The Journal of Geometric Analysis
https://doi.org/10.1007/s12220-021-00634-z

Spectral Estimates for Riemannian Submersions with Fibers of Basic Mean Curvature

Panagiotis Polymerakis[1]

Received: 13 November 2020 / Accepted: 12 February 2021

Abstract
For Riemannian submersions with fibers of basic mean curvature, we compare the spectrum of the total space with the spectrum of a Schrödinger operator on the base manifold. Exploiting this concept, we study submersions arising from actions of Lie groups. In this context, we extend the state-of-the-art results on the bottom of the spectrum under Riemannian coverings. As an application, we compute the bottom of the spectrum and the Cheeger constant of connected, amenable Lie groups.

Keywords Riemannian submersion · Basic mean curvature · Riemannian principal bundle · Amenable Lie group · Bottom of spectrum · Discrete spectrum

Mathematics Subject Classification 58J50 · 35P15 · 53C99

1 Introduction

The study of the spectrum of the Laplacian on a Riemannian manifold has attracted much attention over the last years. In order to comprehend its relations with the geometry of the underlying manifold, it is reasonable to investigate its behavior under maps between Riemannian manifolds that respect the geometry of the manifolds to some extent. In this paper, we study the behavior of the spectrum under Riemannian submersions.

The notion of Riemannian submersion was introduced in the 1960s as a tool to express the geometry of a manifold in terms of the geometry of simpler components, namely, the base space and the fibers. Of course, by geometry of the fibers, we mean both their intrinsic and their extrinsic geometry as submanifolds of the total space. Bearing this in mind, it is natural to describe the spectrum of the total space in terms of the geometry and the spectrum of the base space and the fibers.

✉ Panagiotis Polymerakis
polymerp@mpim-bonn.mpg.de

1 Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

To set the stage, let $p\colon M_2 \to M_1$ be a Riemannian submersion and denote by $F_x := p^{-1}(x)$ the fiber over $x \in M_1$. The spectrum of (the Laplacian on) M_2 has been studied in the case where M_2 is closed (that is, compact and without boundary) and the submersion has totally geodesic, or minimal fibers, or fibers of basic mean curvature (cf. for example the survey [6]). However, the situation is quite more complicated and yet unclear if M_2 is not closed.

Recently, in [23], extending the result of [10], we established a lower bound for the bottom of the spectrum $\lambda_0(M_2)$ of M_2, if the mean curvature of the fibers is bounded in a certain way. More precisely, according to [23, Theorem 1.1], if the (unnormalized) mean curvature of the fibers is bounded by $\|H\| \leq C \leq 2\sqrt{\lambda_0(M_1)}$, then the bottom of the spectrum of M_2 satisfies

$$\lambda_0(M_2) \geq (\sqrt{\lambda_0(M_1)} - C/2)^2 + \inf_{x \in M_1} \lambda_0(F_x).$$

Moreover, if the equality holds and $\lambda_0(M_1) \notin \sigma_{\text{ess}}(M_1)$ (that is, $\lambda_0(M_1)$ is an isolated point of the spectrum of the Laplacian on M_1), then $\lambda_0(F_x)$ is equal to its infimum for almost any $x \in M_1$. Recall that, in general, $\lambda_0(F_x)$ is only upper semi-continuous with respect to $x \in M_1$ (cf. [23, Lemma 2.9]).

In the second part of [23], following [4], we studied Riemannian submersions with closed fibers. In this context, we introduced a Schrödinger operator on M_1, with potential determined by the volume of the fibers, and compared its spectrum with the spectrum of M_2. It should be noticed that if the submersion has fibers of infinite volume, then we are not able to define that operator, at least in the way we did in [23].

In this paper, motivated by the aforementioned results, we introduce a Schrödinger operator on the base space of a Riemannian submersion with fibers of basic mean curvature (see Sect. 2.1) and compare its spectrum with the spectrum of the total space. To be more specific, let $p\colon M_2 \to M_1$ be a Riemannian submersion with fibers of basic mean curvature, and consider the Schrödinger operator

$$S = \Delta + \frac{1}{4}\|p_* H\|^2 - \frac{1}{2}\operatorname{div} p_* H \tag{1}$$

on M_1, where Δ is the (non-negative) Laplacian on M_1. It is worth to point out that S is non-negative, that is, $\lambda_0(S) \geq 0$. Furthermore, it is evident that S coincides with the Laplacian, if the submersion has minimal fibers. Our first result relates the bottom of the spectrum of this operator with the bottom of the spectrum of M_2.

Theorem 1.1 *Let $p\colon M_2 \to M_1$ be a Riemannian submersion with fibers of basic mean curvature, and consider the Schrödinger operator S as above. Then*

$$\lambda_0(M_2) \geq \lambda_0(S) + \inf_{x \in M_1} \lambda_0(F_x).$$

If, in addition, the equality holds and $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$, then $\lambda_0(F_x)$ is almost everywhere equal to its infimum.

It should be emphasized that no assumptions on the geometry or the topology of the manifolds are required in this theorem. In particular, the manifolds do not have to be complete. This, together with the decomposition principle, allows us to derive a similar inequality involving the bottoms of the essential spectra, if the fibers are closed.

It is worth to mention that in some cases, $\lambda_0(S)$ can be estimated in terms of $\lambda_0(M_1)$. For instance, if the mean curvature of the fibers is bounded by $\|H\| \leq C \leq 2\sqrt{\lambda_0(M_1)}$, then the bottoms of the spectra satisfy

$$\lambda_0(S) \geq (\sqrt{\lambda_0(M_1)} - C/2)^2.$$

Thus, Theorem 1.1 provides a sharper lower bound for $\lambda_0(M_2)$ than [23, Theorem 1.1] in the case where both of them are applicable.

It is noteworthy that if the submersion has closed fibers, then the operator S defined in (1) coincides with the Schrödinger operator introduced in [23], and there is a remarkable relation with the work of Bordoni [5] on Riemannian submersions with fibers of basic mean curvature. Given such a submersion $p\colon M_2 \to M_1$ with M_2 closed, Bordoni considered the restrictions Δ_c and Δ_0 of the Laplacian acting on lifted functions and on functions with zero average on any fiber, respectively, and showed in [5, Theorem 1.6] that the spectrum is written as $\sigma(M_2) = \sigma(\Delta_c) \cup \sigma(\Delta_0)$. In this setting, the spectrum of the operator S coincides with the spectrum of Δ_c. It should be observed that expressing the latter one as the spectrum of an operator on M_1 allows us to relate it more easily to the spectrum of M_1. For Riemannian submersions with closed fibers, we obtain the following consequence of Theorem 1.1 (compare with [23, Theorem 1.2]), where we denote by λ_0^{ess} the bottom of the essential spectrum of an operator.

Corollary 1.2 *If $p\colon M_2 \to M_1$ is a Riemannian submersion with closed fibers of basic mean curvature, then $\lambda_0(M_2) = \lambda_0(S)$ and $\lambda_0^{\text{ess}}(M_2) = \lambda_0^{\text{ess}}(S)$. In particular, M_2 has discrete spectrum if and only if the spectrum of S is discrete.*

This corollary generalizes [4, Theorem 1(ii)], which asserts that if $p\colon M_2 \to M_1$ is a Riemannian submersion with closed and minimal fibers, then M_1 has discrete spectrum if and only if M_2 has discrete spectrum. This equivalence has been extended in [23, Corollary 1.4] under the weaker assumption that the fibers are closed and of bounded mean curvature. Corollary 1.2 characterizes the discreteness of the spectrum of M_2 in terms of S instead of the Laplacian, which, nonetheless, is very natural. More precisely, for warped products of the form $M \times_\psi F$ with F closed, this characterization coincides with [1, Theorem 3.3] of Baider.

If, in addition, the manifolds involved in Corollary 1.2 are complete, then we know from [23, Theorem 1.2] that the spectra and the essential spectra satisfy $\sigma(S) \subset \sigma(M_2)$ and $\sigma_{\text{ess}}(S) \subset \sigma_{\text{ess}}(M_2)$. This, together with Theorem 1.1 and Corollary 1.2, shows that it is very reasonable to compare the spectrum of S with the spectrum of M_2, if the submersion has fibers of basic mean curvature.

In the second part of the paper, we study Riemannian principal bundles. To be more specific, let G be a possibly discrete Lie group acting smoothly, freely and properly on a Riemannian manifold M_2 via isometries, where $\dim G < \dim M_2$. Such an action induces on $M_1 = M_2/G$ the structure of Riemannian manifold. If G is non-discrete, the projection $p\colon M_2 \to M_1$ is a Riemannian submersion with fibers of basic mean curvature. We then say that p is a *Riemannian submersion arising from the action of* G. The behavior of the spectrum under such submersions has been studied for instance in [13].

In the case where G is a discrete group, its action gives rise to a normal Riemannian covering. In this context, there are various results establishing relations between properties of the deck transformation group and the behavior of the spectrum. To be more precise, let $q\colon M_2 \to M_1$ be a normal Riemannian covering with deck transformation group Γ. Then the bottoms of the spectra satisfy $\lambda_0(M_2) \geq \lambda_0(M_1)$ (cf. for instance [2] and the references therein). Brooks was the first one to investigate when the equality holds. In [8], he showed that if M_1 is closed, then Γ is amenable if and only if $\lambda_0(M_2) = 0$. It is apparent that in this setting, we also have that $\lambda_0(M_1) = 0$. In [2], we proved that if Γ is amenable, then $\lambda_0(M_2) = \lambda_0(M_1)$, without any assumptions on the topology or the geometry of M_1. It was established in [22] that if, in addition, M_1 is complete, then $\sigma(M_1) \subset \sigma(M_2)$. Conversely, according to [21], if $\lambda_0(M_2) = \lambda_0(M_1)$ and $\lambda_0(M_1) \notin \sigma_{\mathrm{ess}}(M_1)$, then Γ is amenable.

If G is non-discrete, then from the above discussion, it makes sense to compare the spectrum of the Laplacian on the total space with the spectrum of the Schrödinger operator S on the base manifold, defined in (1). Theorem 1.1 implies that $\lambda_0(M_2) \geq \lambda_0(S)$. In the following theorem, we extend the aforementioned results to Riemannian submersions arising from Lie group actions, where we denote by G_0 the connected component of the identity element of G.

Theorem 1.3 *Let $p\colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a Lie group G. Then*

(i) *If G is amenable and G_0 is unimodular, then $\lambda_0(M_2) = \lambda_0(S)$.*
(ii) *If, in addition, M_1 is complete, then $\sigma(S) \subset \sigma(M_2)$.*
(iii) *Conversely, if $\lambda_0(M_2) = \lambda_0(S)$ and $\lambda_0(S) \notin \sigma_{\mathrm{ess}}(S)$, then G is amenable and G_0 is unimodular.*

Recall that there exist connected Lie groups that are amenable but not unimodular (because any solvable group is amenable), and connected Lie groups that are unimodular but not amenable (since any connected, semisimple Lie group is unimodular).

It is notable that if G is compact, then Corollary 1.2 compares the spectra and the essential spectra of the operators. Even though Theorem 1.3 is formulated in terms of spectra, it also provides information about the essential spectra. This follows from the fact that if G is non-compact, then $\sigma(M_2) = \sigma_{\mathrm{ess}}(M_2)$ (cf. for example [22, Theorem 5.2]).

As in the context of Riemannian coverings, it is plausible to wonder if the assumption $\lambda_0(S) \notin \sigma_{\mathrm{ess}}(S)$ can be weakened in Theorem 1.3(iii). We will construct a wide class of examples demonstrating that this assumption is essential. Namely, let M be any Riemannian manifold with $\lambda_0(M) \in \sigma_{\mathrm{ess}}(M)$. We will show that there exists a Riemannian submersion $p\colon M_2 \to M_1 := M$ with minimal fibers, arising from the action of a connected, non-unimodular Lie group G, such that $\lambda_0(M_2) = \lambda_0(M_1)$. Since the submersion has minimal fibers, it is clear that S coincides with the Laplacian on M_1.

In the case where the base manifold is closed, we derive another analog of Brooks' result, which is slightly different. This is because in Theorem 1.3, we investigate the validity of $\lambda_0(M_2) = \lambda_0(S)$, while the following corollary characterizes the stronger property $\lambda_0(M_2) = 0$.

Corollary 1.4 *Let $p\colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a Lie group G, where M_1 is closed. Then G is unimodular and amenable if and only if $\lambda_0(M_2) = 0$.*

Finally, exploiting Theorems 1.1 and 1.3 , we study quotients of Lie groups by normal subgroups. In this setting, we obtain some relations between the mean curvature of the subgroup and the bottom of the spectrum of the group, the subgroup, and the quotient.

Theorem 1.5 *Let G be a connected Lie group endowed with a left-invariant metric and N be a closed (as a subset), connected, normal subgroup of G with mean curvature H. Then*

$$\lambda_0(G) \geq \lambda_0(G/N) + \lambda_0(N) - \frac{1}{4}\|H\|^2 + \frac{1}{2}\operatorname{tr}(\operatorname{ad} H).$$

Moreover, N is unimodular and amenable if and only if

$$\lambda_0(G) = \lambda_0(G/N) - \frac{1}{4}\|H\|^2 + \frac{1}{2}\operatorname{tr}(\operatorname{ad} H).$$

As an application of this theorem, we compute the bottom of the spectrum and the Cheeger constant of connected, amenable Lie groups. This extends the result of [20] in various ways.

Corollary 1.6 *Let G be a connected, amenable Lie group endowed with a left-invariant metric. Then the bottom of its spectrum and its Cheeger constant are given by*

$$\lambda_0(G) = \frac{1}{4}h(G)^2 = \frac{1}{4}\max_{X\in\mathfrak{g},\|X\|=1}(\operatorname{tr}(\operatorname{ad} X))^2.$$

If G is not unimodular, then the maximum is achieved by the unit vector in the direction of the mean curvature (in G) of the commutator subgroup $[S, S]$ of the radical S of G.

The paper is organized as follows: In Sect. 2, we discuss some basic properties of Schrödinger operators, Riemannian submersions and Lie groups. In particular, we provide a spectral theoretic characterization for connected, amenable, and unimodular Lie groups, which is well known for simply connected Lie groups. In Sect. 3, we study Riemannian submersions with fibers of basic mean curvature and prove Theorem 1.1 and Corollary 1.2. In Sect. 4, we focus on submersions arising from Lie group actions and establish Theorem 1.3 and Corollary 1.4. In Sect. 5, we discuss some consequences of our results to Lie groups and show Theorem 1.5 and Corollary 1.6.

2 Preliminaries

Throughout this paper, manifolds are assumed to be connected and without boundary, unless otherwise stated, except for Lie groups. Consider a possibly non-connected Riemannian manifold M. A *Schrödinger operator* on M is an operator of the form $S = \Delta + V$, where Δ is the Laplacian on M and $V \in C^\infty(M)$, such that there exists $c \in \mathbb{R}$ satisfying

$$\langle Sf, f\rangle_{L^2(M)} \geq c\|f\|^2_{L^2(M)}$$

for any $f \in C^\infty_c(M)$. Then the operator

$$S\colon C^\infty_c(M) \subset L^2(M) \to L^2(M)$$

admits a Friedrichs extension, being densely defined, symmetric and bounded from below. It is worth to point out that if M is complete, then this operator is essentially self-adjoint; that is, its closure coincides with its Friedrichs extension (cf. [23, Proposition 2.4]).

The spectrum and the essential spectrum of (the Friedrichs extension of) S are denoted by $\sigma(S)$ and $\sigma_{\mathrm{ess}}(S)$, respectively, and their bottoms (that is, their minimums) by $\lambda_0(S)$ and $\lambda_0^{\mathrm{ess}}(S)$, respectively. In the case of the Laplacian (that is, $V = 0$), we write $\sigma(M)$, $\sigma_{\mathrm{ess}}(M)$ and $\lambda_0(M)$, $\lambda_0^{\mathrm{ess}}(M)$ for these sets and quantities. We have by definition that $\lambda_0^{\mathrm{ess}}(S) = +\infty$ if S has *discrete spectrum*, which means that $\sigma_{\mathrm{ess}}(S)$ is empty. If $\sigma_{\mathrm{ess}}(M)$ is empty, we say that M has discrete spectrum.

The *Rayleigh quotient* of a non-zero $f \in \mathrm{Lip}_c(M)$ with respect to S is defined by

$$\mathcal{R}_S(f) := \frac{\int_M (\|\operatorname{grad} f\|^2 + Vf^2)}{\int_M f^2}.$$

The Rayleigh quotient of f with respect to the Laplacian is denoted by $\mathcal{R}(f)$, or by $\mathcal{R}_g(f)$ if the Riemannian metric g of M is not clear from the context. According to the next proposition, the bottom of the spectrum of S can be expressed as an infimum of Rayleigh quotients (cf. for example [21, Sect. 2] and the references therein).

Proposition 2.1 *Let S be a Schrödinger operator on a Riemannian manifold M. Then the bottom of the spectrum of S satisfies*

$$\lambda_0(S) = \inf_f \mathcal{R}_S(f),$$

where the infimum is taken aver all $f \in C^\infty_c(M) \smallsetminus \{0\}$, or over all $f \in \mathrm{Lip}_c(M) \smallsetminus \{0\}$.

A remarkable property of the essential spectrum of S follows from the decomposition principle, which states that

$$\sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(S, M \smallsetminus K)$$

for any compact domain K of M with smooth boundary. This is well known in the case where M is complete (compare with [12, Proposition 2.1]), but also holds if M is non-complete (cf. for instance [3, Theorem A.17]). The next proposition summarizes the properties of the bottom of the essential spectrum that will be used in the sequel.

Proposition 2.2 *Let S be a Schrödinger operator on a Riemannian manifold M and consider an exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of M consisting of compact domains with smooth boundary. Then the bottom of the essential spectrum of S is given by*

$$\lambda_0^{\text{ess}}(S) = \lim_n \lambda_0(S, M \smallsetminus K_n).$$

In particular, there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M)\smallsetminus\{0\}$ with supp f_n *pairwise disjoint, such that $\mathcal{R}_S(f_n) \to \lambda_0^{\text{ess}}(S)$. Furthermore, for any sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M) \smallsetminus \{0\}$ with* supp f_n *pairwise disjoint, we have that*

$$\lambda_0^{\text{ess}}(S) \leq \liminf_n \mathcal{R}_S(f_n).$$

Proof The third assertion may be found for example in [23, Proposition 2.2]. From this and Proposition 2.1, it is not hard to see that

$$\lambda_0^{\text{ess}}(S) \leq \lim_n \lambda_0(S, M \smallsetminus K_n),$$

while the decomposition principle gives that $\lambda_0^{\text{ess}}(S) = \lambda_0^{\text{ess}}(S, M \smallsetminus K_n) \geq \lambda_0(S, M \smallsetminus K_n)$ for any $n \in \mathbb{N}$, as we wished. The proof is completed by the first part and Proposition 2.1. □

For $\lambda \in \mathbb{R}$, a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M)\smallsetminus\{0\}$ is called a *characteristic sequence* for S and λ if

$$\frac{\|(S-\lambda)f_n\|_{L^2(M)}}{\|f_n\|_{L^2(M)}} \to 0, \text{ as } n \to +\infty.$$

If M is complete, then S is essentially self-adjoint, which allows us to characterize the spectrum of S in terms of compactly supported smooth functions as follows.

Proposition 2.3 *Let S be a Schrödinger operator on a complete Riemannian manifold M and consider $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma(S)$ if and only if there is a characteristic sequence for S and λ.*

Assume now that $\varphi \in C^\infty(M)$ is a positive solution of $S\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{R}$. Denote by $L^2_\varphi(M)$ the L^2-space of M with respect to the measure $\varphi^2 d\,\text{vol}$, where $d\,\text{vol}$ is the volume element of M induced by its Riemannian metric. It is straightforward to verify that the isometric isomorphism $m_\varphi\colon L^2_\varphi(M) \to L^2(M)$, defined by $m_\varphi f = \varphi f$, intertwines $S - \lambda$ with the diffusion operator

$$L := m_\varphi^{-1} \circ (S-\lambda) \circ m_\varphi = \Delta - 2\,\text{grad}\ln\varphi.$$

The operator L is called *renormalization* of S with respect to φ. The Rayleigh quotient of a non-zero $f \in C_c^\infty(M)$ with respect to L is defined by

$$\mathcal{R}_L(f) := \frac{\langle Lf, f\rangle_{L^2_\varphi(M)}}{\|f\|^2_{L^2_\varphi(M)}} = \frac{\int_M \|\operatorname{grad} f\|^2\varphi^2}{\int_M f^2\varphi^2}.$$

Lemma 2.4 *For any non-zero $f \in C_c^\infty(M)$ and $c \in \mathbb{R}$, we have that*

(i) $\mathcal{R}_L(f) = \mathcal{R}_S(\varphi f) - \lambda$,
(ii) $\|(L-c)f\|_{L^2_\varphi(M)} = \|(S-\lambda-c)(\varphi f)\|_{L^2(M)}$.

Proof Follows immediately from the definition of L and the fact that m_φ is an isometric isomorphism. □

2.1 Riemannian Submersions

Let M_1 and M_2 be Riemannian manifolds with $\dim M_1 < \dim M_2$. A surjective smooth map $p\colon M_2 \to M_1$ is called a *submersion* if its differential is surjective at any point. The kernel of p_{*y} is called the *vertical space* at $y \in M_2$, and its orthogonal complement in $T_y M_2$ is called the *horizontal space* at y. These spaces are denoted by $(T_y M_2)^v$ and $(T_y M_2)^h$, respectively. It is evident that the fiber $F_x := p^{-1}(x)$ over $x \in M_1$ is a possibly non-connected submanifold of M_2 and $(T_y M_2)^v$ is the tangent space of F_x at $y \in F_x$. The submersion p is called *Riemannian submersion* if the restriction $p_{*y}\colon (T_y M_2)^h \to T_{p(y)} M_1$ is an isometry for any $y \in M_2$. For more details, see [14] or [15].

Given a Riemannian submersion $p\colon M_2 \to M_1$, a smooth map $s\colon U \to M_2$ defined on an open subset U of M_1, is called *section* if $(p \circ s)(x) = x$ for any $x \in U$. We say that a section $s\colon U \subset M_1 \to M_2$ is *extensible* if it can be extended to a section $s'\colon U' \subset M_1 \to M_2$ with $\bar{U} \subset U'$.

A vector field Y on M_2 is called *horizontal* (*vertical*) if $Y(y)$ belongs to the horizontal (vertical, respectively) space at y for any $y \in M_2$. It is easily checked that any vector field Y on M_2 can be written as $Y = Y^h + Y^v$ with Y^h horizontal and Y^v vertical. Moreover, any vector field X on M_1 has a unique horizontal lift $\tilde{X}$ on M_2; that is, $\tilde{X}$ is horizontal and $p_*\tilde{X} = X$. A vector field Y on M_2 is called *basic* if $Y = \tilde{X}$ for some vector field X on M_1.

The (unnormalized) *mean curvature vector* of the fibers is defined by

$$H(y) := \sum_{i=1}^{k} \alpha(X_i, X_i),$$

where $\alpha(\cdot,\cdot)$ is the second fundamental form of the fiber $F_{p(y)}$ and $\{X_i\}_{i=1}^k$ is an orthonormal basis of $(T_y M_2)^v$. It should be observed that H is a horizontal vector field. We say that the Riemannian submersion p has *minimal fibers* or *fibers of basic mean curvature* if $H = 0$ or H is basic, respectively.

We now discuss some basic examples of Riemannian submersions. It is worth to mention that the manifolds involved in following examples are not assumed to be compact.

Example 2.5 (i) The *warped product* $M_2 = M_1 \times_\psi F$ is the product manifold endowed with the Riemannian metric $g_{M_1} \times \psi^2 g_F$, where $\psi \in C^\infty(M_1)$ is positive. Then the projection to the first factor $p\colon M_2 \to M_1$ is a Riemannian submersion with fibers of basic mean curvature

$$H = -k \operatorname{grad} \ln \tilde{\psi},$$

where $k := \dim F$. It should be noticed that surfaces of revolution are warped products of the form $\mathbb{R} \times_\psi S^1$.

(ii) Another generalization of surfaces of revolution was introduced by Bishop motivated by a result of Clairaut involving such surfaces. A Riemannian submersion $p\colon M_2 \to M_1$ is called *Clairaut submersion* if there exists a positive $f \in C^\infty(M_2)$, such that for any geodesic c on M_2, the function $(f \circ c) \sin\theta$ is constant, where $\theta(t)$ is the angle between $c'(t)$ and $(T_{c(t)}M_2)^h$. Bishop proved that a Riemannian submersion $p\colon M_2 \to M_1$ of complete manifolds with connected fibers is a Clairaut submersion if and only if the fibers are totally umbilical with mean curvature

$$H = -k \operatorname{grad} \ln \tilde{\psi}$$

for some positive $\psi \in C^\infty(M_1)$, where k is the dimension of the fiber (cf. for instance [14, Theorem 1.7]).

(iii) Let G be a Lie group acting smoothly, freely, and properly via isometries on a Riemannian manifold M_2, where $\dim G < \dim M_2$. Then $M_1 := M_2/G$ is a Riemannian manifold and the projection $p\colon M_2 \to M_1$ is a Riemannian submersion with fibers of basic mean curvature. In this case, we say that $p\colon M_2 \to M_1$ is a Riemannian submersion *arising from the action of a Lie group* G.

Given a Riemannian submersion $p\colon M_2 \to M_1$, the *lift* of a function $f \in C^\infty(M_1)$ on M_2 is the smooth function $\tilde{f} := f \circ p$. The next lemma provides a simple expression for the Laplacian and the gradient of a lifted function.

Lemma 2.6 *For any $f \in C^\infty(M_1)$ and its lift $\tilde{f}$ on M_2, we have that*

(i) $\operatorname{grad} \tilde{f} = \widetilde{\operatorname{grad} f}$,
(ii) $\Delta \tilde{f} = \widetilde{\Delta f} + \langle \widetilde{\operatorname{grad} f}, H \rangle$.

Proof Both statements follow from elementary computations, which may be found for example in [4, Sect. 2.2]. ◻

2.2 Lie Groups

In this subsection, we recall some basic definitions and results about Lie groups, and discuss some consequences of the Cheeger and Buser inequalities in this setting.

For a Borel subset A of a Riemannian manifold (M, g), we denote the volume of A by $|A|_g$, or simply by $|A|$ if the Riemannian metric of M is clear from the context. Similarly, for an m-dimensional submanifold N of M, we denote by $|N|$ the m-dimensional volume of N. The *Cheeger constant* of a Riemannian manifold M is defined by

$$h(M) := \inf_K \frac{|\partial K|}{|K|},$$

where the infimum is taken over all compact domains K of M with smooth boundary. It is related to the bottom of the spectrum via the Cheeger inequality (cf. [11])

$$\lambda_0(M) \geq \frac{h(M)^2}{4}.$$

Buser [9] established an inverse inequality for complete manifolds with Ricci curvature bounded from below. In particular, if M is such a manifold, then $\lambda_0(M) = 0$ if and only if $h(M) = 0$. For our purposes, we also need the following lemma from his work, where A^r stands for the r-tubular neighborhood of a subset A of M.

Lemma 2.7 (Compare with [9, Lemma 7.2]; see also [22, Corollary 6.3]). *Let M be a non-compact, complete Riemannian manifold with Ricci curvature bounded from below. If $h(M) = 0$, then for any $\varepsilon, r > 0$, there exists an open, bounded $W \subset M$ such that*

$$|(\partial W)^r| < \varepsilon |W \smallsetminus (\partial W)^r|.$$

Throughout this paper, Lie groups are assumed to be non-discrete and possibly non-connected, unless otherwise stated. A possibly discrete Lie group G is called *amenable* if there exists a left-invariant mean on $L^\infty(G)$; that is, a linear functional $\mu\colon L^\infty(G) \to \mathbb{R}$ such that

$$\operatorname{ess\,inf} f \leq \mu(f) \leq \operatorname{ess\,sup} f \text{ and } \mu(f \circ L_x) = \mu(f),$$

for any $f \in L^\infty(G)$ and $x \in G$, where L_x stands for multiplication from the left with an element $x \in G$. Here, $L^\infty(G)$ is considered with respect to the Haar measure. If G is non-discrete, then its Haar measure is a constant multiple of the volume element of G induced from a left-invariant metric. If G is discrete, then its Haar measure is a constant multiple of the counting measure. For more details, see [16].

Lemma 2.8 *If N is a normal subgroup of a possibly discrete Lie group G, then G is amenable if and only if N and G/N are amenable.*

It is not hard to verify that abelian and compact Lie groups are amenable. Therefore, so are compact extensions of solvable groups. As a matter of fact, a connected Lie group is amenable if and only if it is a compact extension of a solvable group (cf. for example [19, Lemma 2.2]).

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The *radical* $\mathfrak{s}$ of $\mathfrak{g}$ is the largest solvable ideal of $\mathfrak{g}$. The *radical* S of G is the connected subgroup with Lie algebra $\mathfrak{s}$. Then S is a closed, normal subgroup of G and the quotient G/S is semisimple. In this case, we have that G is amenable if and only if G/S is compact (cf. [17, p. 724f] and the references therein).

A Lie group is called *unimodular* if its Haar measure is also right-invariant. For a connected Lie group, this property may be reformulated in terms of its Lie algebra as follows.

Lemma 2.9 [17, Proposition 1.2]. *A connected Lie group G is unimodular if and only if* $\operatorname{tr}(\operatorname{ad} X) = 0$ *for any X in the Lie algebra of G.*

It is worth to point out that connected, nilpotent Lie groups are unimodular and amenable. In addition, compact extensions of connected, unimodular Lie groups are unimodular (cf. [18, Proposition 8]).

Although the aforementioned properties are group theoretic, they are characterized by the spectrum of the Lie group according to the next theorem, which is well known for simply connected Lie groups.

Theorem 2.10 *A connected Lie group G is unimodular and amenable if and only if* $\lambda_0(G) = 0$ *for some/any left-invariant metric on G.*

Proof We know from [17, Theorem 3.8] that a simply connected Lie group $\tilde{G}$ is unimodular and amenable if and only if $h(\tilde{G}) = 0$ with respect to some/any left-invariant metric. By the Cheeger and Buser inequalities, this gives the assertion for simply connected Lie groups. To show its validity for a connected Lie group G, let $q \colon \tilde{G} \to G$ be the universal covering of G. It follows from Lemma 2.9 that $\tilde{G}$ is unimodular if and only if G is unimodular, since their Lie algebras are isomorphic. Furthermore, $\pi_1(G)$ is abelian and isomorphic to the kernel of q as a Lie group homomorphism. Therefore, $\tilde{G}$ is an extension of G by an amenable group, and Lemma 2.8 yields that $\tilde{G}$ is amenable if and only if G is amenable. Taking into account that $\pi_1(G)$ is amenable, we conclude from [2, Theorem 1.2] that $\lambda_0(\tilde{G}) = \lambda_0(G)$. □

By virtue of Buser's lemma, we derive the following consequence of the preceding characterization.

Corollary 2.11 *Let G be a non-compact, connected, unimodular, and amenable Lie group endowed with a left-invariant metric. Then for any $\varepsilon, r > 0$, there exists an open, bounded $W \subset G$ such that*

$$|(\partial W)^r| < \varepsilon |W \smallsetminus (\partial W)^r|.$$

3 Submersions with Fibers of Basic Mean Curvature

The aim of this section is to prove Theorem 1.1. Let $p \colon M_2 \to M_1$ be a Riemannian submersion with fibers of basic mean curvature, and consider the Schrödinger operator

$$S = \Delta + \frac{1}{4}\|p_* H\|^2 - \frac{1}{2} \operatorname{div} p_* H$$

on M_1. As in [4,5,23], the *average* of a function $f \in C_c^\infty(M_2)$ is the smooth function

$$f_{\mathrm{av}}(x) := \int_{F_x} f$$

on M_1 with gradient given by

$$\langle \operatorname{grad} f_{\mathrm{av}}(x), X \rangle = \int_{F_x} \langle \operatorname{grad} f - fH, \tilde{X} \rangle \tag{2}$$

for any $x \in M_1$ and $X \in T_x M_1$, where $\tilde{X}$ is the horizontal lift of X on F_x. The *push-down* of f is the function

$$h(x) := \sqrt{(f^2)_{\mathrm{av}}(x)} = \left(\int_{F_x} f^2 \right)^{1/2}$$

on M_1. Then [23, Lemma 3.1] states that $h \in \operatorname{Lip}_c(M_1)$. Hence, its gradient is defined almost everywhere and vanishes (if defined) in points where h is zero.

Proposition 3.1 *Let $h \in \operatorname{Lip}_c(M_1)$ be the push-down of a function $f \in C_c^\infty(M_2)$ with $\|f\|_{L^2(M_2)} = 1$. Then their Rayleigh quotients are related by*

$$\mathcal{R}(f) \geq \mathcal{R}_S(h) + \int_{M_1} \lambda_0(F_x) h^2(x) dx.$$

Proof For any $x \in M_1$ with $h(x) > 0$, we readily see from formula (2) that

$$\begin{aligned} \operatorname{grad} h(x) =& \frac{1}{2h(x)} \int_{F_x} (p_* \operatorname{grad} f^2 - f^2 p_* H) \\ =& \frac{1}{h(x)} \int_{F_x} f p_* \operatorname{grad} f - \frac{1}{2} h(x) p_* H(x). \end{aligned}$$

In view of this, the fact that $\|h\|_{L^2(M_1)} = 1$, the divergence formula, the Cauchy-Schwarz inequality and that

$$\begin{aligned} \frac{1}{2} \langle \operatorname{grad} h^2(x), p_* H(x) \rangle =& h(x) \langle \operatorname{grad} h(x), p_* H(x) \rangle \\ &= \int_{F_x} f \langle \operatorname{grad} f, H \rangle - \frac{1}{2} h^2(x) \|p_* H(x)\|^2 \end{aligned}$$

for any $x \in M_1$, we compute

$$\begin{aligned} \mathcal{R}_S(h) &= \int_{M_1} \left(\|\operatorname{grad} h\|^2 + \frac{1}{4} \|p_* H\|^2 h^2 - \frac{1}{2} h^2 \operatorname{div} p_* H \right) \\ &= \int_{M_1} \left(\frac{1}{h^2} \left\| \int_{F_x} f p_* \operatorname{grad} f \right\|^2 + \frac{1}{4} h^2 \|p_* H\|^2 - \int_{F_x} f \langle \operatorname{grad} f, H \rangle \right) \end{aligned}$$

$$+\int_{M_1}\left(\frac{1}{4}h^2\|p_*H\|^2+\frac{1}{2}\langle \operatorname{grad} h^2, p_*H\rangle\right)$$
$$\leq \int_{M_1}\int_{F_x}\|(\operatorname{grad} f)^h\|^2=\int_{M_2}\|(\operatorname{grad} f)^h\|^2. \tag{3}$$

Since at any point of M_2, the tangent space of M_2 splits into the orthogonal sum of the horizontal and the vertical space, it is easily checked that (cf. [23, Formula (6)])

$$\begin{aligned}\mathcal{R}(f)&=\int_{M_2}\|(\operatorname{grad} f)^h\|^2+\int_{M_2}\|(\operatorname{grad} f)^v\|^2\\ &\geq \int_{M_2}\|(\operatorname{grad} f)^h\|^2+\int_{M_1}\lambda_0(F_x)h^2(x).\end{aligned}$$

The proof is now completed by formula (3) and Proposition 2.1. □

Proof of Theorem 1.1 From Propositions 2.1 and 3.1 , it is immediate verify the asserted inequality. Suppose now that the equality holds. Then there exists $(f_n)_{n\in\mathbb{N}}\subset C_c^\infty(M_2)$ with $\|f_n\|_{L^2(M_2)}=1$ and $\mathcal{R}(f_n)\to\lambda_0(M_2)$, as follows from Proposition 2.1. Denote by $h_n\in \operatorname{Lip}_c(M_1)$, the push-down of f_n, $n\in\mathbb{N}$. Arguing as in the proof of [23, Theorem 1.1], using Proposition 3.1 instead of [23, Proposition 3.2], we obtain that

$$\mathcal{R}_S(h_n)\to\lambda_0(S) \text{ and } \int_{M_1}(\lambda_0(F_x)-\inf_{y\in M_1}\lambda_0(F_y))h_n^2(x)dx\to 0. \tag{4}$$

Since $\lambda_0(S)\notin\sigma_{\text{ess}}(S)$, we deduce from [21, Proposition 3.5] that after passing to a subsequence if necessary, we may assume that $h_n\to\varphi$ in $L^2(M_1)$ for some function $\varphi\in C^\infty(M_1)$ with $S\varphi=\lambda_0(S)\varphi$. Then φ is positive, by [21, Proposition 3.7]. Arguing as in the proof of [23, Theorem 1.1], we conclude from (4) that

$$\lambda_0(F_x)=\inf_{y\in M_1}\lambda_0(F_y)$$

for almost any $x\in M_1$. □

Proof of Corollary 1.2 If the submersion has closed fibers of basic mean curvature, then S is written as follows:

$$S=\Delta-\frac{\Delta\sqrt{V}}{\sqrt{V}},$$

where $V(x)$ is the volume of F_x (cf. [23, Sect. 4]). Thus, we may consider the renormalization L of S with respect to $\sqrt{V}$. Then Lemmas 2.4 and 2.6 imply that for any non-zero $f\in C_c^\infty(M_1)$, its lift $\tilde{f}\in C_c^\infty(M_2)$ satisfies

$$\mathcal{R}(\tilde{f})=\frac{\int_{M_2}\|\operatorname{grad}\tilde{f}\|^2}{\int_{M_2}\tilde{f}^2}=\frac{\int_{M_1}\|\operatorname{grad} f\|^2V}{\int_{M_1}f^2V}=\mathcal{R}_L(f)=\mathcal{R}_S(f\sqrt{V}). \tag{5}$$

We derive from Proposition 2.1 that $\lambda_0(M_2) \leq \lambda_0(S)$, while the inverse inequality is a consequence of Theorem 1.1.

About the second statement, choose an exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of M_1 consisting of compact domains with smooth boundary. Then $(p^{-1}(K_n))_{n\in\mathbb{N}}$ is an exhausting sequence of M_2 consisting of compact domains with smooth boundary, because the submersion has closed fibers. Applying Theorem 1.1 to the restriction of $p\colon M_2 \smallsetminus p^{-1}(K_n) \to M_1 \smallsetminus K_n$ over any connected component of $M_1 \smallsetminus K_n$, together with Proposition 2.2, gives the estimate

$$\lambda_0^{\text{ess}}(M_2) = \lim_n \lambda_0(M_2 \smallsetminus p^{-1}(K_n)) \geq \lim_n \lambda_0(S, M_1 \smallsetminus K_n) = \lambda_0^{\text{ess}}(S).$$

From Proposition 2.2, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_1) \smallsetminus \{0\}$ with $\operatorname{supp} f_n$ pairwise disjoint, such that $\mathcal{R}_S(f_n) \to \lambda_0^{\text{ess}}(S)$. It is immediate to verify that the lifts $\tilde{h}_n$ of $h_n := f_n/\sqrt{V}$ also have pairwise disjoint supports. Then Proposition 2.2 and formula (5) yield that

$$\lambda_0^{\text{ess}}(M_2) \leq \liminf_n \mathcal{R}(\tilde{h}_n) = \liminf_n \mathcal{R}_S(f_n) = \lambda_0^{\text{ess}}(S),$$

as we wished. □

It should be noticed that if the submersion has minimal fibers, then S coincides with the Laplacian on M_1. Therefore, [23, Example 3.3] is an example of a Riemannian submersion $p\colon M_2 \to M_1$ with minimal fibers, where M_1 is closed and M_2 is complete, such that

$$0 = \lambda_0(M_2) = \lambda_0(S) + \inf_{x\in F_x} \lambda_0(F_x)$$

and there is $x \in M_1$ with $\lambda_0(F_x) > 0$. It is evident that $(\lambda_0(S) \notin \sigma_{\text{ess}}(S), M_1)$being closed. Hence, in general, the asserted equality in the second part of Theorem 1.1 holds almost everywhere, but not everywhere.

According to the next lemma, the Schrödinger operator S defined in (1) is always non-negative. Moreover, it demonstrates that Theorem 1.1 provides a sharper lower bound for $\lambda_0(M_2)$ than [23, Theorem 1.1] in the case where both of them are applicable.

Lemma 3.2 *Let X be a smooth vector field on a Riemannian manifold M. Then the operator*

$$S = \Delta + \frac{1}{4}\|X\|^2 - \frac{1}{2}\operatorname{div} X$$

is non-negative. Furthermore, if $\|X\| \leq C \leq 2\sqrt{\lambda_0(M)}$, then

$$\lambda_0(S) \geq (\sqrt{\lambda_0(M)} - C/2)^2.$$

Proof For any $f \in C_c^\infty(M)$ with $\| f \|_{L^2(M)} = 1$, observe that its Rayleigh quotient is given by

$$\begin{aligned} \mathcal{R}_S(f) &= \int_M \left(\| \operatorname{grad} f \|^2 + \frac{1}{4} \| X \|^2 f^2 + \langle \operatorname{grad} f, f X \rangle \right) \\ &= \int_M \left\| \operatorname{grad} f + \frac{f}{2} X \right\|^2, \end{aligned} \tag{6}$$

where we used the divergence formula. From Proposition 2.1, we readily see that S is non-negative.

Suppose now that $\| X \| \leq C \leq 2\sqrt{\lambda_0(M)}$ and let $f \in C_c^\infty(M)$ with $\| f \|_{L^2(M)} = 1$. An elementary calculation shows that

$$\begin{aligned} \mathcal{R}_S(f) &\geq \int_M \left(\| \operatorname{grad} f \| - \frac{|f|}{2} \| X \| \right)^2 \\ &= \int_M \left(\| \operatorname{grad} f \|^2 + \frac{f^2}{4} \| X \|^2 - \| \operatorname{grad} f \| |f| \| X \| \right) \\ &\geq \mathcal{R}(f) + \frac{1}{4} \int_M f^2 \| X \|^2 - \mathcal{R}(f)^{1/2} \left(\int_M f^2 \| X \|^2 \right)^{1/2} \\ &= \left(\sqrt{\mathcal{R}(f)} - \frac{1}{2} \left(\int_M f^2 \| X \|^2 \right)^{1/2} \right)^2. \end{aligned} \tag{7}$$

By the assumption that $\| X \| \leq C \leq 2\sqrt{\lambda_0(M)}$, the fact that $\| f \|_{L^2(M)} = 1$ and Proposition 2.1, we obtain that

$$\sqrt{\mathcal{R}(f)} - \frac{1}{2} \left(\int_M f^2 \| X \|^2 \right)^{1/2} \geq \sqrt{\lambda_0(M_1)} - C/2 > 0.$$

The proof is completed by Proposition 2.1 and formula (7). □

We end this section by discussing the application of Theorem 1.1 and Corollary 1.2 to the submersions described in Examples 2.5.

Example 3.3 (i) Consider the warped product $M_2 = M_1 \times_\psi F$ and the projection to the first factor $p\colon M_2 \to M_1$. In this case, the operator S defined in (1) is written as follows:

$$S = \Delta - \frac{\Delta \psi^{k/2}}{\psi^{k/2}},$$

and Theorem 1.1 says that

$$\lambda_0(M_2) \geq \lambda_0(S) + \inf_{x \in M_1} \lambda_0(F_x) = \lambda_0(S) + \lambda_0(F) \inf_{x \in M_1} \psi^{-2}(x).$$

If, in addition, F is closed, then we deduce from Corollary 1.2 that $\lambda_0(M_2) = \lambda_0(S)$ and $\lambda_0^{\text{ess}}(M_2) = \lambda_0^{\text{ess}}(S)$. In particular, M_2 has discrete spectrum if and only if the spectrum of S is discrete (compare with [1, Theorem 3.3]). It is not difficult to establish analogous statements for Clairaut submersions.

(ii) Let $p \colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a Lie group G. In view of Theorem 1.1, the bottoms of the spectra are related by $\lambda_0(M_2) \geq \lambda_0(S)$. According to Corollary 1.2, if G is compact, then $\lambda_0(M_2) = \lambda_0(S)$.

4 Submersions Arising from Lie Group Actions

Throughout this section, we consider a Riemannian submersion $p \colon M_2 \to M_1$ arising from the action of a Lie group G. For convenience of the reader, we provide a brief outline of the section and the proof of Theorem 1.3.

In Sect. 4.1, we show that identifying the fiber with G along a section of the submersion gives rise to a smooth family of left-invariant metrics on G. This remark plays a quite important role in our discussion. More precisely, from this and Theorem 1.1, we obtain Theorem 1.3(iii).

The other assertions of Theorem 1.3 are first proved in the case where G is connected. If G is compact, then they follow from Corollary 1.2 and [23, Theorem 1.2]. Thus, we have to focus on the case where G is non-compact and connected. In Sect. 4.2, we construct cut-off functions on such G closely related to the open sets W from Corollary 2.11. In terms of these functions, for a section $s \colon U \subset M_1 \to M_2$, we define cut-off functions in $p^{-1}(U)$ with uniformly (that is, independently from the corresponding W) bounded gradient and Laplacian.

We begin Sect. 4.3 with the proposition that establishes this auxiliary result. The main idea is that given an $f \in C_c^\infty(M_1)$, we may write it as a sum of functions supported in domains admitting sections. Using cut-off functions as above, we are able to pull up these functions, and for suitable choices of W, the sum of these pulled up functions coincides with the lift of f in a relatively large part of its support. In the rest of its support, its gradient and its Laplacian are bounded independently from W.

The proof of Theorem 1.3 is completed after observing that such a submersion p is expressed as the composition of the submersion arising from the action of G_0 with the covering arising from the action of G/G_0.

4.1 Induced Metrics on the Lie Group

Let $p \colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a (possibly non-connected) Lie group G. Given a section $s \colon U \subset M_1 \to M_2$, it is easily checked that the map $\Phi \colon G \times U \to p^{-1}(U)$ defined by $\Phi(x, y) := xs(y)$ is a diffeomorphism, and so is its restriction $\Phi_y := \Phi(\cdot, y) \colon G \to F_y$. Denote by $g_{s(y)}$ the metric induced on G via Φ_y, that is, the pullback via Φ_y of the restriction of the metric of M_2 on F_y. It is straightforward to see that the metric $g_{s(y)}$ depends only on $s(y)$ and not on the behavior of s in a neighborhood of y.

Proposition 4.1 *Let $s\colon U \subset M_1 \to M_2$ be a section. Then the Riemannian metric $g_{s(y)}$ is left-invariant and depends smoothly on $y \in U$.*

Proof For $x_1, x_2 \in G$, it is immediate to verify that

$$x_1 \Phi_y(x_2) = x_1 x_2 s(y) = \Phi_y(x_1 x_2),$$

and therefore, $x_{1*}\Phi_{y_*} = \Phi_{y_*} L_{x_1 *}$. Bearing in mind that G acts on M_2 via isometries, given $x \in G$ and $X, Y \in T_e G$, where e is the neutral element of G, it is now elementary to compute

$$\begin{aligned} g_{s(y)}(L_{x*}X, L_{x*}Y)(x) &= \langle \Phi_{y_*} L_{x*}X, \Phi_{y_*} L_{x*}Y\rangle_{\Phi_y(x)} = \langle x_* \Phi_{y_*}X, x_* \Phi_{y_*}Y\rangle_{x\Phi_y(e)} \\ &= \langle \Phi_{y_*}X, \Phi_{y_*}Y\rangle_{\Phi_y(e)} = g_{s(y)}(X, Y)(e), \end{aligned}$$

which yields that the induced metric on G is left-invariant.

Choose a left-invariant frame field $\{X_i\}_{i=1}^k$ on G. After endowing G with a left-invariant metric and $G \times U$ with the product metric, it is evident that the projection to the first factor $q\colon G \times U \to G$ is a Riemannian submersion. Consider the horizontal lift $\tilde{X}_i$ of X_i on $G \times U$. Notice that $\{\tilde{X}_i\}_{i=1}^k$ is a G-invariant, smooth frame field, and hence, so is $\{\Phi_* \tilde{X}_i\}_{i=1}^k$. Then for $y \in U$ and $x \in G$, we deduce that

$$g_{s(y)}(X_i, X_j)(x) = g_{s(y)}(X_i, X_j)(e) = \langle \Phi_* \tilde{X}_i, \Phi_* \tilde{X}_j\rangle_{s(y)}.$$

Since $\langle \Phi_* \tilde{X}_i, \Phi_* \tilde{X}_j\rangle_z$ is a smooth function (with respect to z) in $p^{-1}(U)$, so is its composition with s, as we wished. □

Corollary 4.2 *Let $s\colon U \subset M_1 \to M_2$ be a section and fix a left-invariant metric g on G. Then there exists $V_s \in C^\infty(U)$ such that for any $y \in U$, the volume element of the induced metric satisfies*

$$d\mathrm{vol}_{g_{s(y)}} = V_s(y) d\mathrm{vol}_g.$$

Proof Follows immediately from Proposition 4.1 and the local expression of the volume element. □

For $y \in M_1$ and $z_1, z_2 \in F_y$, consider the diffeomorphisms $\Phi_i\colon G \to F_y$ defined by $\Phi_i(x) = x z_i$, and the induced metrics $g_i := g_{z_i}$ on G, $i = 1, 2$. Because G acts transitively on F_y, there exists $x_0 \in G$ such that $x_0 z_1 = z_2$. Then it is apparent that

$$\Phi_2(x) = x z_2 = x x_0 z_1 = (\Phi_1 \circ R_{x_0})(x).$$

In particular, if G is unimodular, then we have that

$$d\mathrm{vol}_{g_2} = \Phi_2^*(d\mathrm{vol}_{F_y}) = R_{x_0}^*(\Phi_1^*(d\mathrm{vol}_{F_y})) = R_{x_0}^*(d\mathrm{vol}_{g_1}) = d\mathrm{vol}_{g_1}, \tag{8}$$

where $d\,\mathrm{vol}_{F_y}$ is the volume element of F_y with respect to the induced metric from M_2. This implies that the function V_s from Corollary 4.2 is independent from the section s and can be defined globally.

Corollary 4.3 *Suppose that G is unimodular and fix a left-invariant metric g on G. Then there exists $V \in C^\infty(M_1)$ such that for any section $s: U \subset M_1 \to M_2$ and $y \in U$, the volume element of the induced metric satisfies*

$$d\mathrm{vol}_{g_{s(y)}} = V(y) d\mathrm{vol}_g.$$

Moreover, the gradient of V is given by

$$\operatorname{grad} V = -V p_* H.$$

Proof The existence of the function V is a consequence of Corollary 4.2 and formula (8). About the second statement, let $y \in M_1$ and $s: U \subset M_1 \to M_2$ be a section defined in a neighborhood U of y that is horizontal at y, which means that $s_* T_y M_1$ is the horizontal space of M_2 at $s(y)$. Let $X \in T_y M_1$ and $c: (-\varepsilon, \varepsilon) \to M_1$ be a smooth curve with $c(0) = y$ and $c'(0) = X$. Denote by $F: (-\varepsilon, \varepsilon) \times G \to M_2$ the smooth variation of the isometric immersion $F(0, \cdot): (G, g_{s(y)}) \to M_2$ defined by $F(t, x) = xs(c(t))$, and observe that its variational vector field is the horizontal lift $\tilde{X}$ of X on F_y. The asserted equality follows now from the first variational formula.□

It is well known that if $p: M_2 \to M_1$ is a Riemannian submersion and M_2 is complete, then so is M_1. According to the next corollary, if the submersion arises from the action of a Lie group, the converse implication is also true.

Corollary 4.4 *Let $p: M_2 \to M_1$ be a Riemannian submersion arising from the action of a Lie group G. If M_1 is complete, then M_2 is complete.*

Proof Fix a left-invariant metric g on G and let $(z_n)_{n\in\mathbb{N}} \subset M_2$ be a Cauchy sequence. Then $(p(z_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in M_1, and hence, $p(z_n) \to y$ for some $y \in M_1$. Let $s: U \subset M_1 \to M_2$ be a section defined in a neighborhood U of y, and consider the corresponding diffeomorphism $\Phi: G \times U \to M_2$, as in the beginning of this subsection. Without loss of generality, we may assume that $z_n \in p^{-1}(U)$ for any $n \in \mathbb{N}$. Writing $z_n = \Phi(x_n, p(z_n))$, it remains to prove that $(x_n)_{n\in\mathbb{N}}$ converges. Given a precompact neighborhood U_y of y with $\bar{U}_y \subset U$, it is simple to see that for any sufficiently small $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$, such that for any $n, m \geq n_0$, there exists a smooth curve $c_{n,m}$ from z_n to z_m of length less than ε, with image contained in $p^{-1}(U_y)$. Denoting by $q: G \times U \to G$ the projection to the first factor, it is clear that $\hat{c}_{n,m} := q \circ \Phi^{-1} \circ c_{n,m}$ is a smooth curve from x_n to x_m. Since U_y is precompact, we derive from Proposition 4.1 that there exists $C > 0$ such that $\ell_g(\hat{c}_{n,m}) \leq C\ell(c_{n,m})$ for any $n, m \geq n_0$, where $\ell(\cdot)$ stands for the length of a curve. This shows that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (G, g) and, thus, converges. □

4.2 Cut-Off Functions

The aim of this subsection is to construct some special functions on the Lie group that will be used in the sequel to obtain cut-off functions on M_2. Throughout this subsection, we consider a non-compact, connected Lie group G endowed with a left-invariant metric. Given $r > 0$, choose a sequence $(x_n)_{n \in \mathbb{N}} \subset G$ such that $d(x_n, x_m) \geq r$ for any $n \neq m$ and the open balls $B(x_n, r)$ cover G.

Lemma 4.5 *There exists $n(r) \in \mathbb{N}$ such that any $x \in G$ lies in at most $n(r)$ of the balls $B(x_n, 2r)$, with $n \in \mathbb{N}$.*

Proof Let $x \in G$ and set $E_x := \{n \in \mathbb{N} : x \in B(x_n, 2r)\}$. Notice that for $n \in E_x$, we have that $B(x_n, r/2) \subset B(x, 5r/2)$ and the balls $B(x_n, r/2)$ are disjoint. Bearing in mind that G is a homogeneous space, we compute

$$|B(x, 5r/2)| \geq \sum_{n \in E_x} |B(x_n, r/2)| = |E_x||B(x, r/2)|,$$

where $|E_x|$ is the cardinality of E_x. Since the Ricci curvature of G is bounded from below (say by $(k-1)C$, where k is the dimension of G), the Bishop-Gromov volume comparison theorem gives the estimate

$$|E_x| \leq \frac{|B(x, 5r/2)|}{|B(x, r/2)|} \leq \frac{|B_{5r/2}|}{|B_{r/2}|} =: n(r),$$

where B_ρ is a ball of radius ρ in the k-dimensional space form of sectional curvature C. □

Fix $\psi_e \in C_c^\infty(G)$ with $0 \leq \psi_e \leq 1$, $\operatorname{supp} \psi_e \subset B(e, 3r/2)$ and $\psi_e = 1$ in $B(e, r)$. For $n \in \mathbb{N}$, the function $\psi_n := \psi_e \circ L_{x_n}^{-1}$ satisfies $0 \leq \psi_n \leq 1$, $\operatorname{supp} \psi_n \subset B(x_n, 3r/2)$ and $\psi_n = 1$ in $B(x_n, r)$. We know from Lemma 4.5 that the cover $\{B(x_n, 3r/2)\}_{n \in \mathbb{N}}$ is locally finite, which implies that the function $\psi := \sum_{n \in \mathbb{N}} \psi_n$ is well defined and smooth. It is evident that $\psi \geq 1$, G being covered by the balls $B(x_n, r)$. The smooth partition of unity on G consisting of the functions $\zeta_n := \psi_n/\psi$ with $n \in \mathbb{N}$ is called a *partition of unity corresponding to* r. Clearly, any point of G lies in at most $n(r)$ of the supports of ζ_n, where $n(r)$ is the constant from Lemma 4.5. The cut-off function corresponding to a subset E of $\mathbb{N}$ is defined by

$$\chi_E := \sum_{n \in E} \zeta_n.$$

Let $p \colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a non-compact, connected Lie group G. Consider a relatively compact, open domain $U \subset M_1$ that admits an extensible section $s \colon U \to M_2$, and the corresponding diffeomorphism $\Phi \colon G \times U \to p^{-1}(U)$ defined by $\Phi(x, y) := xs(y)$. For a function $f \colon G \to \mathbb{R}$, we denote by $f_s \colon p^{-1}(U) \to \mathbb{R}$ the function satisfying

$$f_s(\Phi(x, y)) := f(x)$$

for any $x \in G$ and $y \in U$. Given a left-invariant metric on G and $r > 0$, we consider a partition of unity on G corresponding to r and the functions χ_E for $E \subset \mathbb{N}$.

Lemma 4.6 *Let $s : U \to M_2$ be an extensible section defined on a precompact domain U of M_1. Then there exists a constant C independent from $E \subset \mathbb{N}$, such that*

$$|\Delta(\chi_E)_s(z)| \leq C \text{ and } \|\operatorname{grad}(\chi_E)_s(z)\| \leq C$$

for any $z \in p^{-1}(U)$.

Proof Since U is precompact and s is extensible, it is easily checked that the Laplacian and the gradient of $(\psi_e)_s$ are bounded. Since $(\psi_n)_s(z) = (\psi_e)_s(x_n^{-1}z)$ for any $n \in \mathbb{N}$ and $z \in p^{-1}(U)$, we obtain uniform estimates for the Laplacian and the gradient of $(\psi_n)_s$ for all $n \in \mathbb{N}$. Then Lemma 4.5 yields a uniform bound for the Laplacian and the gradient of the functions $\sum_{n\in E}(\psi_n)_s$ for all subsets $E \subset \mathbb{N}$. The proof is completed after observing that

$$(\chi_E)_s = \frac{\sum_{n\in E}(\psi_n)_s}{\sum_{n\in\mathbb{N}}(\psi_n)_s}$$

and that $\sum_{n\in\mathbb{N}}(\psi_n)_s \geq 1$. □

The purpose of considering this partition of unity becomes more clear in the next proposition, where we combine this construction with Corollary 2.11 in the case where G is unimodular and amenable.

Proposition 4.7 *Let G be a non-compact, connected, unimodular and amenable Lie group endowed with a left-invariant metric. Consider $r > 0$ and a partition of unity $\{\zeta_n\}_{n\in\mathbb{N}}$ corresponding to $r/2$. Then for any $\varepsilon > 0$, there exists an open, bounded $W \subset G$ and a finite $E \subset \mathbb{N}$, such that $\chi_E = 1$ in $W \smallsetminus (\partial W)^r$, $\operatorname{supp} \chi_E \subset W^{r/2}$ and*

$$|(\partial W)^{2r}| < \varepsilon |W \smallsetminus (\partial W)^{2r}|.$$

Proof As a consequence of Corollary 2.11, for any $\varepsilon > 0$, there exists an open, bounded $W \subset G$ such that the desired inequality for the volumes holds. Consider the finite set $E := \{n \in \mathbb{N} : x_n \in W \smallsetminus (\partial W)^{r/4}\}$. It is elementary to verify that if $x \in W \smallsetminus (\partial W)^r$, then $n \in E$ for any $n \in \mathbb{N}$ with $x \in B(x_n, 3r/4)$, and therefore, $\chi_E = 1$ in $W \smallsetminus (\partial W)^r$. From the fact that $\operatorname{supp} \zeta_n \subset B(x_n, 3r/4)$, it follows that $\operatorname{supp} \chi_E \subset W^{r/2}$. □

4.3 Pulling Up

Suppose now that G is unimodular and let V be the function from Corollary 4.3. A straightforward calculation shows that

$$S = \Delta + \frac{1}{4}\|p_* H\|^2 - \frac{1}{2}\operatorname{div} p_* H = \Delta - \frac{\Delta\sqrt{V}}{\sqrt{V}}.$$

This allows us to consider the renormalization

$$L = m_{\sqrt{V}}^{-1} \circ S \circ m_{\sqrt{V}} = \Delta - \operatorname{grad} \ln V = \Delta + p_* H$$

of S with respect to $\sqrt{V}$, where we used again Corollary 4.3. According to Lemma 2.6, the Laplacian of the lift $\tilde{f}$ of any $f \in C^\infty(M_1)$ is given by

$$\Delta \tilde{f} = \widetilde{Lf}. \tag{9}$$

Proposition 4.8 *Let $p\colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a non-compact, connected, unimodular, and amenable Lie group G. Then for any $\lambda \in \mathbb{R}$, $\varepsilon > 0$ and $f \in C_c^\infty(M_1) \smallsetminus \{0\}$, there exists $h \in C_c^\infty(M_2) \smallsetminus \{0\}$, such that*

$$\frac{\|(\Delta - \lambda)h\|^2_{L^2(M_2)}}{\|h\|^2_{L^2(M_2)}} \le \frac{\|(L - \lambda)f\|^2_{L^2_{\sqrt{V}}(M_1)}}{\|f\|^2_{L^2_{\sqrt{V}}(M_1)}} + \varepsilon.$$

Proof Cover $\operatorname{supp} f$ with finitely many open, precompact domains U_i that admit extensible sections $s_i\colon U_i \to M_2$, $i = 1, \dots, k$, and choose non-negative $\varphi_i \in C_c^\infty(U_i)$ with $\sum_{i=1}^k \varphi_i = 1$ in $\operatorname{supp} f$. Denote by $x_{ij}\colon U_i \cap U_j \to G$ the transition maps defined by $s_j(y) = x_{ij}(y)s_i(y)$ for all $y \in U_i \cap U_j$, and by $\Phi_i\colon G \times U_i \to p^{-1}(U_i)$ the diffeomorphisms defined by $\Phi_i(x, y) = xs_i(y)$.

Fix a left-invariant metric g on G. Since U_i is precompact and s_i is extensible, notice that there exists $r > 0$ such that $x_{ij}(U_i \cap U_j) \subset B_g(e, r)$ for any $i, j = 1, \dots, k$. Let $\{\zeta_n\}_{n\in\mathbb{N}}$ be a partition of unity on G corresponding to $r/2$, as in Sect. 4.2. For a finite subset E of $\mathbb{N}$, consider the compactly supported, smooth function

$$h_i := (\chi_E)_{s_i} \tilde{\varphi}_i \tilde{f}$$

in $p^{-1}(U_i)$, $i = 1, \dots, k$. Setting $h = \sum_{i=1}^k h_i$, we derive from Lemma 4.6 that there exists a constant C independent from E, such that $|(\Delta - \lambda)h(z)| \le C$ for any $z \in M_2$.

We know from Proposition 4.7 that for any $\varepsilon > 0$, there exists an open, bounded $W \subset G$ and a finite $E \subset \mathbb{N}$, such that $\chi_E = 1$ in $W \smallsetminus (\partial W)^r$, $\operatorname{supp} \chi_E \subset W^{r/2}$ and

$$\frac{|W_0'|_g}{|W_0|_g} < \frac{\|f\|^2_{L^2_{\sqrt{V}}(M_1)}}{C^2 \int_{\operatorname{supp} f} V}, \tag{10}$$

where $W_0 := W \smallsetminus (\partial W)^{2r}$, $W_0' := (\partial W)^{2r}$ and tubular neighborhoods are considered with respect to the background metric g. Denote by $W_i(y)$ and $W_i'(y)$ the images of W_0 and W_0' via $\Phi_i(\cdot, y)$, respectively. Bearing in mind that

$$\Phi_i(x, y) = \Phi_j(xx_{ji}(y), y)$$

for any $y \in U_i \cap U_j$ and $x \in G$, it is not difficult to see that $h(z) = \tilde{f}(z)$ for any $z \in W_i(y)$ and that $\operatorname{supp} h \cap F_y \subset W_i(y) \cup W_i'(y)$ for any $y \in U_i$, $i = 1, \dots, k$. In view of Corollary 4.3, it is now simple to compute

$$\begin{aligned}\|h\|^2_{L^2(M_2)} &= \sum_{i=1}^{k}\int_{M_2}\tilde{\varphi}_i h^2 \geq \sum_{i=1}^{k}\int_{U_i}\int_{W_i(y)}\tilde{\varphi}_i h^2 dy\\ &= \sum_{i=1}^{k}\int_{U_i}\varphi_i(y) f^2(y)|W_0|_{g_{s_i(y)}} dy\\ &= |W_0|_g \sum_{i=1}^{k}\int_{U_i}\varphi_i f^2 V = |W_0|\|f\|^2_{L^2_{\sqrt{V}}(M_1)}.\end{aligned}$$

Furthermore, it is apparent that

$$\begin{aligned}\|(\Delta-\lambda)h\|^2_{L^2(M_2)} =& \sum_{i=1}^{k}\int_{M_2}\tilde{\varphi}_i((\Delta-\lambda)h)^2\\ =& \sum_{i=1}^{k}\int_{U_i}\int_{W_i(y)}\tilde{\varphi}_i((\Delta-\lambda)h)^2 dy\\ &+ \sum_{i=1}^{k}\int_{U_i}\int_{W_i'(y)}\tilde{\varphi}_i((\Delta-\lambda)h)^2 dy.\end{aligned}$$

By virtue of Corollary 4.3 and formula (9), we deduce that

$$\begin{aligned}\sum_{i=1}^{k}\int_{U_i}\int_{W_i(y)}\tilde{\varphi}_i((\Delta-\lambda)h)^2 dy &= \sum_{i=1}^{k}\int_{U_i}\int_{W_i(y)}\tilde{\varphi}_i((\Delta-\lambda)\tilde{f})^2 dy\\ &= \sum_{i=1}^{k}\int_{U_i}\varphi_i(y)((L-\lambda)f(y))^2|W_0|_{g_{s_i(y)}} dy\\ &= |W_0|_g\int_{M_1}((L-\lambda)f)^2 V\\ &= |W_0|_g\|(L-\lambda)f\|^2_{L^2_{\sqrt{V}}(M_1)}.\end{aligned}$$

Finally, Corollary 4.3 implies that

$$\begin{aligned}\sum_{i=1}^{k}\int_{U_i}\int_{W_i'(y)}\tilde{\varphi}_i((\Delta-\lambda)h)^2 \leq& C^2\sum_{i=1}^{k}\int_{\operatorname{supp} f\cap U_i}\varphi_i(y)|W_0'|_{g_{s_i(y)}} dy\\ =& C^2|W_0'|_g\int_{\operatorname{supp} f} V.\end{aligned}$$

From the above estimates, we conclude that

$$\frac{\|(\Delta-\lambda)h\|^2_{L^2(M_2)}}{\|h\|^2_{L^2(M_2)}} \leq \frac{\|(L-\lambda)f\|^2_{L^2_{\sqrt{V}}(M_1)}}{\|f\|^2_{L^2_{\sqrt{V}}(M_1)}} + \frac{|W_0'|_g}{|W_0|_g}\frac{C^2\int_{\operatorname{supp} f} V}{\|f\|^2_{L^2_{\sqrt{V}}(M_1)}},$$

which, together with (10), completes the proof. □

Similarly, exploiting the second inequality of Lemma 4.6, it is not hard to show the following:

Proposition 4.9 *Let $p\colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a non-compact, connected, unimodular, and amenable Lie group G. Then for any $\varepsilon > 0$ and $f \in C^\infty_c(M_1) \smallsetminus \{0\}$, there exists $h \in C^\infty_c(M_2) \smallsetminus \{0\}$ such that $\mathcal{R}(h) \leq \mathcal{R}_L(f) + \varepsilon$.*

Before proceeding to the proof of Theorem 1.3, we establish a part of it in the case where G is a connected Lie group.

Proposition 4.10 *Let $p\colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a connected Lie group G. If G is unimodular and amenable, then the bottoms of the spectra satisfy $\lambda_0(M_2) = \lambda_0(S)$. If, in addition, M_1 is complete, then $\sigma(S) \subset \sigma(M_2)$.*

Proof According to Corollary 1.2, if G is compact, then $\lambda_0(M_2) = \lambda_0(S)$. If, in addition, M_1 is complete, then Corollary 4.4 asserts that so is M_2, and the second statement is a consequence of [23, Theorem 1.2].

Suppose now that G is non-compact, unimodular, and amenable. Then for any $\varepsilon > 0$, there exists a non-zero $f \in C^\infty_c(M_1)$ such that $\mathcal{R}_S(f) < \lambda_0(S) + \varepsilon/2$, by Proposition 2.1. From Propositions 2.4 and 4.9 , it follows that there exists a non-zero $h \in C^\infty_c(M_2)$ with

$$\mathcal{R}(h) \leq \mathcal{R}_L(f/\sqrt{V}) + \varepsilon/2 = \mathcal{R}_S(f) + \varepsilon/2 < \lambda_0(S) + \varepsilon.$$

The proof of the first assertion is completed by Proposition 2.1, $\varepsilon > 0$ being arbitrary.

Assume now that, in addition, M_1 is complete and notice that M_2 is also complete, from Corollary 4.4. Then Proposition 2.3 yields that for any $\lambda \in \sigma(S)$, there exists a characteristic sequence $(f_n)_{n\in\mathbb{N}}$ for S and λ. In view of Proposition 4.8 and Lemma 2.4, for any $n \in \mathbb{N}$, there exists $h_n \in C^\infty_c(M_2) \smallsetminus \{0\}$ satisfying

$$\begin{aligned}\frac{\|(\Delta-\lambda)h_n\|^2_{L^2(M_2)}}{\|h_n\|^2_{L^2(M_2)}} &\leq \frac{\|(L-\lambda)(f_n/\sqrt{V})\|^2_{L^2_{\sqrt{V}}(M_1)}}{\|f_n/\sqrt{V}\|^2_{L^2_{\sqrt{V}}(M_1)}} + \frac{1}{n} \\ &= \frac{\|(S-\lambda)f_n\|^2_{L^2(M_1)}}{\|f_n\|^2_{L^2(M_1)}} + \frac{1}{n} \to 0,\end{aligned}$$

as $n \to +\infty$. That is, $(h_n)_{n\in\mathbb{N}}$ is a characteristic sequence for Δ (on M_2) and λ, and hence, $\lambda \in \sigma(M_2)$, from Proposition 2.3. □

Consider now a Riemannian submersion $p\colon M_2 \to M_1$ arising from the action of a Lie group G. Denote by $p_1 : M_2 \to M$ the Riemannian submersion arising from the action of the connected component G_0 of G. Then the action of G on M_2 descends to a properly discontinuous action of G/G_0 on M, which gives rise to a Riemannian covering $p_2\colon M \to M_1$, and the original submersion is decomposed as $p = p_2 \circ p_1$. It is immediate to verify that the Schrödinger operator

$$S_M := \Delta + \frac{1}{4}\|p_{1*}H\|^2 - \frac{1}{2}\operatorname{div} p_{1*}H$$

on M, defined as in (1), is the lift of the corresponding Schrödinger operator S on M_1.

Proof of Theorem 1.3 Write $p = p_2 \circ p_1$ as above, and suppose that G is amenable and G_0 is unimodular. Then Lemma 2.8 states that G_0 and G/G_0 are also amenable. From Proposition 4.10 and [2, Theorem 1.2], we obtain that

$$\lambda_0(M_2) = \lambda_0(S_M) = \lambda_0(S).$$

If, in addition, M_1 is complete, then so is M, and the spectra are related by

$$\sigma(S) \subset \sigma(S_M) \subset \sigma(M_2),$$

where the first inclusion follows from [22, Corollaries 4.21 and 4.22] and the second one from Proposition 4.10.

Conversely, assume that $\lambda_0(M_2) = \lambda_0(S) \notin \sigma_{\text{ess}}(S)$. By virtue of Theorem 1.1, we have that $\lambda_0(F_x) = 0$ for almost any $x \in M_1$. Recall that F_x is isometric to G endowed with a left-invariant metric, from Lemma 4.1. Taking into account that $\lambda_0(G) = \lambda_0(G_0)$, we derive from Theorem 2.10 that G_0 is unimodular and amenable. Moreover, Theorem 1.1 and [2, Theorem 1.1] show that

$$\lambda_0(M_2) \geq \lambda_0(S_M) \geq \lambda_0(S),$$

and thus, $\lambda_0(S_M) = \lambda_0(S)$. Since $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$, we conclude from [21, Theorem 1.2] that p_2 is an amenable covering, or equivalently, G/G_0 is amenable. The proof is completed by Lemma 2.8. □

Proof of Corollary 1.4 Suppose first that G is unimodular and amenable, and fix a left-invariant metric on G. By formula (6), it is easily checked that $\mathcal{R}_S(\sqrt{V}) = 0$ for the positive $V \in C^\infty(M_1)$ from Corollary 4.3, which together with Theorem 1.3, Proposition 2.1 and Lemma 3.2, implies that $\lambda_0(M_2) = \lambda_0(S) = 0$.

Conversely, assume that $\lambda_0(M_2) = 0$ and write $p = p_2 \circ p_1$ as above. We readily see from Theorem 1.1 that $\lambda_0(S) = 0$. Then G is amenable and G_0 is unimodular, from Theorem 1.3, because $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$, M_1 being closed. We know from Corollary 4.3 that there exists $V \in C^\infty(M)$ with $p_{1*}H = -\operatorname{grad} \ln V$, such that for any section

$s \colon U \subset M \to M_2$, the volume elements of the induced metrics on G_0 (and on G) satisfy

$$d\mathrm{vol}_{g_{s(y)}} = V(y) d\mathrm{vol}_g,$$

where g is a fixed left-invariant metric on G.

Observe that there exists a positive $\varphi \in C^\infty(M_1)$ with $S\varphi = 0$, from [21, Proposition 3.7] and the fact that $\lambda_0(S) = 0 \notin \sigma_{\mathrm{ess}}(S)$. Then $\mathcal{R}_S(\varphi) = 0$, which together with formula (6), gives that $p_* H = -2 \,\mathrm{grad} \ln \varphi$. It is now clear that V is a constant multiple of the lift $\tilde{\varphi}^2$ of φ^2 on M, and in particular, G/G_0-invariant.

Given $z \in M_2$ and $x \in G$, using formula (8), the definition and the G/G_0-invariance of V, we compute

$$\begin{aligned} R_x^*(d \,\mathrm{vol}_{\,g_z}) =& d \,\mathrm{vol}_{\,g_{xz}} = V(p_1(xz)) d \,\mathrm{vol}_{\,g} = V([x] p_1(z)) d \,\mathrm{vol}_{\,g} \\ =& V(p_1(z)) d \,\mathrm{vol}_{\,g} = d \,\mathrm{vol}_{\,g_z}, \end{aligned}$$

where $[x]$ stands for the class of x in G/G_0. Therefore, g_z is a left-invariant metric on G with right-invariant volume element, which means that G is unimodular. □

We end this section with a class of examples demonstrating that the assumption $\lambda_0(S) \notin \sigma_{\mathrm{ess}}(S)$ in Theorem 1.3(iii) cannot be dropped, even if the manifolds are complete and the fibers are minimal.

Example 4.11 Let G be the simply connected Lie group with Lie algebra spanned by two vectors X, Y such that $[X, Y] = Y$. Given $c > 0$, define the left-invariant metric g_c on G by $g_c(X, X) = c^{-1}$, $g_c(X, Y) = 0$ and $g_c(Y, Y) = c$. It is obvious that

$$\langle \nabla_{X_1} X_2, X_3 \rangle = \frac{1}{2} \left(\langle [X_1, X_2], X_3 \rangle - \langle [X_2, X_3], X_1 \rangle + \langle [X_3, X_1], X_2 \rangle \right)$$

for any left-invariant vector fields X_1, X_2, X_3 on G, where the inner products are with respect to g_c and ∇ stands for the Levi-Civita connection of g_c. From this, it is easy to see that (G, g_c) has constant sectional curvature $-c$. Thus, (G, g_c) is isometric to the 2-dimensional space form of sectional curvature $-c$, and in particular, the bottom of its spectrum is given by

$$\lambda_0(G, g_c) = \frac{c^2}{4}. \tag{11}$$

Bearing in mind that G is solvable, observe that G is not unimodular, from Theorem 2.10.

Let M be a Riemannian manifold with $\lambda_0(M) \in \sigma_{\mathrm{ess}}(M)$. For a positive function $\psi \in C^\infty(M)$, endow the product manifold $M_2 := M \times G$ with the Riemannian metric $g(y, x) = g_M(y) \times g_{\psi(y)}(x)$. It is evident that G acts smoothly, freely, and properly via isometries on M_2, and the Riemannian submersion arising from this action is the projection to the first factor $p \colon M_2 \to M$. It is noteworthy that p has minimal fibers, since the volume element of g_c does not depend on c. Hence, the operator S defined as in (1) coincides with the Laplacian on M.

By Proposition 2.2, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M) \setminus \{0\}$ such that $\mathcal{R}(f_n) \to \lambda_0^{\text{ess}}(M) = \lambda_0(M)$ and $\operatorname{supp} f_n \subset U_n$ for some precompact, open domains U_n with $\bar{U}_n$ pairwise disjoint. Clearly, we may choose a positive $\psi \in C^\infty(M)$ with $\psi = c_n < 1/n$ in U_n for any $n \in \mathbb{N}$. Then $p^{-1}(U_n)$ is isometric to the Riemannian product $U_n \times G$, where G is endowed with the Riemannian metricg_{c_n}. In view of Proposition 2.1 and formula (11), it follows that for any $n \in \mathbb{N}$, there exists $h_n \in C_c^\infty(G) \setminus \{0\}$ with $\mathcal{R}_{g_{c_n}}(h_n) < 1/(4n^2)$. Setting $\tilde{h}_n(y,x) = h_n(x)$ and $\tilde{f}_n(y,x) = f_n(y)$, we have that $\tilde{h}_n \tilde{f}_n \in C_c^\infty(M_2)$ and a straightforward calculation implies that

$$\mathcal{R}(\tilde{h}_n \tilde{f}_n) = \mathcal{R}_{g_{c_n}}(h_n) + \mathcal{R}(f_n) \to \lambda_0(M),$$

as $n \to +\infty$. From this, together with Theorem 1.1 and Proposition 2.1, we deduce that $\lambda_0(M_2) = \lambda_0(M) = \lambda_0(S)$, while G is not unimodular.

5 Bottom of Spectrum of Lie Groups

In this section, we discuss some applications of our results to Lie groups. We begin by establishing Theorem 1.5.

Proof of Theorem 1.5 Clearly, the projection $p\colon G \to G/N$ is the Riemannian submersion arising from the (left) action of N on G, and the fiber over $p(z)$ is written as is $F_{p(z)} = Nz = zN$ for any $z \in G$, N being normal. Since multiplication L_x from the left with an element $x \in G$ maps isometrically $F_{p(z)}$ to $F_{p(xz)}$ for any $z \in G$, it is evident that the mean curvature H of the fibers is left-invariant, and so is $p_* H$ on G/N. Then the operator S on G/N defined as in (1) is of the form $S = \Delta + c$ for some $c \in \mathbb{R}$, and the bottom of its spectrum is $\lambda_0(S) = \lambda_0(G/N) + c$.

To determine this constant, let $\{X_i\}_{i=1}^m$ be an orthonormal basis of $T_e G$ with $\{X_i\}_{i=1}^k$ spanning $T_e N$. Considering the left-invariant extension of X_i (also denoted by X_i), it is easily checked that

$$\begin{aligned}
\|H\|^2 &= \sum_{i=1}^k \langle \nabla_{X_i} X_i, H\rangle = -\sum_{i=1}^k \langle \nabla_{X_i} H, X_i\rangle \\
&= -\sum_{i=1}^m \langle \nabla_{X_i} H, X_i\rangle + \sum_{i=k+1}^m \langle \nabla_{X_i} H, X_i\rangle \\
&= \sum_{i=1}^m \langle [H, X_i], X_i\rangle + \sum_{i=k+1}^m \langle \nabla_{p_* X_i} p_* H, p_* X_i\rangle \\
&= \operatorname{tr}(\operatorname{ad} H) + \operatorname{div} p_* H,
\end{aligned} \tag{12}$$

and the operator S is written as follows:

$$S = \Delta - \frac{1}{4}\|H\|^2 + \frac{1}{2}\operatorname{tr}(\operatorname{ad} H).$$

The first statement follows from Theorem 1.1, after noticing that $\lambda_0(F_y) = \lambda_0(N)$ for any $y \in G/N$, F_y being isometric to N. If N is unimodular and amenable, then Theorem 1.3 establishes the asserted equality. Conversely, as a consequence of Theorem 1.1, if

$$\lambda_0(G) = \lambda_0(G/N) - \frac{1}{4}\|H\|^2 + \frac{1}{2}\operatorname{tr}(\operatorname{ad} H),$$

then the infimum of $\lambda_0(F_y)$ with $y \in G/N$ is zero. Then $\lambda_0(N) = 0$, since F_y is isometric to N (endowed with the induced left-invariant metric from G) and Theorem 2.10 yields that N is unimodular and amenable. □

It is worth to point out that in the above setting, the assumption $\lambda_0(S) \notin \sigma_{\mathrm{ess}}(S)$ involved in Theorem 1.3(iii) is not satisfied in general. Indeed, if G/N is non-compact, then $\sigma(S) = \sigma_{\mathrm{ess}}(S)$, S being invariant under multiplication from the left with elements of G/N (cf. for instance [22, Theorem 5.2]). However, the conclusion of Theorem 1.3(iii) holds because the fibers are isometric.

Corollary 5.1 *Let G be a connected, unimodular, and amenable Lie group endowed with a left-invariant metric and N be a closed (as a subset), connected, normal subgroup of G with mean curvature H. Then*

$$\lambda_0(G/N) = \frac{1}{4}\|H\|^2.$$

In particular, G/N is also unimodular (and amenable) if and only if N is minimal.

Proof Since G is unimodular, we obtain from Lemma 2.9 that $\operatorname{tr}(\operatorname{ad} H) = 0$ and that N is also unimodular. According to Lemma 2.8, since G is amenable, so are N and G/N. The proof is completed by Theorems 1.5 and 2.10 . □

Recall that, in general, the quotient of a unimodular and amenable Lie group does not have to be unimodular. The next example demonstrates this fact.

Example 5.2 Let G be the simply connected, solvable Lie group with Lie algebra $\mathfrak{g}$ generated by X, Y, Z satisfying $[X, Y] = Y$, $[X, Z] = -Z$ and $[Y, Z] = 0$. It is obvious that $\operatorname{tr}(\operatorname{ad} X') = 0$ for any $X' \in \mathfrak{g}$, and we deduce from Lemma 2.9 that G is unimodular. Let N be the closed (as a subset), connected, normal subgroup of G with Lie algebra the ideal generated by Z. Denoting by $p\colon G \to G/N$ the projection, it is elementary to verify that $\operatorname{tr}(\operatorname{ad} p_* X) = 1$. We conclude from Lemma 2.9 that G/N is not unimodular, while G is unimodular and amenable.

Before proceeding to the proof of Corollary 1.6, we need some auxiliary results. The next proposition provides a standard way of estimating the Cheeger constant of a Riemannian manifold.

Proposition 5.3 *Let X be a smooth vector field on a Riemannian manifold M with $\|X\| \leq 1$ and $\operatorname{div} X \geq c$ for some $c \in \mathbb{R}$. Then the Cheeger constant of M is bounded by $h(M) \geq c$.*

Proof Using the divergence formula, for any compact domain K of M with smooth boundary, we compute

$$c|K| \leq \int_K \operatorname{div} X = \int_{\partial K} \langle X, \nu \rangle \leq |\partial K|,$$

where ν is the outward pointing unit normal to ∂K. □

Corollary 5.4 *Let G be a connected Lie group endowed with a left-invariant metric. Then the Cheeger constant of G satisfies*

$$h(G) \geq \max_{X \in \mathfrak{g}, \|X\|=1} \operatorname{tr}(\operatorname{ad} X).$$

Proof A straightforward calculation shows that $\operatorname{tr}(\operatorname{ad} X) = -\operatorname{div} X$ for any $X \in \mathfrak{g}$, and the assertion is a consequence of Proposition 5.3. □

Proposition 5.5 *Let G be a connected, amenable Lie group endowed with a left-invariant metric. Suppose that its radical S is not abelian and denote by H the mean curvature (in G) of the commutator subgroup $[S, S]$. Then*

$$\lambda_0(G) = \frac{1}{4}\|H\|^2 = \frac{1}{4}\operatorname{tr}(\operatorname{ad} H).$$

Proof Consider the universal covering $q \colon \tilde{S} \to S$. Since $\tilde{S}$ is simply connected and solvable, it is known that its commutator subgroup $[\tilde{S}, \tilde{S}]$ is closed (as a subset of $\tilde{S}$) and nilpotent (cf. for instance [17, Proposition 1.6] and the references therein). This yields that the commutator subgroup $N := [S, S] = q([\tilde{S}, \tilde{S}])$ is a connected, closed (as a subset), normal, and nilpotent subgroup of G. Since connected, nilpotent groups are unimodular and amenable, Theorem 2.10 gives that

$$\lambda_0(G) = \lambda_0(G/N) - \frac{1}{4}\|H\|^2 + \frac{1}{2}\operatorname{tr}(\operatorname{ad} H).$$

Bearing in mind that G is a compact extension of S, it is evident that G/N is a compact extension of the abelian group S/N. In particular, G/N is unimodular and amenable, and hence, $\lambda_0(G/N) = 0$, from Theorem 2.10. Let $\{X_i\}_{i=1}^m$ be an orthonormal basis of $\mathfrak{g}$ with $\{X_i\}_{i=1}^k$ spanning the Lie algebra of N. Then formula (12) yields that

$$\|H\|^2 = \operatorname{tr}(\operatorname{ad} H) - \operatorname{tr}(\operatorname{ad} p_* H).$$

We derive from Lemma 2.9 that $\operatorname{tr}(\operatorname{ad} p_* H) = 0$, G/N being unimodular, as we wished. □

Proof of Corollary 1.6 If G is unimodular, then the statement follows from Lemma 2.9, Theorem 2.10 and the Cheeger inequality. Suppose now that G is not unimodular and observe that S is not abelian, since G is a compact extension of S. It follows from Theorem 2.10 that $\lambda_0(G) > 0$, and thus, the mean curvature (in G) H of the commutator subgroup $N := [S, S]$ of the radical S of G is non-zero, from Proposition 5.5. In view of Corollary 5.4, Proposition 5.5 and the Cheeger inequality, we conclude that

$$\frac{1}{4}h(G)^2 \geq \frac{1}{4}(\mathrm{tr}(\mathrm{ad}\, H_0))^2 = \frac{1}{4}\,\mathrm{tr}(\mathrm{ad}\, H) = \lambda_0(G) \geq \frac{1}{4}h(G)^2,$$

where $H_0 := \|H\|^{-1}H$. □

According to [7], if the Cheeger constant coincides with the exponential volume growth, then the equality holds in the Cheeger inequality. However, this fails in Corollary 1.6, since there exist unimodular and amenable Lie groups of exponential volume growth (cf. [20, p. 1525] and the references therein).

Acknowledgements I would like to thank Werner Ballmann and Dorothee Schüth for their helpful comments and remarks. I am also grateful to the Max Planck Institute for Mathematics in Bonn for its support and hospitality. I would also like to thank the referee for some helpful suggestions.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Baider, A.: Noncompact Riemannian manifolds with discrete spectra. J. Differ. Geom. **14**(1), 41–57 (1979)
2. Ballmann, W., Matthiesen, H., Polymerakis, P.: On the bottom of spectra under coverings. Math. Z. **288**(3–4), 1029–1036 (2018)
3. Ballmann, W., Polymerakis, P.: Bottom of spectra and coverings. Surv. Differ. Geom. **23**, 1–33 (2020)
4. Bessa, G.P., Montenegro, J.F., Piccione, P.: Riemannian submersions with discrete spectrum. J. Geom. Anal. **22**(2), 603–620 (2012)
5. Bordoni, M.: Spectral estimates for submersions with fibers of basic mean curvature. An. Univ. Vest Timiş. Ser. Mat.-Inform. **44**(1), 23–36 (2006)
6. Bordoni, M.: Spectra of Submersions, Contemporary Geometry and Related Topics, Univ. Belgrade Fac. Math., Belgrade, 2006, pp. 51–62. https://emis.de/proceedings/CGRT2005/
7. Brooks, R.: A relation between growth and the spectrum of the Laplacian. Math. Z. **178**(4), 501–508 (1981)
8. Brooks, R.: The fundamental group and the spectrum of the Laplacian. Comment. Math. Helv. **56**(4), 581–598 (1981)
9. Buser, P.: A note on the isoperimetric constant. Ann. Sci. École Norm. Sup. (4) **15**(2), 213–230 (1982)
10. Cavalcante, M.P., Manfio, F.: On the fundamental tone of immersions and submersions. Proc. Am. Math. Soc. **146**(7), 2963–2971 (2018)

11. Cheeger, J.: A Lower Bound for the Smallest Eigenvalue of the Laplacian, Problems in Analysis (Papers Dedicated to Salomon Bochner, 1969), Princeton University Press, Princeton, NJ, pp. 195–199 (1970)
12. Donnelly, H., Li, P.: Pure point spectrum and negative curvature for noncompact manifolds. Duke Math. J. **46**(3), 497–503 (1979)
13. Dunn, C., Gilkey, P., Park, J.: The spectral geometry of the canonical Riemannian submersion of a compact Lie group. J. Geom. Phys. **57**(10), 2065–2076 (2007)
14. Falcitelli, M., Ianus, S., Pastore, A.M.: Riemannian Submersions and Related Topics. World Scientific Publishing Co., Inc., River Edge, NJ (2004)
15. Gilkey, P., Leahy, J., Park, J.: Spectral Geometry, Riemannian Submersions, and the Gromov- Lawson Conjecture, Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, FL (1999)
16. Greenleaf, F.P.: Invariant Means on Topological Groups and Their Applications. Van Nostrand Mathematical Studies, No. 16, Van Nostrand Reinhold Co., New York (1969)
17. Hoke III, H.F.: Lie groups that are closed at infinity. Trans. Am. Math. Soc. **313**(2), 721–735 (1989)
18. Kallman, R.R.: The existence of invariant measures on certain quotient spaces. Adv. Math. **11**, 387–391 (1973)
19. Milnor, J.: On fundamental groups of complete affinely flat manifolds. Adv. Math. **25**(2), 178–187 (1977)
20. Peyerimhoff, N., Samiou, E.: The Cheeger constant of simply connected, solvable Lie groups. Proc. Am. Math. Soc. **132**(5), 1525–1529 (2004)
21. Polymerakis, P.: Coverings preserving the bottom of the spectrum, MPI-Preprint 2019-3. arXiv:1811.07844
22. Polymerakis, P.: On the spectrum of differential operators under Riemannian coverings. J. Geom. Anal. **30**(3), 3331–3370 (2020)
23. Polymerakis, P.: Spectral estimates and discreteness of spectra under Riemannian submersions. Ann. Glob. Anal. Geom. **57**(2), 349–363 (2020)