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LARGE VALUES OF DIRICHLET L -FUNCTIONS AT ZEROS OF A CLASS OF L -FUNCTIONS

JUNXIAN LI

ABSTRACT. In this paper, we are interested in obtaining large values of Dirichlet L -functions evaluated at zeros of a class of L -functions, that is, $\max_{T \leq \Im \rho \leq 2T} L(\rho, \chi)$, where χ is a primitive Dirichlet character and F belongs to a class of L -functions. The class we consider includes L -functions associated to automorphic representations of $GL(n)$ over \mathbb{Q} .

1. INTRODUCTION

The study of the value distribution of the Riemann zeta function dates back to the work of H. Bohr. Using the theory of almost periodic functions, he showed that $\zeta(s)$ takes any nonzero complex value z infinitely often in any strip $1 < \Re(s) < 1 + \epsilon$. Later in [6], together with B. Jessen, he showed that $\log \zeta(\sigma + it)$ has a continuous limiting distribution on the complex plane for any $\sigma > \frac{1}{2}$. On the critical line, A. Selberg [42, 41] showed that $\log |\zeta(\frac{1}{2} + it)|$ is approximately Gaussian distributed in the sense that

$$\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{\log |\zeta(\frac{1}{2} + it)|}{\sqrt{\frac{1}{2} \log \log T}} \geq \lambda \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-x^2/2} dx, \text{ as } T \rightarrow \infty. \quad (1.1)$$

This implies that the typical size of $|\zeta(\frac{1}{2} + it)|$ is $\exp\left(\sqrt{\frac{1}{2} \log \log T}\right)$. Regarding the exceptional large values of $|\zeta(\frac{1}{2} + it)|$, the Lindelöf Hypothesis asserts that $|\zeta(\frac{1}{2} + it)| = o(t^\epsilon)$ for any $\epsilon > 0$. Assuming the Riemann Hypothesis, one can show ([34, 14]) that

$$|\zeta(\frac{1}{2} + it)| = O\left(\exp\left(c \frac{\log t}{\log \log t}\right)\right), \text{ as } t \rightarrow \infty,$$

for some absolute constant c . D. Farmer, S. Gonek, and C. Hughes [20] conjectured that the maximum value of $\zeta(\frac{1}{2} + it)$ for t in the interval $[0, T]$ is of order $\exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right) \sqrt{\log T \log \log T}\right)$. For omega results, E. C. Titchmarsh [46, Theorem 8.12] first showed that there exist arbitrarily large t with $|\zeta(\frac{1}{2} + it)| \geq \exp(\log^\alpha t)$ for any $\alpha < 1/2$. Under the Riemann Hypothesis, H. Montgomery [37], proved that there exist arbitrarily large values of t such that

$$|\zeta(\frac{1}{2} + it)| \gg \exp\left(\frac{1}{20} \sqrt{\frac{\log t}{\log \log t}}\right).$$

R. Balasubramanian and K. Ramachandra [2] showed unconditionally that there are arbitrarily large t such that

$$|\zeta(\frac{1}{2} + it)| \gg \exp\left(c \sqrt{\frac{\log t}{\log \log t}}\right), \quad (1.2)$$

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for some positive constant c . Later, K. Soundararajan [43] introduced the resonance method and obtained (1.2) for $c = 1 + o(1)$. More recently, A. Bondarenko and K. Seip in a series of papers [9, 10, 11] proved that for any $0 \leq \beta < 1$ and $0 \leq c \leq \sqrt{1 - \beta}$, if T is sufficiently large, then

$$\max_{T^\beta \leq t \leq T} |\zeta(\frac{1}{2} + it)| \gg \exp\left(c \sqrt{\frac{\log T \log \log \log T}{\log \log T}}\right). \quad (1.3)$$

The constant c has been further improved by R. de la Bretèche and G. Tenenbaum [19] by a factor of $\sqrt{2}$. Fewer results have been investigated on large values of degree $\zeta(s)$ at discrete points on the critical line. X. Li and M. Radziwiłł [33] considered the large values of $\zeta(\frac{1}{2} + it)$ in vertical arithmetic progressions on the critical line. J. Kalpokas and P. Sarka [31] considered large values at generalized Gram points. In this paper, we consider the large values of the Riemann zeta function and Dirichlet L -functions at the zeros of a class of L -functions.

Theorem 1.1. *Let χ be a primitive Dirichlet L -functions with conductor $q > 1$. If all non-trivial zeros of $L(s, \chi)$ are on the critical line $\Re(s) = \frac{1}{2}$, then for T sufficiently large,*

$$\max_{\substack{L(\rho, \chi)=0 \\ T \leq \Im \rho \leq 2T}} |\zeta(\rho)| \gg \exp\left(c \sqrt[4]{\frac{\log T}{\phi(q)(\log \log T)^2}}\right),$$

where c is some absolute positive constant.

This can be improved if we assume the Riemann Hypothesis holds for all Dirichlet L -functions.

Theorem 1.2. *Let χ and ψ be two different primitive Dirichlet characters. Under the assumption that the Riemann Hypothesis is true for all Dirichlet L -functions,*

$$\max_{\substack{L(\rho, \psi)=0 \\ T \leq \Im \rho \leq 2T}} |L(\rho, \chi)| \gg \exp\left(c \sqrt{\frac{\log T}{\phi(d) \log \log T}}\right)$$

for some $c > 0$, where d is the least common multiple of the conductors of χ and ψ .¹

It is believed that the values of distinct primitive L -functions are uncorrelated. For example, it is conjectured that different primitive Dirichlet L -functions have no common non-trivial zeros ([23, Conjecture 3]). A. Fujii [22] proved this is true for a positive proportion of distinct primitive Dirichlet characters. Under the Riemann Hypothesis, B. Conrey, A. Ghosh, and S. Gonek [15, 16] showed that at most two-thirds of the zeros of $\zeta(s)$ are also zeros of $L(s, \chi)$, where χ is a non-principal Dirichlet character. They remarked in [16] that similar results hold for Dirichlet L -functions with inequivalent characters under the Generalized Riemann Hypothesis. R. Garunkštis and J. Kalpokas [24] gave a lower bound for the proportion uniformly in the size of the conductors of the characters. Our result shows that under GRH, the values of $\zeta(s)$ at the zeros of another primitive L -function can be almost as large as the extreme large values of $\zeta(s)$ on the critical line without constraints.

Even though we were not able to obtain a bound as good as in (1.3) for individual L -functions, we can show a bound of the same shape as in (1.3) for the value $|\zeta_K(s)|$, the Dedekind zeta function associated to a number field K , on the critical line. When $K = \mathbb{Q}(\zeta_n)$, we know that $\zeta_K(s) = \prod_{\chi \pmod{n}} L(s, \chi)$. For a general number field, $\zeta_K(s)$ can be factored into Artin L -functions associated to irreducible representations of $\text{Gal}(K/\mathbb{Q})$. The Langlands reciprocity conjecture implies that each factor is an L -function for an irreducible cuspidal automorphic representation π of $GL(m)$ over \mathbb{Q} . Thus it make sense to study the values of $\zeta(s)$ at the zeros of automorphic L -functions. We give a result in this direction.

¹Under a weaker larger zero free region assumption, it was claimed [38, Theorem 1.2] that there exists of large value of $\zeta'(\rho)$ of size $\exp\left(c \sqrt{\frac{\log |\Im \rho|}{\log \log |\Im \rho|}}\right)$. However, the argument is problematic as it based on an upper bound for $\mathcal{H}(z)$, whose derivation missed a factor of $\max_{q \leq z} |y_q|$, which too big for the application.

Theorem 1.3. *Let $m \geq 2$ and π be an irreducible automorphic representation of $GL(m)$ over \mathbb{Q} . Assuming that $L(s, \pi)$ has all its non-trivial zeros on the line $\Re(s) = \frac{1}{2}$, then for sufficiently large T*

$$\max_{\substack{L(\rho, \pi)=0 \\ T \leq \Im \rho \leq 2T}} |\zeta(\rho)| \gg \exp \left(c_1 \sqrt[4]{\frac{\log T}{(\log \log T)^2}} \right),$$

for some positive constant c_1 depending on the conductor of π . Let $\chi \pmod{q}$ be a Dirichlet character such that $L(s, \pi \otimes \chi)$ has no pole at $s = 1$. Then under the Grand Riemann Hypothesis, we have for sufficiently large T

$$\max_{\substack{L(\rho, \pi)=0 \\ T \leq \Im \rho \leq 2T}} |L(\rho, \chi)| \gg \exp \left(c_2 \sqrt{\frac{\log T}{\log \log T}} \right),$$

where $c_2 > 0$ is some positive constant depending on π and χ .

As a corollary, we have

Theorem 1.4. *Let f be a holomorphic primitive cusp form of weight $k \geq 1$, level q and let χ be a primitive Dirichlet character. If $L(f, s)$ has all non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$, then for T large enough,*

$$\max_{\substack{L(f, \rho)=0 \\ T \leq \Im \rho \leq 2T}} |\zeta(\rho)| \gg \exp \left(c_3 \sqrt[4]{\frac{\log T}{(\log \log T)^2}} \right),$$

where $c_3 > 0$ is some positive constant depending on f and χ .

Automorphic L -functions are conjectured to belong to the Selberg class. The Riemann zeta function and Dirichlet L -functions are examples of degree 1 L -functions from the Selberg class. Many results mentioned above have been generalized to L -functions in the Selberg class with additional conditions. For example, E. Bombieri and D. Hejhal [7] proved that $\{\log(L_j(\frac{1}{2} + it))\}_{j=1}^N$ behave like independent Gaussian distributed random variables for certain L_j 's in the Selberg class. A short interval analogue was proved by E. Bombieri and A. Perelli [8]. In the same paper [8], they also considered the simultaneous non-vanishing in the setting of the Selberg class under certain additional hypotheses. Some unconditional results for cuspidal automorphic representations have been established by R. Raghunathan [40]. In terms of large values of L -functions in the Selberg class, C. Aistleitner and L. Pańkowaski [1] have some results for L -functions in the Selberg class with polynomial Euler products. Our result could apply to L -functions in the Selberg class with some additional conditions (see Section 3).

2. OUTLINE

We prove a general theorem for a class of functions \mathcal{S}^* . Theorem 1.1- Theorem 1.4 will then follow. The idea is to use the resonance method to compute

$$S_1 = \sum_{\substack{F(\rho)=0 \\ T_1 \leq \Im \rho \leq T_2}} L(\rho, \chi) X(\rho) Y(1 - \rho), \quad (2.1)$$

$$S_0 = \sum_{\substack{F(\rho)=0 \\ T_1 \leq \Im \rho \leq T_2}} X(\rho) Y(1 - \rho), \quad (2.2)$$

where $F(s)$ is an L -function in \mathcal{S} , $L(s, \chi)$ is a Dirichlet L -function, $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$ are chosen to be $\gg 1/\log T$ away from the zeros of F , $X(s) = \sum_{n \leq M} \frac{x_n}{n^s}$ and $Y(s) = \sum_{n \leq M} \frac{y_n}{n^s}$. If $\Re \rho = \frac{1}{2}$ and $y_n = \overline{x_n}$, then $X(\rho) Y(1 - \rho) = |X(\rho)|^2$ and thus

$$\max_{\substack{F(\rho)=0 \\ T \leq \Im \rho \leq 2T}} |L(\rho, \chi)| \geq \frac{|S_1|}{S_0}.$$

To compute the values of $L(\rho, \chi)$ at zeros of F , we use the method of Conrey, Ghosh and Gonek [17] in the study of simple zeros of $\zeta(s)$. To this end, we need some additional conditions on F and thus we restrict ourselves to a subclass \mathcal{S}^* . This was also used by Ng [39] in studying extreme values of $\zeta'(\rho)$. The size of the resonator requires a large zero free region so that the error terms are negligible. When taking F to be a Dirichlet L -function, the classical zero free region allows one to take $M = \exp(c\sqrt{\log T})$ for some positive constant c if there are no Siegel zeros. Even though non-existence of Siegel zeros is still an open problem, we do know that Siegel zeros are very rare if they exist. If we assume the non-trivial zeros of all L -functions \mathcal{S} are on the line $\Re(s) = \frac{1}{2}$, we can take the length of the resonator M to be T^c for some positive constant c under the Ramanujan Conjecture. This will give a bound of the form as in (1.2). An essential part is related to the study of the coefficients of $\frac{F'}{F}(s)$ in arithmetic progressions. We employ a variant of Perron's formula by J. Liu and Y. Ye [35] to avoid assuming the Generalized Ramanujan Conjecture on the coefficients of $F(s)$.

The organization of the rest of the paper is as follows. In Section 3, we define a class of L -functions \mathcal{S} and its subclass \mathcal{S}^* and give their properties. In Section 4, we show that L -functions associated to irreducible cuspidal representations of $GL(n)$ belong to \mathcal{S}^* . In Section 5, we give an estimate of S_0 as defined in (2.2). In Section 6, we give the asymptotic for S_1 as defined in (2.1) for $F \in \mathcal{S}^*$. In Section 7, we define the resonator coefficients and give some properties of the resonator. In Section 8, we complete the proof of Theorem 1.1-Theorem 1.4. Throughout the paper c, β, ϵ denotes positive numbers whose value may change from one line to the next.

3. DEFINITION OF THE CLASS OF L -FUNCTIONS \mathcal{S}

We define a class of functions \mathcal{S} as follows. A function F is in \mathcal{S} if

- 1) Dirichlet Series representation: For $\Re(s) > 1$, $F(s)$ can be represented as an absolutely convergent Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$.
- 2) Analytic continuation: There exists a non-negative integer m such that $(s-1)^m F(s)$ is an entire function of finite order.
- 3) Functional equation: $F(s)$ satisfies the functional equation

$$\Xi_F(s) = w_F \overline{\Xi_F(1-\bar{s})},$$

where

$$\Xi_F(s) := F(s) Q^s \prod_{j=1}^f \Gamma(\lambda_j s + \mu_j) \quad (3.1)$$

with positive real numbers Q, λ_j and complex numbers w_F, μ_j with $|w_F| = 1, \Re \frac{\mu_j}{\lambda_j} > -\frac{1}{2}$.

- 4) Euler product: For $\Re(s)$ sufficiently large, $F(s)$ has the Euler product representation

$$F(s) = \prod_p F_p(s), \quad F_p(s) = \exp \left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}} \right),$$

where $b_F(p^k)$ are some coefficients satisfying $b_F(p^k) \ll p^{k\theta_F}$, for some constant $\theta_F < 1/2$.

For $F \in \mathcal{S}$, we define the degree d_F , weight λ , and conductor q_F as

$$d_F = 2 \sum_{j=1}^f \lambda_j, \quad \lambda = \prod_{j=1}^f \lambda_j^{2\lambda_j}, \quad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^f \lambda_j^{2\lambda_j}. \quad (3.2)$$

Define the analytic conductor of $F(s)$ as

$$q_F(s) = Q^2 \prod_{j=1}^f (|\lambda_j s + \mu_j| + 3)^{2\lambda_j}, \quad Q_F = q_F(0). \quad (3.3)$$

Let ψ be a Dirichlet character. The twisted L -function F_ψ is defined as

$$F_\psi(s) = \sum_{n=1}^{\infty} \frac{a_F(n)\psi(n)}{n^s}, \text{ for } \Re(s) > 1.$$

We define a subclass of \mathcal{S} , denoted by \mathcal{S}^* , which consists of L -functions that satisfy the following additional conditions.

- (i) $F_\psi \in \mathcal{S}$ for any primitive character $\psi \pmod{g}$ and $d_{F_\psi} = d_F, Q_{F_\psi} \ll Q_F g^{d_F}$.
- (ii) F_ψ is entire for all primitive characters ψ with the exception of at most one primitive character $\psi^* \pmod{g^*}$.
- (iii) For any $Q \gg 1$, there exists B_Q which is either 1 or a prime $\gg_F \log_2 Q$, such that

$$1 - \sigma \gg_F \frac{1}{\log(Q(|\Im(s)| + 2))} \quad (3.4)$$

whenever $F_\psi(\sigma + it) = 0$ and $\psi \pmod{g}$ is a Dirichlet character with square-free conductor $\tilde{g} \leq Q$ and $(\tilde{g}, B_Q) = 1$.

- (iv) For $\Re(s) > 1$, denote

$$-\frac{F'}{F}(s) := \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)\lambda_F(n)}{n^s}.$$

Then we have

$$|\Lambda_F(n)| \leq d_F n^{\theta_F} \log n, \quad (3.5)$$

and as $x \rightarrow \infty$, we have

$$\sum_{n \leq x} \Lambda(n) |\lambda_F(n)|^2 = x(1 + o(1)). \quad (3.6)$$

- (v) For $x \gg_{d_F} (Q_F \nu)^c$ with $c = c(d_F)$ is some constant depending only on d_F , we have

$$\sum_{x \leq n \leq x e^{1/\nu}} |\Lambda_F(n)| \ll_{d_F} \frac{x}{\nu}. \quad (3.7)$$

From the definition of \mathcal{S} , we have the following properties.

Lemma 3.1 (Convexity Bound). *Let $F \in \mathcal{S}$ be as above. Define*

$$\mu_F(\sigma) = \limsup_{|t| \rightarrow \infty} \frac{\log |F(\sigma + it)|}{\log |t|}.$$

Then $\mu_F(\sigma)$ is a convex function, and

$$\mu_F(\sigma) \leq \begin{cases} 0, & \text{if } \sigma > 1, \\ \frac{1}{2} d_F (1 - \sigma), & \text{if } 0 \leq \sigma \leq 1, \\ (\frac{1}{2} - \sigma) d_F, & \text{if } \sigma < 0. \end{cases}$$

Proof. This is follows from the general theory of L -functions that can be found in Theorem 6.8 in [45]. \square

Lemma 3.2. *Let ψ be a primitive character. Let $F \in \mathcal{S}^*$, then we have*

- (1) For $\Re(s) \geq 1$, $\frac{F'_\psi(s)}{F_\psi(s)}$ has no poles except for the character ψ^* , where it has a simple pole at $s = 1$.
- (2) For any $\kappa > 1$, we have

$$\sum_{n=1}^{\infty} \frac{|(\Lambda_F * \psi)(n)|}{n^\kappa} \ll \frac{1}{(\kappa - 1)^2}.$$

(3) Let B_Q be defined in condition (iii). Let ψ be any primitive character with square-free conductor $g \leq Q$ such that $(g, B_Q) = 1$. There exists some constant $c = c(F) > 0$ such that for $\Re(s) \geq 1 - c/\log(Q(|\Im s| + 2))$

$$\frac{F'_\psi(s)}{F_\psi(s)} \ll_F \log^2(Q(|\Im s| + 2)).$$

If $F_\psi(z)$ has no zeros for $\Re(s) > 1 - a$ for some $a > 0$, then the above bound hold for $\Re(s) > 1 - a + 1/\log(Q(|\Im s| + 2))$.

Proof. For (1), it follows from the definition of \mathcal{S}^* . For (2), we have

$$\sum_{n=1}^{\infty} |(\Lambda_F * \chi)(n)| n^{-\sigma} \leq \sum_{n=1}^{\infty} |\Lambda_F(n)| n^{-\sigma} \sum_{n=1}^{\infty} n^{-\sigma}. \quad (3.8)$$

From (3.6) and partial summation, we have

$$\sum_{n=1}^{\infty} \frac{|\Lambda_F(n)|}{n^\sigma} \ll \sum_{n=1}^{\infty} \frac{\Lambda(n) + \Lambda(n)|\lambda_F(n)|^2}{n^\sigma} \ll \frac{1}{\sigma - 1}, \quad (3.9)$$

which together with (3.8) yields the desired conclusion. To prove part (3), we can choose $c = c(F)$ such that $\Re(\rho_{F_\psi}) \leq 1 - \frac{2c}{\log(Q(|\Im(s)|+2))}$ for all non-trivial zeros ρ_{F_ψ} of F_ψ by assumption (iii). Similar to [28, Proposition 5.7 (2)], we have

$$\frac{F'_\psi(s)}{F_\psi(s)} + \frac{m}{s} + \frac{m}{s-1} - \sum_{|s+\mu_{\psi,j}|<1} \frac{1}{s + \frac{\mu_{\psi,j}}{\lambda_{\psi,j}}} - \sum_{|s-\rho_{F_\psi}|<1} \frac{1}{s - \rho_{F_\psi}} \ll \log q_{F_\psi}(s)$$

for some absolute constant. Since $\Re(s) \geq 1 - \frac{c}{\log q_{F_\psi}(s)}$, $\Re(\frac{\mu_{\psi,j}}{\lambda_{\psi,j}}) > -\frac{1}{2}$, and $d_{F_\psi} = d_F$ we have

$$\frac{m}{s} + \frac{m}{s-1} - \sum_{|s+\mu_{\psi,j}|<1} \frac{1}{s + \frac{\mu_{\psi,j}}{\lambda_{\psi,j}}} \ll_F d_F \log q_{F_\psi}(s).$$

Since $Q_{F_\psi} \ll Q_F g^{d_F}$, we have

$$\log q_{F_\psi}(s) \ll \log Q_F g^{d_F} (|\Im s| + 2)^{d_F} \ll_F \log Q(|\Im s| + 2).$$

We also have from [28, Proposition 5.7, (1)] that the number of zeros ρ_{F_ψ} such that $|\Im \rho_{F_\psi} - \Im s| < 1$ is bounded by $\log q_{F_\psi}(s)$. Therefore,

$$\begin{aligned} \frac{F'_\psi(s)}{F_\psi(s)} &\ll d_F \log(q_{F_\psi}(s)) + \sum_{|s-\rho_{F_\psi}|<1} \frac{1}{s - \rho_{F_\psi}} \\ &\ll_F \log^2 Q(|\Im s| + 2), \end{aligned}$$

for all ψ with square-free conductor $g \leq Q$ and $(g, B_Q) = 1$. A similar argument can be applied when F_ψ has no zero in the region $\Re(s) > 1 - a$. \square

4. PROPERTIES OF AUTOMORPHIC L-FUNCTIONS

In this section, we will show that L -functions associated to irreducible cuspidal representations of $GL(n)$ belong to the class \mathcal{S}^* .

Let π be an irreducible cuspidal automorphic representation of $GL(d_\pi)$ over \mathbb{Q} , with unitary central character. For $\Re(s) > 1$, let

$$L(s, \pi) := \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} = \prod_p \prod_{j=1}^{d_\pi} \left(1 - \frac{\alpha_\pi(p, j)}{p^s} \right)^{-1}$$

be the global L -function attached to π (cf. R. Godement and H. Jacquet [25], H. Jacquet and J. Shalika [29, 30]). Denote by $\lambda_\pi(p^k)$,

$$\lambda_\pi(p^k) = \sum_{j=1}^{d_\pi} \alpha_\pi(p, j)^k. \quad (4.1)$$

Then for $\Re(s) > 1$, we have

$$-\frac{L'}{L}(s, \pi) := \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)\lambda_\pi(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function. It is known that $L(s, \pi)$ can be analytically continued to an entire function

$$\Phi(s, \pi) = q_\pi^{s/2} \gamma(s, \pi) L(s, \pi), \quad (4.2)$$

which satisfies the functional equation

$$\Phi(s, \pi) = \epsilon_\pi \overline{\Phi}(1-s, \pi),$$

where $\overline{\Phi}(s) = \overline{\Phi(\overline{s})}$ and $\gamma(s, \pi) = \prod_{j=1}^{d_\pi} \Gamma_{\mathbb{R}}(s + \mu_j)$, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$, $\mu_j \in \mathbb{C}$, $|\epsilon_\pi| = 1$. We also have the bound

$$|\lambda_\pi(n)| \leq d_\pi n^{\theta_\pi}, \quad -\Re(\mu_j) \leq \theta_\pi \quad (4.3)$$

for some $\theta_\pi < \frac{1}{2}$. The Generalized Ramanujan Conjecture asserts that $\theta_\pi = 0$. It is known from W. Luo, Z. Rudnick and P. Sarnak [36] that $\theta_\pi \leq \frac{1}{2} - \frac{1}{d_\pi^2 + 1}$. When $K = \mathbb{Q}$ and $d_\pi = 2$, Kim and Sarnak [32] improved the bound to $|\alpha_j(p)| \leq \frac{7}{64}$ based on the work of Kim on the symmetric fourth L -functions. V. Blomer and F. Brumley [4] extended this bound to general number fields and obtained better bounds for $GL(3)$ ($c \leq \frac{5}{14}$) and $GL(4)$ ($c \leq \frac{9}{22}$) L -functions over general number field.

Given a Dirichlet character $\psi \pmod{g}$, where $(g, q_\pi) = 1$, let

$$L(s, \pi \otimes \psi) := \sum_{n=1}^{\infty} \frac{a_\pi(n)\psi(n)}{n^s} = \prod_p \prod_{j=1}^{d_\pi} \left(1 - \frac{\alpha_j(p)\psi(p)}{p^s}\right)^{-1}, \quad \text{for } \Re(s) > 1.$$

We have

$$-\frac{L'}{L}(s, \pi \otimes \psi) = \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)\psi(n)}{n^s}, \quad \Re(s) > 1.$$

It is known that $L(s, \pi \otimes \psi)$ can be analytically continued to an entire function, and furthermore

$$\Phi(s, \pi \otimes \psi) = (g^{d_\pi} q_\pi)^{s/2} \gamma_\psi(s, \pi) L(s, \pi \otimes \psi)$$

is an entire function of order 1 satisfying the functional equation

$$\Phi(s, \pi \otimes \psi) = \epsilon_{\pi, \psi} \overline{\Phi}(1-s, \pi \otimes \psi),$$

where $\overline{\Phi}(s, \pi \otimes \psi) = \overline{\Phi(\overline{s}, \pi \otimes \psi)}$, $\gamma_\psi(s, \pi) = \prod_{j=1}^{d_\pi} \Gamma_{\mathbb{R}}(s + \mu_{j, \psi})$, $\Re \mu_{j, \psi} > -\frac{1}{2}$, $|\epsilon_{\pi, \psi}| = 1$. This shows that $GL(n)$ L -functions belong to \mathcal{S} and it remains to prove conditions (i)-(v).

Condition (i) can be verified by properties of Rankin-Selberg L -functions (see [28, Section 5, p. 97, eq (5.11)]). Condition (ii) is satisfied for $GL(1)$ L -functions and when $d_\pi \geq 2$, we know that $L(s, \pi \otimes \psi)$ is entire and thus condition (ii) is also satisfied. The zero free region of $L(s, \pi \otimes \psi)$ can be found in [5, Proposition 2.11]. In particular, there is only possible one real zero in the region $\sigma > 1 - \frac{c}{\log(Q_\pi(|t|+2))}$ for any irreducible cuspidal automorphic representation π of $GL(d_\pi)$, where c is an absolute positive constant depending only on d_π . The exceptional zero is called a Siegel zero. It is believed that the only possibility of a Siegel zero is from a Dirichlet L -function associated to a quadratic character. In fact, J. Hoffstein and D. Ramakrishnan proved that there is no Siegel zero for cusp forms on $GL(n)$ for $n > 1$ if the functoriality of Langlands holds. This implies that cusp forms on $GL(2)$ admit no Siegel zeros. W. Banks [3] proved the non-existence of Siegel zeros for cusp forms on $GL(3)$. Thus

condition (iii) in the definition of \mathcal{S}^* is satisfied for $d_\pi = 2$ or 3 . For $GL(1)$ L -functions, we know (iii) is true from Theorem 4.1 below. For $GL(n)$ L -functions, we prove an analogue in Theorem 4.2.

Theorem 4.1 (Landau-Page,[21, Corollary 6]). *For $Q \geq 100$, there exists B_Q which is either 1 or a prime $\gg \log_2 Q$ such that $1 - \sigma \gg \frac{1}{\log Q(|t|+1)}$ whenever $L(\sigma + it, \chi) = 0$ and χ is a Dirichlet character modulo q with $q \leq Q, (q, B_Q) = 1$.*

Theorem 4.2. *Let π be an irreducible cuspidal representation of $GL(n)$, and let Q be a sufficiently large integer. Then, there exists a quantity B_Q which is either 1 or a prime of size $\gg \log_2 Q$ such that $L(s, \pi \otimes \psi)$ has no zero in the region*

$$1 - \Re(s) \ll \frac{1}{\log Q(|t| + 1)},$$

whenever the conductor of ψ is squarefree and coprime to B_Q . All implied constants only depend on π .

Lemma 4.3 (J. Hoffstein and D. Ramakrishnan, [27, Theorem A]; [5, Remark 2.12]). *Let π be an irreducible cuspidal automorphic representation of $GL(n)$ with $Q_\pi \leq Q$. Then there is an absolute constant $c > 0$ such that $L(s, \pi)$ has no zeros in the interval $1 - \frac{c}{\log Q} \leq \sigma \leq 1$ with the exception of at most one of such π .*

Lemma 4.4 (F. Brumley, [13, Corollary 6]). *Let π and π' be cuspidal automorphic representations of $GL_n(\mathbb{A})$ with analytic conductor $\leq Q$ and $t \in \mathbb{R}$. There exist constants $c = c(n, n') > 0$ and $A = A(n, n') > 0$ such that $L(\sigma, \pi \times \pi')$ has no zeros in the interval*

$$1 - \frac{c}{Q^A} \leq \sigma \leq 1.$$

Proof. [Proof of Theorem 4.2] Let ψ be a Dirichlet character of squarefree conductor g and π be a cusp form on $GL(n)$ with conductor Q_π . Then $\pi \otimes \psi$ is a cusp form on $GL(n)$ with conductor $\ll Q_\pi g^n$. From [5, Proposition 2.11], we have $L(s, \pi \otimes \psi)$ has no zeros for $\Re(s) \geq_\pi 1 - \frac{c}{\log Q(|t|+1)}$ for all $g \leq Q$, with exception of at most one real zero. From Lemma 4.3, we see that for all primitive characters with conductor at most Q , there is at most one exceptional character $\psi_Q \pmod{g_Q}$ such that it has a real zero β satisfying

$$1 - \beta \ll_\pi \frac{1}{\log Q}.$$

From Lemma 4.4, we see that

$$1 - \beta \gg_\pi \frac{1}{(Q_\pi g_Q^n)^A}.$$

Thus, $g_Q \gg_\pi (\log Q)^{1/A'}$. Since g_Q is squarefree, by prime number theorem, there exists $B_Q \gg \log g_Q \gg \log_2 Q$ such that $L(s, \pi \otimes \psi)$ has no zero in the region

$$\Re(s) \geq 1 - \frac{c}{\log Q},$$

for all Dirichlet characters ψ with squarefree conductor at most Q and coprime to B_Q . \square

Condition (iv) follows from Rankin-Selberg theory. A proof can be found in [28, Theorem 5.13] (see also [35, Lemma 5.2]). Condition (v) is also satisfied for automorphic L -functions of $GL(n)$ (See [44, eq (1.10)]). Therefore, we see that L -functions associated to irreducible automorphic representations of $GL(n)$ belong to \mathcal{S}^* .

5. MOMENT OF THE RESONATOR

Theorem 5.1. *Let $F \in \mathcal{S}$. Suppose $X(s) = \sum_{n \leq M} \frac{x_n}{n^s}$, $Y(s) = \sum_{n \leq M} \frac{y_n}{n^s}$ with $x_n = \overline{y_n}$, and $M \leq T$. Then we have*

$$S_0 = \left(\frac{1}{2\pi} \int_{T_1}^{T_2} \log(\lambda Q^2 t^{d_F}) dt \right) \sum_{m \leq M} \frac{x_m y_m}{m} - \frac{T_2 - T_1}{2\pi} \sum_{m \leq M} \frac{(\Lambda_F * x)(m) y_m + \overline{\Lambda_F} * y(m) x_m}{m} + \mathcal{E}_0,$$

where

$$\mathcal{E}_0 = O\left((\log T)^2 \left(M \left\| \frac{x_n}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 + M^{\theta_F + \epsilon} \|x_n\|_1 \left\| \frac{y_n}{n} \right\|_1 + M^{\theta_F + \epsilon} \|y_n\|_1 \left\| \frac{x_n}{n} \right\|_1 \right) + O\left((\log T)^2 M^{1 + \theta_F + \epsilon} \left(\left\| \frac{x_n}{n} \right\|_1 \|y_n\|_\infty + \left\| \frac{y_n}{n} \right\|_1 \|x_n\|_\infty \right)\right).\right. \quad (5.1)$$

Proof. From the residue theorem, we have for any $c > 1$,

$$S_0 = \frac{1}{2\pi i} \left(\int_{c+iT_1}^{c+iT_2} + \int_{c+iT_2}^{1-c+iT_2} + \int_{1-c+iT_1}^{c+iT_1} + \int_{1-c+iT_2}^{1-c+iT_1} \right) X(s) Y(1-s) \frac{F'}{F}(s) ds = J_R - J_L + J_H, \quad (5.2)$$

where

$$J_R = \frac{1}{2\pi i} \int_{c+iT_1}^{c+iT_2} X(s) Y(1-s) \frac{F'}{F}(s) ds, \quad (5.3)$$

$$J_L = \frac{1}{2\pi i} \int_{1-c+iT_1}^{1-c+iT_2} X(s) Y(1-s) \frac{F'}{F}(s) ds, \quad (5.4)$$

$$J_H = \frac{1}{2\pi i} \left(\int_{c+iT_2}^{1-c+iT_2} + \int_{1-c+iT_1}^{c+iT_1} \right) X(s) Y(1-s) \frac{F'}{F}(s) ds. \quad (5.5)$$

For J_H , we first note that

$$|X(s) Y(1-s)| = \left| \sum_{u \leq M} \frac{x_u}{u^s} \sum_{k \leq M} \frac{y_k}{k^{1-s}} \right| \leq M \left\| \frac{x_n}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 + M^{c-1} \|x_n\|_1 \left\| \frac{y_n}{n} \right\|_1 + M^{c-1} \|y_n\|_1 \left\| \frac{x_n}{n} \right\|_1, \quad (5.6)$$

where each part corresponds to a bound for $0 \leq \Re(s) \leq 1$, $1-c \leq \Re(s) \leq 0$, and $1 \leq \Re(s) \leq c$ respectively. Since $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$ are chosen such that

$$\frac{F'}{F}(\sigma + iT_1) \ll (\log T)^2, \quad \frac{F'}{F}(\sigma + iT_2) \ll (\log T)^2, \quad (5.7)$$

uniformly for $\sigma \in [-1, 2]$, it follows that

$$J_H \ll (\log T)^2 \left(M \left\| \frac{x_n}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 + M^{c-1} \|x_n\|_1 \left\| \frac{y_n}{n} \right\|_1 + M^{c-1} \|y_n\|_1 \left\| \frac{x_n}{n} \right\|_1 \right). \quad (5.8)$$

Taking logarithmic derivative of the functional equation (3.1), we have

$$\frac{F'}{F}(s) = \frac{\Delta'_F}{\Delta_F}(s) - \frac{\overline{F'}}{\overline{F}}(1-s)$$

where $\Delta_L(s) = wQ^{1-2s} \prod_{j=1}^f \frac{\Gamma(\lambda_j(1-s) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)}$. Therefore,

$$\begin{aligned} J_L &= \frac{1}{2\pi i} \int_{1-c+T_1}^{1-c+iT_2} X(s)Y(1-s) \frac{F'}{F}(s) ds \\ &= \frac{1}{2\pi i} \int_{c+iT_1}^{c+iT_2} \left\{ \frac{\Delta'_F}{\Delta_F}(s) - \frac{F'}{F}(s) \right\} \overline{X}(1-s) \overline{Y}(s) ds. \end{aligned} \quad (5.9)$$

We write

$$J_L = K - \overline{I}_R, \quad (5.10)$$

where

$$K = \frac{1}{2\pi i} \int_{c+iT_1}^{c+iT_2} \frac{\Delta'_F}{\Delta_F}(s) \overline{Y}(s) \overline{X}(1-s) ds$$

and

$$I_R = \frac{1}{2\pi i} \int_{c+iT_1}^{c+iT_2} \frac{F'}{F}(s) \overline{Y}(s) \overline{X}(1-s) ds.$$

If $X(s) = \overline{Y}(s)$, then we have

$$I_R = J_R. \quad (5.11)$$

From Stirling's formula, we have

$$\frac{\Delta'_F}{\Delta_F}(s) = -\log(\lambda Q^2 |t|^{d_F}) + O\left(\frac{1}{|t|}\right), \quad (5.12)$$

and thus by (5.6),

$$\begin{aligned} K &= -\frac{1}{2\pi} \int_{T_1}^{T_2} \log(\lambda Q^2 |t|^{d_F}) Y(c-it) X(1-c+it) dt \\ &\quad + O\left(\log T \left(M \left\| \frac{x_n}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 + M^{c-1} \|x_n\|_1 \left\| \frac{y_n}{n} \right\|_1 + M^{c-1} \|y_n\|_1 \left\| \frac{x_n}{n} \right\|_1\right)\right). \end{aligned} \quad (5.13)$$

The main term in K , denoted by K_0 , is given by

$$\begin{aligned} K_0 &= -\frac{1}{2\pi} \int_{T_1}^{T_2} \log(\lambda Q^2 t^{d_F}) \sum_{u \leq M} \frac{x_u}{u^{1-c+it}} \sum_{k \leq M} \frac{y_k}{k^{c-it}} dt \\ &= -\frac{1}{2\pi} \sum_{u \leq M} \frac{x_u}{u^{1-c}} \sum_{k \leq M} \frac{y_k}{k^c} \int_{T_1}^{T_2} \log(\lambda Q^2 t^{d_F}) \left(\frac{k}{u}\right)^{it} dt \\ &= K_d + K_{nd}, \end{aligned} \quad (5.14)$$

where K_d denotes the contribution from the diagonal terms with $k = u$, and K_{nd} denotes the contribution from the off-diagonal terms with $k \neq u$. We have

$$\begin{aligned} K_d &= -\frac{1}{2\pi} \sum_{u \leq M} \frac{x_u y_u}{u} \int_{T_1}^{T_2} \log(\lambda Q^2 t^{d_F}) dt \\ &= -\left(\frac{d_F}{2\pi} T \log T + O(T)\right) \sum_{u \leq M} \frac{x_u y_u}{u}. \end{aligned} \quad (5.15)$$

For K_{nd} , we have

$$\begin{aligned}
K_{nd} &\ll \sum_{\substack{u, k \leq M \\ u \neq k}} \frac{|x_u y_k|}{u^{1-c} k^c} \frac{\log T}{|\log k/u|} \\
&\ll \log T M^{c-1} \sum_{u \leq M} |x_u| \sum_{k \leq M} \frac{|y_k|}{k^c} + \log T \sum_{u \leq M} |x_u| \sum_{u/2 \leq k \leq 2u} \frac{|y_k|}{k^c} \frac{u}{|k-u|} \\
&\ll \log T M^{c-1} \|x_n\|_1 \left\| \frac{y_k}{k^c} \right\|_1 + \log T \|x_n\|_1 \|y_n\|_\infty \log M.
\end{aligned} \tag{5.16}$$

For J_R , we have

$$\begin{aligned}
J_R &= \frac{1}{2\pi i} \int_{c+iT_1}^{c+iT_2} X(s) Y(1-s) \frac{F'}{F}(s) ds \\
&= -\frac{1}{2\pi} \int_{T_1}^{T_2} \sum_{u \leq M} \frac{x_u}{u^{c+it}} \sum_{k \leq M} \frac{y_k}{k^{1-c-it}} \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^{c+it}} dt \\
&= J_d + J_{nd},
\end{aligned} \tag{5.17}$$

where J_d denotes the contribution from the diagonal terms with $k = nu$, and J_{nd} denotes the contribution from the off diagonal terms with $k \neq nu$.

$$J_d = -\frac{T_2 - T_1}{2\pi} \sum_{n=1}^{\infty} \sum_{u \leq M} \frac{\Lambda_F(n) x_u y_{nu}}{nu}, \tag{5.18}$$

and for J_{nd} we have

$$\begin{aligned}
J_{nd} &\ll \log T \sum_{n=1}^{\infty} \frac{|\Lambda_F(n)|}{n^c} \sum_{u \leq M} \frac{|x_u|}{u^c} \sum_{\substack{k \leq M \\ k \neq nu}} \frac{|y_k|}{k^{1-c}} \frac{1}{\log |k/nu|} \\
&\ll \log T \sum_{n=1}^{\infty} \frac{|\Lambda_F(n)|}{n^c} \sum_{u \leq M} \frac{|x_u|}{u^c} M^{c-1} \max_{h \neq k} \sum_{k \leq M} \frac{|y_k|}{|\log(k/h)|} \\
&\ll \log T M^{c-1} \sum_{n=1}^{\infty} \frac{|\Lambda_F(n)|}{n^c} \left\| \frac{x_n}{n^c} \right\|_1 (\|y_n\|_1 + \|y_n\|_\infty M \log M).
\end{aligned} \tag{5.19}$$

Taking $c = 1 + \theta_F + \epsilon$, and combining (5.8), (5.13), (5.15), (5.16), (5.17), (5.18), and (5.19), we complete the proof. \square

6. FIRST MOMENT

In this section, we obtain asymptotic formulas for S_1 defined in (2.1) for $F \in \mathcal{S}^*$ in Theorem 6.1. We will see that Theorem 6.1 is a consequence of (6.11), Theorem 6.3 and Theorem 6.4. We first prove (6.11), then we prove Theorem 6.3 and Theorem 6.4.

Theorem 6.1. *Let $F \in \mathcal{S}^*$, $\psi^* \pmod{g^*}$ be as in (ii). Let χ be a primitive character modulo q . If x_n, y_n are multiplicative and supported on squarefree integers up to M whose prime factors are coprime to B_M (defined in (iii)) and are congruent to 1 modulo $\text{lcm}(q, g^*)$. Then there exists some*

constant $c = c(F) > 0$ such that uniformly for $M \leq \exp(\sqrt{\log T})$,

$$\begin{aligned}
S_1 &= \frac{T}{2\pi} \left(\sum_{nu \leq M} \frac{x_u y_{nu}}{nu} r_0(n) - \sum_{sv \leq M} \frac{x_s y_{sv}}{sv} r_1(v) \right) \\
&\quad - \frac{T}{2\pi} \sum_{u \leq M} \sum_{\substack{v \leq M \\ (v,u)=1}} \frac{x_u y_v r_3(u)}{uv} \sum_{s \leq M} \frac{y_s x_s}{s} \\
&\quad + O \left(T^{\theta_F + \epsilon} \|x_n\|_\infty \left\| \frac{y_n}{n} \right\|_1 + T^\epsilon \left\| \frac{x_n}{n} \right\|_1 (\|y_n\|_\infty M + \|y_n\|_1) \right) \\
&\quad + O \left(q^{1/2} T^{1/2} \mathcal{L}^3 \|x_n\|_1 \left\| \frac{\bar{y}_n}{n} \right\|_1 + q^{1/2 + \theta_F + \epsilon} T^{1/2 + \theta_F + \epsilon} M^{\theta_F + \epsilon} \|y_n\|_\infty \|x_n\|_1 \right) \\
&\quad + \mathcal{E} + \mathcal{E}',
\end{aligned}$$

where

$$\begin{aligned}
r_0(n) &= d_F P_1 \left(\log \left((\lambda Q^2)^{1/d_F} T \right) \right) - (\Lambda_F * 1)(n), \\
r_1(v) &= \log \left(\frac{2qT}{\pi v \epsilon} \right) \bar{f}_{-1} + \frac{\tau(\bar{\chi}) \tau(\psi^*)}{\phi(q)} \overline{\mu(q/\ell_0) \psi^*(q/\ell_0) L(1, \chi \psi^*) \tilde{f}_{-1}}, \quad \ell_0 = \gcd(q, g^*), \\
G(z, \bar{\chi}) &= \sum_{d=1}^{\infty} \frac{\Lambda_F(d) \bar{\chi}(d)}{d^z} = \frac{f_{-1}}{z-1} + f_0 + f_1(z-1) + \dots, \\
G(z, \psi^*) &= \sum_{d=1}^{\infty} \frac{\Lambda_F(d) \psi^*(d)}{d^z} = \frac{\tilde{f}_{-1}}{z-1} + \tilde{f}_0 + \tilde{f}_1(z-1) + \dots, \\
r_3(u) &= -\Lambda(u) \bar{f}_{-1} + \overline{r_4(u)}, \\
r_4(u) &= \sum_{hk=u} \mu(k) \left(\widetilde{X}_1(h, kq) + f_{-1} \widetilde{X}_2(k) \right), \\
\widetilde{X}_1(h, k) &= \sum_{a|(h,k)} \Lambda_F(a) + \sum_{p|h, p|k} \sum_{r=1}^{\infty} \frac{\Lambda_F(p^r)}{p^r} (p-1), \\
\widetilde{X}_2(k) &= f_0 - \eta(1; kq, 1) + \left(\gamma + \sum_{p|kq} \frac{\log p}{p-1} \right) f_{-1}, \\
\eta(1; k, 1) &= \sum_{p|k} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m)}{p^m}, \\
\mathcal{E} &\ll M^{\frac{1}{2} + \theta_F + \epsilon} q^{1 + \theta_F + \epsilon} T \|x_n\|_1 \|y_n\|_\infty \left\| \frac{\tau_3 * |y|(n)}{n} \right\|_1 \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}), \\
\mathcal{E}' &\ll q^{\theta_F + \epsilon} M^{\theta_F + \epsilon} T \left\| \frac{x_s y_s}{s} \right\|_1 \left\| \frac{(\tau_3 * |x|)(n)}{n} \right\|_1 \left\| \frac{y_v}{v} \right\|_1 \exp(-c\sqrt{\log T}).
\end{aligned}$$

If there exist some positive constant $a = a(\chi, F)$ such that both F_ψ and $L(s, \chi\psi)$ have no zeros in the region $\Re(s) \geq 1 - a$ for all ψ , then the term $\exp(-c\sqrt{\log T})$ in the error terms $\mathcal{E}, \mathcal{E}'$ can be replaced by $T^{-\delta + \epsilon}$ for some small enough $\delta = \delta(F, a) > 0$ uniformly for $M \leq \sqrt{T}$.

6.1. **Set up.** First we recall the functional equations and definitions of $F(s)$ and $L(s, \chi)$,

$$F(s) = \Delta_F(s) \overline{F}(1-s), \quad \overline{F}(s) = \overline{F(\overline{s})}, \quad (6.1)$$

$$\Delta_F(s) = w Q^{1-2s} \prod_{j=1}^f \frac{\Gamma(\lambda_j(1-s) + \overline{\mu}_j)}{\Gamma(\lambda_j s + \mu_j)}, \quad (6.2)$$

$$L(s, \chi) = B(s) L(1-s, \overline{\chi}), \quad (6.3)$$

$$B(s) = \frac{\tau(\chi)}{i^\alpha q^{\frac{1}{2}}} \left(\frac{q}{\pi}\right)^{\frac{1}{2}(1-2s)} \frac{\Gamma\left(\frac{1}{2}(1-s+\mathfrak{a})\right)}{\Gamma\left(\frac{1}{2}(s+\mathfrak{a})\right)}, \quad (6.4)$$

where

$$\mathfrak{a} = \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1, \end{cases}, \quad B(s) \overline{B(1-s)} = 1,$$

and

$$\tau(\chi) = \sum_{m=1}^q \chi(m) e_q(m), \quad \tau(\overline{\chi}) = \chi(-1) \overline{\tau(\chi)}.$$

From the definition of S_1 in (2.1), the functional equation (6.3) and the residue theorem, we have

$$\begin{aligned} S_1 &= \sum_{\substack{F(\rho)=0 \\ T_1 \leq \Re \rho \leq T_2}} L(\rho, \chi) X(\rho) Y(1-\rho) \\ &= \sum_{\substack{\overline{F}(1-\rho)=0 \\ T_1 \leq \Re \rho \leq T_2}} B(\rho) L(1-\rho, \overline{\chi}) X(\rho) Y(1-\rho) \end{aligned} \quad (6.5)$$

$$\begin{aligned} &= -\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\overline{F}'}{\overline{F}} (1-s) B(s) L(1-s, \overline{\chi}) X(s) Y(1-s) ds \\ &:= -S_R + S_L - S_H, \end{aligned} \quad (6.6)$$

where \mathcal{C} is the positively oriented rectangle with vertices at $1-\kappa+iT_1, \kappa+iT_1, 1-\kappa+iT_2, \kappa+iT_2$, with $\kappa = 1 + O(\mathcal{L}^{-1})$, $\mathcal{L} = \log(\lambda Q^2 T^{d_F})$, $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$ are chosen so that the nearest zeros of $F(s)$ are $\gg \frac{1}{\log T}$ distance away, and S_R, S_L and S_H are defined as

$$S_R = \int_{\kappa+iT_1}^{\kappa+iT_2} \frac{\overline{F}'}{\overline{F}} (1-s) B(s) L(1-s, \overline{\chi}) X(s) Y(1-s) ds, \quad (6.7)$$

$$S_L = \int_{1-\kappa+iT_1}^{1-\kappa+iT_2} \frac{\overline{F}'}{\overline{F}} (1-s) B(s) L(1-s, \overline{\chi}) X(s) Y(1-s) ds, \quad (6.8)$$

$$\begin{aligned} S_H &= \int_{1-\kappa+iT_1}^{\kappa+iT_1} \frac{\overline{F}'}{\overline{F}} (1-s) B(s) L(1-s, \overline{\chi}) X(s) Y(1-s) ds \\ &\quad - \int_{1-\kappa+iT_2}^{\kappa+iT_2} \frac{\overline{F}'}{\overline{F}} (1-s) B(s) L(1-s, \overline{\chi}) X(s) Y(1-s) ds. \end{aligned} \quad (6.9)$$

By Stirling's formula, for $t > 0$, equation (6.2) becomes

$$\Delta_F(s) = (\lambda Q^2 t^{d_F})^{\frac{1}{2}-\sigma-it} \exp\left(itd_F + \frac{i\pi(\mu-d_F)}{4}\right) \left(w + O\left(\frac{1}{t}\right)\right), \quad (6.10)$$

where

$$\mu = 2 \sum_{j=1}^f (1-2\Re \mu_j), \quad \lambda = \prod_{j=1}^f \lambda_j^{2\lambda_j}.$$

6.2. Horizontal Integral. From Lemma 3.1, Lemma 3.2 and (6.10), we have the bounds

$$\begin{aligned} \frac{F'}{F}(\sigma + it) &\ll_F \log^2 t, \\ B(\sigma + it) &\ll t^{\frac{1}{2}-\sigma}, \\ L(\sigma + it, \chi) &\ll \frac{t^{\frac{1-\sigma}{2}}}{1-\sigma}, 1/2 \leq \sigma \leq 1 - \frac{A}{\log T}, \\ L(\sigma + it, \chi) &\ll \log t, \sigma > 1 - \frac{A}{\log t}, \\ L(\sigma + it, \chi) &= B(\sigma)L(1-\sigma-it, \chi) \ll t^{1/2} \log t, -\frac{A}{\log t} < \sigma < 0, \\ X(s) &\ll M^{1-\sigma} \left\| \frac{x_n}{n} \right\|_1, 1-\kappa \leq \sigma \leq \kappa, \\ Y(1-s) &\ll M^\sigma \left\| \frac{y_n}{n} \right\|_1, 1-\kappa \leq \sigma \leq \kappa. \end{aligned}$$

It follows that the horizontal integral S_H (6.9) is bounded by

$$MT^{\frac{1}{2}} \left\| \frac{x_n}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \log^3 T. \quad (6.11)$$

6.3. Right integral. By taking the logarithmic derivative of the functional equation of $F(s)$, we find that

$$\frac{F'}{F}(s) = \frac{\Delta'_F}{\Delta_F}(s) - \frac{\overline{F}'}{\overline{F}}(1-s),$$

and so the right integral S_R defined in (6.7) becomes

$$S_R = \frac{1}{2\pi i} \int_{\kappa+iT_1}^{\kappa+iT_2} \left\{ \frac{\Delta'_F}{\Delta_F}(s) - \frac{F'}{F}(s) \right\} L(s, \chi) X(s) Y(1-s) ds. \quad (6.12)$$

Next, we use the following lemma to evaluate S_R .

Lemma 6.2. *Set $D(s) := \sum_{n=1}^{\infty} \alpha_n n^{-s}$. Suppose that there exists $\alpha > 0$ such that $\sum_{n=1}^{\infty} |\alpha_n| n^{-\sigma} \ll (\sigma-1)^{-\alpha}$ as $\sigma \rightarrow 1$. Suppose that $|\alpha_n| \ll n^{\theta_D + \epsilon}$. Then for $M \leq T$, we have*

$$\begin{aligned} J_k(T) &:= \frac{1}{2\pi i} \int_{\kappa+iT_1}^{\kappa+iT_2} \left(\frac{\Delta'_F(s)}{\Delta_F(s)} \right)^k D(s) X(s) Y(1-s) ds \\ &= \frac{(-1)^k d_F^k T P_k(\log((\lambda Q^2)^{\frac{1}{d_F}} T))}{2\pi} \sum_{nu \leq M} \frac{\alpha_n x_u y_{nu}}{nu} \\ &\quad + O_k \left(T^{\theta_D + \epsilon} \|x_n\|_{\infty} \left\| \frac{y_n}{n} \right\|_1 + T^{\epsilon} \left\| \frac{x_n}{n} \right\|_1 (\|y_n\|_{\infty} M + \|y_n\|_1) \right). \end{aligned}$$

Proof. From (6.10), we have

$$\frac{\Delta'_F}{\Delta_F}(s) = -\log(\lambda Q^2 t^{d_F}) + O\left(\frac{1}{t}\right),$$

for $1/2 \leq \sigma \leq 2$ and $t \geq 1$, $\kappa = 1 + 1/\mathcal{L}$ and $\mathcal{L} = \log(\lambda Q^2 T^{d_F})$. Thus,

$$J_k(T) = \frac{1}{2\pi i} \int_{\kappa+iT_1}^{\kappa+iT_2} ((-\log(\lambda Q^2 t^{d_F}))^k + O_k(\mathcal{L}^{k-1} t^{-1})) D(s) X(s) Y(1-s) ds.$$

The error terms contribute at most

$$\frac{1}{T} \mathcal{L}^{k-1} \int_{T_1}^{T_2} \sum_{n=1}^{\infty} \frac{|\alpha_n|}{n^{\kappa}} \sum_{u=1}^{\infty} \frac{|x_u|}{u} \sum_{v=1}^{\infty} |y_v| dt \ll \mathcal{L}^{\alpha+k} \left\| \frac{x_n}{n} \right\|_1 \|y_n\|_1.$$

Changing the order of summation and integration gives

$$\begin{aligned} J_k(T) &= \sum_{n,u,v} \frac{\alpha_n x_u y_v (-1)^k}{n^\kappa u^\kappa v^{1-\kappa} 2\pi} \int_{T_1}^{T_2} (\log(\lambda Q^2 t^{d_F}))^k \left(\frac{v}{nu}\right)^{it} dt + O_k\left(T^\epsilon \left\| \frac{x_n}{n} \right\|_1 \|y_n\|_1\right) \\ &:= J_d + J_{nd} + O_k\left(T^\epsilon \left\| \frac{x_n}{n} \right\|_1 \|y_n\|_1\right), \end{aligned}$$

where $s = \kappa + it$, J_d consists of the diagonal terms with $v = nu$, and J_{nd} consists of the off-diagonal terms with $v \neq nu$. Note that

$$\int_{T_1}^{T_2} \log^k(\lambda Q^2 t^{d_F}) dt = d_F^k T P_k(\log((\lambda Q^2)^{1/d_F} T)) + O_k(T^\epsilon),$$

where P_k is a monic polynomial of degree k . Since $|\alpha_n| \ll T^{\theta_D + \epsilon}$, we see that

$$\begin{aligned} J_d &= \frac{(-1)^k}{2\pi} \sum_{nu \leq M} \frac{\alpha_n x_u y_{nu}}{nu} \int_T^{2T} \log^k(\lambda Q^2 t^{d_F}) dt \\ &= \frac{(-1)^k d_F^k T P_k(\log((\lambda Q^2)^{1/d_F} T))}{2\pi} \sum_{nu \leq M} \frac{\alpha_n x_u y_{nu}}{nu} + O_k\left(T^{\theta_D + \epsilon} \|x_n\|_\infty \left\| \frac{y_n}{n} \right\|_1\right). \end{aligned}$$

For the off-diagonal terms,

$$\begin{aligned} J_{nd} &= \sum_{n,u,v,v \neq nu} \frac{(-1)^k \alpha_n x_u y_v}{n^\kappa u^\kappa v^{1-\kappa} 2\pi} \int_T^{2T} \log^k(\lambda Q^2 t^{d_F}) \left(\frac{v}{nu}\right)^{it} dt \\ &\ll \mathcal{L}^{k+\alpha} \left\| \frac{x_n}{n} \right\|_1 \sum_{v \leq M, v \neq nu} \frac{|y_v|}{v^{1-\kappa} |\log(v/nu)|}. \end{aligned}$$

Thus it is enough to consider

$$\max_h \sum_{v \leq M, v \neq h} \frac{|y_v|}{v^{1-\kappa} |\log(v/h)|}. \quad (6.13)$$

For $h \geq 2M$, we see that (6.13) can be bounded by $\|y_n\|$ since $\kappa = 1 + O(\mathcal{L}^{-1})$. For $h \leq 2M$, we have

$$\begin{aligned} \sum_{v \leq M, v \neq h} \frac{|y_v|}{|\log(v/h)|} &\ll \sum_{\substack{v \leq M \\ |v/h| > 3/2 \\ \text{or } |v/h| < 1/2}} |y_v| + \sum_{1/2 \leq v/h \leq 3/2} \frac{|y_v|}{|\log(v/h)|} \\ &\ll \|y_n\|_1 + \|y_n\|_\infty \sum_{s \leq h/2} \left(\frac{1}{|\log(h/(h-s))|} + \frac{1}{|\log(h/(h+s))|} \right) \\ &\ll \|y_n\|_1 + \|y_n\|_\infty \sum_{s \leq v/2} \frac{h}{s} \\ &\ll \|y_n\|_1 + \|y_n\|_\infty M \log M. \end{aligned}$$

This gives

$$J_{nd} \ll \mathcal{L}^{k+\alpha} \left\| \frac{x_n}{n} \right\|_1 (\|y_n\|_1 + \|y_n\|_\infty M \log M),$$

which completes the proof. \square

Theorem 6.3. *Let $F \in \mathcal{S}^*$. If x_n and y_n are coefficients supported on integers $n \leq M \leq T$, then the right integral S_R defined in (6.7) becomes*

$$S_R = \frac{T}{2\pi} \sum_{nu \leq M} \frac{\chi(n)x_u y_{nu}}{nu} \left(-d_F P_1 \left(\log \left((\lambda Q^2)^{1/d_F} T \right) \right) + \sum_{d|n} \Lambda_F(d) \bar{\chi}(d) \right) \\ + O \left(T^{\theta_F + \epsilon} \|x_n\|_\infty \left\| \frac{y_n}{n} \right\|_1 + T^\epsilon \left\| \frac{x_n}{n} \right\|_1 (\|y_n\|_\infty M + \|y_n\|_1) \right).$$

Proof. Apply Lemma 6.2 with $k = 1, \alpha = \chi$ and $k = 0, \alpha = \Lambda_F * \chi$. The assumptions in Lemma 6.2 can be verified from Lemma 3.2. \square

6.4. Left integral. In this section, we prove the following Theorem.

Theorem 6.4. *Let $F, \psi^* \pmod{g^*}$ and $\chi \pmod{q}$ be as before. If x, y are multiplicative functions supported on squarefree integers up to M whose prime factors are coprime to B_M and are congruent to 1 modulo $\text{lcm}(q, g^*)$. Then uniformly for $M \leq \exp(\sqrt{\log T})$,*

$$S_L = -\frac{T}{2\pi} \sum_{su \leq M} \frac{x_{su}}{su} \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{y_{sv}}{v} \left(\delta(u) \log \left(\frac{2qT}{\pi v e} \right) \overline{f_{-1}} - \Lambda(u) \overline{f_{-1}} \right) \\ - \frac{T}{2\pi} \sum_{su \leq M} \frac{x_{su}}{su} \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{y_{sv}}{v} \sum_{hk=u} \mu(k) \left(\overline{\widetilde{X}_1(h, k)} + \overline{f_{-1} \widetilde{X}_2(k)} \right) \\ - \frac{\overline{\tau(\bar{\chi})} \tau(\psi^*) T}{2\pi} \overline{L(1, \chi \psi^*)} \tilde{f}_{-1} \frac{\mu(q/\ell_0) \psi(q/\ell_0)}{\phi(q)} \sum_{s \leq M} \frac{x_s}{s} \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{y_{sv}}{v} \\ + O \left(q^{1/2} T^{1/2} \mathcal{L}^3 \|x_n\|_1 \left\| \frac{\bar{y}_n}{n} \right\|_1 + q^{1/2 + \theta_F + \epsilon} T^{1/2 + \theta_F + \epsilon} M^{\theta_F + \epsilon} \|y_n\|_\infty \|x_n\|_1 \right) \\ + \mathcal{E} + \mathcal{E}',$$

where

$$\ell_0 = \gcd(q, g^*), \quad g^* = \ell_0 g_0,$$

$$\delta(u) = \begin{cases} 1, & \text{if } u = 1, \\ 0, & \text{if } u > 1, \end{cases}$$

$$G(z, \psi) = \sum_{d=1}^{\infty} \frac{\Lambda_F(d) \psi(d)}{d^z}, \quad f_{-1} = \text{Res}_{z=1} G(z, \bar{\chi}), \quad f_0 = \lim_{z \rightarrow 1} G(z, \bar{\chi}) - \frac{f_{-1}}{z-1},$$

$$\eta(z; p; l, \psi) = \sum_{m=0}^{\infty} \frac{\Lambda_F(p^{l+m}) \psi(p^m)}{p^{mz}}, \quad \eta(z; k, \psi) = \sum_{p|k} \eta(z; p, 0, \psi),$$

$$\tilde{f}_{-1} = \text{Res}_{z=1} G(z, \psi^*),$$

$$\widetilde{X}_1(h, k) = \sum_{a|(h, k)} \Lambda_F(a) \bar{\chi}(a) + \sum_{p^l | h, p \nmid k} \bar{\chi}(p^l) \eta(1; p, l, \bar{\chi}) (1 - p^{-1}),$$

$$\widetilde{X}_2(k) = f_0 - \eta(1; kq, \bar{\chi}) + \left(\gamma + \sum_{p|kq} \frac{\log p}{p-1} \right) f_{-1},$$

$$\mathcal{E} \ll M^{\frac{1}{2} + \theta_F + \epsilon} q^{1 + \theta_F + \epsilon} T \|x_n\|_1 \|y_n\|_\infty \left\| \frac{\tau_3 * |y|(n)}{n} \right\|_1 \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}),$$

$$\mathcal{E}' \ll q^{\theta_F + \epsilon} M^{\theta_F + \epsilon} T \left\| \frac{|x_s y_s|}{s} \right\|_1 \left\| \frac{(\tau_3 * |x|)(n)}{n} \right\|_1 \left\| \frac{y_v}{v} \right\|_1 \exp(-c\sqrt{\log T}).$$

If there exist some positive constant $a = a(\chi, F)$ such that both F_ψ and $L(s, \chi\psi)$ have no zeros in the region $\Re(s) \geq 1 - a$ for all ψ , then the term $\exp(-c\sqrt{\log T})$ in the error terms $\mathcal{E}, \mathcal{E}'$ can be replaced by $T^{-\delta+\epsilon}$ for some small enough $\delta = \delta(F, a) > 0$ uniformly for $M \leq \sqrt{T}$.

6.4.1. *Initial Manipulations.* The integral on the left (6.8) is

$$\begin{aligned} S_L &= \frac{1}{2\pi i} \int_{1-\kappa+iT_1}^{1-\kappa+iT_2} \frac{\overline{F}'}{\overline{F}} (1-s)B(s)L(1-s, \overline{\chi})X(s)Y(1-s)ds \\ &= \frac{1}{2\pi} \int_{T_1}^{T_2} \frac{\overline{F}'}{\overline{F}} (\kappa-it)B(1-\kappa+it)L(\kappa-it, \overline{\chi})X(1-\kappa+it)Y(\kappa-it)dt \\ &= \frac{1}{2\pi} \int_{T_1}^{T_2} \frac{F'}{F} (\kappa+it)\overline{B}(1-\kappa-it)L(\kappa+it, \chi)\overline{X}(1-\kappa-it)\overline{Y}(\kappa+it)dt \\ &= \frac{1}{2\pi i} \int_{\kappa+iT_1}^{\kappa+iT_2} \frac{F'}{F} (s)\overline{B}(1-s)L(s, \chi)\overline{X}(1-s)\overline{Y}(s)ds \\ &:= \overline{I}_L, \end{aligned}$$

where $\overline{B}(s) = \overline{B(\overline{s})}$, $\overline{X}(s) = \overline{X(\overline{s})}$ and $\overline{Y}(s) = \overline{Y(\overline{s})}$. Let

$$\frac{F'}{F}(s)L(s, \chi)\overline{Y}(s) = \sum_{m=1}^{\infty} a(m)m^{-s},$$

where

$$a(m) = - \sum_{uvw=m} \Lambda_F(u)\chi(v)\overline{y}_w. \quad (6.14)$$

Then,

$$I_L = \frac{1}{2\pi i} \sum_{k \leq M} \frac{\overline{x}_k}{k} \sum_{m=1}^{\infty} a(m) \int_{\kappa+iT_1}^{\kappa+iT_2} \overline{B}(1-s) \left(\frac{m}{k}\right)^{-s} ds.$$

Lemma 6.5. *Let x_n, y_n be supported on $n \leq M \leq T$. Then,*

$$I_L = \mathcal{M} + O\left(q^{1/2}T^{1/2}\mathcal{L}^3\|x_n\|_1 \left\|\frac{\overline{y}_n}{n}\right\| + q^{1/2+\theta_F+\epsilon}T^{1/2+\theta_F+\epsilon}M^{\theta_F+\epsilon}\|y_n\|_\infty\|x_n\|_1\right),$$

where

$$\mathcal{M} = \frac{\tau(\overline{\chi})}{q} \sum_{k \leq M} \frac{\overline{x}_k}{k} \sum_{m=\lceil \frac{kqT}{2\pi} \rceil}^{\frac{kqT}{\pi}} a(m)e\left(-\frac{m}{kq}\right). \quad (6.15)$$

To prove Lemma 6.5, we need the following lemmas.

Lemma 6.6. *For large A and $A < B \leq 2A$,*

$$\begin{aligned} &\int_A^B \exp\left(it \log\left(\frac{t}{re}\right)\right) \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt \\ &= \begin{cases} (2\pi)^{1-a}r^a e^{-ir+\pi i/4} + E_a(r, A, B), & \text{if } A < r \leq B \leq 2A, \\ E_a(r, A, B), & \text{if } r \leq A \text{ or } r > B, \end{cases} \end{aligned}$$

where a is a fixed real number and

$$E_a(r, A, B) = O\left(A^{a-\frac{1}{2}}\right) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-r|+A^{\frac{1}{2}}}\right) + O\left(\frac{B^{a+\frac{1}{2}}}{|B-r|+B^{\frac{1}{2}}}\right).$$

Proof. This is Lemma 2 in [26]. □

Lemma 6.7. *Let $r, \kappa_0 > 0, T_1 = T + O(1)$ and $T_2 = 2T + O(1)$. Then,*

$$\frac{1}{2\pi i} \int_{\kappa+iT_1}^{\kappa+iT_2} \overline{B}(1-s)r^{-s} ds = \frac{\tau(\overline{\chi})}{q} \delta_q(r) e_q(-r) + O(E(r, \kappa, T) r^{-\kappa} q^{\kappa-1/2}), \quad (6.16)$$

uniformly for $\kappa_0 \leq \kappa \leq 2$, where $\delta_q(r) = 1$ if $T_1/2\pi < r/q \leq T_2/2\pi$ and 0 otherwise, and $E(r, \kappa, T) = E_\kappa\left(\frac{2\pi r}{q}, T_1, T_2\right) \ll T^{\kappa-\frac{1}{2}} + \frac{T^{\kappa+\frac{1}{2}}}{|T-\frac{2\pi r}{q}|+T^{1/2}} + \frac{T^{\kappa+\frac{1}{2}}}{|T-\frac{\pi r}{q}|+T^{1/2}}$.

Proof. From (6.4) and (6.10),

$$B(\sigma + it) = \left(\frac{q|t|}{2\pi}\right)^{\frac{1}{2}-\sigma-it} \exp\left(it + \text{sign}(t) \frac{i\pi(1-2\mathfrak{a})}{4}\right) \left(\frac{\tau(\chi)}{i^\mathfrak{a} q^{\frac{1}{2}}} + O\left(\frac{1}{|t|}\right)\right).$$

Applying Lemma 6.6, for $T_1 < 2\pi r/q \leq T_2$, we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\kappa+iT_1}^{\kappa+iT_2} \overline{B}(1-s)r^{-s} ds \\ &= \frac{\exp(-\frac{i\pi(1-2\mathfrak{a})}{4})}{2\pi} \left(\frac{\tau(\overline{\chi})}{i^{-\mathfrak{a}} q^{\frac{1}{2}}} + O\left(\frac{1}{T}\right)\right) \int_{T_1}^{T_2} \left(\frac{qt}{2\pi}\right)^{\kappa-\frac{1}{2}+it} r^{-\kappa-it} \exp(-it) dt \\ &= \frac{\exp(-\frac{i\pi(1-2\mathfrak{a})}{4})}{2\pi r^\kappa} \left(\frac{\tau(\overline{\chi})}{i^{-\mathfrak{a}} q^{\frac{1}{2}}} + O\left(\frac{1}{T}\right)\right) \int_{T_1}^{T_2} \left(\frac{qt}{2\pi}\right)^{\kappa-\frac{1}{2}} \exp\left(it \log\left(\frac{qt}{2\pi r e}\right)\right) dt \\ &= \frac{i^\mathfrak{a}}{2\pi r^\kappa} q^{\kappa-\frac{1}{2}} (2\pi)^{1-\kappa} \left(\frac{2\pi r}{q}\right)^\kappa \exp(-2\pi i r/q) \frac{\tau(\overline{\chi})}{i^{-\mathfrak{a}} q^{\frac{1}{2}}} + E_\kappa\left(\frac{2\pi r}{q}, T_1, T_2\right) r^{-\kappa} q^{\kappa-1/2} \\ &= \frac{\chi(-1)\overline{\tau(\chi)}}{q} \exp\left(-2\pi i \frac{r}{q}\right) + E_\kappa\left(\frac{2\pi r}{q}, T_1, T_2\right) r^{-\kappa} q^{\kappa-1/2}, \end{aligned}$$

where

$$E_\kappa\left(\frac{2\pi r}{q}, T_1, T_2\right) \ll T^{\kappa-\frac{1}{2}} + \frac{T^{\kappa+\frac{1}{2}}}{|T-\frac{2\pi r}{q}|+T^{1/2}} + \frac{T^{\kappa+\frac{1}{2}}}{|T-\frac{\pi r}{q}|+T^{1/2}},$$

since $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$. □

Proof. [Proof of Lemma 6.5]

Applying Lemma 6.7, we write

$$I_L = \mathcal{M} + \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3,$$

where

$$\mathcal{M} = \frac{\tau(\overline{\chi})}{q} \sum_{k \leq M} \frac{\overline{x}_k}{k} \sum_{m=\lceil \frac{kqT}{2\pi} \rceil}^{\frac{kqT}{\pi}} a(m) e\left(-\frac{m}{kq}\right), \quad (6.17)$$

$$\mathcal{E}_0 \ll q^{-1/2} \sum_{k \leq M} \frac{|\overline{x}_k|}{k} kq(kqT)^{\theta_F+\epsilon} \|y_k\|_\infty \ll q^{1/2+\theta_F+\epsilon} (MT)^{\theta_F+\epsilon} \|x_n\|_1 \|y_n\|_\infty, \quad (6.18)$$

$$\mathcal{E}_1 \ll q^{1/2} T^{1/2} \sum_{k \leq M} |\overline{x}_k| \sum_{m=1}^{\infty} \frac{|a(m)|}{m^\kappa}, \quad (6.19)$$

$$\mathcal{E}_2 \ll q^{1/2} T^{3/2} \sum_{k \leq M} |\overline{x}_k| \sum_{m=1}^{\infty} \frac{|a(m)|}{m^\kappa} \left(\left|T - \frac{2\pi m}{qk}\right| + T^{1/2}\right)^{-1}, \quad (6.20)$$

$$\mathcal{E}_3 \ll q^{1/2} T^{3/2} \sum_{k \leq M} |\overline{x}_k| \sum_{m=1}^{\infty} \frac{|a(m)|}{m^\kappa} \left(\left|T - \frac{\pi m}{qk}\right| + T^{1/2}\right)^{-1}. \quad (6.21)$$

For \mathcal{E}_1 , we have

$$\mathcal{E}_1 \ll q^{1/2} T^{1/2} \|x_n\|_1 \sum_{m \leq M} \frac{|\bar{y}_m|}{m^\kappa} \sum_{n=1}^{\infty} \frac{|\Lambda_F(n)|}{n^\kappa} \zeta(\kappa) \ll q^{1/2} T^{1/2} \mathcal{L}^3 \|x_n\|_1 \left\| \frac{\bar{y}_m}{m} \right\|_1. \quad (6.22)$$

The estimate for \mathcal{E}_2 and \mathcal{E}_3 is similar and we focus on \mathcal{E}_2 . We write $\mathcal{E}_2 = \mathcal{E}_{21} + \mathcal{E}_{22} + \mathcal{E}_{23}$ corresponding to the following cases

- (a) $\left| T - \frac{2\pi m}{qk} \right| > T/2$;
- (b) $\sqrt{T} \leq \left| T - \frac{2\pi m}{qk} \right| \leq T/2$;
- (c) $\left| T - \frac{2\pi m}{qk} \right| \leq \sqrt{T}$.

In case (a), we have

$$\mathcal{E}_{21} \ll q^{1/2} \frac{T^{3/2}}{T} \sum_{k \leq M} |\bar{x}_k| \sum_{m=1}^{\infty} \frac{|a(m)|}{m^\kappa} \ll \mathcal{E}_1 \ll q^{1/2} T^{1/2} \mathcal{L}^3 \|x_n\|_1 \left\| \frac{\bar{y}_m}{m} \right\|_1.$$

In case (b), without loss of generality we can assume $\sqrt{T} \leq \frac{2\pi m}{qk} - T \leq T/2$. We can divide this range into $\ll \log T$ intervals of the form $T + P < 2\pi m/kq \leq T + 2P$ with $\sqrt{T} \ll P \ll T$. Thus, m lies in intervals of the form $I := [\frac{qk}{2\pi}(T + P), \frac{qk}{2\pi}(T + 2P)]$. From (6.14), we have $|a(m)| = |\sum_{uvw=m} \Lambda_F(u)\chi(v)\bar{y}_w| \ll d_F \tau_3(m) m^{\theta_F} \log m \|y_n\|_\infty$, and this gives

$$\begin{aligned} \mathcal{E}_{22} &\ll q^{1/2} T^{3/2} \sum_{k \leq M} |\bar{x}_k| \sum_P \sum_{m \in I} \frac{|a(m)|}{m^\kappa P} \\ &\ll q^{1/2} T^{3/2} \sum_P \sum_{k \leq M} \frac{|\bar{x}_k|}{qk} \sum_{m \in I} \frac{|a(m)|}{TP} \\ &\ll q^{1/2} T^{1/2} \|y_n\|_\infty \sum_P \sum_{k \leq M} \frac{|\bar{x}_k|}{qkP} \sum_{m \in I} \tau_3(m) m^{\theta_F} \log m \\ &\ll q^{1/2} T^{1/2} \|y_n\|_\infty \sum_P \sum_{k \leq M} \frac{|\bar{x}_k|}{qkP} qkP (qMT)^{\theta_F + \epsilon} \\ &\ll q^{1/2 + \theta_F + \epsilon} T^{1/2 + \theta_F + \epsilon} M^{\theta_F + \epsilon} \|y_n\|_\infty \|x_n\|_1. \end{aligned}$$

For case (c), we have $|T - \frac{2\pi m}{qk}| \leq \sqrt{T}$, thus m lies in intervals of the form

$$J := \left[\frac{qk}{2\pi}(T - \sqrt{T}), \frac{qk}{2\pi}(T + \sqrt{T}) \right],$$

which gives

$$\begin{aligned} \mathcal{E}_{23} &\ll q^{1/2} T^{3/2} \sum_{k \leq M} |\bar{x}_k| \sum_{m \in J} \frac{|a(m)|}{qkT} \frac{1}{\sqrt{T}} \\ &\ll q^{1/2} \|y_n\|_\infty \sum_{k \leq M} |\bar{x}_k| \sum_{m \in J} \frac{\tau_3(m) m^{\theta_F} \log m}{qk} \\ &\ll q^{1/2} \|y_n\|_\infty \|x_n\|_1 \frac{1}{qk} qk \sqrt{T} (qMT)^{\theta_F + \epsilon} \\ &\ll q^{1/2 + \theta_F + \epsilon} T^{1/2 + \theta_F + \epsilon} M^{\theta_F + \epsilon} \|y_n\|_\infty \|x_n\|_1. \end{aligned}$$

Combining all three cases, we have

$$\mathcal{E}_2 \ll q^{1/2} T^{1/2} \mathcal{L}^3 \|x_n\|_1 \left\| \frac{\bar{y}_m}{m} \right\|_1 + q^{1/2 + \theta_F + \epsilon} T^{1/2 + \theta_F + \epsilon} M^{\theta_F + \epsilon} \|y_n\|_\infty \|x_n\|_1. \quad (6.23)$$

Lemma 6.5 thus follows from (6.17), (6.22), and (6.23). \square

6.4.2. *Main Term Set up.* We want to evaluate the main term \mathcal{M} , which is given in (6.15) by

$$\mathcal{M} = \frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{m=\lceil \frac{kqT}{2\pi} \rceil}^{\frac{kqT}{\pi}} a(m) e\left(-\frac{m}{kq}\right). \quad (6.24)$$

Write $\frac{m}{kq} = \frac{m'}{k'}$, where $(m', k') = 1$. Then

$$e\left(-\frac{m}{kq}\right) = \frac{1}{\phi(k')} \sum_{\psi \pmod{k'}} \tau(\bar{\psi}) \psi(-m'). \quad (6.25)$$

Next, we write the sum over ψ in terms of primitive characters. If $\psi \pmod{k'}$ is induced by the primitive character $\tilde{\psi} \pmod{g}$, then (cf. [18, p. 67])

$$\tau(\psi) = \mu\left(\frac{k'}{g}\right) \tilde{\psi}\left(\frac{k'}{g}\right) \tau(\tilde{\psi}). \quad (6.26)$$

Thus,

$$\begin{aligned} e\left(-\frac{m}{kq}\right) &= \frac{1}{\phi(k')} \sum_{\psi \pmod{k'}} \tau(\bar{\psi}) \psi(-m') \\ &= \frac{1}{\phi(k')} \sum_{g|k'} \sum_{\tilde{\psi} \pmod{g}}^* \mu\left(\frac{k'}{g}\right) \bar{\tilde{\psi}}\left(\frac{k'}{g}\right) \tau(\bar{\tilde{\psi}}) \tilde{\psi}(-m') \end{aligned} \quad (6.27)$$

$$= \frac{\tau(\chi) \mu\left(\frac{k'}{q}\right) \chi\left(\frac{k'}{q}\right) \bar{\chi}(-m')}{\phi(k')} \mathbf{1}_{q|k'} \quad (6.28)$$

$$+ \frac{\tau(\bar{\psi}^*) \mu\left(\frac{k'}{g^*}\right) \bar{\psi}^*\left(\frac{k'}{g^*}\right) \psi^*(-m')}{\phi(k')} \mathbf{1}_{g^*|k'} \quad (6.29)$$

$$+ \frac{1}{\phi(k')} \sum_{g|k'} \sum_{\substack{\tilde{\psi} \pmod{g} \\ \tilde{\psi} \neq \bar{\chi}, \psi^*}}^* \mu\left(\frac{k'}{g}\right) \bar{\tilde{\psi}}\left(\frac{k'}{g}\right) \tau(\bar{\tilde{\psi}}) \tilde{\psi}(-m'), \quad (6.30)$$

where \sum^* denotes the sum is over primitive characters. Using the Möbius inversion formula for an arbitrary function f (cf. [17, eq. (5.10)]), we have

$$\begin{aligned} f(m', k') &= f\left(\frac{m}{(m, kq)}, \frac{kq}{(m, kq)}\right) \\ &= \sum_{d|(m, kq)} \sum_{e|d} \mu\left(\frac{d}{e}\right) f\left(\frac{m}{e}, \frac{kq}{e}\right), \end{aligned}$$

and we obtain

$$\begin{aligned}
& e\left(-\frac{m}{qk}\right) \frac{\tau(\chi)\mu\left(\frac{k'}{q}\right)\chi\left(\frac{k'}{q}\right)\bar{\chi}(-m')}{\phi(k')} \mathbf{1}_{q|k'} - \frac{\tau(\bar{\psi}^*)\mu\left(\frac{k'}{g}\right)\bar{\psi}^*\left(\frac{k'}{g}\right)\psi^*(-m')}{\phi(k')} \mathbf{1}_{g^*|k'} \\
&= \sum_{\substack{d|m \\ d|kq}} \sum_{e|d} \frac{\mu(d/e)}{\phi(kq/e)} \sum_{g|kq/e} \sum_{\substack{\psi \pmod{g} \\ \psi \neq \bar{\chi} \\ \psi \neq \psi^*}}^* \mu\left(\frac{kq}{eg}\right) \bar{\psi}\left(\frac{kq}{eg}\right) \tau(\bar{\psi})\psi\left(-\frac{m}{e}\right) \\
&= \sum_{g|kq} \sum_{\substack{\psi \pmod{g} \\ \psi \neq \bar{\chi} \\ \psi \neq \psi^*}}^* \tau(\bar{\psi}) \sum_{\substack{d|m \\ d|kq}} \sum_{e|d} \frac{\mu(d/e)}{\phi(kq/e)} \bar{\psi}\left(-\frac{kq}{eg}\right) \psi\left(\frac{m}{e}\right) \mu\left(\frac{kq}{eg}\right) \\
&= \sum_{g|kq} \sum_{\substack{\psi \pmod{g} \\ \psi \neq \bar{\chi} \\ \psi \neq \psi^*}}^* \tau(\bar{\psi}) \sum_{\substack{d|m \\ d|kq}} \psi\left(\frac{m}{d}\right) \delta(g, kq, d, \psi),
\end{aligned}$$

where

$$\delta(g, kq, d, \psi) = \sum_{\substack{e|d \\ e|kq/g}} \frac{\mu(d/e)}{\phi(kq/e)} \bar{\psi}\left(-\frac{kq}{eg}\right) \psi\left(\frac{d}{e}\right) \mu\left(\frac{kq}{eg}\right). \quad (6.31)$$

We write \mathcal{M} as

$$\mathcal{M} = \mathcal{M}_{\bar{\chi}} + \mathcal{M}_{\psi^*} + \mathcal{E} \quad (6.32)$$

where

$$\mathcal{M}_{\bar{\chi}} = \frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{kqT/2\pi \leq m \leq kqT/\pi} a(m) \frac{\tau(\chi)\mu\left(\frac{k'}{q}\right)\chi\left(\frac{k'}{q}\right)\bar{\chi}(-m')}{\phi(k')} \mathbf{1}_{q|k'} \quad (6.33)$$

$$\mathcal{M}_{\psi^*} = \frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{kqT/2\pi \leq m \leq kqT/\pi} a(m) \frac{\tau(\bar{\psi}^*)\mu\left(\frac{k'}{g^*}\right)\bar{\psi}^*\left(\frac{k'}{g^*}\right)\psi^*(-m')}{\phi(k')} \mathbf{1}_{g^*|k'} \quad (6.34)$$

$$\mathcal{E} = \frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{kqT/2\pi \leq m \leq kqT/\pi} a(m) \sum_{\ell|q} \sum_{g|k} \sum_{\psi \pmod{g}} \sum_{\substack{\psi \pmod{g\ell} \\ \psi \neq \bar{\chi} \\ \psi \neq \psi^*}}^* \tau(\bar{\psi}) \sum_{\substack{d|m \\ d|kq}} \psi\left(\frac{m}{d}\right) \delta(g\ell, kq, d, \psi). \quad (6.35)$$

Here we have used the fact that $(k, q) = 1$ to rewrite $\psi \pmod{g}$ with $g | kq$ as $\psi \pmod{g\ell}$ with $\ell | q$ and $g | k$ in \mathcal{E} . To evaluate (6.33), (6.34) and (6.35), we need the following lemma.

Lemma 6.8. *Let $F \in \mathcal{S}^*$, χ be a non-real primitive character modulo $q > 1$ and ψ a primitive character modulo g . If g is squarefree, $g \leq Q$ and $(g, B_Q) = 1$, then for positive integers h, k and $Q \leq \exp(2\sqrt{\log x})$, we have*

$$\sum_{\substack{u \leq x \\ (u, k)=1}} \Lambda_F * \chi(hu)\psi(u) = R(x, h, k, \psi) + O\left(h^{\theta_F} \tau(h)j(h)j(k) \log k (\log h)^3 x \exp(-c\sqrt{\log x})\right),$$

where

$$\begin{aligned}
& R(x, h, k, \psi) \\
&= \begin{cases} \frac{\phi(k)(k, q)}{k\phi((k, q))} \frac{\phi(q)}{q} x (f_{-1}\chi(h) \log(x/e) + X_1(h, k) + \chi(h)X_2(k)), & \text{if } \psi = \bar{\chi}, \\ \chi(h)x\Phi(1; k, \chi\psi)L(1, \chi\psi)f_{-1}, & \text{if } \psi \neq \bar{\chi}, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
G(z, \psi) &= \sum_{d=1}^{\infty} \frac{\Lambda_F(d)\psi(d)}{dz}, \\
f_{-1} &= \operatorname{Res}_{z=1} G(z), \quad f_0 = \lim_{z \rightarrow 1} G(z) - \frac{f_{-1}}{z-1}, \\
\eta(z; p, l, \psi) &= \sum_{m=1}^{\infty} \frac{\Lambda_F(p^{l+m})\psi(p^m)}{p^{mz}}, \quad \eta(z; k, \psi) = \sum_{p|k} \eta(z, p, 0, \psi), \\
X_1(h, k) &= \sum_{a|(h,k)} \Lambda_F(a)\chi(h/a) + \sum_{\substack{p^l|h, p|k \\ (p,k)=1}} \chi(h/p^l)\Lambda_F(p^l) \\
&\quad + \chi(h) \sum_{\substack{p^l|h \\ (p,kq)=1}} \bar{\chi}(p^l)\eta(1; p, l, \psi)(1-p^{-1}), \\
X_2(k) &= f_0 - \eta(1; k, \psi) + \left(\gamma + \sum_{p|kq} \frac{\log p}{p-1} \right) f_{-1}, \\
j(n) &= \prod_{p|n} (1 + 10p^{-1/2}). \tag{6.36}
\end{aligned}$$

Here c is some positive absolute constant depending only on F and χ . If there exists some absolute constant $a > 0$ such that $L(s, \chi\psi)$ and F_ψ have no zeros in the region $\Re(s) \geq 1 - a$ for all ψ , then the error term can be replaced by

$$O(h^{\theta_F} \tau(h) j(h) j(k) \log k (\log h)^3 x^{1-\delta+\epsilon}),$$

for some small enough $\delta = \delta(F, a) > 0$ uniformly for $Q \leq x$.

To prove Lemma 6.8, we need the following lemmas.

Lemma 6.9 (Decomposition of convolutions, [38, Lemma 6.1]). *Let $j, D \in \mathbb{N}$ and let f_1, \dots, f_j be arithmetic functions. Given a decomposition of integers $D = \prod_{i=1}^j d_i$, define the integers $D_i = \prod_{u=1}^{j-i} d_u$ for $1 \leq i \leq j-1$ and $D_j = 1$. Then we have the following identities:*

$$\sum_{\substack{m \leq X \\ (m,k)=1}} (f_1 * \dots * f_j)(mD) = \sum_{d_1 \dots d_j = D} \sum_{\substack{m_1 \dots m_j \leq X \\ (m_i, kD_i)=1}} f_1(m_1 d_j) f_2(m_2 d_{j-1}) \dots f_j(m_j d_1), \tag{6.37}$$

$$\sum_{(m,k)=1} \frac{(f_1 * \dots * f_j)(mD)}{m^s} = \sum_{d_1 d_2 \dots d_j = D} \prod_{i=1}^j \sum_{(m_i, kD_i)=1} \frac{f_i(m_i d_{j-i+1})}{m_i^s}. \tag{6.38}$$

Lemma 6.10 (A variant of Perron's Formula, [35, Theorem 2.1]). *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of absolute convergence σ_a . Let*

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma}, \tag{6.39}$$

for $\sigma > \sigma_a$. Then for $b > \sigma_a$, $x \geq 2$, $U \geq 2$, and $H \geq 2$, we have

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iU}^{b+iU} f(s) \frac{x^s}{s} ds + O\left(\sum_{x-x/H \leq n \leq x+x/H} |a_n| \right) + O\left(\frac{x^b HB(b)}{U} \right). \tag{6.40}$$

Lemma 6.11 (Dirichlet's Hyperbola Principle in short intervals, [12, Lemma 4.1]). *Let $x, y, z \in \mathbb{R}$ such that $1 \leq \max(y, \frac{x}{y}) \leq z \leq x$. Then for any arithmetic function f and g*

$$\begin{aligned} \sum_{x \leq n \leq x+y} (f * g)(n) &= \sum_{d \leq z} f(d) \sum_{\frac{x}{d} \leq k \leq \frac{x+y}{d}} g(k) + \sum_{k \leq x/z} \sum_{\frac{x}{k} \leq d \leq \frac{x+y}{k}} g(d) \\ &\quad + O\left(\max_{k \leq 2x/z} |g(k)| \sum_{z \leq d \leq z(1+y/x)} |f(d)|\right). \end{aligned}$$

Now we are ready to prove Lemma 6.8.

Proof. [Proof of Lemma 6.8] Let

$$A(z; h, \psi) = A(z) = \sum_{\substack{u=1 \\ (u,k)=1}}^{\infty} \Lambda_F * \chi(hu) \psi(u) u^{-z}.$$

From Lemma 3.2, we have for $\kappa = 1 + O((\log x)^{-1})$ and $h \leq x$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|\Lambda_F * \chi(hu) \psi(u)|}{u^\kappa} &\leq \sum_u \frac{(|\Lambda_F| * 1)(hu)}{u^\kappa} = \sum_{h_1 h_2 = h} \sum_{(u_1, h_1)=1} \frac{1}{u_1^\kappa} \sum_{u_2} \frac{|\Lambda_F(u_2 h_1)|}{u_2^\kappa} \\ &\ll \tau(h) h^\kappa \sum_n \frac{(|\Lambda_F| * 1)(n)}{n^\kappa} \ll \tau(h) h (\log x)^2. \end{aligned}$$

Therefore, taking $b = \kappa$ and $H = \sqrt{U}$ in Lemma 6.10, we have

$$\begin{aligned} &\sum_{\substack{u \leq x \\ (u,k)=1}} \Lambda_F * \chi(hu) \psi(u) \\ &= \frac{1}{2\pi i} \int_{\kappa-iU}^{\kappa+iU} A(z) x^z \frac{dz}{z} + O\left(\sum_{x-x/\sqrt{U} \leq u \leq x+x/\sqrt{U}} |\Lambda_F * \chi(hu) \psi(u)|\right) + O\left(\frac{\tau(h) h x (\log x)^2}{\sqrt{U}}\right). \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{x-x/\sqrt{U} \leq u \leq x+x/\sqrt{U}} |\Lambda_F * \chi(hu) \psi(u)| \\ &\ll \sum_{x-x/\sqrt{U} \leq n \leq x+x/\sqrt{U}} \sum_{d|hn} |\Lambda_F(d)| \\ &\ll \sum_{x-x/\sqrt{U} \leq n \leq x+x/\sqrt{U}} \left(\sum_{d|h} |\Lambda_F(d)| + \sum_{d|n} |\Lambda_F(d)| \right) \\ &\ll \frac{x}{\sqrt{U}} \tau(h) h^{\theta_F} \log h + \sum_{x-x/\sqrt{U} \leq n \leq x+x/\sqrt{U}} |(1 * \Lambda_F)(n)|. \end{aligned}$$

Using property (iv) of \mathcal{S}^* and partial summation,

$$\sum_{n \leq x} \frac{|\Lambda_F(n)|}{n} \leq \left(\sum_{n \leq x} \frac{\Lambda(n)}{n} \right)^{1/2} \left(\sum_{n \leq x} \frac{\Lambda(n) |\lambda_F(n)|^2}{n} \right)^{1/2} \ll \log x. \quad (6.41)$$

We also have, from property (v) of \mathcal{S}^* , that for $x \gg_F U^c$

$$\sum_{x \leq n \leq x + \frac{x}{\sqrt{U}}} |\Lambda_F(n)| \ll \sum_{x \leq n \leq x e^{1/\sqrt{U}}} |\Lambda_F(n)| \ll \frac{x}{\sqrt{U}}. \quad (6.42)$$

We apply Lemma 6.11 with $f = |\Lambda_F|, g = 1, y = z = 2x/\sqrt{U}$. Then for $\eta = \eta(F) > 0$ small enough, we have uniformly for $U \leq x^\eta$,

$$x \gg U^c, z \gg \left(\frac{x}{y}\right)^c, \quad (6.43)$$

and thus

$$\begin{aligned} & \sum_{x-x/\sqrt{U} \leq n \leq x+x/\sqrt{U}} |(\Lambda_F * 1)(n)| \\ &= \sum_{d \leq z} |\Lambda_F(d)| \sum_{\frac{x-x/\sqrt{U}}{d} \leq k \leq \frac{x+x/\sqrt{U}}{d}} 1 + \sum_{k \leq x/z} \sum_{\frac{x-x/\sqrt{U}}{k} \leq d \leq \frac{x+x/\sqrt{U}}{k}} |\Lambda_F(d)| + O\left(\sum_{z \leq d \leq z(1+\frac{y}{x-\sqrt{U}})} |\Lambda_F(d)|\right) \\ &\ll y \sum_{d \leq z} \frac{|\Lambda_F(d)|}{d} + \sum_{k \leq x/z} \frac{y}{k} + O\left(\frac{yz}{x-x/\sqrt{U}}\right) \\ &\ll y \log z + y \log x + z \\ &\ll \frac{x}{\sqrt{U}} \log x. \end{aligned}$$

Therefore, for $h \leq x$ and $U \leq x^\eta$,

$$\sum_{\substack{u \leq x \\ (u,k)=1}} \Lambda_F * \chi(hu) \psi(u) = \frac{1}{2\pi i} \int_{\kappa-iU}^{\kappa+iU} A(z) x^z \frac{dz}{z} + O\left(\tau(h) h \frac{x(\log x)^2}{\sqrt{U}}\right). \quad (6.44)$$

Applying Lemma 6.9, we write

$$A(z) = \sum_{ab=h} \sum_{(c,ak)=1} \frac{\chi(bc) \psi(c)}{c^z} \sum_{(d,k)=1} \frac{\Lambda_F(ad) \psi(d)}{d^z} := \sum_{ab=h} A_1(z; a, b) A_2(z; a) \quad (6.45)$$

where

$$A_1(z; a, b) = \chi(b) \sum_{(c,ak)=1} \frac{\chi \psi(c)}{c^z} = \chi(b) L(s, \chi \psi) \prod_{p|ak} (1 - \chi \psi(p) p^{-z}),$$

and

$$A_2(z; a) = \begin{cases} \sum_{(d,k)=1} \frac{\Lambda_F(d) \psi(d)}{d^z}, & \text{if } a = 1, \\ \sum_{k=0}^{\infty} \frac{\Lambda_F(p^{l+k}) \psi(p^k)}{p^{kz}}, & \text{if } a = p^l, p \nmid k, \\ \Lambda_F(a), & \text{if } a = p^l, p \mid k, \\ 0, & \text{else.} \end{cases}$$

Using the following notation

$$\eta(z; p, l, \psi) = \sum_{k=0}^{\infty} \frac{\Lambda_F(p^{l+k}) \psi(p^k)}{p^{kz}}, \quad (6.46)$$

$$\Phi(z; k, \chi) = \prod_{p|k} (1 - \chi(p) p^{-z}), \quad (6.47)$$

$$G(z; \psi) = \sum_{d=1}^{\infty} \frac{\Lambda_F(d) \psi(d)}{d^z} = \sum_p \eta(z; p, 0, \psi), \quad (6.48)$$

$$\eta(z; k, \psi) = \sum_{p|k} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m) \psi(p^m)}{p^{mz}} = \sum_{p|k} \eta(z; p, 0, \psi), \quad (6.49)$$

we can represent the $A(z)$ in (6.45) as

$$\begin{aligned} A(z) &= \chi(h)L(s, \chi\psi)\Phi(z; k, \chi\psi) (G(z; \psi) - \eta(z; k, \psi)) \\ &\quad + \sum_{p^l|h, p^l|k} \chi(h/p^l)L(s, \chi\psi)\Phi(z; p^l k, \chi\psi)\eta(z; p, l, \psi) \\ &\quad + \sum_{p^l|(h, k)} \chi(h/p^l)L(s, \chi\psi)\Phi(z; p^l k, \chi\psi)\Lambda_F(p^l) \\ &:= B_1(z) + B_2(z) + B_3(z). \end{aligned}$$

From $F \in \mathcal{S}^*$ and Lemma 3.2, we deduce that $G(z; \psi)$ has at most a simple pole at $z = 1$. This shows that $A(z)$ has a pole at $z = 1$ of order at most 2. Hence,

$$\frac{1}{2\pi i} \int_{\kappa-iU}^{\kappa+iU} A(z)x^z \frac{dz}{z} = \text{Res}_{z=1} A(z)x^z z^{-1} + \frac{1}{2\pi i} \int_{\sigma_0(U)-iU}^{\sigma_0(U)+iU} A(z)x^z \frac{dz}{z} \quad (6.50)$$

$$+ \left(\int_{\sigma_0(U)+iU}^{\kappa+iU} + \int_{\kappa-iU}^{\sigma_0(U)-iU} \right) A(z)x^z \frac{dz}{z}, \quad (6.51)$$

where $\sigma_0(U) = 1 - \frac{c}{\log(Q(U+2))}$ for some positive constant $c = c(F)$. From Lemma 3.2 and [18, Ch 14, eq (13)], we have

$$\begin{aligned} |G(z, \psi)| &\ll \log^2(Q(|z| + 2)), \quad \text{for } \Re(z) \geq 1 - c/\log(Q(|\Im(z)| + 2)), \\ |L(z, \chi\psi)| &\ll \log(Q(|z| + 2)), \quad \text{for } \Re(z) \geq 1 - c/\log(Q(|\Im(z)| + 2)). \end{aligned}$$

From (6.36) we have $j(n) \ll \exp(o(\sqrt{\log n}))$ and $\Phi(z; k, \chi) \ll j(k)$, for $\Re(z) \geq 1 - c/\log(Q(U+2))$. Moreover, since $|\Lambda_F(p^j)| \ll_F j p^{j\theta_F} \log p$, we see that

$$\eta(z; p, l, \psi) = \sum_{r=0}^{\infty} \frac{\Lambda_F(p^{l+r})\psi(p^r)}{p^{rz}} \ll \sum_{r=0}^{\infty} \frac{(r+l)p^{(r+l)\theta_F} \log p}{p^{rz}} \ll (l+1)p^{l\theta_F} \log p, \quad (6.52)$$

and it follows that

$$\begin{aligned} \eta(z; k, \psi) &\ll \sum_{p^l|k} \eta(z; p, 0, \psi) \ll \sum_{p^l|k} \log p \ll \log k, \\ B_2(z) &\ll \sum_{p^l|h} j(hk)(l+1)p^{l\theta_F} \log p \log(Q(|\Im z| + 2)) \ll j(hk)h^{\theta_F} (\log h)^3 \log(Q(|\Im z| + 2)), \\ B_3(z) &\ll \sum_{p^l|(h, k)} j(hk) \log(Q(|\Im z| + 2)) h^{\theta_F} \log h \ll j(hk)\tau(h)h^{\theta_F} \log h \log(Q(|\Im z| + 2)). \end{aligned}$$

This shows that for $\Re(z) \geq \sigma_0(U)$ and $|\Im z| \leq U$,

$$\begin{aligned} A(z) &= B_1(z) + B_2(z) + B_3(z) \\ &\ll j(k) \log(QU) (\log^2(QU) + \log k) + h^{\theta_F} j(hk) \log(QU) (\log h)^3 + \tau(h)h^{\theta_F} j(hk) \log h \log(QU) \\ &\ll \tau(h)h^{\theta_F} j(h)j(k) \log k (\log h)^3 (\log(QU))^3. \end{aligned}$$

Thus, the horizontal integrals in (6.51) are bounded by

$$\int_{\sigma_0(U)}^{\kappa} \frac{A(\sigma \pm iU)}{|\sigma \pm iU|} x^\sigma d\sigma \ll \frac{x\tau(h)j(h)h^{\theta_F} j(k) \log k (\log h)^3 (\log(QU))^3}{U}.$$

The left vertical integral in (6.50) is bounded by

$$\begin{aligned} x^{\sigma_0(U)} \int_{-U}^U \frac{A(\sigma + iu)}{|\sigma + iu|} du &\ll x^{\sigma_0(U)} h^{\theta_F} \tau(h)j(h)j(k) \log k (\log h)^3 (\log(QU))^4 \\ &\ll x h^{\theta_F} \tau(h)j(h)j(k) \log k (\log h)^3 (\log(QU))^4 \exp\left(-\frac{c \log x}{\log(Q(|U| + 2))}\right). \end{aligned}$$

If we choose $U = \exp(c'\sqrt{\log x})$, then we have uniformly for $Q \leq \exp(2\sqrt{\log x})$ all integrals in (6.50) and (6.51) bounded by

$$O\left(h^{\theta_F} \tau(h) j(h) j(k) \log k (\log h)^3 x \exp(-c''\sqrt{\log x})\right).$$

If $L(s, \chi\psi)$ and F_ψ have no zeros in the region $\Re(s) \geq 1 - a$, then we can choose $\sigma_0(U) = 1 - a + \epsilon$. We have by Lemma 3.2 $\sigma_0(U) \leq \sigma \leq \kappa$,

$$G(z, \psi) \ll (\log QU)^2. \quad (6.53)$$

We also have for $\sigma \geq \sigma_0(U)$, $L(\sigma + it, \chi\psi) \ll (QU)^\epsilon$, if $L(s, \chi\psi)$ has no zeros for $\Re(s) \geq 1 - a$. Thus,

$$A(z) \ll \tau(h) h^{\theta_F} j(k) \log k (\log h)^3 (\log QU)^3 (QU)^\epsilon. \quad (6.54)$$

Therefore, we have

$$\int_{\sigma_0(U)}^{\kappa} \frac{A(\sigma \pm iU)}{|\sigma \pm iU|} x^\sigma d\sigma \ll \frac{x\tau(h)j(h)h^{\theta_F}j(k) \log k (\log h)^3 (\log QU)^3 (QU)^\epsilon}{U}, \quad (6.55)$$

$$x^{\sigma_0(U)} \int_{-U}^U \frac{A(\sigma + iu)}{|\sigma + iu|} du \ll x^{1-a} h^{\theta_F} \tau(h) j(h) j(k) \log k (\log h)^3 (\log QU)^4 (QU)^\epsilon. \quad (6.56)$$

Combining these with (6.44), we see that taking $U = x^{\min(2a, \eta)}$ yields an error term of size

$$O\left(h^{\theta_F} \tau(h) j(h) j(k) \log k (\log h)^3 x^{1-\delta+\epsilon}\right),$$

uniformly for $Q \leq x$. Next we compute the residue at $z = 1$. Suppose we have the Laurent series

$$\begin{aligned} G(z) &= \frac{f_{-1}}{(z-1)} + f_0 + f_1(z-1) + f_2(z-1)^2 + \cdots, \\ L(z, \chi\psi) &= \frac{c_{-1}}{(z-1)} + c_0 + c_1(z-1) + c_2(z-1)^2 + \cdots, \\ \Phi(z; k, \chi\psi) &= \Phi(1; k, \chi\psi) + \Phi'(1; k, \chi\psi)(z-1) + \frac{1}{2}\Phi^{(2)}(1; k, \chi\psi)(z-1)^2 + \cdots, \\ \eta(z; k, \psi) &= \eta(1; k, \psi) + \eta'(1; k, \psi)(z-1) + \eta^{(2)}(1; k, \psi)(z-1)^2 + \cdots, \\ \frac{x^z}{z} &= x \left(1 + \log(x/e)(z-1) + \left(\frac{1}{2} \log^2 x - \log(x/e)\right)(z-1)^2 + \cdots \right). \end{aligned}$$

From the fact that

$$\Phi(1; p^l k, \chi\psi) = \begin{cases} \Phi(1; k, \chi\psi), & p \mid k, \\ \Phi(1; k, \chi\psi)(1 - \chi\psi(p)p^{-1}), & p \nmid k. \end{cases}$$

we see that,

$$\begin{aligned} & \text{Res}_{z=1} A(z) x^z z^{-1} \\ &= \chi(h) x \Phi(1; k, \chi\psi) (c_{-1}(f_0 - \eta(1; k, \psi)) + c_0 f_{-1}) \\ &+ \chi(h) x (\log(x/e) \Phi(1; k, \chi\psi) + \Phi'(1; k, \chi\psi)) f_{-1} c_{-1} \\ &+ x \Phi(1; k, \chi\psi) c_{-1} \left(\sum_{a \mid (h, k)} \Lambda_F(a) \chi(h/a) + \sum_{p^l \mid h, p \nmid k} \chi(h/p^l) \eta(1; p, l, \psi) (1 - \chi\psi(p)p^{-1}) \right). \end{aligned}$$

If $\chi\psi$ is principal, then $L(s, \chi\psi) = \prod_{p|q} (1-p^{-s})\zeta(s)$ and thus $c_{-1} = \frac{\phi(q)}{q}$ and $c_0 = \frac{\phi(q)}{q}(\gamma + \sum_{p|q} \frac{\log p}{p-1})$. From (6.47), we have

$$\begin{aligned}\Phi(1; k, \chi\psi) &= \prod_{p|k, p \nmid q} (1-p^{-1}) = \frac{\phi(k)(k, q)}{k\phi((k, q))}, \\ \Phi'(1; k, \chi\psi) &= \Phi(1; k, \chi\psi) \sum_{p|k, p \nmid q} \frac{\log p}{p-1}.\end{aligned}$$

Since $\psi(p) = 0$ for $p \mid q$, we see that for $p \mid q$,

$$\eta(z; p, l, \psi) = \Lambda_F(p^l), \quad (6.57)$$

and thus the residue of $A(z)$ at $z = 1$ can be written as

$$\begin{aligned}& \text{Res}_{z=1} A(z)x^z z^{-1} \\ &= x\chi(h) \frac{\phi(k)(k, q)}{k\phi((k, q))} \frac{\phi(q)}{q} \left(f_0 - \eta(1; k, \psi) + \left(\gamma + \sum_{p|q} \frac{\log p}{p-1} \right) f_{-1} \right) \\ &+ x \frac{\phi(k)(k, q)}{k\phi((k, q))} \frac{\phi(q)}{q} \left(\chi(h) \left(\log(x/e) + \sum_{\substack{p|k \\ p \nmid q}} \frac{\log p}{p-1} \right) f_{-1} + \sum_{a|(h, k)} \Lambda_F(a)\chi(h/a) \right) \\ &+ x \frac{\phi(k)(k, q)}{k\phi((k, q))} \frac{\phi(q)}{q} \left(\sum_{\substack{p^l|h, p|q \\ (p, k)=1}} \chi(h/p^l)\Lambda_F(p^l) + \sum_{\substack{p^l|h \\ (p, kq)=1}} \chi(h/p^l)\eta(1; p, l, \psi)(1-p^{-1}) \right).\end{aligned}$$

If $\chi\psi$ is non-principal, the only possible pole of $A(z)$ arises from $G(z; \psi)$, in which case it is a simple pole at $z = 1$, and

$$\text{Res}_{z=1} A(z)x^z z^{-1} = x\chi(h)\Phi(1; k, \chi\psi)L(1, \chi\psi)f_{-1}.$$

□

6.5. Proof of Theorem 6.4. To evaluate (6.33), (6.34) and (6.35), we apply Lemma 6.8 with $x = \frac{kqT}{2\pi dv}$. First we note that $\frac{kqT}{2\pi dv} \gg T^{1/2}$ since $d \leq M \leq \sqrt{T}$. By the support of x_n , we have $G(z, \psi)$ have zero free region of the form $1 - \sigma \ll \frac{1}{\log M(|\Im z| + 2)}$ for all ψ with conductor $\leq M$. Therefore, we all terms with $\exp(-c\sqrt{\log x})$ (or $x^{-\delta+\epsilon}$) can be adjusted to some absolute constant c (or δ) depending on F and $L(s, \chi)$ uniformly for $M \leq \exp(\sqrt{\log x})$ (or $M \leq \sqrt{T}$).

6.5.1. Error Terms. We change the order of summation and expand the definition of $a(m)$ (6.14) to obtain

$$\begin{aligned}\mathcal{E} &= \frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{\ell|q} \sum_{g|k} \sum_{\psi} \sum_{\substack{(\text{mod } g\ell) \\ \psi \neq \bar{\chi} \\ \psi \neq \psi^*}}^* \tau(\bar{\psi}) \sum_{d|kq} \delta(g\ell, kq, d, \psi) \sum_{kqT/2\pi d \leq m \leq kqT/\pi d} a(md)\psi(m) \\ &= -\frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{\ell|q} \sum_{g|k} \sum_{\psi} \sum_{\substack{(\text{mod } g\ell) \\ \psi \neq \bar{\chi} \\ \psi \neq \psi^*}}^* \tau(\bar{\psi}) \sum_{d|kq} \delta(g\ell, kq, d, \psi) \\ &\times \sum_{sh=d} \sum_{\substack{sv \leq M \\ (v, h)=1}} \bar{y}_{sv}\psi(v) \sum_{kqT/2\pi dv \leq u \leq kqT/\pi dv} (\Lambda_F * \chi)(hu)\psi(u).\end{aligned} \quad (6.58)$$

Since $R(x, h, k, \psi) = 0$ for all $\psi \neq \bar{\chi}, \psi^*$, we can bound \mathcal{E} by

$$\begin{aligned} \mathcal{E} &\ll \frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{|\bar{x}_k|}{k} \sum_{\ell|q} \sum_{g|k} \sum_{\psi \pmod{g\ell}}^* |\tau(\bar{\psi})| \sum_{d|kq} |\delta(g\ell, kq, d, \psi)| \\ &\times \sum_{sh=d} \sum_{\substack{sv \leq M \\ (v, \bar{h})=1}} |\bar{y}_{sv}| \frac{qkT}{dv} h^{\theta_F} \tau(h) j(h) \log k (\log h)^3 \exp(-c\sqrt{\log T}). \end{aligned}$$

To estimate the error \mathcal{E} , we need the following lemmas on arithmetic functions.

Lemma 6.12. *For $d, k, q, \ell \in \mathbb{N}$, ψ a primitive character modulo $g\ell$ and $kq \ll T$ with $(k, q) = 1$, we have the inequality*

$$\delta(g\ell, kq, d, \psi) \ll \frac{(d, kq/g\ell)}{\phi(kq)}.$$

Proof. From equation (6.31), we have

$$\delta(g\ell, kq, d, \psi) \ll \sum_{\substack{e|d \\ e|kq/g\ell}} \frac{\phi(e)}{\phi(kq)} = \frac{(d, kq/g\ell)}{\phi(kq)}$$

since $1 * \phi = id$. □

Lemma 6.13. *Let h be a positive multiplicative function and let $1 \leq k, q \leq M$ and $(k, q) = 1$. Then,*

$$\sum_{d|kq} \frac{(d, k)h(d)}{d} \ll (1 * |h|)(k) \left\| \frac{h(n)}{n} \right\|_1.$$

Proof. Let $g = (d, k)$, and write $d = gd_1$ and $k = gk_1$. Then,

$$\sum_{d|kq} \frac{(d, k)h(d)}{d} \ll \sum_{g|k} \sum_{d_1|kq/g} \frac{g}{gd_1} |h(gd_1)| \ll \sum_{g|k} |h(g)| \left\| \frac{h(n)}{n} \right\|_1 = (1 * |h|)(k) \left\| \frac{h(n)}{n} \right\|_1.$$

□

Now we are ready to estimate \mathcal{E} . From Lemma 6.12 and $(k, q) = 1$,

$$\delta(g\ell, kq, d, \psi) \ll \frac{(d, kq/g\ell)}{\phi(kq)} \ll \frac{(d, kq/g\ell)}{\phi(k)\phi(q)}.$$

We also have

$$\sum_{sh=d} \sum_{\substack{sv \leq M \\ (v, \bar{h})=1}} \frac{|y_{sv}|}{v} \tau(h) j(h) \ll (\tau * |y|)(d) j(d) \left\| \frac{y_n}{n} \right\|_1.$$

Therefore, using $h, k \leq qM \ll T$, we have

$$\begin{aligned}
\mathcal{E} &\ll \frac{|\tau(\bar{\chi})|}{q} \sum_{k \leq M} \frac{|\bar{x}_k|}{k} \sum_{\ell|q} \sum_{g|k} \sum_{\psi}^* \sum_{(\text{mod } g\ell)} |\tau(\bar{\psi})| \sum_{d|kq} |\delta(g\ell, kq, d, \psi)| \\
&\quad \times \sum_{sh=d} \sum_{\substack{sv \leq M \\ (v, h)=1}} |\bar{y}_{sv}| \frac{qkT}{dv} h^{\theta_F} \tau(h) j(h) \log k (\log h)^3 \exp(-c\sqrt{\log T}) \\
&\ll \frac{(qM)^{\theta_F} q (\log M)^3 (\log q)^3 T}{\sqrt{q}} \sum_{k \leq M} |x_k| \sum_{\ell|q} \sum_{g|k} \sqrt{g\ell} \phi(g\ell) \\
&\quad \times \sum_{d|kq} \frac{(d, kq/g\ell)}{\phi(kq)} \frac{(\tau * |y|)(d) j(d)}{d} \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}) \\
&\ll M^{\theta_F + \epsilon} q^{\frac{1}{2} + \theta_F + \epsilon} T \sum_{\ell|q} \sum_{g \leq M} \sum_{k \leq M/g} |x_{gk}| (g\ell)^{3/2} \frac{(\tau_3 * |y|)(kq) j(k) j(q)}{\phi(gkq)} \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}) \\
&\ll M^{\theta_F + \epsilon} q^{1 + \theta_F + \epsilon} T \sum_{g \leq M} |x_g| \sqrt{g} \sum_{\substack{k \leq M/g \\ (k, q)=1}} |x_k| \frac{(\tau_3 * |y|)(k) j(k)}{\phi(k)} \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}) \\
&\ll M^{\frac{1}{2} + \theta_F + \epsilon} q^{1 + \theta_F + \epsilon} \|y_n\|_{\infty} \|x_n\|_1 \left\| \frac{\tau_3 * |y|(n)}{n} \right\|_1 \left\| \frac{(\tau * |y|)(n)}{n} \right\|_1 \left\| \frac{y_n}{n} \right\|_1 \exp(-c\sqrt{\log T}). \tag{6.59}
\end{aligned}$$

6.5.2. *Main Term Evaluation.* From (6.33), we have

$$\mathcal{M}_{\bar{\chi}} = \frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{kqT/2\pi \leq m \leq kqT/\pi} a(m) \frac{\tau(\chi) \mu(\frac{k'}{q}) \chi(\frac{k'}{q}) \bar{\chi}(-m')}{\phi(k')} \mathbf{1}_{q|k'},$$

where $k' = \frac{kq}{(m, kq)}$ and $m' = \frac{m}{(m, kq)}$ and $q | k'$. Since x, y are supported on integers that are coprime to q , we have $(k, q) = 1$ and it follows that $(m, kq) = (m, k)(m, q)$. Thus if $q | k'$, we must have $(m, q) = 1$, so that $k' = qk/(m, k)$ and thus $m' = m/(m, k)$. We write $l = (m, k)$, $k = lk_1$, and $m = lm_1$, and then replace k_1 by k and m_1 by m . Using that $\tau(\chi)\tau(\bar{\chi})\chi(-1) = q$, we have

$$\begin{aligned}
\mathcal{M}_{\bar{\chi}} &= \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{\substack{kqT/2\pi \leq m \leq kqT/\pi \\ (m, q)=1}} a(m) \frac{1}{\phi(\frac{k}{(m, k)}) \phi(q)} \mu\left(\frac{k}{(m, k)}\right) \chi\left(\frac{k}{(m, k)}\right) \bar{\chi}\left(\frac{m}{(m, k)}\right) \\
&= \sum_{lk \leq M} \frac{\bar{x}_{lk}}{lk} \frac{\mu(k) \chi(k)}{\phi(k)} \sum_{\substack{kqT/2\pi \leq m \leq kqT/\pi \\ (m, kq)=1}} a(lm) \bar{\chi}(m) \\
&= -\frac{1}{\phi(q)} \sum_{lk \leq M} \frac{\bar{x}_{lk}}{lk} \frac{\mu(k) \chi(k)}{\phi(k)} \sum_{sh=l} \sum_{\substack{sv \leq M \\ (v, hkg)=1}} \bar{y}_{sv} \bar{\chi}(v) \sum_{\substack{kqT/2\pi \leq uv \leq kqT/\pi \\ (u, kq)=1}} \Lambda_F * \chi(hu) \bar{\chi}(u) \tag{6.60}
\end{aligned}$$

After an application of Lemma 6.8 to (6.60), we obtain

$$\begin{aligned}
\mathcal{M}_{\bar{\chi}} &= -\frac{1}{\phi(q)} \sum_{lk \leq M} \frac{\bar{x}_{lk}}{lk} \frac{\mu(k) \chi(k)}{\phi(k)} \sum_{sh=l} \sum_{\substack{sv \leq M \\ (v, hkg)=1}} \bar{y}_{sv} \bar{\chi}(v) \\
&\quad \times \frac{\phi(q)}{q} \left\{ \frac{\phi(k)}{k} \frac{kqT}{2\pi v} \left(\chi(h) \log\left(\frac{2kqT}{\pi v e}\right) f_{-1} + X_1(h, kq) + \chi(h) X_2(kq) \right) \right. \\
&\quad \left. + O\left(h^{\theta_F} \tau(h) j(h) (\log h)^3 j(kq) \log(kq) \frac{kqT}{v} \exp(-c\sqrt{\log T}) \right) \right\}. \tag{6.61}
\end{aligned}$$

Let $\mathcal{E}_{\bar{\chi}}$ denote the contribution of the O -terms in (6.61). After replacing l by sh and using $h \leq qM$, we have

$$\begin{aligned} \mathcal{E}_{\bar{\chi}} &\ll q^{\theta_F + \epsilon} M^{\theta_F + \epsilon} T \sum_{shk \leq M} \frac{|\bar{x}_{shk}|}{shk} \frac{k}{\phi(k)} \tau(h) j(h) \sum_{sv \leq M} \frac{|\bar{y}_{sv}|}{v} \exp(-c\sqrt{\log T}) \\ &\ll q^{\theta_F + \epsilon} M^{\theta_F + \epsilon} T \left\| \frac{|x_s y_s|}{s} \right\|_1 \left\| \frac{(\tau_3 * |x|)(n)}{n} \right\|_1 \left\| \frac{y_v}{v} \right\|_1 \exp(-c\sqrt{\log T}). \end{aligned}$$

Since x_k are supported on integers k with $(k, q) = 1$, thus $l \nmid q$ and $h \nmid q$. Upon writing $l = sh$ and $hk = u$, we find that

$$\begin{aligned} \mathcal{M}_{\bar{\chi}} &= -\frac{T}{2\pi} \sum_{su \leq M} \frac{\bar{x}_{su} \chi(u)}{su} \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{\bar{y}_{sv} \bar{\chi}(v)}{v} \sum_{hk=u} \mu(k) \left(\log \left(\frac{2kqT}{\pi ve} \right) f_{-1} + \widetilde{X}_1(h, k) + \widetilde{X}_2(k) \right) + \mathcal{E}_{\bar{\chi}} \\ &= -\frac{T}{2\pi} \sum_{su \leq M} \frac{\bar{x}_{su} \chi(u)}{su} \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{\bar{y}_{sv} \bar{\chi}(v)}{v} \sum_{hk=u} \mu(k) \left(\log \left(\frac{2qT}{\pi ve} \right) f_{-1} + f_{-1} \log k \right) \\ &\quad - \frac{T}{2\pi} \sum_{su \leq M} \frac{\bar{x}_{su} \chi(u)}{su} \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{\bar{y}_{sv} \bar{\chi}(v)}{v} \sum_{hk=u} \mu(k) \left(\widetilde{X}_1(h, k) + f_{-1} \widetilde{X}_2(k) \right) + \mathcal{E}_{\bar{\chi}} \\ &= -\frac{T}{2\pi} \sum_{su \leq M} \frac{\bar{x}_{su} \chi(u)}{su} \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{\bar{y}_{sv} \bar{\chi}(v)}{v} \left(\delta(u) \log \left(\frac{2qT}{\pi ve} \right) f_{-1} - \Lambda(u) f_{-1} \right) \\ &\quad - \frac{T}{2\pi} \sum_{su \leq M} \frac{\bar{x}_{su} \chi(u)}{su} \sum_{\substack{sv \leq M \\ (v, uq)=1}} \frac{\bar{y}_{sv} \bar{\chi}(v)}{v} \sum_{hk=u} \mu(k) \left(\widetilde{X}_1(h, k) + f_{-1} \widetilde{X}_2(k) \right) + \mathcal{E}_{\bar{\chi}}, \end{aligned} \tag{6.62}$$

where

$$\begin{aligned} \delta(u) &= \begin{cases} 1, & \text{if } u = 1, \\ 0, & \text{if } u > 1, \end{cases} \\ \widetilde{X}_1(h, k) &= \sum_{a|(h, k)} \Lambda_F(a) \bar{\chi}(a) + \sum_{p^l | h, p \nmid k} \bar{\chi}(p^l) \eta(1; p, l, \bar{\chi}) (1 - p^{-1}), \\ \widetilde{X}_2(k) &= f_0 - \eta(1; kq, \bar{\chi}) + \left(\gamma + \sum_{p|kq} \frac{\log p}{p-1} \right) f_{-1}. \end{aligned}$$

For \mathcal{M}_{ψ^*} , we write the modulus of ψ^* as $g^* = g_0 \ell_0$, where $\ell_0 = (g^*, q)$ and $(g_0, q) = 1$.

$$\mathcal{M}_{\psi^*} = \frac{\tau(\bar{\chi})}{q} \sum_{k \leq M} \frac{\bar{x}_k}{k} \sum_{kqT/2\pi < m \leq kqT/\pi} a(m) \frac{1}{\phi(k')} \mu \left(\frac{k'}{g_0 \ell_0} \right) \bar{\psi}^* \left(\frac{k'}{g_0 \ell_0} \right) \tau(\bar{\psi}^*) \psi^*(-m') \mathbf{1}_{g_0 \ell_0 | k'},$$

where $k' = \frac{kq}{(kq, m)}$, $m' = \frac{m}{(kq, m)}$ and $g_0 \ell_0 | k'$. We further choose x, y supported on integers that are coprime to qg_0 . If $g_0 \ell_0 | k' | kq$, we must have $g_0 | k$ since $(k, q) = 1$, $(g_0, q) = 1$. From the support of x_k we have $(k, g_0) = 1$ and thus we must have $g_0 = 1$. Therefore, we only need to consider the case when ψ^* is a character modulo ℓ_0 with $\ell_0 | q$. Since $(k, q) = 1$, we see that $\ell_0 | k'$ implies that $\ell_0 | \frac{q}{(q, m)}$. If we write $l = (m, k)$, $\ell = (q, m)$, $k = lk_1$, $m = \ell m_1$, $q = \ell \ell_0 q_1$, then $(m_1, k_1) = 1$, $(m_1, q_1 \ell_0) = 1$ and

$\ell_0 \mid q/\ell$. Thus,

$$\begin{aligned}
\mathcal{M}_{\psi^*} &= \frac{\tau(\bar{\chi})}{q} \sum_{\ell|q/\ell_0} \sum_{l \leq M} \sum_{k_1 \leq M/l} \frac{\bar{x}_{lk_1}}{lk_1} \frac{1}{\phi(\ell_0 k_1 q_1)} \mu(k_1 q_1) \bar{\psi}^*(k_1 q_1) \tau(\bar{\psi}^*) \\
&\quad \times \sum_{\substack{\ell_0 k_1 q_1 T/2\pi < m_1 \leq \ell_0 k_1 q_1 T/\pi \\ (m_1, \ell_0 k_1 q_1) = 1}} a(\ell m_1) \psi^*(-m_1) \\
&= -\frac{\tau(\bar{\chi})}{q} \sum_{\ell|q/\ell_0} \sum_{l \leq M} \sum_{k_1 \leq M/l} \frac{\bar{x}_{lk_1}}{lk_1} \frac{1}{\phi(\ell_0 k_1 q_1)} \mu(k_1 q_1) \bar{\psi}^*(k_1 q_1) \tau(\bar{\psi}^*) \psi^*(-1) \\
&\quad \times \sum_{sh=\ell l} \sum_{\substack{sv \leq M \\ (v, h \ell_0 k_1 q_1) = 1}} \bar{y}_{sv} \psi^*(v) \sum_{\substack{k_1 q T/2\pi \ell < uv \leq k_1 q T/\pi \ell \\ (u, \ell_0 k_1 q_1) = 1}} (\Lambda_F * \chi)(hu) \psi^*(u) \\
&= -\frac{\tau(\bar{\chi})}{q} \sum_{\ell|q/\ell_0} \sum_{l \leq M} \sum_{k_1 \leq M/l} \frac{\bar{x}_{lk_1}}{lk_1} \frac{1}{\phi(k_1 q/\ell)} \mu\left(k_1 \frac{q}{\ell \ell_0}\right) \bar{\psi}^*\left(k_1 \frac{q}{\ell \ell_0}\right) \tau(\bar{\psi}^*) \\
&\quad \times \sum_{sh=\ell l} \sum_{\substack{sv \leq M \\ (v, h k_1 q/\ell) = 1}} \bar{y}_{sv} \psi^*(v) \left\{ \frac{k_1 q T}{2\pi \ell v} \Phi(1; k_1 q/\ell, \chi \psi^*) L(1, \chi \psi^*) \chi(h) \tilde{f}_{-1} \right. \\
&\quad \left. + O\left(h^{\theta_F} \tau(h) j(h) (\log h)^3 j(qk) \log(qk) \frac{k_1 q T}{\ell v} \exp(-c\sqrt{\log T}) \right) \right\}, \tag{6.63}
\end{aligned}$$

where $\tilde{f}_{-1} = \text{Res}_{z=1} G(z, \psi^*)$. Let \mathcal{E}_{ψ^*} denote the contribution of O -terms in (6.63). Since $(sl, q) = 1, \ell \mid q, sh = \ell l$, we must have $\ell \mid h$. After replacing sh by ℓl and h by ℓh , we obtain a bound for \mathcal{E}_{ψ^*} as

$$\begin{aligned}
\mathcal{E}_{\psi^*} &\ll q^{\theta_F + \epsilon} M^{\theta_F + \epsilon} \sum_{\ell|q/\ell_0} \sum_{l \leq M} \sum_{k_1 \leq M/l} \frac{|\bar{x}_{lk_1}|}{lk_1} \frac{1}{\phi(\ell_0 k_1 q_1)} \sum_{sh=\ell l} \sum_{\substack{sv \leq M \\ (v, h \ell_0 k_1 q_1) = 1}} |\bar{y}_{sv}| \frac{k_1 q T}{\ell v} \exp(-c\sqrt{\log T}) \\
&\ll q^{\theta_F + \epsilon} M^{\theta_F + \epsilon} T \sum_{\ell|q/\ell_0} \sum_{sh \leq M} \sum_{k_1 \leq M/sh} \frac{|\bar{x}_{shk_1}|}{shk_1} \sum_{v \leq M/s} \frac{|\bar{y}_{sv}|}{v} \tau(h) j(h) \exp(-c\sqrt{\log T}) \\
&\ll q^{\theta_F + \epsilon} M^{\theta_F + \epsilon} T \left\| \frac{|\bar{x}_s \bar{y}_s|}{s} \right\|_1 \left\| \frac{(\tau_3 * |\bar{x}|)(n)}{n} \right\|_1 \left\| \frac{\bar{y}_v}{v} \right\|_1 \exp(-c\sqrt{\log T}). \tag{6.64}
\end{aligned}$$

Now we further require that x, y be supported on squarefree integers whose prime factors are $\equiv 1 \pmod{qg_0}$. It follows that if $x_{k_1} \neq 0$, then $\chi(k_1) = \psi^*(k_1) = 1$ and $(k_1, qg_0) = 1$, thus

$$\Phi(z; \ell_0 k_1 q_1, \chi \psi^*) = \prod_{p|\ell_0 k_1 q_1} (1 - \chi \psi^*(p) p^{-1}) = \prod_{\substack{p|k_1 \\ p \nmid qg_0}} (1 - p^{-1}) = \frac{\phi(k_1)}{k_1}.$$

Using this in (6.63), we simplify \mathcal{M}_{ψ^*} as

$$\begin{aligned}
\mathcal{M}_{\psi^*} &= -\frac{\tau(\bar{\chi})}{q} \sum_{\ell|q/\ell_0} \sum_{l \leq M} \sum_{k_1 \leq M/l} \frac{\bar{x}_{lk_1}}{lk_1} \frac{1}{\phi(k_1 q/\ell)} \mu\left(k_1 \frac{q}{\ell \ell_0}\right) \bar{\psi}^*\left(k_1 \frac{q}{\ell \ell_0}\right) \tau(\bar{\psi}^*) \sum_{sh=\ell l} \sum_{\substack{sv \leq M \\ (v, h k_1 q/\ell) = 1}} \bar{y}_{sv} \psi^*(v) \\
&\quad \times \frac{k_1 q T}{2\pi \ell v} \frac{\phi(k_1)}{k_1} L(1, \chi \psi^*) \chi(h) \tilde{f}_{-1} + \mathcal{E}_{\psi^*},
\end{aligned}$$

where \mathcal{E}_{ψ^*} is estimated by (6.64). Now we replace ℓl by sh , since $(k_1, q) = 1$, the main term in \mathcal{M}_{ψ^*} becomes

$$-\tau(\bar{\chi}) \tau(\bar{\psi}^*) \sum_{\ell|q/\ell_0} \sum_{sh \leq M} \sum_{k_1 \leq M\ell/sh} \frac{\bar{x}_{shk_1/\ell}}{shk_1/\ell} \frac{\mu\left(k_1 \frac{q}{\ell \ell_0}\right) \bar{\psi}^*\left(k_1 \frac{q}{\ell \ell_0}\right)}{\phi(q/\ell)} \sum_{\substack{sv \leq M \\ (v, h k_1 q/\ell) = 1}} \frac{\bar{y}_{sv}}{v} \psi^*(v) \frac{T}{2\pi \ell} L(1, \chi \psi^*) \chi(h) \tilde{f}_{-1}.$$

Since $(sk_1, q) = 1$, we must have $\ell \mid h$. After replacing h by ℓh , the above becomes

$$- \tau(\bar{\chi})\overline{\tau(\psi^*)} \sum_{\ell|q/\ell_0} \sum_{sh \leq M/\ell} \sum_{k_1 \leq M\ell/sh} \frac{\bar{x}_{shk_1} \mu(k_1 \frac{q}{\ell_0}) \overline{\psi^*}(k_1 \frac{q}{\ell_0})}{shk_1 \phi(q/\ell)} \sum_{\substack{sv \leq M \\ (v, hk_1 q) = 1}} \frac{\bar{y}_{sv} \psi^*(v)}{v} \frac{T}{2\pi\ell} L(1, \chi\psi^*) \chi(h\ell) \tilde{f}_{-1}.$$

Since $\ell \mid q$, the non zero contribution comes only from $\ell = 1$. By the assumption on the support of x_n , we have $\chi(h) = 1$ if $x_h \neq 0$. After writing $hk_1 = u$, we see that the main term in \mathcal{M}_{ψ^*} is simplified to

$$\begin{aligned} & - \tau(\bar{\chi})\overline{\tau(\psi^*)} \sum_{sh \leq M} \sum_{k_1 \leq M/sh} \frac{\bar{x}_{shk_1} \mu(k_1 q/\ell_0) \overline{\psi^*}(k_1 q/\ell_0)}{shk_1 \phi(q)} \sum_{\substack{sv \leq M \\ (v, hk_1 q) = 1}} \frac{\bar{y}_{sv} \psi^*(v)}{v} \frac{T}{2\pi} L(1, \chi\psi^*) \tilde{f}_{-1} \\ &= - \frac{\tau(\bar{\chi})\overline{\tau(\psi^*)} T}{2\pi\phi(q)} \sum_{s \leq M} \sum_{u \leq M/s} \frac{\bar{x}_{su}}{su} \sum_{k_1|u} \mu(k_1) \mu(q/\ell_0) \overline{\psi^*}(k_1) \overline{\psi^*}(q/\ell_0) \sum_{\substack{sv \leq M \\ (v, uq) = 1}} \frac{\bar{y}_{sv} \psi^*(v)}{v} L(1, \chi\psi^*) \tilde{f}_{-1} \\ &= - \frac{\tau(\bar{\chi})\overline{\tau(\psi^*)} T}{2\pi\phi(q)} \mu(q/\ell_0) \overline{\psi^*}(q/\ell_0) \sum_{s \leq M} \sum_{u \leq M/s} \frac{\bar{x}_{su}}{su} \sum_{k_1|u} \mu(k_1) \sum_{\substack{sv \leq M \\ (v, uq) = 1}} \frac{\bar{y}_{sv} \psi^*(v)}{v} L(1, \chi\psi^*) \tilde{f}_{-1} \\ &= - \frac{\tau(\bar{\chi})\overline{\tau(\psi^*)} T}{2\pi\phi(q)} \mu(q/\ell_0) \overline{\psi^*}(q/\ell_0) \sum_{s \leq M} \sum_{u \leq M/s} \frac{\bar{x}_s}{s} \sum_{\substack{sv \leq M \\ (v, q) = 1}} \frac{\bar{y}_{sv}}{v} L(1, \chi\psi^*) \tilde{f}_{-1}. \end{aligned}$$

Therefore,

$$\mathcal{M}_{\psi^*} = - \frac{\tau(\bar{\chi})\overline{\tau(\psi^*)} T}{2\pi} \frac{\mu(q/\ell_0) \overline{\psi^*}(q/\ell_0)}{\phi(q)} L(1, \chi\psi^*) \tilde{f}_{-1} \sum_{s \leq M} \frac{\bar{x}_s}{s} \sum_{\substack{sv \leq M \\ (v, q) = 1}} \frac{\bar{y}_{sv}}{v} + \mathcal{E}_{\psi^*}. \quad (6.65)$$

If there exist some positive constant $a = a(\chi, F)$ such that both F_ψ and $L(s, \chi\psi)$ have no zeros in the region $\Re(s) \geq 1 - a$ for all ψ , then we can replace the term $\exp(-c\sqrt{\log T})$ in the error terms $\mathcal{E}, \mathcal{E}'$ can be replaced by $T^{-\delta+\epsilon}$ for some small enough $\delta = \delta(F, a)$, since ψ has conductor $\leq M \leq \sqrt{T}$.

7. RESONATOR COEFFICIENTS

In this section, we define the resonator coefficients and give their properties.

Let χ be a Dirichlet character and denote $\psi^* \pmod{g^*}$ as the Dirichlet character $\neq \bar{\chi}$ such that $F_{\psi^*}(s)$ has a pole at $s = 1$. If no such character exists, set $g^* = 1$. Let $f(n)$ be multiplicative and supported on squarefree integers $n \leq M$. Let $d = \text{lcm}(q, g^*)$, $K = \sqrt{\phi(d) \log M \log_2 M}$, and B_M be as in (iii) in the definition of \mathcal{S}^* . Define

$$f(p) = \frac{K}{\log p}, \text{ if } K^2 \leq p \leq \exp((\log K)^2), p \neq B_M, p \equiv 1 \pmod{d}, \quad (7.1)$$

and 0 otherwise. Then, we have the following estimates of norms involving f .

Lemma 7.1. *Let $f(n)$ be defined by (7.1). Then for M sufficiently large, we have*

- (1) $|f(n)| \leq n^{1/2}$,
- (2) $\left\| \frac{f(n)}{n} \right\|_1 \ll \exp\left(\frac{1}{\phi(d)}(1+o(1))\frac{K}{\log K^2}\right)$,
- (3) $\|f(n)\|_1 \ll M \exp\left(\frac{1}{\phi(d)}(1+o(1))\frac{K}{\log K^2}\right)$,
- (4) $\left\| \frac{f(n)^2}{n} \right\|_1 \ll \exp\left(\frac{1}{2\phi(d)}(1+o(1))\frac{K^2}{(\log K^2)^2}\right)$,
- (5) $\|f(n)^2\|_1 \ll M \exp\left(\frac{1}{2\phi(d)}(1+o(1))\frac{K^2}{(\log K^2)^2}\right)$,
- (6) $\|j(n)(\tau_r * f)(n)f(n)/n\|_1 \ll \exp\left(\frac{1}{2\phi(d)}(1+o(1))\frac{K^2}{(\log K^2)^2}\right)$,

where $j(n)$ is defined in (6.36).

- Proof.* (1) Since $K^2 \leq p$, we see that $f(p) \leq \sqrt{p} \frac{K}{\sqrt{p} \log p} \leq \sqrt{p}$. Thus $f(n) \leq \sqrt{n}$ by multiplicativity.
(2) Using the multiplicativity of $f(n)$, we have an upper bound

$$\|f(n)/n\|_1 = \sum_{n=1}^M \frac{f(n)}{n} \leq \prod_p \left(1 + \frac{f(p)}{p}\right) \leq \exp\left(\sum_p \frac{f(p)}{p}\right).$$

From the prime number theorem in an arithmetic progression,

$$\begin{aligned} \sum_p \frac{f(p)}{p} &\leq \sum_{\substack{K^2 \leq p \leq \exp((\log K)^2) \\ p \equiv 1 \pmod{d}}} \frac{K}{p \log p} = (1 + o(1)) \frac{1}{\phi(d)} \int_{K^2}^{\exp((\log K)^2)} \frac{K}{x \log^2 x} dx \\ &= (1 + o(1)) \frac{1}{\phi(d)} \frac{K}{\log K^2}. \end{aligned}$$

- (3) Using $n \leq M$, Rankin's trick and part (2), we have

$$\|f(n)\|_1 = \sum_{n=1}^M f(n) \leq M \sum_{n=1}^M \frac{f(n)}{n} \leq M \left\| \frac{f(n)}{n} \right\|_1 \ll M \exp\left(\frac{(1 + o(1)) K}{\phi(d) \log K^2}\right),$$

as $M \rightarrow \infty$.

- (4) Similarly to (2), we have

$$\sum_{n=1}^M \frac{f(n)^2}{n} \leq \prod_p \left(1 + \frac{f(p)^2}{p}\right) \leq \exp\left(\sum_p \frac{f(p)^2}{p}\right).$$

Using the prime number theorem in an arithmetic progression again, we derive that

$$\begin{aligned} \sum_{K^2 \leq p \leq \exp((\log K)^2)} \frac{f(p)^2}{p} &\leq \sum_{\substack{K^2 \leq p \leq \exp((\log K)^2) \\ p \equiv 1 \pmod{d}}} \frac{K^2}{p \log^2 p} \\ &= \frac{1}{\phi(d)} (1 + o(1)) \int_{K^2}^{\exp((\log K)^2)} \frac{K^2}{x \log^3 x} dx \\ &= (1 + o(1)) \frac{1}{2\phi(d)} \frac{K^2}{(\log K^2)^2}, \end{aligned}$$

as $M \rightarrow \infty$.

- (5) Similarly to part (3), we have $\|f^2\|_1 \leq M \left\| \frac{f^2(n)}{n} \right\|_1$.
(6) Note that since n is squarefree and $j(n)$ and $\tau_k(n)$ are multiplicative,

$$\sum_{n \leq M} \frac{j(n)(\tau_k * f)(n)f(n)}{n} \leq \prod_p \left(1 + \frac{j(p)(\tau_k * f)(p)f(p)}{p}\right).$$

Since $j(p) = 1 + O(\sqrt{p})$, $f(p) = \frac{K}{\log p}$, and $(\tau_k * f)(p) = \frac{K}{\log p} + k$, we obtain

$$\begin{aligned} &\sum_{\substack{K^2 \leq p \leq \exp((\log K)^2) \\ p \equiv 1 \pmod{d}}} \frac{j(p)(\tau_k * f)(p)f(p)}{p} \\ &\leq \sum_{\substack{K^2 \leq p \leq \exp((\log K)^2) \\ p \equiv 1 \pmod{d}}} \left(\frac{K^2}{p \log^2 p} + O\left(\frac{K}{p \log p} + \frac{K^2}{p^{3/2-\epsilon}}\right)\right) \\ &= \frac{1}{2\phi(d)} (1 + o(1)) \frac{K^2}{(\log K^2)^2}, \quad M \rightarrow \infty. \end{aligned}$$

□

Lemma 7.2. *Let $f(p)$ be defined in (7.1). Set*

$$\mathcal{Q}_0 = \prod_p \left(1 + \frac{f(p)^2}{p}\right), \quad \mathcal{Q}_1 = \prod_p \left(1 + \frac{f(p)^2}{p} + \frac{f(p)}{p}\right). \quad (7.2)$$

Then, as $M \rightarrow \infty$,

- (1) $\sum_{nu \leq M} \frac{f(u)f(nu)}{nu} = \mathcal{Q}_1(1 + o(1))$,
- (2) $\sum_{n \leq M} \frac{f(n)^2}{n} \leq \mathcal{Q}_0$,
- (3) $\frac{\mathcal{Q}_1}{\mathcal{Q}_0} = \exp\left(\frac{1}{\phi(d)} \frac{K}{\log K^2} (1 + o(1))\right)$,
- (4) $\sum_{n \leq M} \frac{|\Lambda_F(n)|f(n)}{n(1 + f(n)^2 n^{-1})} \ll K \log_2 M$,
- (5) $\sum_{n \leq M} \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|f(p)}{p^m(1 + f(p)^2 p^{-1})} \ll K \log_2 M$,
- (6) $\sum_{nu \leq M} \frac{\log n f(u)f(nu)}{nu} \ll \mathcal{Q}_1 K \log_2 M$.

Proof. (1) Since f is multiplicative and supported on squarefree numbers,

$$\begin{aligned} \sum_{nu \leq M} \frac{f(n)f(nu)}{nu} &= \sum_{n \leq M} \frac{f(n)}{n} \sum_{\substack{u \leq M/n \\ (u,n)=1}} \frac{f(u)^2}{u} \\ &= \sum_{n \leq M} \frac{f(n)}{n} \left(\prod_{(p,n)=1} \left(1 + \frac{f(p)^2}{p}\right) - \sum_{\substack{u \geq M/n \\ (n,u)=1}} \frac{f(u)^2}{u} \right). \end{aligned}$$

By Rankin's trick, the contribution from $u > M/n$ is bounded by

$$\sum_{n \leq M} \frac{f(n)}{n} \left(\frac{n}{M}\right)^\alpha \sum_{\substack{u=1 \\ (u,n)=1}}^{\infty} \frac{f(u)^2 u^\alpha}{u} \leq \frac{1}{M^\alpha} \prod_p (1 + f(p)^2 p^{\alpha-1} + f(p) p^{\alpha-1}) \quad (7.3)$$

for any $\alpha > 0$. By Rankin's trick again, the main term becomes

$$\prod_p \left(1 + \frac{f(p)^2}{p} + \frac{f(p)}{p}\right) + O\left(\frac{1}{M^\alpha} \prod_p \left(1 + \frac{f(p)^2}{p} + \frac{f(p) p^\alpha}{p}\right)\right). \quad (7.4)$$

Combining (7.4) and (7.3), we deduce that

$$\sum_{nu \leq M} \frac{f(u)f(nu)}{nu} = \mathcal{Q}_1 + O\left(\frac{1}{M^\alpha} \prod_p (1 + f(p)^2 p^{\alpha-1} + f(p) p^{\alpha-1})\right).$$

Taking $\alpha = 1/(\log K)^3$, we see that the ratio of the error to the main term is bounded by

$$\begin{aligned} & \ll \exp \left(-\alpha \log M + \sum_{\substack{K^2 \leq p \leq \exp((\log K)^2) \\ p \equiv 1 \pmod{d}}} (p^\alpha - 1) \left(\frac{K^2}{p \log^2 p} + \frac{K}{p \log p} \right) \right) \\ & \ll \exp \left(-\alpha \log M + \alpha \frac{K^2}{\phi(d) \log K^2} - \alpha \frac{K^2}{\phi(d) (\log K)^2} \right) \\ & \ll \exp \left(-\alpha \frac{\log M}{\log_2 M} \right). \end{aligned} \tag{7.5}$$

Note that we used the fact that $\frac{K^2}{\phi(d) \log K^2} \leq \log M$ in the last step to ensure (7.5) is $o(1)$. Therefore,

$$\sum_{nu \leq M} \frac{f(u)f(nu)}{nu} = \mathcal{Q}_1(1 + o(1)).$$

(2) We have the inequality

$$\sum_{n \leq M} \frac{f(n)^2}{n} \leq \sum_n \frac{f(n)^2}{n} = \prod_p \left(1 + \frac{f(p)^2}{p} \right) = \mathcal{Q}_0.$$

(3) From the definitions of \mathcal{Q}_0 and \mathcal{Q}_1 defined in (7.2), it can be seen that

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_0} = \prod_p \left(1 + \frac{f(p)}{p(1 + f(p)^2 p^{-1})} \right).$$

From (iii), we have $B_M \gg \log_2 M \gg \log K$. Since $\log K \ll \log_2 T$, we have

$$\frac{K}{B_M \log B_M} \ll \frac{K}{\log K \log_2 K} = o\left(\frac{K}{\log K^2}\right).$$

It follows that

$$\begin{aligned} \sum_{K \leq p \leq \exp((\log K)^2)} \frac{f(p)}{p(1 + f(p)^2 p^{-1})} &= \sum_{\substack{K^2 \leq p \leq \exp((\log K)^2) \\ p \equiv 1 \pmod{d}}} \frac{K}{p \log p} (1 + o(1)) + o\left(\frac{K}{\log K^2}\right) \\ &= (1 + o(1)) \frac{1}{\phi(d) \log K^2}. \end{aligned}$$

(4) Since f is supported on squarefree integers, and $\Lambda_F(n)$ is supported on prime powers,

$$\begin{aligned} \sum_{n \leq M} \frac{|\Lambda_F(n)|f(n)}{n(1 + f(n)^2 p^{-1})} &= K \sum_{p \leq M} \frac{|\lambda_F(p)|}{p(1 + f(p)^2 p^{-1})} \\ &\ll K \left(\sum_{p \leq M} \frac{1}{p} \right)^{1/2} \left(\sum_{p \leq M} \frac{|\lambda_F(p)|^2}{p} \right)^{1/2} \\ &\ll K \log_2 M, \end{aligned}$$

where the last inequality follows from (3.6) and partial summation.

(5) Since we have $\Lambda_F(n) \ll n^{1/2}$, we see that

$$\sum_p \sum_{m=4}^{\infty} \frac{|\Lambda_F(p^m)|f(p)}{p^m} \ll K \sum_p \frac{1}{p^2} \frac{1}{1 - p^{-1/2}} \ll K.$$

This shows that

$$\begin{aligned}
\sum_{p \leq M} \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)| f(p)}{p^m} &\ll \sum_{p \leq M} \sum_{m \leq 4} \frac{|\Lambda_F(p^m)| f(p)}{p} + K \\
&\ll K \sum_{n \leq M^4} \frac{|\lambda_F(n)|}{n} + K \ll K \log_2 M \\
&\ll K \left(\sum_{n \leq M^4} \frac{|\lambda_F(n)|^2}{n} \right)^{1/2} \left(\sum_p \sum_{m \leq 4} \frac{1}{p^m} \right)^{1/2} \\
&\ll K \log_2 M.
\end{aligned}$$

(6) From the identity $\log n = \sum_{d|n} \Lambda(d)$,

$$\begin{aligned}
\sum_{nu \leq M} \frac{\log n f(u) f(nu)}{nu} &= \sum_{k \leq M} \frac{\Lambda(k) f(k)}{k} \sum_{\substack{nu \leq M/k \\ (nu, k)=1}} \frac{f(u) f(nu)}{nu} \\
&\leq \sum_{k \leq M} \frac{\Lambda(k) f(k)}{k} \sum_{\substack{n \leq M/k \\ (n, k)=1}} \frac{f(n)}{n} \prod_{(p, nk)=1} \left(1 + \frac{f(p)^2}{p} \right) \\
&\leq \mathcal{Q}_0 \sum_{k \leq M} \frac{\Lambda(k) f(k)}{k} \sum_{\substack{n \leq M \\ (n, k)=1}} \frac{f(n)}{n} \prod_{p|nk} \left(1 + \frac{f(p)^2}{p} \right)^{-1} \\
&\leq \mathcal{Q}_0 \sum_{n \leq M} \frac{f(n)}{n} \prod_{p|n} \left(1 + \frac{f(p)^2}{n} \right)^{-1} \sum_{k \leq M} \frac{\Lambda(k) f(k)}{k} \prod_{p|k} \left(1 + \frac{f(p)^2}{p} \right)^{-1} \\
&\leq \mathcal{Q}_1 K \log_2 M.
\end{aligned}$$

□

8. PROOF OF THEOREM 1.1-1.4

Now we are ready to prove Theorem 1.1-Theorem 1.4. Let $F \in \mathcal{S}^*$ and $\psi^* \pmod{g^*}$ be the Dirichlet character $\neq \bar{\chi}$ such that $F_{\psi^*}(s)$ has a pole at $s = 1$. Let $\chi \neq \psi^*$ be a Dirichlet character. Define $x_n = y_n = f(n)$, where $f(n), n = 1, \dots, M$ are the resonator coefficients defined in Section 7. Using Theorem 6.1 and Lemma 7.1, we see that the error term becomes $o(T)$ upon taking $M = \exp(c' \sqrt{\log T})$ for some $c' > 0$. If there exist positive constants a depending on F , and χ such that all three of $L(s, \chi\psi)$, F , and F_{ψ} have no zeros in the region $\Re(s) \geq 1 - a$ for all ψ , then we can take $M = T^{\delta'}$ for some small positive constant $\delta' = \delta'(F, \chi)$. Thus it remains to compute the main terms. Let g be the multiplicative function supported on squarefree integers with $g(p) = 1 + \frac{f(p)^2}{p}$. From the

multiplicativity of f and g , we have

$$\begin{aligned}
\sum_{nu \leq M} \frac{x_u y_{nu}}{nu} (\Lambda_F * 1)(n) &= \sum_{nu \leq M} \frac{f(u)f(nu)(\Lambda_F * 1)(n)}{nu} \\
&= \sum_{k \leq M} \frac{\Lambda_F(k)f(k)}{k} \sum_{\substack{nu \leq M/k \\ (n,u,k)=1}} \frac{f(u)f(nu)}{nu} \\
&\leq \sum_{k \leq M} \frac{\Lambda_F(k)f(k)}{k} \sum_{\substack{n \leq M/k \\ (n,k)=1}} \frac{f(n)}{n} \prod_{(p,nk)=1} \left(1 + \frac{f(p)^2}{p}\right) \\
&\leq \mathcal{Q}_0 \sum_{k \leq M} \frac{\Lambda_F(k)f(k)}{k} \sum_{\substack{n \leq M \\ (n,k)=1}} \frac{f(n)}{n} \prod_{p|nk} \left(1 + \frac{f(p)^2}{p}\right)^{-1} \\
&\leq \mathcal{Q}_0 \sum_{n \leq M} \frac{f(n)}{ng(n)} \sum_{k \leq M} \frac{\Lambda_F(k)f(k)}{kg(k)} \\
&\leq \mathcal{Q}_1 \sum_{p \leq M} \frac{\Lambda_F(p)f(p)}{pg(p)} \ll \mathcal{Q}_1 K \log_2 M, \tag{8.1}
\end{aligned}$$

where we used (2) and (4) from Lemma 7.2. Next, we give an upper bound for the terms involving $r_3(u)$. From the assumption that $F_{\bar{\chi}}$ has no pole at $z = 1$, we see that $f_{-1} = 0$, and thus

$$\begin{aligned}
r_3(u) = r_4(u) &= \sum_{hk=u} \mu(k) \left(f_0 - \sum_{p|k} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m)}{p^m} + \sum_{p|(h,k)} \Lambda_F(a) + \sum_{p|h, p \nmid k} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m)}{p^m} (p-1) \right) \\
&= \left(f_0 - \sum_{p|u} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m)}{p^m} \right) \sum_{k|u} \mu(k) + \sum_{hk=u} \mu(k) \sum_{p|h} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m)}{p^{m-1}} \\
&=: f_0 \delta(u) - \sum_{p|u} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m)}{p^{m-1}} \sum_{k|\frac{u}{p}} \mu(k),
\end{aligned}$$

which is non zero only when $u = 1$ or u is a prime. Thus,

$$\begin{aligned}
\sum_{u \leq M} \sum_{\substack{v \leq M \\ (v,u)=1}} \frac{x_u y_v r_3(u)}{uv} \sum_{\substack{s \leq M \\ (s,uv)=1}} \frac{x_s y_s}{s} &\leq \mathcal{Q}_0 \sum_{u \leq M} \sum_{\substack{v \leq M \\ (v,u)=1}} \frac{x_u y_v r_3(u)}{uv g(uv)} \\
&\leq \mathcal{Q}_0 \sum_{v \leq M} \frac{y_v}{v g(v)} \sum_{\substack{u \leq M \\ (u,v)=1}} \frac{x_u}{u g(u)} r_3(u) \\
&\ll \mathcal{Q}_1 \sum_{p \leq M} \frac{x_p}{p g(p)} \sum_{m=1}^{\infty} \frac{\Lambda(p) |\lambda_F(p^m)|}{p^{m-1}} \\
&\leq \mathcal{Q}_1 K \log_2 M. \tag{8.2}
\end{aligned}$$

where the last inequality follows from (5) of Lemma 7.2. Therefore, from Theorem 6.1, (8.1) and (8.2), we have

$$S_1 = \frac{T}{2\pi} d_F P_1(\log(\lambda Q^2)^{1/d_F} T)(1 + o(1)) \mathcal{Q}_1 + O(T \mathcal{Q}_1 K \log_2 M).$$

Next we consider S_0 (cf. (2.2)). Using Lemma 7.2, we have that

$$\begin{aligned} \sum_{nu \leq M} \frac{\Lambda_F(n)f(n)f(nu)}{nu} &= \sum_{n \leq M} \frac{\Lambda_F(n)f(n)}{n} \sum_{\substack{u \leq M/n \\ (u,n)=1}} \frac{f(u)^2}{u} \\ &\leq \prod_p \left(1 + \frac{f(p)^2}{p}\right) \sum_{n \leq M} \frac{\Lambda_F(n)f(n)}{ng(n)} \\ &\ll \mathcal{Q}_0 K \log_2 M. \end{aligned} \tag{8.3}$$

Combing (8.3) with Theorem 5, as $T \rightarrow \infty$, we have

$$\begin{aligned} S_0 &= \frac{T}{2\pi} d_F P_1(\log(\lambda Q^2)^{1/d_F} T) \mathcal{Q}_0(1 + o(1)) - \frac{T}{2\pi} \sum_{nu \leq M} \frac{\Lambda_F(n)x_u y_{nu}}{nu} + o(T) \\ &= \frac{d_F T}{2\pi} \log T \mathcal{Q}_0(1 + o(1)) + O(T \mathcal{Q}_0 K \log_2 M). \end{aligned}$$

Therefore,

$$\max_{\substack{F(\rho)=0 \\ T \leq \Im \rho \leq 2T}} |\zeta(\rho)| \geq \frac{|S_1|}{S_2} \gg \frac{\mathcal{Q}_1}{\mathcal{Q}_0} = \exp\left(\frac{1}{\phi(\text{lcm}(q, g^*))} (1 + o(1)) \frac{K}{\log K^2}\right),$$

by (3) in Lemme 7.2. If $M = \exp(c' \sqrt{\log T})$, then we can chose $K = \sqrt{\phi(\text{lcm}(q, g^*)) \log M \log_2 M}$, which gives

$$\frac{K}{\log K^2} \gg \sqrt{\frac{\log M}{\log_2 M}} = c'' \frac{(\log T)^{1/4}}{(\log_2 T)^{1/2}},$$

with $c'' = \sqrt{c'}/2$. In the second part, we can take $M = T^{\delta'}$ for some small $\delta' = \delta'(F, a) > 0$ and

$$\frac{K}{\log K^2} \gg \sqrt{\frac{\log M}{\log_2 M}} \gg \sqrt{\delta' \frac{\log T}{\log_2 T}}.$$

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