A Pair of Pants for the Apparent Horizon

Daniel Pook-Kolb,[1,2] Robie A. Hennigar,[3,4,5] and Ivan Booth[3]

[1]*Max-Planck-Institut für Gravitationsphysik (Albert Einstein Institute), Callinstr. 38, 30167 Hannover, Germany*
[2]*Leibniz Universität Hannover, 30167 Hannover, Germany*
[3]*Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland and Labrador, A1C 5S7, Canada*
[4]*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1*
[5]*Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, Ontario, Canada N2L 3C5*

We resolve the fate of the two original apparent horizons during the head-on merger of two non-spinning black holes, showing that these horizons exist for a finite amount of time before they individually "turn around" and move backward in time. This completes the understanding of the "pair of pants" diagram for the apparent horizon. Our result is facilitated by a new method for locating marginally outer trapped surfaces (MOTSs) based on a generalized shooting method. We also discuss the role played by the MOTS stability operator in discerning which among a multitude of MOTSs should be considered as black hole boundaries.

It is common practice to picture the merger of two black holes according to the famous "pair of pants" diagram, which describes the evolution of the event horizon during the merger. But what does the analogous picture for the apparent horizon look like? While the evolution of the event horizon has been understood for nearly half a century [1], the complete evolution of the apparent horizon during a merger has remained unresolved.

The event horizon is well-suited to theoretical analyses, but its teleological nature makes it less useful in highly dynamical or practical situations, such as in numerical simulations of black hole collisions. In these cases, it is much cleaner to use marginally trapped surfaces. A closed two-dimensional space-like surface $\mathcal{S}$ in a four-dimensional spacetime is said to be trapped if light rays emanating from the surface are converging. In the case where the outgoing light rays are neither converging nor diverging it is said to be a marginally outer trapped surface (MOTS).

MOTSs possess a number of desirable properties that make them natural candidates as quasi-local black hole boundaries. The notion of a trapped surface was first introduced by Penrose and is a key ingredient in the singularity theorems of general relativity [2]. MOTSs can be assigned physical quantities such as mass and angular momentum, allowing for the tracking of these quantities along the world tube traced out by the MOTS during an evolution [3–5], and variations in these quantities obey mechanical laws akin to the laws of black hole thermodynamics [6–8]. Whether or not a MOTS can be evolved into the future, generating a smooth, three-dimensional horizon-like structure is related to its stability properties: if the principal (smallest) eigenvalue of the stability operator is positive, then the MOTS is guaranteed to evolve smoothly into the future [9, 10]. The term MOTS is sometimes used interchangeably with apparent horizon, although here, and in the accompanying papers of this sequence [11, 12], we shall use the term apparent horizon to refer to a MOTS that is stable in this sense.

Despite their importance as a quasi-local characterization of black holes, there remain a number of unresolved questions pertaining to the evolution of interior MOTSs during a merger. While it is true that the details of what occurs within the event horizon (where all MOTSs are located) is causally disconnected from the rest of the universe, this does not mean this question is without relevance. At the very least, this is important for conceptual purposes, to understand to what extent MOTSs provide a physically reasonable description of the merger. Furthermore, the existence of a connected sequence of MOTSs between the initial and final states of the merger provides a means by which physical properties can be tracked throughout the full evolution of the system. Finally, one may hope that there exist correlations between the dynamics in the strong field regime and properties of the distant spacetime. Indeed, such correlations have been shown to exist under certain circumstances [5, 13–16].

The behaviour of apparent horizons during the initial stages of a merger is well-known [1]. Initially there are two individual apparent horizons corresponding to two separate black holes. When these holes become sufficiently close to one another, a common apparent horizon forms surrounding the individual horizons, which continue to exist. This common horizon immediately splits into an inner and outer branch. The outer branch grows in area and becomes more symmetric, ultimately asymptoting to the event horizon. The inner common MOTS moves inward, becoming increasingly distorted.

The bifurcation of the common horizon, combined with the fact that there are known exact solution examples of MOTSs weaving back and forth through time [17, 18], led to the speculative idea that all MOTSs involved in the merger may in fact be different components of a single world-tube that weaves its way through time [19, 20]. However, in most situations of interest MOTSs must be located numerically. Therefore, improvements in the understanding of the evolution of MOTSs during a merger have been in lockstep with improvements in the numerical methods used to locate them. For this reason, progress beyond this qualitative picture was limited.

With the advent of more robust numerical finders for MOTSs [21], it has been possible to go beyond these initial stages of the merger and better understand the interior dynamics. The inner common MOTS continues to move inward and merges (non-smoothly) with the two apparent horizons of the individual black holes at the moment these horizons touch [22–25]. However, all three of these surfaces continue to exist past this point, with the two individual horizons interpenetrating and the inner common MOTS developing self-intersections. Identifying these self-intersections was not possible with previous MOTS finders, as these had implicitly built in the assumption that the MOTSs are 'star-shaped'. Such seemingly exotic surfaces have subsequently been shown to be rather generic. For example, there are an infinite number of self-intersecting MOTSs present within the horizon of the Schwarzschild black hole [26]. This raises the question of whether or not additional, exotic MOTSs are present in the spacetime of a merger event, and what (if any) role is played by these surfaces.

In this Letter, we report on three closely connected results. First, using a novel horizon finder that we develop, we identify for the first time an apparently infinite number of MOTSs present in Brill-Lindquist initial data. We then discuss the role these new MOTSs play in resolving the final fate of the apparent horizons of the two original black holes. Finally, we discuss the stability of these surfaces. As we will see, stability provides a natural way to discern which among the multitude of MOTSs present during the merger can reasonably be called apparent horizons.

A Multitude of MOTSs. To locate MOTSs, we employ a novel shooting method. The procedure, while in the tradition of methods first developed in the 1970s [27], is more versatile and applies to general axisymmetric configurations [11]. This method has been implemented in [28] and can be applied to both analytically known initial data as well as to slices obtained from numerical simulations. Moreover, it overcomes a limitation of the method introduced in [21] as it does not require an initial guess for the shape of the surface to be located. As such, our approach is ideally suited for locating MOTSs with geometries that are not only unexpected but also arbitrarily complicated.

The approach, which is detailed in [11], exploits axisymmetry to reduce the problem of locating a 2-surface to that of determining a curve, which we refer to as a *MOTSodesic*. The full MOTS is then obtained as the surface of revolution of the MOTSodesic. Given an axisymmetric 3-surface $\{\Sigma, h_{ij}, D_i\}$, a curve $\gamma(s)$ in the two-dimensional space orthogonal to the rotational Killing field $\varphi = \frac{\partial}{\partial\phi}$ is a MOTSodesic provided that

$$T^a D_a T^b = (N^c D_c(\ln R) + k_u) N^b = \kappa^+ N^b \,. \qquad (1)$$

Here T^a is the unit-length tangent vector to γ, N^a is its unit-length normal, R is the circumferential radius, and k_u is the trace of the extrinsic curvature with respect to the unit time-like normal u to Σ, and we have chosen an arclength parameterization for γ. Eq. (1) comprises two coupled second-order ODEs that allow one to determine the MOTSodesics.

Henceforth, we restrict our considerations to the head-on merger of two non-spinning black holes. For this purpose, we use Brill-Lindquist (BL) initial data [29]. These describe a Cauchy slice Σ which is time symmetric, i.e. with vanishing extrinsic curvature. The three-metric is conformally flat, $h_{ij} = \psi^4 \delta_{ij}$, where δ_{ij} is the flat metric and the conformal factor is given by

$$\psi = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2} \,, \qquad (2)$$

where $m_{1,2}$ are the bare masses of the black holes and $r_{1,2}$ are the (coordinate) distances to the respective puncture.

For BL initial data, we work in cylindrical coordinates (ρ, z, ϕ) on the spatial slice and consider the curve γ : $(\rho, z) = (P(s), Z(s))$. For this, the MOTSodesic equations reduce to

$$\ddot{P} = \frac{\dot{Z}^2}{P} + \frac{4\psi_\rho}{\psi^5} - \frac{6\dot{P}(\dot{P}\psi_\rho + \dot{Z}\psi_z)}{\psi} \,, \qquad (3)$$

$$\ddot{Z} = -\frac{\dot{Z}\dot{P}}{P} + \frac{4\psi_z}{\psi^5} - \frac{6\dot{Z}(\dot{P}\psi_\rho + \dot{Z}\psi_z)}{\psi} \,, \qquad (4)$$

where subscripts denote partial derivatives and the arclength parameterization reads $\psi^4(\dot{P}^2 + \dot{Z}^2) = 1$. The equations are solved by first obtaining a series solution in the vicinity of the axis to ensure regularity there. The series solution is used as initial conditions to solve the system using Mathematica's NDSolve. We use the shooting method, tuning the initial conditions until the surface can be considered to approximately close, a condition we take to be an approach to the axis to within a distance of about 10^{-6} or better. These surfaces are then confirmed to be MOTSs using the methods of [21].

Using this method, we find in addition to the four 'standard' MOTSs a large number of more exotic MOTSs — see Fig. 1 for an example. All new MOTSs are found between the outer apparent horizon and the two original apparent horizons and can enclose either, both, or neither of the two punctures. These surfaces tend to 'hug' closely the outer apparent horizon and/or the individual apparent horizons and can fold or closely wrap around these surfaces a seemingly arbitrary number of times. Indeed, nothing we have found indicates that there are a finite number of additional MOTSs, and the number may very well be infinite.

The Fate of the Apparent Horizons. The existence of new MOTSs in BL initial data raises a number of important questions. First, are these surfaces generic during the merger, or are they artefacts of the high degree of symmetry present in the initial data? Second, if generic, what role (if any) do these surfaces play in the merger? Finally, with a seemingly infinite number of MOTSs present in a merger, how can one discern physically relevant surfaces that demarcate black hole boundaries?

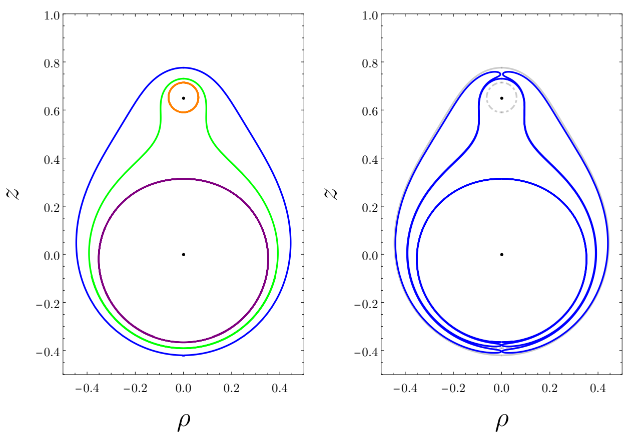

FIG. 1. MOTSs present in Brill-Lindquist initial data for the case $m_1 = 0.2$, $m_2 = 0.8$ with a separation of $d = 0.65$. Left: The 'standard' horizons consisting of the outer apparent horizon (blue), inner common MOTS (green), and apparent horizons of individual black holes (orange and purple) reproduced using the shooting method. Right: An example of a new exotic MOTS.

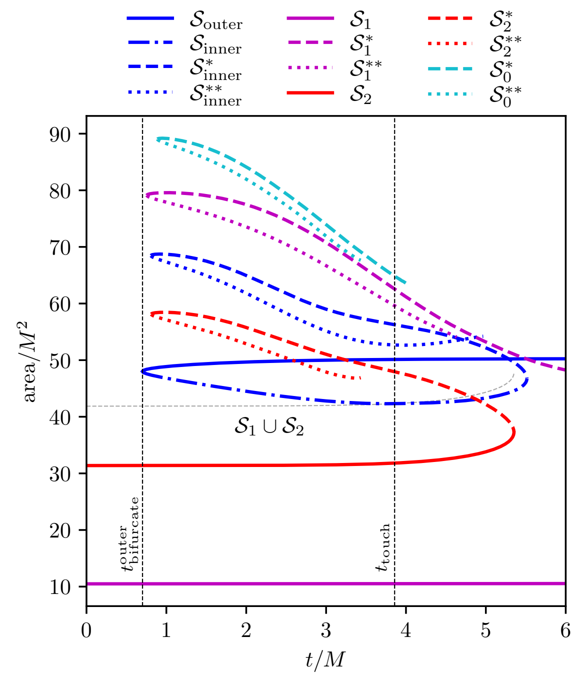

FIG. 2. The area of the various MOTSs as a function of time. We indicate using lines of the same colour continuous world tubes moving forward and backward in time. The dashed line indicates the sum of areas of $\mathcal{S}_{1,2}$. For purely numerical reasons, we lose track of some of the MOTSs — this is the case for the curves that abruptly terminate in the figure.

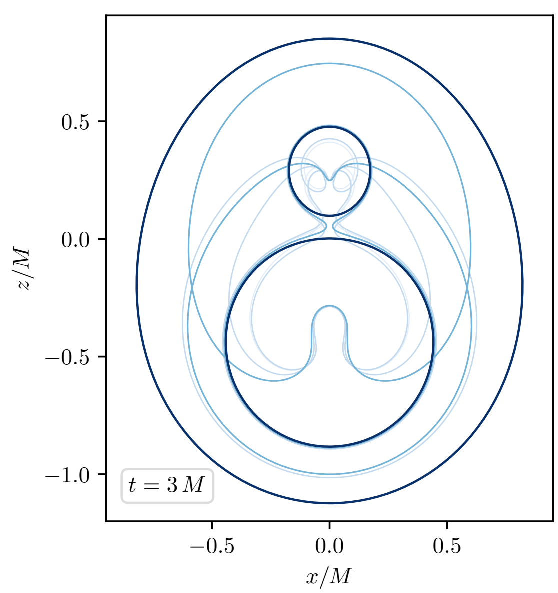

FIG. 3. A "snapshot" of Fig. 2 showing a variety of MOTSs. The three dark lines correspond to $\mathcal{S}_{\rm outer}$ and $\mathcal{S}_{1,2}$. Two of the shown surfaces exhibit self-intersections. Lighter colors indicate a larger number of negative eigenvalues of the stability operator (see below and [11, 12] for details).

To address these questions, we perform numerical evolutions of the initial data. We focus primarily on the configuration with total ADM mass $M = m_1 + m_2 = 1$, a mass ratio of $q = m_2/m_1 = 2$ and a distance parameter of $d = 0.9$ corresponding to two black holes that are initially separate with no common apparent horizon present. To track the MOTSs in the simulations we use the method described in [21, 23] and available from [28]. To locate the MOTSs, we use two approaches: One is the shooting method described earlier and detailed in [11], while the other is based on the assumption that MOTSs appear or disappear only in bifurcations. To this end, we try to track each MOTS to the future and to the past. Whenever a MOTS cannot be tracked further in either direction, we look for a "close by" one with which it might annihilate or bifurcate, respectively.

We perform our simulations with the *Einstein Toolkit* [30, 31] and set up the initial conditions using *TwoPunctures* [32]. The Einstein equations are evolved in the BSSN formulation using an axisymmetric version of *McLachlan* [33], which uses *Kranc* [34, 35] to generate efficient C++ code. We work with a $1 + \log$ slicing and a Γ-driver shift condition [36, 37]. Most of our results are obtained with a spatial grid resolution of $1/\Delta x = 720$. Additional resolutions $1/\Delta x = 240$, 360 and 480 and shorter simulations with $1/\Delta x = 960$, 1440 and 1920 are used to assess convergence and resolve certain features. Additional details of our numerical setup are described in [12, 23].

Our main results are illustrated in Fig. 2, which shows the evolution of the area of several relevant MOTSs, and Fig. 3 which shows the different MOTSs at a particular moment of time. Of the curves shown, $\mathcal{S}_{\rm outer}$ is the common apparent horizon, $\mathcal{S}_{\rm inner}$ is the inner common MOTS, and $\mathcal{S}_{1,2}$ correspond to the apparent horizons of the individual black holes. In addition to these standard MOTSs, we find many new MOTSs; the evolution of a selection of these surfaces is shown in Fig. 2. The surfaces all form through bifurcations, splitting into an outer and inner branch. They form dynamically during the course of the simulation, indicating that the exotic MOTSs present in the initial data are not artefacts of that configuration. The shown surfaces all form *after* the outer apparent horizon has formed, and despite several MOTSs having larger area than $\mathcal{S}_{\rm outer}$, all are contained inside of it.

The figure makes clear that the new MOTSs are essential to understanding the final fate of the apparent horizons of the individual black holes. In the figure shown, both $\mathcal{S}_{\rm inner}$ and $\mathcal{S}_2$ are seen to be independently annihilated by new MOTSs. We have good indications that $\mathcal{S}_1$ is annihilated by $\mathcal{S}_1^*$, but this could not be fully resolved in our simulation due to the MOTSs becoming too close to the punctures, where the resolution is necessarily worse. Nonetheless, it seems clear that the new MOTSs we have located provide a mechanism by which the apparent horizons of the original black holes are annihilated. To illustrate this in greater detail, we present in Fig. 4 several "snapshots" of the evolution of the horizon $\mathcal{S}_2$.

The Role of Stability. The stability operator is key in understanding the details of the picture so far described. For a non-spinning axisymmetric MOTS in vacuum, the stability operator takes the form [11]

$$L_\Sigma \psi = -\Delta_{\mathcal{S}}\psi + \left(\frac{\mathcal{R}}{2} - 2|\sigma_+|^2\right)\psi\,, \qquad (5)$$

where $\Delta_{\mathcal{S}}$ is the Laplacian on the MOTS $\mathcal{S}$, $\mathcal{R}$ is its scalar curvature, and σ_+^{AB} is the trace-free part of the extrinsic curvature of the outward-directed null normal. The eigenvalues $L_\Sigma\psi = \lambda_{l,m}\psi$ of the stability operator determine the stability properties of the associated MOTS. We find that, associated to every bifurcation/annihilation is the vanishing of an eigenvalue of the stability operator — see Fig. 5 for the case of the bifurcations. While it is known that the vanishing of the principal eigenvalue corresponds to a bifurcation/annihilation event [38], what we see here is that for all but $\mathcal{S}_2/\mathcal{S}_2^*$ and $\mathcal{S}_{\rm outer}/\mathcal{S}_{\rm inner}$ it is one of the *higher* eigenvalues that vanishes at the bifurcation or annihilation. This provides robust numerical evidence for the smoothness of the world tubes shown in Fig. 2.

Of all the MOTSs we have located, only three have a positive principal eigenvalue: $\mathcal{S}_{\rm outer}$, $\mathcal{S}_1$ and $\mathcal{S}_2$. As a result, these surfaces act as barriers for trapped and untrapped surfaces in their vicinity, and they are also the only ones to have everywhere expanding space-like world tubes. These properties are precisely what one would associate with horizons. Therefore stability provides an unambiguous criterion by which the MOTSs corresponding to black hole boundaries may be numerically identified. It is this result that underlies our choice of terminology for the apparent horizon advocated in the introduction. All other MOTSs we have located have negative eigenvalues. In fact, the picture that has emerged can be understood rather simply: each time a given world tube turns around in time, the number of negative eigenvalues increases.

Interestingly, the stability of the MOTSs can be understood entirely geometrically through the concept of MOTSodesics introduced earlier [11]. To do so, one first relaxes the requirement of closed MOTSodesics, i.e. allowing also for marginally outer trapped *open* surfaces. It is then possible to analyse the deviation between nearby MOTSodesics in complete analogy with geodesic deviation. As it happens, the MOTSodesic deviation equation contains precisely the same information as the stability operator (5). Negative eigenvalues of the stability operator for a given MOTS have a geometric interpretation as the number of intersections of nearby MOTSodesics with the MOTS of interest. This provides a completely geometric and visual way by which the stability of a given MOTS can be determined, and highlights the fact that the stability operator is to MOTSs what the Jacobi equation is to geodesics.

Summary. Here we have shown that the interior of a black hole merger is far richer than previously thought, containing a large (possibly infinite) number of hitherto unidentified MOTSs. These MOTSs were initially located using a new generalized shooting method that sidesteps the drawback in existing finders of requiring an initial guess for the surface of interest. The additional surfaces play a crucial role in the interior dynamics of the merger, and are responsible for the annihilation of the apparent horizons of the original black holes. As such, these new MOTSs make possible for the first time to understand how two black holes become one, giving the analog of the "pair of pants" diagram for the apparent horizon. The picture is considerably more complex than the equivalent picture for the event horizon and involves several world tubes that weave their way back and forth in time. Rather than obscuring the utility of the quasi-local horizon framework, the multitude of MOTSs present during the merger actually highlights the rarity of stable MOTSs. Of all the MOTSs we have located, only three are stable, and these are precisely those that are most naturally associated with black hole boundaries.

Acknowledgements. We would like to express our gratitude to Graham Cox, Jose Luis Jaramillo, Badri Krishnan, Hari Kunduri and the members of the Memorial University Gravity Journal Club for valuable discussions and suggestions. IB was supported by the Natural Science and Engineering Research Council of Canada Discovery Grant 2018-0473. The work of RAH was supported by the Natural Science and Engineering Research Council of Canada through the Banting Postdoctoral Fellowship program and also by AOARD Grant FA2386-19-1-4077.

[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Monographs on Mathemat-

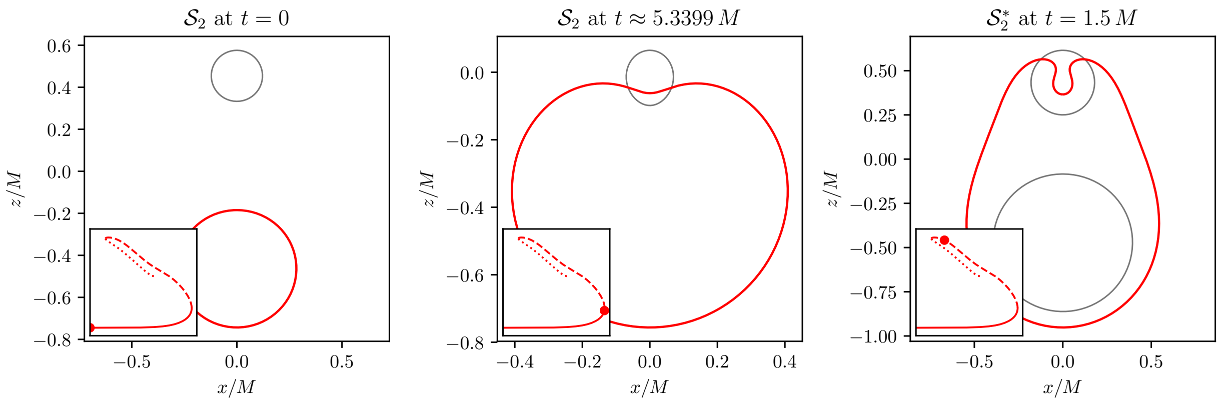

FIG. 4. The annihilation of $\mathcal{S}_2$. The inset shows, with a red dot, where along the world tube the shown MOTS occurs [see Fig. 2].

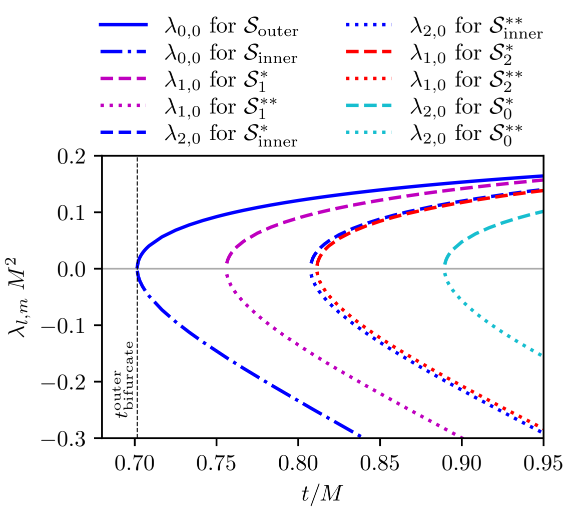

FIG. 5. Eigenvalues (with $m = 0$) of the stability operator for the ten MOTSs participating in the five bifurcations shown in Fig. 2. For each MOTS, we show the respective eigenvalue which tends to zero.

ical Physics (Cambridge University Press, 1973).
[2] Roger Penrose, “Gravitational collapse and space-time singularities,” Phys. Rev. Lett. **14**, 57–59 (1965).
[3] Olaf Dreyer, Badri Krishnan, Deirdre Shoemaker, and Erik Schnetter, “Introduction to isolated horizons in numerical relativity,” Phys. Rev. D **67**, 024018 (2003), arXiv:gr-qc/0206008.
[4] Badri Krishnan, “Fundamental properties and applications of quasi-local black hole horizons,” Class. Quant. Grav. **25**, 114005 (2008), arXiv:0712.1575 [gr-qc].
[5] Anshu Gupta, Badri Krishnan, Alex Nielsen, and Erik Schnetter, “Dynamics of marginally trapped surfaces in a binary black hole merger: Growth and approach to equilibrium,” Phys. Rev. **D97**, 084028 (2018), arXiv:1801.07048 [gr-qc].
[6] Abhay Ashtekar and Badri Krishnan, “Dynamical horizons: Energy, angular momentum, fluxes and balance laws,” Phys. Rev. Lett. **89**, 261101 (2002), arXiv:gr-qc/0207080.
[7] Abhay Ashtekar and Badri Krishnan, “Dynamical horizons and their properties,” Phys. Rev. **D68**, 104030 (2003), arXiv:gr-qc/0308033.
[8] Abhay Ashtekar and Badri Krishnan, “Isolated and dynamical horizons and their applications,” Living Rev. Rel. **7**, 10 (2004), arXiv:gr-qc/0407042.
[9] Lars Andersson, Marc Mars, and Walter Simon, “Local existence of dynamical and trapping horizons,” Phys.Rev.Lett. **95**, 111102 (2005), arXiv:gr-qc/0506013 [gr-qc].
[10] Lars Andersson, Marc Mars, and Walter Simon, “Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes,” Adv.Theor.Math.Phys. **12** (2008), arXiv:0704.2889 [gr-qc].
[11] Ivan Booth, Robie A. Hennigar, and Daniel Pook-Kolb, “Ultimate fate of apparent horizons during a binary black hole merger I: locating and understanding axisymmetric marginally outer trapped surfaces,” (2021).
[12] Daniel Pook-Kolb, Ivan Booth, and Robie A. Hennigar, “Ultimate fate of apparent horizons during a binary black hole merger II: Horizons weaving back and forth in time,” (2021).
[13] Luciano Rezzolla, Rodrigo P. Macedo, and Jose Luis Jaramillo, “Understanding the ’anti-kick’ in the merger of binary black holes,” Phys. Rev. Lett. **104**, 221101 (2010), arXiv:1003.0873 [gr-qc].
[14] Jose Luis Jaramillo, Rodrigo Panosso Macedo, Philipp Moesta, and Luciano Rezzolla, “Black-hole horizons as probes of black-hole dynamics I: post-merger recoil in head-on collisions,” Phys. Rev. D **85**, 084030 (2012), arXiv:1108.0060 [gr-qc].
[15] J. L. Jaramillo, R. P. Macedo, P. Moesta, and L. Rezzolla, “Towards a cross-correlation approach to strong-field dynamics in Black Hole spacetimes,” AIP Conf. Proc. **1458**, 158–173 (2012), arXiv:1205.3902 [gr-qc].
[16] Vaishak Prasad, Anshu Gupta, Sukanta Bose, Badri Krishnan, and Erik Schnetter, “News from horizons in binary black hole mergers,” Phys. Rev. Lett. **125**, 121101 (2020), arXiv:2003.06215 [gr-qc].
[17] Ishai Ben-Dov, “The Penrose inequality and apparent horizons,” Phys. Rev. D **70**, 124031 (2004), arXiv:gr-qc/0408066.
[18] Ivan Booth, Lionel Brits, Jose A. Gonzalez, and Chris Van Den Broeck, “Marginally trapped tubes and dynamical hori-

zons," Class. Quant. Grav. **23**, 413–440 (2006), arXiv:gr-qc/0506119.
[19] Sean A. Hayward, "Black holes: New horizons," in *Recent developments in theoretical and experimental general relativity, gravitation and relativistic field theories. Proceedings, 9th Marcel Grossmann Meeting, MG'9, Rome, Italy, July 2-8, 2000. Pts. A-C* (2000) pp. 568–580, arXiv:gr-qc/0008071 [gr-qc].
[20] Ishai Ben-Dov, "Penrose inequality and apparent horizons," Phys. Rev. D **70**, 124031 (2004).
[21] Daniel Pook-Kolb, Ofek Birnholtz, Badri Krishnan, and Erik Schnetter, "Existence and stability of marginally trapped surfaces in black-hole spacetimes," Phys. Rev. D **99**, 064005 (2019).
[22] Daniel Pook-Kolb, Ofek Birnholtz, Badri Krishnan, and Erik Schnetter, "Interior of a binary black hole merger," Phys. Rev. Lett. **123**, 171102 (2019).
[23] Daniel Pook-Kolb, Ofek Birnholtz, Badri Krishnan, and Erik Schnetter, "Self-intersecting marginally outer trapped surfaces," Phys. Rev. D **100**, 084044 (2019).
[24] Daniel Pook-Kolb, Ofek Birnholtz, José Luis Jaramillo, Badri Krishnan, and Erik Schnetter, "Horizons in a binary black hole merger I: Geometry and area increase," (2020), arXiv:2006.03939 [gr-qc].
[25] Daniel Pook-Kolb, Ofek Birnholtz, José Luis Jaramillo, Badri Krishnan, and Erik Schnetter, "Horizons in a binary black hole merger II: Fluxes, multipole moments and stability," (2020), arXiv:2006.03940 [gr-qc].
[26] Ivan Booth, Robie A. Hennigar, and Saikat Mondal, "Marginally outer trapped surfaces in the Schwarzschild spacetime: Multiple self-intersections and extreme mass ratio mergers," Phys. Rev. D **102**, 044031 (2020), arXiv:2005.05350 [gr-qc].
[27] Andrej Čadež, "Apparent horizons in the two-black-hole problem," Annals of Physics **83**, 449 – 457 (1974).
[28] Daniel Pook-Kolb, Ofek Birnholtz, Ivan Booth, Robie A. Hennigar, José Luis Jaramillo, Badri Krishnan, Erik Schnetter, and Victor Zhang, "MOTS Finder version 1.5," (2021).
[29] Dieter R. Brill and Richard W. Lindquist, "Interaction energy in geometrostatics," Phys. Rev. **131**, 471–476 (1963).
[30] Frank Löffler, Joshua Faber, Eloisa Bentivegna, Tanja Bode, Peter Diener, Roland Haas, Ian Hinder, Bruno C. Mundim, Christian D. Ott, Erik Schnetter, Gabrielle Allen, Manuela Campanelli, and Pablo Laguna, "The Einstein Toolkit: A Community Computational Infrastructure for Relativistic Astrophysics," Class. Quantum Grav. **29**, 115001 (2012), arXiv:1111.3344 [gr-qc].
[31] EinsteinToolkit, "Einstein Toolkit: Open software for relativistic astrophysics," `http://einsteintoolkit.org/`.
[32] Marcus Ansorg, Bernd Brügmann, and Wolfgang Tichy, "A single-domain spectral method for black hole puncture data," Phys. Rev. D **70**, 064011 (2004), arXiv:gr-qc/0404056.
[33] J. David Brown, Peter Diener, Olivier Sarbach, Erik Schnetter, and Manuel Tiglio, "Turduckening black holes: an analytical and computational study," Phys. Rev. D **79**, 044023 (2009), arXiv:0809.3533 [gr-qc].
[34] Sascha Husa, Ian Hinder, and Christiane Lechner, "Kranc: a Mathematica application to generate numerical codes for tensorial evolution equations," Comput. Phys. Commun. **174**, 983–1004 (2006), arXiv:gr-qc/0404023.
[35] Kranc, "Kranc: Kranc assembles numerical code," .
[36] Miguel Alcubierre, Gabrielle Allen, Bernd Brügmann, Thomas Dramlitsch, Jose A. Font, Philippos Papadopoulos, Edward Seidel, Nikolaos Stergioulas, Wai-Mo Suen, and Ryoji Takahashi, "Towards a stable numerical evolution of strongly gravitating systems in general relativity: The Conformal treatments," Phys. Rev. **D62**, 044034 (2000), arXiv:gr-qc/0003071 [gr-qc].
[37] Miguel Alcubierre, Bernd Brügmann, Peter Diener, Michael Koppitz, Denis Pollney, Edward Seidel, and Ryoji Takahashi, "Gauge conditions for long term numerical black hole evolutions without excision," Phys. Rev. **D67**, 084023 (2003), arXiv:gr-qc/0206072 [gr-qc].
[38] Lars Andersson, Marc Mars, Jan Metzger, and Walter Simon, "The Time evolution of marginally trapped surfaces," Class. Quant. Grav. **26**, 085018 (2009), arXiv:0811.4721 [gr-qc].