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ON THE NONLINEAR ENERGY TRANSFER IN A WAVE SPECTRUM**INTRODUCTION**

The final stage of development of a wind-generated sea is probably determined largely by two nonlinear processes: (1) the dissipation of wave energy due to wave breaking, and (2) the energy flux in the wave spectrum resulting from the nonlinear interactions among different wave components. The present paper is restricted to an analysis of the second process. The nonlinear interactions are in general small and can thus be evaluated by conventional perturbation techniques. The perturbation analysis of a random sea surface to the second order has already been carried out in detail by Mr. L. J. Tick. The perturbation solutions were found to be steady, sinusoidal waves yielding a constant correction term for the wave spectrum. However, the analysis did not disclose any dynamical interactions involving energy transfer among different wave components, as these are described by perturbation equations of a higher order than the second.

By extending the perturbation equations to the third order, Dr. O. M. Phillips explained the mechanism of these interactions and examined the conditions under which energy transfer is possible between two discrete wave trains. In order to determine the energy transfer in the case of a continuous spectrum, however, the perturbation analysis has to be extended still further to the fifth order. Moreover, supplementary statis-

tical analysis of the response of an undamped oscillator to certain random forcing functions is also necessary. The final result can be interpreted in terms of quadruple interactions in which, under certain resonance conditions, energy is transferred from three "active" components controlling the transfer rate to a "passive" fourth component. The process is proportional to the fourth power of the wave slope, giving an estimated order of magnitude comparable to the magnitudes of the generating and dissipating processes in a seaway.

PERTURBATION ANALYSIS

As the derivation of the perturbation equations for a random sea surface with small mean square wave slope has been described in detail by Mr. Tick and Dr. Phillips, we can restrict ourselves here to a brief outline of the method with indications of the additional aspects encountered in extending the analysis to the fifth order.

The sea surface $z = h(x, y, t)$ is assumed to be a homogeneous random function, which can thus be represented as a Fourier sum

$$h(x, y, t) = \sum_{\mathbf{k}} H_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (1)$$

where $\mathbf{x} = (x, y)$ and $x, y,$ and z denote cartesian coordinates with the z -axis directed vertically upwards. Approximating Fourier-sums will be used throughout rather than exact Fourier-Stieltjes

integrals as they allow a more condensed presentation of the rather complicated multiple integrals occurring later in the analysis.

In the linear approximation (denoted by a subscript 1) the equations of motion and the boundary conditions lead to the differential equation:

$${}_1\ddot{H}_{\mathbf{k}}(t) + \omega_{\mathbf{k}}^2 {}_1H_{\mathbf{k}}(t) = 0 \quad (2)$$

where $\omega_{\mathbf{k}} = \sqrt{gk}$, $k = |\mathbf{k}|$ (neglecting surface tension). The general solution is

$${}_1H_{\mathbf{k}}(t) = {}_1H_{\mathbf{k}}^+ e^{-i\omega_{\mathbf{k}}t} + {}_1H_{\mathbf{k}}^- e^{i\omega_{\mathbf{k}}t}, \quad {}_1H_{\mathbf{k}}^{\pm} \text{ const} \quad (3)$$

so that from Equation (1)

$${}_1h = \sum_{\mathbf{k}} [{}_1H_{\mathbf{k}}^+ e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_{\mathbf{k}}t} + {}_1H_{\mathbf{k}}^-, e^{i\mathbf{k}\cdot\mathbf{x} + i\omega_{\mathbf{k}}t}] \quad (4)$$

The sign-indices denote the direction of propagation of the waves relative to the wave-number \mathbf{k} . As ${}_1h$ is real, the amplitudes satisfy the relation

$${}_1H_{\mathbf{k}}^+ = ({}_1H_{-\mathbf{k}}^-)^* \quad (5)$$

In the nonlinear case the amplitudes $H_{\mathbf{k}}(t)$ can be developed in a perturbation series

$$H_{\mathbf{k}}(t) = {}_1H_{\mathbf{k}}(t) + {}_2H_{\mathbf{k}}(t) + {}_3H_{\mathbf{k}}(t) + \dots \quad (6)$$

The n th amplitude ${}_nH_{\mathbf{k}}(t)$ satisfies the same differential equation as ${}_1H_{\mathbf{k}}(t)$, except that the equation is no longer homogeneous, the inhomogeneous, exciting term depending on amplitudes of an order lower than n . Assuming these to be already determined in terms of the first-order amplitudes, the resulting equation for ${}_nH_{\mathbf{k}}(t)$ has the general form

$${}_n\ddot{H}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 {}_nH_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 + \dots + \mathbf{k}_n = \mathbf{k} \\ s_1, s_2, s_3, \dots, s_n}} C_{\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n}^{s_1, s_2, \dots, s_n} (t) {}_1H_{\mathbf{k}_1}^{s_1} \cdot {}_1H_{\mathbf{k}_2}^{s_2} \cdot \dots \cdot {}_1H_{\mathbf{k}_n}^{s_n} \quad (7)$$

where s_j represents a sign-index.

For $n = 2$ the coefficient $C_{\mathbf{k}_1, \mathbf{k}_2}^{s_1, s_2}$ is sinusoidal and the equation becomes

$${}_2\ddot{H}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 {}_2H_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} \\ s_1, s_2}} A_{\mathbf{k}_1, \mathbf{k}_2}^{s_1, s_2} \cdot {}_1H_{\mathbf{k}_1}^{s_1} \cdot {}_1H_{\mathbf{k}_2}^{s_2} \cdot e^{-i(s_1\omega_{\mathbf{k}_1} + s_2\omega_{\mathbf{k}_2})t} \quad (8)$$

where the coefficients $A_{\mathbf{k}_1, \mathbf{k}_2}^{s_1, s_2}$ are constant.

The exciting terms on the right-hand side generate steady, harmonic oscillations if

$$(s_1\omega_{\mathbf{k}_1} + s_2\omega_{\mathbf{k}_2})^2 \neq \omega_{\mathbf{k}}^2$$

and nonstationary oscillations with linearly in-

creasing amplitudes if $(s_1\omega_{\mathbf{k}_1} + s_2\omega_{\mathbf{k}_2})^2 = \omega_{\mathbf{k}}^2$. For gravity waves with $\omega_{\mathbf{k}} = \sqrt{gk}$ it follows from the inequalities

$$\sqrt{k_1 + k_2} \leq \sqrt{k_1} + \sqrt{k_2}$$

$$\text{and} \quad |\mathbf{k}_1 + \mathbf{k}_2| \leq k_1 + k_2$$

$$\text{that} \quad (\omega_{\mathbf{k}} =) \omega_{\mathbf{k}_1 + \mathbf{k}_2} \leq \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}$$

The substitution $\mathbf{k}' = \mathbf{k}_1 + \mathbf{k}_2$, $\mathbf{k}'' = -\mathbf{k}_2$ yields further

$$\omega_{\mathbf{k}'} - \omega_{\mathbf{k}''} \leq \omega_{\mathbf{k}' + \mathbf{k}''}$$

so that finally

$$\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} \leq \omega_{\mathbf{k}_1 + \mathbf{k}_2} (= \omega_{\mathbf{k}}) \leq \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} \quad (9)$$

The equality sign on the left holds only for $\mathbf{k}_2 = 0$ or $\mathbf{k}_1 + \mathbf{k}_2 = 0$ and the equality sign on the right only for $\mathbf{k}_1 = 0$ or $\mathbf{k}_2 = 0$. It can be shown that the coefficient $A_{\mathbf{k}_1, \mathbf{k}_2}^{s_1, s_2}$ in Equation (8) vanishes for these cases. It thus follows that none of the exciting terms in (8) can satisfy the resonance condition $(s_1\omega_{\mathbf{k}_1} + s_2\omega_{\mathbf{k}_2})^2 = \omega_{\mathbf{k}}^2$, so that the solution ${}_2H_{\mathbf{k}}(t)$ is composed entirely of steady harmonic oscillations. It follows further that the exciting terms in the differential equation for the next perturbation amplitude ${}_3H_{\mathbf{k}}$ again consist solely of harmonic components

$${}_3\ddot{H}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 {}_3H_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k} \\ s_1, s_2, s_3}} A_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3}^{s_1, s_2, s_3} \cdot {}_1H_{\mathbf{k}_1}^{s_1} \cdot {}_1H_{\mathbf{k}_2}^{s_2} \cdot {}_1H_{\mathbf{k}_3}^{s_3} \cdot e^{-i(s_1\omega_{\mathbf{k}_1} + s_2\omega_{\mathbf{k}_2} + s_3\omega_{\mathbf{k}_3})t} \quad (10)$$

where $A_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3}^{s_1, s_2, s_3}$ is constant.

Equation (10) is the first perturbation equation in which resonance occurs, as it can easily be verified that the resonance condition

$$(s_1\omega_{\mathbf{k}_1} + s_2\omega_{\mathbf{k}_2} + s_3\omega_{\mathbf{k}_3})^2 = \omega_{\mathbf{k}}^2$$

can be satisfied by a suitable choice of \mathbf{k}_j and s_j . The perturbation amplitude ${}_3H_{\mathbf{k}}(t)$ thus contains unsteady, continuously increasing, resonant components representing a continuous transfer of energy from the wave components ${}_1H_{\mathbf{k}_1}^{s_1}$, ${}_1H_{\mathbf{k}_2}^{s_2}$, and ${}_1H_{\mathbf{k}_3}^{s_3}$ to the resultant component ${}_3H_{\mathbf{k}}$. Equation (10) was derived independently by Dr. Phillips for the case of two interacting, discrete wave trains for which two of the three wave numbers \mathbf{k}_1 , \mathbf{k}_2 , and \mathbf{k}_3 coincide.

Although the third order perturbation equation illustrates the main features of the resonance mechanism responsible for the nonlinear energy transfer; we shall find that the perturbation equa-

tions have to be developed still further to the fourth and fifth orders in order to actually evaluate the energy flux. The equations for ${}_4H_{\mathbf{k}}(t)$ and ${}_5H_{\mathbf{k}}(t)$ are not as simple in structure as those for the first three orders as the exciting terms depend partly on the nonstationary perturbations ${}_3H_{\mathbf{k}}$ and are thus no longer simply harmonic functions of t . This leads to "higher-order" resonant excitation of the amplitudes ${}_4H_{\mathbf{k}}$ and ${}_5H_{\mathbf{k}}$, i.e., resonant excitation by terms which are themselves already nonstationary resonant oscillations.

The mechanism of these interactions cannot be discussed in detail here, but the implications for the resulting energy flux will be considered briefly in the next section.

THE ENERGY FLUX

The first step in determining the energy flux resulting from the nonlinear interactions must obviously be to determine the total energy of the sea in terms of the perturbation amplitudes considered in the previous section. The total energy of the sea per unit projection area is

$$E = E_{\text{pot}} + E_{\text{kin}} = \rho g \frac{\bar{h}^2}{2} + \rho \int_{-\infty}^h \frac{(\text{grad } \varphi)^2}{2} dz \quad (11)$$

where φ is the velocity potential and the bars denote ensemble means.

We develop E again in a perturbation series:

$$E = {}_2E + {}_3E + {}_4E + {}_5E + \dots \quad (12)$$

In the linear approximation the mean kinetic energy is equal to the mean potential energy, and we obtain, applying Equations (4) and (5),

$${}_2E = \rho g \bar{h}^2 = 2\rho g \sum_{\mathbf{k}} |{}_1H_{\mathbf{k}}^+|^2 \quad (13)$$

In the limiting case of a continuous spectrum Equation (13) takes the form

$${}_2E = \int \int_{-\infty}^{+\infty} {}_2F(\mathbf{k}) dk_x dk_y, \quad (14)$$

where ${}_2F(\mathbf{k}) dk_x dk_y$ is defined as the energy (in the linear approximation) of all waves travelling in the positive \mathbf{k} -direction whose wave numbers \mathbf{k}' lie in the rectangular interval

$$\mathbf{k} \leq \mathbf{k}' \leq \mathbf{k} + d\mathbf{k}$$

We introduce now the basic hypothesis that the

sea is a Gaussian process in the linear approximation. In this case the linear wave components ${}_1H_{\mathbf{k}}^{\pm}$ are statistically independent, and the sea is completely specified statistically by the spectrum ${}_2F(\mathbf{k})$. Further, it can easily be shown that for this case all odd components in the perturbation series (12) vanish.

In the higher-order expressions for E the mean kinetic energy is no longer equal to the mean potential energy, so that in general we have

$$E = \rho g \bar{h}^2 + \Delta E \quad (15)$$

where the perturbation series of ΔE begins with a fourth-order term. ΔE is no longer quadratic in the surface elevation, which, as pointed out by Mr. Tick, precludes expressing E as a spectral integral in the general nonlinear case. However, we shall find that the nonstationary components of E representing the continuous energy flux can nonetheless still be expressed completely in terms of the linear spectrum ${}_2F(\mathbf{k})$.

The next (non-vanishing) term in the perturbation series (12) is found from Equations (15), (1), and (6) to be

$${}_4E = \rho g \sum_{\mathbf{k}} [|\bar{{}_2H_{\mathbf{k}}(t)}|^2 + 2 \text{Re}(\overline{{}_1H_{\mathbf{k}_3} {}_1H_{-\mathbf{k}}})] + {}_4(\Delta E) \quad (16)$$

It can be shown that ${}_4(\Delta E)$ contains only periodic terms and thus does not contribute to the nonlinear energy flux. The first term in the sum is also constant. The remaining term contains the nonstationary component ${}_3H_{-\mathbf{k}}$, but the mean product $\text{Re}(\overline{{}_1H_{\mathbf{k}_3} {}_1H_{-\mathbf{k}}})$ can also be shown to be constant as the resonant components of ${}_3H_{-\mathbf{k}}$ are 90 degrees out of phase with the corresponding components of ${}_1H_{\mathbf{k}}$. ${}_4E$ thus contains only stationary components yielding a constant correction term for the total energy of the sea. The term has been analysed in detail by Mr. Tick (with the exception of terms depending on the third-order perturbation amplitude).

The next term in the perturbation series (12) is

$${}_6E = \rho g \sum_{\mathbf{k}} [|\bar{{}_3H_{\mathbf{k}}}|^2 + 2 \text{Re}(\overline{{}_2H_{\mathbf{k}} {}_4H_{-\mathbf{k}}}) + 2 \text{Re}(\overline{{}_1H_{\mathbf{k}} {}_5H_{-\mathbf{k}}})] + {}_6(\Delta E) \quad (17)$$

It can again be shown that ${}_6(\Delta E)$ contains only constant terms. The terms in the sum, however, are now no longer constant, and it is these terms that describe the nonlinear energy flux that we are seeking.

The problem is thus reduced to the determination of the three co-variance products entering in

the sum in Equation (17), in which all perturbation amplitudes from the first to the fifth order occur. The perturbations are all solutions of harmonic differential equations in which the exciting terms are random functions of time, and the basic statistical problem is thus to express the covariance products of these solutions in terms of the statistical properties of the corresponding exciting functions. The required relations can be derived from certain asymptotic properties of the solutions. However, it will not be possible to cite or prove these relations here completely, and we shall restrict ourselves to only a few characteristic results.

Let ψ be a solution of the differential equation

$$\ddot{\psi} + \omega_0^2 \psi = p(t) \quad (18)$$

where $p(t)$ is a stationary, random (not necessarily real) function with a continuous spectral density function $f(\omega)$. It can then be shown that for large t

$$|\overline{\psi}|^2 = \frac{\pi t}{2\omega_0^2} [f(\omega_0) + f(-\omega_0)] \quad (19)$$

An expression similar to Equation (19), in which the spectral density was expressed in terms of the correlation function of $p(t)$, has been derived by Dr. Phillips. In our specific problem $f(\omega)$ is given implicitly in terms of a multidimensional wave-number spectrum $\tilde{f}(\mathbf{k})$, where $\omega = \omega(\mathbf{k})$, and it is then convenient to express (19) in the more general form

$$|\overline{\psi}|^2 = \frac{\pi t}{2\omega_0^2} \int_{-\infty}^{+\infty} \tilde{f}(\mathbf{k}) [\delta(\omega(\mathbf{k}) - \omega_0) + \delta(\omega(\mathbf{k}) + \omega_0)] dk_1 dk_2 \dots dk_n, \quad (20)$$

where δ denotes the Dirac δ -function.

As the exciting term in the differential equation for ${}_3H_{\mathbf{k}}(t)$ is a stationary random function, Equation (20) can be applied immediately to obtain the rate of increase of the mean product $|\overline{{}_3H_{\mathbf{k}}}|^2$. The linear increase in energy of ${}_3H_{\mathbf{k}}$ must be balanced by a corresponding loss in energy of the interacting components representing the exciting term in Equation (10). This energy loss is expressed by the remaining terms in ${}_6E$, i.e., the mean products of the stationary perturbations ${}_1H_{\mathbf{k}}$ and ${}_2H_{\mathbf{k}}$ with the nonstationary perturbations ${}_5H_{-\mathbf{k}}$ and ${}_4H_{-\mathbf{k}}$ excited by "higher-order" resonant interactions. For large t expressions similar in structure to Equation (20) can also be derived for these products, but as nonstationary exciting forces are involved, the results cannot be formu-

lated as simply as Equation (20) and will not be given here. Once these asymptotic relations have been derived, the determination of the nonlinear energy transfer is then simply a matter of (rather complicated) algebra. The formulation of the final result depends decisively on two properties:

1. As the linear wave components are assumed to be statistically independent, the rate of change of the mean products in ${}_6E$, representing the energy transfer, can be expressed completely in terms of the linear spectrum ${}_2F(\mathbf{k})$.
2. The rate of change of the mean products in ${}_6E$ is determined entirely by resonant interactions in which the perturbation components ${}_nH_{\mathbf{k}}(t)$ are excited with their natural frequencies $\omega_{\mathbf{k}}$. The higher-harmonic, nonlinear components with frequencies unequal to $\omega_{\mathbf{k}}$ do not participate in the energy transfer. The unsteady components of the perturbations can thus be interpreted directly as perturbations of the linear wave components and, consequently, the mean products in ${}_6E$ as perturbations of the linear spectrum ${}_2F(\mathbf{k})$. On account of these two properties the nonlinear energy transfer can be expressed entirely in terms of ${}_2F(\mathbf{k})$. Dropping the subscript 2, the final result is

$$\begin{aligned} \frac{\partial F(\mathbf{k})}{\partial t} = & \iiint \iiint_{-\infty}^{+\infty} \tilde{F}(\mathbf{k}') F(\mathbf{k}'') F(\mathbf{k}' + \mathbf{k}'') \\ & - \mathbf{k}) T_1(\mathbf{k}', \mathbf{k}'', \mathbf{k}' + \mathbf{k}'') dk'_x dk'_y dk''_x dk''_y \\ & - F(\mathbf{k}) \iiint \iiint_{-\infty}^{+\infty} F(\mathbf{k}') F(\mathbf{k}'') T_2(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \\ & dk'_x dk'_y dk''_x dk''_y, \quad (21) \end{aligned}$$

where

$$T_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{9\pi}{4\rho^2 g k_1 k_2 k_3} (A_{\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_3}^{+, +, -})^2 \delta(\omega_{\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3}) \quad (22)$$

$$\begin{aligned} T_2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = & \frac{9\pi}{4\rho^2 g k_1 k_2} \left\{ \frac{A_{\mathbf{k}, -\mathbf{k}_1, -\mathbf{k}_2}^{+, -, -} \cdot A_{-\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2, -\mathbf{k}_1, -\mathbf{k}_2}^{+, -, -}}{\omega_{\mathbf{k}} (\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})} \right. \\ & \delta(\omega_{\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2} + \omega_{\mathbf{k}}, -\omega_{\mathbf{k}_1}, -\omega_{\mathbf{k}_2}) \\ & + \frac{A_{\mathbf{k}, \mathbf{k}_1, -\mathbf{k}_2}^{+, +, -} \cdot A_{-\mathbf{k} - \mathbf{k}_1 + \mathbf{k}_2, -\mathbf{k}_1, -\mathbf{k}_2}^{-, +, -}}{\omega_{\mathbf{k}} (\omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})} \\ & \delta(\omega_{\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}) \\ & \left. + \frac{A_{\mathbf{k}, -\mathbf{k}_1, +\mathbf{k}_2}^{+, -, +} \cdot A_{-\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2, -\mathbf{k}_1, \mathbf{k}_2}^{-, -, +}}{\omega_{\mathbf{k}} (\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2})} \right\} \delta(\omega_{\mathbf{k} - \mathbf{k}_1 + \mathbf{k}_2} - \omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}) \quad (23) \end{aligned}$$

and

$$A_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3}^{s_1, s_2, s_3} = \frac{1}{3} \{ C_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3}^{s_1, s_2, s_3} + C_{\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3}^{s_2, s_1, s_3} + C_{\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_1}^{s_2, s_3, s_1} \} \quad (24)$$

with

$$\begin{aligned} C_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3}^{s_1, s_2, s_3} = & \frac{2[\mathbf{k}_2 \cdot \mathbf{k}_3 - \dots - k_2 k_3] (\omega_2 + \omega_3) (\omega_1 + \omega_2 + \omega_3) [k_1 |k_2 + k_3| - \mathbf{k}_1 \cdot (\mathbf{k}_2 + \mathbf{k}_3)]}{(\omega_{k_2 + k_3})^2 - (\omega_2 + \omega_3)^2} \\ & + [\mathbf{k}_2 \cdot \mathbf{k}_3 - \dots - k_2 k_3] \left[\frac{\omega_1 (\omega_2 + \omega_3)}{2g^2} (\omega_2^2 + \omega_3^2 + \omega_2 \omega_3) + \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{2g^2} - \frac{\omega_1^3 (\omega_2 + \omega_3)}{2g^2} \right. \\ & \left. - \frac{\omega_1 (\omega_2 + \omega_3) |k_2 + k_3|}{g} - \frac{\mathbf{k}_1 \cdot (\mathbf{k}_2 + \mathbf{k}_3)}{2} \right] \\ & + \frac{\omega_2 \omega_3}{2g} (k_2 + k_3) \mathbf{k}_1 \cdot (\mathbf{k}_2 + \mathbf{k}_3) - \frac{\omega_1^3}{2g^4} \omega_2 \omega_3 (\omega_2^2 + \omega_3^2) (\omega_2 + \omega_3) \\ & + \frac{\omega_1 \omega_2 \omega_3^2}{2g^4} [\omega_2 \omega_3 (\omega_2 + \omega_3) - \omega_2^3 - \omega_3^3], \quad (\text{where } \omega_j = s_j \omega_{\mathbf{k}_j}). \end{aligned} \quad (25)$$

Equations (21)–(25) can be checked by applying the law of energy conservation, which requires that the total energy of the sea remains constant. This gives the condition

$$\sum_{\text{Permut. of } \mathbf{k}_j} \{ T_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) - T_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \} = 0 \quad (26)$$

interactions between four components $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and \mathbf{k}_4 occur if, and only if, two pairs of wave numbers, say $(\mathbf{k}_1, \mathbf{k}_2)$ and $(\mathbf{k}_3, \mathbf{k}_4)$, exist such that

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \quad (27)$$

and

$$\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} = \omega_{\mathbf{k}_3} + \omega_{\mathbf{k}_4} \quad (28)$$

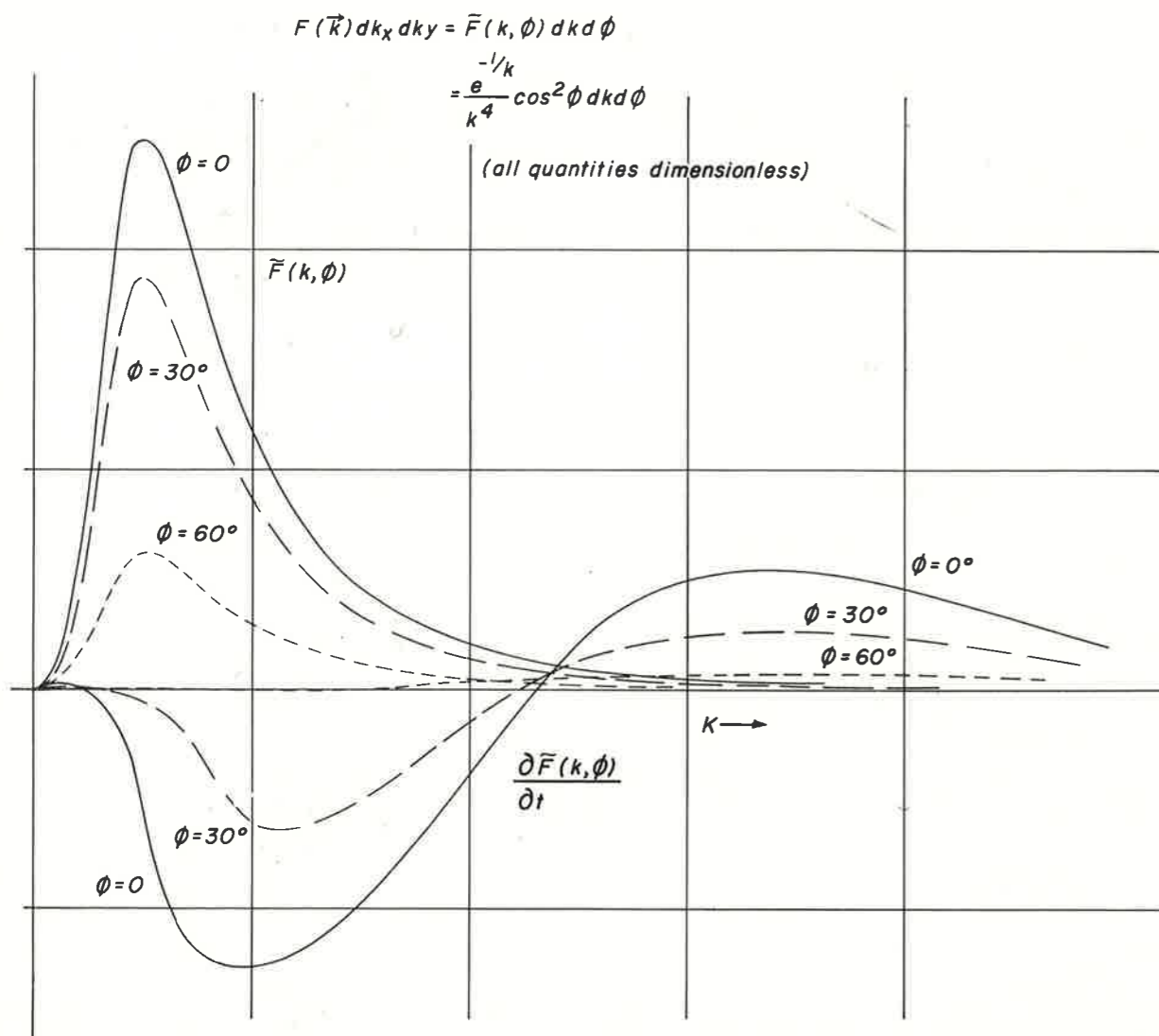
If these conditions are fulfilled, all four interactions corresponding to the four possibilities of choosing three active components and one passive component occur. It can then be shown that if all four values of the spectrum are equal, the energy transfer is exactly balanced. It thus vanishes generally for an isotropic, white spectrum. The net transfer for four wave numbers also vanishes if both wave number pairs are equal. This is the case, for instance, for all interactions in a uni-directional spectrum, as it can be shown that in this case Equations (27) and (28) have only trivial solutions. However, the unidirectional state is unstable and breaks down for small two-dimensional disturbances and would furthermore probably be found to be nonstationary if the perturbation analysis were extended to higher orders.

In an actual wave spectrum the limiting case of an isotropic, white spectrum is, of course, never attained, as the nonlinear energy flux has to be considered in conjunction with the other processes determining the energy balance of the spectrum. The nonlinear energy flux in a wave spectrum is in many respects very similar to the nonlinear energy flux in a homogeneous turbulence spectrum, which is also generally assumed to favour a more uniform distribution of energy over all

¹ In accordance with the definition of ${}_2F(k)$ the "component \mathbf{k} " refers always to the wave component travelling in the positive \mathbf{k} -direction.

wave numbers. This suggests an equilibrium structure in a fully developed wave spectrum analogous to the "cascade" structure of a turbulence spectrum. The dissipation of wave energy due to wave breaking and turbulence is probably concentrated in the high wave-number region of the spectrum contributing significantly to the mean-square wave slope. The generation of wave energy by the wind forces, on the other hand, will probably be concentrated more in the region of small wave numbers corresponding to waves with phase velocities nearly equal to the wind velocity. The energy flux from low to high wave numbers required to maintain equilibrium is then supplied by the nonlinear interactions, which transfer energy from the high spectral peak in the energy-generating region to the low-energy region of dissipation at higher wave numbers. For a non-equilibrium spectrum in the development period

Figure 4-3-1. Energy flux for a fully-developed spectrum.



the situation will generally be different, as the spectral maximum in this case lies closer to the higher wave numbers. It is possible that in the initial period considerable energy then also flows in the direction of lower wave numbers. It is hoped to investigate these cases further by computation of the right-hand side of Equation (21) for several spectra.

An indication of the order of magnitude of the nonlinear energy flux can be obtained by dimensional analysis of Equation (21). If T is the characteristic time of the energy flux, ϑ the root mean-square wave slope and T_w a suitably defined mean wave period we find

$$T \sim T_w \cdot \vartheta^{-4} \quad (29)$$

Assuming the proportionality factor equal to one [a very crude procedure on account of the complicated functions T_1 and T_2 in Equation (21)],

we find that for $T_w = 10$ sec. and $\vartheta = \frac{1}{10}$, say, $T = 28h$, which is comparable in magnitude to the development periods of wave spectra.

A more accurate estimate of the order of magnitude of the energy flux was obtained by computing the energy flux for the fully developed spectrum

$$\bar{F}(k, \phi) = \begin{cases} e^{-1/k} \cdot k^{-4} \cdot \cos^2 \phi, & |\phi| \leq \frac{\pi}{2} \\ 0 & |\phi| > \frac{\pi}{2} \end{cases}$$

where k and θ are nondimensional cylindrical coordinates, which corresponds to a frequency spectrum

$$e^{-v^2} \cdot v^{-7} \cdot \cos^2 \phi.$$

The computations were made on an IBM 650,² which proved to be rather too small for accurate integration of the integrals in Equation (21). The results should therefore be considered as only qualitative with an inaccuracy factor of the order of two. The resulting energy flux for three values of ϕ is shown in Figure 4-3-1. As expected, the transfer of energy is from longer to shorter waves. As the transfer functions are weighted towards higher wave numbers, the maximal energy loss occurs at higher wave numbers than the maximum

² The author is indebted to Miss C. Schwarz and Mr. G. Krause for making the computations.

DISCUSSION

Dr. Longuet-Higgins (prepared comment on the papers concerning nonlinear aspects of the spectrum): Both Mr. Tick and Dr. Pierson have raised the question of whether the third-order interaction terms discovered by Dr. Phillips, which grow with time, represent a real transfer of energy or whether they could be accommodated by slightly perturbing the frequency of the original components.

Now two wave components

$$a_1 \cos(\mathbf{k}_1 \cdot \mathbf{x} - \sigma_1 t) \quad \text{and} \quad a_2 \cos(\mathbf{k}_2 \cdot \mathbf{x} - \sigma_2 t)$$

were shown to give rise to a secular term

$$At \sin[(2\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{x} - (2\sigma_1 - \sigma_2)t], \quad (1)$$

where A is independent of \mathbf{x} and t . Generally the frequency $(2\sigma_1 - \sigma_2)$ of this secular term differs

of the spectrum. The tendency to isotropy or wave scattering is small. This may be due partly to the strongly peaked form of the spectrum. A little energy is scattered at angles greater than 90° and even at an angle $\phi = -180^\circ$, but the energy flux is of the order of 0.1 per cent and less of the maximum energy flux for $\phi = 0^\circ$. For a wind speed of 10 m/s, the characteristic times of the energy flux in the region $|\phi| < \pi/2$ were found to be a few fractions of an hour for the shorter waves and a few hours for the longer waves, which is comparable with the characteristic times of the generating and dissipating processes in a wind-generated sea.

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quency $\frac{1}{2}\sigma_0$ and wave number $-\frac{1}{2}k_0$. This suggested the following simple experiment:

In a long wave channel let the wavemaker be given a motion with two harmonic components σ_0 and $\frac{3}{2}\sigma_0$ (for finite depth the ratio must differ slightly from $\frac{3}{2}$), and let the waves be absorbed at the far end of the tank by a "beach" or other wave absorber. Then let the two primary wave trains be cut off, for example by inserting a thin, rigid barrier vertically into the wave channel quite close to the wavemaker. After the primary waves have travelled away from the barrier there will be left a region of almost calm water, in which the tertiary wave, which travels in the opposite direction, may be observed. According to Phillips's analysis, the amplitude of the tertiary wave (in deep water) should be

$$a = C(a_1k_1)^2 (a_2k_2)L \quad (3)$$

where L is the distance from beach to barrier and C is a constant of order 1. A somewhat lengthy calculation (unchecked) gave $C = \frac{5}{12}$. The crest-to-trough height of the tertiary waves is $2a$, and this value is further doubled by reflection from the vertical barrier inserted in the tank.

I carried out an experiment on these lines in a wave tank 70 feet long at the Hydraulics Research Station, Wallingford, Connecticut, with the kind permission of Mr. Russell. The parameters were: $a_1 = 0.65$ inch; $a_2 = 0.045$ inch; $a_1k_1 = 0.045$, $a_2k_2 = 0.088$; h (mean depth) = 21.5 inches. On the basis of the above formula one would expect $a = 0.12$ inches, which would be readily observable.

The experiment showed no trace of such a wave, and the accuracy of the measurements was such that, if a tertiary wave existed, its amplitude a was less than 0.02 inches. This is an apparent contradiction.

[In carrying out the calculation for finite depth it was discovered that in the previous calculation for infinite depth a term had been omitted. Taking this into account, one finds $C = 0$ in Equation (3), although for waves intersecting at any other angle (except 180 degrees), the coupling does not vanish. This explains the null result in the experiment. It also means that in order to establish the interaction experimentally, observations must be made with wave trains that intersect obliquely.]

Both theory and experiment suggest that the observations mentioned by Dr. Barber are due to a different cause. *Note added on 12 June 1961.*]

Dr. Eckart: Usually, if one is working with a hydrodynamic problem, it is a good idea to close Lamb and ask oneself what Rayleigh did in this field. I don't know whether Rayleigh did anything with nonlinear gravity waves, but he did do something with a problem that involves nonlinearity, viscosity, and perturbation theory. The problem is an amusing one, and it has been quite thoroughly investigated experimentally. It illustrates the question of the difference between perturbation and numerical order. The experiment is an old-fashioned one.

You have a helmholtz resonator. This is a cross-section. Here you have a lighted candle. If you bring in a vibrating tuning fork to the resonator, the candle flame is blown out. In the more sophisticated experiment that lends itself to a qualitative measurement you have a cylinder closed by cellophane ends containing a fluid. Here you place a crystal oscillator, which can generate ultrasonic waves and a beam of waves to send through the fluid. If the fluid is air, then in a fraction of a second a ring vortex is formed. If it is water, it will take somewhat longer for the vortex to grow to a steady state. The velocity in the vortex turns out to be proportional to the energy in the beam of sound waves.

The formula for this turns out to be independent of the viscosity of the fluid. The time required to reach a steady state, however, is dependent on viscosity, as I have illustrated by air and water. If you make the appropriate perturbation calculation to second order, setting viscosity equal to zero, you get nothing. If you leave the viscosity in, you get the result. You get a driving force on this vortex that is proportional to viscosity. You also get a resistance at the wall which is proportional to viscosity so the steady state is independent of viscosity.

Moreover, the vorticity in this steady state is quite high. You can calculate the motion essentially as an incompressible fluid of constant density but you have to take solutions having no potential. So this case again refers to one of the problems that was brought up today — whether or not it is permissible to neglect vorticity in these higher-order calculations.

The most remarkable thing about this is that the velocity in the vortex is hundreds or even thousands of times greater than the velocity in the sound wave. Yet you can deal with it by perturbation theory. So I think we need not be too afraid if some of the second-order terms turn

out to be rather large. The perturbation theory may still apply. The reason is fairly clear in this case. The frequencies are so vastly different that the interaction terms are not of any great importance. There are also many other second-order terms in this problem, some of which are of very small magnitude. At least I hope they are all of a small order of magnitude because this is the one that I calculated at the time these experiments were performed.

Mr. Harris: It seems to me that in any series solutions, such as this perturbation high-order of terms, it is essential to know what the convergence is. Maybe the seventh or ninth order destroys the perturbation theory. I think this should be explained.

Dr. Hasselmann: There is an existence theorem for the convergence of the perturbation series I have used here, but I cannot remember now what the conditions are.

Mr. Harris: There would have to be a function of something. I don't think it could be convergent for all time, for example.

Dr. Hasselmann: I am not quite sure of this point actually. Possibly it does not converge. It is always practically semi-convergent. That is enough. The mathematicians can trouble themselves with this if they want to. Physicists have always assumed that the method works unless the perturbation solutions show some form of irregularity, and I know of no case where this attitude has led them into trouble.

Mr. Tick: I think there are a number of mathematical difficulties here. Since I was one of the originators of this nonlinear attack, I feel justified in cautioning of the danger. In order for perturbation to have any meaning in the continuous spectrum case, there must be some sort of uniformity of correctness of the expansion since the size of the terms depends on wave length. That is to say, the convergence or semi-convergence depends on the wave number being considered.

In discussing the random case of continuous spectra we may have various rates of convergence for different parts of the problem, in which case I think there is chaos, and I think we know very little at this point. Certainly there is no place to turn in the mathematical literature for any discussion of this problem because it has never arisen before.

Second, concerning the proof of existence or nonexistence of certain stationary solutions, I think we want to be careful to de-limit a couple of problems here. The first problem is: is there a stationary solution to the equations? It will certainly be hard to prove. This is the probabilistic version of the LEVI-CIVITA problem.

Second, is there a stationary solution to the third order — whatever that means. I want to stress this point. Here it is best to go back to Rayleigh. In obtaining the solution he had to resort to a certain mathematical technique, namely, to choose a co-ordinate system moving with the phase speed. Such a technique is not available in the two-wave case. It is also not clear to me that the way to remove a secular trend is by frequency perturbation.

If you look through the literature, the secular trend removal techniques were evolved to deal with a particular problem. They provide no guide to other classes of problems. It just works, or it doesn't. First, we don't know whether frequency perturbation is the method. I have never suggested that it is. I suggested that people who have been concerned with the secular method have used this technique.

Second, we don't know if the frequency perturbation is of the form suggested by Dr. Longuet-Higgins. Again I caution you that order is a very tricky notion, which I don't understand.

Mr. Harris: If the perturbation theory gives rise to something that increases with time, it obviously can't apply forever. Whether or not it is physically important can be evaluated when the convergence condition is stated, so we can see whether it becomes divergent very quickly or acts in such a time that it is not important.

There is also the problem that the term of second order may be one not included in the original equation. Or some second-order term not included in the original equation is far more important than the one that arises in the original equation.

Dr. Hasselmann: You get the rate of change of the spectrum at a particular time t , and this rate of change is then valid for all t provided the original assumption that you have a Gaussian process remains valid. This is, in fact, a critical point. The question is: how long does the process, assuming it to be Gaussian originally, retain this property when you begin integrating the integro-

differential equation with respect to time? There is no doubt that if you wait long enough, the process will no longer be Gaussian. However, it can be shown that the rate at which the process loses its Gaussian character (apart from the stationary, second-order terms) is at least two orders of magnitude smaller than the rate of energy transfer, so that for the times that enter in integrating the equation the process can be considered to be Gaussian.

Mr. Harris: You may say that theoretically this is not important, but from a practical point of view it is very important. You can't calculate anything unless you have some notion of what length time steps you can use.

Dr. Eckart: I believe that there is one other point of view on this problem that may shed some light on it. If one considers it not in the Eulerian system, but as Pierson proposes, the problem is simplified from a mathematical point of view in that the boundary condition at least is linear.

A number of perturbation techniques become available. But now let us suppose that these do converge. Let us suppose that there are no secular terms. You may still get into trouble because in this case the solution may become multi-valued. The wave profile may take on a double-valued shape. This is clearly a breaker, but the whole question of the nonlinear effects associated with breakers is completely outside the differential equations.

In order to take this into account, one has to introduce techniques that are similar to the shock wave techniques in compressible fluid theory. You won't avoid this difficulty of the breaker and the shock wave technique by staying with the Eulerian method.

Even granting convergence or granting that there is a reasonable interpretation of the secular term, there are still some difficulties that have

not been touched upon by any of the speakers today.

Dr. Longuet-Higgins: I feel we can reduce these difficulties of breaking by taking the waves at one-tenth of the amplitude.

Dr. Eckart: Yes. If you restrict yourself in amplitude, which you can on paper, this is fine.

Dr. Longuet-Higgins: Of course you can on paper. But it means that the probabilities of breaking are reduced exponentially. If you have a more or less Gaussian probability of distribution of slopes — which gets more and more Gaussian as you reduce the amplitude — and if you reduce it by half, I imagine you really do greatly reduce the probability of breaking.

Dr. Eckart: Then I will stick my neck out and make this prediction. If you do work on the nonlinear theory and neglect the breakers, the white caps, and the spray, then I think the nonlinear effects will turn out to be negligible.

Dr. Longuet-Higgins: The times taken for the non-linear transfer of energy to become appreciable are inversely proportional to the square of amplitude. On the other hand, the probabilities of breaking are reduced by a very much larger factor.

Dr. Eckart: Let me modify my prediction. You may end up with an explanation of the wind-driven currents or something of this sort. But I don't think you will add very much to the theory of the surface waves themselves. You may add a good deal to the theory of the motions of the sea. Are we in agreement then?

Dr. Longuet-Higgins: There may be a range of wave amplitudes in which the tertiary interactions are appreciable though the wave breaking is not.