Dihedral evaluations of hypergeometric functions with the Kleinian projective monodromy

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Abstract

Algebraic hypergeometric functions can be compactly expressed as radical or dihedral functions on pull-back curves where the monodromy group is much simpler. This article considers the classical ${}_3\mathrm{F}_2$-functions with the projective monodromy group $\mathrm{PSL}(2,\mathbb{F}_7)$ and their pull-back transformations of degree 21 that reduce the projective monodromy to the dihedral group D_4 of 8 elements.

1 Introduction

One way to obtain a workable expression for an algebraic hypergeometric function is to pull-back it to an algebraic curve where the (finite) monodromy group would be simpler, say, a finite cyclic group [Vid13]. For example,

$$ {}_2\mathrm{F}_1\left(\left. \begin{array}{c} 5/24,\ 13/24 \\ 5/4 \end{array} \right| \frac{108x\,(x-1)^4}{(x^2+14x+1)^3} \right) = \frac{1}{1-x}\,\left(1+14x+x^2\right)^{5/8}, \tag{1} $$

$$ {}_2\mathrm{F}_1\left(\left. \begin{array}{c} 1/6,\ 5/6 \\ 5/4 \end{array} \right| \frac{27x\,(x+1)^4}{2(x^2+4x+1)^3} \right) = \frac{\left(1+2x\right)^{1/4}}{1+x}\,\sqrt{1+4x+x^2} \tag{2} $$

around $x=0$. Here the ${}_2\mathrm{F}_1$-functions have the octahedral group $\cong S_4$ as the projective monodromy group (of the hypergeometric differential equation). The rational arguments of degree 6 reduce the monodromy to small cyclic groups, as evidenced by the radical (i.e., algebraic power) functions on the right-hand sides of these identities.

If a Fuchsian differential equation E on the Riemann sphere $\mathbb{CP}^1$ has an algebraic solution f, then E can be transformed by a pull-back transformation with respect to an algebraic covering $\varphi : B \to \mathbb{CP}^1$ so that f becomes a rational or radical solution on the curve B. The monodromy representation of the transformed equation then has an invariant subspace generated by f, and φ is a *Darboux covering* as defined in [Vid13]. The explicit expression of f as a radical function on B is called a *Darboux evaluation* of f. In [Vid13], all tetrahedral,

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octahedral and icosahedral Schwarz types [Sch73] of algebraic ${}_2\mathrm{F}_1$-functions are exemplified by Darboux evaluations.

Reduction of a finite monodromy group to a dihedral (rather than cyclic) group is worth attention as well. The degree of the pull-back covering would be generally smaller, and dihedral expressions are still compact and practically workable. For example, a dihedral expression of octahedral function (1) is obtained after a cubic transformation [Vid09, (21)]:

$$\begin{aligned}
{}_2\mathrm{F}_1\left(\begin{matrix} 5/24,\ 13/24 \\ 5/4 \end{matrix} \,\middle|\, \frac{27x\,(x-1)^2}{(3x+1)^3} \right) &= (1+3x)^{5/8}\, {}_2\mathrm{F}_1\left(\begin{matrix} 5/8,\ 9/8 \\ 5/4 \end{matrix} \,\middle|\, x \right) \\
&= \frac{(1+3x)^{5/8}}{\sqrt{1-x}}\, {}_2\mathrm{F}_1\left(\begin{matrix} 5/8,\ 1/8 \\ 5/4 \end{matrix} \,\middle|\, x \right) \\
&= \frac{(1+3x)^{5/8}}{\sqrt{1-x}} \left(\frac{1+\sqrt{1-x}}{2} \right)^{-1/4}.
\end{aligned} \tag{3}$$

Here the second ${}_2\mathrm{F}_1$-function has a dihedral monodromy group, and is converted using standard transformations [Vid09, (17), (2)]. It can be evaluated directly using [Vid11, (3.1) with $k=-1$].

Algebraic generalized hypergeometric functions ${}_p\mathrm{F}_{p-1}$ are classified by Beukers and Heckman [BH89]. One particularly interesting case [Kat11], [vdPU00] is algebraic ${}_3\mathrm{F}_2$-functions such that the projective monodromy group (of their third order Fuchsian equations) is the simple group

$$\Lambda = \mathrm{PSL}(2,\mathbb{F}_7) \cong \mathrm{GL}(3,\mathbb{F}_2) \tag{4}$$

with 168 elements. Third order Fuchsian equations with this projective monodromy group were anticipated by Klein [Kle79, a footnote in §9], and first constructed by Halphen [Hal84] and Hurwitz [Hur86].

In [BH89, Table 8.3], classes of ${}_3\mathrm{F}_2$-functions with the projective monodromy group Λ are labelled by the numbers 2, 3, 4. The customary monodromy group (inside $\mathrm{GL}(3,\mathbb{C})$) of their Fuchsian equations is the complex reflection group ST24 in the Shephard–Todd classification [ST54], isomorphic to the central extension $\Lambda \times (\mathbb{Z}/2\mathbb{Z})$. A classification up to contiguous relations of (Fuchsian equations for) ${}_3\mathrm{F}_2$-functions with the projective monodromy group Λ is given in [Vid18, Proposition 2.1]. It give these six classes and representative ${}_3\mathrm{F}_2$-functions:

$$\begin{aligned}
&(3A): \quad {}_3\mathrm{F}_2\left(\begin{matrix} -\frac{3}{14}, \frac{1}{14}, \frac{9}{14} \\ \frac{1}{3}, \frac{2}{3} \end{matrix} \,\middle|\, z \right); \qquad && (3B): \quad {}_3\mathrm{F}_2\left(\begin{matrix} -\frac{1}{14}, \frac{3}{14}, \frac{5}{14} \\ \frac{1}{3}, \frac{2}{3} \end{matrix} \,\middle|\, z \right); \\
&(4A): \quad {}_3\mathrm{F}_2\left(\begin{matrix} -\frac{3}{14}, \frac{1}{14}, \frac{9}{14} \\ \frac{1}{4}, \frac{3}{4} \end{matrix} \,\middle|\, z \right); \qquad && (4B): \quad {}_3\mathrm{F}_2\left(\begin{matrix} -\frac{1}{14}, \frac{3}{14}, \frac{5}{14} \\ \frac{1}{4}, \frac{3}{4} \end{matrix} \,\middle|\, z \right); \\
&(7A): \quad {}_3\mathrm{F}_2\left(\begin{matrix} -\frac{1}{14}, \frac{1}{14}, \frac{5}{14} \\ \frac{1}{7}, \frac{5}{7} \end{matrix} \,\middle|\, z \right); \qquad && (7B): \quad {}_3\mathrm{F}_2\left(\begin{matrix} -\frac{1}{14}, \frac{1}{14}, \frac{9}{14} \\ \frac{2}{7}, \frac{6}{7} \end{matrix} \,\middle|\, z \right).
\end{aligned} \tag{5}$$

Equations of type (3A) are directly related to the modular curve $\mathcal{X}(7)$, and to Klein's quadric curve

$$X^3Y + Y^3Z + Z^3X = 0. \tag{6}$$

This is a Riemann surface of genus $g = 3$, with the group of holomorphic symmetries isomorphic to Λ.

As shown in [Vid18], the projective monodromy Λ of considered Fuchsian equations can be reduced to $\mathbb{Z}/7\mathbb{Z}$ by pull-back transformations of degree $\#\Lambda/7 = 24$. The monodromy representation of transformed equations is completely reducible, hence the pulled-back equations have a basis of radical (i.e., algebraic power) solutions. This gives Darboux evaluations for all solutions of a considered Fuchsian equation in terms of the basis radical solutions.

This article presents Darboux coverings of degree 21 that reduce the projective monodromy Λ to the dihedral group D_4 with 8 elements. The dihedral group is a 2-Sylow subgroup of Λ; see [Elk98, p. 66, 92]. The degree 21 covering exists by the Galois correspondence associated to degree 168 Galois coverings with the monodromy Λ.

Let E_0, E_1 denote third order differential equations with the projective monodromies Λ, D_4, respectively, that and related by a pull-back transformation of degree 21. The customary monodromy group of E_1 is a central extension of D_4, thus a dihedral group as well. As there are no irreducible 3-dimensional representations of dihedral groups [Keo75], the monodromy representation of E_1 is reducible. Its one-dimensional invariant subspace gives a radical solution of E_1, hence a Darboux evaluation of a solution of E_0. The two-dimensional invariant subspace of the monodromy of E_1 leads to *dihedral evaluations* of solutions of E_0. This article presents Darboux and dihedral evaluations of representative ${}_3\mathrm{F}_2$-functions with respect to the degree 21 Darboux coverings.

2 Preliminaries

Section 2.1 recalls basic knowledge about differential equations for ${}_3\mathrm{F}_2$-functions, their pull-back transformations, and contiguous relations. Section 2.2 characterizes the classification 5 of ${}_3\mathrm{F}_2$-functions with the projective monodromy Λ. Section 2.3 introduces Darboux coverings following [Vid13] and [Vid18, §2.4].

2.1 Hypergeometric functions

The hypergeometric function ${}_3\mathrm{F}_2\left(\left.\begin{array}{c}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{array}\right|\, z\right)$ satisfies the differential equation

$$\begin{aligned}\left(z\frac{d}{dz}+\alpha_1\right)\left(z\frac{d}{dz}+\alpha_2\right)\left(z\frac{d}{dz}+\alpha_3\right)Y(z)\qquad\qquad&\\ =\frac{d}{dz}\left(z\frac{d}{dz}+\beta_1-1\right)\left(z\frac{d}{dz}+\beta_2-1\right)Y(z),&\end{aligned} \tag{7}$$

minding the commutativity rule $\frac{d}{dz}z = z\frac{d}{dz}+1$. This is a third order Fucshian equation with three singular points $z=0$, $z=1$, $z=\infty$. The singularities and local exponents at them are encoded by the generalized Riemann's P-symbol:

$$P\left(\begin{array}{ccc|c} z=0 & z=1 & z=\infty & \\ \hline 0 & 0 & \alpha_1 & \\ 1-\beta_1 & 1 & \alpha_2 & z \\ 1-\beta_2 & \gamma & \alpha_3 & \end{array}\right) \tag{8}$$

with $\gamma=\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2$. Generically, a basis of local solutions at $z=0$ and $z=\infty$ can be written in terms of ${}_3\mathrm{F}_2$-series. Let $M=(m_{i,j})$ denote the matrix

$$M=\left(\begin{array}{ccc} \alpha_1 & \alpha_1-\beta_2+1 & \alpha_1-\beta_1+1 \\ \alpha_2 & \alpha_2-\beta_2+1 & \alpha_2-\beta_1+1 \\ \alpha_3 & \alpha_3-\beta_2+1 & \alpha_3-\beta_1+1 \end{array}\right). \tag{9}$$

A generic basis of local solutions at $z=0$ is

$$\begin{gathered} {}_3\mathrm{F}_2\left(\left.\begin{array}{c} m_{1,1}\,,\, m_{2,1}\,,\, m_{3,1} \\ \beta_1,\ \beta_2 \end{array}\right| z\right), \quad z^{1-\beta_2}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} m_{1,2}\,,\, m_{2,2}\,,\, m_{3,2} \\ 2-\beta_2,\ \beta_1-\beta_2+1 \end{array}\right| z\right), \\ z^{1-\beta_1}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} m_{1,3}\,,\, m_{2,3}\,,\, m_{3,3} \\ 2-\beta_1,\ \beta_2-\beta_1+1 \end{array}\right| z\right), \end{gathered} \tag{10}$$

while a generic basis of local solutions at $z=\infty$ is

$$z^{-\alpha_j}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} m_{j,1}\,,\ m_{j,2}\,,\ m_{j,3} \\ \alpha_j-\alpha_k+1, \alpha_j-\alpha_\ell+1 \end{array}\right| z\right) \tag{11}$$

with $j\in\{1,2,3\}$ and $\{k,\ell\}=\{1,2,3\}\setminus\{j\}$. We refer to the set of 6 functions formed by 3 local hypergeometric solutions (disregarding a power factor) at $z=0$ and 3 such local solutions at $z=\infty$ of the same third order Fuchsian equation as *companion hypergeometric functions* to each other.

Let B denote an algebraic curve. Let $\varphi(\ldots)$ denote a rational function on B; it defines an algebraic covering $\varphi: B\to\mathbb{CP}^1$. A *pull-back transformation* with respect to φ of a differential equation for $y(z)$ in d/dz has the form

$$z\longmapsto\varphi(\ldots), \qquad y(z)\longmapsto Y(\ldots)=\theta(\ldots)\,y(\varphi(\ldots)), \tag{12}$$

where $\theta(\ldots)$ is a radical function on B. The equations that differ by a pull-back transformation with respect to the trivial covering $\varphi(\ldots)=z$ are called *projectively equivalent*.

There are several algebraic transformations for ${}_3\mathrm{F}_2$-functions [Kat08]. Here are quadratic and cubic transformations:

$$\begin{aligned} &{}_3\mathrm{F}_2\left(\left.\begin{array}{c} a, a+\frac14, a+\frac12 \\ b+\frac14, 3a-b+1 \end{array}\right| -\frac{4z}{(z-1)^2}\right) \\ &\qquad\qquad = (1-z)^{2a}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} 2a, 2a-b+\frac34, b-a \\ b+\frac14, 3a-b+1 \end{array}\right| z\right), \end{aligned} \tag{13}$$

$$
{}_3\mathrm{F}_2\left(\left. \begin{matrix} a, a+\frac{1}{3}, a+\frac{2}{3} \\ b+\frac{1}{2}, 3a-b+1 \end{matrix} \right| \frac{27z^2}{(4-z)^3} \right) \tag{14}
$$
$$
= \left(1-\frac{z}{4}\right)^{3a} {}_3\mathrm{F}_2\left(\left. \begin{matrix} 3a, b, 3a-b+\frac{1}{2} \\ 2b, 6a-2b+1 \end{matrix} \right| z \right).
$$

They can be understood as pull-back transformations between Fuchsian equations (particularly, (7)) for hypergeometric functions [Vid09], [Kat08]. The involved equations have the local exponent $c = 1/2$ at $z = 1$, and the quadratic or cubic arguments $\varphi(z)$ on the left-hand sides have properly branching points in the fiber $\varphi = 1$. This helps the number of singularities of the pulled-back equation to equal merely 3.

Two ${}_3\mathrm{F}_2$-functions whose parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ differ respectively by integers are called *contiguous* to each other. This defines a *contiguity equivalence* relation on the ${}_3\mathrm{F}_2$-functions. For example, differentiating a ${}_3\mathrm{F}_2$-function gives a contiguous function, generically:

$$
\frac{d}{dz}\,{}_3\mathrm{F}_2\left(\left. \begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \right| z \right) = \frac{\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2}\,{}_3\mathrm{F}_2\left(\left. \begin{matrix} \alpha_1+1, \alpha_2+1, \alpha_3+1 \\ \beta_1+1, \beta_2+1 \end{matrix} \right| z \right), \tag{15}
$$

Fuchsian equations of contiguous functions have the same monodromy, generically. For a generic set of four contiguous ${}_3\mathrm{F}_2$-functions there is a linear *contiguous relation* between them [Rai45]. For example, differential equation (7) can be rewritten as a contiguous relation between ${}_3\mathrm{F}_2\left(\left. \begin{smallmatrix} \alpha_1+n,\, \alpha_2+n,\, \alpha_3+n \\ \beta_1+n,\, \beta_2+n \end{smallmatrix} \right| z \right)$ with $n \in \{0, 1, 2, 3\}$. Consequently, a contiguous function to a generic ${}_3\mathrm{F}_2$-function F can be expressed linearly in terms of F and its first and second derivatives (thus, as a gauge transformation). In particular, we have

$$
{}_3\mathrm{F}_2\left(\left. \begin{matrix} \alpha_1+1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \right| z \right) = \left(1+\frac{z}{\alpha_1}\frac{d}{dz}\right) {}_3\mathrm{F}_2\left(\left. \begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \right| z \right), \tag{16}
$$
$$
{}_3\mathrm{F}_2\left(\left. \begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1-1, \beta_2 \end{matrix} \right| z \right) = \left(1+\frac{z}{\beta_1-1}\frac{d}{dz}\right) {}_3\mathrm{F}_2\left(\left. \begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \right| z \right), \tag{17}
$$

2.2 Monodromy groups

Let E denote a 3rd order Fuchsian equation on the Riemann sphere $\mathbb{CP}^1$. Suppose that f_1, f_2, f_3 is a basis of its solutions. If either the (conventional) monodromy group or the differential Galois group [vdPU00] of E are finite, those two groups coincide with the classical Galois group of the finite field extension $\mathbb{C}(z, f_1, f_2, f_3) \supset \mathbb{C}$. In that case, the *projective monodromy group* refers to the the Galois group of the finite extension $\mathbb{C}(z, f_2/f_1, f_3/f_1) \supset \mathbb{C}(z)$. Both extensions of $\mathbb{C}(z)$ are Galois extensions, because the monodromy representation gives linear transformations of f_1, f_2, f_3 (in $\mathrm{GL}(3,\mathbb{C})$) or fractional-linear transformations of $f_2/f_1, f_3/f_1$ (in $\mathrm{PGL}(3,\mathbb{C})$).

Standard transformations that preserve the projective monodromy are:

(i) Gauge transformations of the contiguity equivalence;

(ii) Projective equivalence transformations;

(iii) Being companion hypergeometric functions (i.e., being solutions of the same third order Fuchsian equation);

(iv) Multiplying the hypergeometric parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \in \mathbb{Q}$ by an integer coprime to their denominators.

These transformations characterize the classification of algebraic hypergeometric functions in [BH89, Theorem 7.1].

The classification in (5) of ${}_3\mathrm{F}_2$-functions with the projective monodromy Λ is given up to the transformations *(i)–(iii)*; see [Vid18, Proposition 2.1]. Transformation *(iv)* relates the types (3A), (3B), or the types (4A), (4B), or the types (7A), (7B).

2.3 Darboux coverings

The notions of *Darboux curves*, *Darboux coverings* and *Darboux evaluations* are introduced in [Vid99, Ch. 4] and [Vid13]. The terminology is motivated by integration theory of vector fields [LZ09], where *Darboux polynomials* determine invariant hypersurfaces. In differential Galois theory [Wei95], Darboux polynomials are specified by algebraic solutions of an associated Riccati equation. Here is a formulation of [Vid13, Definition 3.1].

Definition 2.1. *Consider a linear homogeneous differential equation*

$$\frac{d^n}{dz^n} + a_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}} + \ldots + a_1(z)\frac{d}{dz} + a_0(z) = 0 \tag{18}$$

on $\mathbb{CP}^1$*, thus with* $a_i(z) \in \mathbb{C}(z)$*. We say that an algebraic covering* $\varphi : B \to \mathbb{CP}^1$ *is a* Darboux covering *for* (18) *if a pull-back transformation* (12) *of it with respect to* φ *has a solution* Y *such that:*

(a) the logarithmic derivative $u = Y'/Y$ *is a rational function on the algebraic curve* B*;*

(b) the algebraic degree of u *over* $\mathbb{C}(z)$ *equals the degree of* φ*.*

The algebraic curve B *is then called a* Darboux curve*.*

Condition *(a)* means that the monodromy representation of the pulled-back equation has a one-dimensional invariant subspace (generated by Y). Determination of Darboux coverings is made easier by their basic properties. The following lemma underlines that Darboux coverings are "invariant" under transformations of hypergeometric equations that preserve the monodromy.

Lemma 2.2. *Let* E_1 *denote a hypergeometric equation* (7) *with a finite primitive monodromy group. Suppose that other hypergeometric equation* E_2 *is related to* E_1 *by transformations described in (i), (iv) in §2.2. If* $\varphi : B \to \mathbb{CP}^1$ *is a Darboux covering for* E_1*, then* φ *is a Darboux covering for* E_2 *as well.*

Proof. The transformations *(i)*, *(iv)* do not affect the primitive monodromy group, thus equations E_1, E_2 have isomorphic monodromies. Let E_1^*, E_2^* denote the Fuchsian equations obtained from E_1, E_2, respectively, by applying the same pull-back transformation with respect to φ. The monodromies of E_1^*, E_2^* are isomorphic. Therefore, if one has a radical solution so does the other. $\square$

Corollary 2.3. *The same degree* 21 *Darboux coverings* φ *or* $1/\varphi$ *for reduction of the projective monodromy* $\Lambda = \mathrm{PSL}(2, \mathbb{F}_7)$ *of hypergeometric equation* (7) *to the dihedral group* D_4 *apply to all hypergeometric functions of the types* (3A) *and* (3B)*; or to all* ${}_3\mathrm{F}_2$*-functions of the types* (4A) *and* (4B)*; or to all* ${}_3\mathrm{F}_2$*-functions of the types* (7A) *and* (7B)*.*

Proof. Each of the six types describes an equivalence class under the contiguity equivalence. The pairs of types (3A), (3B); or (4A), (4B); or (7A), (7B) are related by transformation *(iv)* in §2.2. $\square$

To match hypergeometric functions with radical or dihedral solutions of a pulled-back Fuchsian equation, the next lemma is useful. It applies to equations obtained by the considered degree 21 pull-back transformations.

Lemma 2.4. *Suppose that differential equation* (18) *has a finite dihedral monodromy group. If the local exponents* $\lambda_1, \ldots, \lambda_n$ *at a point* $P \in \mathbb{CP}^1$ *are all different modulo* $\mathbb{Z}$*, then for each local exponent* λ_i *(*$i \in \{1, \ldots, n\}$*) there is exactly one (up to scalar multiplication) radical solution with the vanishing order* λ *at* P*.*

Proof. (Compare with [Vid18, Lemma 2.5].) A monodromy representation of the finite dihedral group reduced to a direct sum of 1-dimensional and 2-dimensional invariant subspaces [Keo75]. Each 1-dimensional subspace gives a radical solution, while each 2-dimensional space gives two independent solutions of the form $(p \pm q\sqrt{g})^e$. This gives a basis of the whole solution space. If two of those solutions had the same local exponent λ^* at P, their linear combination would have the vanishing order $\lambda^* + k$ with integer $k > 0$. Hence there is a basis solution for each of the n local exponents. $\square$

3 Three Darboux coverings

Here we compute the Darboux coverings of degree 21 that reduce the projective monodromy group $\Lambda = \mathrm{PSL}(2, \mathbb{F}_7)$ of third order hypergeometric equations (7) to the dihedral group D_4 of 8 elements. They turn out to be Belyi maps [LZ04] with the following branching patterns (and genus g):

$$\text{for the types (3A) and (3B):} \quad [7^3/3^7/2^8 1^5] \quad (g = 0); \tag{19}$$

$$\text{for the types (4A) and (4B):} \quad [7^3/4^4 2^2 1/2^8 1^5] \quad (g = 0); \tag{20}$$

$$\text{for the types (7A) and (7B):} \quad [7^3/7^3/2^8 1^5] \quad (g = 2). \tag{21}$$

As we will see, there is exactly one Darboux covering $\Psi_3 : \mathbb{P}^1 \to \mathbb{P}^1$ (up to Möbius transformations on either $\mathbb{P}^1$) with the branching pattern $[7^3/3^7/2^8 1^5]$. Just as the degree 24 Darboux coverings in [Vid18, §2.5], the degree 21 Darboux coverings are related to each other by quadratic and cubic transformations (13)–(14) of hypergeometric solutions. Consequently, there are unique Darboux coverings $\Psi_4 : \mathbb{P}^1 \to \mathbb{P}^1$ and $\Psi_7 : H_7 \to \mathbb{P}^1$ (up to holomorphic symmetries of the Darboux curves and $\mathbb{P}^1$) with the branching patterns in (20)–(21), respectively.

Let $\mathcal{K}$ denote Klein's curve (6). As is well known [Kat11, §5.1.1], hypergeometric equations of type (3A) are directly related to this curve. In particular, the identity [FM16, Proposition 30]

$$\begin{aligned} {}_3\mathrm{F}_2\left(\left.\begin{matrix}\frac{5}{42},\ \frac{19}{42},\ \frac{11}{14}\\ \frac{5}{7},\ \frac{8}{7}\end{matrix}\,\right|\,x\right)^3 {}_3\mathrm{F}_2\left(\left.\begin{matrix}-\frac{1}{42},\ \frac{13}{42},\ \frac{9}{14}\\ \frac{4}{7},\ \frac{6}{7}\end{matrix}\,\right|\,x\right) = \\ {}_3\mathrm{F}_2\left(\left.\begin{matrix}-\frac{1}{42},\ \frac{13}{42},\ \frac{9}{14}\\ \frac{4}{7},\ \frac{6}{7}\end{matrix}\,\right|\,x\right)^3 {}_3\mathrm{F}_2\left(\left.\begin{matrix}\frac{17}{42},\ \frac{31}{42},\ \frac{15}{14}\\ \frac{9}{7},\ \frac{10}{7}\end{matrix}\,\right|\,x\right) \\ + \frac{x}{1728}\,{}_3\mathrm{F}_2\left(\left.\begin{matrix}\frac{17}{42},\ \frac{31}{42},\ \frac{15}{14}\\ \frac{9}{7},\ \frac{10}{7}\end{matrix}\,\right|\,x\right)^3 {}_3\mathrm{F}_2\left(\left.\begin{matrix}\frac{5}{42},\ \frac{19}{42},\ \frac{11}{14}\\ \frac{5}{7},\ \frac{8}{7}\end{matrix}\,\right|\,x\right) \end{aligned} \tag{22}$$

gives a projective parametrization of $\mathcal{K}$ by hypergeometric functions, and the degree 168 Galois covering $\mathcal{K} \to \mathbb{P}^1$. The Darboux coverings of types (3A), (3B) can be found by investigating degree 21 subcoverings of the Galois covering. More concretely [Elk98, §4], the modular curve $\mathcal{X}(7)$ is isomorphic to $\mathcal{K}$, and we may look at the Galois covering $\mathcal{X}(7) \to \mathcal{X}(1)$. Its subcoverings are listed in the Cummins-Pauli tables [CP03] as corresponding level 7 congruence subgroups. There we find the congruence subgroup $7\mathrm{D}^0$ of index 21, with $c_2 = 5$, $c_3 = 0$, and the cusps 7^3. This gives the branching pattern $[7^3/3^7/2^8 1^5]$. Besides, the covering is unique (up to Möbius transformations) and defined over $\mathbb{Q}$, as the table entry in [CP03] indicates the conjugation orbit of size con $= 1$. There are no other entries of level 7 and index 21 (also no entries with the branching $[7^3/3^7/2^{10}1]$ of genus $g = 1$). Besides, $7\mathrm{D}^0$ is a subgroup of $7\mathrm{A}^0$. The larger congruence subgroup $7\mathrm{A}^0$ is of index 7, and gives the branching pattern $[2^2 1^3/3^2 1/7]$ (with con $= 2$). Hence, the degree 21 coverings are compositions of degree 7 Belyi maps with this branching and cubic coverings. The degree 7 maps are straightforward to compute [Vid09, Step 4 in §3]. They are defined over $\mathbb{Q}(\sqrt{-7})$, say:

$$\psi_7(z) = \frac{z}{1728}\left(z + \sqrt{-7}\right)^3 \left(z + \frac{7 + 5\sqrt{-7}}{2}\right)^3. \tag{23}$$

Then $\psi_7(z) - 1$ equals

$$\frac{z^2 + (2{+}4\sqrt{-7})z - 27}{1728}\left(z + \frac{9{+}5\sqrt{-7}}{2}\right)\left(z^2 + (2{+}2\sqrt{-7})z + \frac{-5{+}\sqrt{-7}}{2}\right)^2.$$

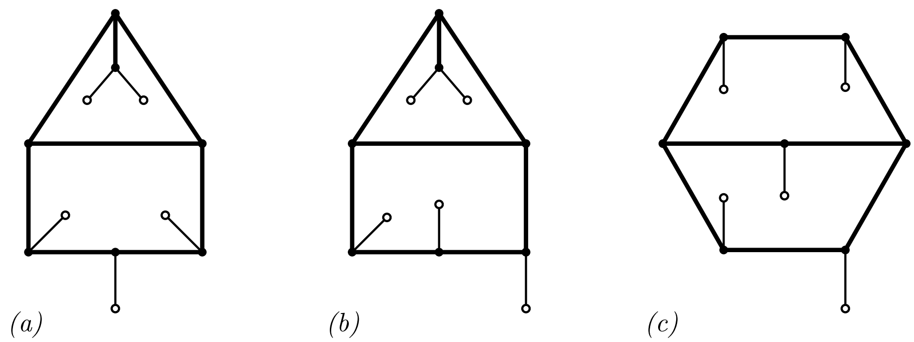

Figure 1: The dessins d'enfant with the branching pattern $[7^3/3^7/2^8 1^5]$.

To produce the branching $[7^3/3^7/2^8 1^5]$, the cubic covering must branch above $z = 0$ (with order 3) and above 2 of the 3 simple roots of $\psi_7(z) - 1$. The discriminant of $z^2 + (2 + 4\sqrt{-7})z - 27$ equals $16\sqrt{-7}$, hence this polynomial does not factor over $\mathbb{Q}(\sqrt{-7})$. We must have simple branching points above its roots to have a chance of obtaining a composition defined over $\mathbb{Q}$ as expected. A correct composition is

$$\Psi_3(x) = \psi_7\left(\frac{(4x - 7 - \sqrt{-7})^3}{(20 - 4\sqrt{-7})(x^3 - 7x + 7)} \right). \tag{24}$$

We obtain

$$\Psi_3 = \frac{x^3(3x^2-7)^3(2x^2-7x+7)^3(11x^2-35x+28)^3}{1728(x^3-7x+7)^7}. \tag{25}$$

A dessin d'enfant of this covering is depicted in Figure 1 *(a)*. The covering Ψ_3 was computed in [Vid99, §4.3] by lengthy computations with Maple starting from the branching pattern alone. It is computed in [Elk98, (4.35)] as well, with $x = 2\phi/(\phi+1)$. (The factor $5\phi^2 - 15\phi - 7$ there must be corrected to $5\phi^2 - 14\phi - 7$.)

Remark 3.1. Evidently, there is at least one other Galois orbit of Belyi maps with the same branching $[7^3/3^7/2^8 1^5]$. They are alternative compositions of Ψ_7 with a cubic covering. Those Belyi maps are defined over $\mathbb{Q}\big(\sqrt{\sqrt{-7}}\,\big)$, that is, the splitting field of the polynomial $z^2+(2+4\sqrt{-7})z-27$. The dessins d'enfant of these composite Belyi maps are represented (up to mirroring) in Figure 1 *(b)*, *(c)*. As in [vHV15], thick edges represent two dessin edges with a white vertex of order 2 in the middle. The Belyi maps are not quotients of the Galois covering $\mathcal{K} \to \mathbb{P}^1$ because their monodromy (i.e., the *cartographic group* [LZ04]) is not homomorphic to Λ but is a larger group of order $2^6 3^8 7$. Modern computations (following [vHV15]) show that there are no other Belyi maps with the branching pattern $[7^3/3^7/2^8 1^5]$.

By Corollary 2.3, the Belyi map Ψ_3 is a Darboux covering for type (3B) hypergeometric functions as well. To compute degree 21 Darboux coverings for the other types, we follow their algebraic correspondence relations to Ψ_3 that are consequences of quadratic and cubic transformations of ${}_3\mathrm{F}_2$-functions [Kat08]. In particular, representative ${}_3\mathrm{F}_2$-functions of types (3A) and (7A) are related by cubic transformation (14). For example,

$$ {}_3\mathrm{F}_2\left(\left. \begin{array}{c} -\frac{1}{42},\, \frac{13}{42},\, \frac{9}{14} \\ \frac{4}{7},\, \frac{6}{7} \end{array} \right| \frac{27z^2}{(4-z)^3} \right) = \left(1-\frac{z}{4}\right)^{-1/14} {}_3\mathrm{F}_2\left(\left. \begin{array}{c} -\frac{1}{14},\, \frac{1}{14},\, \frac{5}{14} \\ \frac{1}{7},\, \frac{5}{7} \end{array} \right| z \right). \tag{26} $$

The argument of the ${}_3\mathrm{F}_2$-function on the left equals $1/\Psi_3$ in its Darboux or dihedral evaluation (so that $\Psi_3 = 0$ and $\Psi_3 = \infty$ match correctly singularities of the hypergeometric equation). The Darboux covering Ψ_7 is therefore constructed by parametrizing the fiber product

$$ \frac{1}{\Psi_3(x)} = \frac{27z^2}{(4-z)^3} \tag{27} $$

(compare with [Vid13, Lemma 3.5]). Irreducibility of this equation means that the Darboux covering $\Psi_7 : H_7 \to \mathbb{P}^1$ is unique (up to holomorphic symmetries of both curves). To parametrize the fiber product by a simple equation, we substitute $z = 8\hat{z}/(2\hat{z}+3)$ so that the right-hand side becomes $\hat{z}^{\,2}(2\hat{z}+3)$. After the next substitution $\hat{z} = \tilde{z}P_3^2/(xQ_3)$ we get the equation

$$ 27P_3^3 = 49\tilde{z}^2(2\tilde{z}P_3^2 + 3xQ_3) \tag{28} $$

of degree 9 in $x, \tilde{z}$. This equation defines a curve of genus 2 isomorphic to

$$ v^2 = u\left(1 - \tfrac{63}{4}u + 91u^2 - 231u^2 + 224u^4\right). \tag{29} $$

This is our standard model of H_7. Eventually, we can find this parametrization of (27) by H_7:

$$ x = \frac{2v + (6u-1)(56u^2-21u+2)}{(4u-1)(56u^2-16u+1)}, \qquad z = \Psi_7, \tag{30} $$

where Ψ_7 is the Darboux curve:

$$ \Psi_7 = \frac{P_7^2\,(2v-3u+14u^2)^7}{u\,(1-4u)^7\,(-2v-3u+14u^2)^7} \tag{31} $$

with

$$ P_7 = v + \tfrac{7}{2}u - 35u^2 + 112u^3 - 128u^4. \tag{32} $$

As a function on H_7, the Belyi map in (27) has the branching $[14^3 7^3/3^{21}/2^{29}1^5]$ (see [Vid13, §A.7] for details). Comparing with the cubic covering $[2^1 1/3/2^1 1]$, the map Ψ_7 branches over the three points distinguished in $[2^1 1/\star/\star 1]$. It follows that Ψ_7 is a Belyi map with the branching pattern $[7^3/7^3/2^8 1^5]$. This can be checked by computing the divisors of Ψ_7 and Ψ_7-1 on H_7; see §4.5.

Representative ${}_3\mathrm{F}_2$-functions of types (4B) and (7B) are related by quadratic transformation (13). For example,

$$ {}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{28},\, \frac{3}{14},\, \frac{13}{28} \\ \frac{2}{7},\, \frac{6}{7} \end{array}\right| -\frac{4z}{(1-z)^2}\right) = (1-z)^{-1/14}\, {}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{14},\, \frac{1}{14},\, \frac{9}{14} \\ \frac{2}{7},\, \frac{6}{7} \end{array}\right| z\right). \tag{33} $$

Consequently, the Darboux covering Ψ_4 is constructed from the fiber product

$$ \frac{1}{\Psi_4} = -\frac{4\,\Psi_7}{(\Psi_7-1)^2}. \tag{34} $$

The right-hand side is a Belyi map with the branching $[7^6/4^8 2^5/2^{21}]$. The Darboux covering is obtained by factoring out the symmetry $\Psi_7 \mapsto 1/\Psi_7$. This symmetry turns out to be the hypergeometric involution $(u,v)\mapsto(u,-v)$ on H_7, as the function in (34) is a rational function in u. The substitution $u=1/x$ gives

$$ \Psi_4 = \frac{x\,(2x^2-21+56)^2\,(x^2-28)^4\,(x^2-7x+14)^4}{4\,(x^3-14x^2+56x-56)^7}. \tag{35} $$

This is a Belyi function with the branching pattern $[7^3/4^4 2^2 1/2^8 1^5]$.

4 Darboux evaluations

The remainder of this article gives representative Darboux evaluations for algebraic ${}_3\mathrm{F}_2$-functions with the projective monodromy group $\Lambda \cong \mathrm{PSL}(2,\mathbb{F}_7)$ using the degree 21 Darboux coverings that reduce the monodromy to the dihedral group D_4 of 8 elements.

4.1 The case (3A)

The Darboux covering Ψ_3 is given in (25). The point $x=0$ is (up to projective equivalence) a regular point after a pull-back transformation. Accordingly, these Darboux evaluations express hypergeometric functions as linear combinations of radical and dihedral solutions. Note that

$$ \Psi_3 - 1 = \frac{49(2x-3)(4x^4-21x^2+28)\,H_1^2\,H_2^2}{1728(x^3-7x+7)^7}, \tag{36} $$

where H_1, H_2 are degree 4 polynomials. Let us denote

$$ P_3 = 1 - x + \frac{x^3}{7}, \tag{37} $$

$$ Q_3 = \left(1-\frac{3x^2}{7}\right)\left(1-x+\frac{2x^2}{7}\right)\left(1-\frac{5x}{4}+\frac{11x^2}{28}\right), \tag{38} $$

so that

$$\Psi_3 = \frac{49\,x^3\,Q_3^3}{27\,P_3^7}. \tag{39}$$

We use these polynomials (and normalize radical or dihedral solutions) for local consideration at $x=0$.

Theorem 4.1. *Let us define the functions*

$$Y_0 = \sqrt{1-\frac{2x}{3}}, \tag{40}$$

$$Y_1 = \left(\frac{21-8x^2+8\sqrt{7-\frac{21}{4}x^2+x^4}}{21+8\sqrt{7}}\right)^{1/4}, \tag{41}$$

$$Y_2 = \left(\frac{21-8x^2-8\sqrt{7-\frac{21}{4}x^2+x^4}}{21-8\sqrt{7}}\right)^{1/4}. \tag{42}$$

The following identities hold in a neighborhood of $x=0$:

$$P_3^{1/6}\;{}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{42},\ \frac{5}{42},\ \frac{17}{42} \\ \frac{1}{3},\ \frac{2}{3} \end{array}\right|\,\Psi_3\right) = \frac{1}{2}\,Y_0 + \frac{3+\sqrt{7}}{12}\,Y_1 + \frac{3-\sqrt{7}}{12}\,Y_2, \tag{43}$$

$$x\,Q_3 P_3^{-13/6}\;{}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{13}{42},\ \frac{19}{42},\ \frac{31}{42} \\ \frac{2}{3},\ \frac{4}{3} \end{array}\right|\,\Psi_3\right) = -3\,Y_0 + \frac{3+\sqrt{7}}{2}\,Y_1 + \frac{3-\sqrt{7}}{2}\,Y_2, \tag{44}$$

$$x^2 Q_3^2 P_3^{-9/2}\;{}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{9}{14},\ \frac{11}{14},\ \frac{15}{14} \\ \frac{4}{3},\ \frac{5}{3} \end{array}\right|\,\Psi_3\right) = -2\sqrt{7}\,Y_1 + 2\sqrt{7}\,Y_2. \tag{45}$$

Proof. A pull-back of Hurwitz equation [Hur86] with respect to Ψ_3 was considered in the PhD thesis [Vid99, §4.3]. The conclusion is that the functions

$$\sqrt{2x-3} \quad\text{and}\quad \left(x^3-7x+7\right)^{1/6}\;{}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{42},\ \frac{5}{42},\ \frac{17}{42} \\ \frac{1}{3},\ \frac{2}{3} \end{array}\right|\,\Psi_3\right) \tag{46}$$

satisfy the same differential equation of order 3. The pulled-back equation [Vid99, (4.27)] can be fully solved by straightforwardly applying Maple's command dsolve. Other two solutions of the same equation are

$$Y_{\pm} = \left(21-8x^2 \pm 4\sqrt{28-21x^2+4x^4}\right)^{1/4}. \tag{47}$$

The functions Y_0, Y_1, Y_2 are the solutions $\sqrt{2x-3}$, Y_+, Y_- normalized to the value 1 at $x=0$. Formulas (43)–(44) express linearly a set of three companion ${}_3\mathrm{F}_2$-functions in the solution basis Y_0, Y_1, Y_2. □

Conversely, the solutions Y_0, Y_1, Y_2 can be expressed linearly in terms of hypergeometric functions. For example,

$$Y_0 = P_3^{1/6}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{42},\,\frac{5}{42},\,\frac{17}{42} \\ \frac{1}{3},\,\frac{2}{3} \end{array}\right|\,\Psi_3\right) - \frac{x}{6}\,Q_3 P_3^{-13/6}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{13}{42},\,\frac{19}{42},\,\frac{31}{42} \\ \frac{2}{3},\,\frac{4}{3} \end{array}\right|\,\Psi_3\right) \tag{48}$$

in a neighborhood of $x=0$. Note that $Y_2 = Y_1^{-1}$.

The same hypergeometric, radical and dihedral functions can be similarly compared around a root of Q_3. For example, here are the solutions Y_0, Y_1, Y_2 normalized to have the value 1 at $x = \sqrt{7/3}$:

$$\widetilde{Y}_0 = \sqrt{\big(9+2\sqrt{21}\big)(2x-3)}, \tag{49}$$

$$\widetilde{Y}_1 = \left(3\cdot\frac{21-8x^2+4\sqrt{4x^4-21x^2+28}}{7+4\sqrt{7}}\right)^{1/4}, \tag{50}$$

$$\widetilde{Y}_2 = \left(3\cdot\frac{21-8x^2-4\sqrt{4x^4-21x^2+28}}{7-4\sqrt{7}}\right)^{1/4}. \tag{51}$$

We normalize $\widetilde{P}_3 = -3\big(9+2\sqrt{21}\big)\,P_3$ so that $\widetilde{P}_3\big(\sqrt{7/3}\big) = 1$ as well. Then, for instance,

$$\begin{aligned} &3\big(\sqrt{7}-\sqrt{3}\big)\widetilde{P}_3^{1/6}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{42},\,\frac{5}{42},\,\frac{17}{42} \\ \frac{1}{3},\,\frac{2}{3} \end{array}\right|\,\Psi_3\right) \\ &\qquad\qquad = \big(2\sqrt{7}-3\sqrt{3}\big)\,\widetilde{Y}_0 + \frac{\sqrt{7}+1}{2}\,\widetilde{Y}_1 + \frac{\sqrt{7}-1}{2}\,\widetilde{Y}_2 \end{aligned} \tag{52}$$

in a neighborhood of $x = \sqrt{7/3}$.

Similar expressions in terms of re-normalized Y_0, Y_1, Y_2 can be obtained for the companion hypergeometric solutions

$$\begin{aligned} &\widehat{Q}_3^{1/14}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{42},\,\frac{13}{42},\,\frac{9}{14} \\ \frac{4}{7},\,\frac{6}{7} \end{array}\right|\,\frac{1}{\Psi_3}\right), \quad P_3\,\widehat{Q}_3^{-5/14}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{5}{42},\,\frac{19}{42},\,\frac{11}{14} \\ \frac{5}{7},\,\frac{8}{7} \end{array}\right|\,\frac{1}{\Psi_3}\right), \\ &P_3^3\,\widehat{Q}_3^{-17/14}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{17}{42},\,\frac{31}{42},\,\frac{15}{14} \\ \frac{9}{7},\,\frac{10}{7} \end{array}\right|\,\frac{1}{\Psi_3}\right) \end{aligned} \tag{53}$$

around a root x_0 of $P_3 = 0$ (therefore $\Psi_3 = \infty$). Here $\widehat{Q}_3 = C x\,Q_3$ with a constant C chosen so that $\widehat{Q}_3(x_0) = 1$. One can take $x_0 = -\xi^2 - 2\xi + 1$, where $\xi = 2\cos\frac{2\pi}{7}$.

4.2 The case (3B)

By differentiating (43)–(45) and using contiguous relations, a basis of hypergeometric solutions of any differential equation (7) of type (3A) can be expressed in terms of radical and dihedral solutions. Up to projective equivalence, the dihedral solutions will be products of Y_1 or Y_2 with rational functions in $\mathbb{Q}\big(x, \sqrt{4x^4 - 21x^2 + 28}\big)$. The radical solutions can be obtained by considering Riemann's P-symbols of pulled-backed equations as in [Vid18, Proofs of Theorems 3.1, 3.3]. That is, radical solutions are constructed by picking a local exponent at each singular point and appending a polynomial part (to match a local exponent at $x = \infty$).

As with degree 24 Darboux evaluations of type (3B) hypergeometric functions in [Vid18, §3.2], type (3B) Darboux evaluations of degree 21 appear to always require those extraneous factors to Y_1, Y_2 and Y_0. In particular, let

$$W_1 = \frac{2x^2 - 3x + 2\sqrt{7 - \frac{21}{4}x^2 + x^4}}{2\sqrt{7}}, \tag{54}$$

$$W_2 = \frac{3x - 2x^2 + 2\sqrt{7 - \frac{21}{4}x^2 + x^4}}{2\sqrt{7}}. \tag{55}$$

Note that the conjugation of $\sqrt{7}$ interchanges W_1 and W_2. A basis of solutions of a relevant pulled-back equation is

$$xY_0, \qquad W_1Y_1, \qquad W_2Y_2. \tag{56}$$

We obtain the following identities in a neighborhood of $x = 0$:

$$\sqrt{P_3}\;{}_3\mathrm{F}_2\left(\left.\begin{matrix} -\frac{1}{14}, \frac{3}{14}, \frac{5}{14} \\ \frac{1}{3}, \frac{2}{3} \end{matrix}\,\right|\, \Psi_3\right) = \frac{3+\sqrt{7}}{6}\,W_1Y_1 + \frac{3-\sqrt{7}}{6}\,W_2Y_2, \tag{57}$$

$$x\,Q_3P_3^{-11/6}\;{}_3\mathrm{F}_2\left(\left.\begin{matrix} \frac{11}{42}, \frac{23}{42}, \frac{29}{42} \\ \frac{2}{3}, \frac{4}{3} \end{matrix}\,\right|\, \Psi_3\right) = \frac{1}{2}\,xY_0 - \frac{\sqrt{7}}{6}\,W_1Y_1 + \frac{\sqrt{7}}{6}\,W_2Y_2, \tag{58}$$

$$x^2Q_3^2P_3^{-25/6}\;{}_3\mathrm{F}_2\left(\left.\begin{matrix} \frac{25}{42}, \frac{37}{42}, \frac{43}{42} \\ \frac{4}{3}, \frac{5}{3} \end{matrix}\,\right|\, \Psi_3\right) = 6\,xY_0 + 2\sqrt{7}\,W_1Y_1 - 2\sqrt{7}\,W_2Y_2. \tag{59}$$

4.3 The case (4A)

The Darboux covering Ψ_4 is given in (35). The point $x = 0$ is a singular point after a pull-back transformation, with no integer differences of local exponents. Lemma 2.4 implies that a basis of companion hypergeometric solutions is matched bijectively (up to a constant factor) with the radical and dihedral solutions. For shorthand, let us introduce the polynomials

$$P_4 = 1 - x + \tfrac{1}{4}\,x^2 - \tfrac{1}{56}\,x^3, \tag{60}$$

$$Q_4 = \big(1 - \tfrac{1}{28}\,x^2\big)\big(1 - \tfrac{1}{2}\,x + \tfrac{1}{14}\,x^2\big), \tag{61}$$

$$Q_8 = \big(1 + 3x - \tfrac{9}{4}x^2 + \tfrac{1}{2}x^3 - \tfrac{1}{28}x^4\big)\big(1 - \tfrac{5}{8}x + \tfrac{3}{16}x^2 - \tfrac{1}{32}x^3 + \tfrac{1}{448}x^4\big), \tag{62}$$

$$R_1 = 1 - \tfrac{3}{8}\,x + \tfrac{1}{28}\,x^2, \tag{63}$$

$$R_2 = 1 - \tfrac{33}{32}\,x + \tfrac{13}{22}\,x^2 - \tfrac{9}{128}\,x^3 + \tfrac{1}{224}\,x^4. \tag{64}$$

Then

$$\Psi_4 = -\frac{343\,x\,R_1^2\,Q_4^4}{32\,P_4^7} = 1 - \frac{R_2\,Q_8^2}{P_4^7}. \tag{65}$$

Let us also define the functions

$$K_1 = \left(\tfrac{1}{2}\sqrt{R_1R_2} + \tfrac{1}{2} - \tfrac{45}{128}x + \tfrac{5}{64}x^2 - \tfrac{5}{896}x^3\right)^{1/4}, \tag{66}$$

$$K_2 = \left(\frac{\sqrt{R_1R_2} - 1 + \frac{45}{64}x - \frac{5}{32}x^2 + \frac{5}{448}x^3}{\frac{625}{57344}\,x^2}\right)^{1/4}. \tag{67}$$

They both have expansions $1+O(x)$ around $x=0$. The conjugation $\sqrt{R_1R_2} \mapsto -\sqrt{R_1R_2}$ of K_1 equals $\frac{5}{8}\,(-x^2/28)^{1/4}\,K_2$.

Theorem 4.2. *The following identities hold in a neighborhood of $x=0$:*

$$ {}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{28},\ \frac{3}{28},\ \frac{19}{28} \\ \frac{1}{2},\ \frac{3}{4} \end{array}\right|\, \Psi_4\right) = \frac{K_1}{P_4^{1/4}}, \tag{68}$$

$$ {}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{3}{14},\ \frac{5}{14},\ \frac{13}{14} \\ \frac{3}{4},\ \frac{5}{4} \end{array}\right|\, \Psi_4\right) = \frac{P_4^{3/2}}{Q_4\sqrt{R_1}}, \tag{69}$$

$$ {}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{13}{28},\ \frac{17}{28},\ \frac{33}{28} \\ \frac{5}{4},\ \frac{3}{2} \end{array}\right|\, \Psi_4\right) = \frac{P_4^{13/4}K_2}{Q_4^2\,R_1}. \tag{70}$$

Proof. The third order linear differential equation for

$$P_4^{1/4}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{28},\ \frac{3}{28},\ \frac{19}{28} \\ \frac{1}{2},\ \frac{3}{4} \end{array}\right|\, \Psi_4\right). \tag{71}$$

is satisfied by the following 3 functions:

$$x^{1/4}, \qquad \left(\pm\sqrt{R_1R_2} + 1 - \tfrac{45}{64}x + \tfrac{5}{32}x^2 - \tfrac{5}{448}x^3\right)^{1/4}. \tag{72}$$

This can be established by deriving the differential equation for the 3 functions, and then checking it for (71). Alternatively, deriving the equation for (71) is lengthier, but Maple's routine dsolve provides the solutions in (72). □

Surely, the more attractive are purely radical Darboux evaluations as in (69). Here are two more examples:

$$ {}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{14},\ \frac{3}{14},\ \frac{5}{14} \\ \frac{1}{4},\ \frac{3}{4} \end{array}\right|\, \Psi_4\right) = \frac{\sqrt{R_1}}{\sqrt{P_4}}, \tag{73}$$

$$ {}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{3}{14},\ \frac{5}{14},\ \frac{13}{14} \\ \frac{1}{4},\ \frac{3}{4} \end{array}\right|\, \Psi_4\right) = \frac{P_4^{3/2}\,R_1}{Q_8}\left(1 - \frac{1}{4}\,x^2 + \frac{1}{28}\,x^3\right). \tag{74}$$

They can be found rather quickly by dividing out a befitting ${}_3\mathrm{F}_2$-function by finitely many possibilities of predictable (by Riemann's P-symbols) powers of the irreducible polynomials in (60)–(64), and checking which combination gives the power series that appears to be a polynomial of predictable degree.

4.4 The case (4B)

As with degree 24 Darboux evaluations of type (4B) hypergeometric functions in [Vid18, §6], the Darboux evaluations of degree 21 appear to always require extraneous factors to "basic" radical or dihedral expressions. The following identities (around $x = 0$) are obtained after lengthy computations and simplifications:

$$ {}_3F_2\left(\left. \begin{matrix} -\frac{3}{28}, \frac{1}{28}, \frac{9}{28} \\ \frac{1}{4}, \frac{1}{2} \end{matrix} \right| \Psi_4 \right) = \frac{\frac{1}{2}\sqrt{R_1R_2} + \frac{1}{2} - \frac{15}{32}x + \frac{53}{448}x^2 - \frac{1}{112}x^3}{P_4^{3/4}K_1}, \tag{75} $$

$$ {}_3F_2\left(\left. \begin{matrix} \frac{11}{28}, \frac{15}{28}, \frac{23}{28} \\ \frac{3}{4}, \frac{3}{2} \end{matrix} \right| \Psi_4 \right) = \frac{P_4^{11/4}}{Q_4^2R_1} \frac{(\sqrt{R_1R_2} - 1 + \frac{15}{16}x - \frac{53}{224}x^2 + \frac{1}{56}x^3)}{\frac{15}{64}\,x\,K_2}, \tag{76} $$

$$ {}_3F_2\left(\left. \begin{matrix} \frac{9}{14}, \frac{11}{14}, \frac{15}{14} \\ \frac{5}{4}, \frac{7}{4} \end{matrix} \right| \Psi_4 \right) = \frac{P_4^{9/2}}{Q_4^3R_1^{3/2}} \left(1 - \frac{3}{14}\,x\right). \tag{77} $$

Here are two other examples of the more attractive radical evaluations:

$$ {}_3F_2\left(\left. \begin{matrix} \frac{1}{14}, \frac{9}{14}, \frac{11}{14} \\ \frac{3}{4}, \frac{5}{4} \end{matrix} \right| \Psi_4 \right) = \frac{\sqrt{P_4}}{Q_4\sqrt{R_1}} \left(1 - \frac{3}{5}x + \frac{1}{8}x^2 - \frac{1}{112}x^3\right), \tag{78} $$

$$ {}_3F_2\left(\left. \begin{matrix} -\frac{3}{14}, \frac{1}{14}, \frac{9}{14} \\ \frac{1}{4}, \frac{3}{4} \end{matrix} \right| \Psi_4 \right) = \frac{\sqrt{R_1}}{P_4^{3/2}} \left(1 - \frac{3}{4}x + \frac{1}{4}x^2 - \frac{3}{112}x^3\right). \tag{79} $$

4.5 The cases (7A) and (7B)

As discussed in §3, the Darboux curve for the types (7A), (7B) is the genus 2 curve H_7 given by

$$ v^2 = u\left(1 - \tfrac{63}{4}u + 91u^2 - 231u^2 + 224u^4\right). \tag{80} $$

The degree 21 Darboux covering Ψ_7 is given by formulas (31)–(32). Its principal divisor on H_7 is

$$ \operatorname{div}(\Psi_7) = 7U_1 + 7U_2 + 7U_3 - 7\widetilde{U}_1 - 7\widetilde{U}_2 - 7\widetilde{U}_3, \tag{81} $$

where the u-coordinates of U_1, U_2, U_3 satisfy $14u(2u-1)^2 = 1$. The points $\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3$ are obtained by the hyperelliptic involution $(u,v) \mapsto (u,-v)$. This follows from the following divisors of the polynomial components:

$$ \begin{aligned} \operatorname{div}(2v - 3u + 14u^2) &= (0,0) + (\tfrac{1}{4}, -\tfrac{1}{16}) + U_1 + U_2 + U_3 - 5\mathcal{O}, \\ \operatorname{div}(2v + 3u - 14u^2) &= (0,0) + (\tfrac{1}{4}, \tfrac{1}{16}) + \widetilde{U}_1 + \widetilde{U}_2 + \widetilde{U}_3 - 5\mathcal{O}, \\ \operatorname{div}(P_7) &= (0,0) + 7\,(\tfrac{1}{4}, \tfrac{1}{16}) - 8\mathcal{O}. \end{aligned} \tag{82} $$

The Darboux or dihedral evaluations have to be evaluated at one of the points $U_1, U_2, U_3, \widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3$. They are defined over $\mathbb{Q}(\cos\frac{2\pi}{7})$. Therefore we settle for the following proposition. Handy Darboux evaluations of degree 24 for the types (7A), (7B) are given in [Vid18, §5].

Proposition 4.3. *The following four functions W_7, W_0, W_+, W_- on the hyperelliptic curve H_7 satisfy the same linear differential equation of order 3:*

$$W_7 = \sqrt{2v+3u-14u^2}\; P_7^{1/14}\, u^{-2/7}\, {}_3\mathrm{F}_2\left(\left. \begin{array}{c} -\frac{1}{14}, \frac{1}{14}, \frac{5}{14} \\ \frac{1}{7}, \frac{5}{7} \end{array} \right| \Psi_7 \right), \tag{83}$$

$$W_0 = 4u - 1, \tag{84}$$

$$W_\pm = \left(P_1 \pm Q_1 \sqrt{56u - 21 + \frac{2}{u}} \right)^{1/4} \tag{85}$$

with

$$\begin{aligned} P_1 =& 1344u^5 - 1540u^4 + 784u^3 - 204u^2 + \tfrac{53}{2}u - \tfrac{11}{8} - 3(28u^2 - 10u + 1)v, \\ Q_1 =& -112u^4 + 84u^3 - 25u^2 + \tfrac{5}{2}u + (16u^2 - 4u + 1)v. \end{aligned}$$

Proof. From the proof of Theorem 4.1 we have the four functions in formulas (46)–(47) satisfying the same linear differential equation of order 3. By switching the ${}_3\mathrm{F}_2$-function to a companion function, we conclude that

$$\sqrt{2x-3},\; Y_+,\; Y_-,\; \Psi_3^{1/42}\left(x^3 - 7x + 7\right)^{1/6} {}_3\mathrm{F}_2\left(\left. \begin{array}{c} -\frac{1}{42}, \frac{13}{42}, \frac{9}{14} \\ \frac{4}{7}, \frac{6}{7} \end{array} \right| \frac{1}{\Psi_3} \right) \tag{86}$$

satisfy the same equation. We use quadratic transformation (26) and the parametrization (30) to arrive at the claimed functions after a lengthy simplification. (We can recognize R_1, R_2 of §4.3 in (85), (80) after the substitution $u = 1/x$.) □

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