

Intrinsic Sensitivity Limits for Multiparameter Quantum Metrology

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The quantum Cramér-Rao bound is a cornerstone of modern quantum metrology, as it provides the ultimate precision in parameter estimation. In the multiparameter scenario, this bound becomes a matrix inequality, which can be cast to a scalar form with a properly chosen weight matrix. Multiparameter estimation thus elicits tradeoffs in the precision with which each parameter can be estimated. We show that, if the information is encoded in a unitary transformation, we can naturally choose the weight matrix as the metric tensor linked to the geometry of the underlying algebra $\mathfrak{su}(n)$. This ensures an intrinsic bound that is independent of the choice of parametrization.

Introduction.— A central challenge in quantum metrology is to develop strategies for enhancing the precision of parameter estimation. The quantum Fisher information (QFI), and the associated quantum Cramér-Rao bound (QCRB), are invaluable tools for this task [1, 2], as they characterize the ultimate precision attainable for different classes of probe states. This is crucial to identifying quantum resources that lead to an enhancement in sensitivity versus classical strategies.

It is well established that the maximal sensitivities achievable for the estimation of a single parameter using particle-separable and arbitrary quantum probe states are the shot noise and Heisenberg limits, respectively [3]. However, much less is known about the corresponding bounds for the simultaneous estimation of multiple parameters. Multiparameter quantum metrology finds many important applications in fields as diverse as imaging [4–6], field sensing [7–9], sensor networks [10–12], and remote sensing [13] to cite but a few examples. In this case, the QCRB is a matrix inequality and the ultimate bound is generally not saturable for all parameters. This is because the corresponding optimal observables may be incompatible [14–16] and one cannot simultaneously achieve the optimal precision for each individual parameter. Several comprehensive reviews highlight recent progress in this subject area [17–22].

These difficulties have fueled the search for tighter bounds, which prove to be handy for practical implementations. The Holevo Cramér-Rao bound (HCRB) [2] epitomizes the conundrums associated with incompatible observables: its main tenet is to map the matrix QCRB onto a *scalar* inequality by using a positive-definite weight matrix and then optimize this scalar bound over all physically viable measurement procedures for a given probe state. In this manner, one obtains a weighted mean square error that has to be minimized. This is considered hard to evaluate, even numerically, because it is defined through a constrained minimization over a set of operators. Closed results are known only for very simple models [23–25] and the numerical tractability of calculating the HCRB for finite-dimensional systems has been recently considered [26, 27].

Surprisingly, little attention has been paid to the properties of the weight matrix: its only role is to give relative weights to

the different parameter estimate variances, so different choices lead to different optimal probe states and experimental designs. In this Letter, we discuss whether that matrix can be chosen in a *natural* way.

To provide a proper answer, we first assume that the parameters are imprinted onto the probe state via a unitary transformation; i.e., we assume that the parameters belong to the group $SU(n)$. From a geometrical viewpoint, the associated Lie algebra $\mathfrak{su}(n)$ can be endowed with a metric tensor, the Killing-Cartan form, with all the desirable properties [28]. When this metric is used as our weight matrix, we obtain a QCRB with intrinsic properties, independent of the choice of parametrization. Such a QCRB can significantly facilitate finding the corresponding optimal states. We demonstrate the power of this bound for the broad scenario of estimating all of the parameters of an $SU(n)$ operation, which generalizes known multiparameter estimation results for rotation [9] and multiphase [29] estimation problems.

Setting the scenario.— We are interested in estimating multiple parameters simultaneously. The typical scheme requires some d -dimensional vector of parameters $\theta = (\theta_1, \dots, \theta_d)^\top \in \mathbb{R}^d$ to be imprinted on a (pure) probe state $|\psi\rangle$, which is shifted by applying a corresponding unitary transformation $U(\theta) \in SU(n)$ that encodes the parameters θ (the superscript \top denotes the transpose). The output state $|\psi_\theta\rangle = U(\theta)|\psi\rangle$ is then detected via a set of measurements, represented by a positive operator-valued measure (POVM) [1] $\{\Pi_x\}$, where the POVM elements are labeled by an index x that represents the possible outcomes (discrete or continuous) according to Born's rule $p(x|\theta) = \langle\psi_\theta|\Pi_x|\psi_\theta\rangle$.

Often, the protocol is repeated N times using identical copies of the state. From the output results $\mathbf{x} = (x_1, \dots, x_N)^\top$, which we assume to be independent and identically distributed, one can construct a joint probability distribution $p(\mathbf{x}|\theta) = \prod_{j=1}^N p(x_j|\theta)$ and then infer the vector parameter via an estimator $\hat{\theta}$. In the following, we restrict our attention to locally unbiased estimators, for which $\langle\hat{\theta}_j\rangle = \theta_j$ and $d\langle\hat{\theta}_j\rangle/d\theta_k = \delta_{jk}$, and we compute average values $\langle\cdot\rangle$ using the probability distribution $p(\mathbf{x}|\theta)$ [1].

The performance of the estimator can be properly assessed

in terms of the covariance matrix $\mathbf{C}_\psi(\hat{\boldsymbol{\theta}})$, defined as

$$[\mathbf{C}_\psi(\hat{\boldsymbol{\theta}})]_{jk} = \langle (\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k) \rangle, \quad (1)$$

where $j, k = 1, \dots, d$. The diagonal elements are the variances of the different parameters, whereas the nondiagonal elements characterize the possible correlations between various parameters.

To guide the design of real experiments, it is possible to calculate theoretical lower bounds for $\mathbf{C}_\psi(\hat{\boldsymbol{\theta}})$. The ultimate limit for any possible POVM is given by the time-honoured QCRB, which stipulates that

$$\mathbf{C}_\psi(\hat{\boldsymbol{\theta}}) \geq \mathbf{Q}_\psi^{-1}(\boldsymbol{\theta}), \quad (2)$$

where the matrix inequality $\mathbf{A} \geq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is a positive semidefinite matrix. Here, the quantum Fisher information matrix (QFIM) for pure states and unitary evolution takes the particularly simple form [18]

$$[\mathbf{Q}_\psi(\boldsymbol{\theta})]_{jk} = 4 \mathbf{C}_\psi(H_j, H_k). \quad (3)$$

The operators H_j are the generators of the transformation, determined through $H_j = iU^\dagger(\boldsymbol{\theta})\partial_{\theta_j}U(\boldsymbol{\theta})$, and we define the symmetrized covariance between two operators as $\mathbf{C}_\psi(A, B) = \frac{1}{2}\langle AB + BA \rangle - \langle A \rangle \langle B \rangle$ and expectation values with respect to $|\psi\rangle$.

Given a covariance matrix, we can balance the precision of the various parameters by using a weight matrix $\mathbf{W} > 0$; this leads to the inequality

$$\text{wMSE}(\hat{\boldsymbol{\theta}}) \equiv \text{Tr}[\mathbf{W}\mathbf{C}_\psi(\hat{\boldsymbol{\theta}})] \geq \text{Tr}[\mathbf{W}\mathbf{Q}_\psi^{-1}(\boldsymbol{\theta})] \equiv C_S(\boldsymbol{\theta}). \quad (4)$$

The left-hand side is the so-called weighted mean square error of the estimator wMSE, whereas $C_S(\boldsymbol{\theta})$ is a scalar cost function, much in the same spirit as the risk functions used in Bayesian estimation [30]. For a given \mathbf{W} , the standard approach is to minimize $C_S(\boldsymbol{\theta})$ to derive better parameter estimates. In contradistinction, we address here whether there is an intrinsic choice for the matrix \mathbf{W} .

Intrinsic bound.— Let us assume that the unitary process can be represented as $U(\boldsymbol{\theta}) = \exp[i\boldsymbol{\Omega}_{(\boldsymbol{\theta})} \cdot \mathbf{X}]$, where the vector $\boldsymbol{\Omega}_{(\boldsymbol{\theta})}$ encodes the pertinent set of parameters $\boldsymbol{\theta}$ and $\{X_i\}$ are the generators of $\text{SU}(n)$; that is, $\{X_i\}$ comprise a basis of the Lie algebra $\mathfrak{su}(n)$, which consists of traceless Hermitian $n \times n$ complex matrices with $i[\cdot, \cdot]$ as their Lie bracket [31] ($[\cdot, \cdot]$ denoting the regular commutator). For our final results to hold, we require that the vector $\boldsymbol{\Omega}_{(\boldsymbol{\theta})}$ is a smooth injective function of the parameters $\boldsymbol{\theta}$.

A straightforward method for calculating $\partial_{\theta_j}U(\boldsymbol{\theta})$ is through Wilcox's formula [32] for a unitary operator $U = \exp(A)$:

$$\frac{\partial U}{\partial \lambda} = \int_0^1 \exp[(1-\beta)A] \frac{\partial A}{\partial \lambda} \exp(\beta A) d\beta. \quad (5)$$

By recalling the definition of the generators, we find

$$H_j = -\frac{\partial \boldsymbol{\Omega}_{\boldsymbol{\theta}}}{\partial \theta_j} \cdot \int_0^1 U^{-\beta} \mathbf{X} U^\beta d\beta. \quad (6)$$

Since $U^{-1} \mathbf{X} U \in \mathfrak{su}(n)$, we conclude that H_j can be generically expressed as a linear combination of the generators $\{X_i\}$ through

$$H_j = \mathbf{h}_j \cdot \mathbf{X}, \quad (7)$$

where the real vectors of coefficients $\mathbf{h}_j(\boldsymbol{\theta})$ are computed from Eq. (6) and depend on $\boldsymbol{\Omega}_{\boldsymbol{\theta}}$. The far-reaching result that the generators H_j of the transformation are equal to linear combinations of the generators $\{X_i\}$ is essential to finding an intrinsic QCRB that is independent of parametrization. Moreover, this relation will always hold for the same vectors \mathbf{h}_j regardless of the representation of the group that we use.

In this way, we can immediately work out a compact expression for the QFIM:

$$\mathbf{Q}_\psi(\boldsymbol{\theta}) = 4\mathbf{H}^\top(\boldsymbol{\theta}) \mathbf{C}_\psi(\mathbf{X}) \mathbf{H}(\boldsymbol{\theta}) \quad (8)$$

and the scalar QCRB now reads

$$\text{Tr}[\mathbf{W}\mathbf{C}_\psi(\hat{\boldsymbol{\theta}})] \geq \frac{1}{4} \text{Tr}[\mathbf{H}^{-1\top}(\boldsymbol{\theta}) \mathbf{W} \mathbf{H}^{-1}(\boldsymbol{\theta}) \mathbf{C}_\psi^{-1}(\mathbf{X})]. \quad (9)$$

The remarkable property of these expressions is that we have separated the parameter dependence $\mathbf{H}(\boldsymbol{\theta}) = (\mathbf{h}_1, \dots, \mathbf{h}_d)^\top$ from the state dependence that is embodied in $[\mathbf{C}_\psi(\mathbf{X})]_{jk} = \mathbf{C}_\psi(X_j, X_k)$. This form allows us to easily identify when the QFIM becomes singular, which implies that all d parameters cannot be simultaneously estimated for some probe state and parametrization. For example, when $\mathbf{C}_\psi(\mathbf{X})$ is singular, the probe state will never be useful for estimating all d parameters, while, when \mathbf{H} is singular, the coordinate system defined by the parametrization is singular at that specific set of parameters $\boldsymbol{\theta}$ regardless of the probe state. As discussed in Ref. [33], singularities in one coordinate system can be alleviated for specific parameters by switching to a new coordinate system; here, we show how a proper choice of weight matrix \mathbf{W} removes all ambiguities that can arise from the choice of coordinate system.

To proceed, we note that the parameters $\boldsymbol{\theta}$, are coordinates of the group manifold (the parameters $\boldsymbol{\Omega}_{(\boldsymbol{\theta})}$ form an alternative set of coordinates of the manifold), which is compact and simply connected [31]. One can define therein a local metric tensor and through that covariant operations and invariant quantities. As $\mathfrak{su}(n)$ is compact and semisimple, the Killing form $B(X, Y) = 2n \text{Tr}(X^\dagger Y)$ is nondegenerate and defines a scalar product (X, Y) that, in turn, induces a natural metric $ds^2 = (dU, dU)$, called the Cartan metric. Actually, by expressing the metric as

$$ds^2 = \sum_{jk} g_{jk} d\theta_j d\theta_k \quad (10)$$

and computing $dU = -iU \sum_j H_j d\theta_j$, we find that the metric explicitly depends on the generators from Eq. (7) through

$$g_{jk} = \text{Tr}(H_j H_k). \quad (11)$$

We can then use the orthonormality of the generators $(X_j, X_k) = \delta_{ij}$ to determine the components of the metric

tensor to be (up to an inessential global constant)

$$g_{jk} = \mathbf{h}_j \cdot \mathbf{h}_k, \quad (12)$$

or, in vector form, $\mathbf{g} = \mathbf{H}^\top \mathbf{H}$. It follows that, if we choose the Cartan metric as the weight matrix $\mathbf{W} = \mathbf{g}$, all of the parametrization dependence cancels from Eq. (9) and the QCRB becomes

$$\text{Tr}[\mathbf{g} \mathbf{C}_\psi(\hat{\boldsymbol{\theta}})] \geq \frac{1}{4} \text{Tr}[\mathbf{C}_\psi^{-1}(\mathbf{X})]. \quad (13)$$

The only remaining ingredients are the covariances of the generators of $\mathfrak{su}(n)$ with respect to the original state, or any state along the unitary orbit, as $\text{Tr}[\mathbf{C}_\psi^{-1}(\mathbf{X})] = \text{Tr}[\mathbf{C}_{\psi'}^{-1}(\mathbf{X})]$, with $|\psi'\rangle = U|\psi\rangle$. This is exactly what one requires physically and seems quite elegant. It nicely complements the single-parameter scenario in which the QFI for unitary operations takes the same value anywhere along the unitary orbit.

Discussion.— To see how our new result works, let us first consider the simplest case of $\text{SU}(2)$, which describes rotations. A general element is characterized by the Euler angle parametrization

$$U(\Phi, \Theta, \Psi) = e^{-i\Phi J_z} e^{-i\Theta J_y} e^{-i\Psi J_z}, \quad (14)$$

where we have used the standard angular momentum notation \mathbf{J} for the generators. Derivatives with respect to the parameters lead us to

$$\begin{aligned} \mathbf{h}_\Phi &= (0, 0, 1)^\top, \\ \mathbf{h}_\Theta &= (-\sin \Phi, \cos \Theta, 0)^\top, \\ \mathbf{h}_\Psi &= (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)^\top, \end{aligned} \quad (15)$$

which makes the matrix \mathbf{H} singular when, for example, $\Theta = 0$. The metric for this parametrization is

$$ds^2 = d\Phi^2 + 2 \cos \Theta d\Phi d\Psi + d\Theta^2 + d\Psi^2, \quad (16)$$

which directly cancels the singularities in \mathbf{H} through $\mathbf{H}^{-1\top} \mathbf{g} \mathbf{H}^{-1} = \mathbb{1}$, as promised.

It is straightforward to show that in this case $\mathbf{C}_\psi(\mathbf{J})$ is singular if and only if the probe state is an eigenstate of some angular momentum projection; that is, proportional to $U(\boldsymbol{\theta})|Jm\rangle$. States with any definite angular momentum projection cannot be used for simultaneously estimating all three parameters of a rotation.

To find the most sensitive states we have to minimize $\text{Tr}[\mathbf{C}_\psi^{-1}(\mathbf{J})]$. This is straightforward to optimize because, for any symmetric, positive semidefinite, invertible matrix \mathbf{M} , $\text{Tr}(\mathbb{1})^2 = \text{Tr}(\mathbf{M}^{1/2} \mathbf{M}^{-1/2})^2 \leq \text{Tr}(\mathbf{M}) \text{Tr}(\mathbf{M}^{-1})$, with equality if and only if \mathbf{M} is proportional to the identity matrix. Since $\text{Tr}[\mathbf{C}_\psi(\mathbf{J})] = J(J+1) - \sum_k \langle J_k \rangle^2$, we find

$$\text{Tr}[\mathbf{C}_\psi^{-1}(\mathbf{J})] \geq \frac{9}{J(J+1)}, \quad (17)$$

with the trace of the inverse achieving the minimum only when the state is first-order unpolarized [34] (that is, $\langle \mathbf{J} \rangle = 0$), and

when $\mathbf{C}_\psi(\mathbf{J}) \propto \mathbb{1}$, as is the case for the so called Kings of Quantumness [35–37]. We see that having isotropic angular momentum up until second order makes a state most sensitive to arbitrary rotations about arbitrary axes. Our minimum intrinsic QCRB is given by the square of the number of parameters divided by the quadratic Casimir invariant $\mathcal{C}_2 = J_x^2 + J_y^2 + J_z^2 = J(J+1)\mathbb{1}$.

The conditions guaranteeing the saturation of the QCRB constitute a touchy business [38]. Fortunately, for the case of pure states, a sufficient condition is $\langle [H_j, H_k] \rangle = 0$, $\forall j, k$. These expectation values will vanish for all states with $\langle \mathbf{J} \rangle = 0$ so the optimal state will have a saturable QCRB in this case.

These results can be directly extended to the case of a full $\text{SU}(n)$ estimation. Using the same trick, we get that $\text{Tr}[\mathbf{C}_\psi(\mathbf{X})]$ is bounded by the quadratic Casimir invariant $\mathcal{C}_2 = \sum_{jk} g_{jk} X_j X_k$ of $\text{SU}(n)$ and, therefore, the second-order unpolarized states

$$\langle \mathbf{X} \rangle = 0, \quad \mathbf{C}_\psi(\mathbf{X}) = \frac{\mathcal{C}_2}{d}, \quad (18)$$

saturate the optimal QCRB for the simultaneous estimation of all d parameters. This yields a minimum *total* variance d^2/\mathcal{C}_n , generalizing Eq. (17). All of the parameters should be simultaneously estimable with Heisenberg-scaling precisions, because the square root of the Casimir invariant $\sqrt{\mathcal{C}_2}$, which corresponds to physical quantities such as the number of particles, is quadratic in the labels of the irreducible representation.

Such a protocol outperforms sequential estimation protocols by a factor on the order of d . Sequential estimation protocols require splitting the physical resources $\mathcal{O}(\sqrt{\mathcal{C}_2})$ into d parts, then at best estimating *each* of the d parameters with variance $\mathcal{O}[(\sqrt{d}/\mathcal{C}_2)^2]$, so that the *total* variance scales as $\mathcal{O}(d^3/\mathcal{C}_2)$. This has been seen for the specific examples of multiphase estimation [29], which satisfies $U(1)^{\otimes n} \subset \text{SU}(n)$ up to a global phase (i.e., modulo the centre of the group), and rotation estimation [9], which satisfies $\text{SU}(2) \subset \text{SU}(n)$. Estimating the components of any subset of $\text{SU}(n)$ can now be similarly optimized using our results.

Two final comments pertain. First, it is easy to realize that the elegant form (8) for the QFIM also holds for mixed states, where we must employ the replacement [39]

$$\mathbf{C}_\psi(\mathbf{X}) \mapsto \frac{1}{2} \int_0^\infty ds \text{Tr} [(\varrho \mathbf{X} - \mathbf{X} \varrho) e^{-\varrho s} (\mathbf{X}^\top \varrho - \varrho \mathbf{X}^\top) e^{-\varrho s}], \quad (19)$$

so the parameter dependencies factor out in terms of $\mathbf{H}(\boldsymbol{\theta})$ and will always disappear in the QCRB if we weight the covariances using the metric. Second, the treatment can be repeated much in the same way when the information is encoded in channels with symmetries beyond $\text{SU}(n)$.

Concluding remarks.— We have used geometric arguments to show how to obtain an intrinsic QCRB for unitary process. This directly generalizes many earlier results for multiparameter estimation, notably generalizing the single-parameter estimation result that estimation precision is unchanged along a unitary orbit. An important advantage of these geometric argu-

ments is that they provide deep connections between quantum estimation and general quantum information tasks.

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