Identifying weak values with intrinsic dynamical properties in Modal theories (Supplemental Information)

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Appendix A: QUANTUM BACKACTION CANNOT BE AVOIDED IN TWO-TIME MEASUREMENT EXPERIMENTS

In this Appendix we use the generalized von Neumann Orthodox protocol to describe the measurement of a qunatum system without assuming a strong (or projective) measurement. In particular, we will show that (Born) probabilities of two-time measurements are always contaminated by the measuring apparatus.

1. Semi-weak measurements

Let us first consider the expectation value of a property of a quantum system at time t, i.e.:

$$\langle y(t_1)\rangle = \int dy_k y_k P(y_k), \tag{A.1}$$

where $P(y_k)$ is the probability of reading-out a particular value $y_k = z_k$, and integrals are definite integrals over all possible values of the variables x, y and z (i.e., $-\infty, +\infty$). In order to relate the probability distribution of outcomes $P(y_k)$ with the degrees of freedom of the system x, we here follow a generalized quantum Von Neumann measurement protocol for weak (generalized) measurements [1–3]. We assume the full state of the system-ancilla-pointer to be initially described by a separable state vector:

$$|\Psi(0)\rangle = \sum_i c_i(0)|s_i\rangle \otimes \int a(y,0)|y\rangle dy \otimes \int f(z,0)|z\rangle dz, \tag{A.2}$$

where the system state vector $|\psi(0)\rangle = \sum_i c_i(0)|s_i\rangle$ has been defined using the eigenstates $|s_i\rangle$ of the operator $\hat{S}$ of interest, with $\hat{S}|s_i\rangle = s_i|s_i\rangle$. Without the loss of generality, we chose here a discrete and nondegenerate spectrum $\{s_1, s_2, s_3....\}$ of the operator $\hat{S}$. The (ancilla) state vector $|\phi_W(0)\rangle = \int a(y,0)|y\rangle dy$ interacts with the system and also with the (pointer) state vector $|\phi_P(0)\rangle = \int f(z,0)|z\rangle dz$.

First, a pre-measurement (unitary) evolution from $t = 0$ to t_1 entangles the ancilla with the system and the pointer with the ancilla as follows:

$$|\Psi(t_1)\rangle = \sum_i c_i|s_i\rangle \otimes \int a(y-\lambda s_i)|y\rangle dy \otimes \int f(z-y)|z\rangle dz. \tag{A.3}$$

The original ancilla wave function $a(y,0)$ splits into several wave functions $a(y-\lambda s_i)$ with $i = 1, 2, ...$ We have defined λ as a macroscopic parameter with dimensions of $[y]/[S]$ which can be defined as the effective coupling constant [4]. The shape of $a(y_k - \lambda s_i)$ is arbitrary and includes, in particular, strong (direct measurement) interactions when

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$\int\, dy\; a(y-\lambda s_i)a(y-\lambda s_j) = \delta_{i,j}$ and weaker (indirect measurement) interactions when $\int\, dy\; a(y-\lambda s_i)a(y-\lambda s_j) \neq \delta_{i,j}$. We have defined $\delta_{i,j}$ as a Kronecker delta function. The only two conditions imposed on the ancilla wave functions $a(y-\lambda s_i)$ to be representative of an indirect or weak measurement are: (i) $\int y|a(y-\lambda s_i)|^2 dy = \lambda s_i \;\; \forall i$, which implies that the center of mass of $|a(y-\lambda s_i)|^2$ is λs_i, and (ii) $\int |a(y-\lambda s_i)|^2 dy = 1 \;\; \forall i$, which simply states that the ancilla wave function is well normalized.

Second, the read-out process is described by the non-unitary operator $\hat{I}_S \otimes \hat{I}_W \otimes \hat{\mathbb{P}}_{z_k}$, where $\hat{I}_S$ is the identity operator defined in the system Hilbert space, $\hat{I}_W$ is the identity operator defined in the ancilla Hilbert space, and $\hat{\mathbb{P}}_{z_k} = |z_k\rangle\langle z_k|$ is a projector acting on the Hilbert space of the pointer. As mentioned previously, it becomes now evident that an indirect measurement of a system is just a direct measurement of an ancilla that is entangled with the system. The non-unitary operator $\hat{I}_S \otimes \hat{I}_W \otimes \hat{\mathbb{P}}_{z_k}$ causes the collapse of the pointer wave function providing the read-out value $z_k = y_k$ and the measured state becomes $|\Psi_k(t_1)\rangle = \sum_i c_i a(y_k - \lambda s_i)|s_i\rangle \otimes |y_k\rangle \otimes |z_k\rangle$. Therefore, the state of the system can be effectively represented by:

$$|\psi_k(t_1)\rangle = \sum_i a(y_k - \lambda s_i) c_i |s_i\rangle, \tag{A.4}$$

where the subscript k indicates (the perturbation that the system has suffered due to) the measurement of the pointer value $z_k = y_k$ [5].

The probability of measuring a particular pointer position y_k can be then easily evaluated from Born's rule $P(y_k) = \langle\Psi_k(t_1)|\Psi_k(t_1)\rangle = \langle\psi_k(t_1)|\psi_k(t_1)\rangle$ applied to the non-normalized state in Eq. (A.4). While the probability distribution $P(y_k) = \sum_i |c_i|^2 |a(y_k - \lambda s_i)|^2$ clearly depends on the type of ancilla that we are considering, the expectation value in Eq. (A.1),

$$\langle y(t_1)\rangle = \int y_k \sum_i |c_i|^2 |a(y_k - \lambda s_i)|^2 dy_k = \sum_i |c_i|^2 \lambda s_i = \lambda\, \langle\hat{S}\rangle, \tag{A.5}$$

only depends on the system state $|\psi(t_1)\rangle = \sum_i c_i |s_i\rangle$. As we already anticipated, expectation values of static (one-time) properties provide information of the system that is not contaminated by the measuring apparatus.

One can now generalize the above measurement scheme to account for a second measurement of another observable G at time $t_2 > t_1$. By repeated read-out of the positions y_k (at t_1) and y_ω (at t_2) for a large number of identically prepared experiments, we can compute the corresponding two-time correlation function $\langle y(t_2)y(t_1)\rangle$ as:

$$\langle y(t_2)y(t_1)\rangle = \int dy_k \int dy_\omega\; y_k y_\omega P(y_\omega, y_k), \tag{A.6}$$

where $P(y_\omega, y_k)$ is the joint probability of subsequent read-out of the values y_k and y_ω at times t_1 and t_2, respectively.

To evaluate $P(y_\omega, y_k)$ in Eq. (A.6) we simply need to apply the above protocol to the final state in (A.4). We first let the state in Eq. (A.4) to evolve freely from t_1 till t_2 according to the time-evolution operator $\hat{U} = \exp(i\hat{H}(t_2 - t_1)/\hbar)$, where $\hat{H}$ is the Hamiltonian that dictates the evolution of the system degrees of freedom x in the absence of any interaction with the ancilla and pointer degrees of freedom. For convenience, we write the state of the system in terms of the eigenstates of the operator $\hat{G}$, i.e., $|g_j\rangle$, using the transformation $|s_k\rangle = \sum_j \beta_{k,j}|g_j\rangle$. We then rewrite the state of the system $\hat{U}|s_i\rangle = \sum_k \gamma_{i,k}|s_k\rangle$ in terms of the new basis as $\hat{U}|s_i\rangle = \sum_{k,j} \gamma_{i,k}\beta_{k,j}|g_j\rangle$. More compactly, $\hat{U}|s_i\rangle = \sum_j c_{j,i}|g_j\rangle$, where $c_{j,i} = \sum_k \gamma_{i,k}\beta_{k,j}$. Therefore, the state of the system right before the second pre-measurement can be written as the (non-normalized) state:

$$|\psi_k(t_2)\rangle = \sum_{i,j} a(y_k - \lambda s_i) c_i c_{j,i} |g_j\rangle. \tag{A.7}$$

Subsequently, under the assumption that there is no correlation between the ancilla degrees of freedom at times t_1 and t_2, the system state vector in (A.7) undergoes a second pre-measurement evolution and the system becomes entangled again with the ancilla and the pointer wave functions, i.e.:

$$|\Psi_k(t_2)\rangle = \sum_{i,j} a(y_k - \lambda s_i) c_i c_{j,i} |g_j\rangle \otimes \int a(y - \lambda g_j)|y\rangle dy \otimes \int f(z-y)|z\rangle dz, \tag{A.8}$$

where now $a(y - \lambda g_j)$ is the ancilla wave function displaced by λg_j.

The read-out of the pointer position (for an output value y_ω) at time t_2 is described again by a non-unitary operator $\hat{I}_S \otimes \hat{I}_W \otimes \hat{\mathbb{P}}_{z_w}$ with $\hat{\mathbb{P}}_{z_w} = |z_w\rangle\langle z_w|$. This non-unitary operator causes the collapse of the state in Eq. (A.8) into $|\Psi_{k,\omega}(t_2)\rangle = \sum_{i,j} c_i c_{j,i} a(y_k - \lambda s_i) a(y_\omega - \lambda g_j)|g_j\rangle \otimes |y_\omega\rangle \otimes |z_w\rangle$, and so the state of the system can be effectively written as:

$$|\psi_{k,\omega}(t_2)\rangle = \sum_{i,j} c_i c_{j,i} a(y_k - \lambda s_i) a(y_\omega - \lambda g_j)|g_j\rangle. \tag{A.9}$$

Born's rule can be used again to write the probability $P(y_\omega, y_k) = \langle\Psi_{k,\omega}(t_2)|\Psi_{k,\omega}(t_2)\rangle = \langle\psi_{k,\omega}(t_2)|\psi_{k,\omega}(t_2)\rangle$ of subsequently measuring y_k and y_ω as:

$$P(y_\omega, y_k) = \sum_j \sum_{i,i'} c^*_{i'} c_i c^*_{j,i'} c_{j,i} a^*(y_k - \lambda s_{i'})\; a(y_k - \lambda s_i)|a(y_\omega - \lambda g_j)|^2. \tag{A.10}$$

By introducing the probability $P(y_\omega, y_k)$ in Eq. (A.10) into Eq. (A.6) we finally get:

$$\langle y(t_2)y(t_1)\rangle = \lambda \sum_{i,i'} \int dy_k y_k a(y_k - \lambda s_i) a^*(y_k - \lambda s_{i'})\langle\psi(t_1)|s_{i'}\rangle\langle s_{i'}|\hat{U}^\dagger \hat{G} \hat{U}|s_i\rangle\langle s_i|\psi(t_1)\rangle, \tag{A.11}$$

where we have used $\int dy_\omega\; y_\omega\; |a(y_\omega - \lambda g_j)|^2 = \lambda g_j$ and $\hat{G} = \sum_j g_j |g_j\rangle\langle g_j|$ together with $c_i = \langle s_i|\psi(t_1)\rangle$ and $c_{j,i} = \langle g_j|U|s_i\rangle$.

Expression (A.11) is completely general and describes the two-time correlation function of $\hat{S}$ and $\hat{G}$ at times t_1 and t_2. At this point what is significant is that in (A.11) we have not been able to eliminate the dependence of the ancilla degrees of freedom $a(y_k - \lambda s_i)$ and $a^*(y_k - \lambda s_{i'})$ on $\langle y(t_2)y(t_1)\rangle$. Therefore, in contrast to what happens to the one-time expectation values in Eq. (A.5), different types of measurements (ancillas) will produce different time correlation functions. Therefore, multiple-time correlation functions such as $\langle y(t_2)y(t_1)\rangle$ are not universal properties of quantum systems, but are dependent on the measuring apparatus.

The only scenario where the outcome of the second measurement does not depend on the first measurement is when the initial state of the system is an eigenstate of the operator $\hat{S}$, i.e., $|\psi(t_1)\rangle = |s_k\rangle$. Then the first measurement always yields the same output result $y_k = \lambda s_k$ without having perturbed the state of the system, and hence the second measurement happens to be independent of the first measurement. Mathematically this can be stated as:

$$\langle y(t_2)y(t_1)\rangle = \lambda^2 s_k \langle s_k|\hat{U}^\dagger \hat{G} \hat{U}|s_k\rangle = \lambda^2 \langle \hat{G}(t_2)\rangle\langle\hat{S}(t_1)\rangle, \tag{A.12}$$

where we have used that $\langle\psi(t_1)|s_{i'}\rangle\langle s_i|\psi(t_1)\rangle = \langle s_k|s_{i'}\rangle\langle s_i|s_k\rangle = \delta_{i,k'}\delta_{i',k'}$ and that $\int dy y|a(y-\lambda s)|^2 = \lambda s$. Equivalently, if $\langle s_{i'}|\hat{U}^\dagger \hat{G}\hat{U}|s_i\rangle = g_w \delta_{i',i}$ which means that the evolved state $\hat{U}|s_i\rangle$ is an eigenstate of $\hat{G}$, then Eq. (A.11) can be also written as $\langle y(t_2)y(t_1)\rangle = \lambda^2 g_w \langle\psi(t_1)|\hat{S}|\psi(t_1)\rangle = \lambda^2\langle\hat{G}(t_2)\rangle\langle\hat{S}(t_1)\rangle$. In these two scenarios, since the results $\langle\hat{G}(t_2)\rangle$ and $\langle\hat{S}(t_1)\rangle$ are apparatus-independent, the two-time correlation function in Eq. (A.12) also represents an apparatus-independent correlation function. Unfortunately, this result is not general enough and is invalid in many practical situations where the initial state is a coherent superposition of the observable eigenstates [6].

2. Ideally-weak measurements

We define an ideally-weak measurement as the one where the system-ancilla coupling is minimized. This is mathematically equivalent to making the support of the ancilla wave function much larger than the support of system wave function, i.e. $y \gg \lambda s_k$. In this limit we can assume a first order Taylor approximation so that the ancilla wave packet can be written as $a(y_k - \lambda s_i) \approx a(y_k) - \lambda s_i \frac{\partial a(y_k)}{\partial y_k}$.

The measuring protocol satisfying the above Taylor expansion is what we call *ideally-weak measurement* in the text. The above condition for its definition written above can be equivalently written as $\left|\frac{\partial a(y_k)}{\partial y_k}\right| \gg \left|\frac{\lambda}{2}\frac{\partial^2 a(y_k)}{\partial y_k^2} s\right|$.

The evaluation of the main result in Eq. (A.11) for the ideal weak measurements used here, requires the evaluation of the integral $\int dy_k\; y_k a(y_k - \lambda s_i) a(y_k - \lambda s_{i'})^*$, which can be written as:

$$\begin{aligned}
\int dy_k\; y_k a(y_k - \lambda s_i) a^*(y_k - \lambda s_{i'}) &= \int dy_k\; y_k \left(a(y_k) - \lambda \frac{\partial a(y_k)}{\partial y_k} s_i\right)\left(a^*(y_k) - \lambda \frac{\partial a^*(y_k)}{\partial y_k} s_{i'}\right) \\
&= \int dy_k\; y_k a(y_k) a^*(y_k) - \lambda s_i \int_{-\infty}^{\infty} dy_k y_k a^*(y_k)\frac{\partial a(y_k)}{\partial y_k} \\
&- \lambda s_{i'} \int dy_k y_k a(y_k)\frac{\partial a^*(y_k)}{\partial y_k} + \lambda^2 s_i\; s_{i'} \int dy_k y_k \frac{\partial a^*(y_k)}{\partial y_k}\frac{\partial a(y_k)}{\partial y_k} \\
&= \lambda \frac{1}{2}(s_i + s_{i'}).
\end{aligned} \tag{A.13}$$

In the evaluation of Eq. (A.13) we have considered that ancilla wave function is real (not complex), $a(y_k - \lambda s_i) = a^*(y_k - \lambda s_i)$. We have also used the identities $\int dy\; y\; a(y)\frac{\partial a(y)}{\partial y} = -1/2, \int dy\; y\; \frac{\partial a(y)}{\partial y}\frac{\partial a(y)}{\partial y} = 0$, and $\int dy\; y\; a(y)a(y) = 0$. Finally, using Eq. (A.13) into the integral in Eq. (A.11), we get:

$$\begin{aligned}\langle y(t_2)y(t_1)\rangle &= \frac{\lambda^2}{2}\langle\Psi(t_1)|\hat{U}^\dagger\hat{G}\hat{U}\hat{S}|\psi(t_1)\rangle + \frac{\lambda^2}{2}\langle\Psi(t_1)|\hat{S}\hat{U}^\dagger\hat{G}\hat{U}|\psi(t_1)\rangle \\ &= \lambda^2\mathrm{Re}[\langle\Psi(t_1)|\hat{U}^\dagger\hat{G}\hat{U}\hat{S}|\psi(t_1)\rangle],\end{aligned} \tag{A.14}$$

where we have used the identities $\hat{S} = \sum_i s_i|s_i\rangle\langle s_i|$, $\hat{G} = \sum_j g_j|g_j\rangle\langle g_j|$, $\sum_i |s_i\rangle\langle s_i| = 1$ and $\hat{G} = \sum_j |g_j\rangle\langle g_j|$. Now, by simply defining the Heisenberg operators $\hat{G}(t_2) = \hat{U}^\dagger\hat{G}\hat{U}$ and $\hat{S}(t_1) = \hat{S}$, then can write the two-time correlation function in the ideally-weak measurement regime as:

$$\langle y(t_2)y(t_1)\rangle = \lambda^2\mathrm{Re}[\langle\psi(t_1)|\hat{G}(t_2)\hat{S}(t_1)|\psi(t_1)\rangle]. \tag{A.15}$$

3. Even ideally-weak measurements involve quantum backaction

To understand whether the result Eq. (A.15) is contaminated or not by the measurement, let us rewrite the general state in the system space in Eq. (A.9), according to the Taylor series used above for the ancilla wave function:

$$\begin{aligned}|\psi_{k,\omega}(t_2)\rangle &= \sum_{j,i}|g_j\rangle a(y_\omega - \lambda g_j)\langle g_j|\hat{U}|s_i\rangle a(y_k - \lambda s_i)\langle s_i|\psi(t_1)\rangle \\ &= \sum_{j,i}|g_j\rangle\left(a(y_\omega) - \lambda\frac{\partial a(y_\omega)}{\partial y_\omega}g_j\right)\langle g_j|\hat{U}|s_i\rangle\left(a(y_k) - \lambda\frac{\partial a(y_k)}{\partial y_k}s_i\right)\langle s_i|\psi(t_1)\rangle,\end{aligned} \tag{A.16}$$

which can be easily expanded to read:

$$\begin{aligned}|\psi_{k,\omega}(t_2)\rangle &= a(y_\omega)a(y_k)\sum_{j,i}|g_j\rangle\langle g_j|\hat{U}|s_i\rangle\langle s_i|\psi(t_1)\rangle - \lambda\frac{\partial a(y_\omega)}{\partial y_\omega}a(y_k)\sum_{j,i}|g_j\rangle\langle g_j|\hat{U}|s_i\rangle s_i\langle s_i|\psi(t_1)\rangle \\ &-\lambda\frac{\partial a(y_k)}{\partial y_k}a(y_\omega)\sum_{j,i}|g_j\rangle g_j\langle g_j|\hat{U}|s_i\rangle\langle s_i|\psi(t_1)\rangle + \lambda^2\frac{\partial a(y_\omega)}{\partial y_\omega}\frac{\partial a(y_k)}{\partial y_k}\sum_{j,i}|g_j\rangle g_j\langle g_j|\hat{U}|s_i\rangle s_i\langle s_i|\psi(t_1)\rangle.\end{aligned} \tag{A.17}$$

Using now $\hat{S} = \sum_i s_i|s_i\rangle\langle s_i|$ and $\hat{G} = \sum_i g_i|s_i\rangle\langle s_i|$ Eq. (A.17) reduces to:

$$|\psi_{k,\omega}(t_2)\rangle = \left(a(y_\omega)a(y_k)\hat{U} + \lambda^2\frac{\partial a(y_\omega)}{\partial y_\omega}\frac{\partial a(y_k)}{\partial y_k}\hat{G}\hat{U}\hat{S} - \lambda\frac{\partial a(y_\omega)}{\partial y_\omega}a(y_k)\hat{G}\hat{U} - \lambda\frac{\partial a(y_k)}{\partial y_k}a(y_\omega)\hat{U}\hat{S}\right)|\psi(t_1)\rangle. \tag{A.18}$$

For simplicity, we defined $\partial a/\partial y \equiv \partial a(y)/\partial y$. Erroneously assuming $\partial a(y)/\partial y = 0$, one could then think that the state of the system after the two measurements can be approximated only by the first term in Eq. (A.18) as $|\psi_{k,\omega}(t_2)\rangle \approx |\tilde{\psi}_{k,\omega}(t_2)\rangle \equiv a(y_\omega)a(y_k)\hat{U}|\psi(t_1)\rangle$. This approximation would indeed imply that the state of the system has not been perturbed during the two-time measurement. Thus, the Born's probability associated with this unperturbed wavefunction is $P(y_\omega, y_k) = \langle\tilde{\psi}_{k,\omega}(t_2)|\tilde{\psi}_{k,\omega}(t_2)\rangle \approx |a(y_\omega)|^2|a(y_k)|^2$. Computing two time correlations with this two time probability, obtained after neglecting the contributions $\partial a(y)/\partial y$, leads to $\langle y(t_2)y(t_1)\rangle = 0$.

The approximation $|\psi_{k,\omega}(t_2)\rangle \approx |\tilde{\psi}_{k,\omega}(t_2)\rangle$ yields a wrong result because the unperturbed state $|\tilde{\psi}_{k,\omega}(t_2)\rangle$ signifies no correlation at all, between system and measuring apparatus. Thus, since we want some type of correlation, the terms $\partial a(y)/\partial y$ in Eq. (A.18) cannot be neglected. Even if these terms are very small in general, for the rare events associated to $y \to \infty$, they become comparable or larger than the main term that gives zero correlation, $\partial a(y)/\partial y \approx a(y)$ and hence, these terms $\partial a(y)/\partial y$ are indeed the responsible for providing non-zero correlations in Eq. (A.15) due to the rare events $y \to \infty$.

In conclusion, two-time measurements do entail, in general, non-negligible perturbation on the state of the system, and therefore cannot provide intrinsic (or unperturbed) information of the dynamics of quantum systems. This is an important result that is in contrast with the naive thought that ideally-weak measurements can be used to avoid the quantum backaction of the measurement apparatus. The reader can argue that we have not discuss results from individual experiments, but only results from an average over an ensemble of identically prepared quantum systems. The key point is that if the ensemble average shows backaction is because some individual experiments (not all perhaps) suffer from backaction.

Appendix B: WEAK VALUES OF SPIN-$\frac{1}{2}$ PARTICLE

We consider a spin-$\frac{1}{2}$ particle (electron). The three cartesian components of the spin operator are given by $\hat{S}_x = \frac{\hbar}{2}\sigma_x$, $\hat{S}_y = \frac{\hbar}{2}\sigma_y$ and $\hat{S}_z = \frac{\hbar}{2}\sigma_z$, with the Pauli matrices defined as:

$$(\sigma_x, \sigma_y, \sigma_z) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{B.1}$$

We consider a particle whose *guiding state* is described by the spinor

$$|\psi\rangle = \alpha|z,+\rangle + \beta|z,-\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{B.2}$$

with $|z,+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the eigenstate of the z-spin operator $\hat{S}_z$ with eigenvalue $+\hbar/2$ and $|z,-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the eigenstate of the $\hat{S}_z$ operator with eigenvalue $-\hbar/2$. The constants α and β provides the correct normalization $|\alpha|^2 + |\beta|^2 = 1$ of the superposition state. Now, from expression Eq. (7) in the manuscript, we can easily compute the following local-in-position weak value of the spin in the z-direction:

$$\text{Re}\left(\frac{\langle x|\hat{S}_z|\psi\rangle}{\langle x|\psi\rangle}\right) = \text{Re}\left(\frac{(+\frac{\hbar}{2})\alpha\langle x|z,+\rangle + (-\frac{\hbar}{2})\beta\langle x|z,-\rangle}{\alpha\langle x|z,+\rangle + \beta\langle x|z,-\rangle)}\right) = \frac{\hbar}{2}\text{Re}\left(\frac{\alpha\langle x|z,+\rangle - \beta\langle x|z,-\rangle}{\alpha\langle x|z,+\rangle + \beta\langle x|z,-\rangle)}\right). \tag{B.3}$$

For the same state in Eq. (B.2), we can compute the local-in-position weak value of the spin in the y-direction. By using $|z,+\rangle = 1/\sqrt{2}(|y,+\rangle + |y,-\rangle)$ and $|z,-\rangle = -i/(\sqrt{2}(|y,+\rangle - |y,-\rangle)$, we rewrite the *guiding state* in Eq. (B.2) as

$$|\psi\rangle = \alpha|z,+\rangle + \beta|z,-\rangle = \frac{\alpha - i\beta}{\sqrt{2}}|y,+\rangle + \frac{\alpha + i\beta}{\sqrt{2}}|y,-\rangle. \tag{B.4}$$

Using again Eq. (7) in the manuscript, we compute the following local-in-position weak value of the spin in the y-direction:

$$\text{Re}\left(\frac{\langle x|\hat{S}_y|\psi\rangle}{\langle x|\psi\rangle}\right) = \text{Re}\left(\frac{(+\frac{\hbar}{2})\frac{\alpha-i\beta}{\sqrt{2}}\langle x|y,+\rangle + (-\frac{\hbar}{2})\frac{\alpha+i\beta}{\sqrt{2}}\langle x|y,-\rangle}{\frac{\alpha-i\beta}{\sqrt{2}}\langle x|y,+\rangle + \frac{\alpha+i\beta}{\sqrt{2}}\langle x|y,-\rangle)}\right) = \frac{\hbar}{2}\text{Re}\left(\frac{\frac{\alpha-i\beta}{\sqrt{2}}\langle x|y,+\rangle - \frac{\alpha+i\beta}{\sqrt{2}}\langle x|y,-\rangle}{\frac{\alpha-i\beta}{\sqrt{2}}\langle x|y,+\rangle + \frac{\alpha+i\beta}{\sqrt{2}}\langle x|y,-\rangle)}\right). \tag{B.5}$$

Finally, from the same state in Eq. (B.2), we can also compute the local-in-position weak value of the spin in the x-direction too. By using $|z,+\rangle = 1/\sqrt{2}(|x,+\rangle + |x,-\rangle)$ and $|z,-\rangle = -1/(\sqrt{2}(|x,+\rangle - |x,-\rangle)$, we rewrite the *guiding state* as

$$|\psi\rangle = \alpha|z,+\rangle + \beta|z,-\rangle = \frac{\alpha - \beta}{\sqrt{2}}|x,+\rangle + \frac{\alpha + \beta}{\sqrt{2}}|x,-\rangle \tag{B.6}$$

From expression (7) in the manuscript, we again compute straightforwardly the following local-in-position weak value of the spin in the x-direction:

$$\text{Re}\left(\frac{\langle x|\hat{S}_x|\psi\rangle}{\langle x|\psi\rangle}\right) = \text{Re}\left(\frac{(+\frac{\hbar}{2})\frac{\alpha-\beta}{\sqrt{2}}\langle x|x,+\rangle + (-\frac{\hbar}{2})\frac{\alpha+\beta}{\sqrt{2}}\langle x|x,-\rangle}{\frac{\alpha-\beta}{\sqrt{2}}\langle x|x,+\rangle + \frac{\alpha+\beta}{\sqrt{2}}\langle x|x,-\rangle)}\right) = \frac{\hbar}{2}\text{Re}\left(\frac{\frac{\alpha-\beta}{\sqrt{2}}\langle x|x,+\rangle - \frac{\alpha+\beta}{\sqrt{2}}\langle x|x,-\rangle}{\frac{\alpha-\beta}{\sqrt{2}}\langle x|x,+\rangle + \frac{\alpha+\beta}{\sqrt{2}}\langle x|x,-\rangle)}\right). \tag{B.7}$$

1. Weak values that are not always dynamic intrinsic properties of a modal theory

The above result in Eq. (B.3) has two interesting limits. When $\alpha = 1$ and $\beta = 0$, $|\psi\rangle = |z,+\rangle$, the local-in position weak value of the spin in the z-direction is,

$$\text{Re}\left(\frac{\langle x|\hat{S}_z|\psi\rangle}{\langle x|\psi\rangle}\right) = \text{Re}\left(\frac{(+\frac{\hbar}{2})\langle x|z,+\rangle}{\langle x|z,+\rangle}\right) = \frac{\hbar}{2} \tag{B.8}$$

Identically, for $\alpha = 0$ and $\beta = 1$, $|\psi\rangle = |z, +\rangle$, the local-in position weak value of the spin in the z-direction is,

$$\text{Re}\left(\frac{\langle x|\hat{S}_z|\psi\rangle}{\langle x|\psi\rangle}\right) = \text{Re}\left(\frac{(-\frac{\hbar}{2})\langle x|z,-\rangle}{\langle x|z,-\rangle}\right) = -\frac{\hbar}{2} \tag{B.9}$$

In both cases, the states that defined the quantum system are eiegnestate of the $\hat{S}_z$. Thus, according to the orthodox eigenstate-eigenvalue-link, such quantum systems have well-defined properties of the spin in the z direction: the eiegnvalues $\hbar/2$ and $-\hbar/2$, respectively.

In the general case, the spin in the z-direction of the state $|\psi\rangle = \alpha|z,+\rangle + \beta|z,-\rangle$, is not a well-defined Orthodox property. The relevant point in the present subsection is discussing if the spin in the z direction (or in the three cartesians directions) is a well-defined intrinsic property in Modals theories or not. Certainly, the position x has been used as the *property state* in Eq. (B.3), Eq. (B.5) and Eq. (B.7). Thus, we are discussing if the z-component of the spin is an intrinsic property $\text{Re}\left({}_x\langle\hat{S}_z\rangle_\psi\right)$ in the type of Modal theories known as Bohmian mechanics (or also known as de Broglie-Bohm theories). If a property reaches the status of intrinsic property in a Modal theory is because such property is part of ontology of such Modal theory. Is spin an ontological property of Bohmian mechanics? Well, the answer depends on how the ontology of Bohmian mechanics is internally defined. Regarding the spin in Bohmian mechanics, there are basically two opposite schools:

- The first school argues that the z component of spin is an additional (hidden) ontological variable of a particle in the Bohmian theory, as is the position of the particle [7–9]. The y and x components of the spin, as defined above, are also well-defined ontic properties. This school keeps the *classical* idea of spin as a little spinning ball of charge with definite spin angular momentum vector. Such picture requires, not only the z component of the spin to be an ontic property at all times, but also the x and y components to be ontic variables. They imaging the spin as a well defined vector in the 3D physical space. Then, for them, the weak values in Eq. (B.3), in Eq. (B.3) and Eq. (B.5) are the measurement of a well-defined intrinsic properties that defines the "spin vector" of three components at all times:

$$\left(\text{Re}\left(\frac{\langle x|\hat{S}_x|\psi\rangle}{\langle x|\psi\rangle}\right), \text{Re}\left(\frac{\langle x|\hat{S}_y|\psi\rangle}{\langle x|\psi\rangle}\right), \text{Re}\left(\frac{\langle x|\hat{S}_z|\psi\rangle}{\langle x|\psi\rangle}\right)\right) = \left(\text{Re}\left({}_x\langle\hat{S}_x\rangle_\psi\right), \text{Re}\left({}_x\langle\hat{S}_y\rangle_\psi\right), \text{Re}\left({}_x\langle\hat{S}_z\rangle_\psi\right)\right) \tag{B.10}$$

  In conclusion, for this first school, when the system is detached from a measurement context, the "spin vector" of three components is indeed an intrinsic property (given by the above expression) with a deep physical meaning.

- The second school argues that spin is not an onticological property of Bohmian mechanics [10–12]. Only the position (not the spin) is the ontological (hidden) variables in this school (at least for non-relativistic quantum mechanics). In fact, this school argues that, even when the spin is measured, the spin is not an ontic property [10–12]. The value of the laboratory that we call spin is, in fact, the value of a position of a pointer in a Stern-Gerlach experiment [10]. Such pointer positions will obviously depend on the coefficients α and β of the quantum state in Eq. (B.3). For this school, even being considered a Modal theory under the umbrella of Bohmian mechanics, the experimental value given by the weak values Eq. (B.3), Eq. (B.5) and Eq. (B.7) are just juggling with experimental data with no ontological meaning. In conclusion, for this second school, the "spin vector" of three components It cannot be an intrinsic property, because it is not even an ontic property.

It is far from the scope of this paper to judge (or show our preferences between) both Bohmian definitions of spin. In this subsection, we do only want to show a clear example indicating that the selection of what are the intrinsic (ontic) properties of a given Modal theory is just an arbitrary decision of each Modal theory, without been influenced by the experimental accessibility of weak values. In simple words, for the first Modal theory, the weak values computed in Eq. (B.3), Eq. (B.5) and Eq. (B.7) are very relevant values with a deep physical meaning. For the second Modal theory, the weak values computed in Eq. (B.3), Eq. (B.5) and Eq. (B.7) are just a manipulation of experimental data of an ensemble of experiments, without any deep physical meaning. Notice that this second school will insist that spin is not an ontological property even on the scenarios given by Eq. B.8 and B.9 where the Orthodox theory accepts that spin is a well defined property (following the eigenstate eiegnvalue link). In any case, the empirical results of both Bohmian schools on spin are in perfect agreement with all empirical results and basic theorems of the Orthodox theory.

2. Weak values and intrinsic properties are context-free

The concept of conextuality revolves around the role of the measuring apparatus as an active element in the definition of the properties of a system. As such, any discussion of contextuality requires to clarify which degrees of freedom are understood as the system and which degrees of freedom are the measuring apparatus. Once such distinction is made clear, it is obvious that dynamic intrinsic properties discussed in this paper are context-free because, by construction, they are defined without involving the degrees of freedom of the measuring apparatus (only the degrees of freedom of the system are relevant with the unitary evolution of a closed system) for such property. In other words, there is no measuring context involved in the definition of the intrinsic properties. Once we select a Modal theory and a dynamical intrinsic property as part of the ontology of such theory, then, the associated weak value must also be context-free because the weak value and the dynamic intrinsic property are, both, the same.

But, as discussed in the next appendix C, weak values are measured in a laboratory. Then, why we consider them as context-free? The solution to this apparent paradox is that the weak values are obtained under a very special protocol involving an ensemble of identical experiments. In each experiment, the quantum system is perturbed by the two-times measurement (in a contextual way) as we have shown in appendix A. However, when doing the post-processing of the data, the individual perturbation present in each experiment is compensated so that the final weak value has eliminated the back-action (contamination) due to the measuring apparatus in each experiment. Only when the experimental procedure ensures that we are dealing with a back-action free value, we get a weak value. In the subsection C 2 we clarify under which measuring circumstances we can ensure that we have an experimental weak value equal to that of the context-free weak value.

We want to provide some additional clarification about the context-free intrinsic properties and the associated weak values. We have been able in Eq. (B.3), Eq. (B.5) and Eq. (B.7) to find the weak values for the three Cartesian components a the spin of a particle. Such values are understood as the real intrinsic values of the spin of the particle for the first Bohmian school mentioned above. Notice that the order of the computation of the different weak values of spin is irrelevant. For example, the sequence (1st) weak value of S_z; (2nd) weak value of S_y and (3nd) weak value of S_x provide the same results as any other sequence, for example, (1st) weak value of S_x; (2nd) weak value of S_z and (3nd) weak value of S_y. The reason why this values are independent of the order are obviously that we have used always the same state in Eq. (B.2) for the computation of Eq. (B.3), Eq. (B.5) and Eq. (B.7), either for defining an intrinsic property or a weak value. The Modal theory defines the intrinsic properties of one component of spin without implying any manipulation of the state for a posterior definition of other component of spin. For the empirical result of weak values, they are always computed from an ensemble of experiments with identical preparation of the state in Eq. (B.2) and involve the weak measurement of only one type of spin component. In conclusion, intrinsic properties and weak values are context-free by construction.

It is well-known, however, that the three components that define the intrinsic value of the spin (as defended by the first school of the Bohmian theory) will not be the three components measured in the laboratory in a unique experiment. The reason is clear. In a laboratory, the first measurement of the z component of the spin will collapse the initial state Eq. (B.2) into one of the two eigenstate of the $\hat{S}_z$ operator. Then, we can no longer use state Eq. (B.2) to evaluate the subsequent results of the measurement of other component of spin. Therefore, the order in the measurement affect the values that we get for each spin component. In conclusion, the measured components of spin properties are contextual by construction. For this reason, the Orthodox theory (and also the second school of the Bohmian theory) prefer to say that the property of S_x, S_y and S_x are not defined prior to its measurement. On the contrary, the first Bohmian school prefer to say that the spin components are always well-defined at the ontological level (ontic variable) but they change its numerical value when interacting with the measuring apparatus (the context). Thus, ontic variables have to contextual to satisfy empirical results (this is exactly the case for all Bohmian ontic variables discussed here). But, we emphasize that an ontic variable can be understood as an intrinsic property only when we detach the system from the measuring apparatus, which is not the case in this example.

The important point in this section is that there is no contradiction at all between the first conclusion, *intrinsic properties and weak values of the components of spin are context-free by construction* and the second conclusion *the measured components of spin (and the ontic variables) are contextual by construction.* We are talking about two different things.

Appendix C: EXPERIMENTAL ACCESSIBILITY OF WEAK VALUES

In the manuscript, we have defined weak values as intrinsic properties of Modal theories. We show in this appendix that this definition is compatible with what the original development done by Aharonov, Albert and Vaidman in 1988 [13] which was focused on its experimental accessibility. Next, we reproduce such original development of the weak value done by Aharonov, Albert and Vaidman in 1988 [13] with our own notation used in the paper. We identifying that no collapse (no back-action) is considered in the development. Then, we provide a path for discussing the experimental accessibility of weak values directly for the probability of two measurements done in appendix A.

1. Original derivation of weak values

The weak value requires two measurements: a first weak measurement, plus a second strong measurement. We know that a weak measurement of the system is, in fact, a strong measurement of another complementary system coupled to what we have defined initially as the system. We define such complementary system as the ancilla described by the degrees of freedom y. Thus, the first weak measurement requires an entanglement between the system and the ancilla. Such entanglement can be obtained through a unitary interaction of the ancilla with the system given by the Von Neumann time-evolution operator $\exp(\frac{-i}{\hbar}\lambda\hat{S}\otimes\hat{P}_a)$ where $\hat{S}$ is the system operator (the observable we want to measure), $\hat{P}_a$ is the momentum operator for the ancilla and λ is the effective coupling constant quantifying the coupling strength between the system and ancilla.

We assume the full state of the system-ancilla to be initially described by a separable state vector:

$$|\Psi(0)\rangle = |\psi(0)\rangle \otimes |\phi(0)\rangle = |\psi(0)\rangle \otimes \int a(y,0)|y\rangle dy, \tag{C.1}$$

where the system state vector is $|\psi(0)\rangle$ and the ancilla state is $|\phi(0)\rangle = \int a(y,0)|y\rangle dy$ with $a(y_k,0) = \langle y_k|\phi(0)\rangle$ the ancilla wave packet in the y-position representation at time $t=0$. The process of the pre-measurement of the first weak measurement can be mathematically described as follows,

$$|\Psi(t)\rangle = \exp\left(\frac{-i}{\hbar}\lambda\hat{S}\otimes\hat{P}_a\right)|\Psi(0)\rangle. \tag{C.2}$$

Now, assuming a small coupling between the system and the ancilla, the previous pre-measurement evolution of the weak measurement can be defined as:

$$\exp\left(\frac{-i}{\hbar}\lambda\hat{S}\otimes\hat{P}_a\right) \approx 1 - \frac{i}{\hbar}\lambda\hat{S}\otimes\hat{P}_a \tag{C.3}$$

assuming λ is small. Because of (C.1) and (C.3), one can write (C.2) as:

$$\begin{aligned} |\Psi(t)\rangle &= \exp\left(\frac{-i}{\hbar}\lambda\hat{S}\otimes\hat{P}_a\right)|\Psi(0)\rangle \approx \left(1-\frac{i}{\hbar}\lambda\hat{S}\otimes\hat{P}_a\right)|\Psi(0)\rangle \\ &\approx \left(1-\frac{i}{\hbar}\lambda\hat{S}\otimes\hat{P}_a\right)|\psi(0)\rangle\otimes|\phi(0)\rangle \approx |\psi(0)\rangle\otimes|\phi(0)\rangle - \frac{i}{\hbar}\lambda\hat{S}\;|\psi(0)\rangle\otimes\hat{P}_a\;|\phi(0)\rangle \end{aligned} \tag{C.4}$$

Now, one would have to implement the collapse of the ancilla during the first weak measurement. However, the typical development done in the literature neglects such back-action on the system due to the first weak measurement and proceeds with the second strong (post-selection) measurement.

We have noticed in the previous section A 3 of this appendix that there is a dramatic difference between assuming $\exp(i\lambda\hat{S}\otimes\hat{P}_a/\hbar) \approx 1 - i\lambda\hat{S}\otimes\hat{P}_a/\hbar$ in Eq. (C.3) and assuming $\exp(i\lambda\hat{S}\otimes\hat{P}_a/\hbar) = 1$. The first in Eq. (C.3), iplies a non-zero coupling between the system and the ancilla, while $\exp(i\lambda\hat{S}\otimes\hat{P}_a/\hbar) = 1$ means no coupling at all, i.e. no measurement at all, and a state after the measurement given by $|\Psi(t)\rangle = \exp(i\lambda\hat{S}\otimes\hat{P}_a/\hbar)|\Psi(0)\rangle \approx |\Psi(0)\rangle$. We argue that neglecting the back-action of the first weak measurement on the system is not possible. We will return to this point in section C 2 when discussing the computation of weak values directly from two-times probabilities.

The second strong measurement is done in a different measuring apparatus with pointer y_ω. Thus, in principle, we would have to include a pre-measurement entanglement of the system with a new degree of freedom of the second apparatus and then collapse the system. However, we can simplify the discussion without introducing any relevant approximation by just assuming that the state of the system and ancilla after the post selection given by the eigenvalue g_w is given by the projector $|g_w\rangle\langle g_w|\otimes\hat{1}_a$ multiplied by the state in (C.4). We define $\hat{1}_a$ as the unitary operator for the ancilla that is not affected by this second measurement. Thus, after the post-selection the state of the system and ancilla is:

$$\begin{aligned} \left(|g_w\rangle\langle g_w|\otimes\hat{1}_a\right)|\Psi(t)\rangle &\approx |g_w\rangle\langle g_w|\psi(0)\rangle\otimes|\phi(0)\rangle - \frac{i}{\hbar}\lambda|g_w\rangle\langle g_w|\hat{S}|\psi(0)\rangle\otimes\hat{P}_a\;|\phi(0)\rangle \\ &\approx \langle g_w|\psi(0)\rangle\;|g_w\rangle\otimes\left(1-\frac{i}{\hbar}\lambda\frac{\langle g_w|\hat{S}|\psi(0)\rangle}{\langle g_w|\psi(0)\rangle}\hat{P}_a\right)|\phi(0)\rangle \\ &\approx \langle g_w|\psi(0)\rangle\;|g_w\rangle\otimes\exp\left(-\frac{i}{\hbar}\lambda\frac{\langle g_w|\hat{S}|\psi(0)\rangle}{\langle g_w|\psi(0)\rangle}\hat{P}_a\right)|\phi(0)\rangle \end{aligned} \tag{C.5}$$

Now, by writing the ancilla wave packet of the first measurement as in (C.1), $\phi(0) = \int a(y,0)|y\rangle dy$ and using the identity $\hat{1}_a = \int |y'\rangle\langle y'|dy'$ we have can easily write the ancilla wave packet $a(y_k,t) = \langle y_k|\phi(t)\rangle$ in the position representation as at time t:

$$a(y_k,t) = \langle y_k| \exp\left(-\frac{i}{\hbar}\lambda\frac{\langle g_w|\hat{S}|\psi(0)\rangle}{\langle g_w|\psi(0)\rangle}\hat{P}_a\right)\int |y'\rangle\langle y'|dy' \int \; a(y,0)|y\rangle dy =$$

$$\int dy \left(1-\frac{i}{\hbar}\lambda\frac{\langle g_w|\hat{S}|\psi(0)\rangle}{\langle g_w|\psi(0)\rangle}\langle y_k|\hat{P}_a|y\rangle + ...\right)\; a(y,0) =$$

$$\left(1-\lambda\frac{\langle g_w|\hat{S}|\psi(0)\rangle}{\langle g_w|\psi(0)\rangle}\frac{\partial}{\partial y} + ...\right)\; a(y,0)|_{y=y_k} = a\left(y_k - \lambda\frac{\langle g_w|\hat{S}|\psi(0)\rangle}{\langle g_w|\psi(0)\rangle}, 0\right) \tag{C.6}$$

where we have used $\langle y_k|\hat{P}_a|y\rangle a(y,0) = -i\hbar\int dy\delta(y_k-y)\frac{\partial a(y,0)}{\partial y}$. We use here a Taylor series to show that the ancilla wave packet at time t, after the weak and post selected measurements, is not centered at zero, but to a new position $\lambda\frac{\langle g_w|\hat{S}|\psi(0)\rangle}{\langle g_w|\psi(0)\rangle}$. We clearly recognize the definition of the weak value in the displacement of the ancilla wave packet $g_\omega\langle\hat{S}\rangle_{\psi(t)} \equiv \frac{\langle g_\omega|\hat{S}|\psi(t)\rangle}{\langle g_\omega|\psi(t)\rangle}$. The collapse of the ancilla due to the weak measurement is still missing in the previous development. We have only developed the pre-measurement (unitary) evolution of the first weak measurement. At each weak measurement, the ancilla will be collapsed to a unique position y_k. We can assume that the probability of the different y_k is given by the Born law $|a(y_k,t)|^2 = \left|a\left(y_k - \lambda\frac{\langle g_w|\hat{S}|\psi(0)\rangle}{\langle g_\omega|\psi(0)\rangle}, 0\right)\right|^2$. In fact, $|a(y_k,t)|^2$ is the probability of getting y_k conditioned to the fact that we have also measured g_ω given by $P(y_k|q_\omega,t) = |a(y_k,t)|^2$. Thus, the central position of ancilla wave packet in Eq. (C.6) is gievn by the ensemble of weak measurements y_k when properly post-selected by g_ω as:

$$\begin{aligned}\mathrm{Re}\left(g_\omega\langle\hat{S}\rangle_{\psi(t)}\right) \;&\equiv\; \mathrm{Re}\left(\frac{\langle g_\omega|\hat{S}|\psi(t)\rangle}{\langle g_\omega|\psi(t)\rangle}\right) = \frac{1}{\lambda}\int_{-\infty}^{\infty} dy_k|a(y_k,y_\omega,t)|^2 y_k = \frac{1}{\lambda}\int_{-\infty}^{\infty} dy_k P(y_k|y_\omega,t)|y_k \\ &= \;\frac{1}{\lambda}\int_{-\infty}^{\infty} dy_k\frac{P(y_k,y_\omega,t)}{P(y_\omega)}y_k = \frac{1}{\lambda}\frac{\int_{-\infty}^{\infty} dy_k P(y_k,y_\omega,t)y_k}{\int_{-\infty}^{\infty} dy_k P(y_k,y_\omega,t)}\end{aligned} \tag{C.7}$$

where we have used that the definition of conditional probability $P(y_k|y_\omega,t) = \frac{P(y_k,y_\omega,t)}{P(y_\omega,t)} = \frac{P(y_k,y_\omega,t)}{\int_{-\infty}^{\infty} dy_k P(y_k,y_\omega,t)}$ with $P(y_\omega) = \int_{-\infty}^{\infty} dy_k P(y_k,y_\omega,t)$. Notice that we have identified $g_\omega = \frac{y_\omega}{\lambda}$ where g_ω is the eigenvalue of the system and y_ω is the pointer of the second measuring apparatus. Such identification is obvious for the second strong measurement.

The results in Eq. (C.7) are correct, but they hide the difficulties of accessing experimentally to the weak values. In other words, special conditions depending on the operator $\hat{S}$ are required to satisfy the identity Eq. (C.7). The reasons, as we have indicated, is because the above development of weak value avoids the back-action on the system due to the first weak measurement. We have shown in the previous appendix A that in the two times experiments, one cannot neglect the back action on the system due to the first weak measurement never. The fact that the weak measurement provides no perturbation on the state of the system is only true when assuming $\exp\left(\frac{-i}{\hbar}\lambda\hat{S}\otimes\hat{P}_a\right) \approx 1$ in Eq. (C.3). But, then, we have shown that this approximation means no measurement at all. Next, we will discussion the experimental accessibility of weak values without such assumption.

2. Derivation of weak values based on two-time genralized Von-Neuman measurements

We want here to start from the two time probability $P(y_\omega,y_k)$ in Eq. (A.10) without any approximation about the back-action of the first weak measurement on the system, to see under which conditions we are able to reach a weak value experimentally. In simpler words, we want to check expression Eq. (C.7) from a more general development. We rewrite Eq. (C.7) here as an approximation:

$$\mathrm{Re}\left({}_{g_\omega}\langle\hat{S}\rangle_{\psi(t_2)}\right) = \mathrm{Re}\left(\frac{\langle g_\omega|\hat{S}|\psi(t_2)\rangle}{\langle g_\omega|\psi(t_2)\rangle}\right) \approx \frac{1}{\lambda}\frac{\int dy_k\; y_k P(y_\omega,y_k)}{\int dy_k P(y_\omega,y_k)}. \tag{C.8}$$

We can use the first order Taylor expansion for the first ancilla wave function $a(y_k-\lambda s_i) \approx a(y_k) - \lambda s_i\frac{\partial a(y_k)}{\partial y_k}$ and a Kronecker delta function for the second ancilla wave function $|a(y_\omega - \lambda g_j)|^2 = |\delta_{y_\omega,\lambda g_j}|^2$. Using this in Eq. (A.10) and defining $g_w = \frac{y_w}{\lambda}$ we get

$$P(y_\omega, y_k) \approx \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} \left(a^*(y_k) - \lambda s_{i'} \frac{\partial a^*(y_k)}{\partial y_k} \right) \left(a(y_k) - \lambda s_i \frac{\partial a(y_k)}{\partial y_k} \right). \tag{C.9}$$

Using Eq. (A.13) in Eq. (C.9) we can evaluate the numerator in Eq. (C.8) as

$$\begin{aligned}
\int dy_k\ y_k P(y_\omega, y_k) &\approx \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} \lambda \frac{1}{2}(s_i + s'_i) \\
&\approx \sum_{i,i'} \langle \psi(t_1)|s_{i'}\rangle\langle s_i|\psi(t_1)\rangle\langle s_{i'}|\hat{U}^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}|s_i\rangle \lambda \frac{1}{2}(s_i + s_{i'}) \\
&\approx \frac{\lambda}{2}\langle\psi(t_1)|U^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}\hat{S}|\psi(t_1)\rangle + \frac{\lambda}{2}\langle g_\omega|\hat{U}|\psi(t_1)\rangle\langle\psi(t_1)|\hat{S}\hat{U}^\dagger|g_\omega\rangle,
\end{aligned} \tag{C.10}$$

where the dependence on the second measurement is indicated by the term $|g_w\rangle$. We have used the identity developed in Eq. (A.13) and also $\hat{S} = \sum_i |s_i\rangle s_i \langle s_i| = \sum_i |s_{i'}\rangle s_{i'} \langle s_{i'}|$ and $\sum_i |s_i\rangle\langle s_i| = \sum_{i'} |s'_i\rangle\langle s_{i'}| = 1$ and we allow a unitary evolution described by $\hat{U}$ between the measurements. The commutation of the operators S and U in Eq. (C.10) requires some further discussion.

Our goal of achieving the approximation in Eq. (C.8) necessitates that the operator $\hat{S}$ should commute with the unitary operator $\hat{U}$. For the case of work, the operator $\hat{S}$ is the Hamiltonian $\hat{H}$ which commutes with the unitary operator. In the case of the momentum operator $\hat{P}$ using the assumption of a flat potential leads to the commutation of the momentum operator $\hat{P}$ with the unitary operator (as far as the time between the weak and projective measurements are done within a short time interval $t_2 - t_1$). Under this assumption, we get

$$\int dy_k\ y_k P(y_\omega, y_k) \approx \mathrm{Re}\left[\lambda\langle\psi(t_2)|g_\omega\rangle\langle g_\omega|\hat{S}|\psi(t_2)\rangle\right]. \tag{C.11}$$

In any case, we emphasize that the fact that the operator $\hat{S}$ commutes with the unitary operator $\hat{U}$ is not always satisfied.

We evaluate now the denominator of Eq. (C.8)as follows:

$$\int dy_k P(y_\omega, y_k) \approx \int dy_k \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} \left(a^*(y_k) - \lambda s_{i'} \frac{\partial a^*(y_k)}{\partial y_k} \right) \left(a(y_k) - \lambda s_i \frac{\partial a(y_k)}{\partial y_k} \right). \tag{C.12}$$

Noting that $\int dy_k |a(y_k)|^2 = 1$ and $\int dy_k a(y_k) \frac{\partial a(y_k)}{\partial y_k} = 0$ we get

$$\int dy_k P(y_\omega, y_k) \approx \int dy_k \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} \left(|a(y_k)|^2 + \lambda^2 s_i s_{i'} \left| \frac{\partial a(y_k)}{\partial y_k} \right|^2 \right). \tag{C.13}$$

Again, one is tempted to argue that in general $c^*_{i'} c_i |a(y_k)|^2 \gg \lambda^2 c^*_{i'} c_i s_i s_{i'} \left| \frac{\partial a(y_k)}{\partial y_k} \right|^2$ so that the last term can be neglected. However, as a test, by using $a(y_k) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} \exp\left(\frac{-y_k^2}{2\sigma^2}\right)$ we can check for which values of $y = y_k$ this is true. We get that the second coefficient becomes comparable to the first one when $y \approx \sigma^2/\lambda$.

The solution to the above-mentioned source of contamination in the denominator is quite simple from an experimental point of view. In a real experiment, the rare events corresponding to $y > \sigma^2/\lambda$ will not provide a significant contribution. Notice that we are approximating here the marginal probability in Eq. (C.13) with $P(y_\omega, y_k) \approx 0$ for $y_k \to \infty$. The same approximation cannot be done when dealing with the correlation functions computed in the text and in A 3. Then, we can evaluate the denominator as

$$\begin{aligned}
\int dy_k P(y_\omega, y_k) &\approx \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} = \sum_{i,i'} \langle\psi(t_1)|s_{i'}\rangle\langle s_i|\psi(t_1)\rangle\langle s_{i'}|\hat{U}^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}|s_i\rangle \\
&\approx \sum_{i,i'} \langle\psi(t_1)|s_{i'}\rangle\langle s_{i'}|\hat{U}^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}|s_i\rangle\langle s_i|\psi(t_1)\rangle = \langle\psi(t_1)|\hat{U}^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}|\psi(t_1)\rangle.
\end{aligned} \tag{C.14}$$

Finally, using Eq. (C.14) and Eq. (C.11), we arrive at expression (C.8).

In conclusion, we have been able to reproduce Eq. (C.7) from the two time probability $P(y_\omega, y_k)$ in Eq. (A.10) without neglecting the back-action on the system due to the first weak measurement. Importantly, we have seen that the fact that the operator $\hat{S}$ commutes with the unitary operator $\hat{U}$ is not always satisfied. This implies that it is not always possible to reach the weak value (or intrinsic properties) from the experimental values of the two times probability $P(y_\omega, y_k)$. These difficulties are not evidenced in the development done in Sec. C 1 because it was (erroneously) assumed that a weak measurement has no perturbation on the system.

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