2} g^{\mu\nu} \nabla_\mu \tilde{S} \nabla_\nu \tilde{S} - \epsilon s g^{\mu\nu} B_\mu \nabla_\nu \tilde{S} + \mathscr{O}(\epsilon^2). \end{aligned} \tag{2.100}$$

The phase $\tilde{S}$ represents the overall phase factor, up to order $\mathscr{O}(\epsilon^2)$, of a circularly polarized WKB solution, $\mathscr{A}_\alpha = \mathrm{Re}(\sqrt{\mathscr{I}} m_\alpha e^{i\gamma} e^{iS/\epsilon})$ or $\mathscr{A}_\alpha = \mathrm{Re}(\sqrt{\mathscr{I}} \bar{m}_\alpha e^{-i\gamma} e^{iS/\epsilon})$, depending on the state of circular polarization. As discussed in Ref. [33], the Berry phase γ, which comes as a correction to the overall phase of the WKB solution, is responsible for the spin Hall effect of light. The corresponding Hamiltonian function on T^*M is

$$H(x, p) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu - \epsilon s g^{\mu\nu} p_\mu B_\nu(x, p), \tag{2.101}$$

and we have the following Hamilton's equations:

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = g^{\mu\nu} p_\nu - \epsilon s \left(B^\mu + p^\alpha \overset{v}{\nabla}{}^\mu B_\alpha \right), \tag{2.102}$$

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2} \partial_\mu g^{\alpha\beta} p_\alpha p_\beta + \epsilon s p_\alpha \left(\partial_\mu g^{\alpha\beta} B_\beta + g^{\alpha\beta} \partial_\mu B_\beta \right). \tag{2.103}$$

These equations contain polarization-dependent corrections to the null geodesic Eqs. (2.91) and (2.92), representing the gravitational spin Hall effect of light. For $\epsilon = 0$, one recovers the standard geodesic equation in canonical coordinates.

We can also write these ray equations in a more compact form

$$\begin{pmatrix} \dot{x}^\mu \\ \dot{p}_\mu \end{pmatrix} = \begin{pmatrix} 0 & \delta^\mu_\nu \\ -\delta^\nu_\mu & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x^\nu} \\ \frac{\partial H}{\partial p_\nu} \end{pmatrix}, \tag{2.104}$$

where the constant matrix on the right-hand side is the inverse of the symplectic 2-form, or the Poisson tensor [121].

Noncanonical coordinates

The Hamiltonian (2.101) contains the Berry connection B_μ, which is gauge dependent. The latter means that B_μ depends on the choice of m_α and $\bar{m}_\alpha$; for example, the transformation $m_\alpha \mapsto m_\alpha e^{i\phi}$ causes the following transformation of the Berry connection:

$$B_\mu \mapsto B_\mu - \nabla_\mu \phi. \tag{2.105}$$

This kind of gauge dependence was considered by Littlejohn and Flynn in Ref. [117], where they also proposed how to make the Hamiltonian and the equations of motion gauge invariant. The main idea is to introduce noncanonical coordinates such that the Berry connection is removed from the Hamiltonian and the symplectic form acquires the corresponding Berry curvature, which is gauge invariant. This is similar to the description of a charged particle in an electromagnetic field in terms of either the canonical or the kinetic momentum of the particle. The Berry connection and Berry curvature play a similar role as the electromagnetic vector potential and the electromagnetic tensor [51].

We start by rewriting the Hamiltonian (2.101) as

$$H(x,p) = H_0(x,p) - \epsilon s g^{\mu\nu} p_\mu B_\nu(x,p), \tag{2.106}$$

where $H_0 = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$. Following Ref. [117], the Berry connection can be written in the following way, by using the definition of the horizontal derivative:

$$\begin{aligned} p^\mu B_\mu(x,p) &= i p^\mu \bar{m}^\alpha \overset{h}{\nabla}_\mu m_\alpha \\ &= i p^\mu \bar{m}^\alpha \nabla_\mu m_\alpha + i p^\mu p_\sigma \Gamma^\sigma_{\mu\rho} \bar{m}^\alpha \overset{v}{\nabla}{}^\rho m_\alpha \\ &= i \frac{\partial H_0}{\partial p_\mu} \bar{m}^\alpha \nabla_\mu m_\alpha - i \frac{\partial H_0}{\partial x^\mu} \bar{m}^\alpha \overset{v}{\nabla}{}^\rho m_\alpha. \end{aligned} \tag{2.107}$$

The Berry connection can be eliminated formally from the Hamiltonian (2.101) by considering the following substitution on T^*M:

$$X^\mu = x^\mu + i\epsilon s \bar{m}^\alpha \overset{v}{\nabla}{}^\mu m_\alpha, \tag{2.108}$$

$$P_\mu = p_\mu - i\epsilon s \bar{m}^\alpha \nabla_\mu m_\alpha. \tag{2.109}$$

It is possible to obtain this substitution as the linearization of a change of coordinates. For more details, see Appendix A.4.

Since the symplectic form transforms nontrivially under this substitution, (X, P) are noncanonical coordinates. The Hamiltonian (2.101) is a scalar, so we obtain

$$\begin{aligned} H'(X,P) &= H(x,p) \\ &= H\left(X^\mu - i\epsilon s \bar{m}^\alpha \overset{v}{\nabla}{}^\mu m_\alpha, P_\mu + i\epsilon s \bar{m}^\alpha \nabla_\mu m_\alpha\right) \\ &= H(X,P) - i\epsilon s \frac{\partial H_0}{\partial x^\mu} \bar{m}^\alpha \overset{v}{\nabla}{}^\mu m_\alpha + i\epsilon s \frac{\partial H_0}{\partial p_\mu} \bar{m}^\alpha \nabla_\mu m_\alpha \\ &= H_0(X,P). \end{aligned} \tag{2.110}$$

In the new coordinate system (X, P), we obtain the following Hamiltonian:

$$H'(X,P) = \frac{1}{2} g^{\mu\nu}(X) P_\mu P_\nu. \tag{2.111}$$

The corresponding Hamilton's equations can be written in a matrix form as

$$\begin{pmatrix} \dot{X}^\mu \\ \dot{P}_\mu \end{pmatrix} = T' \begin{pmatrix} \frac{\partial H'}{\partial X^\nu} \\ \frac{\partial H'}{\partial P_\nu} \end{pmatrix}, \tag{2.112}$$

where T' is the Poisson tensor in the new variables. Following Marsden and Ratiu [121, p. 343], we obtain

$$T' = \begin{pmatrix} \epsilon s \left(F_{pp}\right)^{\nu\mu} & \delta^\mu_\nu + \epsilon s \left(F_{xp}\right)_\nu{}^\mu \\ -\delta^\nu_\mu - \epsilon s \left(F_{xp}\right)^\nu{}_\mu & -\epsilon s \left(F_{xx}\right)_{\nu\mu} \end{pmatrix}, \tag{2.113}$$

where we have the following Berry curvature terms:

$$\begin{aligned} \left(F_{pp}\right)^{\nu\mu} &= i\Big(\overset{v}{\nabla}{}^\mu \bar{m}^\alpha \overset{v}{\nabla}{}^\nu m_\alpha - \overset{v}{\nabla}{}^\nu \bar{m}^\alpha \overset{v}{\nabla}{}^\mu m_\alpha + \bar{m}^\alpha \overset{v}{\nabla}{}^{[\mu} \overset{v}{\nabla}{}^{\nu]} m_\alpha - m_\alpha \overset{v}{\nabla}{}^{[\mu} \overset{v}{\nabla}{}^{\nu]} \bar{m}^\alpha\Big), \\ \left(F_{xx}\right)_{\nu\mu} &= i\Big(\nabla_\mu \bar{m}^\alpha \nabla_\nu m_\alpha - \nabla_\nu \bar{m}^\alpha \nabla_\mu m_\alpha + \bar{m}^\alpha \nabla_{[\mu} \nabla_{\nu]} m_\alpha - m_\alpha \nabla_{[\mu} \nabla_{\nu]} \bar{m}^\alpha\Big), \\ \left(F_{px}\right)_\nu{}^\mu &= -\left(F_{xp}\right)^\mu{}_\nu = i\left(\overset{v}{\nabla}{}^\mu \bar{m}^\alpha \nabla_\nu m_\alpha - \nabla_\nu \bar{m}^\alpha \overset{v}{\nabla}{}^\mu m_\alpha\right). \end{aligned} \tag{2.114}$$

The Poisson tensor in noncanonical coordinates T' automatically satisfies the Jacobi identity, since it is a covariant quantity obtained from the Poisson tensor in canonical coordinates T through a change of variables on the cotangent bundle.

Simplified expressions for the Berry curvature terms can be found in Appendix A.3. Now we can write Hamilton's equations in the new variables:

$$\dot{X}^\mu = P^\mu + \epsilon s P^\nu \left(F_{px}\right)_\nu{}^\mu + \epsilon s \Gamma^\alpha_{\beta\nu} P_\alpha P^\beta \left(F_{pp}\right)^{\nu\mu}, \tag{2.115}$$
$$\dot{P}_\mu = \Gamma^\alpha_{\beta\mu} P_\alpha P^\beta - \epsilon s P^\nu \left(F_{xx}\right)_{\nu\mu} - \epsilon s \Gamma^\alpha_{\beta\nu} P_\alpha P^\beta \left(F_{xp}\right)^\nu{}_\mu . \tag{2.116}$$

The last term on the right-hand side of Eq. (2.115) is the covariant analogue of the spin Hall effect correction obtained in optics, $\left(\dot{\mathbf{p}} \times \mathbf{p}\right)/|\mathbf{p}|^3$, due to the Berry curvature in momentum space [37, 148]. This term is also the source of the gravitational spin Hall effect in the work of Gosselin *et al.* [95]. In Eq. (2.116), the second term on the right-hand side contains the Riemann tensor and resembles the curvature term obtained in the Mathisson-Papapetrou-Dixon equations [59].

Given a null covector P_μ, the class of Lorentz transformations leaving P_μ invariant define the little group, which is isomorphic to SE(2), the symmetry group of the two-dimensional Euclidean plane [163]. In terms of a null tetrad $\{P, n, m, \bar{m}\}$, the action of the little group can be split into the following types of transformations [47, p. 53]:

$$\begin{aligned} \text{Type 1:} \quad & P \mapsto P, \quad n \mapsto n, \\ & m \mapsto m e^{i\phi}, \quad m \mapsto \bar{m} e^{-i\phi}, \\ \text{Type 2:} \quad & P \mapsto P, \quad n \mapsto n + \bar{a} m + a \bar{m} + a\bar{a} P, \\ & m \mapsto m + aP, \quad m \mapsto \bar{m} + \bar{a} P, \end{aligned} \tag{2.117}$$

where ϕ is a real scalar function and a is a complex scalar function. The transformations of Type 1 are the spin rotations mentioned in Sec. 2.3.4, while the transformations of Type 2 can be considered as a change of observer t_μ, based on Eq. (2.56). It can easily be checked that the Berry curvature terms in Eq. (2.114) are invariant under Type 1 transformations. However, the Berry curvature terms are not invariant under Type 2 transformations. As a consequence, the ray equations (2.115) and (2.116) depend on the choice of observer. It is shown in the following section how this observer dependence is related to the problem of localizing massless spinning particles [38, 163].

2.4.3. Lagrangian formulation of the ray equations

The ray equations obtained in Eqs. (2.102) and (2.103) or in Eqs. (2.115) and (2.116) can also be obtained from a Lagrangian formulation. The Lagrangian formulation can be helpful for deriving conserved quantities when considering certain spacetimes.

In the case of the ray equations (2.102) and (2.103), the action and the Lagrangian can be written as

$$S(x^\mu, \dot{x}^\mu, p_\mu, \dot{p}_\mu) = \int d\tau \, L(x^\mu, \dot{x}^\mu, p_\mu, \dot{p}_\mu), \tag{2.118}$$

where the Lagrangian L is the Legendre transformation of the Hamiltonian given in Eq. (2.101):

$$L(x^\mu, \dot{x}^\mu, p_\mu, \dot{p}_\mu) = \dot{x}^\mu p_\mu - \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + \epsilon s g^{\mu\nu} p_\mu B_\nu(x, p). \tag{2.119}$$

Note that in this case, the Lagrangian is defined as a function on TT^*M, and the corresponding Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} = 0, \tag{2.120}$$

$$\frac{\partial L}{\partial p_\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{p}_\mu} = 0, \tag{2.121}$$

give the ray equations in Eqs. (2.102) and (2.103). Furthermore, the Lagrangian in Eq. (2.119) can be extended to describe the dynamics of the Jones vector z, as in Eq. (2.66). We can write

$$L = \dot{x}^\mu p_\mu + \frac{i\epsilon}{2}(z^\dagger \dot{z} - \dot{z}^\dagger z) - \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + \epsilon s g^{\mu\nu} p_\mu B_\nu(x, p) + \lambda(z^\dagger z - 1), \tag{2.122}$$

where λ is a Lagrange multiplier, used for constraining the Jones vector to satisfy $z^\dagger z = 1$. In this case, the Lagrangian is a scalar function defined on $TT^*M \times T\mathbb{C}^2$, and the corresponding Euler-Lagrange equations are

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} = 0, \tag{2.123}$$

$$\frac{\partial L}{\partial p_\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{p}_\mu} = 0, \tag{2.124}$$

$$\frac{\partial L}{\partial z^\dagger} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{z}^\dagger} = 0, \tag{2.125}$$

$$\frac{\partial L}{\partial z} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{z}} = 0, \tag{2.126}$$

$$z^\dagger z = 1. \tag{2.127}$$

Here, Eqs. (2.123) and (2.124) are the same as the ray equations (2.102) and (2.103). Keeping in mind that $s = z^\dagger \sigma_3 z$, Eqs. (2.125) and (2.126) are the same as Eq. (2.66) and its complex conjugate.

A Lagrangian formulation of the ray equations (2.115) and (2.116) in noncanonical coordinates can be obtained in the following way. We start with the same Lagrangian as in Eq. (2.119), but now we use capital letters X and P for the coordinates on TT^*M (this is just for notation consistency; we are not doing a coordinate transformation). At the lowest order in ϵ, the Lagrangian is

$$L = \dot{X}^\mu P_\mu - \frac{1}{2} g^{\mu\nu} P_\mu P_\nu, \tag{2.128}$$

and the corresponding Euler-Lagrange equations are

$$\dot{X}^\mu = P^\mu, \tag{2.129}$$

$$\dot{P}_\mu = \Gamma^\alpha_{\beta\mu} P_\alpha P^\beta. \tag{2.130}$$

Keeping terms of order ϵ^1, the Lagrangian is

$$L = \dot{X}^\mu P_\mu - \frac{1}{2} g^{\mu\nu} P_\mu P_\nu + \epsilon s g^{\mu\nu} P_\mu B_\nu(X, P). \tag{2.131}$$

Using the definition of the Berry connection B_μ and the definition of the horizontal derivative, we expand the order ϵ^1 term in the above Lagrangian:

$$\begin{aligned} g^{\mu\nu} P_\mu B_\nu &= i P^\mu \bar{m}^\alpha \overset{h}{\nabla}_\mu m_\alpha \\ &= i P^\mu \bar{m}^\alpha \left(\partial_\mu m_\alpha - \Gamma^\beta_{\mu\alpha} m_\beta \right) + i P^\mu \bar{m}^\alpha \Gamma^\sigma_{\mu\beta} P_\sigma \overset{v}{\nabla}^\rho m_\alpha \\ &= i P^\mu \bar{m}^\alpha \nabla_\mu m_\alpha + i P^\mu \Gamma^\sigma_{\mu\beta} P_\sigma \bar{m}^\alpha \overset{v}{\nabla}^\rho m_\alpha \end{aligned} \tag{2.132}$$

Using the lowest-order ray equations (2.129) and (2.130), we can rewrite the above equation as

$$\begin{aligned} g^{\mu\nu} P_\mu B_\nu &= i P^\mu \bar{m}^\alpha \nabla_\mu m_\alpha + i P^\mu \Gamma^\sigma_{\mu\beta} P_\sigma \bar{m}^\alpha \overset{v}{\nabla}^\rho m_\alpha \\ &= \dot{X}^\mu i \bar{m}^\alpha \nabla_\mu m_\alpha + \dot{P}_\mu i \bar{m}^\alpha \overset{v}{\nabla}^\mu m_\alpha \\ &= \dot{X}^\mu (B_x)_\mu + \dot{P}_\mu (B_p)^\mu, \end{aligned} \tag{2.133}$$

where we introduced the notation $(B_x)_\mu = i \bar{m}^\alpha \nabla_\mu m_\alpha$ and $(B_p)^\mu = i \bar{m}^\alpha \overset{v}{\nabla}^\mu m_\alpha$. The Lagrangian can now be written as

$$L = \dot{X}^\mu \left(P_\mu + \epsilon s (B_x)_\mu \right) - \frac{1}{2} g^{\mu\nu} P_\mu P_\nu + \epsilon s \dot{P}_\mu (B_p)^\mu, \tag{2.134}$$

and the corresponding Euler-Lagrange equations

$$\frac{\partial L}{\partial X^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{X}^\mu} = 0, \tag{2.135}$$

$$\frac{\partial L}{\partial P_\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{P}_\mu} = 0, \tag{2.136}$$

are the same as the ray equations (2.115) and (2.116). This Lagrangian can also be extended to include the dynamics of the Jones vector z, exactly as in the previous case.

2.4.4. Comparison with Mathisson-Papapetrou-Dixon equations

In this section, we show how the effective ray equations in noncanonical coordinates can be written in a similar form as the Mathisson-Papapetrou-Dixon equations. We start with the effective ray equations (2.115) and (2.116), together with the evolution equations for z and $\bar{z}$:

$$\begin{aligned}
\dot{X}^\mu &= P^\mu + \epsilon s P^\nu \left(F_{px}\right)_\nu{}^\mu + \epsilon s \Gamma^\alpha_{\beta\nu} P_\alpha P^\beta \left(F_{pp}\right)^{\nu\mu}, && (2.137)\\
\dot{P}_\mu &= \Gamma^\alpha_{\beta\mu} P_\alpha P^\beta - \epsilon s P^\nu \left(F_{xx}\right)_{\nu\mu} - \epsilon s \Gamma^\alpha_{\beta\nu} P_\alpha P^\beta \left(F_{xp}\right)^\nu{}_\mu, && (2.138)\\
\dot{z} &= i P^\mu B_\mu \sigma_3 z, && (2.139)\\
\dot{\bar{z}} &= -i P^\mu B_\mu \bar{z} \sigma_3. && (2.140)
\end{aligned}$$

where $s = \bar{z}\sigma_3 z = \pm 1$ (last equality is based on the choice of initial conditions). Keeping in mind the fact that P_α is a coordinate on phase space, we have $\nabla_\mu P_\alpha = \frac{\partial}{\partial X^\mu} P_\alpha - \Gamma^\rho_{\alpha\mu} P_\rho$. Then, based on the results in Appendix A.3, the components of the Berry curvature can be rewritten as

$$\begin{aligned}
\left(F_{pp}\right)^{\nu\mu} &= \frac{2i}{(t_\alpha P^\alpha)^2} \bar{m}^{[\mu} m^{\nu]},\\
\left(F_{xx}\right)_{\nu\mu} &= i R_{\alpha\beta\mu\nu} \bar{m}^\alpha m^\beta + \frac{2i}{(t_\sigma P^\sigma)^2} \bar{m}^{[\alpha} m^{\beta]} \Big[P_\rho P_\sigma \Gamma^\rho_{\mu\alpha} \Gamma^\sigma_{\nu\beta}\\
&\qquad - t_\sigma P^\sigma (P^\rho \Gamma^\rho_{\mu\alpha} \nabla_\nu t_\beta + P_\sigma \Gamma^\sigma_{\nu\beta} \nabla_\mu t_\alpha) \Big],\\
\left(F_{px}\right)_\nu{}^\mu = - \left(F_{xp}\right)^\mu{}_\nu &= \frac{2i}{(t_\alpha P^\alpha)^2} \bar{m}^{[\mu} m^{\beta]} \left(-P_\rho \Gamma^\rho_{\nu\beta} + t_\alpha P^\alpha \nabla_\nu t_\beta \right),
\end{aligned} \tag{2.141}$$

where t^α represents the 4-velocity of a family of timelike observers. Inserting these expressions into the ray equations, we obtain

$$\begin{aligned}
\dot{X}^\mu &= P^\mu + \frac{2i\epsilon s}{t_\alpha P^\alpha} \bar{m}^{[\mu} m^{\beta]} P^\nu \nabla_\nu t_\beta && (2.142)\\
\dot{P}_\mu &= \Gamma^\alpha_{\beta\mu} P_\alpha P^\beta - \epsilon s P^\nu i R_{\alpha\beta\mu\nu} \bar{m}^\alpha m^\beta + \frac{2i\epsilon s}{t_\sigma P^\sigma} \bar{m}^{[\alpha} m^{\beta]} P_\rho \Gamma^\rho_{\mu\alpha} P^\nu \nabla_\nu t_\beta, && (2.143)
\end{aligned}$$

$$\dot{z} = iP^\mu B_\mu \sigma_3 z, \tag{2.144}$$
$$\dot{\bar{z}} = -iP^\mu B_\mu \bar{z}\sigma_3. \tag{2.145}$$

In order to compare these ray equations with the MPD equations, we define the spin tensor as

$$S^{\alpha\beta} = i\epsilon\bar{z}\sigma_3 z(\bar{m}^\alpha m^\beta - \bar{m}^\beta m^\alpha) = 2i\epsilon s\bar{m}^{[\alpha}m^{\beta]}. \tag{2.146}$$

We take the derivative of $S^{\alpha\beta}$ (note that this is not the covariant dot used in the MPD equations):

$$\dot{S}^{\alpha\beta} = 2i\epsilon\dot{s}\bar{m}^{[\alpha}m^{\beta]} + 2i\epsilon s\frac{d}{d\tau}\bar{m}^{[\alpha}m^{\beta]}. \tag{2.147}$$

Using Eqs (2.144) and (2.145), it is straightforward to show that $\dot{s} = 0$. Keeping in mind that m^α and $\bar{m}^\alpha$ are functions of X^μ and P_μ, we obtain

$$\begin{aligned}\dot{S}^{\alpha\beta} &= 2i\epsilon s\frac{d}{d\tau}\bar{m}^{[\alpha}m^{\beta]}\\ &= 2i\epsilon s\left[\dot{X}^\mu\frac{\partial}{\partial x^\mu}(\bar{m}^{[\alpha}m^{\beta]}) + \dot{P}_\mu\frac{\partial}{\partial p_\mu}(\bar{m}^{[\alpha}m^{\beta]})\right]\\ &= 2i\epsilon s\left[\dot{X}^\mu\nabla_\mu(\bar{m}^{[\alpha}m^{\beta]}) - \dot{X}^\mu\Gamma^\alpha_{\mu\rho}\bar{m}^{[\rho}m^{\beta]} - \dot{X}^\mu\Gamma^\beta_{\mu\rho}\bar{m}^{[\alpha}m^{\rho]} + \dot{P}_\mu\overset{v}{\nabla}{}^\mu(\bar{m}^{[\alpha}m^{\beta]})\right].\end{aligned} \tag{2.148}$$

We can now introduce the covariant dot used in the MPD equations as

$$\frac{D}{D\tau}S^{\alpha\beta} = \dot{X}^\mu\nabla_\mu S^{\alpha\beta} = \dot{S}^{\alpha\beta} + \dot{X}^\mu\Gamma^\alpha_{\mu\rho}S^{\rho\beta} + \dot{X}^\mu\Gamma^\beta_{\mu\rho}S^{\alpha\rho}, \tag{2.149}$$

and we obtain

$$\frac{D}{D\tau}S^{\alpha\beta} = 2i\epsilon s\dot{X}^\mu\nabla_\mu(\bar{m}^{[\alpha}m^{\beta]}) + 2i\epsilon s\dot{P}_\mu\overset{v}{\nabla}{}^\mu(\bar{m}^{[\alpha}m^{\beta]}). \tag{2.150}$$

Using the expansion of the vertical and covariant derivatives of m^α and $\bar{m}^\alpha$ introduced in Appendix A.3, we obtain:

$$\frac{D}{D\tau}S^{\alpha\beta} = \frac{2i\epsilon s}{t_\sigma P^\sigma}\left(P^\mu\nabla_\mu t_\rho\right)\left(P^\alpha\bar{m}^{[\beta}m^{\rho]} - P^\beta\bar{m}^{[\alpha}m^{\rho]}\right). \tag{2.151}$$

Furthermore, using Eq. (2.142), the equation above can be rewritten in the following form:

$$\frac{D}{D\tau}S^{\alpha\beta} = P^\alpha\dot{X}^\beta - P^\beta\dot{X}^\alpha \tag{2.152}$$

We can now rewrite the ray equations in a similar form as the MPD equations:

$$\frac{D}{D\tau}X^\mu = P^\mu + \frac{1}{t_\alpha P^\alpha} S^{\mu\beta} P^\nu \nabla_\nu t_\beta \tag{2.153}$$

$$\frac{D}{D\tau}P_\mu = -\frac{1}{2} P^\nu R_{\alpha\beta\mu\nu} S^{\alpha\beta}, \tag{2.154}$$

$$\frac{D}{D\tau}S^{\alpha\beta} = P^\alpha \dot{X}^\beta - P^\beta \dot{X}^\alpha, \tag{2.155}$$

where

$$\frac{D}{D\tau}X^\mu = \dot{X}^\mu, \qquad \frac{D}{D\tau}P_\mu = \dot{P}_\mu - \Gamma^\alpha_{\beta\mu} P_\alpha \dot{X}^\beta. \tag{2.156}$$

This form of the effective ray equations resembles what one obtains by starting with the MPD equations

$$\frac{D}{D\tau}P_\mu = -\frac{1}{2} P^\nu R_{\alpha\beta\mu\nu} S^{\alpha\beta}, \tag{2.157}$$

$$\frac{D}{D\tau}S^{\alpha\beta} = P^\alpha \dot{X}^\beta - P^\beta \dot{X}^\alpha, \tag{2.158}$$

and fixes the evolution equation for the worldline X^μ by imposing the spin supplementary condition

$$S^{\alpha\beta} t_\beta = 0. \tag{2.159}$$

2.5. Examples and applications

In this section, we study the modified ray equations describing the gravitational spin Hall effect of light on several background spacetimes. The examples including the relativistic Hall effect and Wigner translations, the optical metric, and the cosmological spacetimes are treated analytically. The examples exploring the ray equations on a Schwarzschild and Kerr background, as well as the section on black hole shadows are treated numerically, based on the Mathematica code presented in Appendix A.7.

When working with the modified ray equations, in either the canonical form given in Eqs. (2.102) and (2.103) or the noncanonical form given in Eqs. (2.115) and (2.116), one needs to specify the background metric $g_{\mu\nu}$, and the choice of polarization vectors m^α and $\bar{m}^\alpha$. The polarization vectors are needed to compute the Berry connection and the Berry curvature. A particular choice of polarization vectors can easily be constructed by introducing an orthonormal tetrad $(e_a)^\mu$, with $(e_0)^\mu = t^\mu$ representing our choice of a family of timelike observers. Adapting the polarization vectors used in optics [148], we can write $p^\mu = P^a(e_a)^\mu$, $v^\mu = V^a(e_a)^\mu$, and $w^\mu = W^a(e_a)^\mu$, where the components of these vectors are given by

$$P^a = \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}, \qquad V^a = \frac{1}{P_p}\begin{pmatrix} 0 \\ -P^2 \\ P^1 \\ 0 \end{pmatrix}, \qquad W^a = \frac{1}{P_p P_s}\begin{pmatrix} 0 \\ P^1P^3 \\ P^2P^3 \\ -(P_p)^2 \end{pmatrix}, \tag{2.160}$$

where

$$\begin{aligned} P_p &= \sqrt{(P^1)^2 + (P^2)^2}, \\ P_s &= \sqrt{(P^1)^2 + (P^2)^2 + (P^3)^2}. \end{aligned} \tag{2.161}$$

The vectors v^μ and w^μ are real unit spacelike vectors that represent a linear polarization basis satisfying Eq. (A.19). They are related to the circular polarization vectors m^α and $\bar{m}^\alpha$ by Eq. (A.18). Using this particular choice of polarization vectors, the Berry connection and the Berry curvature terms can be computed, and the modified ray equations can be integrated, either analytically or numerically.

2.5.1. Relativistic Hall effect and Wigner translations

The relativistic Hall effect [38] is a special relativistic effect that occurs when Lorentz transformations are applied to objects carrying angular momentum. In particular, consider a localized wave packet carrying intrinsic angular momentum and propagating in the z direction in Minkowski spacetime. If a Lorentz boost is applied in the x direction, then the location of the Lorentz-transformed energy density centroid is shifted in the y direction, depending on the orientation of the angular momentum. This shift corresponds to the Wigner translation [163, 69, 40].

The following example shows that an effect analogous to the Wigner translation discussed in Ref. [163] appears in the effective ray equations (2.115) and (2.116). We consider the Minkowski spacetime in Cartesian coordinates (t, x, y, z), with

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{2.162}$$

and we want to compare the effective rays obtained from Eqs. (2.115) and (2.116) with two different choices of observer. In the first case, we consider the standard orthonormal tetrad

$$e_0 = \partial_t, \qquad e_1 = \partial_x, \qquad e_2 = \partial_y, \qquad e_3 = \partial_z, \tag{2.163}$$

where $(e_0)^\mu$ is our first choice of observer. With this orthonormal tetrad, the polarization vectors are defined as in Eq. (2.160), and the Berry curvature terms can be computed. The ray equations reduce to the geodesic equations

$$\dot{X}^\mu = P^\mu, \qquad \dot{P}_\mu = 0. \tag{2.164}$$

In order to describe light rays traveling in the z direction, we impose initial conditions $X^\mu(0) = (0,0,0,0)$ and $P_\mu(0) = (-1,0,0,1)$, and we obtain

$$\begin{aligned} X^\mu(\tau) &= (\tau, 0, 0, \tau)\,, \\ P_\mu(\tau) &= (-1, 0, 0, 1)\,. \end{aligned} \tag{2.165}$$

As a second case, we apply a time-dependent boost in the x direction to the standard orthonormal tetrad in Eq. (2.162). We obtain

$$\begin{aligned} e'_0 &= \cosh t\, \partial_t - \sinh t\, \partial_x, \qquad & e'_2 &= \partial_y \\ e'_1 &= -\sinh t\, \partial_t + \cosh t\, \partial_x, \qquad & e'_3 &= \partial_z, \end{aligned} \tag{2.166}$$

where $(e'_0)^\mu$ is our second choice of observer. Note that $(e'_0)^\mu$ represents a family of observers boosted in the x direction, with the rapidity of the boost represented by the time coordinate t. The polarization vectors are chosen as in Eq. (2.160), but this time with respect to the orthonormal tetrad in Eq. (2.166). The Berry curvature terms in Eqs. (2.115) and (2.116) can be explicitly computed, and we obtain

$$\dot{X}^\mu = P^\mu + \epsilon s P^\nu \left(F_{px}\right)_\nu{}^\mu\,, \tag{2.167}$$

$$\dot{P}_\mu = 0, \tag{2.168}$$

where

$$P^\nu \left(F_{px}\right)_\nu{}^\mu = \frac{P_t}{[(e'_0)^\mu P_\mu]^2} \begin{pmatrix} 0 \\ 0 \\ P_z \\ -P_y \end{pmatrix}. \tag{2.169}$$

We impose the same initial conditions as in the previous case: $X^\mu(0) = (0,0,0,0)$ and $P_\mu(0) = (-1,0,0,1)$. Since the frequency is defined as $\omega = -(e'_0)^\mu P_\mu/\epsilon$, the small parameter ϵ can be identified with the wavelength of the initial light ray, as measured by the observer $(e'_0)^\mu$ at the spacetime point $x^\mu = X^\mu(0)$. Then, the ray equations can be analytically integrated, and we obtain

$$\begin{aligned} X^\mu(\tau) &= (\tau, 0, -s\epsilon \tanh\tau, \tau)\,, \\ P_\mu(\tau) &= (-1, 0, 0, 1)\,. \end{aligned} \tag{2.170}$$

Thus, given a circularly polarized light ray traveling in the z direction and two families of observers $(e_0)^\mu$ and $(e'_0)^\mu$, which are related by boosts in the x direction, we obtained the polarization-dependent Wigner translation in the y direction, $\Delta y = s\epsilon \tanh \tau$, in agreement with [163, Eq. 28]. Note that the Wigner translation is always smaller than one wavelength.

Recovering the results of Ref. [163] suggests that a worldline $X^\mu(\tau)$ representing a solution of Eqs. (2.115) and (2.116) could be interpreted as the location of the energy density centroid of a localized wave packed with definite circular polarization, as measured by the chosen family of observers.

2.5.2. The optical metric

Consider a fixed spacetime metric $\tilde{g}_{\mu\nu}$ and a dielectric medium with a varying refractive index n sitting in the rest frame of a timelike observer u^μ. It has been shown in Ref. [92] (see also Ref. [164]) that the combined effect of the background spacetime and the dielectric medium on light rays can be studied by using the optical metric

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + (1 - n^{-2}) u_\mu u_\nu. \tag{2.171}$$

Here, we show how the effective ray equations (2.115) and (2.116), together with the optical metric, can be used to recover the well-known equations describing the spin Hall effect of light in an inhomogeneous medium [63, 114, 131, 33, 34, 67, 68, 36, 148].

We write the optical metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + (1 - n^{-2}) u_\mu u_\nu, \tag{2.172}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $n = n(t, x, y, z)$ is the refractive index of the medium, and $u = \partial_t$ represents the rest frame of the medium. We also consider the following orthonormal tetrad associated with the optical metric $g_{\mu\nu}$:

$$e_0 = n\partial_t, \quad e_1 = \partial_x, \quad e_2 = \partial_y, \quad e_3 = \partial_z. \tag{2.173}$$

The choice of orthonormal tetrad is mainly motivated by the choice of a timelike observer $t^\mu = (e_0)^\mu$ performing the optical experiment. Note that, outside the optical medium where $n = 1$, this timelike observer reduces to the standard Minkowski observer ∂_t, which is generally assumed in the optics literature, when studying the spin Hall effect of light. In this case, the effective ray equations (2.115) and (2.116) in noncanonical coordinates reduce to

$$\dot{X}^\mu = P^\mu + \epsilon s \Gamma^\alpha_{\beta\nu} P_\alpha P^\beta \left(F_{pp}\right)^{\nu\mu}, \tag{2.174}$$

$$\dot{P}_\mu = \Gamma^\alpha_{\beta\mu} P_\alpha P^\beta, \tag{2.175}$$

where we have the following coordinate components:

$$P^\mu = g^{\mu\nu} P_\nu = \begin{pmatrix} -n^2 P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} -n^2 P_0 \\ \mathbf{P} \end{pmatrix}, \tag{2.176}$$

$$\Gamma^\alpha_{\beta\nu} P_\alpha P^\beta \, (F_{pp})^{\nu\mu} = \begin{pmatrix} 0 \\ \frac{\dot{\mathbf{P}} \times \mathbf{P}}{[(P_1)^2 + (P_2)^2 + (P_3)^2]^{3/2}} \end{pmatrix}, \tag{2.177}$$

$$\Gamma^\alpha_{\beta\mu} P_\alpha P^\beta = n (P_0)^2 \partial_\mu n = n (P_0)^2 \begin{pmatrix} \partial_t n \\ \nabla n \end{pmatrix}. \tag{2.178}$$

We can restrict to the case where the refractive index is time-independent: $n = n(x, y, z)$. Furthermore, we can use the Hamiltonian to remove the equation for $\dot{P}_0$:

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu = 0 \quad \Rightarrow \quad (P_0)^2 n^2 = (P_1)^2 + (P_2)^2 + (P_3)^2 = P^2, \tag{2.179}$$

where we introduced the notation $P = \sqrt{(P_1)^2 + (P_2)^2 + (P_3)^2}$. The ray equations reduce to

$$\dot{X}^\mu = \begin{pmatrix} nP \\ \mathbf{P} + \epsilon s \frac{\dot{\mathbf{P}} \times \mathbf{P}}{P^3} \end{pmatrix}, \tag{2.180}$$

$$\dot{P}_i = \frac{P^2}{n} \nabla_i n, \tag{2.181}$$

To obtain the same form of the ray equations as in optics, we have to reparametrize the rays in terms of $X^0 = T$ instead of τ. We have $\dot{X}^0 = nP$, and

$$\frac{dX^i}{dT} = \frac{d\mathbf{X}}{dT} = \frac{\mathbf{P}}{nP} + \epsilon s \frac{\frac{d\mathbf{P}}{dT} \times \mathbf{P}}{P^3}, \tag{2.182}$$

$$\frac{dP_i}{dT} = \frac{d\mathbf{P}}{dT} = \frac{P}{n^2} \nabla n, \tag{2.183}$$

These are the effective ray equations describing the spin Hall effect of light in an inhomogeneous medium, obtained in Ref. [148, Eqs. 90 and 91]. These ray equations can also be rewritten in the form presented in Ref. [31] by rescaling the momentum and time, as mentioned in Ref. [148].

2.5.3. Cosmological spacetimes

Robertson-Walker spacetime

The propagation of polarized light in a Robertson-Walker spacetime has been studied in [66] by using the Souriau-Saturnini equations. The authors obtained polarization-dependent ray trajectories, but with the difference between the polarized rays and the null geodesic being smaller than one wavelength.

Here, we perform a similar analysis using the ray equations (2.115) and (2.116). We consider the Robertson-Walker line element

$$ds^2 = -dt^2 + A^2(t,x,y,z)\left(dx^2 + dy^2 + dz^2\right), \tag{2.184}$$

where

$$A(t,x,y,z) = \frac{a(t)}{1 + \frac{K}{4}(x^2 + y^2 + z^2)}. \tag{2.185}$$

As in [66], we consider the flat Λ-CDM model, with $K = 0$ and

$$a(t) = a_0 \left(\frac{\cosh\left(\sqrt{3\Lambda} t\right) - 1}{\cosh\left(\sqrt{3\Lambda} t_0\right) - 1} \right)^{1/3}. \tag{2.186}$$

Furthermore, the following orthonormal tetrad is considered:

$$e_0 = \partial_t, \quad e_1 = (A^2)^{-1/2}\partial_x, \quad e_2 = (A^2)^{-1/2}\partial_y, \quad e_3 = (A^2)^{-1/2}\partial_z, \tag{2.187}$$

where e_0 is our choice of observer. With this choice, the polarization vectors can be defined as in (2.160), and the Berry curvature terms can be computed. The modified ray equations reduce to the null geodesic equations

$$\dot{x}^\mu = p^\mu, \tag{2.188}$$

$$\dot{p}_\mu = \Gamma^\alpha_{\beta\mu} p_\alpha p^\beta. \tag{2.189}$$

Note that, while there are no polarization-dependent effects for the particular choice of observer used here, picking some other accelerated observer could result in a nonzero effect, as discussed in Sec. 2.5.1. Such effects would still be limited to displacements smaller than one wavelength. This suggest that the polarization-dependent effect obtained in [66] is of the same nature. However, the choice of the observer is not particularly clear in [66], since it is encoded in the choice of spin supplementary condition used for deriving the Souriau-Saturnini equations.

Kasner spacetime

As another cosmological example, we consider the Kasner line element

$$ds^2 = -dt^2 + t^{2c_1}dx^2 + t^{2c_2}dy^2 + t^{2c_3}dz^2, \tag{2.190}$$

where c_1, c_2 and c_3 are real constants, satisfying

$$c_1 + c_2 + c_3 = (c_1)^2 + (c_2)^2 + (c_3)^2 = 1. \tag{2.191}$$

Furthermore, we consider the orthonormal tetrad

$$e_0 = \partial_t, \quad e_1 = (t^{2c_1})^{-1/2}\partial_x, \quad e_2 = (t^{2c_2})^{-1/2}\partial_y, \quad e_3 = (t^{2c_3})^{-1/2}\partial_z, \tag{2.192}$$

where $(e_0)^\mu$ represents our choice of a family of timelike observers. In this case, the effective ray equations (2.115) and (2.116) become

$$\dot{X}^\mu = P^\mu + \epsilon s P^\nu \left(F_{px}\right)_\nu{}^\mu, \tag{2.193}$$

$$\dot{P}_\mu = \Gamma^\alpha_{\beta\mu} P_\alpha P^\beta, \tag{2.194}$$

where

$$P^\mu = \begin{pmatrix} -P_0 \\ t^{-2c_1}P_1 \\ t^{-2c_2}P_2 \\ t^{-2c_3}P_3 \end{pmatrix}, \tag{2.195}$$

$$P^\nu \left(F_{px}\right)_\nu{}^\mu = \frac{-(p_0)^2 t^2}{[(P_1)^2 t^{2(c_2+c_3)} + (P_2)^2 t^{2(c_3+c_1)} + (P_3)^2 t^{2(c_1+c_2)}]} \begin{pmatrix} 0 \\ (c_2 - c_3)P_2P_3 \\ (c_3 - c_1)P_3P_1 \\ (c_1 - c_2)P_1P_2 \end{pmatrix}, \tag{2.196}$$

$$\Gamma^\alpha_{\beta\mu} P_\alpha P^\beta = \frac{c_1(P_1)^2 t^{-2c_1} + c_2(P_2)^2 t^{-2c_2} + c_3(P_3)^2 t^{-2c_3}}{t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.197}$$

Using the Hamiltonian, we have

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu = 0 \quad \Rightarrow \quad (P_0)^2 = t^{-2c_1}(P_1)^2 + t^{-2c_2}(P_2)^2 + t^{-2c_3}(P_3)^2. \tag{2.198}$$

Thus, we can eliminate the equation for $\dot{P}_0$, and we obtain:

$$\dot{X}^\mu = \begin{pmatrix} \sqrt{(P_1)^2(X^0)^{-2c_1} + (P_2)^2(X^0)^{-2c_2} + (P_3)^2(X^0)^{-2c_3}} \\ P_1(X^0)^{-2c_1} + \frac{\epsilon s(c_3-c_2)p_3p_2}{(P_1)^2(X^0)^{1-c_1+c_2+c_3} + (P_2)^2(X^0)^{1+c_1-c_2+c_3} + (P_3)^2(X^0)^{1+c_1+c_2-c_3}} \\ P_2(X^0)^{-2c_2} + \frac{\epsilon s(c_1-c_3)p_1p_3}{(P_1)^2(X^0)^{1-c_1+c_2+c_3} + (P_2)^2(X^0)^{1+c_1-c_2+c_3} + (P_3)^2(X^0)^{1+c_1+c_2-c_3}} \\ P_3(X^0)^{-2c_3} + \frac{\epsilon s(c_2-c_1)p_2p_1}{(P_1)^2(X^0)^{1-c_1+c_2+c_3} + (P_2)^2(X^0)^{1+c_1-c_2+c_3} + (P_3)^2(X^0)^{1+c_1+c_2-c_3}} \end{pmatrix}, \tag{2.199}$$

$$\dot{P}_i = 0. \tag{2.200}$$

If we set $c_1 = 1, c_2 = c_3 = 0$, the equations reduce to

$$\dot{X}^\mu = \begin{pmatrix} \sqrt{(P_1)^2(X^0)^{-2} + (P_2)^2 + (P_3)^2} \\ P_1(X^0)^{-2} \\ P_2 + \frac{\epsilon s P_1 P_3}{(P_1)^2+(P_2)^2(X^0)^2+(P_3)^2(X^0)^2} \\ P_3 - \frac{\epsilon s P_2 P_1}{(P_1)^2+(P_2)^2(X^0)^2+(P_3)^2(X^0)^2} \end{pmatrix}, \tag{2.201}$$

$$\dot{P}_i = 0, \tag{2.202}$$

This system of ordinary differential equations can be integrated, and we obtain

$$\begin{aligned} X^0(\tau) &= \sqrt{\frac{-(P_1)^2 + [(P_2)^2 + (P_3)^2]^2(\tau + C_1)^2}{(P_2)^2 + (P_3)^2}}, \\ X^1(\tau) &= -\operatorname{arctanh}\left[\frac{(P_2)^2 + (P_3)^2}{P_1}(\tau + C_1)\right] + C_2, \\ X^2(\tau) &= P_2(\tau + C_1) - \epsilon s \frac{P_1 P_3}{[(P_2)^2 + (P_3)^2]^2(\tau + C_1)} + C_3, \\ X^3(\tau) &= P_3(\tau + C_1) + \epsilon s \frac{P_1 P_2}{[(P_2)^2 + (P_3)^2]^2(\tau + C_1)} + C_4, \\ P_1(\tau) &= P_1(0) = P_1, \\ P_2(\tau) &= P_2(0) = P_2, \\ P_3(\tau) &= P_3(0) = P_3, \end{aligned} \tag{2.203}$$

where C_1, C_2, C_3, C_4 are integration constants. In order to analyze the $\mathcal{O}(\epsilon^1)$ terms, it is convenient to reparametrize the rays in terms of X^0. From the first equation, we obtain:

$$\tau + C_1 = \pm\sqrt{\frac{(X^0)^2[(P_2)^2 + (P_3)^2] + (P_1)^2}{[(P_2)^2 + (P_3)^2]^2}}, \tag{2.204}$$

and the $\mathcal{O}(\epsilon^1)$ terms are proportional to $1/X^0$. Thus, the polarization-dependent correction terms decay as $1/X^0$, so any polarization-dependent effects would quickly become unobservable.

In the more general case, where $c_1 \neq c_2 \neq c_3 \neq 0$, it is not so straightforward to integrate the ray equations (2.199) and (2.200) analytically. The main difficulty is represented by the computation of the integral

$$\int \frac{1}{\sqrt{(P_1)^2 t^{-2c_1} + (P_2)^2 t^{-2c_2} + (P_3)^2 t^{-2c_3}}} dt, \tag{2.205}$$

where P_1, P_2 and P_3 are constants.

2.5.4. Schwarzschild spacetime

To illustrate how the polarization-dependent correction terms modify the ray trajectories on a Schwarzschild background, let us provide some numerical examples. For convenience, we perform the numerical computations in canonical coordinates (x, p) and treat x^0 as a parameter along the rays (the same results can also be obtained using the ray equations in noncanonical coordinates (X, P), using the code presented in Appendix A.7). Hence, Eqs. (2.102) and (2.103) become

$$\dot{x}^0 = 1, \tag{2.206}$$

$$\dot{x}^i = \frac{g^{i\nu}p_\nu - \epsilon s \left(B^i + p^\alpha \overset{v}{\nabla}{}^i B_\alpha \right)}{g^{0\nu}p_\nu - \epsilon s \left(B^0 + p^\alpha \overset{v}{\nabla}{}^0 B_\alpha \right)}, \tag{2.207}$$

$$\dot{p}_i = \frac{-\frac{1}{2}\partial_i g^{\alpha\beta}p_\alpha p_\beta + \epsilon s p_\alpha \left(\partial_i g^{\alpha\beta} B_\beta + g^{\alpha\beta}\partial_i B_\beta \right)}{g^{0\nu}p_\nu - \epsilon s \left(B^0 + p^\alpha \overset{v}{\nabla}{}^0 B_\alpha \right)}, \tag{2.208}$$

and p_0 is calculated from

$$\frac{1}{2} g^{\mu\nu} p_\mu p_\nu - \epsilon s g^{\mu\nu} p_\mu B_\nu(x, p) = 0. \tag{2.209}$$

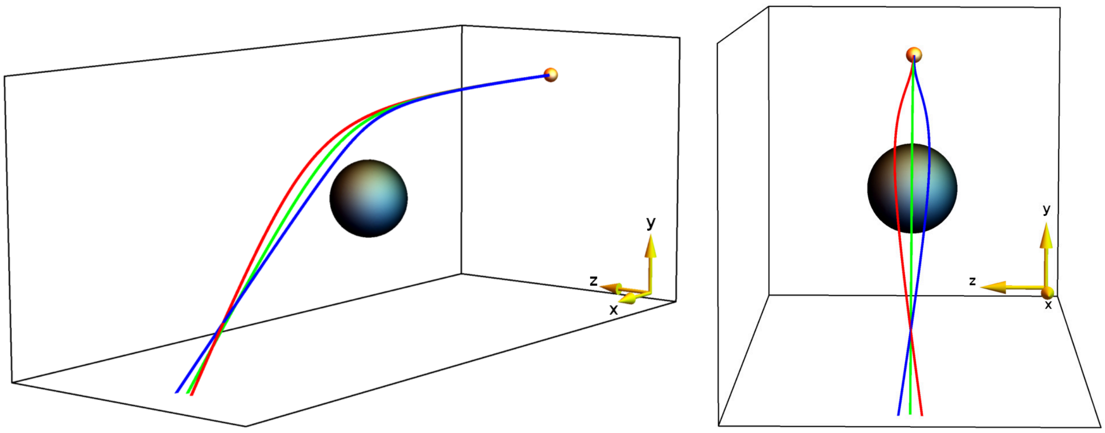
This equation can be solved explicitly, using the fact that the velocity $\dot{x}^\alpha$ is future oriented:

$$\begin{aligned} p_0 = \frac{1}{g^{00}} \Bigg[&- \left(g^{0i}p_i - \epsilon s g^{0\mu} B_\mu \right) \\ &+ \sqrt{\left(g^{0i}p_i - \epsilon s g^{0\mu}B_\mu \right)^2 - g^{00}\left(g^{ij}p_i p_j - 2\epsilon s p_i g^{i\mu} B_\mu \right)} \Bigg]. \end{aligned} \tag{2.210}$$

Note that in general B_μ depends on p_0. However, since this is an $\mathcal{O}(\epsilon^1)$ term, we can replace the $\mathcal{O}(\epsilon^0)$ expression for p_0 in B_μ.

To compare with the results of Gosselin *et al.* [95], we consider a Schwarzschild spacetime in Cartesian isotropic coordinates (t, x, y, z):

$$\mathrm{d}s^2 = -\left(\frac{1 - \frac{r_s}{4R}}{1 + \frac{r_s}{4R}} \right)^2 \mathrm{d}t^2 + \left(1 + \frac{r_s}{4R} \right)^4 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{2.211}$$



(a) (b)

Figure 2.1.: Results of numerical simulations illustrating the gravitational spin Hall effect of light around a Schwarzschild black hole. The effect is exaggerated for visualization purposes. The two figures present the same rays from different viewing angles. The central sphere represents the Schwarzschild black hole, and the small orange sphere represents a source of light. The blue and the red trajectories correspond to rays of opposite circular polarization, $s = \pm 1$, while the green trajectory represents a null geodesic. We take $r_s = 1$, and we start with the initial position $x^i(0) = (-50r_s, 15r_s, 0)$, and initial normalized momentum $p_i = (1, 0, 0)$. The wavelength λ is set to a sufficiently large value to make the effect visible on this plot.

where $r_s = 2GM/c^2$ is the Schwarzschild radius and $R = \sqrt{x^2 + y^2 + z^2}$. We also define the following orthonormal tetrad:

$$\begin{aligned} e_0 &= \frac{1 + \frac{r_s}{4R}}{1 - \frac{r_s}{4R}}\, \partial_t, & e_1 &= \left(1 + \frac{r_s}{4R}\right)^{-2} \partial_x, \\ e_2 &= \left(1 + \frac{r_s}{4R}\right)^{-2} \partial_y, & e_3 &= \left(1 + \frac{r_s}{4R}\right)^{-2} \partial_z, \end{aligned} \tag{2.212}$$

where $t^\mu = (e_0)^\mu$ is our choice of observer.

The Berry connection B_μ can be explicitly computed by introducing a particular choice of polarization vectors, using Eq. (2.160) and the orthonormal tetrad (2.212). We now have all the elements required for the numerical integration of Eqs. (2.206)-(2.208). For this purpose, we used the NDSolve function of *Mathematica* [105]. For these examples, we used the default settings for the integration method, precision and accuracy.

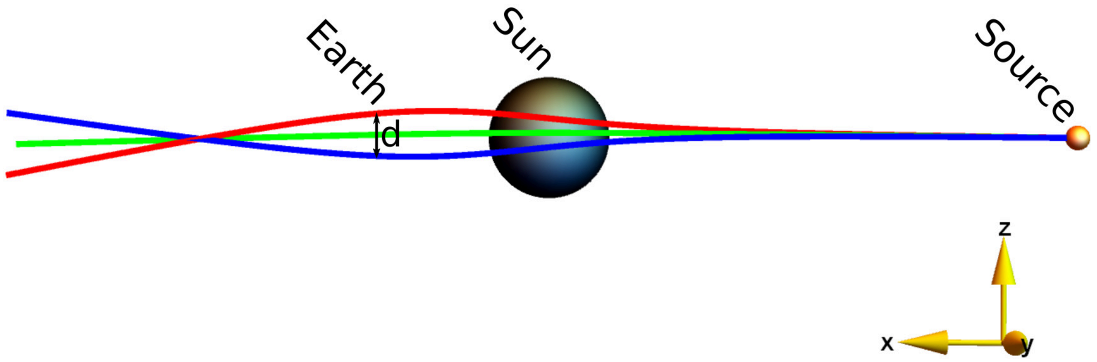


Figure 2.2.: Results of numerical simulations illustrating the gravitational spin Hall effect of light around the Sun. The effect is exaggerated for visualization purposes. The separation distance d is observed from the Earth. The blue and the red trajectories correspond to rays of opposite circular polarization, $s = \pm 1$, while the green trajectory represents a null geodesic. We take $r_s = 3$ km, and we start with the initial position $x^i(0) = (-10^7 r_s, 3 \times 10^5 r_s, 0)$, and initial normalized momentum $p_i = (1, 0, 0)$.

After obtaining a numerical solution $(x(t), p(t))$ to Eqs. (2.206)-(2.208), in order to ensure the gauge invariance of our results, we have to evaluate the gauge-invariant noncanonical quantities $(X(t), P(t))$, as given in Eqs. (2.108) and (2.109). These are the quantities used to represent the trajectories in Figs. 2.1 and 2.2. A comparative discussion between the use of canonical and noncanonical ray equations in optics, together with numerical examples, can be found in Ref. [148].

As the first step, we numerically compare our ray Eqs. (2.206)-(2.208) with those predicted by Gosselin *et al.* [95]. This is done by numerically integrating Eqs. (2.206)-(2.208), as well as Eq. (23) from Ref. [95]. Up to numerical errors, we obtain the same ray trajectories with both sets of equations. However, while the equations obtained by Gosselin *et al.* only apply to static spacetimes, Eqs. (2.206)-(2.208) do not have this limitation.

The results of our numerical simulations are shown in Fig. 2.1, which illustrates the general behavior of the gravitational spin Hall effect of light around a Schwarzschild black hole. [The actual effect is small, so the figure is obtained by numerical integration of Eqs. (2.206)-(2.208) for unrealistic parameters.] Here, we consider rays of opposite circular polarization ($s = \pm 1$) passing close to a Schwarzschild black hole, together with a reference null geodesic ($s = 0$). Except for the value of s, we are considering the same initial conditions, $(x^i(0), p_i(0))$, for these rays. Unlike the null geodesic, for which the motion is planar, the circularly polarized rays are not confined to a plane.

As another example, we used initial conditions $(x^i(0), p_i(0))$ such that the rays are initialized as radially ingoing or outgoing. In this case (not illustrated, since it is trivial), the gravitational spin Hall effect vanishes, and the circularly polarized rays coincide with the radial null geodesic.

Using these numerical methods, we can also estimate the magnitude of the gravitational spin Hall effect. As a particular example, we consider a similar situation to the one presented in Fig. 2.1, where the black hole is replaced with the Sun. More precisely, we model this situation by considering a Schwarzschild black hole with $r_s \approx 3$ km. We consider the deflection of circularly polarized rays coming from a light source far away, passing close to the surface of the Sun, and then observed on the Earth. This situation is illustrated in Fig. 2.2. The numerical results are based on the initial data presented in the caption of Fig. 2.2. When reaching the Earth, the separation distance between the rays of opposite circular polarization depends on the wavelength. For example, taking wavelengths of the order $\lambda \approx 10^{-9}$ m results in a separation distance of the order $d \approx 10^{-15}$ m, while for wavelengths of the order of $\lambda \approx 1$ m we obtain a separation distance of the order $d \approx 10^{-6}$ m. Although the ray separation is small (about six orders of magnitude smaller than the wavelength), what really matters is that the rays are scattered by a finite angle. Therefore, the ray separation grows linearly with distance after the reintersection point. This means that the effect should be robustly observable if one measures it sufficiently far from the Sun. Furthermore, massive compact astronomical objects, such as black holes or neutron stars, are expected to produce a larger gravitational spin Hall effect.

As a consistency check, we also performed the numerical computations using different coordinates, such as the standard Schwarzschild spherical coordinates and Gullstrand–Painlevé coordinates. The results are independent of the choice of coordinates. However, the polarized rays are not invariant under a change of observer. This is due to an effect analogous to the Wigner translations discussed in Sec. 2.5.1. For example, instead of the static observer introduced in Eq. (2.212), one could consider a free-falling observer. In this case, the ray trajectories presented in Figs. 2.1 and 2.2 are slightly modified, due to the Wigner translations, and preliminary investigations indicate that these modifications are smaller than one wavelength, as in the case discussed in Sec. 2.5.1. It is not clear how to separate the purely gravitational effect from the observer-dependent Wigner translations. However, this is not a problem. The Wigner translation represents the observer-dependent ambiguity in defining the location of the ray on a single-wavelength scale and remains bounded. In contrast, the purely gravitational effect can affect the angle of light scattering off a gravitating object and thus the ray displacement associated with this effect accumulates linearly with the distance. This means that the latter effect dominates at large distances.

2.5.5. Kerr spacetime

Here, we present some numerical examples of polarized light rays experiencing the gravitational spin Hall effect in Kerr spacetimes. In this case, due to the complexity of the ray equations, it is more convenient to use the ray equations in noncanonical coordinates. The Kerr metric is considered in Boyer-Lindquist coordinates (t, r, θ, ϕ), with the components of the metric tensor $g_{\mu\nu}$ given as

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{2mr}{\Sigma} & 0 & 0 & -\frac{2mar\sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{2mar\sin^2\theta}{\Sigma} & 0 & 0 & \frac{\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]\sin^2\theta}{\Sigma} \end{pmatrix}, \tag{2.213}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2mr + a^2, \tag{2.214}$$

where m is the mass and a is the spin parameter. Furthermore, we consider the following orthonormal tetrad $(e_a)^\mu$:

$$\begin{aligned} e_0 &= \frac{r^2 + a^2}{\sqrt{\Sigma\Delta}}\partial_t + \frac{a}{\sqrt{\Sigma\Delta}}\partial_\phi, \\ e_1 &= \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \\ e_2 &= \frac{1}{\sqrt{\Sigma}}\partial_\theta, \\ e_3 &= \frac{a\sin\theta}{\sqrt{\Sigma}}\partial_t + \frac{1}{\sin\theta\sqrt{\Sigma}}\partial_\phi. \end{aligned} \tag{2.215}$$

This orthonormal tetrad is used to define the polarization vectors as in Eq. (2.160), and the Berry curvature terms can be explicitly computed. This example is implemented in the Mathematica code provided in Appendix A.7.

As initial conditions for the equations of motion (2.115) and (2.116) we need to prescribe $X^\mu(\tau = 0)$, $P_\mu(\tau = 0)$ and $s = \pm 1$. Note that $\dot{s} = 0$, so the initial state of circular polarization is preserved. In order to ensure that $P_\mu(0)$ is null, we write it as

$$\begin{aligned} P_\mu(0) &= k_0(e^0)_\mu + k_1(e^1)_\mu + k_2(e^2)_\mu + k_3(e^3)_\mu \\ &= -(e^0)_\mu + k_1(e^1)_\mu + k_2(e^2)_\mu + k_3(e^3)_\mu, \end{aligned} \tag{2.216}$$

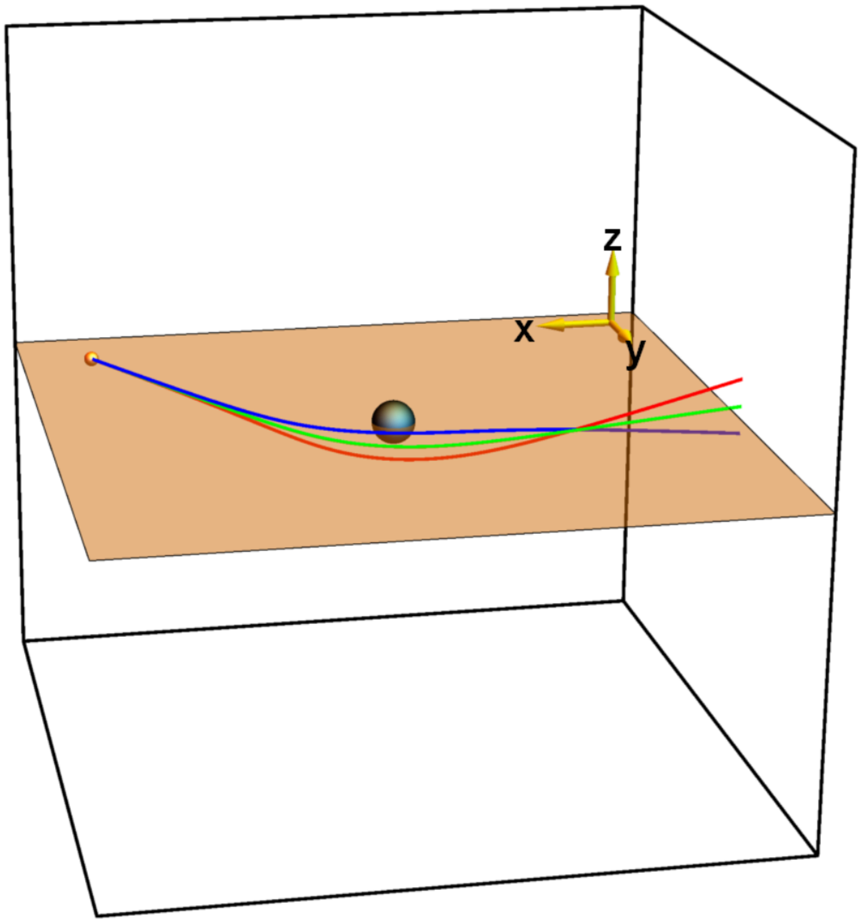
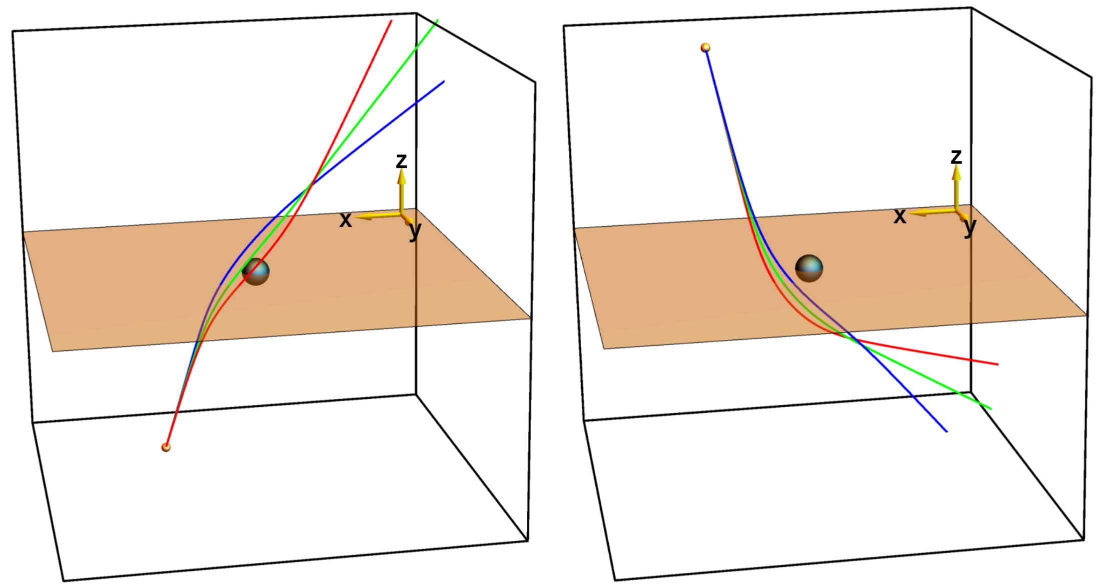


Figure 2.3.: Results of numerical simulations illustrating the gravitational spin Hall effect of light around a Kerr black hole. The effect is exaggerated for visualization purposes. The initial conditions are prescribed such that the reference null geodesic remains in the equatorial plane. The Kerr black hole spin parameter $a = 0.99$, and the angular momentum of the black hole is directed along the positive z axis. The equatorial plane is also plotted.

with $(k_1)^2 + (k_2)^2 + (k_3)^2 = 1$. Here, (k_1, k_2, k_3) represent coordinates on the celestial sphere of the observer $(e_0)^\mu$, located at the spacetime point $x^\mu(0)$. Alternatively, one can also use spherical coordinates (ρ, ψ) on the celestial sphere, such that $(k_1, k_2, k_3) = (\sin\rho\cos\psi, \sin\rho\sin\psi, \cos\rho)$.

Note that $(e_0)^\mu P_\mu(0) = 1$, so the frequency measured by the observer $(e_0)^\mu$ at



(a) (b)

Figure 2.4.: Results of numerical simulations illustrating the gravitational spin Hall effect of light around a Kerr black hole. The effect is exaggerated for visualization purposes. The Kerr black hole spin parameter $a = 0.99$, and the angular momentum of the black hole is directed along the positive z axis. The equatorial plane is also plotted.

the spacetime point $X^\mu(0)$ is

$$\omega = -\frac{(e_0)^\mu P_\mu(0)}{\epsilon} = \frac{1}{\epsilon} \tag{2.217}$$

Thus, with the given choice of initial data, the small parameter ϵ can be identified with the wavelength of the light ray.

The equations of motion (2.115) and (2.116) are also constrained by the Hamiltonian function $H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu = 0$. We can use this constraint to eliminate one of the equations of motion, and to ensure that P_μ remains null, despite possible numerical errors. We choose to use the Hamiltonian to solve for P_0:

$$P_0 = \frac{1}{g^{00}} \Big\{ - g^{0i} P_i \quad + \Big[\left(g^{0i} P_i \right)^2 - g^{00} \left(g^{ij} P_i P_j \right) \Big]^{1/2} \Big\}, \tag{2.218}$$

and eliminate the $\dot{P}_0$ equation of motion.

Based on these assumptions, and using the code presented in Appendix A.7, we present here some polarized ray trajectory examples. In Fig. 2.3 we have an example where the initial conditions are prescribed such that the reference null geodesic remains in the equatorial plane. As in the Schwarzschild case, the polarized rays are not planar, but there is still a symmetry between right-handed and left-handed circular polarized rays when reflecting with respect to the equatorial plane.

In the more general case, where the reference null geodesic is not restricted to the equatorial plane, the behavior of the polarized rays becomes more complicated, and there is no obvious symmetry between right-handed and left-handed circular polarized rays. An example is presented in Fig. 2.4. Further examples can be easily explored using the Mathematica code provided in Appendix A.7.

2.5.6. Black hole shadows with polarized light

The shadow of a black hole can be defined on the celestial sphere of an observer as the set of ray trajectories on which no light from a background source can reach the observer [120, 132, 133, 56]. Given the recent observation of the black hole shadow of the supermassive black hole M87 [77], there is great interest in the study of black hole shadows, and it is natural to ask if the polarization of light can have any effect on the shape of black hole shadows.

We can use the Mathematica code which computes the ray trajectories of polarized light to obtain the shape of black hole shadows, as seen by certain observers. We numerically determine the black hole shadow, as seen by an observer $t^\mu = (e_0)^\mu$ at the spacetime point $X^\mu(\tau = 0)$, by performing backwards ray tracing. More specifically, this means that we compute multiple ray trajectories using the Mathematica code presented in Appendix A.7, with initial conditions $X^\mu(\tau = 0)$ and $P_\mu(\tau = 0)$ as in Eq. (2.216), where $P_\mu(\tau = 0)$ is determined by different points (k_1, k_2, k_3) on the celestial sphere of the observer. Given a particular point (k_1, k_2, k_3), if the corresponding ray falls into the black hole (i.e. hits the event horizon), then (k_1, k_2, k_3) is part of the black hole shadow on the celestial sphere of the observer.

As a first example, we investigate the black hole shadow of a Schwarzschild black hole. In this case, we immediately find that the black hole shadow obtained with the polarized rays is identical to the black hole shadow determined by the null geodesics, so polarization has no effect on the shape of Schwarzschild black hole shadows. The shadow is spherical, as discussed in [133].

In the case of Kerr black holes, the polarization of light will have a nontrivial effect on the shape of the black hole shadow. Consider a Kerr black hole with

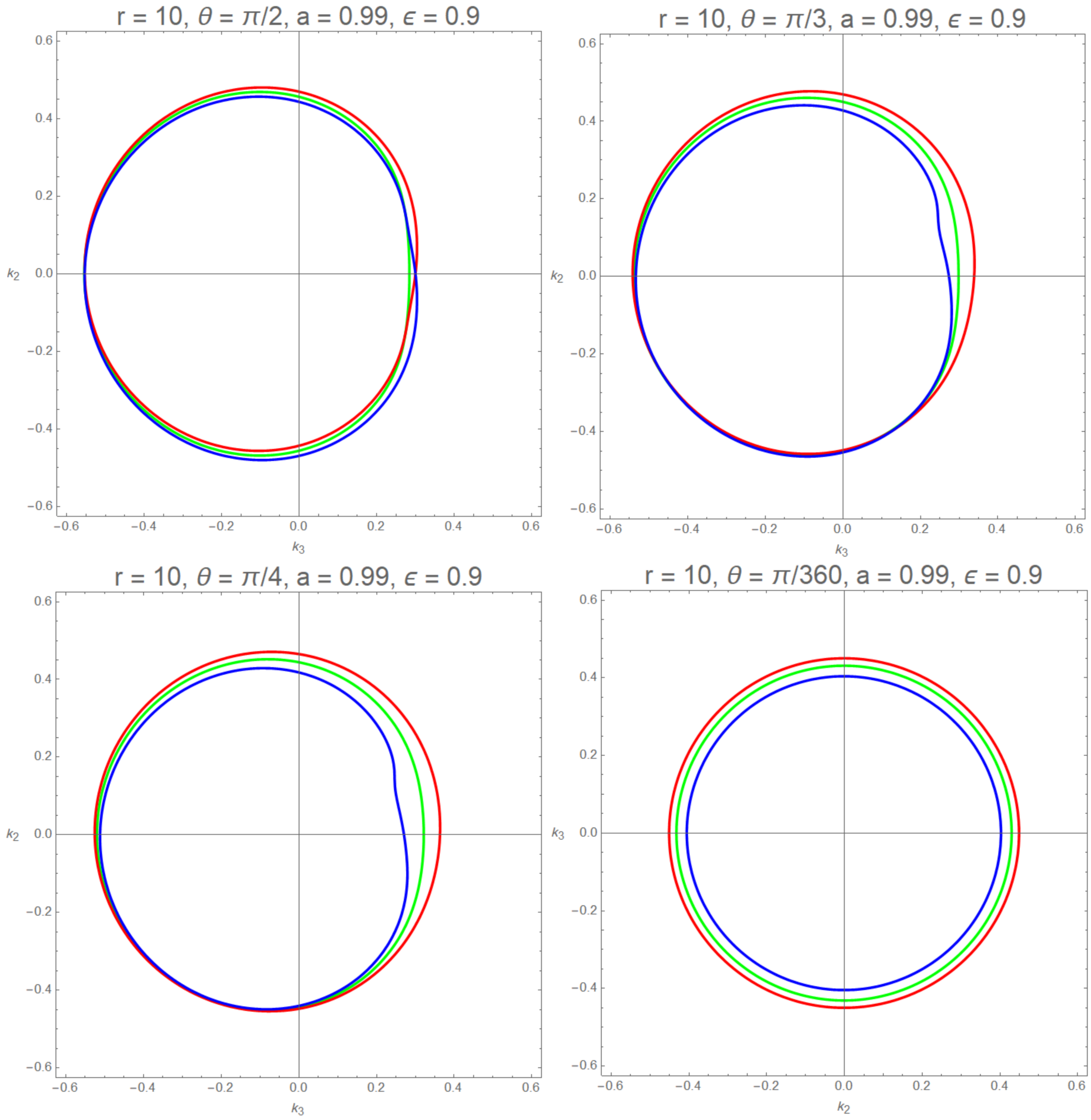


Figure 2.5.: Black hole shadows with polarized light for a Kerr black hole

$a = 0.99$, and we set $\epsilon = 0.9$ (ϵ is suppose to be much smaller than one, but we take this value for the sake of visualization). In Fig. 2.5 we present the obtained black hole shadows, as seen by observers t^μ located at $r = 10$, and at different radial positions $\theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{360}$. The black hole shadows are plotted in the $k_2 - k_3$ plane. The green lines represent the edge of the black hole shadow computed with the geodesic equations ($s = 0$), while the red and the blue lines represent the edge of the black hole shadows computed with Eqs. (2.115) and (2.116), with $s = \pm 1$.

3. Gravitational spin Hall effect of Dirac fields

Contents

In this chapter, we perform a similar WKB analysis of the Dirac equation. In contrast to Ch. 2, where the WKB ansatz was used to obtain a high-frequency approximation of Maxwell's equations, here we are using the WKB ansatz to perform a semiclassical analysis of the Dirac equation. In this case, the expansion parameter in the WKB ansatz is Planck's constant $\hbar$.

We start by introducing the Dirac equation on a fixed spacetime background, together with a fixed electromagnetic field $\mathscr{F}_{\mu\nu} = 2\nabla_{[\mu}\mathscr{A}_{\nu]}$. Using $\hbar$ as an expansion parameter, we define a WKB ansatz for the Dirac field, and a semiclassical approximation of the Dirac action is obtained. At the lowest order in $\hbar$, we obtain ray equations that are timelike geodesics, while at the next order in $\hbar$ we obtain a transport equation for the internal spin degree of freedom along the corresponding timelike geodesics. This transport equation is described in terms of the Berry connection. Following similar steps as in Ch. 2, an effective dispersion relation is derived and the corresponding effective ray equations describe the gravitational spin Hall effect of Dirac fields. The correction terms in the effective ray equations are expressed in terms of the Berry curvature. The effective ray equations in noncanonical coordinates have a similar form to the Mathisson-Papapetrou-Dixon equations at linear order in spin. Thus, our WKB analysis is in agreement with the results presented in Refs. [9, 147].

3.1. The Dirac equation

Consider a Lorentzian manifold $(M, g_{\mu\nu})$, which is a solution of the Einstein field equations, admitting a spin structure [50, p. 416]. A Dirac field Ψ is a section of a vector bundle with fiber $\mathbb{C}^4$, associated with the spin frame principal bundle $\mathrm{Spin}_{3,1}(M)$ via the representation $\rho(\Lambda) = \Lambda$, where $\Lambda \in \mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C})$ [50, p. 418]. The Dirac field Ψ, of charge e and mass m, satisfies the Dirac equation

$$(i\hbar\gamma^\mu \nabla_\mu - e\gamma^\mu \mathscr{A}_\mu - m)\,\Psi = 0, \tag{3.1}$$

where $\mathscr{A}_\mu$ is the electromagnetic vector potential, and γ^μ are the spacetime gamma matrices, which are related to the flat spacetime gamma matrices, γ^a, by the tetrad fields $(e_a)^\mu$: $\gamma^\mu = (e_a)^\mu \gamma^a$. The spinor covariant derivative ∇_μ is defined by a spin connection on the spin frame bundle $\mathrm{Spin}_{3,1}(M)$. Given a spin structure on M, the Levi-Civita connection on the Lorentz frame bundle $\mathrm{L}(M)$ determines a spin connection on the spin frame bundle $\mathrm{Spin}_{3,1}(M)$ [50, p. 419]. The spinor covariant derivative ∇_μ acts on the spinor fields as

$$\nabla_\mu \Psi = \left(\partial_\mu - \frac{i}{4}{\omega_\mu}^{ab}\sigma_{ab}\right)\Psi, \tag{3.2}$$

where $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$, and ${\omega_\mu}^{ab}$ is defined as

$${\omega_\mu}^{ab} = (e^a)_\nu \nabla_\mu (e^b)^\nu. \tag{3.3}$$

We are using the same symbol, ∇_μ, for both spinor covariant derivatives and the covariant derivatives of tensor fields, associated with the Levi-Civita connection. This should not cause any confusion, since the type of covariant derivative is determined by the object on which it acts. The vector potential $\mathscr{A}_\mu$ is not included in the definition of the spinor covariant derivative, because these are of different order in $\hbar$.

The Dirac equation is the Euler-Lagrange equation for the following action:

$$J = \int_M \mathrm{d}^4 x \, \sqrt{g} \, \bar{\Psi} \hat{D} \Psi, \tag{3.4}$$

where $\bar{\Psi} = \Psi^\dagger \gamma^0$, and the Dirac operator is

$$\hat{D} = i\hbar \gamma^\mu \nabla_\mu - e\mathscr{A}_\mu \gamma^\mu - m. \tag{3.5}$$

Since the action is invariant under $U(1)$-transformations $\Psi \mapsto e^{i\theta}\Psi$, the following Dirac current j^μ is conserved:

$$\nabla_\mu j^\mu = 0, \qquad\qquad j^\mu = \bar{\Psi}\gamma^\mu \Psi. \tag{3.6}$$

3.2. WKB approximation

We assume that the Dirac field admits a WKB expansion of the form

$$\begin{aligned} \Psi(x) &= \psi[x, k_\mu(x), \hbar] e^{iS(x)/\hbar}, \\ \psi[x, k_\mu(x), \hbar] &= \psi_0[x, k_\mu(x)] + \hbar \psi_1[x, k_\mu(x)] + \mathscr{O}(\hbar^2), \end{aligned} \tag{3.7}$$

where S is a real scalar function, ψ is a complex amplitude spinor, and Planck's constant $\hbar$ represents a small expansion parameter. The gradient of S is denoted as $k_\mu(x) = \nabla_\mu S(x)$. Note that we are allowing the amplitude ψ to depend on $k_\mu(x)$. This is justified by the mathematical formulation of the WKB approximation [18, 76], where $k_\mu(x)$ determines a Lagrangian submanifold $x \mapsto (x, k(x)) \in T^*M$, and the amplitude ψ is defined on the Lagrangian submanifold.

3.3. Semiclassical expansion

The semiclassical analysis of the Dirac equation is usually performed by inserting the WKB ansatz (3.7) into the Dirac equation (3.1), and analyzing the results order-by-order in $\hbar$ [146, 9, 147, 162]. However, we find it more convenient to perform the semiclassical analysis at the level of the action (3.4). The advantages of this variational approach are extensively discussed in Ref. [166].

3.3.1. WKB approximation of the Dirac action

A variational formulation of the WKB approximation for the Dirac field is obtained by inserting the WKB ansatz (3.7) into the action (3.4):

$$\begin{aligned} J &= \int_M \mathrm{d}^4x \sqrt{g} \left(\bar{\psi} e^{-iS/\hbar}\right) \hat{D} \left(\psi e^{iS/\hbar}\right) \\ &= \int_M \mathrm{d}^4x \sqrt{g}\, \bar{\psi} \left(D + i\hbar\gamma^\mu \nabla_\mu\right) \psi + \mathscr{O}(\hbar^2). \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} D &= -\gamma^\mu v_\mu - m, \\ v_\mu &= k_\mu + e\mathscr{A}_\mu. \end{aligned} \tag{3.9}$$

The action depends on the phase function $S(x)$, the amplitudes $\psi(x, \nabla S)$ and $\bar{\psi}(x, \nabla S)$. Performing a variation of the action with respect to these fields (since the amplitude ψ depends on ∇S, the variation of the action must be performed as in [129, Appendix B]), we obtain the following Euler-Lagrange equations:

$$D\psi + i\hbar\gamma^\mu \nabla_\mu \psi = \mathscr{O}(\hbar^2), \tag{3.10}$$
$$\bar{\psi} D - i\hbar(\nabla_\mu \bar{\psi})\gamma^\mu = \mathscr{O}(\hbar^2), \tag{3.11}$$
$$\nabla_\mu \left(\bar{\psi}\gamma^\mu \psi\right) = \mathscr{O}(\hbar^2). \tag{3.12}$$

Equations (3.10) and (3.11) can also be obtained by directly inserting the WKB ansatz into the Dirac equation, and Eq. (3.12) represents the WKB approximation of the conservation law given in Eq. (3.6).

3.3.2. WKB equations at order $\hbar^0$

At the lowest order in $\hbar$, the Euler-Lagrange equations (3.10)-(3.12) reduce to

$$D\psi_0 = 0, \tag{3.13}$$
$$\bar{\psi}_0 D = 0, \tag{3.14}$$
$$\nabla_\mu {j_0}^\mu = 0, \tag{3.15}$$

where we introduced the notation ${j_0}^\mu = \bar{\psi}_0 \gamma^\mu \psi_0$ for the conserved Dirac current at the lowest order in $\hbar$. Using Eqs. (3.13) and (3.14), we can obtain an alternative expression for ${j_0}^\mu$. We start by writing the following identities

$$\begin{aligned} \bar{\psi}_0 \gamma^\mu \left(m\psi_0\right) &= -\bar{\psi}_0 \gamma^\mu v_\nu \gamma^\nu \psi_0, \\ \left(\bar{\psi}_0 m\right) \gamma^\mu \psi_0 &= -v_\nu \bar{\psi}_0 \gamma^\nu \gamma^\mu \psi_0. \end{aligned} \tag{3.16}$$

Adding these two equations and using the anticommutation property of gamma matrices, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}$, we obtain

$$j_0{}^\mu = \bar{\psi}_0\gamma^\mu\psi_0 = \frac{1}{m}\mathscr{I}_0 v^\mu, \tag{3.17}$$

where we defined the lowest-order intensity as $\mathscr{I}_0 = \bar{\psi}_0\psi_0$. The transport equation (3.15) can be rewritten as

$$\nabla_\mu\left(\mathscr{I}_0 v^\mu\right) = 0. \tag{3.18}$$

Using Eqs. (3.13) and (3.14), we can write

$$0 = -\bar{\psi}_0 D\psi_0 = \mathscr{I}_0\left(\frac{1}{m}v_\mu v^\mu + m\right). \tag{3.19}$$

Thus, we obtained the dispersion relation

$$v_\mu v^\mu = -m^2. \tag{3.20}$$

Observing that

$$2\nabla_{[\nu}v_{\mu]} = -e\mathscr{F}_{\mu\nu}, \tag{3.21}$$

and we can differentiate the dispersion relation (3.20) to obtain the Lorentz force law:

$$v^\mu\nabla_\mu v_\nu = ev^\mu\mathscr{F}_{\mu\nu}. \tag{3.22}$$

Equations (3.13) and (3.14) are homogeneous systems of linear algebraic equations for the unknown amplitude ψ_0 [146, 9, 147]. In order for these systems to admit nontrivial solutions, the determinant of the matrix D must be zero. This condition is equivalent to the dispersion relation (3.20):

$$\det(D) = 0 \quad \Leftrightarrow \quad v_\mu v^\mu = -m^2. \tag{3.23}$$

Under the restriction $v_\mu v^\mu = -m^2$, the matrix D has rank 2. We can introduce a 4-spinor basis $\{\Sigma_1, \Sigma_2, \Pi_1, \Pi_2\}$, where Σ_1 and Σ_2 are eigenspinors of D, with eigenvalue zero, and Π_1 and Π_2 are eigenspinors of D, with eigenvalue $-2m$ [146, 9, 147]:

$$D\Sigma_A = 0, \qquad \bar{\Sigma}_A D = 0, \tag{3.24}$$

$$D\Pi_A = -2m\Pi_A, \qquad \bar{\Pi}_A D = -2m\bar{\Pi}_A, \tag{3.25}$$

where $A, B = 1, 2$. Furthermore, the 4-spinors satisfy the orthogonality relations

$$\bar{\Sigma}_A\Sigma_B = -\bar{\Pi}_A\Pi_B = \delta_{AB}, \tag{3.26}$$

and the resolution of identity

$$\Sigma_A \bar{\Sigma}_A - \Pi_A \bar{\Pi}_A = \mathbb{I}_4. \tag{3.27}$$

Here, and in the following, we are assuming an additional summation convention over repeated capital indices, such as

$$\Sigma_A \bar{\Sigma}_A = \sum_{A=1}^{2} (\Sigma_A \bar{\Sigma}_A) = \Sigma_1 \bar{\Sigma}_1 + \Sigma_2 \bar{\Sigma}_2. \tag{3.28}$$

Then, Eqs. (3.13) and (3.14) are satisfied if the amplitude ψ_0 is an eigenspinor of D, with eigenvalue zero. The most general form for ψ_0 is

$$\psi_0(x, v) = \sqrt{\mathscr{I}_0(x)} \left[z_1(x) \Sigma_1(x, v) + z_2(x) \Sigma_2(x, v) \right] = \sqrt{\mathscr{I}_0(x)} z_A \Sigma_A, \tag{3.29}$$

where z_1 and z_2 are scalar coefficients, satisfying the constraint

$$\bar{z}_1 z_1 + \bar{z}_2 z_2 = \bar{z}_A z_A = 1. \tag{3.30}$$

Note that, since the matrix D explicitly depends on v_μ, its eigenspinors will in general also depend on v_μ. As mentioned in Sec. 3.2, this can be viewed as a consequence of the amplitude ψ being defined on the Lagrangian submanifold determined by v_μ.

3.3.3. Ray equations

Equations (3.18) and (3.20) is a system of partial differential equations

$$\frac{1}{2} g^{\alpha\beta} \left(k_\alpha + e\mathscr{A}_\alpha \right) \left(k_\beta + e\mathscr{A}_\beta \right) = -\frac{m^2}{2}, \tag{3.31}$$

$$\nabla_\alpha \left[\mathscr{I}_0 \left(k^\alpha + e\mathscr{A}^\alpha \right) \right] = 0, \tag{3.32}$$

where $k_\alpha = \nabla_\alpha S$, and the unknowns are S and $\mathscr{I}_0$. The first equation is a Hamilton-Jacobi equation for the phase function S, and the second equation is a transport equation for the intensity $\mathscr{I}_0$ [125]. The Hamilton-Jacobi equation can be solved using the method of characteristics. This is done by defining a Hamiltonian function H on T^*M, such that

$$H(x, \nabla S) = \frac{1}{2} g^{\alpha\beta} \left(k_\alpha + e\mathscr{A}_\alpha \right) \left(k_\beta + e\mathscr{A}_\beta \right) = -\frac{m^2}{2}. \tag{3.33}$$

In this case, the Hamiltonian is

$$H(x, p) = \frac{1}{2} g^{\alpha\beta} \left(p_\alpha + e\mathscr{A}_\alpha \right) \left(p_\beta + e\mathscr{A}_\beta \right), \tag{3.34}$$

and Hamilton's equations take the following form:

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = p^\mu + e\mathscr{A}^\mu, \tag{3.35}$$

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2} g^{\alpha\beta}{}_{,\mu} \left(p_\alpha + e\mathscr{A}_\alpha\right)\left(p_\beta + e\mathscr{A}_\beta\right) - e\left(p^\alpha + e\mathscr{A}^\alpha\right)\mathscr{A}_{\alpha,\mu}. \tag{3.36}$$

Introducing the kinetic momentum $v_\alpha = p_\alpha + e\mathscr{A}_\alpha$, Hamilton's equations can be written in the more compact form

$$\dot{x}^\mu = v^\mu, \tag{3.37}$$

$$\dot{v}_\mu = -\frac{1}{2} g^{\alpha\beta}{}_{,\mu} v_\alpha v_\beta + e v^\alpha \mathscr{F}_{\alpha\mu}. \tag{3.38}$$

Given a solution $\{x^\mu(\tau), p_\mu(\tau)\}$ for Hamilton's equations, we obtain a solution of the Hamilton-Jacobi equation (3.33) by taking [90]:

$$S(x^\mu(\tau_1), p_\mu(\tau_1)) = \int_{\tau_0}^{\tau_1} d\tau L(x, \dot{x}, p, \dot{p}) + \text{const.}, \tag{3.39}$$

where

$$L(x, \dot{x}, p, \dot{p}) = \dot{x}^\mu p_\mu - H(x, p) \tag{3.40}$$

is the corresponding Lagrangian. The ray equations (3.35) and (3.36) can also be obtained as the Euler-Lagrange equations corresponding to the Lagrangian L.

Once the Hamilton-Jacobi equation is solved, the transport equation (3.32) can also be analyzed (See Ref. [125]). However, our main interest is in the ray equations governed by the Hamiltonian (3.34) or by the Lagrangian (3.40). The ray equations (3.35) and (3.36) describe timelike trajectories of massive charged particles. These equations can easily be rewritten as

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta - e\dot{x}^\alpha \mathscr{F}_\alpha{}^\mu = 0, \tag{3.41}$$

or in the explicitly covariant form:

$$v^\alpha \nabla_\alpha v^\mu = \dot{x}^\alpha \nabla_\alpha \dot{x}^\mu = e\dot{x}^\alpha \mathscr{F}_\alpha{}^\mu. \tag{3.42}$$

3.3.4. WKB equations at order $\hbar^1$

Considering the Euler-Lagrange equations (3.10) and (3.11) at order $\hbar^1$ only, we obtain

$$D\psi_1 = -i\gamma^\mu \nabla_\mu \psi_0, \tag{3.43}$$

$$\bar{\psi}_1 D = i\nabla_\mu \bar{\psi}_0 \gamma^\mu. \tag{3.44}$$

Given ψ_0, these are two inhomogeneous systems of linear algebraic equations, where the unknowns are ψ_1 and $\bar{\psi}_1$. For any inhomogeneous system, the general solution can be written as the sum of the solution for the homogeneous system and a particular solution for the inhomogeneous system. We can write ψ_1 as

$$\psi_1(x, v) = b_1(x)\Sigma_1(x, v) + b_2(x)\Sigma_2(x, v) + \psi_p(x, v), \tag{3.45}$$

where $b_{1,2}$ are scalar coefficients, and ψ_p is a particular solution of the inhomogeneous system. The system will admit nontrivial solutions if and only if the right hand side of the inhomogeneous equation is orthogonal to all solutions of the transposed homogeneous equation. Such solutions are always a linear combination of Σ_1 and Σ_2. Therefore, we have the following solvability conditions [146, 9, 147], which impose constraints on ψ_0:

$$\bar{\Sigma}_1 \gamma^\mu \nabla_\mu \psi_0 = \bar{\Sigma}_2 \gamma^\mu \nabla_\mu \psi_0 = 0, \tag{3.46}$$
$$\nabla_\mu \bar{\psi}_0 \gamma^\mu \Sigma_1 = \nabla_\mu \bar{\psi}_0 \gamma^\mu \Sigma_2 = 0. \tag{3.47}$$

We can rewrite these equations by using the expansion of ψ_0 given in Eq. (3.29) and the transport equation (3.18) as

$$\begin{aligned} v^\mu \nabla_\mu z_A &= \frac{1}{2}\delta_{AB} z_B \nabla_\mu v^\mu - m\bar{\Sigma}_A \gamma^\mu \nabla_\mu \Sigma_B z_B, \\ v^\mu \nabla_\mu \bar{z}_B &= \frac{1}{2}\bar{z}_A \delta_{AB} \nabla_\mu v^\mu - m\bar{z}_A \bar{\Sigma}_A \gamma^\mu \nabla_\mu \Sigma_B. \end{aligned} \tag{3.48}$$

Using the identities

$$\bar{\Sigma}_A \gamma^\mu \Sigma_B = \frac{1}{m}\delta_{AB} v^\mu \quad \Rightarrow \quad \delta_{AB} \nabla_\mu v^\mu = m\left(\nabla_\mu \bar{\Sigma}_A \gamma^\mu \Sigma_B + \bar{\Sigma}_A \gamma^\mu \nabla_\mu \Sigma_B\right), \tag{3.49}$$

we obtain

$$\begin{aligned} v^\mu \nabla_\mu z_A &= iM_{AB} z_B, \\ v^\mu \nabla_\mu \bar{z}_B &= -i\bar{z}_A M_{AB}, \end{aligned} \tag{3.50}$$

where the 2×2 hermitian matrix M has components

$$M_{AB} = \frac{im}{2}\left(\bar{\Sigma}_A \gamma^\mu \nabla_\mu \Sigma_B - \nabla_\mu \bar{\Sigma}_A \gamma^\mu \Sigma_B\right). \tag{3.51}$$

Using the properties of the eigenspinors, given in Eqs. (3.24)-(3.27), the matrix components M_{AB} can be rewritten in the following way:

$$M_{AB} = \frac{i}{2} v^\mu \left(\bar{\Sigma}_A \nabla_\mu \Sigma_B - \nabla_\mu \bar{\Sigma}_A \Sigma_B\right) - \frac{e}{4}\mathscr{F}_{\mu\nu} \bar{\Sigma}_A \sigma^{\mu\nu} \Sigma_B. \tag{3.52}$$

Here, the first term represents the Berry connection, and the second term represents the so-called "no-name" term. The "no-name" term was first introduced in a general context by Littlejohn and Flynn in Ref. [117], and its role in the WKB approximation to the Dirac equation was discussed in Ref. [162].

We can write Eq. (3.50) in a more compact form by introducing the following 2-dimensional unit complex vectors:

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \bar{z} = \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \end{pmatrix}. \tag{3.53}$$

We also introduce the following notation for the Berry connection and the spin tensor:

$$\begin{aligned} B_{\mu AB}(x,v) &= \frac{i}{2}\left(\bar{\Sigma}_A \nabla_\mu \Sigma_B - \nabla_\mu \bar{\Sigma}_A \Sigma_B\right), \\ S^{\mu\nu}{}_{AB}(x,v) &= \frac{1}{2}\bar{\Sigma}_A \sigma^{\mu\nu}\Sigma_B, \end{aligned} \tag{3.54}$$

where $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. The Berry connection B_μ is a 2×2 matrix-valued one-form, while $S^{\mu\nu}$ is a 2×2 matrix-valued tensor. Depending on the context, we will sometimes omit the matrix indices A, B. Then, Eq. (3.50) can be written as

$$\begin{aligned} v^\mu \nabla_\mu z &= i\left(v^\mu B_\mu - \frac{e}{2}\mathscr{F}_{\mu\nu}S^{\mu\nu}\right) z, \\ v^\mu \nabla_\mu \bar{z} &= -i\bar{z}\left(v^\mu B_\mu - \frac{e}{2}\mathscr{F}_{\mu\nu}S^{\mu\nu}\right). \end{aligned} \tag{3.55}$$

If we restrict z to a worldline $x^\mu(\tau)$, which is a solution of the ray equations (3.37) and (3.38), we can write

$$\begin{aligned} \dot{z} &= i\left(v^\mu B_\mu - \frac{e}{2}\mathscr{F}_{\mu\nu}S^{\mu\nu}\right) z, \\ \dot{\bar{z}} &= -i\bar{z}\left(v^\mu B_\mu - \frac{e}{2}\mathscr{F}_{\mu\nu}S^{\mu\nu}\right). \end{aligned} \tag{3.56}$$

These equations describe the evolution of the spin degree of freedom along the worldline $x^\mu(\tau)$.

It is important to emphasize how the covariant derivatives act on the eigenspinors Σ_A, which are defined on the Lagrangian submanifold. Using the horizontal and vertical derivatives defined in Appendix A.1, we obtain

$$\begin{aligned} v^\mu \nabla_\mu \Sigma_A &= v^\mu \nabla_\mu \left[\Sigma_A(x,v)\right] \\ &= v^\mu \left(\overset{h}{\nabla}_\mu \Sigma_A\right)(x,v) + v^\mu \left(\nabla_\mu v_\nu\right)\left(\overset{v}{\nabla}{}^\nu \Sigma_A\right)(x,v) \\ &= v^\mu \overset{h}{\nabla}_\mu \Sigma_A + e v^\mu \mathscr{F}_{\mu\nu}\overset{v}{\nabla}{}^\nu \Sigma_A. \end{aligned} \tag{3.57}$$

The expression for the Berry connection becomes

$$v^\mu B_{\mu AB} = \frac{i}{2} v^\mu \left(\bar{\Sigma}_A \overset{h}{\nabla}_\mu \Sigma_B - \overset{h}{\nabla}_\mu \bar{\Sigma}_A \Sigma_B \right) + \frac{ie}{2} v^\mu \mathscr{F}_{\mu\nu} \left(\bar{\Sigma}_A \overset{v}{\nabla}^\nu \Sigma_B - \overset{v}{\nabla}^\nu \bar{\Sigma}_A \Sigma_B \right). \tag{3.58}$$

3.3.5. Geometric definition of the Berry connection

A general discussion about the geometry of the transport equation arising from the WKB approximation of multicomponent wave equations can be found in Ref. [76] (see also Refs. [117, 75, 41]). Here, we specialize this discussion to the case of the Dirac equation, focusing on the geometry of the Berry connection and the corresponding Berry curvature.

The WKB approximation of multicomponent wave equations generally results in a Hamilton-Jacobi equation for the phase S and a transport equation for the amplitude ψ_0. In the present case, the Hamilton-Jacobi equation for S was discussed in Sec. 3.3.3, and the transport equation for ψ_0 is split into two parts. The first one describes the evolution of the intensity $\mathscr{I}_0$ and is given in Eq. (3.18), while the second one describes the evolution of the spin degree of freedom, as presented in Eq. (3.56).

Since the amplitude ψ_0 is defined on the Lagrangian submanifold, the Berry connection has to be defined as a connection on an appropriate 4-spinor bundle with the base space the Lagrangian submanifold. Furthermore, the amplitude ψ_0 is an eigenspinor of D, with eigenvalue $\lambda = 0$, so the appropriate bundle is then the λ-eigenbundle of the 4-spinor bundle with base space the Lagrangian submanifold. Then, the Berry connection has to be a Lie algebra-valued one-form defined on the Lagrangian submanifold. This is clearly not the case for B_μ, which is a Lie algebra-valued one-form on spacetime.

A connection one-form defined on the Lagrangian submanifold has to be contracted with a tangent vector to the Lagrangian submanifold. By construction, the tangent vectors to the Lagrangian submanifold are the Hamiltonian vector fields. Working in (x, v) coordinates, the Hamiltonian vector field corresponding to the ray equations (3.37) and (3.38) is

$$X_H = v^\mu \frac{\partial}{\partial x^\mu} + \left(\Gamma^\rho_{\nu\mu} v_\rho v^\nu + e v^\nu \mathscr{F}_{\nu\mu} \right) \frac{\partial}{\partial v_\mu}. \tag{3.59}$$

We can obtain the appropriate definition of the Berry connection as follows. Using the definition of the horizontal derivative, we can rewrite Eq. (3.58) as

$$\begin{aligned} v^\mu B_{\mu AB} &= \frac{i}{2} v^\mu \left(\bar{\Sigma}_A \nabla_\mu \Sigma_B - \nabla_\mu \bar{\Sigma}_A \Sigma_B \right) \\ &\quad + \frac{i}{2} \left(\Gamma^\rho_{\nu\mu} v_\rho v^\nu + e v^\nu \mathscr{F}_{\nu\mu} \right) \left(\bar{\Sigma}_A \overset{v}{\nabla}{}^\mu \Sigma_B - \overset{v}{\nabla}{}^\mu \bar{\Sigma}_A \Sigma_B \right) \\ &= \mathscr{B}_{AB}(X_H), \end{aligned} \tag{3.60}$$

where

$$\mathscr{B}_{AB} = \frac{i}{2} \left(\bar{\Sigma}_A \nabla_\mu \Sigma_B - \nabla_\mu \bar{\Sigma}_A \Sigma_B \right) dx^\mu + \frac{i}{2} \left(\bar{\Sigma}_A \overset{v}{\nabla}{}^\mu \Sigma_B - \overset{v}{\nabla}{}^\mu \bar{\Sigma}_A \Sigma_B \right) dv_\mu \tag{3.61}$$

are the components of the appropriately defined Berry connection, which is a Lie algebra-valued one-form defined on the Lagrangian submanifold. The corresponding Lie algebra is $\mathfrak{u}(2)$, since the one-form $\mathscr{B}$ takes values in the space of two-dimensional Hermitian matrices.

The curvature of the connection $\mathscr{B}$ can be calculated using the standard definition [51, Sec. 2.3.2]

$$F = d\mathscr{B} - i[\mathscr{B}, \mathscr{B}]. \tag{3.62}$$

This is the Berry curvature, and it plays an important role in the spin Hall effect correction to the ray equations, as we will discuss in the next sections. In coordinates, the expression of the Berry curvature is

$$F = (F_{xx})_{\mu\nu} dx^\mu dx^\nu + (F_{px})_\mu{}^\nu dx^\mu dv_\nu + (F_{xp})^\mu{}_\nu dv_\mu dx^\nu + (F_{pp})_{\mu\nu} dv^\mu dv^\nu, \tag{3.63}$$

where

$$(F_{xx})_{\mu\nu} = \frac{\partial (\mathscr{B}_x)_\nu}{\partial x^\mu} - \frac{\partial (\mathscr{B}_x)_\mu}{\partial x^\nu} - i \left[(\mathscr{B}_x)_\mu, (\mathscr{B}_x)_\nu \right], \tag{3.64}$$

$$(F_{pp})^{\mu\nu} = \frac{\partial (\mathscr{B}_p)^\nu}{\partial v_\mu} - \frac{\partial (\mathscr{B}_p)^\mu}{\partial v_\nu} - i \left[(\mathscr{B}_p)^\mu, (\mathscr{B}_p)^\nu \right], \tag{3.65}$$

$$(F_{px})_\mu{}^\nu = - (F_{xp})^\nu{}_\mu = \frac{\partial (\mathscr{B}_p)^\nu}{\partial x^\mu} - \frac{\partial (\mathscr{B}_x)_\mu}{\partial v_\nu} - i \left[(\mathscr{B}_x)_\mu, (\mathscr{B}_p)^\nu \right], \tag{3.66}$$

and

$$(\mathscr{B}_x)_{\mu AB} = \frac{i}{2} \left(\bar{\Sigma}_A \nabla_\mu \Sigma_B - \nabla_\mu \bar{\Sigma}_A \Sigma_B \right), \tag{3.67}$$

$$(\mathscr{B}_p)^\mu{}_{AB} = \frac{i}{2} \left(\bar{\Sigma}_A \overset{v}{\nabla}{}^\mu \Sigma_B - \overset{v}{\nabla}{}^\mu \bar{\Sigma}_A \Sigma_B \right). \tag{3.68}$$

Using the properties of the eigenspinors, given in Eqs. (3.24)-(3.27), the components of the Berry curvature can be explicitly computed, as shown in Appendix A.5:

$$(F_{xx})_{\mu\nu} = -\frac{1}{2}R_{\mu\nu\alpha\beta}S^{\alpha\beta} + \frac{1}{m^2}v_\rho v_\sigma \Gamma^\rho_{\alpha\mu}\Gamma^\sigma_{\beta\nu}S^{\alpha\beta}, \tag{3.69}$$

$$(F_{pp})^{\mu\nu} = \frac{1}{m^2}S^{\mu\nu}, \tag{3.70}$$

$$(F_{px})_\mu{}^\nu = -(F_{xp})^\nu{}_\mu = -\frac{1}{m^2}v_\rho \Gamma^\rho_{\mu\alpha}S^{\alpha\nu}. \tag{3.71}$$

3.4. Effective dispersion relation and spin-orbit coupling

In the standard WKB treatment, the equations for each individual order in $\hbar$ are set to zero. The resulting ray equations (3.37) and (3.38) are used for determining the transport of the spin degree of freedom, through Eq. (3.56). However, with this approach there is no backreaction from the dynamics of $z(\tau)$ on the rays $x^\mu(\tau)$ and $v(\tau)$. In order to properly take into account the spin-orbit coupling between the spin dynamics and the ray dynamics, we derive here an effective dispersion relation, containing $\mathcal{O}(\hbar^1)$ corrections to the dispersion relation given in Eq. (3.20). This represents a weaker condition when compared to the standard WKB treatment, where terms of different order in $\hbar$ are set to zero individually. Instead, here we only require that the combined sum of the terms of order $\hbar^0$ and $\hbar^1$ vanishes. The effective dispersion relation is obtained by taking the Euler-Lagrange equations (3.10)-(3.12), but without treating terms of different orders in $\hbar$ separately. The effective dispersion relation is then treated as an effective Hamilton-Jacobi equation, and the resulting ray equations contain spin-dependent correction terms, describing the gravitational spin Hall effect of Dirac particles.

3.4.1. Effective dispersion relation

Starting with Eqs. (3.10)-(3.12), we can write

$$\bar{\psi}\left(\gamma^\mu v_\mu + m\right)\psi - i\hbar\bar{\psi}\gamma^\mu\nabla_\mu\psi = \mathcal{O}(\hbar^2), \tag{3.72}$$

$$\bar{\psi}\left(\gamma^\mu v_\mu + m\right)\psi + i\hbar\nabla_\mu\bar{\psi}\gamma^\mu\psi = \mathcal{O}(\hbar^2). \tag{3.73}$$

By adding these two equations, we obtain

$$v_\mu\bar{\psi}\gamma^\mu\psi + m\bar{\psi}\psi - \frac{i\hbar}{2}\left(\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi\right) = \mathcal{O}(\hbar^2) \tag{3.74}$$

The Dirac current ${j_1}^\mu = \bar{\psi}\gamma^\mu\psi + \mathcal{O}(\hbar^2)$ can be rewritten in a different form. Using Eqs. (3.10) and (3.11), we can write

$$\begin{aligned}\bar{\psi}\gamma^\mu\left(m\psi\right) &= \bar{\psi}\gamma^\mu\left(-\gamma^\nu v_\nu\psi + i\hbar\gamma^\nu\nabla_\nu\psi\right) + \mathcal{O}(\hbar^2)\\ \left(\bar{\psi}m\right)\gamma^\mu\psi &= \left(-\bar{\psi}\gamma^\nu v_\nu - i\hbar\nabla_\nu\bar{\psi}\gamma^\nu\right)\gamma^\mu\psi + \mathcal{O}(\hbar^2)\end{aligned} \tag{3.75}$$

Adding these two equations, we obtain

$$\begin{aligned}{j_1}^\mu &= -\frac{1}{2m}v_\nu\bar{\psi}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\psi + \frac{i\hbar}{2m}\left(\bar{\psi}\gamma^\mu\gamma^\nu\nabla_\nu\psi - \nabla_\nu\bar{\psi}\gamma^\nu\gamma^\mu\psi\right) + \mathcal{O}(\hbar^2)\\ &= \frac{1}{m}v^\mu\bar{\psi}\psi - \frac{i\hbar}{2m}g^{\mu\nu}\left(\bar{\psi}\nabla_\nu\psi - \nabla_\nu\bar{\psi}\psi\right) + \frac{\hbar}{2m}\nabla_\nu\left(\bar{\psi}\sigma^{\mu\nu}\psi\right) + \mathcal{O}(\hbar^2).\end{aligned} \tag{3.76}$$

Using the above form of ${j_1}^\mu$, Eq. (3.74) can be rewritten as

$$\begin{aligned}\frac{1}{m}\bar{\psi}\psi v_\mu v^\mu - \frac{i\hbar}{2m}v^\mu\left(\bar{\psi}_0\nabla_\mu\psi_0 - \nabla_\mu\bar{\psi}_0\psi_0\right) + \frac{\hbar}{2m}v_\mu\nabla_\nu\left(\bar{\psi}_0\sigma^{\mu\nu}\psi_0\right)&\\ - \frac{i\hbar}{2}\left(\bar{\psi}_0\gamma^\mu\nabla_\mu\psi_0 - \nabla_\mu\bar{\psi}_0\gamma^\mu\psi_0\right) = -m\bar{\psi}\psi + \mathcal{O}(\hbar^2).&\end{aligned} \tag{3.77}$$

We expand the $\mathcal{O}(\hbar)$ terms using Eq. (3.29):

$$\frac{i\hbar}{2m}v^\mu\left(\bar{\psi}_0\nabla_\mu\psi_0 - \nabla_\mu\bar{\psi}_0\psi_0\right) = \frac{i\hbar\mathcal{I}_0}{2m}v^\mu\left(z^\dagger\nabla_\mu z - \nabla_\mu z^\dagger z\right) + \frac{\hbar\mathcal{I}_0}{m}v^\mu z^\dagger B_\mu z. \tag{3.78}$$

Using Eqs. (3.13) and (3.14), we can write:

$$\begin{aligned}\frac{\hbar}{2m}v_\mu\nabla_\nu\left(\bar{\psi}_0\sigma^{\mu\nu}\psi_0\right) &= \frac{\hbar}{2m}\nabla_\nu\left(v_\mu\bar{\psi}_0\sigma^{\mu\nu}\psi_0\right) - \frac{\hbar}{2m}\bar{\psi}_0\sigma^{\mu\nu}\psi\nabla_\nu v_\mu\\ &= \frac{i\hbar}{4m}\nabla_\nu\left(v_\mu\bar{\psi}_0\gamma^\mu\gamma^\nu\psi_0 - v_\mu\bar{\psi}_0\gamma^\nu\gamma^\mu\psi_0\right) + \frac{e\hbar}{4m}\mathscr{F}_{\mu\nu}\bar{\psi}_0\sigma^{\mu\nu}\psi\\ &= -\frac{i\hbar}{4m}\nabla_\nu\left(m\bar{\psi}_0\gamma^\nu\psi_0 - m\bar{\psi}_0\gamma^\nu\psi_0\right) + \frac{e\hbar}{4m}\mathscr{F}_{\mu\nu}\bar{\psi}_0\sigma^{\mu\nu}\psi\\ &= \frac{e\hbar\mathcal{I}_0}{2m}\mathscr{F}_{\mu\nu}z^\dagger S^{\mu\nu}z.\end{aligned} \tag{3.79}$$

We also have

$$\begin{aligned}\frac{i\hbar}{2}\left(\bar{\psi}_0\gamma^\mu\nabla_\mu\psi_0 - \nabla_\mu\bar{\psi}_0\gamma^\mu\psi_0\right) = \frac{i\hbar\mathcal{I}_0}{2m}v^\mu\left(z^\dagger\nabla_\mu z - \nabla_\mu z^\dagger z\right)&\\ + \frac{\hbar\mathcal{I}_0}{m}v^\mu z^\dagger B_\mu z - \frac{e\hbar\mathcal{I}_0}{2m}\mathscr{F}_{\mu\nu}z^\dagger S^{\mu\nu}z.&\end{aligned} \tag{3.80}$$

Combining the above equations, we obtain the effective dispersion relation

$$\frac{1}{2}v_\mu v^\mu - \frac{i\hbar}{2}v^\mu\left(\bar{z}\nabla_\mu z - \nabla_\mu\bar{z}z\right) - \hbar v^\mu\bar{z}B_\mu z + \frac{e\hbar}{2}\mathscr{F}_{\mu\nu}\bar{z}S^{\mu\nu}z = -\frac{m^2}{2}. \tag{3.81}$$

3.5. Effective ray equations

In this section we derive effective ray equations containing spin-dependent correction terms. These equations are meant to describe the gravitational spin Hall effect of electrons. We start with the effective dispersion relation (3.81) and treat it as an effective Hamilton-Jacobi equation for the phase function S.

$$\begin{aligned} &\frac{1}{2}g^{\alpha\beta}\left(k_\alpha + e\mathscr{A}_\alpha\right)\left(k_\beta + e\mathscr{A}_\beta\right) - \frac{i\hbar}{2}\left(\bar{z}\dot{z} - \dot{\bar{z}}z\right) \\ &\qquad - \hbar\left(k^\alpha + e\mathscr{A}^\alpha\right)\bar{z}B_\alpha z + \frac{e\hbar}{2}\mathscr{F}_{\alpha\beta}\bar{z}S^{\alpha\beta}z = -\frac{m^2}{2} + \mathscr{O}(\hbar^2). \end{aligned} \tag{3.82}$$

We define the corresponding Hamiltonian function

$$\begin{aligned} H(x,p,z,\bar{z},\dot{z},\dot{\bar{z}}) = &\frac{1}{2}g^{\alpha\beta}\left(p_\alpha + e\mathscr{A}_\alpha\right)\left(p_\beta + e\mathscr{A}_\beta\right) - \frac{i\hbar}{2}\left(\bar{z}\dot{z} - \dot{\bar{z}}z\right) \\ &- \hbar\left(p^\alpha + e\mathscr{A}^\alpha\right)\bar{z}B_\alpha z + \frac{e\hbar}{2}\mathscr{F}_{\alpha\beta}\bar{z}S^{\alpha\beta}z, \end{aligned} \tag{3.83}$$

and we solve for the phase function S as in Sec. 3.3.3:

$$S(x^\alpha(\tau_1), p_\alpha(\tau_1), z(\tau_1), \bar{z}(\tau_1)) = \int_{\tau_0}^{\tau_1} d\tau L(x,\dot{x},p,\dot{p},z,\dot{z},\bar{z},\dot{\bar{z}}) + \text{const.}, \tag{3.84}$$

where the Lagrangian is

$$\begin{aligned} L = \dot{x}^\alpha p_\alpha - &\frac{1}{2}g^{\alpha\beta}\left(p_\alpha + e\mathscr{A}_\alpha\right)\left(p_\beta + e\mathscr{A}_\beta\right) + \frac{i\hbar}{2}\left(\bar{z}\dot{z} - \dot{\bar{z}}z\right) \\ &+ \hbar\left(p^\alpha + e\mathscr{A}^\alpha\right)\bar{z}B_\alpha z - \frac{e\hbar}{2}\mathscr{F}_{\alpha\beta}\bar{z}S^{\alpha\beta}z. \end{aligned} \tag{3.85}$$

Note that the Lagrangian is a scalar function defined on $T(T^*M \times \mathbb{C}^2)$, and the effective ray dynamics is given by the Euler-Lagrange equations

$$\frac{\partial L}{\partial u} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{u}} = 0, \tag{3.86}$$

where $u \in \{x^\mu, p_\mu, z, \bar{z}\}$. The Euler-Lagrange equations are

$$\dot{x}^\mu = v^\mu - \hbar\bar{z}B^\mu z - \hbar v^\alpha \bar{z}\frac{\partial B_\alpha}{\partial p_\mu}z + \frac{e\hbar}{2}\mathscr{F}_{\alpha\beta}\bar{z}\frac{\partial S^{\alpha\beta}}{\partial p_\mu}z, \tag{3.87}$$

$$\begin{aligned} \dot{p}_\mu = -\frac{1}{2}g^{\alpha\beta}{}_{,\mu}v_\alpha v_\beta - ev^\alpha\mathscr{A}_{\alpha,\mu} + \hbar v^\alpha\bar{z}B_{\alpha,\mu}z \\ + e\hbar\mathscr{A}^\alpha{}_{,\mu}\bar{z}B_\alpha z - \frac{e\hbar}{2}\bar{z}(\mathscr{F}_{\alpha\beta}S^{\alpha\beta})_{,\mu}z, \end{aligned} \tag{3.88}$$

$$\dot{z} = i\left(v^\mu B_\mu - \frac{e}{2}\mathscr{F}_{\mu\nu}S^{\mu\nu}\right)z, \tag{3.89}$$
$$\dot{\bar{z}} = -i\bar{z}\left(v^\mu B_\mu - \frac{e}{2}\mathscr{F}_{\mu\nu}S^{\mu\nu}\right). \tag{3.90}$$

These equations contain spin-dependent correction terms of $\mathscr{O}(h^1)$ to the ray equations obtained in Eqs. (3.35) and (3.36). The $\mathscr{O}(h^1)$ terms reflect the spin-orbit coupling between the external and internal degrees of freedom, resulting in the gravitational spin Hall effect of localized Dirac wave packets.

3.5.1. Noncanonical coordinates

The effective ray equations (3.87)-(3.90) can also be formulated as a Hamiltonian system on the symplectic manifold $T^*M \times \mathbb{C}^2$. Considering canonical coordinates $(x, p, z, \bar{z})$, the corresponding Hamiltonian function is

$$H(x,p,z,\bar{z}) = \frac{1}{2}g^{\alpha\beta}\left(p_\alpha + e\mathscr{A}_\alpha\right)\left(p_\beta + e\mathscr{A}_\beta\right) - \hbar\left(p^\alpha + e\mathscr{A}^\alpha\right)\bar{z}B_\alpha z + \frac{e\hbar}{2}\mathscr{F}_{\alpha\beta}\bar{z}S^{\alpha\beta}z, \tag{3.91}$$

and the symplectic 2-form is

$$\Omega = dx^\alpha \wedge dp_\alpha + i\hbar dz \wedge d\bar{z}. \tag{3.92}$$

In this symplectic setup, Hamilton's equations are [1, Sec. 3.3]

$$\Omega(X_H, \,\cdot\,) = dH, \tag{3.93}$$

where the Hamiltonian vector field X_H can be expressed in coordinate form as

$$X_H = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}_\mu \frac{\partial}{\partial p_\mu} + \dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}. \tag{3.94}$$

By solving for the components of the Hamiltonian vector field, we obtain the effective ray equations (3.87)-(3.90) in the following form:

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial p_\mu}, & \dot{p}_\mu &= -\frac{\partial H}{\partial x^\mu}, \\ \dot{z} &= -\frac{i}{\hbar}\frac{\partial H}{\partial \bar{z}}, & \dot{\bar{z}} &= \frac{i}{\hbar}\frac{\partial H}{\partial z}. \end{aligned} \tag{3.95}$$

As a first step towards noncanonical coordinates, we rewrite the Hamiltonian, symplectic form and effective ray equations in the new coordinate system $(x, v, z, \bar{z})$, where $v_\mu = p_\mu + e\mathscr{A}_\mu$. In these new coordinates, the Hamiltonian is

$$H(x,v,z,\bar{z}) = \frac{1}{2}g^{\alpha\beta}v_\alpha v_\beta - \hbar v_\alpha \bar{z}B_\alpha z + \frac{e\hbar}{2}\mathscr{F}_{\alpha\beta}\bar{z}S^{\alpha\beta}z. \tag{3.96}$$

Applying the standard coordinate transformation rules for 2-forms, the symplectic form can be expressed in the new coordinates $(x, v, z, \bar{z})$ as

$$\Omega = e\mathscr{F}_{\alpha\beta}dx^\alpha dx^\beta + dx^\alpha \wedge dv_\alpha + i\hbar dz \wedge d\bar{z}. \tag{3.97}$$

Using Eq. (3.93), we can obtain the effective ray equations as the components of the Hamiltonian vector field in the new coordinates:

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial v_\mu}, & \dot{v}_\mu &= -\frac{\partial H}{\partial x^\mu} + e\dot{x}^\nu \mathscr{F}_{\nu\mu}, \\ \dot{z} &= -\frac{i}{\hbar}\frac{\partial H}{\partial \bar{z}}, & \dot{\bar{z}} &= \frac{i}{\hbar}\frac{\partial H}{\partial z}. \end{aligned} \tag{3.98}$$

In order to eliminate the Berry connection from the Hamiltonian, we perform the following coordinate transformation:

$$X^\mu = x^\mu + \hbar\bar{z}(\mathscr{B}_p)^\mu z, \tag{3.99}$$

$$P_\mu = v_\mu - \hbar\bar{z}(\mathscr{B}_x)_\mu z - e\hbar\mathscr{F}_{\mu\nu}\bar{z}(\mathscr{B}_p)^\nu z. \tag{3.100}$$

The Hamiltonian function becomes

$$\begin{aligned} H(x^\mu, v_\mu, z, \bar{z}) &= H(X^\mu - \hbar\bar{z}(\mathscr{B}_p)^\mu z, P_\mu + \hbar\bar{z}(\mathscr{B}_x)_\mu z + e\hbar\mathscr{F}_{\mu\nu}\bar{z}(\mathscr{B}_p)^\nu z, z, \bar{z}) \\ &= H(X^\mu, P_\mu, z, \bar{z}) - \hbar\frac{\partial H}{\partial X^\mu}\bar{z}(\mathscr{B}_p)^\mu z \\ &\quad + \hbar\frac{\partial H}{\partial P_\mu}\left[\bar{z}(\mathscr{B}_x)_\mu z + e\mathscr{F}_{\mu\nu}\bar{z}(\mathscr{B}_p)^\nu z\right] + \mathscr{O}(\hbar^2). \end{aligned} \tag{3.101}$$

Thus, the Hamiltonian in the noncanonical coordinates $(X, P, z, \bar{z})$ is

$$H(X, P, z, \bar{z}) = \frac{1}{2}g^{\alpha\beta}P_\alpha P_\beta + \frac{e\hbar}{2}\mathscr{F}_{\alpha\beta}\bar{z}S^{\alpha\beta}z + \mathscr{O}(\hbar^2). \tag{3.102}$$

Applying the same coordinate transformation to the symplectic form, we obtain:

$$\begin{aligned} \Omega = {} & e\mathscr{F}_{\alpha\beta}dX^\alpha dX^\beta + dX^\alpha \wedge dP_\alpha + i\hbar dz \wedge d\bar{z} \\ & - \hbar\bar{z}\left[\frac{\partial(\mathscr{B}_x)_\beta}{\partial X^\alpha} - \frac{\partial(\mathscr{B}_x)_\alpha}{\partial X^\beta}\right] z dX^\alpha dX^\beta - \hbar\bar{z}\left[\frac{\partial(\mathscr{B}_p)^\beta}{\partial X^\alpha} - \frac{\partial(\mathscr{B}_x)_\alpha}{\partial P_\beta}\right] z dX^\alpha dP_\beta \\ & - \hbar\bar{z}\left[\frac{\partial(\mathscr{B}_x)_\beta}{\partial P_\alpha} - \frac{\partial(\mathscr{B}_p)^\alpha}{\partial X^\beta}\right] z dP_\alpha dX^\beta - \hbar\bar{z}\left[\frac{\partial(\mathscr{B}_p)^\beta}{\partial P_\alpha} - \frac{\partial(\mathscr{B}_p)^\alpha}{\partial P_\beta}\right] z dP_\alpha dP_\beta \\ & + \hbar\bar{z}(\mathscr{B}_x)_\alpha dX^\alpha \wedge dz + \hbar(\mathscr{B}_x)_\alpha z dX^\alpha \wedge d\bar{z} \\ & + \hbar\bar{z}(\mathscr{B}_p)^\alpha dP_\alpha \wedge dz + \hbar(\mathscr{B}_p)^\alpha z dP_\alpha \wedge d\bar{z} + \mathscr{O}(\hbar^2). \end{aligned} \tag{3.103}$$

The effective ray equations in noncanonical coordinates $(X, P, z, \bar{z})$ are

$$\begin{aligned}
\dot{X}^\mu &= \frac{\partial H}{\partial P_\mu} + \hbar \dot{X}^\nu \bar{z} \left[\frac{\partial (\mathscr{B}_p)^\mu}{\partial X^\nu} - \frac{\partial (\mathscr{B}_x)_\nu}{\partial P_\mu} \right] z \\
&\quad + \hbar \dot{P}_\nu \bar{z} \left[\frac{\partial (\mathscr{B}_p)^\mu}{\partial P_\nu} - \frac{\partial (\mathscr{B}_p)^\nu}{\partial P_\mu} \right] z + \hbar \bar{z} (\mathscr{B}_p)^\mu \dot{z} + \hbar \dot{\bar{z}} (\mathscr{B}_p)^\mu z \qquad (3.104) \\
\dot{P}_\mu &= -\frac{\partial H}{\partial X_\mu} + e \dot{X}^\nu \mathscr{F}_{\nu\mu} - \hbar \dot{X}^\nu \bar{z} \left[\frac{\partial (\mathscr{B}_x)_\mu}{\partial X^\nu} - \frac{\partial (\mathscr{B}_x)_\nu}{\partial X^\mu} \right] z \\
&\quad - \hbar \dot{P}_\nu \bar{z} \left[\frac{\partial (\mathscr{B}_x)_\mu}{\partial P_\nu} - \frac{\partial (\mathscr{B}_p)^\nu}{\partial X^\mu} \right] z - \hbar \bar{z} (\mathscr{B}_x)_\mu \dot{z} - \hbar \dot{\bar{z}} (\mathscr{B}_x)_\mu z \qquad (3.105) \\
\dot{z} &= i \left[\dot{X}^\alpha (\mathscr{B}_x)_\alpha + \dot{P}_\alpha (\mathscr{B}_p)^\alpha - \frac{e}{2} \mathscr{F}_{\mu\nu} S^{\mu\nu} \right] z, \qquad (3.106) \\
\dot{\bar{z}} &= -i \bar{z} \left[\dot{X}^\alpha (\mathscr{B}_x)_\alpha + \dot{P}_\alpha (\mathscr{B}_p)^\alpha - \frac{e}{2} \mathscr{F}_{\mu\nu} S^{\mu\nu} \right]. \qquad (3.107)
\end{aligned}$$

Inserting the expressions of $\dot{z}$ and $\dot{\bar{z}}$ into Eqs. (3.104) and (3.105), we obtain:

$$\dot{X}^\mu = \frac{\partial H}{\partial P_\mu} + \hbar \dot{X}^\nu \bar{z} (F_{px})_\nu{}^\mu z + \hbar \dot{P}_\nu \bar{z} (F_{pp})^{\nu\mu} z - \frac{ie\hbar}{2} \bar{z} [(\mathscr{B}_p)^\mu, \mathscr{F}_{\alpha\beta} S^{\alpha\beta}] z \qquad (3.108)$$

$$\dot{P}_\mu = -\frac{\partial H}{\partial X_\mu} + e \dot{X}^\nu \mathscr{F}_{\nu\mu} - \hbar \dot{X}^\nu \bar{z} (F_{xx})_{\nu\mu} z - \hbar \dot{P}_\nu \bar{z} (F_{xp})^\nu{}_\mu z + \frac{ie\hbar}{2} \bar{z} [(\mathscr{B}_x)_\mu, \mathscr{F}_{\alpha\beta} S^{\alpha\beta}] z. \qquad (3.109)$$

Since the ray equations are correct up to error terms of $\mathscr{O}(\hbar^2)$, we can replace $\dot{X}^\mu = P^\mu + \mathscr{O}(\hbar^1)$ and $\dot{P}_\mu = \Gamma^\alpha_{\beta\mu} P_\alpha P^\beta + e P^\nu \mathscr{F}_{\nu\mu} + \mathscr{O}(\hbar^1)$ on the right hand side in the above equations. Furthermore, using the expressions for the components of the Berry curvature given in Eqs. (3.69)-(3.71), we can simplify some terms:

$$\begin{aligned}
\hbar \dot{X}^\nu \bar{z} (F_{px})_\nu{}^\mu z + \hbar \dot{P}_\nu \bar{z} (F_{pp})^{\nu\mu} z &= \hbar P^\nu \bar{z} (F_{px})_\nu{}^\mu z + \hbar \Gamma^\rho_{\sigma\nu} P_\rho P^\sigma \bar{z} (F_{pp})^{\nu\mu} z \\
&\quad + e\hbar P^\alpha \mathscr{F}_{\alpha\nu} \bar{z} (F_{pp})^{\nu\mu} z + \mathscr{O}(\hbar^2) \\
&= -\frac{\hbar}{m^2} \Gamma^\rho_{\nu\sigma} P_\rho P^\nu \bar{z} S^{\sigma\mu} z + \frac{\hbar}{m^2} \Gamma^\rho_{\sigma\nu} P_\rho P^\sigma \bar{z} S^{\nu\mu} z \\
&\quad + \frac{e\hbar}{m^2} P^\alpha \mathscr{F}_{\alpha\nu} \bar{z} S^{\nu\mu} z + \mathscr{O}(\hbar^2) \\
&= \frac{e\hbar}{m^2} P^\alpha \mathscr{F}_{\alpha\nu} \bar{z} S^{\nu\mu} z + \mathscr{O}(\hbar^2).
\end{aligned} \qquad (3.110)$$

$$\begin{aligned}
\hbar \dot{X}^\nu \bar{z}(F_{xx})_{\nu\mu} z + \hbar \dot{P}_\nu \bar{z}(F_{xp})^\nu{}_\mu z &= \hbar P^\nu \bar{z}(F_{xx})_{\nu\mu} z + \hbar \Gamma^\rho_{\sigma\nu} P_\rho P^\sigma \bar{z}(F_{xp})^\nu{}_\mu z \\
&\quad + e\hbar P^\alpha \mathscr{F}_{\alpha\nu} \bar{z}(F_{xp})^\nu{}_\mu z + \mathcal{O}(\hbar^2) \\
&= -\frac{\hbar}{2} P^\nu R_{\nu\mu\alpha\beta} \bar{z} S^{\alpha\beta} z + \frac{\hbar}{m^2} P^\nu P_\rho P_\sigma \Gamma^\rho_{\alpha\nu} \Gamma^\sigma_{\beta\mu} \bar{z} S^{\alpha\beta} z \\
&\quad + \frac{\hbar}{m^2} P_\rho P^\sigma P_\delta \Gamma^\rho_{\sigma\nu} \Gamma^\delta_{\mu\alpha} \bar{z} S^{\alpha\nu} z \\
&\quad + \frac{e\hbar}{m^2} P^\alpha P_\rho \Gamma^\rho_{\mu\beta} \mathscr{F}_{\alpha\nu} \bar{z} S^{\beta\nu} z + \mathcal{O}(\hbar^2) \\
&= -\frac{\hbar}{2} P^\nu R_{\nu\mu\alpha\beta} \bar{z} S^{\alpha\beta} z + \frac{e\hbar}{m^2} P^\alpha P_\rho \Gamma^\rho_{\mu\beta} \mathscr{F}_{\alpha\nu} \bar{z} S^{\beta\nu} z + \mathcal{O}(\hbar^2).
\end{aligned} \tag{3.111}$$

Thus, the effective ray equations in noncanonical coordinates can be written in the simplified form

$$\dot{X}^\mu = P^\mu + \frac{e\hbar}{m^2} P^\alpha \mathscr{F}_{\alpha\nu} \bar{z} S^{\nu\mu} z + \frac{e\hbar}{2} \mathscr{F}_{\alpha\beta} \bar{z} \left(\overset{v}{\nabla}{}^\mu S^{\alpha\beta} - i[(\mathscr{B}_p)^\mu, S^{\alpha\beta}] \right) z \tag{3.112}$$

$$\begin{aligned}
\dot{P}_\mu &= \Gamma^\alpha_{\beta\mu} P_\alpha P^\beta + e\dot{X}^\nu \mathscr{F}_{\nu\mu} + \frac{\hbar}{2} P^\nu R_{\nu\mu\alpha\beta} \bar{z} S^{\alpha\beta} z + \frac{e\hbar}{m^2} P^\alpha P_\rho \Gamma^\rho_{\mu\beta} \mathscr{F}_{\alpha\nu} \bar{z} S^{\beta\nu} z \\
&\quad - \frac{e\hbar}{2} \mathscr{F}_{\alpha\beta,\mu} \bar{z} S^{\alpha\beta} z - \frac{e\hbar}{2} \mathscr{F}_{\alpha\beta} \bar{z} \left(S^{\alpha\beta}{}_{,\mu} - i[(\mathscr{B}_x)_\mu, S^{\alpha\beta}] \right) z,
\end{aligned} \tag{3.113}$$

$$\dot{z} = i \left(P^\alpha B_\alpha - \frac{e}{2} \mathscr{F}_{\mu\nu} S^{\mu\nu} \right) z, \tag{3.114}$$

$$\dot{\bar{z}} = -i\bar{z} \left(P^\alpha B_\alpha - \frac{e}{2} \mathscr{F}_{\mu\nu} S^{\mu\nu} \right). \tag{3.115}$$

In Eqs. (3.112) and (3.113), the terms involving the derivatives of $S^{\alpha\beta}$ and the "no-name" term $\mathscr{F}_{\alpha\beta} S^{\alpha\beta}$ can be rewritten as in Appendix A.6. Furthermore, in the absence of an external electromagnetic field $\mathscr{F}_{\alpha\beta}$, Eqs. (3.112) and (3.113) take the same form as the Mathisson-Papapetrou-Dixon equations at linear order in spin. This is in agreement with the results of the WKB analysis presented in Refs. [9, 147].

A. Appendix

A.1. Horizontal and vertical derivatives on T^*M

Let (x^μ, p_μ) be canonical coordinates on T^*M. Considering fields defined on T^*M, such as $u_\alpha(x,p)$ and $v^\alpha(x,p)$, the horizontal and vertical derivatives are defined as follows [153, Sec. 3.5]:

$$\overset{v}{\nabla}{}^\mu u_\alpha = \frac{\partial}{\partial p_\mu} u_\alpha, \tag{A.1a}$$

$$\overset{h}{\nabla}{}_\mu u_\alpha = \frac{\partial}{\partial x^\mu} u_\alpha - \Gamma^\sigma_{\alpha\mu} u_\sigma + \Gamma^\sigma_{\mu\rho} p_\sigma \frac{\partial}{\partial p_\rho} u_\alpha, \tag{A.1b}$$

$$\overset{v}{\nabla}{}^\mu v^\alpha = \frac{\partial}{\partial p_\mu} v^\alpha, \tag{A.2a}$$

$$\overset{h}{\nabla}{}_\mu v^\alpha = \frac{\partial}{\partial x^a} v^\alpha + \Gamma^\alpha_{\sigma\mu} v^\sigma + \Gamma^\sigma_{\mu\rho} p_\sigma \frac{\partial}{\partial p_\rho} v^\alpha. \tag{A.2b}$$

The extension to general tensor fields on T^*M is straightforward. Note that, in contrast to Ref. [153, Sec. 3.5], we have the opposite sign for the last term in the definition of the horizontal derivative. This is because our fields, $u_\alpha(x,p)$ and $v^\alpha(x,p)$, are defined on T^*M, and not on TM, as is the case in the reference mentioned before. The horizontal and vertical derivatives can also be extended to spinor fields $\Psi(x,p)$ defined on T^*M:

$$\overset{v}{\nabla}{}^\mu \Psi = \frac{\partial}{\partial p_\mu} \Psi, \tag{A.3}$$

$$\overset{h}{\nabla}{}_\mu \Psi = \frac{\partial}{\partial x^\mu} \Psi - \frac{i}{4} \omega_\mu{}^{ab} \sigma_{ab} \Psi + \Gamma^\sigma_{\mu\rho} p_\sigma \frac{\partial}{\partial p_\rho} \Psi, \tag{A.4}$$

We can make use of the following properties:

$$\begin{aligned} &[\overset{h}{\nabla}{}_\mu, \overset{v}{\nabla}{}^\nu] = 0, \quad [\overset{v}{\nabla}{}^\mu, \overset{v}{\nabla}{}^\nu] = 0, \\ &\overset{h}{\nabla}{}_\mu p_\alpha = \overset{h}{\nabla}{}_\mu g_{\alpha\beta} = \overset{v}{\nabla}{}^\mu g_{\alpha\beta} = 0. \end{aligned} \tag{A.5}$$

A.2. Variation of the action

Here, we derive the Euler-Lagrange equations that correspond to the action

$$J = \int_M \mathrm{d}^4x \sqrt{g}\, \mathscr{L}, \tag{A.6}$$

where the Lagrangian density is of the following form:

$$\begin{aligned}\mathscr{L} = \mathscr{L}\Big(&S(x), \nabla_\mu S(x), A_\alpha[x, \nabla S(x)], \nabla_\mu \left\{A_\alpha[x, \nabla S(x)]\right\}, \\ &A^{*\alpha}[x, \nabla S(x)], \nabla_\mu \left\{A^{*\alpha}[x, \nabla S(x)]\right\}\Big).\end{aligned} \tag{A.7}$$

Here, $S(x)$ is an independent field, while A_α and $A^{*\alpha}$ cannot be considered independent, since they depend on $\nabla_\mu S$. Following Hawking and Ellis [102, p. 65], we define the variation of a field Ψ_i as a one-parameter family of fields $\Psi_i(u, x)$, with $u \in (-\varepsilon, \varepsilon)$ and $x \in M$. We use the following notation:

$$\left.\frac{\partial \Psi_i(u, x)}{\partial u}\right|_{u=0} = \Delta \Psi_i. \tag{A.8}$$

Note that the derivative with respect to the parameter u commutes with the covariant derivative, so we have:

$$\frac{\mathrm{d}}{\mathrm{d}u} \nabla_\mu S(u, x) = \nabla_\mu \left(\frac{\partial S}{\partial u}\right), \tag{A.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} A_\alpha \left(u, x, \nabla S(u, x)\right) = \frac{\partial A_\alpha}{\partial u} + \frac{\partial A_\alpha}{\partial \nabla_\nu S} \nabla_\nu \left(\frac{\partial S}{\partial u}\right), \tag{A.10}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u} \nabla_\mu \left[A_\alpha \left(u, x, \nabla S(u, x)\right)\right] &= \nabla_\mu \left[\frac{\mathrm{d}}{\mathrm{d}u} A_\alpha \left(u, x, \nabla S(u, x)\right)\right] \\ &= \nabla_\mu \left[\frac{\partial A_\alpha}{\partial u} + \frac{\partial A_\alpha}{\partial \nabla_\nu S} \nabla_\nu \left(\frac{\partial S}{\partial u}\right)\right].\end{aligned} \tag{A.11}$$

We consider the variation of the action, taking special care when applying the chain rule:

$$\begin{aligned}0 = \left.\frac{\mathrm{d}J}{\mathrm{d}u}\right|_{u=0} &= \int_M \mathrm{d}^4x \sqrt{g} \left\{ \frac{\partial \mathscr{L}}{\partial S} \Delta S + \frac{\partial \mathscr{L}}{\partial \nabla_\mu S} \Delta \left(\nabla_\mu S\right) \right. \\ &+ \frac{\partial \mathscr{L}}{\partial A_\alpha} \left[\Delta A_\alpha + \frac{\partial A_\alpha}{\partial \nabla_\mu S} \nabla_\mu \left(\Delta S\right)\right] + \frac{\partial \mathscr{L}}{\partial \nabla_\mu A_\alpha} \nabla_\mu \left[\Delta A_\alpha + \frac{\partial A_\alpha}{\partial \nabla_\nu S} \nabla_\nu \left(\Delta S\right)\right] \\ &+ \left. \frac{\partial \mathscr{L}}{\partial A^{*\alpha}} \left[\Delta A^{*\alpha} + \frac{\partial A^{*\alpha}}{\partial \nabla_\mu S} \nabla_\mu \left(\Delta S\right)\right] + \frac{\partial \mathscr{L}}{\partial \nabla_\mu A^{*\alpha}} \nabla_\mu \left[\Delta A^{*\alpha} + \frac{\partial A^{*\alpha}}{\partial \nabla_\nu S} \nabla_\nu \left(\Delta S\right)\right] \right\}.\end{aligned} \tag{A.12}$$

Integrating by parts, and assuming the boundary terms vanish, we obtain:

$$0 = \left.\frac{\mathrm{d}J}{\mathrm{d}u}\right|_{u=0} = \int_M \mathrm{d}^4x\,\sqrt{g}\left\{\left(\frac{\partial\mathscr{L}}{\partial A_\alpha} - \nabla_\mu\frac{\partial\mathscr{L}}{\partial\nabla_\mu A_\alpha}\right)\Delta A_\alpha + \left(\frac{\partial\mathscr{L}}{\partial A^{*\alpha}} - \nabla_\mu\frac{\partial\mathscr{L}}{\partial\nabla_\mu A^{*\alpha}}\right)\Delta A^{*\alpha}\right.$$
$$\left. + \frac{\partial\mathscr{L}}{\partial S}\Delta S - \nabla_\mu\left[\frac{\partial\mathscr{L}}{\partial\nabla_\mu S} + \frac{\partial A_\alpha}{\partial\nabla_\mu S}\left(\frac{\partial\mathscr{L}}{\partial A_\alpha} - \nabla_\nu\frac{\partial\mathscr{L}}{\partial\nabla_\nu A_\alpha}\right) + \frac{\partial A^{*\alpha}}{\partial\nabla_\mu S}\left(\frac{\partial\mathscr{L}}{\partial A^{*\alpha}} - \nabla_\nu\frac{\partial\mathscr{L}}{\partial\nabla_\nu A^{*\alpha}}\right)\right]\Delta S\right\}. \tag{A.13}$$

Since the above equation must be satisfied for all variations ΔS, ΔA_α, and $\Delta A^{*\alpha}$, we obtain the following Euler-Lagrange equations:

$$\frac{\partial\mathscr{L}}{\partial A^{*\alpha}} - \nabla_\mu\frac{\partial\mathscr{L}}{\partial\nabla_\mu A^{*\alpha}} = \mathscr{O}(\epsilon^2), \tag{A.14}$$

$$\frac{\partial\mathscr{L}}{\partial A_\alpha} - \nabla_\mu\frac{\partial\mathscr{L}}{\partial\nabla_\mu A_\alpha} = \mathscr{O}(\epsilon^2), \tag{A.15}$$

$$\frac{\partial\mathscr{L}}{\partial S} - \nabla_\mu\left[\frac{\partial\mathscr{L}}{\partial\nabla_\mu S} + \frac{\partial A_\alpha}{\partial\nabla_\mu S}\left(\frac{\partial\mathscr{L}}{\partial A_\alpha} - \nabla_\nu\frac{\partial\mathscr{L}}{\partial\nabla_\nu A_\alpha}\right) + \frac{\partial A^{*\alpha}}{\partial\nabla_\mu S}\left(\frac{\partial\mathscr{L}}{\partial A^{*\alpha}} - \nabla_\nu\frac{\partial\mathscr{L}}{\partial\nabla_\nu A^{*\alpha}}\right)\right] = \mathscr{O}(\epsilon^2). \tag{A.16}$$

Furthermore, Eq. (A.16) can be simplified by using Eqs. (A.14) and (A.15). Thus, as a final result, we have the following set of Euler-Lagrange equations:

$$\begin{aligned}
\frac{\partial\mathscr{L}}{\partial A^{*\alpha}} - \nabla_\mu\frac{\partial\mathscr{L}}{\partial\nabla_\mu A^{*\alpha}} &= \mathscr{O}(\epsilon^2), \\
\frac{\partial\mathscr{L}}{\partial A_\alpha} - \nabla_\mu\frac{\partial\mathscr{L}}{\partial\nabla_\mu A_\alpha} &= \mathscr{O}(\epsilon^2), \\
\frac{\partial\mathscr{L}}{\partial S} - \nabla_\mu\frac{\partial\mathscr{L}}{\partial\nabla_\mu S} &= \mathscr{O}(\epsilon^2).
\end{aligned} \tag{A.17}$$

A.3. Berry curvature - electromagnetic case

In order to calculate the Berry curvature terms (2.114), it is enough to use a tetrad $\{t^\alpha, p^\alpha, v^\alpha, w^\alpha\}$, where t^α is a future-oriented timelike vector field representing a family of observers and p^α is a generic vector, not necessarily null, representing the momentum of a point particle (ray). The vectors v^α and w^α are real spacelike vectors related to m^α and $\bar{m}^\alpha$ by the following relations:

$$m^\alpha = \frac{1}{\sqrt{2}}\left(v^\alpha + i w^\alpha\right), \qquad \bar{m}^\alpha = \frac{1}{\sqrt{2}}\left(v^\alpha - i w^\alpha\right). \tag{A.18}$$

The elements of the tetrad $\{t^\alpha, p^\alpha, v^\alpha, w^\alpha\}$ satisfy the following relations:

$$\begin{gathered} t_\alpha t^\alpha = -1, \qquad p_\alpha p^\alpha = \kappa, \qquad t_\alpha p^\alpha = -\epsilon\omega, \qquad v_\alpha v^\alpha = w_\alpha w^\alpha = 1, \\ t_\alpha v^\alpha = t_\alpha w_\alpha = p_\alpha v^\alpha = p_\alpha w^\alpha = v_\alpha w^\alpha = 0. \end{gathered} \tag{A.19}$$

Note that the vectors v^α and w^α depend of p^μ through the orthogonality condition, while t^α is independent of p^μ. We start by computing the vertical derivatives of the vectors v^α and w^α. Using the tetrad, we can write:

$$\overset{v}{\nabla}{}^\mu v^\alpha = \frac{\partial v^\alpha}{\partial p_\mu} = c_1{}^\mu t^\alpha + c_2{}^\mu p^\alpha + c_3{}^\mu v^\alpha + c_4{}^\mu w^\alpha, \tag{A.20}$$

$$\overset{v}{\nabla}{}^\mu w^\alpha = \frac{\partial w^\alpha}{\partial p_\mu} = d_1{}^\mu t^\alpha + d_2{}^\mu p^\alpha + d_3{}^\mu v^\alpha + d_4{}^\mu w^\alpha, \tag{A.21}$$

where $c_i{}^\mu$ and $d_i{}^\mu$ are unknown vector fields that need to be determined. Using the properties from Eq. (A.19), we obtain

$$\begin{aligned} \overset{v}{\nabla}{}^\mu v^\alpha &= \frac{\epsilon\omega}{\epsilon^2\omega^2+\kappa} v^\mu t^\alpha - \frac{1}{\epsilon^2\omega^2+\kappa} v^\mu p^\alpha + c_4{}^\mu w^\alpha, \\ \overset{v}{\nabla}{}^\mu w^\alpha &= \frac{\epsilon\omega}{\epsilon^2\omega^2+\kappa} w^\mu t^\alpha - \frac{1}{\epsilon^2\omega^2+\kappa} w^\mu p^\alpha + d_3{}^\mu v^\alpha. \end{aligned} \tag{A.22}$$

Applying the same arguments to the terms $\nabla_\mu v_\alpha$ and $\nabla_\mu w_\alpha$, we also obtain

$$\begin{aligned} \nabla_\mu v_\alpha = &- \frac{1}{\epsilon^2\omega^2+\kappa}\left(\epsilon\omega p_\sigma \nabla_\mu v^\sigma + \kappa t_\sigma \nabla_\mu v^\sigma\right) t_\alpha \\ &+ \frac{1}{\epsilon^2\omega^2+\kappa}\left(p_\sigma \nabla_\mu v^\sigma - \epsilon\omega t_\sigma \nabla_\mu v^\sigma\right) p_\alpha + f_{4\mu} w_\alpha, \\ \nabla_\mu w_\alpha = &- \frac{1}{\epsilon^2\omega^2+\kappa}\left(\epsilon\omega p_\sigma \nabla_\mu w^\sigma + \kappa t_\sigma \nabla_\mu w^\sigma\right) t_\alpha \\ &+ \frac{1}{\epsilon^2\omega^2+\kappa}\left(p_\sigma \nabla_\mu w^\sigma - \epsilon\omega t_\sigma \nabla_\mu w^\sigma\right) p_\alpha + g_{3\mu} v_\alpha. \end{aligned} \tag{A.23}$$

Note that the fields $c_{4\mu}$, $d_{3\mu}$, $f_{4\mu}$, and $g_{3\mu}$ are undetermined within this approach, but this is not a problem, because they do not affect the Berry curvature.

A.3.1. F_{pp}

We compute $(F_{pp})^{\nu\mu}$ by using Eq. (A.22) and setting $\kappa = 0$. Since the vertical derivatives commute (see Eq. (A.5)), we can write

$$\begin{aligned}(F_{pp})^{\nu\mu} &= i\left(\overset{v}{\nabla}{}^{\mu}\bar{m}^{\alpha}\overset{v}{\nabla}{}^{\nu}m_{\alpha} - \overset{v}{\nabla}{}^{\nu}\bar{m}^{\alpha}\overset{v}{\nabla}{}^{\mu}m_{\alpha}\right)\\ &= \overset{v}{\nabla}{}^{\nu}v^{\alpha}\overset{v}{\nabla}{}^{\mu}w_{\alpha} - \overset{v}{\nabla}{}^{\mu}v^{\alpha}\overset{v}{\nabla}{}^{\nu}w_{\alpha}\\ &= \frac{2}{\epsilon^2\omega^2}v^{[\nu}w^{\mu]}\\ &= \frac{2i}{\epsilon^2\omega^2}m^{[\nu}\bar{m}^{\mu]}.\end{aligned} \tag{A.24}$$

A.3.2. F_{xx}

We have

$$(F_{xx})_{\nu\mu} = i\Big(\nabla_\mu\bar{m}^\alpha\nabla_\nu m_\alpha - \nabla_\nu\bar{m}^\alpha\nabla_\mu m_\alpha + \bar{m}^\alpha\nabla_{[\mu}\nabla_{\nu]}m_\alpha - m_\alpha\nabla_{[\mu}\nabla_{\nu]}\bar{m}^\alpha\Big). \tag{A.25}$$

The last two terms can be expressed in terms of the Riemann tensor:

$$i\Big(\bar{m}^\alpha\nabla_{[\mu}\nabla_{\nu]}m_\alpha - m_\alpha\nabla_{[\mu}\nabla_{\nu]}\bar{m}^\alpha\Big) = -iR_{\alpha\beta\mu\nu}m^\alpha\bar{m}^\beta. \tag{A.26}$$

The first two terms can be computed using Eq. (A.23) and $\kappa = 0$:

$$\begin{aligned}(\tilde{F}_{xx})_{\nu\mu} &= i\Big(\nabla_\mu\bar{m}^\alpha\nabla_\nu m_\alpha - \nabla_\nu\bar{m}^\alpha\nabla_\mu m_\alpha\Big)\\ &= \nabla_\nu v^\alpha\nabla_\mu w_\alpha - \nabla_\mu v^\alpha\nabla_\nu w_\alpha\\ &=\frac{1}{\epsilon^2\omega^2}\Big(p_\sigma\nabla_\mu v^\sigma p_\rho\nabla_\nu w^\rho - p_\sigma\nabla_\nu v^\sigma p_\rho\nabla_\mu w^\rho\\ &\quad - \epsilon\omega p_\sigma\nabla_\mu v^\sigma t_\rho\nabla_\nu w^\rho + \epsilon\omega p_\sigma\nabla_\nu v^\sigma t_\rho\nabla_\mu w^\rho\\ &\quad - \epsilon\omega t_\sigma\nabla_\mu v^\sigma p_\rho\nabla_\nu w^\rho + \epsilon\omega t_\sigma\nabla_\nu v^\sigma p_\rho\nabla_\mu w^\rho\Big)\\ &=\frac{i}{\epsilon^2\omega^2}\Big(p_\sigma\nabla_\mu m^\sigma p_\rho\nabla_\nu\bar{m}^\rho - p_\sigma\nabla_\nu m^\sigma p_\rho\nabla_\mu\bar{m}^\rho\\ &\quad - \epsilon\omega p_\sigma\nabla_\mu m^\sigma t_\rho\nabla_\nu\bar{m}^\rho + \epsilon\omega p_\sigma\nabla_\nu m^\sigma t_\rho\nabla_\mu\bar{m}^\rho\\ &\quad - \epsilon\omega t_\sigma\nabla_\mu m^\sigma p_\rho\nabla_\nu\bar{m}^\rho + \epsilon\omega t_\sigma\nabla_\nu m^\sigma p_\rho\nabla_\mu\bar{m}^\rho\Big).\end{aligned} \tag{A.27}$$

A.3.3. F_{px} and F_{xp}

Since $(F_{px})_{\nu}{}^{\mu} = -(F_{xp})^{\mu}{}_{\nu}$, it is enough to compute only one term. Using Eqs. (A.22) and (A.23), and setting $\kappa = 0$, we obtain

$$\begin{aligned}(F_{px})_{\nu}{}^{\mu} &= i\left(\overset{v}{\nabla}{}^{\mu}\bar{m}^{\alpha}\nabla_{\nu}m_{\alpha} - \nabla_{\nu}\bar{m}^{\alpha}\overset{v}{\nabla}{}^{\mu}m_{\alpha}\right)\\ &= \nabla_{\nu}v^{\alpha}\overset{v}{\nabla}{}^{\mu}w_{\alpha} - \overset{v}{\nabla}{}^{\mu}v^{\alpha}\nabla_{\nu}w_{\alpha}\\ &= \frac{1}{\epsilon^2\omega^2}\Big[\left(p_{\sigma}\nabla_{\nu}w^{\sigma} - \epsilon\omega t_{\sigma}\nabla_{\nu}w^{\sigma}\right)v^{\mu} - \left(p_{\sigma}\nabla_{\nu}v^{\sigma} - \epsilon\omega t_{\sigma}\nabla_{\nu}v^{\sigma}\right)w^{\mu}\Big]\\ &= \frac{i}{\epsilon^2\omega^2}\Big[\left(p_{\sigma}\nabla_{\nu}\bar{m}^{\sigma} - \epsilon\omega t_{\sigma}\nabla_{\nu}\bar{m}^{\sigma}\right)m^{\mu} - \left(p_{\sigma}\nabla_{\nu}m^{\sigma} - \epsilon\omega t_{\sigma}\nabla_{\nu}m^{\sigma}\right)\bar{m}^{\mu}\Big].\end{aligned} \tag{A.28}$$

A.4. Coordinate transformation

The substitution from Eqs. (2.108) and (2.109) can be obtained, up to terms of order ϵ^2, as a linearization of the following composition of changes of coordinates on the cotangent bundle T^*M. Consider the family of diffeomorphisms (Φ_ϵ) generated by the vector field on M

$$Y = is\bar{m}^{\alpha}\overset{v}{\nabla}{}^{\mu}m_{\alpha}\partial_{x^{\mu}}, \tag{A.29}$$

that is to say

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi_{\epsilon}(x) = Y(\Phi_{\epsilon}(x)) \text{ with } \Phi_0(x) = x. \tag{A.30}$$

By construction, the Taylor expansion in a coordinate chart of Φ_ϵ at order ϵ^1 leads to Eq. (2.108). Φ_ϵ naturally lifts to the cotangent bundle using the pullback Φ_ϵ^*:

$$\Phi_\epsilon^* : (x,p) \mapsto (\Phi_\epsilon(x), p \circ d\Phi_\epsilon^{-1}|_{\Phi_\epsilon(x)}). \tag{A.31}$$

Note that the choice of the lift is not unique. The mapping Φ_ϵ^* is, at order one in ϵ, in coordinates,

$$(x^{\mu}, p_{\mu}) \mapsto (x^{\mu} + is\epsilon\bar{m}^{\alpha}\overset{v}{\nabla}{}^{\mu}m_{\alpha}, p_{\mu} - i\epsilon s p_{\beta}\partial_{x^{\mu}}(\bar{m}^{\alpha}\overset{v}{\nabla}{}^{\beta}m_{\alpha})). \tag{A.32}$$

Consider next the translation of the momentum variable defined by

$$\Psi_\epsilon : (x,p) \mapsto (x, p - \epsilon\sigma), \tag{A.33}$$

where $\sigma = is(\bar{m}^{\alpha}\nabla_{\mu}m_{\alpha} + p_{\beta}\partial_{x^{\mu}}(\bar{m}^{\alpha}\overset{v}{\nabla}{}^{\beta}m_{\alpha}))dx^{\mu}$. The linearization in ϵ of the diffeomorphism $\Psi_\epsilon \circ \Phi_\epsilon{}^*$ provides by construction the change of variables in Eqs. (2.108) and (2.109).

A.5. Berry curvature - Dirac case

Using the definition of the covariant derivative for spinor fields, given in Eq. (3.2), we can rewrite the components of the Berry curvature as

$$(F_{xx})_{\mu\nu} = \nabla_\mu (\mathscr{B}_x)_\nu - \nabla_\nu (\mathscr{B}_x)_\mu - i\left[(\mathscr{B}_x)_\mu, (\mathscr{B}_x)_\nu\right], \tag{A.34}$$

$$(F_{pp})^{\mu\nu} = \overset{v}{\nabla}{}^\mu (\mathscr{B}_p)^\nu - \overset{v}{\nabla}{}^\nu (\mathscr{B}_p)^\mu - i\left[(\mathscr{B}_p)^\mu, (\mathscr{B}_p)^\nu\right], \tag{A.35}$$

$$(F_{px})_\mu{}^\nu = -(F_{xp})^\nu{}_\mu = \nabla_\mu (\mathscr{B}_p)^\nu - \overset{v}{\nabla}{}^\nu (\mathscr{B}_x)_\mu - i\left[(\mathscr{B}_x)_\mu, (\mathscr{B}_p)^\nu\right]. \tag{A.36}$$

We insert into the above equations the definition of the components of the Berry connection

$$(\mathscr{B}_x)_{\mu AB} = \frac{i}{2}\left(\bar{\Sigma}_A \nabla_\mu \Sigma_B - \nabla_\mu \bar{\Sigma}_A \Sigma_B\right), \tag{A.37}$$

$$(\mathscr{B}_p)^\mu{}_{AB} = \frac{i}{2}\left(\bar{\Sigma}_A \overset{v}{\nabla}{}^\mu \Sigma_B - \overset{v}{\nabla}{}^\mu \bar{\Sigma}_A \Sigma_B\right), \tag{A.38}$$

and we use the orthogonality properties of the eigenspinors:

$$\bar{\Sigma}_A \Sigma_B = \delta_{AB} \Rightarrow \begin{cases} \nabla_\mu \bar{\Sigma}_A \Sigma_B + \bar{\Sigma}_A \nabla_\mu \Sigma_B = 0, \\ \overset{v}{\nabla}{}^\mu \bar{\Sigma}_A \Sigma_B + \bar{\Sigma}_A \overset{v}{\nabla}{}^\mu \Sigma_B = 0. \end{cases} \tag{A.39}$$

We obtain

$$(F_{xx})_{\mu\nu AB} = i\bar{\Sigma}_A (\nabla_{[\mu} \nabla_{\nu]} \Sigma_B) - i(\nabla_{[\mu} \nabla_{\nu]} \bar{\Sigma}_A)\Sigma_B \tag{A.40}$$

$$+ 2i(\nabla_{[\mu} \bar{\Sigma}_A)(\mathbb{I}_4 - \Sigma_C \bar{\Sigma}_C)(\nabla_{\nu]} \Sigma_B), \tag{A.41}$$

$$(F_{pp})^{\mu\nu}{}_{AB} = 2i(\overset{v}{\nabla}{}^{[\mu} \bar{\Sigma}_A)(\mathbb{I}_4 - \Sigma_C \bar{\Sigma}_C)(\overset{v}{\nabla}{}^{\nu]} \Sigma_B), \tag{A.42}$$

$$(F_{px})_\mu{}^\nu{}_{AB} = -(F_{xp})^\nu{}_{\mu AB} = 2i(\nabla_{[\mu} \bar{\Sigma}_A)(\mathbb{I}_4 - \Sigma_C \bar{\Sigma}_C)(\overset{v}{\nabla}{}^{\nu]} \Sigma_B). \tag{A.43}$$

Furthermore, using the resolution of identity given in Eq. (3.27), we obtain

$$(F_{xx})_{\mu\nu AB} = i\bar{\Sigma}_A (\nabla_{[\mu} \nabla_{\nu]} \Sigma_B) - i(\nabla_{[\mu} \nabla_{\nu]} \bar{\Sigma}_A)\Sigma_B \tag{A.44}$$

$$- 2i(\nabla_{[\mu} \bar{\Sigma}_A)\Pi_C \bar{\Pi}_C (\nabla_{\nu]} \Sigma_B), \tag{A.45}$$

$$(F_{pp})^{\mu\nu}{}_{AB} = -2i(\overset{v}{\nabla}{}^{[\mu} \bar{\Sigma}_A)\Pi_C \bar{\Pi}_C (\overset{v}{\nabla}{}^{\nu]} \Sigma_B), \tag{A.46}$$

$$(F_{px})_\mu{}^\nu{}_{AB} = -(F_{xp})^\nu{}_{\mu AB} = -2i(\nabla_{[\mu} \bar{\Sigma}_A)\Pi_C \bar{\Pi}_C (\overset{v}{\nabla}{}^{\nu]} \Sigma_B). \tag{A.47}$$

The commutator of spinor covariant derivatives can be expressed in terms of the Riemann tensor as

$$\begin{aligned} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\Psi &= -\frac{1}{4} R_{\mu\nu\rho\sigma} \gamma^\rho \gamma^\sigma \Psi, \\ (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\bar{\Psi} &= \frac{1}{4} R_{\mu\nu\rho\sigma} \bar{\Psi} \gamma^\rho \gamma^\sigma. \end{aligned} \tag{A.48}$$

Using these relations, we can write

$$i(\bar{\Sigma}_A\nabla_{[\mu}\nabla_{\nu]}\Sigma_B - \nabla_{[\mu}\nabla_{\nu]}\bar{\Sigma}_A\Sigma_B) = -\frac{1}{2}R_{\mu\nu\alpha\beta}S^{\alpha\beta}{}_{AB}. \tag{A.49}$$

The remaining terms that need to be computed are of the form $\bar{\Pi}_C\nabla_\mu\Sigma_B$ or $\bar{\Pi}_C\overset{v}{\nabla}{}^\mu\Sigma_B$. For this purpose, we can use the fact that Σ_A is an eigenspinor of D, with eigenvalue zero, while Π_A is an eigenspinor of D, with eigenvalue $-2m$. We start with

$$\bar{\Pi}_A D\Sigma_B = 0. \tag{A.50}$$

Taking the vertical derivative of this expression, we obtain:

$$\begin{aligned} 0 &= (\overset{v}{\nabla}{}^\mu\bar{\Pi}_A)D\Sigma_B + \bar{\Pi}_A(\overset{v}{\nabla}{}^\mu D)\Sigma_B + \bar{\Pi}_A D(\overset{v}{\nabla}{}^\mu\Sigma_B) \\ &= \bar{\Pi}_A(\overset{v}{\nabla}{}^\mu D)\Sigma_B - 2m\bar{\Pi}_A(\overset{v}{\nabla}{}^\mu\Sigma_B) \\ &= -\bar{\Pi}_A\gamma^\mu\Sigma_B - 2m\bar{\Pi}_A(\overset{v}{\nabla}{}^\mu\Sigma_B), \end{aligned} \tag{A.51}$$

and we can finally write

$$\bar{\Pi}_A\overset{v}{\nabla}{}^\mu\Sigma_B = -\frac{1}{2m}\bar{\Pi}_A\gamma^\mu\Sigma_B \tag{A.52}$$

Similarly, by taking a covariant derivative of $\bar{\Pi}_A D\Sigma_B = 0$, we obtain

$$\begin{aligned} 0 &= (\nabla_\mu\bar{\Pi}_A)D\Sigma_B + \bar{\Pi}_A(\nabla_\mu D)\Sigma_B + \bar{\Pi}_A D(\nabla_\mu\Sigma_B) \\ &= \bar{\Pi}_A(\nabla_\mu D)\Sigma_B - 2m\bar{\Pi}_A(\nabla_\mu\Sigma_B) \\ &= -(\nabla_\mu v_\alpha)\bar{\Pi}_A\gamma^\alpha\Sigma_B - 2m\bar{\Pi}_A(\nabla_\mu\Sigma_B), \end{aligned} \tag{A.53}$$

However, since we are working on T^*M, with coordinates (x^μ, v_μ), we have

$$\nabla_\mu v_\alpha = \frac{\partial}{\partial x^\mu}v_\alpha - \Gamma^\sigma_{\mu\alpha}v_\sigma = -\Gamma^\sigma_{\mu\alpha}v_\sigma. \tag{A.54}$$

Thus, we can write

$$\bar{\Pi}_A\nabla_\mu\Sigma_B = -\frac{1}{2m}(\nabla_\mu v_\alpha)\bar{\Pi}_A\gamma^\alpha\Sigma_B = \frac{1}{2m}v_\sigma\Gamma^\sigma_{\mu\alpha}\bar{\Pi}_A\gamma^\alpha\Sigma_B \tag{A.55}$$

Using Eqs. (A.52) and (A.55), we arrive at the final form for the components of the Berry curvature:

$$(F_{xx})_{\mu\nu} = -\frac{1}{2}R_{\mu\nu\alpha\beta}S^{\alpha\beta} + \frac{1}{m^2}v_\rho v_\sigma\Gamma^\rho_{\alpha\mu}\Gamma^\sigma_{\beta\nu}S^{\alpha\beta}, \tag{A.56}$$

$$(F_{pp})^{\mu\nu} = \frac{1}{m^2}S^{\mu\nu}, \tag{A.57}$$

$$(F_{px})_{\mu}{}^{\nu} = -(F_{xp})^{\nu}{}_{\mu} = -\frac{1}{m^2} v_\rho \Gamma^{\rho}_{\mu\alpha} S^{\alpha\nu}. \tag{A.58}$$

The components of the Berry curvature were also calculated in Ref. [162], although only for the case of Minkowski spacetime. Restricting to Minkowski spacetime, the only nonzero component of the Berry curvature is $(F_{pp})^{\mu\nu}$, and in this case our result agrees with the result presented in Ref. [162, Eq. 30].

A.6. “No-name” terms

Using the definition $S^{\alpha\beta}$ and $(\mathscr{B}_p)^\mu$, we can rewrite the last term in Eq. (3.112) as

$$\begin{aligned}
\overset{v}{\nabla}{}^\mu S^{\alpha\beta} - i[(\mathscr{B}_p)^\mu, S^{\alpha\beta}] &= \frac{1}{2}\left[(\overset{v}{\nabla}{}^\mu\bar{\Sigma}_A)\sigma^{\alpha\beta}\Sigma_B + \bar{\Sigma}_A\sigma^{\alpha\beta}(\overset{v}{\nabla}{}^\mu\Sigma_B)\right] \\
&\quad + \frac{1}{4}\left[\bar{\Sigma}_A(\overset{v}{\nabla}{}^\mu\Sigma_C) - (\overset{v}{\nabla}{}^\mu\bar{\Sigma}_A)\Sigma_C\right]\bar{\Sigma}_C\sigma^{\alpha\beta}\Sigma_B \\
&\quad - \frac{1}{4}\bar{\Sigma}_A\sigma^{\alpha\beta}\Sigma_C\left[\bar{\Sigma}_C(\overset{v}{\nabla}{}^\mu\Sigma_B) - (\overset{v}{\nabla}{}^\mu\bar{\Sigma}_C)\Sigma_B\right] \\
&= \frac{1}{4}\left[2(\overset{v}{\nabla}{}^\mu\bar{\Sigma}_A) + \bar{\Sigma}_A(\overset{v}{\nabla}{}^\mu\Sigma_C)\bar{\Sigma}_C - (\overset{v}{\nabla}{}^\mu\bar{\Sigma}_A)\Sigma_C\bar{\Sigma}_C\right]\sigma^{\alpha\beta}\Sigma_B \\
&\quad + \frac{1}{4}\bar{\Sigma}_A\sigma^{\alpha\beta}\left[2(\overset{v}{\nabla}{}^\mu\Sigma_B) - \Sigma_C\bar{\Sigma}_C(\overset{v}{\nabla}{}^\mu\Sigma_B) + \Sigma_C(\overset{v}{\nabla}{}^\mu\bar{\Sigma}_C)\Sigma_B\right]
\end{aligned} \tag{A.59}$$

Using Eqs. (A.39) and (3.27), the above expression simplifies to

$$\begin{aligned}
\overset{v}{\nabla}{}^\mu S^{\alpha\beta} - i[(\mathscr{B}_p)^\mu, S^{\alpha\beta}] &= \frac{1}{2}\left[(\overset{v}{\nabla}{}^\mu\bar{\Sigma}_A) - (\overset{v}{\nabla}{}^\mu\bar{\Sigma}_A)\Sigma_C\bar{\Sigma}_C\right]\sigma^{\alpha\beta}\Sigma_B \\
&\quad + \frac{1}{2}\bar{\Sigma}_A\sigma^{\alpha\beta}\left[(\overset{v}{\nabla}{}^\mu\Sigma_B) - \Sigma_C\bar{\Sigma}_C(\overset{v}{\nabla}{}^\mu\Sigma_B)\right] \\
&= \frac{1}{2}(\overset{v}{\nabla}{}^\mu\bar{\Sigma}_A)(\mathbb{I}_4 - \Sigma_C\bar{\Sigma}_C)\sigma^{\alpha\beta}\Sigma_B \\
&\quad + \frac{1}{2}\bar{\Sigma}_A\sigma^{\alpha\beta}(\mathbb{I}_4 - \Sigma_C\bar{\Sigma}_C)(\overset{v}{\nabla}{}^\mu\Sigma_B) \\
&= -\frac{1}{2}(\overset{v}{\nabla}{}^\mu\bar{\Sigma}_A)\Pi_C\bar{\Pi}_C\sigma^{\alpha\beta}\Sigma_B - \frac{1}{2}\bar{\Sigma}_A\sigma^{\alpha\beta}\Pi_C\bar{\Pi}_C(\overset{v}{\nabla}{}^\mu\Sigma_B)
\end{aligned} \tag{A.60}$$

Inserting Eq. (A.52) and its complex conjugate into the above expression, we obtain

$$\begin{aligned}
\overset{v}{\nabla}{}^{\mu} S^{\alpha\beta} - i[(\mathscr{B}_p)^{\mu}, S^{\alpha\beta}] &= \frac{1}{4m}\bar{\Sigma}_A\gamma^{\mu}\Pi_C\bar{\Pi}_C\sigma^{\alpha\beta}\Sigma_B + \frac{1}{4m}\bar{\Sigma}_A\sigma^{\alpha\beta}\Pi_C\bar{\Pi}_C\gamma^{\mu}\Sigma_B \\
&= \frac{1}{4m}\bar{\Sigma}_A\gamma^{\mu}(\Sigma_C\bar{\Sigma}_C - \mathbb{I}_4)\sigma^{\alpha\beta}\Sigma_B + \frac{1}{4m}\bar{\Sigma}_A\sigma^{\alpha\beta}(\Sigma_C\bar{\Sigma}_C - \mathbb{I}_4)\gamma^{\mu}\Sigma_B \\
&= \frac{1}{4m}\bar{\Sigma}_A\gamma^{\mu}(\Sigma_C\bar{\Sigma}_C - \mathbb{I}_4)\sigma^{\alpha\beta}\Sigma_B + \frac{1}{4m}\bar{\Sigma}_A\sigma^{\alpha\beta}(\Sigma_C\bar{\Sigma}_C - \mathbb{I}_4)\gamma^{\mu}\Sigma_B \\
&= \frac{1}{m^2}P^{\mu}S^{\alpha\beta} - \frac{1}{4m}\bar{\Sigma}_A(\gamma^{\mu}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\gamma^{\mu})\Sigma_B
\end{aligned} \tag{A.61}$$

The anticommutator $\gamma^{\mu}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\gamma^{\mu}$ can be rewritten in a different form by using the properties of the gamma matrices. We consider the flat spacetime gamma matrices γ^a, which are related to the spacetime gamma matrices by the orthonormal tetrad as $\gamma^{\mu} = (e_a)^{\mu}\gamma^a$. For the flat spacetime gamma matrices, we can write the following relation:

$$\gamma^a\gamma^b\gamma^c = -\eta^{ab}\gamma^c - \eta^{bc}\gamma^a + \eta^{ac}\gamma^b + i\epsilon^{dabc}\gamma_d\gamma^5, \tag{A.62}$$

where η^{ab} is the Minkowski metric tensor, with signature $- + ++$, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Using this relation, we obtain

$$\gamma^c\sigma^{ab} + \sigma^{ab}\gamma^c = 2\epsilon^{dabc}\gamma_d\gamma^5, \tag{A.63}$$

and

$$\begin{aligned}
\gamma^{\mu}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\gamma^{\mu} &= 2(e_a)^{\alpha}(e_b)^{\beta}(e_c)^{\mu}\epsilon^{dabc}\gamma_d\gamma^5 \\
&= 2(e_a)^{\alpha}(e_b)^{\beta}(e_c)^{\mu}(e_d)^{\nu}\epsilon^{dabc}\gamma_{\nu}\gamma^5,
\end{aligned} \tag{A.64}$$

We can perform a similar calculation for the last term in Eq. (3.113). First, note that by using the properties of the covariant derivative, we can write

$$\mathscr{F}_{\alpha\beta,\mu}\bar{z}S^{\alpha\beta}z + \mathscr{F}_{\alpha\beta}\bar{z}S^{\alpha\beta}{}_{,\mu}z = (\nabla_{\mu}\mathscr{F}_{\alpha\beta})\bar{z}S^{\alpha\beta}z + \mathscr{F}_{\alpha\beta}\bar{z}\nabla_{\mu}S^{\alpha\beta}z. \tag{A.65}$$

The last term in Eq. (3.113) becomes $\nabla_{\mu}S^{\alpha\beta} - i[(\mathscr{B}_x)_{\mu}, S^{\alpha\beta}]$, and we can apply the same steps as before. We obtain

$$\nabla_{\mu}S^{\alpha\beta} - i[(\mathscr{B}_x)_{\mu}, S^{\alpha\beta}] = -\frac{1}{2}(\nabla_{\mu}\bar{\Sigma}_A)\Pi_C\bar{\Pi}_C\sigma^{\alpha\beta}\Sigma_B - \frac{1}{2}\bar{\Sigma}_A\sigma^{\alpha\beta}\Pi_C\bar{\Pi}_C(\nabla_{\mu}\Sigma_B). \tag{A.66}$$

Using Eq. (A.55) and its complex conjugate, we obtain

$$\begin{aligned}
\nabla_{\mu}S^{\alpha\beta} - i[(\mathscr{B}_x)_{\mu}, S^{\alpha\beta}] &= -\frac{1}{4m}P_{\sigma}\Gamma^{\sigma}_{\mu\rho}\left(\bar{\Sigma}_A\gamma^{\rho}\Pi_C\bar{\Pi}_C\sigma^{\alpha\beta}\Sigma_B + \bar{\Sigma}_A\sigma^{\alpha\beta}\Pi_C\bar{\Pi}_C\gamma^{\rho}\Sigma_B\right) \\
&= -\frac{1}{m^2}P_{\sigma}P^{\rho}\Gamma^{\sigma}_{\mu\rho}S^{\alpha\beta} + \frac{1}{4m}P_{\sigma}\Gamma^{\sigma}_{\mu\rho}\bar{\Sigma}_A(\gamma^{\rho}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\gamma^{\rho})\Sigma_B.
\end{aligned} \tag{A.67}$$

A.7. Mathematica code for computing ray trajectories

We provide the Mathematica code used for the numerical integration of the non-canonical ray equations (2.115) and (2.116). The present version of the code is written for the case of a Kerr spacetime, together with the orthonormal tetrad introduced in Sec. 2.5.5. However, it can easily be adapted for other cases by modifying the metric and the orthonormal tetrad.

In[1]:=

```
(* BLOCK 1 - Definitions and initializations *)

Clear[a];

(* Assume all variables are real *)
$Assumptions = Element[x0[τ], Reals] &&
    Element[x1[τ], Reals] && Element[x2[τ], Reals] && Element[x3[τ], Reals] &&
    Element[p0[τ], Reals] && Element[p1[τ], Reals] && Element[p2[τ], Reals] &&
    Element[p3[τ], Reals] && Element[s, Reals] && x1[τ] > 0 && Element[a, Reals];

(* Coordinates *)

(* X^μ *)
X = {x0[τ], x1[τ], x2[τ], x3[τ]};

(* P_μ *)
P = {p0[τ], p1[τ], p2[τ], p3[τ]};

(* P_i *)
p = {p1[τ], p2[τ], p3[τ]};


(* BL coordinates in Kerr (t, r, θ, ϕ) = (x0, x1, x2, x3) *)

M = 1; (* mass *)
rs = 2 M; (* Schwarzschild radius *)

Σ = r^2 + a^2 Cos[θ]^2 /. r → x1[τ] /. θ → x2[τ] /. ϕ → x3[τ];
Δ = r^2 - rs r + a^2 /. r → x1[τ] /. θ → x2[τ] /. ϕ → x3[τ];
A = (r^2 + a^2)^2 - a^2 Δ Sin[θ]^2 /. r → x1[τ] /. θ → x2[τ] /. ϕ → x3[τ];

(* Metric g_μν *)
```

$$g = \begin{pmatrix} -\left(1-\frac{rs\,r}{\Sigma}\right) & 0 & 0 & -\frac{rs\,a\,r\,\mathrm{Sin}[\theta]^2}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{rs\,a\,r\,\mathrm{Sin}[\theta]^2}{\Sigma} & 0 & 0 & A\,\frac{\mathrm{Sin}[\theta]^2}{\Sigma} \end{pmatrix} \;/.\; r \to x1[\tau] \;/.\; \theta \to x2[\tau] \;/.\; \phi \to x3[\tau];$$

```
(* Inverse metric g^μν *)

ig = Inverse[g] // Simplify;

(* Orthonormal tetrad (e_a)^μ and dual tetrad (d^a)_μ, with (e_a)^μ = t^μ *)
```

$$e0 = \left\{\frac{r^2 + a^2}{\sqrt{\Sigma\,\Delta}}, 0, 0, \frac{a}{\sqrt{\Sigma\,\Delta}}\right\} \;/.\; r \to x1[\tau] \;/.\; \theta \to x2[\tau] \;/.\; \phi \to x3[\tau];$$

```
e1 = {0, Sqrt[Δ/Σ], 0, 0} /. r → x1[τ] /. θ → x2[τ] /. ϕ → x3[τ];
e2 = {0, 0, Sqrt[1/Σ], 0} /. r → x1[τ] /. θ → x2[τ] /. ϕ → x3[τ];
e3 = {a Sin[θ]^2/(Sqrt[Σ] Sin[θ]), 0, 0, 1/(Sqrt[Σ] Sin[θ])} /. r → x1[τ] /. θ → x2[τ] /. ϕ → x3[τ];
d0 = (e0.g) // Simplify;
d1 = (e1.g) // Simplify;
d2 = (e2.g) // Simplify;
d3 = (e3.g) // Simplify;
```

(* Christoffel $\Gamma^{i}{}_{jk} = \frac{1}{2}g^{im}(\ \partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk}\ )$ *)

```
Γ =
  1/2 Table[(Sum[(ig[[i]][[m]] (D[g[[m]][[j]], X[[k]]] + D[g[[m]][[k]], X[[j]]] - D[g[[j]][[
          k]], X[[m]]])), {m, 1, 4}]) // Simplify, {i, 1, 4}, {j, 1, 4}, {k, 1, 4}];
```

(* Riemann $R^{i}{}_{jkl} = \partial_k\Gamma^{i}{}_{lj} - \partial_l\Gamma^{i}{}_{kj} + \Gamma^{i}{}_{km}\Gamma^{m}{}_{lj} - \Gamma^{i}{}_{lm}\Gamma^{m}{}_{kj}$ *)

```
Riem = Table[(D[Γ[[i]][[l]][[j]], X[[k]]] - D[Γ[[i]][[k]][[j]], X[[l]]] +
      Sum[(Γ[[i]][[k]][[m]] Γ[[m]][[l]][[j]] - Γ[[i]][[l]][[m]] Γ[[m]][[k]][[j]]), {m, 1, 4}]) //
    Simplify, {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}] // Parallelize;
```

(* $P^{\mu} = g^{\mu\alpha}P_{\alpha}$ *)

```
Pu = Table[(Sum[(ig[[μ]][[α]] P[[α]]), {α, 1, 4}]) // Simplify, {μ, 1, 4}] // Parallelize;
```

(* $\dot{P}_{\mu}$ at lowest order in ϵ: $\dot{P}_{\mu} = -\frac{1}{2}g^{\alpha\beta}{}_{,\mu}P_{\alpha}P_{\beta} = \Gamma^{\alpha}{}_{\beta\mu}P_{\alpha}P^{\beta}$ *)

```
Pd =
  Table[(Sum[Sum[(Γ[[α]][[β]][[μ]] P[[α]] Pu[[β]]), {β, 1, 4}], {α, 1, 4}]) // Simplify, {μ, 1, 4}] // Parallelize;
```

(* $H = \frac{1}{2}g^{\mu\nu}P_{\mu}P_{\nu} = 0 \implies P_0 = \frac{1}{g^{00}}\left(-g^{0i}P_i + \sqrt{(g^{0i}P_i)^2 - g^{00}g^{ij}P_iP_j}\right)$ *)

```
pt0 =
  ig[[1]][[1]]^-1 (-(Sum[(ig[[1]][[i]] P[[i]]), {i, 2, 4}]) + ((Sum[(ig[[1]][[i]] P[[i]]), {i, 2, 4}])^2 - ig[[1]][[
          1]] (Sum[Sum[(ig[[i]][[j]] P[[i]] P[[j]]), {j, 2, 4}], {i, 2, 4}]))^(1/2)) // Simplify;
```

```
(* Polarization vectors *)
(* P^μ = k^a (e_a)^μ  =>  k^i = (d^i)_μ P^μ *)
k1 = Sum[(d1[[μ]] Pu[[μ]]), {μ, 1, 4}] // Simplify;
k2 = Sum[(d2[[μ]] Pu[[μ]]), {μ, 1, 4}] // Simplify;
k3 = Sum[(d3[[μ]] Pu[[μ]]), {μ, 1, 4}] // Simplify;
(* Definition of the linear polarization covectors v, w and vectors vu, wu *)
v = (-k2/Sqrt[k1^2 + k2^2] d1 + k1/Sqrt[k1^2 + k2^2] d2) // Simplify;
w = (k1 k3/(Sqrt[k1^2 + k2^2] Sqrt[k1^2 + k2^2 + k3^2]) d1 + k2 k3/(Sqrt[k1^2 + k2^2] Sqrt[k1^2 + k2^2 + k3^2]) d2 - Sqrt[k1^2 + k2^2]/Sqrt[k1^2 + k2^2 + k3^2] d3) //
   Simplify;
vu = Table[(Sum[(ig[[j]][[i]] v[[i]]), {i, 1, 4}]) // Simplify, {j, 1, 4}] // Parallelize;
wu = Table[(Sum[(ig[[j]][[i]] w[[i]]), {i, 1, 4}]) // Simplify, {j, 1, 4}] // Parallelize;

(* ∇_μ v^α = ∂_μ v^α + Γ^α_μρ v^ρ *)
Dvu = Table[(D[vu[[α]], X[[μ]]] + Sum[(Γ[[α]][[μ]][[ρ]] vu[[ρ]]), {ρ, 1, 4}]) // Simplify,
    {μ, 1, 4}, {α, 1, 4}] // Parallelize;
(* ∂/∂p_μ v^α *)
Dkvu = Table[D[vu[[α]], P[[μ]]] // Simplify, {μ, 1, 4}, {α, 1, 4}] // Parallelize;
(* ∇_μ w_α = ∂_μ w_α - Γ^ρ_μα w_ρ *)
Dw = Table[(D[w[[α]], X[[μ]]] - Sum[(Γ[[ρ]][[μ]][[α]] w[[ρ]]), {ρ, 1, 4}]) // Simplify,
    {μ, 1, 4}, {α, 1, 4}] // Parallelize;
(* ∂/∂p_μ w_α *)
Dkw = Table[D[w[[α]], P[[μ]]] // Simplify, {μ, 1, 4}, {α, 1, 4}] // Parallelize;

(* (F_pp)^νμ = ∂v^α/∂p_ν ∂w_α/∂p_μ - ∂v^α/∂p_μ ∂w_α/∂p_ν *)
Fpp =
  Table[(Sum[(Dkvu[[ν]][[α]] Dkw[[μ]][[α]] - Dkvu[[μ]][[α]] Dkw[[ν]][[α]]), {α, 1, 4}]) // Simplify,
    {ν, 1, 4}, {μ, 1, 4}] // Parallelize;
(* Ṗ_ν (F_pp)^νμ *)
```

```
fpp = Table[(Sum[(Pd[[ν]] Fpp[[ν]][[μ]]), {ν, 1, 4}]) // Simplify, {μ, 1, 4}] // Parallelize;

(* (F_xx)_νμ = R^α_βμν w_α v^β + ∇_ν v^α ∇_μ w_α - ∇_μ v^α ∇_ν w_α *)

Fxx1 = Table[(Sum[Sum[(Riem[[α]][[β]][[μ]][[ν]] w[[α]] vu[[β]]), {β, 1, 4}], {α, 1, 4}]),
    {ν, 1, 4}, {μ, 1, 4}] // Parallelize;
Fxx2 = Table[(Sum[(Dvu[[ν]][[α]] Dw[[μ]][[α]] - Dvu[[μ]][[α]] Dw[[ν]][[α]]), {α, 1, 4}]),
    {ν, 1, 4}, {μ, 1, 4}] // Parallelize;

fxx1 = Table[(Sum[(Pu[[ν]] Fxx1[[ν]][[μ]]), {ν, 1, 4}]) // Simplify, {μ, 1, 4}] // Parallelize;
fxx2 = Table[(Sum[(Pu[[ν]] Fxx2[[ν]][[μ]]), {ν, 1, 4}]), {μ, 1, 4}] // Parallelize;
fxx = fxx1 + fxx2;

(* (F_px)_ν^μ = ∇_ν v^α ∂w_α/∂p_μ - ∂v^α/∂p_μ ∇_ν w_α *)

Fpx = Table[(Sum[(Dvu[[ν]][[α]] Dkw[[μ]][[α]] - Dkvu[[μ]][[α]] Dw[[ν]][[α]] ), {α, 1, 4}]) // Simplify,
    {ν, 1, 4}, {μ, 1, 4}] // Parallelize;
fpx = Table[(Sum[(Pu[[ν]] Fpx[[ν]][[μ]]), {ν, 1, 4}]), {μ, 1, 4}] // Parallelize;
fxp = Table[-(Sum[(Pd[[ν]] Fpx[[μ]][[ν]]), {ν, 1, 4}]), {μ, 1, 4}] // Parallelize;
```

In[286]:=

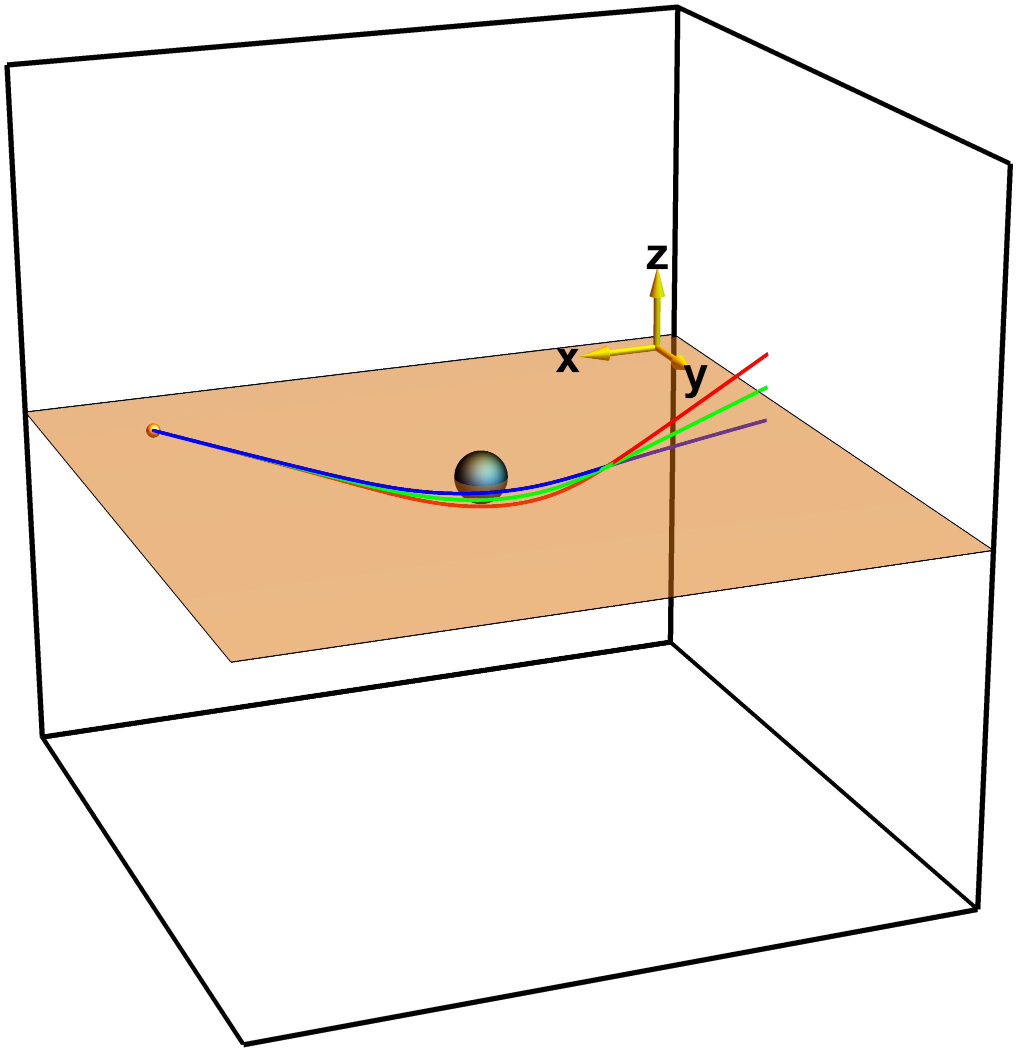
```
(* Trajectories *)

Clear[a, s, ϵ];
pu = Table[Pu[[i]], {i, 2, 4}];
pd = Table[Pd[[i]], {i, 2, 4}];
fxxi = Table[fxx[[i]], {i, 2, 4}];
fxpi = Table[fxp[[i]], {i, 2, 4}];

(* EOM *)

EOM0 = {D[X, τ] == Pu, D[p, τ] == pd} /. p0[τ] → pt0;
EOM1 = {D[X, τ] == Pu + s ϵ fpx + s ϵ fpp, D[p, τ] == pd - s ϵ fxxi - s ϵ fxpi} /. p0[τ] → pt0;
EOM2 = {D[X, τ] == Pu - s ϵ fpx - s ϵ fpp, D[p, τ] == pd + s ϵ fxxi + s ϵ fxpi} /. p0[τ] → pt0;

(* Initial conditions *)
(* integration time τmax, small parameter/wavelength ϵ, Kerr parameter a  *)
τ0 = 0;
τmax = 5 × 10^1;
```

```
ϵ = 7 × 10^-1;
a = 90 / 100;

(* spin/polarization *)
s = 1;

(* initial position *)
x0i = 0;
x1i = 10 rs;
x2i = π / 2;
x3i = π / 2 + π / 4 + π;

(* initial normalized momentum Pid *)
ψ = π / (24 / 10);
ρ = π / 2;
Pid = (-d0 + Sin[ψ] Sin[ρ] d1 + Sin[ψ] Cos[ρ] d2 + Cos[ψ] d3);
p1i = -Pid[[2]] /. τ → τ0;
p2i = Pid[[3]] /. τ → τ0;
p3i = Pid[[4]] /. τ → τ0;

(* initial data vector *)
id = (X /. τ → τ0) == {x0i, x1i, x2i, x3i} && (p /. τ → τ0) == {p1i, p2i, p3i};

(* stop if integration hits event horizon x1 = 2rs *)
τint0 = τmax;
horizon0 = WhenEvent[x1[τ] ≤ 1.01 (M + Sqrt[M^2 - a^2]), {"StopIntegration", τint0 = τ}];
τint1 = τmax;
horizon1 = WhenEvent[x1[τ] ≤ 1.01 (M + Sqrt[M^2 - a^2]), {"StopIntegration", τint1 = τ}];
τint2 = τmax;
horizon2 = WhenEvent[x1[τ] ≤ 1.01 (M + Sqrt[M^2 - a^2]), {"StopIntegration", τint2 = τ}];

(* Integration *)

sol0 = NDSolve[EOM0 && id && horizon0, {x0, x1, x2, x3, p1, p2, p3}, {τ, τ0, τmax}];
sol1 = NDSolve[EOM1 && id && horizon1, {x0, x1, x2, x3, p1, p2, p3}, {τ, τ0, τmax}];
sol2 = NDSolve[EOM2 && id && horizon2, {x0, x1, x2, x3, p1, p2, p3}, {τ, τ0, τmax}];



(* Plots *)

(* Plot range *)
range = 10 rs;
τplot0 = Min[τint0, τmax];
τplot1 = Min[τint1, τmax];
τplot2 = Min[τint2, τmax];
```

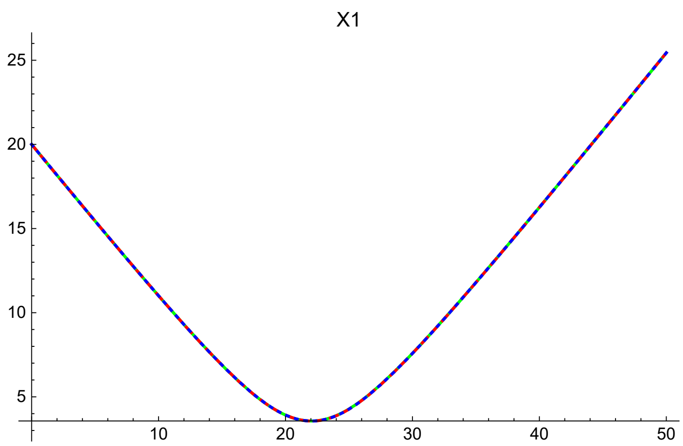
```
(* 3D Plot *)
arrow1 = {Arrowheads[0.025], Arrow[
    Tube[{{-0.9 range, -0.9 range, 0}, {-0.9 range + range / 4, -0.9 range, 0}}, 0.2], -0]};
arrow2 = {Arrowheads[0.025], Arrow[Tube[{{-0.9 range, -0.9 range, 0},
      {-0.9 range, -0.9 range + range / 4, 0}}, 0.2], -0]};
arrow3 = {Arrowheads[0.025], Arrow[Tube[{{-0.9 range, -0.9 range, 0},
      {-0.9 range, -0.9 range, range / 4}}, 0.2], -0]};
arrowtext1 = Text[Style["x", Large, Bold], {-0.9 range + range / 3.5, -0.9 range, 0}];
arrowtext2 = Text[Style["y", Large, Bold], {-0.9 range, -0.9 range + range / 3.5, 0}];
arrowtext3 = Text[Style["z", Large, Bold], {-0.9 range, -0.9 range, range / 3.5}];
frame =
  Graphics3D[{Yellow, arrow1, arrow2, arrow3, Black, arrowtext1, arrowtext2, arrowtext3}];
wall1 = Graphics3D[{Transparent, EdgeForm[Thick], Polygon[{{-range, -range, -range},
      {-range, range, -range}, {range, range, -range}, {range, -range, -range}}]}];
wall2 = Graphics3D[{Transparent, EdgeForm[Thick], Polygon[{{-range, range, -range},
      {-range, -range, -range}, {-range, -range, range}, {-range, range, range}}]}];
wall3 = Graphics3D[{Transparent, EdgeForm[Thick], Polygon[{{-range, -range, -range},
      {-range, -range, range}, {range, -range, range}, {range, -range, -range}}]}];
source = Graphics3D[{Specularity[White, 5], Orange,
    Sphere[{x1i Sin[x2i] Cos[x3i], x1i Sin[x2i] Sin[x3i], x1i Cos[x2i]}, 0.2 rs]}];
plane = ContourPlot3D[z == 0, {x, -range, range}, {y, -range, range},
   {z, -range, range}, Mesh → None, ContourStyle → Opacity[0.5]];
BH = Graphics3D[{Specularity[White, 3], Black, Sphere[{0, 0, 0}, M + Sqrt[M^2 - a^2]]}];

g0 = ParametricPlot3D[{Evaluate[(x1[τ] Sin[x2[τ]] Cos[x3[τ]]) /. sol0][[1]],
    Evaluate[(x1[τ] Sin[x2[τ]] Sin[x3[τ]]) /. sol0][[1]],
    Evaluate[(x1[τ] Cos[x2[τ]]) /. sol0][[1]]},
   {τ, τ0, τplot0}, PlotStyle → {Thick, Green}, PlotPoints → 3000];
g1 = ParametricPlot3D[{Evaluate[(x1[τ] Sin[x2[τ]] Cos[x3[τ]]) /. sol1][[1]],
    Evaluate[(x1[τ] Sin[x2[τ]] Sin[x3[τ]]) /. sol1][[1]],
    Evaluate[(x1[τ] Cos[x2[τ]]) /. sol1][[1]]},
   {τ, τ0, τplot1}, PlotStyle → {Thick, Red}, PlotPoints → 3000];
g2 = ParametricPlot3D[{Evaluate[(x1[τ] Sin[x2[τ]] Cos[x3[τ]]) /. sol2][[1]],
    Evaluate[(x1[τ] Sin[x2[τ]] Sin[x3[τ]]) /. sol2][[1]],
    Evaluate[(x1[τ] Cos[x2[τ]]) /. sol2][[1]]},
   {τ, 0, τplot2}, PlotStyle → {Thick, Blue}, PlotPoints → 3000];

Style[Show[g0, g1, g2, BH, source, plane, wall1, wall2, wall3,
  frame, PlotRange → {{-range, range}, {-range, range}, {-range, range}},
  Boxed → False, Axes → False, ViewPoint → {1.5, 4, 1.5}], Antialiasing → True]

(* Individual Plots *)
Show[Plot[{Evaluate[(x1[τ]) /. sol0][[1]]}, {τ, τ0, τplot0},
  PlotStyle → Green, PlotLegends → Automatic, PlotLabel → "X1", PlotRange → All],
 Plot[{Evaluate[(x1[τ]) /. sol1][[1]]}, {τ, τ0, τplot1}, PlotStyle → {Red, Dashed},
  PlotLegends → Automatic, PlotLabel → "X1", PlotRange → All],
 Plot[{Evaluate[(x1[τ]) /. sol2][[1]]}, {τ, τ0, τplot2}, PlotStyle → {Blue, Dotted},
  PlotLegends → Automatic, PlotLabel → "X1", PlotRange → All]]
```

```
Show[Plot[{Evaluate[(x2[τ]) /. sol0][[1]]}, {τ, τ0, τplot0},
   PlotStyle → Green, PlotLegends → Automatic, PlotLabel → "X2", PlotRange → All],
  Plot[{Evaluate[(x2[τ]) /. sol1][[1]]}, {τ, τ0, τplot1}, PlotStyle → {Red, Dashed},
   PlotLegends → Automatic, PlotLabel → "X2", PlotRange → All],
  Plot[{Evaluate[(x2[τ]) /. sol2][[1]]}, {τ, τ0, τplot2}, PlotStyle → {Blue, Dotted},
   PlotLegends → Automatic, PlotLabel → "X2", PlotRange → All]]

Show[Plot[{Evaluate[(x3[τ]) /. sol0][[1]]}, {τ, τ0, τplot0},
   PlotStyle → Green, PlotLegends → Automatic, PlotLabel → "X3", PlotRange → All],
  Plot[{Evaluate[(x3[τ]) /. sol1][[1]]}, {τ, τ0, τplot1}, PlotStyle → {Red, Dashed},
   PlotLegends → Automatic, PlotLabel → "X3", PlotRange → All],
  Plot[{Evaluate[(x3[τ]) /. sol2][[1]]}, {τ, τ0, τplot2}, PlotStyle → {Blue, Dotted},
   PlotLegends → Automatic, PlotLabel → "X3", PlotRange → All]]

Show[Plot[{Evaluate[(p1[τ]) /. sol0][[1]]}, {τ, τ0, τplot0},
   PlotStyle → Green, PlotLegends → Automatic, PlotLabel → "P1", PlotRange → All],
  Plot[{Evaluate[(p1[τ]) /. sol1][[1]]}, {τ, τ0, τplot1}, PlotStyle → {Red, Dashed},
   PlotLegends → Automatic, PlotLabel → "P1", PlotRange → All],
  Plot[{Evaluate[(p1[τ]) /. sol2][[1]]}, {τ, τ0, τplot2}, PlotStyle → {Blue, Dotted},
   PlotLegends → Automatic, PlotLabel → "P1", PlotRange → All]]

Show[Plot[{Evaluate[(p2[τ]) /. sol0][[1]]}, {τ, τ0, τplot0},
   PlotStyle → Green, PlotLegends → Automatic, PlotLabel → "P2", PlotRange → All],
  Plot[{Evaluate[(p2[τ]) /. sol1][[1]]}, {τ, τ0, τplot1}, PlotStyle → {Red, Dashed},
   PlotLegends → Automatic, PlotLabel → "P2", PlotRange → All],
  Plot[{Evaluate[(p2[τ]) /. sol2][[1]]}, {τ, τ0, τplot2}, PlotStyle → {Blue, Dotted},
   PlotLegends → Automatic, PlotLabel → "P2", PlotRange → All]]

Show[Plot[{Evaluate[(p3[τ]) /. sol0][[1]]}, {τ, τ0, τplot0},
   PlotStyle → Green, PlotLegends → Automatic, PlotLabel → "P3", PlotRange → All],
  Plot[{Evaluate[(p3[τ]) /. sol1][[1]]}, {τ, τ0, τplot1}, PlotStyle → {Red, Dashed},
   PlotLegends → Automatic, PlotLabel → "P3", PlotRange → All],
  Plot[{Evaluate[(p3[τ]) /. sol2][[1]]}, {τ, τ0, τplot2}, PlotStyle → {Blue, Dotted},
   PlotLegends → Automatic, PlotLabel → "P3", PlotRange → All]]
```

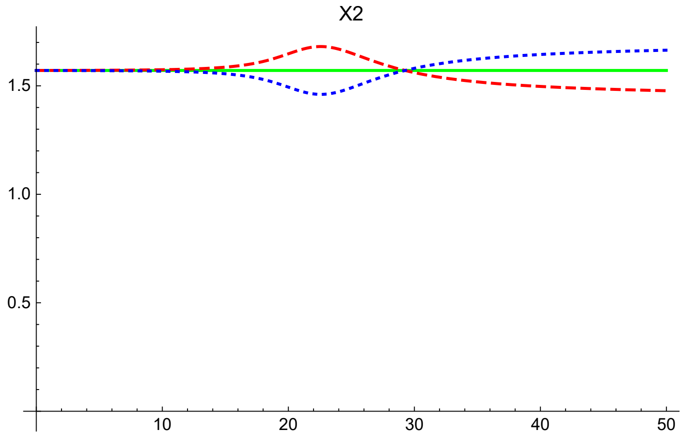
Out[339]=



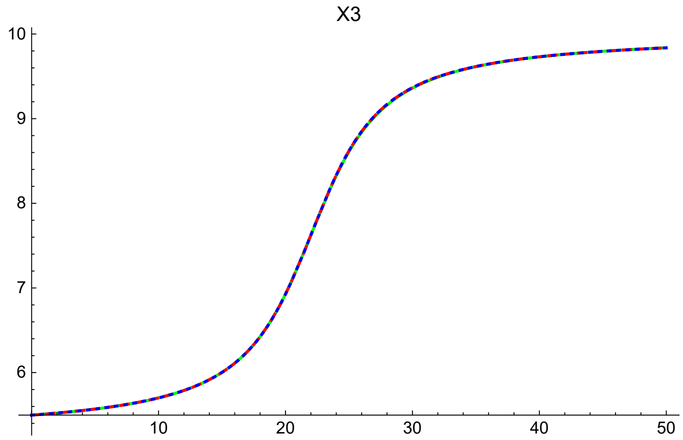
Out[340]=



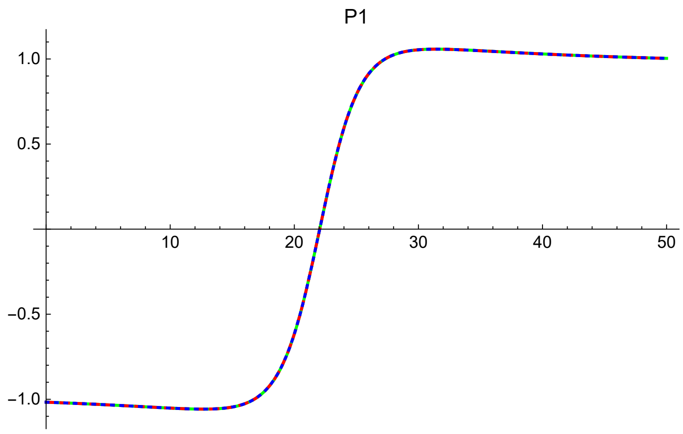
Out[341]=



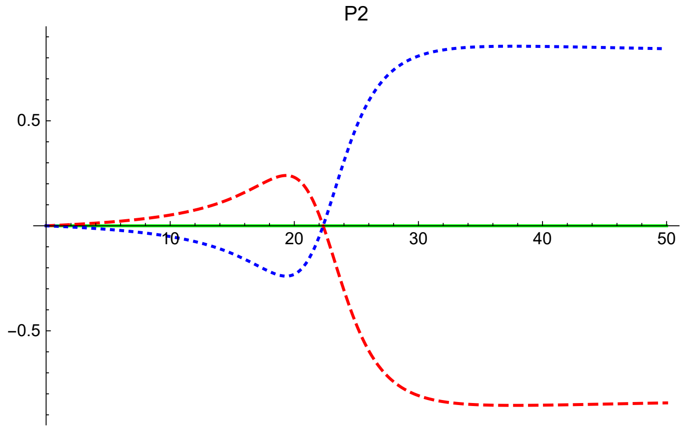
Out[342]=



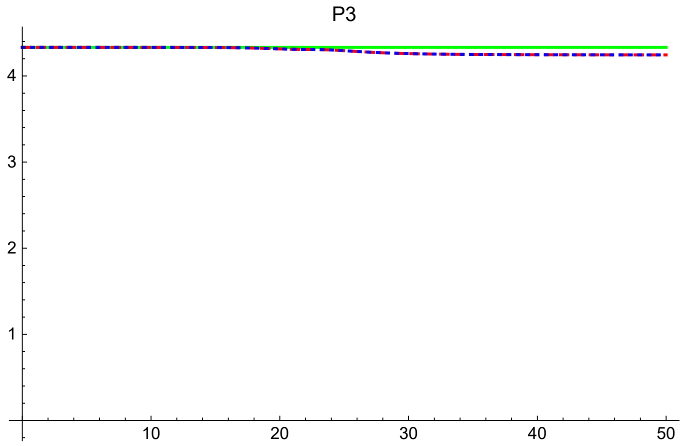
Out[343]=



Out[344]=



Out[345]=



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Bibliography

[1] Ralph Abraham and Jerrold E. Marsden. *Foundations of Mechanics*. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978. ISBN 0-8053-0102-X. URL https://doi.org/10.1090/chel/364. Second edition.

[2] Y. Aharonov and J. Anandan. Phase change during a cyclic quantum evolution. *Phys. Rev. Lett.*, 58:1593–1596, Apr 1987. doi: 10.1103/PhysRevLett.58.1593. URL https://link.aps.org/doi/10.1103/PhysRevLett.58.1593.

[3] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman. Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes. *Phys. Rev. A*, 45:8185–8189, Jun 1992. doi: 10.1103/PhysRevA.45.8185. URL https://link.aps.org/doi/10.1103/PhysRevA.45.8185.

[4] J. Anandan. Non-adiabatic non-abelian geometric phase. *Phys. Lett. A*, 133(4):171–175, 1988. ISSN 0375-9601. doi: https://doi.org/10.1016/0375-9601(88)91010-9. URL http://www.sciencedirect.com/science/article/pii/0375960188910109.

[5] J. Anandan. Comment on geometric phases for classical field theories. *Phys. Rev. Lett.*, 60:2555–2555, Jun 1988. doi: 10.1103/PhysRevLett.60.2555. URL https://link.aps.org/doi/10.1103/PhysRevLett.60.2555.

[6] J. Anandan. Topological and geometrical phases due to gravitational field with curvature and torsion. *Phys. Lett. A*, 195(5):284–292, 1994. ISSN 0375-9601. doi: https://doi.org/10.1016/0375-9601(94)90030-2. URL http://www.sciencedirect.com/science/article/pii/0375960194900302.

[7] L. Andersson, J. Joudioux, M. A. Oancea, and A. Raj. Propagation of polarized gravitational waves. *Physical Review D*, 103:044053, Feb 2021. doi: 10.1103/PhysRevD.103.044053. URL https://link.aps.org/doi/10.1103/PhysRevD.103.044053.

[8] D. L. Andrews and M. Babiker. *The Angular Momentum of Light*. Cambridge University Press, 2012. doi: 10.1017/CBO9780511795213.

[9] J. Audretsch. Trajectories and spin motion of massive spin-$\frac{1}{2}$ particles in gravitational fields. *J. Phys. A*, 14:411–422, 1981. doi: 10.1088/0305-4470/14/2/017.

[10] K. Bakke, C. Furtado, and J. R. Nascimento. Gravitational geometric phase in the presence of torsion. *The European Physical Journal C*, 60(3):501, Mar 2009. ISSN 1434-6052. doi: 10.1140/epjc/s10052-009-0944-z. URL https://doi.org/10.1140/epjc/s10052-009-0944-z.

[11] A. A. Bakun, B. P. Zakharchenya, A. A. Rogachev, M. N. Tkachuk, and V. G. Fleĭsher. Observation of a surface photocurrent caused by optical orientation of electrons in a semiconductor. *JETP Lett.*, 40:1293, December 1984. URL http://www.jetpletters.ac.ru/ps/1262/article_19087.shtml.

[12] P. Baral, A. Ray, R. Koley, and P. Majumdar. Gravitational waves with orbital angular momentum. *arXiv preprint arXiv:1901.08804*, 2019.

[13] E. Barausse, E. Racine, and A. Buonanno. Hamiltonian of a spinning test particle in curved spacetime. *Phys. Rev. D*, 80(10):104025, 2009.

[14] C. Barceló, S. Liberati, and M. Visser. Analogue gravity. *Living Reviews in Relativity*, 14(1):3, May 2011. ISSN 1433-8351. doi: 10.12942/lrr-2011-3. URL https://doi.org/10.12942/lrr-2011-3.

[15] V. Bargmann and E. P. Wigner. Group theoretical discussion of relativistic wave equations. *Proceedings of the National Academy of Sciences*, 34(5): 211–223, 1948. ISSN 0027-8424. doi: 10.1073/pnas.34.5.211. URL http://www.pnas.org/content/34/5/211.

[16] S. M. Barnett. Relativistic electron vortices. *Phys. Rev. Lett.*, 118:114802, Mar 2017. doi: 10.1103/PhysRevLett.118.114802. URL https://link.aps.org/doi/10.1103/PhysRevLett.118.114802.

[17] S. M. Barnett, L. Allen, and M. J. Padgett. *Optical angular momentum*. CRC Press, 2016.

[18] Sean Bates and Alan Weinstein. *Lectures on the Geometry of Quantization*, volume 8 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1997. ISBN 0-8218-0798-6. URL https://bookstore.ams.org/bmln-8.

[19] A. Bérard and H. Mohrbach. Spin Hall effect and Berry phase of spinning particles. *Phys. Lett. A*, 352(3):190–195, 2006. ISSN 0375-9601. doi: https://doi.org/10.1016/j.physleta.2005.11.071. URL http://www.sciencedirect.com/science/article/pii/S0375960105018116.

[20] M. V. Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 392(1802):45–57, 1984. ISSN 0080-4630. doi: 10.1098/rspa.1984.0023. URL http://rspa.royalsocietypublishing.org/content/392/1802/45.

[21] M. V. Berry. The adiabatic phase and Pancharatnam's phase for polarized light. *Journal of Modern Optics*, 34(11):1401–1407, 1987. doi: 10.1080/09500348714551321. URL https://doi.org/10.1080/09500348714551321.

[22] R. Bhandari and J. Samuel. Observation of topological phase by use of a laser interferometer. *Phys. Rev. Lett.*, 60:1211–1213, Mar 1988. doi: 10.1103/PhysRevLett.60.1211. URL https://link.aps.org/doi/10.1103/PhysRevLett.60.1211.

[23] I. Bialynicki-Birula. On the wave function of the photon. *Acta Phys. Pol. A*, 86(1):97–116, 7 1994.

[24] I. Bialynicki-Birula. V Photon wave function. volume 36 of *Progress in Optics*, pages 245–294. Elsevier, 1996. doi: https://doi.org/10.1016/S0079-6638(08)70316-0. URL http://www.sciencedirect.com/science/article/pii/S0079663808703160.

[25] I. Bialynicki-Birula and Z. Bialynicka-Birula. Berry's phase in the relativistic theory of spinning particles. *Phys. Rev. D*, 35:2383–2387, Apr 1987. doi: 10.1103/PhysRevD.35.2383. URL https://link.aps.org/doi/10.1103/PhysRevD.35.2383.

[26] I. Bialynicki-Birula and Z. Bialynicka-Birula. Exponential beams of electromagnetic radiation. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 39(15):S545, 2006. URL http://stacks.iop.org/0953-4075/39/i=15/a=S03.

[27] I. Bialynicki-Birula and Z. Bialynicka-Birula. Gravitational waves carrying orbital angular momentum. *New J. Phys.*, 18(2):023022, 2016. URL http://stacks.iop.org/1367-2630/18/i=2/a=023022.

[28] I. Bialynicki-Birula and Z. Bialynicka-Birula. Relativistic electron wave packets carrying angular momentum. *Phys. Rev. Lett.*, 118:114801, Mar 2017. doi: 10.1103/PhysRevLett.118.114801. URL https://link.aps.org/doi/10.1103/PhysRevLett.118.114801.

[29] Iwo Bialynicki-Birula and Szymon Charzyński. Trapping and guiding bodies by gravitational waves endowed with angular momentum. *Phys. Rev. Lett.*, 121(17):171101, 2018. doi: 10.1103/PhysRevLett.121.171101.

[30] K. Y. Bliokh. Geometrical optics of beams with vortices: Berry phase and orbital angular momentum Hall effect. *Phys. Rev. Lett.*, 97:043901, Jul 2006. doi: 10.1103/PhysRevLett.97.043901. URL https://link.aps.org/doi/10.1103/PhysRevLett.97.043901.

[31] K. Y. Bliokh. Geometrodynamics of polarized light: Berry phase and spin Hall effect in a gradient-index medium. *J. Opt. A: Pure Appl. Opt.*, 11 (9):094009, aug 2009. doi: 10.1088/1464-4258/11/9/094009. URL https://doi.org/10.1088%2F1464-4258%2F11%2F9%2F094009.

[32] K. Y. Bliokh and A. Aiello. Goos–Hänchen and Imbert–Fedorov beam shifts: An overview. *Journal of Optics*, 15(1):014001, 2013. URL http://stacks.iop.org/2040-8986/15/i=1/a=014001.

[33] K. Y. Bliokh and Y. P. Bliokh. Modified geometrical optics of a smoothly inhomogeneous isotropic medium: The anisotropy, Berry phase, and the optical Magnus effect. *Phys. Rev. E*, 70:026605, Aug 2004. doi: 10.1103/PhysRevE.70.026605. URL https://link.aps.org/doi/10.1103/PhysRevE.70.026605.

[34] K. Y. Bliokh and Y. P. Bliokh. Topological spin transport of photons: The optical Magnus effect and Berry phase. *Phys. Lett. A*, 333(3):181–186, 2004. ISSN 0375-9601. doi: https://doi.org/10.1016/j.physleta.2004.10.035. URL http://www.sciencedirect.com/science/article/pii/S037596010401480X.

[35] K. Y. Bliokh and F. Nori. Transverse and longitudinal angular momenta of light. *Physics Reports*, 592:1–38, 2015. ISSN 0370-1573. doi: https://doi.org/10.1016/j.physrep.2015.06.003. URL http://www.sciencedirect.com/science/article/pii/S0370157315003336.

[36] K. Y. Bliokh, A. Niv, V. Kleiner, and E. Hasman. Geometrodynamics of spinning light. *Nat. Photonics*, 2:748, Nov 2008. URL http://dx.doi.org/10.1038/nphoton.2008.229.

[37] K. Y. Bliokh, F. J. Rodríguez-Fortuño, F. Nori, and A. V. Zayats. Spin-orbit interactions of light. *Nat. Photonics*, 9(12):796–808, 2015. doi: 10.1038/nphoton.2015.201.

[38] Konstantin Y. Bliokh and Franco Nori. Relativistic Hall Effect. *Phys. Rev. Lett.*, 108:120403, Mar 2012. doi: 10.1103/PhysRevLett.108.120403. URL https://link.aps.org/doi/10.1103/PhysRevLett.108.120403.

[39] Konstantin Y Bliokh, Justin Dressel, and Franco Nori. Conservation of the spin and orbital angular momenta in electromagnetism. *New Journal of Physics*, 16(9):093037, sep 2014. doi: 10.1088/1367-2630/16/9/093037. URL https://doi.org/10.1088%2F1367-2630%2F16%2F9%2F093037.

[40] K. Bolonek-Lasoń, P. Kosiński, and P. Maślanka. Lorentz transformations, sideways shift and massless spinning particles. *Physics Letters B*, 769:117 – 120, 2017. ISSN 0370-2693. doi: https://doi.org/10.1016/j.physletb.2017.03.034. URL http://www.sciencedirect.com/science/article/pii/S0370269317302228.

[41] Jens Bolte and Stefan Keppeler. A semiclassical approach to the Dirac equation. *Annals of Physics*, 274(1):125 – 162, 1999. ISSN 0003-4916. doi: https://doi.org/10.1006/aphy.1999.5912. URL http://www.sciencedirect.com/science/article/pii/S0003491699959122.

[42] A. Brodutch and D. R. Terno. Polarization rotation, reference frames, and Mach's principle. *Phys. Rev. D*, 84:121501, Dec 2011. doi: 10.1103/PhysRevD.84.121501. URL https://link.aps.org/doi/10.1103/PhysRevD.84.121501.

[43] A. Brodutch, T. F. Demarie, and D. R. Terno. Photon polarization and geometric phase in general relativity. *Phys. Rev. D*, 84:104043, Nov 2011. doi: 10.1103/PhysRevD.84.104043. URL https://link.aps.org/doi/10.1103/PhysRevD.84.104043.

[44] Y. Q. Cai and G. Papini. Applying Berry's phase to problems involving weak gravitational and inertial fields. *Classical and Quantum Gravity*, 7(2):269, 1990. URL http://stacks.iop.org/0264-9381/7/i=2/a=021.

[45] Y. Q. Cai, G. Papini, and W. R. Wood. Berry's phase for photons and topology in Maxwell's theory. *Journal of Mathematical Physics*, 31(8):1942–1946, 1990. doi: 10.1063/1.528641. URL https://doi.org/10.1063/1.528641.

[46] K. M. Case. Some generalizations of the Foldy-Wouthuysen transformation. *Physical Review*, 95:1323–1328, Sep 1954. doi: 10.1103/PhysRev.95.1323. URL https://link.aps.org/doi/10.1103/PhysRev.95.1323.

[47] S. Chandrasekhar. *The Mathematical Theory of Black Holes*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, 1998. ISBN 0-19-850370-9. Reprint of the 1992 edition.

[48] R. Y. Chiao and Y.-S. Wu. Manifestations of Berry's topological phase for the photon. *Phys. Rev. Lett.*, 57:933–936, Aug 1986. doi: 10.1103/PhysRevLett.57.933. URL https://link.aps.org/doi/10.1103/PhysRevLett.57.933.

[49] Yvonne Choquet-Bruhat. *General Relativity and the Einstein Equations*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2009. ISBN 978-0-19-923072-3. doi: 10.1093/acprof:oso/9780199230723.001.0001.

[50] Yvonne Choquet-Bruhat, Cécile DeWitt-Morette, and Margaret Dillard-Bleick. *Analysis, manifolds and physics*. North-Holland Publishing Co., Amsterdam-New York, second edition, 1982. ISBN 0-444-86017-7.

[51] D. Chruściński and A. Jamiołkowski. *Geometric Phases in Classical and Quantum Mechanics*, volume 36. Springer Science & Business Media, 2012. doi: 10.1007/978-0-8176-8176-0.

[52] A. Corichi and M. Pierri. Gravity and geometric phases. *Phys. Rev. D*, 51:5870–5875, May 1995. doi: 10.1103/PhysRevD.51.5870. URL https://link.aps.org/doi/10.1103/PhysRevD.51.5870.

[53] L. F. Costa, C. Herdeiro, J. Natário, and M. Zilhao. Mathisson's helical motions for a spinning particle: Are they unphysical? *Phys. Rev. D*, 85(2): 024001, 2012.

[54] L. F. O. Costa and J. Natário. Center of mass, spin supplementary conditions, and the momentum of spinning particles. In *Equations of Motion in Relativistic Gravity*, pages 215–258. Springer, 2015.

[55] L. F. O. Costa, G. Lukes-Gerakopoulos, and O. Semerák. Spinning particles in general relativity: Momentum-velocity relation for the Mathisson-Pirani spin condition. *Phys. Rev. D*, 97(8):084023, 2018.

[56] Pedro VP Cunha and Carlos AR Herdeiro. Shadows and strong gravitational lensing: a brief review. *Gen. Relativ. Gravit.*, 50(4):42, 2018.

[57] W. G. Dixon. A covariant multipole formalism for extended test bodies in general relativity. *Il Nuovo Cimento (1955-1965)*, 34(2):317–339, 1964. doi: https://doi.org/10.1007/BF02734579.

[58] W. G. Dixon. The definition of multipole moments for extended bodies. *Gen. Relativ. Gravit.*, 4(3):199–209, 1973.

[59] W. G. Dixon. *The New Mechanics of Myron Mathisson and Its Subsequent Development*, pages 1–66. Springer International Publishing, Cham, 2015. ISBN 978-3-319-18335-0. doi: 10.1007/978-3-319-18335-0_1. URL https://doi.org/10.1007/978-3-319-18335-0_1.

[60] I. Y. Dodin, D. E. Ruiz, K. Yanagihara, Y. Zhou, and S. Kubo. Quasioptical modeling of wave beams with and without mode conversion. I. Basic theory. *Phys. Plasmas*, 26(7):072110, 2019. doi: 10.1063/1.5095076. URL https://doi.org/10.1063/1.5095076.

[61] S. R. Dolan. Higher-order geometrical optics for circularly-polarized electromagnetic waves. *arXiv:1801.02273*, 2018.

[62] S. R. Dolan. Geometrical optics for scalar, electromagnetic and gravitational waves on curved spacetime. *Int. J. Mod. Phys. D*, 27:1843010, 2018. doi: 10.1142/S0218271818430101. URL https://www.worldscientific.com/doi/abs/10.1142/S0218271818430101.

[63] A. V. Dooghin, N. D. Kundikova, V. S. Liberman, and B. Ya. Zel'dovich. Optical Magnus effect. *Phys. Rev. A*, 45:8204–8208, Jun 1992. doi: 10.1103/PhysRevA.45.8204. URL https://link.aps.org/doi/10.1103/PhysRevA.45.8204.

[64] J. J. Duistermaat. *Fourier Integral Operators*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2011. ISBN 978-0-8176-8107-4. doi: 10.1007/978-0-8176-8108-1. URL https://doi.org/10.1007/978-0-8176-8108-1.

[65] C. Duval. Polarized spinoptics and symplectic physics. *arXiv preprint arXiv:1312.4486*, 2013.

[66] C. Duval and T. Schücker. Gravitational birefringence of light in Robertson-Walker cosmologies. *Phys. Rev. D*, 96:043517, Aug 2017. doi: 10.1103/PhysRevD.96.043517. URL https://link.aps.org/doi/10.1103/PhysRevD.96.043517.

[67] C. Duval, Z. Horváth, and P. A. Horváthy. Fermat principle for spinning light. *Phys. Rev. D*, 74:021701(R), Jul 2006. doi: 10.1103/PhysRevD.74.021701. URL https://link.aps.org/doi/10.1103/PhysRevD.74.021701.

[68] C. Duval, Z. Horváth, and P. A. Horváthy. Geometrical spinoptics and the optical Hall effect. *J. Geom. Phys.*, 57(3):925–941, 2007. ISSN 0393-0440. doi: https://doi.org/10.1016/j.geomphys.2006.07.003. URL http://www.sciencedirect.com/science/article/pii/S0393044006000878.

[69] C. Duval, M. Elbistan, P.A. Horváthy, and P.-M. Zhang. Wigner–Souriau translations and Lorentz symmetry of chiral fermions. *Physics Letters B*, 742:322 – 326, 2015. ISSN 0370-2693. doi: https://doi.org/10.1016/j.physletb.2015.01.048. URL http://www.sciencedirect.com/science/article/pii/S0370269315000581.

[70] C. Duval, L. Marsot, and T. Schücker. Gravitational birefringence of light in Schwarzschild spacetime. *Phys. Rev. D*, 99:124037, Jun 2019. doi: 10.1103/PhysRevD.99.124037. URL https://link.aps.org/doi/10.1103/PhysRevD.99.124037.

[71] M. I. Dyakonov and A. V. Khaetskii. *Spin Hall Effect*, pages 211–243. Springer Berlin Heidelberg, 2008. ISBN 978-3-540-78820-1. doi: 10.1007/978-3-540-78820-1_8. URL https://doi.org/10.1007/978-3-540-78820-1_8.

[72] M. I. Dyakonov and V. I. Perel. Possibility of orienting electron spins with current. *JETP Lett.*, 13:467, June 1971.

[73] M. I. Dyakonov and V. I. Perel. Current-induced spin orientation of electrons in semiconductors. *Phys. Lett. A*, 35(6):459–460, 1971. ISSN 0375-9601. doi: https://doi.org/10.1016/0375-9601(71)90196-4. URL http://www.sciencedirect.com/science/article/pii/0375960171901964.

[74] A. S. Eddington. *Space, time and gravitation: An outline of the general relativity theory*. Cambridge University Press, 1987.

[75] C. Emmrich and H. Römer. Multicomponent Wentzel–Kramers–Brillouin approximation on arbitrary symplectic manifolds: A star product approach. *Journal of Mathematical Physics*, 39(7):3530–3546, 1998. doi: 10.1063/1.532450. URL https://doi.org/10.1063/1.532450.

[76] C. Emmrich and A. Weinstein. Geometry of the transport equation in multicomponent WKB approximations. *Communications in Mathematical Physics*, 176(3):701–711, Mar 1996. ISSN 1432-0916. doi: 10.1007/BF02099256. URL https://doi.org/10.1007/BF02099256.

[77] The Event Horizon Telescope Collaboration, K Akiyama, A Alberdi, et al. First M87 Event Horizon Telescope Results. I. the Shadow of the Supermassive Black Hole. *The Astrophysical Journal Letters*, 875(L1):17pp, 2019.

[78] M. Fathi and R. T. Thompson. Cartographic distortions make dielectric spacetime analog models imperfect mimickers. *Phys. Rev. D*, 93:124026, Jun 2016. doi: 10.1103/PhysRevD.93.124026. URL https://link.aps.org/doi/10.1103/PhysRevD.93.124026.

[79] F. I. Fedorov. To the theory of total reflection. *Journal of Optics*, 15(1): 014002, 2013.

[80] L.-L. Feng and W. Lee. Gravitomagnetism and the Berry phase of photon in an rotating gravitational field. *International Journal of Modern Physics D*, 10(06):961–969, 2001. doi: 10.1142/S0218271801001402. URL https://doi.org/10.1142/S0218271801001402.

[81] R. Fickler, G. Campbell, B. Buchler, P. K. Lam, and A. Zeilinger. Quantum entanglement of angular momentum states with quantum numbers up to 10,010. *Proceedings of the National Academy of Sciences*, 113(48):13642–13647, 2016. ISSN 0027-8424. doi: 10.1073/pnas.1616889113. URL http://www.pnas.org/content/113/48/13642.

[82] L. L. Foldy and S. A. Wouthuysen. On the Dirac theory of spin 1/2 particles and its non-relativistic limit. *Physical Review*, 78:29–36, Apr 1950. doi: 10.1103/PhysRev.78.29. URL https://link.aps.org/doi/10.1103/PhysRev.78.29.

[83] G. R. Fowles. *Introduction to Modern Optics*. Courier Corporation, 1989.

[84] V. P. Frolov. Maxwell equations in a curved spacetime: Spin optics approximation. *Phys. Rev. D*, 102:084013, Oct 2020. doi: 10.1103/PhysRevD.102.084013. URL https://link.aps.org/doi/10.1103/PhysRevD.102.084013.

[85] V. P. Frolov and A. A. Shoom. Spinoptics in a stationary spacetime. *Phys. Rev. D*, 84:044026, Aug 2011. doi: 10.1103/PhysRevD.84.044026. URL https://link.aps.org/doi/10.1103/PhysRevD.84.044026.

[86] V. P. Frolov and A. A. Shoom. Scattering of circularly polarized light by a rotating black hole. *Phys. Rev. D*, 86:024010, Jul 2012. doi: 10.1103/PhysRevD.86.024010. URL https://link.aps.org/doi/10.1103/PhysRevD.86.024010.

[87] S. A. Fulling. Pseudodifferential operators, covariant quantization, the inescapable van Vleck–Morette determinant, and the $\frac{R}{6}$ controversy. *Int. J. Mod. Phys. D*, 05(06):597–608, 1996. doi: 10.1142/S0218271896000382. URL https://doi.org/10.1142/S0218271896000382.

[88] S. A. H. Gangaraj, M. G. Silveirinha, and G. W. Hanson. Berry phase, Berry connection, and Chern number for a continuum bianisotropic material from a classical electromagnetics perspective. *IEEE Journal on Multiscale and Multiphysics Computational Techniques*, 2:3–17, 2017. doi: 10.1109/JMMCT.2017.2654962.

[89] J. C. Garrison and R. Y. Chiao. Geometrical phases from global gauge invariance of nonlinear classical field theories. *Phys. Rev. Lett.*, 60:165–168, Jan 1988. doi: 10.1103/PhysRevLett.60.165. URL https://link.aps.org/doi/10.1103/PhysRevLett.60.165.

[90] H. Goldstein, C. P. Poole, and J. L. Safko. *Classical Mechanics*. Addison Wesley, 2002. ISBN 9780201657029. URL https://books.google.de/books?id=tJCuQgAACAAJ.

[91] F. Goos and H. Hänchen. Ein neuer und fundamentaler versuch zur totalreflexion. *Annalen der Physik*, 436(7–8):333–346, 1947. doi: 10.1002/andp.19474360704. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/andp.19474360704.

[92] W. Gordon. Zur lichtfortpflanzung nach der relativitätstheorie. *Annalen der Physik*, 377(22):421–456, 1923. doi: 10.1002/andp.19233772202. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/andp.19233772202.

[93] P. Gosselin, A. Bérard, and H. Mohrbach. Semiclassical diagonalization of quantum Hamiltonian and equations of motion with Berry phase corrections. *The European Physical Journal B*, 58(2):137–148, Jul 2007. ISSN 1434-6036. doi: 10.1140/epjb/e2007-00212-6. URL https://doi.org/10.1140/epjb/e2007-00212-6.

[94] P. Gosselin, A. Bérard, and H. Mohrbach. Semiclassical dynamics of Dirac particles interacting with a static gravitational field. *Phys. Lett. A*, 368(5):356–361, 2007. ISSN 0375-9601. doi: https://doi.org/10.1016/j.physleta.2007.04.022. URL http://www.sciencedirect.com/science/article/pii/S0375960107005646.

[95] P. Gosselin, A. Bérard, and H. Mohrbach. Spin Hall effect of photons in a static gravitational field. *Phys. Rev. D*, 75:084035, Apr 2007.

doi: 10.1103/PhysRevD.75.084035. URL https://link.aps.org/doi/10.1103/PhysRevD.75.084035.

[96] W. Greiner and D.A. Bromley. *Relativistic quantum mechanics. Wave equations*. Springer, 2000. ISBN 9783540674573.

[97] D. Haefner, S. Sukhov, and A. Dogariu. Spin Hall effect of light in spherical geometry. *Phys. Rev. Lett.*, 102:123903, Mar 2009. doi: 10.1103/PhysRevLett.102.123903. URL https://link.aps.org/doi/10.1103/PhysRevLett.102.123903.

[98] F. D. M. Haldane. Path dependence of the geometric rotation of polarization in optical fibers. *Optics Letters*, 11(11):730–732, Nov 1986. doi: 10.1364/OL.11.000730. URL http://ol.osa.org/abstract.cfm?URI=ol-11-11-730.

[99] F. D. M. Haldane. Comment on "Observation of Berry's topological phase by use of an optical fiber". *Phys. Rev. Lett.*, 59:1788–1788, Oct 1987. doi: 10.1103/PhysRevLett.59.1788. URL https://link.aps.org/doi/10.1103/PhysRevLett.59.1788.

[100] E. H. Hall. On a new action of the magnet on electric currents. *American Journal of Mathematics*, 2(3):287–292, 1879. ISSN 00029327, 10806377. URL http://www.jstor.org/stable/2369245.

[101] A. I. Harte. Gravitational lensing beyond geometric optics: I. Formalism and observables. *Gen. Relativ. Gravit.*, 51(1):14, Jan 2019. ISSN 1572-9532. doi: 10.1007/s10714-018-2494-x. URL https://doi.org/10.1007/s10714-018-2494-x.

[102] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973. doi: 10.1017/CBO9780511524646.

[103] O. Hosten and P. Kwiat. Observation of the spin Hall effect of light via weak measurements. *Science*, 319(5864):787–790, 2008. ISSN 0036-8075. doi: 10.1126/science.1152697. URL http://science.sciencemag.org/content/319/5864/787.

[104] C. Imbert. Calculation and experimental proof of the transverse shift induced by total internal reflection of a circularly polarized light beam. *Phys. Rev. D*, 5(4):787, 1972.

[105] Wolfram Research, Inc. Mathematica, Version 12.1. URL https://www.wolfram.com/mathematica. Champaign, IL, 2020.

[106] J. D. Jackson. *Classical electrodynamics*. John Wiley & Sons, 2012.

[107] J. Jayaraman. A note on the recent Foldy-Wouthuysen transformations for particles of arbitrary spin. *J. Phys. A*, 8(1):L1, 1975. URL http://stacks.iop.org/0305-4470/8/i=1/a=001.

[108] T. Jungwirth, J. Wunderlich, and K. Olejník. Spin hall effect devices. *Nature Materials*, 11(5):382–390, May 2012. ISSN 1476-4660. doi: 10.1038/nmat3279. URL https://doi.org/10.1038/nmat3279.

[109] T. Kato. On the Adiabatic Theorem of Quantum Mechanics. *Journal of the Physical Society of Japan*, 5(6):435–439, 1950. doi: 10.1143/JPSJ.5.435. URL https://doi.org/10.1143/JPSJ.5.435.

[110] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom. Observation of the spin Hall effect in semiconductors. *Science*, 306(5703): 1910–1913, 2004. ISSN 0036-8075. doi: 10.1126/science.1105514. URL http://science.sciencemag.org/content/306/5703/1910.

[111] Yougang Ke, Yachao Liu, Junxiao Zhou, Yuanyuan Liu, Hailu Luo, and Shuangchun Wen. Optical integration of pancharatnam-berry phase lens and dynamical phase lens. *Applied Physics Letters*, 108(10):101102, 2016. doi: 10.1063/1.4943403. URL https://doi.org/10.1063/1.4943403.

[112] M. Krenn and A. Zeilinger. On small beams with large topological charge: II. Photons, electrons and gravitational waves. *New J. Phys.*, 20(6):063006, 2018. URL http://stacks.iop.org/1367-2630/20/i=6/a=063006.

[113] H. Larocque, I. Kaminer, V. Grillo, G. Leuchs, M. J. Padgett, R. W. Boyd, M. Segev, and E. Karimi. 'Twisted' electrons. *Contemporary Physics*, 59(2): 126–144, 2018. doi: 10.1080/00107514.2017.1418046. URL https://doi.org/10.1080/00107514.2017.1418046.

[114] V. S. Liberman and B. Y. Zel'dovich. Spin-orbit interaction of a photon in an inhomogeneous medium. *Phys. Rev. A*, 46:5199–5207, Oct 1992. doi: 10.1103/PhysRevA.46.5199. URL https://link.aps.org/doi/10.1103/PhysRevA.46.5199.

[115] X. Ling, X. Yi, X. Zhou, Y. Liu, W. Shu, H. Luo, and S. Wen. Realization of tunable spin-dependent splitting in intrinsic photonic spin Hall effect. *Applied Physics Letters*, 105(15):151101, 2014. doi: 10.1063/1.4898190. URL https://doi.org/10.1063/1.4898190.

[116] X. Ling, X. Zhou, K. Huang, Y. Liu, C.-W. Qiu, H. Luo, and S. Wen. Recent advances in the spin Hall effect of light. *Rep. Prog. Phys.*, 80(6):066401, 2017. URL http://stacks.iop.org/0034-4885/80/i=6/a=066401.

[117] R. G. Littlejohn and W. G. Flynn. Geometric phases in the asymptotic theory of coupled wave equations. *Phys. Rev. A*, 44:5239–5256, Oct 1991. doi: 10.1103/PhysRevA.44.5239. URL https://link.aps.org/doi/10.1103/PhysRevA.44.5239.

[118] Yachao Liu, Yougang Ke, Hailu Luo, and Shuangchun Wen. Photonic spin hall effect in metasurfaces: a brief review. *Nanophotonics*, 6(1):51–70, 2017.

[119] H. Luo, X. Zhou, W. Shu, S. Wen, and D. Fan. Enhanced and switchable spin Hall effect of light near the Brewster angle on reflection. *Phys. Rev. A*, 84:043806, Oct 2011. doi: 10.1103/PhysRevA.84.043806. URL https://link.aps.org/doi/10.1103/PhysRevA.84.043806.

[120] M. Mars, C. F. Paganini, and M. A. Oancea. The fingerprints of black holes–shadows and their degeneracies. *Classical and Quantum Gravity*, 35(2):025005, 2018. URL http://stacks.iop.org/0264-9381/35/i=2/a=025005.

[121] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, volume 17. Springer Science & Business Media, 2013. doi: 10.1007/978-0-387-21792-5.

[122] M. Mathisson. Republication of: New mechanics of material systems. *Gen. Relativ. Gravit.*, 42(4):1011–1048, 2010. doi: https://doi.org/10.1007/s10714-010-0939-y.

[123] B. J. McMorran, A. Agrawal, I. M. Anderson, A. A. Herzing, H. J. Lezec, J. J. McClelland, and J. Unguris. Electron vortex beams with high quanta of orbital angular momentum. *Science*, 331(6014):192–195, 2011. ISSN 0036-8075. doi: 10.1126/science.1198804. URL http://science.sciencemag.org/content/331/6014/192.

[124] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman San Francisco, 1973. ISBN 0716703343 0716703440.

[125] B. Ben Moussa and G. T. Kossioris. On the System of Hamilton–Jacobi and Transport Equations Arising in Geometrical Optics. *Commun. Part. Diff. Eq.*, 28(5-6):1085–1111, 2003. doi: 10.1081/PDE-120021187. URL https://doi.org/10.1081/PDE-120021187.

[126] N. Mukunda and R. Simon. Quantum Kinematic Approach to the Geometric Phase. I. General Formalism. *Annals of Physics*, 228(2):205–268, 1993. ISSN 0003-4916. doi: https://doi.org/10.1006/aphy.1993.1093. URL http://www.sciencedirect.com/science/article/pii/S0003491683710936.

[127] S. Murakami. *Intrinsic Spin Hall Effect*, pages 197–209. Springer Berlin Heidelberg, 2006. ISBN 978-3-540-32430-0. doi: 10.1007/11423256_16. URL https://doi.org/10.1007/11423256_16.

[128] M. A. Oancea, C. F. Paganini, J. Joudioux, and L. Andersson. An overview of the gravitational spin Hall effect. *arXiv:1904.09963*, 2019.

[129] M. A. Oancea, J. Joudioux, I. Y. Dodin, D. E. Ruiz, C. F. Paganini, and L. Andersson. Gravitational spin hall effect of light. *Phys. Rev. D*, 102: 024075, Jul 2020. doi: 10.1103/PhysRevD.102.024075. URL https://link.aps.org/doi/10.1103/PhysRevD.102.024075.

[130] Tomoki Ohsawa and Melvin Leok. Symplectic semiclassical wave packet dynamics. *Journal of Physics A: Mathematical and Theoretical*, 46(40): 405201, sep 2013. doi: 10.1088/1751-8113/46/40/405201. URL https://doi.org/10.1088%2F1751-8113%2F46%2F40%2F405201.

[131] M. Onoda, S. Murakami, and N. Nagaosa. Hall Effect of Light. *Phys. Rev. Lett.*, 93:083901, Aug 2004. doi: 10.1103/PhysRevLett.93.083901. URL https://link.aps.org/doi/10.1103/PhysRevLett.93.083901.

[132] C. F. Paganini and M. A. Oancea. Smoothness of the future and past trapped sets in Kerr–Newman–Taub-NUT spacetimes. *Classical and Quantum Gravity*, 35(6):067001, 2018. URL http://stacks.iop.org/0264-9381/35/i=6/a=067001.

[133] C. F. Paganini, B. Ruba, and M. A. Oancea. Characterization of null geodesics on Kerr spacetimes. *arXiv preprint arXiv:1611.06927*, 2016.

[134] M. C. Palmer, M. Takahashi, and H. F. Westman. Localized qubits in curved spacetimes. *Annals of Physics*, 327(4):1078–1131, 2012. ISSN 0003-4916. doi: https://doi.org/10.1016/j.aop.2011.10.009. URL http://www.sciencedirect.com/science/article/pii/S0003491611001692.

[135] S. Pancharatnam. Generalized theory of interference, and its applications. *Proceedings of the Indian Academy of Sciences - Section A*, 44(5):247–262, Nov 1956. ISSN 0370-0089. doi: 10.1007/BF03046050. URL https://doi.org/10.1007/BF03046050.

[136] A. Papapetrou. Spinning test-particles in general relativity. I. *Proc. R. Soc. A*, 209(1097):248–258, 1951. doi: https://doi.org/10.1098/rspa.1951.0200.

[137] A. K. Pati. Geometric aspects of noncyclic quantum evolutions. *Phys. Rev. A*, 52:2576–2584, Oct 1995. doi: 10.1103/PhysRevA.52.2576. URL https://link.aps.org/doi/10.1103/PhysRevA.52.2576.

[138] R. Penrose and W. Rindler. *Spinors and Space-Time: Two-Spinor Calculus and Relativistic Fields*, volume 1. Cambridge University Press, 1987. doi: https://doi.org/10.1017/CBO9780511564048.

[139] V. Perlick. *Ray optics, Fermat's principle, and applications to general relativity*, volume 61. Springer Science & Business Media, 2000.

[140] V. Perlick. Gravitational lensing from a spacetime perspective. *Living Reviews in Relativity*, 7(1):9, Sep 2004. ISSN 1433-8351. doi: 10.12942/lrr-2004-9. URL https://doi.org/10.12942/lrr-2004-9.

[141] F. A. E. Pirani. Republication of: On the physical significance of the Riemann tensor. *Gen. Relativ. Gravit.*, 41(5):1215–1232, 2009.

[142] J. Plebanski. Electromagnetic waves in gravitational fields. *Physical Review*, 118:1396–1408, Jun 1960. doi: 10.1103/PhysRev.118.1396. URL https://link.aps.org/doi/10.1103/PhysRev.118.1396.

[143] J. Ralston. Gaussian beams and the propagation of singularities. *Studies in Partial Differential Equations*, 23(206):C248, 1982.

[144] W. G. Ramírez and A. A. Deriglazov. Lagrangian formulation for Mathisson-Papapetrou-Tulczyjew-Dixon equations. *Phys. Rev. D*, 92(12):124017, 2015.

[145] J. N. Ross. The rotation of the polarization in low birefringence monomode optical fibres due to geometric effects. *Optical and Quantum Electronics*, 16 (5):455–461, Sep 1984. ISSN 1572-817X. doi: 10.1007/BF00619638. URL https://doi.org/10.1007/BF00619638.

[146] S. I. Rubinow and Joseph B. Keller. Asymptotic solution of the dirac equation. *Phys. Rev.*, 131:2789–2796, Sep 1963. doi: 10.1103/PhysRev.131.2789. URL https://link.aps.org/doi/10.1103/PhysRev.131.2789.

[147] R. Rüdiger. The Dirac equation and spinning particles in general relativity. *Proc. R. Soc. A*, 377:417–424, 1981. doi: https://doi.org/10.1098/rspa.1981.0132.

[148] D. E. Ruiz and I. Y. Dodin. First-principles variational formulation of polarization effects in geometrical optics. *Phys. Rev. A*, 92:043805, Oct 2015. doi: 10.1103/PhysRevA.92.043805. URL https://link.aps.org/doi/10.1103/PhysRevA.92.043805.

[149] D. E. Ruiz and I. Y. Dodin. Extending geometrical optics: A Lagrangian theory for vector waves. *Phys. Plasmas*, 24(5):055704, 2017. doi: 10.1063/1.4977537. URL https://doi.org/10.1063/1.4977537.

[150] S. M. Rytov. On the transition from wave to geometrical optics. In *Doklady Akademii Nauk SSSR*, volume 18, pages 263–267, 1938.

[151] J. Samuel and R. Bhandari. General setting for Berry's phase. *Phys. Rev. Lett.*, 60:2339–2342, Jun 1988. doi: 10.1103/PhysRevLett.60.2339. URL https://link.aps.org/doi/10.1103/PhysRevLett.60.2339.

[152] P. Saturnini. *Un modèle de particule à spin de masse nulle dans le champ de gravitation*. Thesis, Université de Provence, 1976. URL https://hal.archives-ouvertes.fr/tel-01344863/.

[153] V. A. Sharafutdinov. *Integral Geometry of Tensor Fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994. ISBN 90-6764-165-0. doi: 10.1515/9783110900095. URL https://doi.org/10.1515/9783110900095.

[154] Andrey A Shoom. Gravitational Faraday and Spin-Hall Effects of Light. *arXiv preprint arXiv:2006.10077*, 2020.

[155] B. Simon. Holonomy, the quantum adiabatic theorem, and Berry's phase. *Phys. Rev. Lett.*, 51:2167–2170, Dec 1983. doi: 10.1103/PhysRevLett.51.2167. URL https://link.aps.org/doi/10.1103/PhysRevLett.51.2167.

[156] D. Singh. An analytic perturbation approach for classical spinning particle dynamics. *Gen. Relativ. Gravit.*, 40(6):1179–1192, 2008.

[157] N. A. Sinitsyn. Semiclassical theories of the anomalous Hall effect. *Journal of Physics: Condensed Matter*, 20(2):023201, dec 2007. doi: 10.1088/0953-8984/20/02/023201. URL https://doi.org/10.1088%2F0953-8984%2F20%2F02%2F023201.

[158] J. Sinova, S. O. Valenzuela, J. Wunderlich, C. H. Back, and T. Jungwirth. Spin Hall effects. *Rev. Mod. Phys.*, 87:1213–1260, Oct 2015. doi: 10.1103/RevModPhys.87.1213. URL https://link.aps.org/doi/10.1103/RevModPhys.87.1213.

[159] A. P. Slobozhanyuk, A. N. Poddubny, I. S. Sinev, A. K. Samusev, Y. F. Yu, A. I. Kuznetsov, A. E. Miroshnichenko, and Y. S. Kivshar. Enhanced photonic spin Hall effect with subwavelength topological edge states. *Laser & Photonics Reviews*, 10(4):656–664, 2016. doi: 10.1002/lpor.201600042. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/lpor.201600042.

[160] J.-M. Souriau. Modele de particule a spin dans le champ électromagnétique et gravitationnel. *Ann. Inst. Henri Poincaré A*, 20:315–364, 1974. URL http://www.numdam.org/item/AIHPA_1974__20_4_315_0/.

[161] J. Steinhoff. Spin and quadrupole contributions to the motion of astrophysical binaries. In *Equations of Motion in Relativistic Gravity*, pages 615–649. Springer, 2015.

[162] Michael Stone, Vatsal Dwivedi, and Tianci Zhou. Berry phase, lorentz covariance, and anomalous velocity for dirac and weyl particles. *Phys. Rev. D*, 91:025004, Jan 2015. doi: 10.1103/PhysRevD.91.025004. URL https://link.aps.org/doi/10.1103/PhysRevD.91.025004.

[163] Michael Stone, Vatsal Dwivedi, and Tianci Zhou. Wigner Translations and the Observer Dependence of the Position of Massless Spinning Particles. *Phys. Rev. Lett.*, 114:210402, May 2015. doi: 10.1103/PhysRevLett.114.210402. URL https://link.aps.org/doi/10.1103/PhysRevLett.114.210402.

[164] Robert T. Thompson. Covariant electrodynamics in linear media: Optical metric. *Phys. Rev. D*, 97:065001, Mar 2018. doi: 10.1103/PhysRevD.97.065001. URL https://link.aps.org/doi/10.1103/PhysRevD.97.065001.

[165] A. Tomita and R. Y. Chiao. Observation of Berry's topological phase by use of an optical fiber. *Phys. Rev. Lett.*, 57:937–940, Aug 1986. doi: 10.1103/PhysRevLett.57.937. URL https://link.aps.org/doi/10.1103/PhysRevLett.57.937.

[166] E. R. Tracy, A. J. Brizard, A. S. Richardson, and A. N. Kaufman. *Ray Tracing and Beyond: Phase Space Methods in Plasma Wave Theory*. Cambridge University Press, 2014. doi: 10.1017/CBO9780511667565.

[167] W. Tulczyjew. Motion of multipole particles in general relativity theory. *Acta Phys. Pol.*, 18:393, 1959.

[168] M. Uchida and A. Tonomura. Generation of electron beams carrying orbital angular momentum. *Nature*, 464:737, Apr 2010. URL https://doi.org/10.1038/nature08904.

[169] J. Vines, D. Kunst, J. Steinhoff, and T. Hinderer. Canonical Hamiltonian for an extended test body in curved spacetime: To quadratic order in spin. *Phys. Rev. D*, 93(10):103008, 2016.

[170] V. V. Vladimirskii. The rotation of polarization plane for curved light ray. In *Doklady Akademii Nauk SSSR*, volume 31, pages 222–226, 1941.

[171] K. Volke-Sepulveda, V. Garcés-Chávez, S. Chávez-Cerda, J. Arlt, and K. Dholakia. Orbital angular momentum of a high-order Bessel light beam. *Journal of Optics B: Quantum and Semiclassical Optics*, 4(2):S82, 2002. URL http://stacks.iop.org/1464-4266/4/i=2/a=373.

[172] R. M. Wald. *General Relativity*. University of Chicago Press, Chicago, IL, 1984. ISBN 0-226-87032-4; 0-226-87033-2. doi: 10.7208/chicago/9780226870373.001.0001. URL https://doi.org/10.7208/chicago/9780226870373.001.0001.

[173] F. Wilczek and A. Zee. Appearance of gauge structure in simple dynamical systems. *Phys. Rev. Lett.*, 52:2111–2114, Jun 1984. doi: 10.1103/PhysRevLett.52.2111. URL https://link.aps.org/doi/10.1103/PhysRevLett.52.2111.

[174] Jörg Wunderlich, Byong-Guk Park, Andrew C. Irvine, Liviu P. Zârbo, Eva Rozkotová, Petr Nemec, Vít Novák, Jairo Sinova, and Tomás Jungwirth. Spin hall effect transistor. *Science*, 330(6012):1801–1804, 2010. ISSN 0036-8075. doi: 10.1126/science.1195816. URL https://science.sciencemag.org/content/330/6012/1801.

[175] D. Xiao, M.-C. Chang, and Q. Niu. Berry phase effects on electronic properties. *Rev. Mod. Phys.*, 82:1959–2007, Jul 2010. doi: 10.1103/RevModPhys.82.1959. URL https://link.aps.org/doi/10.1103/RevModPhys.82.1959.

[176] N. Yamamoto. Spin Hall effect of gravitational waves. *Phys. Rev. D*, 98:061701(R), Sep 2018. doi: 10.1103/PhysRevD.98.061701. URL https://link.aps.org/doi/10.1103/PhysRevD.98.061701.

[177] C.-M. Yoo. Notes on spinoptics in a stationary spacetime. *Phys. Rev. D*, 86:084005, Oct 2012. doi: 10.1103/PhysRevD.86.084005. URL https://link.aps.org/doi/10.1103/PhysRevD.86.084005.

[178] Junxiao Zhou, Haoliang Qian, Guangwei Hu, Hailu Luo, Shuangchun Wen, and Zhaowei Liu. Broadband photonic spin hall meta-lens. *ACS Nano*, 12

(1):82–88, 2018. doi: 10.1021/acsnano.7b07379. URL https://doi.org/10.1021/acsnano.7b07379. PMID: 29211443.

[179] Xinxing Zhou, Xiaohui Ling, Hailu Luo, and Shuangchun Wen. Identifying graphene layers via spin hall effect of light. *Applied Physics Letters*, 101(25): 251602, 2012. doi: 10.1063/1.4772502. URL https://doi.org/10.1063/1.4772502.

[180] Tengfeng Zhu, Yijie Lou, Yihan Zhou, Jiahao Zhang, Junyi Huang, Yan Li, Hailu Luo, Shuangchun Wen, Shiyao Zhu, Qihuang Gong, Min Qiu, and Zhichao Ruan. Generalized spatial differentiation from the spin hall effect of light and its application in image processing of edge detection. *Phys. Rev. Applied*, 11:034043, Mar 2019. doi: 10.1103/PhysRevApplied.11.034043. URL https://link.aps.org/doi/10.1103/PhysRevApplied.11.034043.