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ANALOGUES OF CYCLIC INSERTION-TYPE IDENTITIES FOR MULTIPLE ZETA STAR VALUES

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Abstract. We prove an identity for multiple zeta star values, which generalizes some identities due to Imatomi, Tanaka, Tasaka and Wakabayashi. This identity gives an analogue of cyclic insertion-type identities, for multiple zeta star values, and connects the block decomposition with Zhao's generalized 2–1 formula.

1. Introduction

For integers $s_1, \ldots, s_r \geq 1$, the multiple zeta values (MZVs) and multiple zeta star values (MZSVs) are defined by the following series, respectively,

$$\zeta(s_1, \ldots, s_r) := \sum_{0<n_1<n_2<\cdots<n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}},$$

$$\zeta^\star(s_1, \ldots, s_r) := \sum_{0<n_1\leq n_2\leq\cdots\leq n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$

These series are convergent for $s_r > 1$. In each case r is called the *depth*, and $s_1 + \cdots + s_r$ is called the *weight*. We make use of the shorthand 'wt' for the weight of the MZVs appearing in an identity, and write $\{s\}^n$ to mean s repeated n times.

In [**8**], the following identities for multiple zeta star values are conjectured.

CONJECTURE 1.1. [**8**, Conjectures 4.1 and 4.3] *For any integers $a_0, \ldots, a_{2n} \geq 0$, we have*

$$\sum_{\text{permute } a_0,\ldots,a_{2n}} \zeta^\star(\{2\}^{a_0+1}, 1, \{2\}^{a_1}, 3, \{2\}^{a_2}, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}}) \overset{?}{\in} \mathbb{Q}\pi^{\mathrm{wt}}, \tag{1}$$

$$\sum_{\text{permute } a_1,\ldots,a_{2n}} \zeta^\star(1, \{2\}^{a_1}, 3, \{2\}^{a_2}, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}}) \overset{?}{\in} \mathbb{Q}\pi^{\mathrm{wt}}. \tag{2}$$

Notice the blocks of 2 all have lengths a_i, except for the initial one; it has length $a_0 + 1$ in the first identity and length 0 in the second identity.

These identities are similar in structure to the cyclic insertion conjecture of [**1**], on classical MZVs, and should perhaps be regarded as an analogue. The cyclic insertion conjecture states the following.

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CONJECTURE 1.2. [**1**, Conjecture 1] *For any integers* $a_0, \ldots, a_{2n} \geq 0$, *we have*

$$\sum_{\text{cycle } a_0,\ldots,a_{2n}} \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \{2\}^{a_2}, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}}) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}.$$

In [**4**], a symmetrized version of Conjecture 1.2 was proven by the author, up to a rational, using the motivic MZV framework of Brown [**2, 3**]. This symmetrized result was generalized by the author to a wider class of MZVs using the *(alternating) block decomposition* of iterated integrals [**5**], along with a conjectural cyclic version. A proof of a generalization of the cyclic version has since been claimed by Hirose and Sato [**6**], under the name of *block-shuffle* identity.

Zhao's generalized 2–1 formula, Theorem 1.4 in [**9**], gives an expression for an arbitrary MZSV as a sum of alternating MZVs, with arguments from a certain indexing set $\Pi(\mathbf{s}^{(1)})$. As a consequence, in Theorem 5.2 of [**9**] Zhao gives a concise proof of Conjecture 1.1. The goal of this paper is to generalize Zhao's proof of Conjecture 1.1, by connecting Zhao's construction $\mathbf{s}^{(1)}$ with the block decomposition of the multiple zeta value $\zeta(\mathbf{s})$. This allows us to give analogues of other MZV cyclic insertion identities in the MZSV case.

Before stating the main result, we must first recall the construction of the block decomposition from [**5**]. Any word in $\{0,1\}^\times$ can be written as a concatenation of some number of 'alternating words' 0, 1, 01, 10, 010, 101, 0101, 1010, ... By deconcatenating w at a repeated letter, one obtains the (unique) decomposition of w into the minimal possible number of such words. Moreover, by assuming w starts with 0, the lengths of the alternating words uniquely determine w since the concatenation occurs at a repeated letter.

Definition 1.3. (Block decomposition [**5**]) For $w \in \{0,1\}^\times$, starting with a 0, write w as a concatenation of the fewest alternating words $w_1, \ldots, w_n$, with lengths $\ell_1, \ldots, \ell_n$, respectively. The block decomposition of w is

$$\mathrm{bl}(w) := (\ell_1, \ldots, \ell_n).$$

Note that the block decompositions $(\ell_1, \ldots, \ell_n)$ corresponding to words describing an MZV via the integral representation (i.e. first letter 0, last letter 1 for the bounds of integration) satisfy the parity condition

$$n - \sum_{i=1}^{n} \ell_i \equiv 1 \pmod 2.$$

Convergence reasons mean that such a block decomposition will also satisfy $\ell_1 > 1$ and $\ell_n > 1$.

Example 1.4. For $w = 01100101010010101$ (corresponding to $\zeta(1,3,2,2,3,2,2)$), we have

$$\mathrm{bl}(w) = \mathrm{bl}(\underbrace{01}_{2} \mid \underbrace{10}_{2} \mid \underbrace{0101010}_{7} \mid \underbrace{010101}_{6}) = (2,2,7,6).$$

From the lengths (2, 2, 7, 6) we recover a unique word starting with 0, by writing

$$\mathrm{bl}^{-1}(2,2,7,6) = 0\,\underbrace{1 \mid 1}_{\text{repeat}}\,\underbrace{0 \mid 0}_{\text{repeat}}\,10101\,\underbrace{0 \mid 0}_{\text{repeat}}\,10101 = w.$$

We also make use of the following notation around partitions. For $n \geq 1$ a positive integer, write $\mathrm{Part}(n)$ for the set of all unordered partitions $\mathbf{r} = \{r_1, \ldots, r_k\}$ of the set $\{1, \ldots, n\}$ into subsets $r_1, \ldots, r_k$. Write $\mathrm{Part}_{\mathrm{odd}}(n)$ for those such that the cardinality $\# r_i$ of each subset r_i is *odd*. These partitions are unordered and consist of subsets, which means $\mathbf{r} = \{\{1\}, \{2, 4, 5\}, \{3\}\}$ and $\mathbf{r}' = \{\{4, 5, 2\}, \{3\}, \{1\}\}$ both represent the same element of $\mathrm{Part}_{\mathrm{odd}}(5)$. The subsets in the partition may be canonically indexed in lexicographic order. For notational simplicity, I may drop all of the set brackets when writing a partition, and simply use | to separate the subsets of this partition. So I can write the above as $\mathbf{r} = 1|245|3$, say.

We can now state the main result of this paper.

THEOREM 1.5. *For integers $\ell_i > 1$, the following identity on MZSVs holds:*

$$\sum_{\sigma \in S_n} \zeta^\star(\mathrm{bl}^{-1}(2 \circ \ell_{\sigma(1)}, \ldots, \ell_{\sigma(n)})) = \sum_{\substack{\mathbf{r} = \{r_1, \ldots, r_k\} \\ \in \mathrm{Part}_{\mathrm{odd}}(n)}} 2^{\# \mathbf{r}} \prod_{i=1} (\# r_i - 1)! \prod_{i=1} \widetilde{\zeta}\Big(\sum_{j \in r_i} \ell_j\Big).$$

Here

$$\zeta^\star(0;\, 10^{t_1 - 1} \cdots 10^{t_d - 1};\, 1) := \zeta^\star(t_1, \ldots, t_d),$$

as in the iterated integral representation of an MZV, including also the bounds *of integration. Moreover we define $\widetilde{\zeta}$ and $\circ$ by*

$$\widetilde{\zeta}(n) = \begin{cases} \zeta(n) & \text{if } n \text{ odd,} \\ \frac{1}{2}\zeta^\star(\{2\}^{n/2}) & \text{if } n \text{ even,} \end{cases} \quad \text{and} \quad \circ = \begin{cases} \text{`}+\text{'} & \text{if } n \not\equiv \sum \ell_i \pmod 2, \\ \text{`},\text{'} & \text{if } n \equiv \sum \ell_i \pmod 2. \end{cases}$$

In particular, the sum is always a polynomial in Riemann zeta values, since

$$\zeta^\star(\{2\}^m) \in \mathbb{Q}\pi^{2m}.$$

In the case all ℓ_i even, we recover the identities in Conjecture 1.1, and can give explicit terms for the right-hand side in various cases.

Example 1.6. If $(\ell_1, \ell_2, \ell_3) = (2a + 2, 2b + 2, 2c + 2)$, we are in the case $n = 3$, and $\circ = \text{`}+\text{'}$ since $\sum_i \ell_i \not\equiv 3 \pmod 2$. Then

$$\zeta^\star(\mathrm{bl}^{-1}(2 + 2a + 2, 2b + 2, 2c + 2)) = \zeta^\star(\{2\}^{a+1}, 1, \{2\}^b, 3, \{2\}^c).$$

To give the corresponding identity, Theorem 1.5 tells us that we need to sum over $\mathbf{r} \in \mathrm{Part}_{\mathrm{odd}}(3) = \{1|2|3, 123\}$, and so for the right-hand side we obtain

$$\begin{aligned} &2^3(1-1)!^3 \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{a+1}) \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{b+1}) \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{c+1}) \\ &\quad + 2^1(3-1)! \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{a+b+c+3}). \end{aligned}$$

The first line corresponds to the partition $\mathbf{r} = 1|2|3$, and the second to the partition $\mathbf{r} = 123$. This combination simplifies to

$$\zeta^\star(\{2\}^{a+1})\zeta^\star(\{2\}^{b+1})\zeta^\star(\{2\}^{c+1}) + 2\zeta^\star(\{2\}^{a+b+c+3}).$$

This gives the identity

$$\begin{aligned} &\sum_{\text{permute } a,\, b,\, c} \zeta^\star(\{2\}^{a+1}, 1, \{2\}^b, 3, \{2\}^c) \\ &\qquad = \zeta^\star(\{2\}^{a+1})\zeta^\star(\{2\}^{b+1})\zeta^\star(\{2\}^{c+1}) + 2\zeta^\star(\{2\}^{a+b+c+3}) \in \mathbb{Q}\pi^{\mathrm{wt}}, \end{aligned}$$

as in case (1) of Conjecture 1.1.

If $(\ell_1, \ldots, \ell_4) = (2a+2, 2b+2, 2c+2, 2d+2)$, we are in the case $n = 4$, and $\circ =$ ‘,’ since $\sum_i \ell_i \equiv 4 \pmod 2$. Then

$$\zeta^\star(\mathrm{bl}^{-1}(2, 2a+2, 2b+2, 2c+2, 2d+2)) = \zeta^\star(1, \{2\}^a, 3, \{2\}^b, 1, \{2\}^c, 3, \{2\}^d).$$

We sum over $\mathbf{r} \in \mathrm{Part}_{\mathrm{odd}}(4) = \{1|234, 134|2, 124|3, 123|4, 1|2|3|4\}$, and obtain

$$\begin{aligned} &\sum_{\text{permute } a,b,c,d} \zeta^\star(1, \{2\}^a, 3, \{2\}^b, 1, \{2\}^c, 3, \{2\}^d) \\ &\quad = 2\big(\zeta^\star(\{2\}^{a+b+c+3})\zeta^\star(\{2\}^{d+1}) + \zeta^\star(\{2\}^{a+b+d+3})\zeta^\star(\{2\}^{c+1}) \\ &\qquad + \zeta^\star(\{2\}^{a+c+d+3})\zeta^\star(\{2\}^{b+1}) + \zeta^\star(\{2\}^{b+c+d+3})\zeta^\star(\{2\}^{a+1})\big) \\ &\qquad + \zeta^\star(\{2\}^{a+1})\zeta^\star(\{2\}^{b+1})\zeta^\star(\{2\}^{c+1})\zeta^\star(\{2\}^{d+1}) \in \mathbb{Q}\pi^{\mathrm{wt}}, \end{aligned}$$

as in case (2) of Conjecture 1.1.

Example 1.7. (Hoffman's identity) For integers $a, b, c \geq 0$, Hoffman's identity on MZVs (generalized and proven up to $\mathbb{Q}$ by the author in [**5**], and the generalization itself proven exactly by Hirose and Sato in [**6**]) states

$$\begin{aligned} &\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) \\ &\quad = -\zeta(\{2\}^{a+b+c+3}). \end{aligned}$$

It arises from the block decomposition $(\ell_1, \ell_2, \ell_3) = (2a+3, 2b+3, 2c+2)$ of the first MZV above.

We can apply Theorem 1.5 to $(\ell_1, \ell_2, \ell_3) = (2a+3, 2b+3, 2c+2)$ to obtain an analogue on MZSVs. We are in the case $\circ =$ ‘$+$’, since $\sum_i \ell_i \not\equiv 3 \pmod 2$, and we obtain the following combination of MZSVs:

$$\begin{aligned} &\zeta^\star(\{2\}^{a+1}, 3, \{2\}^b, 3, \{2\}^c) + \zeta^\star(\{2\}^{b+1}, 3, \{2\}^a, 3, \{2\}^c) \\ &\quad + \zeta^\star(\{2\}^{b+1}, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta^\star(\{2\}^{a+1}, 3, \{2\}^c, 1, 2, \{2\}^b) \\ &\quad + \zeta^\star(\{2\}^{c+1}, 1, 2, \{2\}^a, 1, 2, \{2\}^b) + \zeta^\star(\{2\}^{c+1}, 1, 2, \{2\}^b, 1, 2, \{2\}^a). \end{aligned}$$

Theorem 1.5 tells us that we need to sum over $\mathbf{r} \in \mathrm{Part}_{\mathrm{odd}}(3) = \{1|2|3, 123\}$, and so we obtain

$$\begin{aligned} &2^3(1-1)!^3\zeta(2a+3)\zeta(2b+3) \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{c+1}) \\ &\quad + 2^1(3-1)! \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{a+b+c+4}), \end{aligned}$$

where the first line corresponds to $\mathbf{r} = 1|2|3$ and the second line to $\mathbf{r} = 123$. This combination simplifies to

$$4\zeta(2a+3)\zeta(2b+3)\zeta^\star(\{2\}^{c+1}) + 2\zeta^\star(\{2\}^{a+b+c+4}).$$

Similar identities can be given for a wide range of initial block lengths, allowing one to produce identities for many MZSVs with indices 1, 2 and 3. Consider the following example.

Example 1.8. Starting with $\zeta^\star(1, 3, 3, \{2\}^m)$, one reads off the block decomposition

$$\mathrm{bl}^{-1}(0;\ 1\,100\,100\,(10)^m;\ 1) = (2, 2, 3, 2m+2).$$

By taking $(\ell_1, \ell_2, \ell_3) = (2, 3, 2m+2)$, we are in the case $\circ =$ ‘,’ as $\sum \ell_i \equiv 3 \pmod 2$. Theorem 1.5 gives us the following identity containing $\zeta^\star(1, 3, 3, \{2\}^m)$:

$$\begin{aligned} &\zeta^\star(1, 3, 3, \{2\}^m) + \zeta^\star(1, 2, 1, \{2\}^m, 3) + \zeta^\star(1, \{2\}^m, 3, 3) \\ &\quad + \zeta^\star(1, 2, 1, 3, \{2\}^m) + \zeta^\star(1, 3, \{2\}^m, 1, 2) + \zeta^\star(1, \{2\}^m, 3, 1, 2) \\ &\qquad = 2\zeta(2)\zeta(3)\zeta^\star(\{2\}^{m+1}) + 4\zeta(2m+7). \end{aligned}$$

2. Background on Zhao’s generalized 2–1 formula

Warning: since I use the opposite convention for MZVs, the version of ‘$\mathbf{s}^{(1)}$’ defined here is the reverse of the one obtained from Zhao’s definition. In fact, I will construct $\mathbf{s}^{(i)}$ by forward induction, so that the last element $\mathbf{s}^{(k)}$ is the relevant one. These changes are incorporated into the text below and the proofs thereafter.

Much of the proof relies on Zhao’s generalization of the 2–1 formula, proven in [**9**]. In this section we recall the necessary notation and concepts from [**9**] in order to apply the generalized 2–1 formula.

Introduce $\mathbb{D} = \mathbb{Z}_{>0} \cup \overline{\mathbb{Z}_{>0}}$ and $\mathbb{D}_0 = \mathbb{Z}_{\geq 0} \cup \overline{\mathbb{Z}_{\geq 0}}$, where $\overline{\mathbb{Z}_{>0}} = \{\overline{n} \mid n > 0\}$ is the set of signed positive numbers and $\overline{\mathbb{Z}_{\geq 0}} = \{\overline{n} \mid n \geq 0\}$ is the set of signed non-negative numbers. The absolute value and sign functions are extended to $\overline{\mathbb{Z}_{\geq 0}}$ via $|\overline{a}| = |a|$ and $\operatorname{sgn}(\overline{a}) = -1$, for $\overline{a} \in \overline{\mathbb{Z}_{\geq 0}}$. (Note that $\operatorname{sgn}(0) = 1$.) Under the operation ‘$\oplus$’ defined by

$$a \oplus b = \begin{cases} \overline{|a| + |b|} & \text{if exactly one of } a \text{ and } b \text{ is in } \mathbb{Z}_{\geq 0}, \\ |a| + |b| & \text{if } a, b \in \mathbb{Z}_{\geq 0} \text{ or if } a, b \in \overline{\mathbb{Z}_{\geq 0}}, \end{cases}$$

the set $\mathbb{D}_0$ forms a semi-group.

Since 0 and $\overline{0}$ do not play a role for us, we can just think that $\overline{n}$ is essentially $-n$, and then the operation $\oplus$ is addition of absolute values and multiplication of the signs.

The multiple zeta values $\zeta(s_1, \ldots, s_k)$ then extend to so-called alternating MZVs, with arguments $s_i \in \mathbb{D}$, via

$$\zeta(s_1, \ldots, s_r) := \sum_{0<n_1<n_2<\cdots<n_r} \frac{\operatorname{sgn}(s_1)^{n_1} \cdots \operatorname{sgn}(s_r)^{n_r}}{n_1^{|s_1|} \cdots n_r^{|s_r|}}.$$

This series is convergent provided $s_r \neq 1$, even including the case where $s_r = \overline{1}$.

Zhao’s generalized 2–1 theorem [**9**, Theorem 1.4] gives a relation between a truncated MZSV and a sum over a certain indexing set $\Pi(\mathbf{s}^{(k)})$ of a certain mollified companion to the truncated (alternating) MZVs. We recall first the construction of $\mathbf{s}^{(i)}$. Let $\mathbf{s} = (s_1, \ldots, s_\ell)$ be an argument string, then with my reversed MZV convention, we construct $\mathbf{s}^{(i)}$, for $i = 1, 2, \ldots, \ell$, by forward induction. (Zhao would define $\mathbf{s}^{(i)}$, $i = \ell, \ldots, 2, 1$, by backward induction.) Set

$$\mathbf{s}^{(1)} = \begin{cases} (1) & \text{if } s_1 = 1, \\ (\{1\}^{s_1-2}, \overline{2}) & \text{if } s_1 \geq 2. \end{cases}$$

Then for $1 < i \leq \ell$ define

$$\mathbf{s}^{(i)} = \begin{cases} s^{(i-1)} \cdot (1) & \text{if } s_i = 1, \\ s^{(i-1)} \oplus (2) & \text{if } s_i = 2, \\ s^{(i-1)} \oplus (\overline{1}, \{1\}^{s_i-3}, \overline{2}) & \text{if } s_i \geq 3. \end{cases}$$

Here, for $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_t)$, one sets

$$\begin{aligned}\mathbf{a} \cdot (1) &= (a_1, \dots, a_r, 1), \\ \mathbf{a} \oplus \mathbf{b} &= (a_1, \dots, a_{r-1}, a_r \oplus b_1, b_2, \dots, b_t).\end{aligned}$$

From Zhao's generalized 2–1 theorem involving truncated MZ(S)Vs, one obtains the following result directly relating MZSVs and alternating MZVs. (Here I use $\mathbf{s}^{(k)}$, whereas Zhao would use $\mathbf{s}^{(1)}$ with the other MZV convention.)

LEMMA 2.1. (Zhao) *For any arguments $\mathbf{s} = (s_1, \dots, s_k) \in (\mathbb{Z}_{>0})^k$ with $s_k > 1$, we have*

$$\zeta^\star(\mathbf{s}) = \varepsilon(\mathbf{s}) \sum_{\mathbf{p} \in \Pi(\mathbf{s}^{(k)})} 2^{\#\mathbf{p}} \zeta(\mathbf{p}),$$

where $\Pi(t_1, \dots, t_\ell)$ is the set of all indices of the form $(t_1 \circ \dots \circ t_\ell)$, where $\circ$ is either ',' or '$\oplus$', and $\varepsilon(\mathbf{s}) = 1$ if $s_1 = 1$, and $\varepsilon(\mathbf{s}) = -1$ if $s_1 \geq 2$.

Proof. Apply Theorem 1.4 of Zhao [**9**], and pass to the limit $n \to \infty$ using Lemma 4.5 of Zhao [**9**]. Lemma 4.5 requires that the last argument of the alternating harmonic sum has absolute value > 1; this is the case since the last entry of $\mathbf{s}^{(k)}$ has absolute value ≥ 2 when $s_k \geq 2$. □

Example 2.2. For clarity, we give an illustration of this result in the case of $\zeta^\star(1, 3, 3, \{2\}^m)$, from Example 1.8. Since $s_1 = 1$, we set

$$\mathbf{s}^{(1)} = (1).$$

Then since $s_2 = s_3 = 3$ we obtain

$$\begin{aligned}\mathbf{s}^{(2)} &= (1) \oplus (\overline{1}, \{1\}^0, \overline{2}) = (\overline{2}, \overline{2}), \\ \mathbf{s}^{(3)} &= (\overline{2}, \overline{2}) \oplus (\overline{1}, \{1\}^0, \overline{2}) = (\overline{2}, 3, \overline{2}).\end{aligned}$$

Finally since $s_i = 2$, for $i \geq 4$, we get

$$\mathbf{s}^{(i)} = (\overline{2}, 3, \overline{2}) \oplus (2)^{\oplus i-3} = (\overline{2}, 3, \overline{2i-4}),$$

so in particular $\mathbf{s}^{(m+3)} = (\overline{2}, 3, \overline{2m+2})$. Since $s_1 = 1$, we find $\varepsilon(\mathbf{s}) = 1$, and applying the generalized 2–1 theorem gives

$$\zeta^\star(1, 3, 3, \{2\}^m) = 2\zeta(2m+7) + 4\zeta(\overline{5}, \overline{2m+2}) + 4\zeta(\overline{2}, \overline{2m+5}) + 8\zeta(\overline{2}, 3, \overline{2m+2}).$$

Recall that the block decomposition of $\zeta(1, 3, 3, \{2\}^m)$ is $(2, 2, 3, 2m+2)$; this already suggests a close relationship between the block decomposition and Zhao's $\mathbf{s}^{(i)}$ construction.

3. Proof of Theorem 1.5

The key step in the proof is to relate the block decomposition to Zhao's $\mathbf{s}^{(i)}$, with the following lemma. This will allow us to directly apply Zhao's generalized 2–1 theorem and obtain the result.

LEMMA 3.1. *Let* $\mathbf{s} = (s_1, \ldots, s_k)$ *be a sequence of MZV arguments (not necessarily convergent). Let* $\mathbf{L} = (\ell_1, \ldots, \ell_n)$ *be the corresponding block decomposition. If* $\ell_1 = 2$*, then the last* $\mathbf{s}^{(k)}$ *associated to* $\mathbf{s}$ *in Zhao's* $\mathbf{s}^{(i)}$ *construction is given by*

$$\mathbf{s}^{(k)} = (\widetilde{\ell_2}, \ldots, \widetilde{\ell_n}),$$

otherwise $\ell_1 > 2$ *and then*

$$\mathbf{s}^{(k)} = (\widetilde{\ell_1 - 2}, \ldots, \widetilde{\ell_n}),$$

where

$$\widetilde{\ell_i} = \begin{cases} \ell_i & \text{if } \ell_i \text{ odd}, \\ \overline{\ell_i} & \text{if } \ell_i \text{ even}. \end{cases}$$

Proof. The proof of this proceeds by induction on the depth of $\mathbf{s}$. We directly check the claim for depth 1, when $\mathbf{s} = (s_1)$.

Case $s_1 = 1$: then $\mathbf{s}^{(1)} = (1)$ and $\mathrm{bl}(1) = (2, 1)$.

Case $s_1 = 2$: then $\mathbf{s}^{(1)} = (\overline{2})$ and $\mathrm{bl}(2) = (4)$.

Case $s_1 \geq 3$: then $\mathbf{s}^{(1)} = (\{1\}^{s_1-2}, \overline{2})$ and $\mathrm{bl}(s_1) = (3, \{1\}^{s_1-3}, 2)$, by writing the word for s_1 out

$$0;\ 10\{0\}^{s_1-3}0;\ 1 = 010 \mid \{0 \mid\}^{s_1-3} \mid 01.$$

Now suppose the result holds for all depth-d argument sequences $\mathbf{s}$. Notice that whether $\ell_1 = 2$ or $\ell_1 > 2$ does not change when adding a new argument, since it is tied to whether the first argument $s_1 = 1$ or $s_1 > 1$, respectively. We can also consider both cases together, by viewing the first as a degenerate version of the second, where $\widetilde{\ell_1 - 2} = \widetilde{0} := \emptyset$ makes no contribution.

Let $(\ell_1, \ldots, \ell_n)$ be the block decomposition of $(s_1, \ldots, s_k)$. By induction we know that

$$\mathbf{s}^{(k)} = (\widetilde{\ell_1 - 2}, \widetilde{\ell_2}, \ldots, \widetilde{\ell_n}).$$

In general, observe that the integral word corresponding to $(s_1, \ldots, s_k, s_{k+1})$ is obtained from the integral word for $(s_1, \ldots, s_k)$ by appending $\{0\}^{s_{k+1}-1}1$. This is just a streamlined version of removing the upper bound 1 of integration, appending the string $1\{0\}^{s_{k+1}-1}$ which corresponds to s_{k+1}, then re-appending the upper bound 1 of integration.

Case $s_{k+1} = 1$: then

$$\begin{aligned} \mathbf{s}^{(k+1)} &= (\widetilde{\ell_1 - 2}, \widetilde{\ell_2}, \ldots, \widetilde{\ell_n}) \cdot (1) \\ &= (\widetilde{\ell_1 - 2}, \widetilde{\ell_2}, \ldots, \widetilde{\ell_n}, 1) \\ &= (\widetilde{\ell_1 - 2}, \widetilde{\ell_2}, \ldots, \widetilde{\ell_n}, \widetilde{1}). \end{aligned}$$

Since the upper bound of integration is 1, appending the extra $\{0\}^0 1 = 1$ produces a new block with length 1, as

$$\cdots \mid \underbrace{\cdots 01010 \cdots}_{\ell_{n-1}} \mid \underbrace{\cdots 0101}_{\ell_n} \rightsquigarrow \cdots \mid \underbrace{\cdots 01010 \cdots}_{\ell_{n-1}} \mid \underbrace{\cdots 0101}_{\ell_n} \mid 1.$$

So the block decomposition of $(s_1, \ldots, s_k, 1)$ is $(\ell_1, \ldots, \ell_n, 1)$, which matches the result of $\mathbf{s}^{(k+1)}$.

Case $s_{k+1} = 2$: then

$$\begin{aligned}\mathbf{s}^{(k+1)} &= (\widetilde{\ell_1 - 2}, \ldots, \widetilde{\ell_n}) \oplus (2) \\ &= (\widetilde{\ell_1 - 2}, \ldots, \widetilde{\ell_{n-1}}, \widetilde{\ell_n} \oplus 2) \\ &= (\widetilde{\ell_1 - 2}, \ldots, \widetilde{\ell_{n-1}}, \widetilde{\ell_n + 2}).\end{aligned}$$

This is because $\oplus\, 2$ does not change the parity or sign of the result, so $\widetilde{\ell_n} \oplus 2 = \widetilde{\ell_n + 2}$.

Since the upper bound of integration is 1, appending the extra $\{0\}^1 1 = 01$ increases the length of the last block by 2, as

$$\cdots | \underbrace{\cdots 01010 \cdots}_{\ell_{n-1}} | \underbrace{\cdots 0101}_{\ell_n} \rightsquigarrow \cdots | \underbrace{\cdots 01010 \cdots}_{\ell_{n-1}} | \overbrace{\underbrace{\cdots 0101}_{\ell_n}\, 01}^{\ell_n + 2}.$$

So the block decomposition of $(s_1, \ldots, s_k, 2)$ is $(\ell_1, \ldots, \ell_{n-1}, \ell_n + 2)$, which matches the result of $\mathbf{s}^{(k+1)}$.

Case $s_{k+1} \geq 3$: then

$$\begin{aligned}\mathbf{s}^{(k+1)} &= (\widetilde{\ell_1 - 2}, \ldots, \widetilde{\ell_n}) \oplus (\overline{1}, \{1\}^{s_{k+1}-3}, \overline{2}) \\ &= (\widetilde{\ell_1 - 2}, \ldots, \widetilde{\ell_{n-1}}, \widetilde{\ell_n} \oplus \overline{1}, \{1\}^{s_{k+1}-3}, \overline{2}) \\ &= (\widetilde{\ell_1 - 2}, \ldots, \widetilde{\ell_{n-1}}, \widetilde{\ell_n + 1}, \{\widetilde{1}\}^{s_{k+1}-3}, \widetilde{2}).\end{aligned}$$

This is because $\oplus\, \overline{1}$ changes both the sign and the parity, so $\widetilde{\ell_n} \oplus 1 = \widetilde{\ell_n + 1}$.

Since the upper bound of integration is 1, appending the extra $\{0\}^{s_{k+1}-1} 1 = 0\{0\}^{s_{k+1}-3} 01$ increases the length of the last block by 1, and adds new blocks as

$$\cdots | \underbrace{\cdots 01010 \cdots}_{\ell_{n-1}} | \underbrace{\cdots 0101}_{\ell_n} \rightsquigarrow \cdots | \underbrace{\cdots 01010 \cdots}_{\ell_{n-1}} | \overbrace{\underbrace{\cdots 0101}_{\ell_n}\, 0}^{\ell_n + 1} | \{\underbrace{0}_{1} |\}^{s_{k+1}-3} | \underbrace{01}_{2}.$$

So the block decomposition of $(s_1, \ldots, s_k, s_{k+1})$ is $(\ell_1, \ldots, \ell_{n-1}, \ell_n + 1, \{1\}^{s_{k+1}-3}, 2)$, which matches the result of $\mathbf{s}^{(k+1)}$.

In each case the result matches, so by induction the block decomposition of $\mathbf{s} = (s_1, \ldots, s_k)$ and $\mathbf{s}^{(k)}$, the last $\mathbf{s}^{(i)}$ in Zhao's construction, are related as claimed. □

Given integers $\ell_1, \ldots, \ell_n$, with $\ell_n > 1$, we can form a block decomposition $(2 \circ \ell_1, \ldots, \ell_n)$ where

$$\circ = \begin{cases} `+\text{'} & \text{if } n \not\equiv \sum \ell_i \pmod 2, \\ `,\text{'} & \text{if } n \equiv \sum \ell_i \pmod 2. \end{cases}$$

This corresponds to some MZV argument string $\mathbf{s} = (s_1, \ldots, s_k)$, by the parity condition. From the lemma we find that $\mathbf{s}^{(k)} = (\widetilde{\ell_1}, \ldots, \widetilde{\ell_n})$.

We now give a result which allows us to symmetrize the result of Zhao's generalized 2–1 theorem.

PROPOSITION 3.2. *Let* $\mathbf{s} = (s_1, \ldots, s_n)$ *be given, and assume* S_n *acts on the indices* $1, \ldots, n$ *in the standard way. Then*

$$\sum_{\sigma \in S_n} \sum_{\mathbf{p} \in \Pi(\sigma \cdot \mathbf{s})} 2^{\#\mathbf{p}} \zeta(\mathbf{p}) = \sum_{\substack{\mathbf{q}=\{q_1,\ldots,q_t\} \\ \in \mathrm{Part}(n)}} 2^{\#\mathbf{q}} \left(\prod_{i=1}^{\#\mathbf{q}} \#q_i! \right) \zeta_{\mathrm{sym}} \left(\bigoplus_{j \in q_1} s_j, \ldots, \bigoplus_{j \in q_t} s_j \right),$$

where $\zeta_{\mathrm{sym}}(a_1, \ldots, a_t) := \sum_{\tau \in S_t} \zeta(a_{\tau(1)}, \ldots, a_{\tau(t)})$ *symmetrizes the arguments of the MZV.*

That is, one can move the S_n *action from* $\mathbf{s}$ *to the arguments of the zeta, at the expense of some coefficients.*

Proof. Firstly, observe that $p \in \Pi(\sigma \cdot \mathbf{s})$ means $p = s_{\sigma(1)} \circ \cdots \circ s_{\sigma(n)}$, where each $\circ$ is ',' or '$\oplus$'. But this is equivalent to $p = \sigma \cdot (s_1 \circ \cdots \circ s_n)$, under the induced S_n action, and $(s_1 \circ \cdots \circ s_n) \in \Pi(s_1, \ldots, s_n)$. So we can write

$$\sum_{\sigma \in S_n} \sum_{p \in \Pi(\mathbf{s})} 2^{\#\mathbf{p}} \zeta(\sigma \cdot \mathbf{p}).$$

Warning: σ acts on the elements s_i inside $\mathbf{p} = (s_1 \circ \cdots \circ s_n)$, and not on the comma-separated blocks. So $\zeta(\sigma \cdot \mathbf{p})$ is not simply $\zeta_{\mathrm{sym}}(\mathbf{p})$. We need to do further manipulation to obtain the desired form.

An element $\mathbf{p} \in \Pi(\mathbf{s})$ is of the form $s_1 \circ \cdots \circ s_n$ for some choices $\circ =$ ',' or '$\oplus$'. That is

$$\mathbf{p} = \big(s_1 \oplus \cdots \oplus s_{i_1}, s_{i_1+1} \oplus \cdots \oplus s_{i_2}, \ldots, s_{i_{t-1}+1} \oplus \cdots \oplus s_{\underbrace{i_t}_{=n}}\big),$$

and

$$\sigma \cdot \mathbf{p} = \big(s_{\sigma(1)} \oplus \cdots \oplus s_{\sigma(i_1)}, s_{\sigma(i_1+1)} \oplus \cdots \oplus s_{\sigma(i_2)}, \ldots, s_{\sigma(i_{t-1}+1)} \oplus \cdots \oplus s_{\underbrace{\sigma(i_t)}_{i_t=n}}\big).$$

We can therefore define a surjective map

$$\begin{aligned} \phi \colon (\Pi(\mathbf{p}), S_n) &\to \mathrm{Part}^*(n), \\ (\mathbf{p}, \sigma) &\mapsto [\{\sigma(1), \ldots, \sigma(i_1)\}, \{\sigma(i_1+1), \ldots, \sigma(i_2)\}, \\ &\qquad \ldots, \{\sigma(i_{t-1}+1), \ldots, \sigma(i_t)\}], \end{aligned}$$

where $i_1, \ldots, i_t$ are given by the expression for $\sigma \cdot \mathbf{p}$ above. Here $\mathrm{Part}^*(n)$ is the set of ordered partitions $\mathbf{r} = [r_1, \ldots, r_k]$ of the set $\{1, \ldots, n\}$, into subsets $r_1, r_2, \ldots, r_k$. The order of the elements of the parts is not important, but the order of the parts themselves is. That is $[\{a, b\}, \{c\}] = [\{b, a\}, \{c\}]$, but these are different from $[\{c\}, \{a, b\}]$.

If $\mathbf{q} = \phi(\mathbf{p}, \sigma) = [q_1, \ldots, q_t]$, then $\zeta(\sigma \cdot \mathbf{p}) = \zeta(\bigoplus_{j \in q_1} s_j, \ldots, \bigoplus_{j \in q_t} s_j)$, and $\#\mathbf{q} = \#\mathbf{p}$. Notice that $\#\phi^{-1}(\mathbf{q}) = \#q_1! \cdots \#q_t!$, since any permutation which respects the parts of $\mathbf{p}$ maps to the same $\mathbf{q}$. So we can write that the desired sum is

$$= \sum_{\substack{\mathbf{q}=[q_1,\ldots,q_t] \\ \in \mathrm{Part}^*(n)}} 2^{\#\mathbf{q}} \left(\prod_{i=1}^{\#\mathbf{q}} \#q_i! \right) \zeta \left(\bigoplus_{j \in q_1} s_j, \ldots, \bigoplus_{j \in q_t} s_j \right).$$

Finally, we have a surjective map

$$\begin{aligned}\psi\colon\ &\mathrm{Part}^*(n) \to \mathrm{Part}(n),\\ &[q_1,\ldots,q_t] \mapsto \{q_1,\ldots,q_t\},\end{aligned}$$

with $\psi^{-1}(\{q_1,\ldots,q_t\}) = \{[q_{\sigma(1)},\ldots,q_{\sigma(t)}] \mid \sigma \in S_t\}$.

So the sum can be written

$$= \sum_{\substack{\mathbf{q}=\{q_1,\ldots,q_t\}\\ \in \mathrm{Part}(n)}} 2^{\#\mathbf{q}} \#q_1! \cdots \#q_t! \underbrace{\sum_{\sigma\in S_t} \zeta\Big(\bigoplus_{j\in q_{\sigma(1)}} s_j,\ldots,\bigoplus_{j\in q_{\sigma(t)}} s_j\Big)}_{=:\zeta_{\mathrm{sym}}}.$$

This completes the proof. □

Using the symmetric sum formula [**7**] (or rather Zhao's generalization to alternating MZVs, a special case of which is stated in Lemma 5.1 of [**9**]), one can evaluate the right-hand side above. This symmetric sum formula states that

$$\zeta_{\mathrm{sym}}(s_1,\ldots,s_k) = \sum_{\mathbf{b}\in\mathrm{Part}(k)} (-1)^{k-\#\mathbf{b}} \prod_{i=1}^{\#\mathbf{b}} (\#b_i - 1)! \prod_{i=1}^{\#\mathbf{b}} \zeta\Big(\bigoplus_{j\in b_i} s_j\Big).$$

We obtain the following.

PROPOSITION 3.3. *The following evaluation holds:*

$$\begin{aligned}&\sum_{\mathbf{q}\in\mathrm{Part}(n)} 2^{\#\mathbf{q}}\Big(\prod_{i=1}^{\#\mathbf{q}} \#q_i!\Big)\zeta_{\mathrm{sym}}\Big(\bigoplus_{j\in q_1} s_j,\ldots,\bigoplus_{j\in q_{\#\mathbf{q}}} s_j\Big)\\ &\quad= \sum_{\mathbf{r}\in\mathrm{Part}_{\mathrm{odd}}(n)} 2^{\#\mathbf{r}} \prod_{i=1}^{\#\mathbf{r}} (\#r_i - 1)! \prod_{i=1}^{\#\mathbf{r}} \zeta\Big(\bigoplus_{j\in r_i} s_j\Big).\end{aligned}$$

Proof. Firstly, we must apply the symmetric sum formula to evaluate the left-hand side. It gives

$$\sum_{\mathbf{q}\in\mathrm{Part}(n)} 2^{\#\mathbf{q}}\Big(\prod_{i=1}^{\#\mathbf{q}} \#q_i!\Big) \sum_{\mathbf{t}\in\mathrm{Part}(\#\mathbf{q})} (-1)^{\#\mathbf{q}-\#\mathbf{t}}\Big(\prod_{j=1}^{\#\mathbf{t}} (\#t_j - 1)!\Big) \prod_{j=1}^{\#\mathbf{t}} \zeta\Big(\bigoplus_{\alpha\in t_j}\bigoplus_{\beta\in q_\alpha} s_\beta\Big).$$

As the parts q_α are disjoint, the ζ argument

$$\bigoplus_{\alpha\in t_j}\bigoplus_{\beta\in q_\alpha} s_\beta$$

can be written as

$$\bigoplus_{\alpha\in r_j} s_\alpha,$$

for some partition $\mathbf{r}\in\mathrm{Part}(n)$. This partition is obtained from $(\mathbf{q},\mathbf{t})$ by 'flattening' in the following sense:

$$f(\mathbf{q},\mathbf{t}) := \mathbf{r} = \{r_1,\ldots,r_{\#\mathbf{t}}\}, \quad r_i = \bigcup_{j\in t_i} q_j.$$

(Recall that we have indexed the subsets in $\mathbf{q}$ in lexicographical order.) For example

$$f(\{\overbrace{\{1,2,4\}}^{q_1}, \overbrace{\{3,5\}}^{q_2}, \overbrace{\{6,8\}}^{q_3}, \overbrace{\{7\}}^{q_4}\}, \{\overbrace{\{1,3,4\}}^{t_1}, \overbrace{\{2\}}^{t_2}\})$$
$$= \{q_1 \cup q_3 \cup q_4, q_2\} = \{\{1,2,4,6,7,8\}, \{3,5\}\}.$$

So we may formally write the sum as

$$\sum_{\mathbf{r}\in\mathrm{Part}(n)} \sum_{(\mathbf{q},\mathbf{t})\in f^{-1}(\mathbf{r})} 2^{\#\mathbf{q}}(-1)^{\#\mathbf{q}-\#\mathbf{t}}\biggl(\prod_{i=1}^{\#\mathbf{q}} \#q_i!\biggr)\biggl(\prod_{j=1}^{\#\mathbf{t}}(\#t_j - 1)!\biggr)\prod_{k=1}^{\#\mathbf{r}} \zeta\biggl(\bigoplus_{\alpha\in r_k} s_\alpha\biggr).$$

We thus need to evaluate the coefficient

$$c_{\mathbf{r}} := \sum_{(\mathbf{q},\mathbf{t})\in f^{-1}(\mathbf{r})} 2^{\#\mathbf{q}}(-1)^{\#\mathbf{q}-\#\mathbf{t}}\biggl(\prod_{i=1}^{\#\mathbf{q}} \#q_i!\biggr)\biggl(\prod_{j=1}^{\#\mathbf{t}}(\#t_j - 1)!\biggr).$$

We want to show two things: firstly that, if the partition $\mathbf{r}$ has any even size parts, then the coefficient is 0. Secondly, if the partition only has odd size parts, the coefficient is as indicated in the statement of the proposition.

We can describe $f^{-1}(\mathbf{r})$ more explicitly, as follows. The elements $\mathbf{q}$ which flatten to $\mathbf{r}$ are obtained as $\mathbf{q} = \bigcup_{j=1}^{\#\mathbf{r}} \mathbf{T}_j$, where $\mathbf{T}_j$ is any partition of r_j. This choice of partitions $\mathbf{T}_1, \ldots, \mathbf{T}_{\#\mathbf{r}}$ determines $\mathbf{t}$, since $r_j = \bigcup \mathbf{T}_j$. For example, if

$$\mathbf{r} = \{\{1,3,4,5,7\}, \{2,6,8\}\},$$

then $f^{-1}(\mathbf{r}) \ni (\mathbf{q},\mathbf{t})$, where

$$\mathbf{q} = \bigcup \{\underbrace{\{1,4 \mid 3 \mid 5,7\}}_{\mathbf{T}_1}, \underbrace{\{2,6 \mid 8\}}_{\mathbf{T}_2}\} = \{\{1,4\},\{2,6\},\{3\},\{5,7\},\{8\}\},$$
$$\mathbf{t} = \{\{1,3,4\},\{2,5\}\},$$

for the partitions $\mathbf{T}_1 = 14 \mid 3 \mid 57$ of $r_1 = \{1,3,4,5,7\}$ and $\mathbf{T}_2 = 26 \mid 8$ of $r_2 = \{2,6,8\}$.

Under this construction we have

$$\#\mathbf{t} = \#\mathbf{r}, \quad \#t_i = \#\mathbf{T}_i, \quad \#\mathbf{q} = \sum_{i=1}^{\#\mathbf{r}} \#\mathbf{T}_i.$$

Moreover $q_j = T_{k,\ell} \in \mathbf{T}_k$ for some k, ℓ, so that

$$\prod_{j=1}^{\#\mathbf{q}} \#q_j! = \prod_{k=1}^{\#\mathbf{r}} \prod_{\ell=1}^{\#\mathbf{T}_k} \#T_{k,\ell}!.$$

Thus

$$c_{\mathbf{r}} = (-1)^{\#\mathbf{r}} \sum_{\mathbf{T}_1\in\mathrm{Part}(\#r_1)} \cdots \sum_{\mathbf{T}_{\#\mathbf{r}}\in\mathrm{Part}(\#r_{\#\mathbf{r}})} (-2)^{\sum_i \#\mathbf{T}_i} \cdot \prod_{k=1}^{\#\mathbf{r}} \prod_{\ell=1}^{\#\mathbf{T}_k} \#T_{k,\ell}! \cdot \prod_{j=1}^{\#\mathbf{r}} (\#\mathbf{T}_j - 1)!.$$

This sum can now be factored into a product of the form

$$c_{\mathbf{r}} = (-1)^{\#\mathbf{r}} \prod_{i=1}^{\#\mathbf{r}} g(\#r_i),$$

where

$$g(i) := \sum_{\mathbf{w}\in\mathrm{Part}(i)} (-2)^{\#\mathbf{w}}(\#\mathbf{w}-1)! \cdot \prod_{\ell=1}^{\#\mathbf{w}} \#w_\ell!.$$

I claim that g can be evaluated as follows:

$$g(i) = \begin{cases} -2(i-1)! & \text{if } i \text{ odd}, \\ 0 & \text{if } i \text{ even}. \end{cases}$$

If this claim does hold, then

$$\begin{aligned} c_{\mathbf{r}} &= \begin{cases} 0 & \text{if some } \#r_i \text{ even}, \\ (-1)^{\#\mathbf{r}} \prod_{i=1}^{\#\mathbf{r}} (-2)(\#r_i - 1)! & \text{if all } \#r_i \text{ odd}, \end{cases} \\ &= \begin{cases} 0 & \text{if some } \#r_i \text{ even}, \\ 2^{\#\mathbf{r}} \prod_{i=1}^{\#\mathbf{r}} (\#r_i - 1)! & \text{if all } \#r_i \text{ odd}. \end{cases} \end{aligned}$$

So the proposition will follow. □

For the proof to be complete, we need to show the following claim.

CLAIM 3.4. *Let*

$$g(n) := \sum_{\mathbf{w}\in\mathrm{Part}(n)} (-2)^{\#\mathbf{w}}(\#\mathbf{w}-1)! \cdot \prod_{\ell=1}^{\#\mathbf{w}} \#w_\ell!,$$

then

$$g(n) = \begin{cases} -2(n-1)! & \textit{if } n \textit{ odd}, \\ 0 & \textit{if } n \textit{ even}. \end{cases}$$

Proof. We show the generalized identity

$$\frac{1}{n!} g(n, x) = -\frac{1}{n} + \frac{1}{n}(1+x)^n,$$

where

$$g(n, x) := \sum_{\mathbf{w}\in\mathrm{Part}(n)} x^{\#\mathbf{w}}(\#\mathbf{w}-1)! \cdot \prod_{\ell=1}^{\#\mathbf{w}} \#w_\ell!.$$

Hence for $x = -2$, we obtain

$$g(n, -2) = -(n-1)! + (n-1)!(-1)^n = \begin{cases} -2(n-1)! & n \text{ odd}, \\ 0 & n \text{ even}, \end{cases}$$

as claimed.

To show the generalized identity, we can first show that the derivatives agree. Then integrating gives

$$\frac{1}{n!} g(n, x) = c + \frac{1}{n}(1+x)^n,$$

for some constant c. One sees that $c = -1/n$ by setting $x = 0$; the left-hand side is 0 and the right-hand side is $c + 1/n$, which proves the claim.

To see the derivatives agree, we must show

$$\frac{1}{n!}g'(n, x) = (1 + x)^{n-1},$$

equivalently,

$$g'(n, x) = n!(1 + x)^{n-1}. \tag{3}$$

Term-by-term differentiation of $g(n, x)$ gives

$$g'(n, x) = \sum_{\mathbf{w}\in \mathrm{Part}(n)} x^{\#\mathbf{w}-1}\#\mathbf{w}! \cdot \prod_{\ell=1}^{\#\mathbf{w}} \#w_\ell!.$$

The coefficient of x^{i-1} on the right-hand side of equation (3) is

$$n!\binom{n-1}{i-1}.$$

The coefficient of x^{i-1} on the left-hand side is

$$\sum_{\substack{\mathbf{w}\in \mathrm{Part}(n)\\ \#\mathbf{w}=i}} \underbrace{\#\mathbf{w}!}_{i!} \cdot \prod_{\ell=1}^{\#\mathbf{w}} w_\ell!.$$

These two expressions give two different ways to count the number of ordered partitions of $[1, \dots, n]$ into i non-empty ordered parts, and hence are equal. Here an ordered partition with ordered parts means that $\big[\,[1, 2], [3]\,\big]$, $\big[\,[2, 1], [3]\,\big]$, $\big[\,[3], [1, 2]\,\big]$ and $\big[\,[3], [2, 1]\,\big]$ are all counted as distinct. In the following, we refer to such a partition as an ordered/ordered partition.

We can form an ordered/ordered partition of $[1, \dots, n]$ into i parts by first taking any permutation of $[1, \dots, n]$, then inserting $i - 1$ bars into any choice of the $n - 1$ gaps, breaking the i non-empty parts, for example

$$\begin{aligned}[1, \dots, 8] &\xrightarrow{\text{permute}} [4\underset{\uparrow}{,}5, 2, 3\underset{\uparrow}{,}7, 6\underset{\uparrow}{,}1, 8]\\ &\xrightarrow{\text{insert bars}} \big[\,[4], [5, 2, 3], [7, 6], [1, 8]\,\big].\end{aligned}$$

There are $n!$ permutations, and $\binom{n-1}{i-1}$ ways of choosing $i - 1$ positions from the $n - 1$ gaps. This gives the right-hand side.

Alternatively, we can form an ordered/ordered partition of $[1, \dots, n]$ by taking a partition in $\mathrm{Part}(n)$ of $\{1, \dots, n\}$ into i parts, then reordering the i parts arbitrarily, as well as arbitrarily reordering the elements of each part. Every such ordered/ordered partition of $[1, \dots, n]$ arises in this way, for some unique $\mathbf{w}$, as forgetting about both orderings gives a surjection onto $\mathrm{Part}(n)$, for example

$$\begin{aligned}\mathrm{Part}(8) \ni \mathbf{w} &= \{\{1, 8\}, \{2, 3, 5\}, \{4\}, \{6, 7\}\}\\ &\xrightarrow[\text{and elements}]{\text{permute parts}} \big[\,[4], [5, 2, 3], [7, 6], [1, 8]\,\big].\end{aligned}$$

Let $\mathbf{w} \in \operatorname{Part}(n)$ be a partition of $\{1, \ldots, n\}$ into i parts, with sizes of each part $\#w_1, \ldots, \#w_i$, respectively. Then there are $i! \prod_{\ell=1}^{\#\mathbf{w}} \#w_\ell!$ such ordered/ordered partitions arising from $\mathbf{w}$. We must sum over all such $\mathbf{w} \in \operatorname{Part}(n)$, giving

$$\sum_{\substack{\mathbf{w}\in\operatorname{Part}(n)\\ \#\mathbf{w}=i}} i! \cdot \prod_{\ell=1}^{\#\mathbf{w}} \#w_\ell!.$$

This is the left-hand side.

The coefficients of both sides agree, hence we get the required equality of derivatives, and so the claim follows. □

Finally, we can use these results to prove Theorem 1.5.

Proof of Theorem 1.5. Using Lemma 2.1 and Lemma 3.1 we have that

$$\sum_{\sigma\in S_n} \zeta^\star(\mathrm{bl}^{-1}(2 \circ \ell_{\sigma(1)}, \ldots, \ell_{\sigma(n)})) = \varepsilon(\circ) \sum_{\sigma\in S_n} \sum_{\mathbf{p}\in\Pi(\sigma\cdot(\widetilde{\ell_1},\ldots,\widetilde{\ell_n}))} 2^{\#\mathbf{p}} \zeta(\mathbf{p}),$$

for $\varepsilon(+) := -1$ and $\varepsilon(,) := 1$. This is because each $\varepsilon(\mathbf{s})$ agrees with $\varepsilon(\circ)$, for the following reason. If $\circ =$ ‘,’, then

$$\mathrm{bl}^{-1}(2, \ell_{\sigma(1)}, \ldots) = 01 \mid 1 \cdots .$$

This means the MZSV $\zeta^\star(\mathrm{bl}^{-1}(2, \ell_{\sigma(1)}, \ldots)) = \zeta^\star(1, \ldots)$ and $\varepsilon(\mathbf{s}) = 1$ since $s_1 = 1$. Otherwise $\circ =$ ‘+’, and

$$\mathrm{bl}^{-1}(2 + \ell_{\sigma(1)}, \ldots) = 0101 \cdots$$

since every $\ell_i > 1$. This means the MZSV $\zeta^\star(\mathrm{bl}^{-1}(2 + \ell_{\sigma(1)}, \ldots)) = \zeta^\star(2, \ldots)$, and $\varepsilon(\mathbf{s}) = -1$ since $s_1 = 2$.

Now interchange the summations, and write the result as a sum over odd-sized partitions using Proposition 3.2 and Proposition 3.3:

$$= \varepsilon(\circ) \sum_{\mathbf{r}\in\operatorname{Part}_{\mathrm{odd}}(n)} 2^{\#\mathbf{r}} \prod_{i=1}^{\#\mathbf{r}} (\#r_i - 1)! \prod_{i=1}^{\#\mathbf{r}} \zeta\Big(\bigoplus_{j\in r_i} \widetilde{\ell_j}\Big).$$

Since the size of each partition is odd, we can explicitly evaluate $\bigoplus_{j\in p_i} \widetilde{\ell_j}$, and the resulting ζ as follows.

Case $\#\{\ell_i \mid \mathit{even}\} \equiv 0 \pmod 2$: then

$$\bigoplus_{j\in p_i} \widetilde{\ell_j} = \sum_{j\in p_i} \ell_j,$$

and this sum is odd. This is because the number of bars is additive (i.e. the sign is multiplicative), and there are an even number of bars in total. So the $\oplus$ sum agrees with the sum of the undecorated ℓ_i. Moreover, the total is odd since we sum an odd number of odd numbers.

Overall, this means

$$\zeta\Big(\bigoplus_{j\in p_i} \widetilde{\ell_j}\Big) = \zeta\Big(\sum_{j\in p_i} \ell_j\Big).$$

Case $\#\{\ell_i \mid even\} \equiv 1 \pmod 2$: then

$$\bigoplus_{j \in p_i} \widetilde{\ell_j} = \overline{\sum_{j \in p_i} \ell_j},$$

and this sum is even. This is because there are an odd number of bars in total, so one remains after doing the $\oplus$ sum. Consequently the $\oplus$ sum agrees with the bar of the undecorated sum. Moreover, the total is even, since we add an even number of odd numbers.

This means

$$\zeta\Bigl(\bigoplus_{j \in p_i} \widetilde{\ell_j}\Bigr) = \zeta\Bigl(\overline{\sum_{j \in p_i} \ell_j}\Bigr).$$

We can now use Zlobin's evaluation [**10**] of $\zeta^\star(\{2\}^n) = -2\zeta(\overline{2n})$ (which is also contained in Zhao's generalized 2–1 theorem) to write

$$\zeta\Bigl(\overline{\sum_{j \in p_i} \ell_j}\Bigr) = -\frac{1}{2}\zeta^\star\Bigl(\{2\}^{(1/2)\sum_{j \in p_i} \ell_j}\Bigr).$$

This is almost our definition of $\widetilde{\zeta}$. I claim that the number of -1 signs between $\varepsilon(\circ)$ and all the $-\zeta^\star(\{2\}^n)$ is even. We may discard it to obtain an equivalent formula with our original definition of $\widetilde{\zeta}$.

Why is the total number of '-1's even?

Case $\circ =$ ',': here $\varepsilon(,) = 1$, since the MZSVs begin

$$\zeta^\star(\mathrm{bl}^{-1}(2, \ell_{\sigma(1)}, \ldots)) = \zeta^\star(01 \mid 10 \cdots) = \zeta^\star(1, \ldots).$$

I claim that the number of 'even-sum' parts is even, hence the total number of '-1's is even as claimed. In this case $n \equiv \sum \ell_i \pmod 2$, and we can check n odd or even separately.

Suppose n is odd, then $\sum \ell_i$ is also odd. By counting the number n of ℓ_i, we see $\mathbf{p} \in \mathrm{Part}_{\mathrm{odd}}(\ell_i)$ has an odd number of parts. If an odd number of parts have even sum, we would have $\sum \ell_i$ even, a contradiction.

Similarly if n is even, then $\sum \ell_i$ is also even. By counting the number n of ℓ_i, we see $\mathbf{p} \in \mathrm{Part}_{\mathrm{odd}}(\ell_i)$ has an even number of parts. If an odd number of parts have even sum, we would again have $\sum \ell_i$ odd, a contradiction.

Case $\circ =$ '$+$': here $\varepsilon(+) = -1$, since all $\ell_i > 1$, meaning the MZSVs begin

$$\zeta^\star(\mathrm{bl}^{-1}(2 + \ell_{\sigma(1)}, \ldots)) = \zeta^\star(0101 \cdots) = \zeta^\star(2, \ldots).$$

I claim that the number of 'even-sum' parts is odd, hence the total number of '-1's is even as claimed. In this case $n \not\equiv \sum \ell_i \pmod 2$, so just check n odd or even separately.

Suppose n is odd, then $\sum \ell_i$ is even. By counting the number n of ℓ_i, we see that $\mathbf{p} \in \mathrm{Part}_{\mathrm{odd}}(\ell_i)$ has an odd number of parts. If an even number of parts have even sum, we would obtain $\sum \ell_i$ odd.

Finally n is even, so $\sum \ell_i$ is odd. By counting the number n of ℓ_i, we see that $\mathbf{p} \in \mathrm{Part}_{\mathrm{odd}}(\ell_i)$ has an even number of parts. If an even number of parts have even sum, we would obtain $\sum \ell_i$ even.

In all cases the overall number of '-1's is even and we can drop the -1 from the definition of $\widetilde{\zeta}$, to obtain the required result. This completes the proof of Theorem 1.5. $\square$

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References

[1] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisoněk. Combinatorial aspects of multiple zeta values. Electron. J. Combin. **5** (1998), #R38.

[2] F. Brown. Mixed Tate motives over $\mathbb{Z}$. Ann. of Math. (2) **175**(2) (2012), 949–976.

[3] F. C. S. Brown. On the decomposition of motivic multiple zeta values. In Galois-Teichmüller Theory and Arithmetic Geometry (Advanced Studies in Pure Mathematics, 63), Mathematical Society of Japan, 2012, pp. 31–58.

[4] S. Charlton. $\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m)/\pi^{4n+2m(2n+1)}$ is rational. J. Number Theory **148** (2015), 463–477.

[5] S. Charlton. The alternating block decomposition of iterated integrals, and cyclic insertion on multiple zeta values. Preprint, 2017, arXiv:1703.03784.

[6] M. Hirose and N. Sato. Hoffman's conjectural identity. Int. J. Number Theory **15**(1) (2019), 167–171.

[7] M. Hoffman. Multiple harmonic series. Pacific J. Math. **152**(2) (1992), 275–290.

[8] K. Imatomi, T. Tanaka, K. Tasaka and N. Wakabayashi. On some combinations of multiple zeta-star values. Acta Humanistica et Scientifica Universitatis Sangio Kyotiensis (Natural Science Series, 42). Kyoto Sangyo University, 2013, pp. 1–20.

[9] J. Zhao. Identity families of multiple harmonic sums and multiple zeta star values. J. Math. Soc. Japan **68**(4) (2016), 1669–1694.

[10] S. A. Zlobin. Generating functions for the values of a multiple zeta function. Moscow Univ. Math. Bull. **60**(2) (2005), 44–48.

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