Transmutation of protons in a strong electromagnetic field

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The process of turning a proton into a neutron, positron and electron-neutrino in a strong plane-wave electromagnetic field, is studied. This process is forbidden in vacuum and is seen to feature an exponential suppression factor which is non-perturbative in the field amplitude. The suppression is alleviated when the proton experiences a field strength of about 13 times the Schwinger critical field in its rest frame. Around this threshold the lifetime of the proton, in its rest frame, is comparable to the usual neutron decay lifetime. As the field strength is increased, the proton lifetime becomes increasingly short. We investigate possible scenarios where this process may be observed in the laboratory using an ultra-intense laser and a high-energy proton beam with the conclusion, however, that it would be very challenging to observe this effect in the near future.

I. INTRODUCTION

In the Standard Model, the proton is regarded as a stable particle and experimentally it is shown that the half-life is at least 10^{29} years [1]. The proton is stable in the Standard Model due to baryon number conservation and to the fact that there is no lighter baryon to which the proton can decay. The presence of a strong electromagnetic field, however, allows absorption of four-momentum from the field, thus allowing the lighter proton to turn into heavier products. Electromagnetic field strengths on the order of the Schwinger critical field given by $E_c = m_e^2 c^3/e\hbar$, where m_e is the electron mass, c the speed of light, $e > 0$ the elementary charge and $\hbar$ Planck constant, sets the scale at which nonlinear quantum effects in electrodynamics become important [2, 3]. Among these, we mention the production of an electron-positron pair by a single photon in a strong electromagnetic field [4–29], or the non-perturbative Schwinger mechanism, where electric fields on the order of or larger than E_c will start to spontaneously produce electron-positron pairs from vacuum [30–41]. To be specific we will study the process where a proton turns into a neutron, a positron, and an electron-neutrino, i.e.,

$$p \to n + e^+ + \nu_e. \tag{1}$$

We will show that this "proton-transmutation" process "turns on" when the proton experiences an electromagnetic field of about 13 times the Schwinger field strength in its rest frame and that this process features a similar non-perturbative exponential suppression as the Schwinger mechanism. As we will elaborate quantitatively below, one can intuitively understand the similar field scale in proton transmutation and in electron-positron pair production as the energy gaps to be overcome are $\sim (m_N + m_e - m_P)c^2 \approx 1.8$ MeV and $\sim 2m_e c^2 \approx 1$ MeV, respectively, with m_N, m_e, and m_P being the neutron, the electron/positron, and the proton mass. The process has been considered before [42–46], however always with some significant simplifications such as assuming the particles to be spin-0 instead of spin-$\frac{1}{2}$, or using an interaction like the electromagnetic interaction, preserving parity. Ritus in [2], who mainly studied modification of processes already allowed in vacuum, also makes semi-quantitative estimates of this process by analytic continuation of those results. In this paper, we treat the process using the Fermi beta-decay point interaction characterized by the Fermi constant $G_F \approx 1.2 \times 10^{-5}$ GeV^{-2} [47–49] and the particles as spin-$\frac{1}{2}$ point particles in the presence of a plane-wave field, i.e., we use the Volkov states to describe charged particles [50, 51]. We may use the Fermi point interaction because the energy-momentum transfer in the process is on the order of the difference between the neutron and the proton mass, which is much smaller than the masses of the intermediate W boson. Below, we will also discuss when the approximation of point particle for the proton and the neutron is acceptable. Finally, for studies about how decay processes due to the weak interaction are influenced by a strong plane wave we refer to the reviews Refs. [2, 52].

The metric tensor $\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ is used throughout and the Feynman slash notation indicates the contraction of a four-vector with the Dirac gamma matrices γ^μ (the matrix γ^5 is defined as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$) [51]. Finally, units with $\hbar = c = 1$ are employed.

II. THEORY

In order to avoid restrictions on the plane-wave intensity, we describe both the proton and the positron by using Volkov states, which are the exact solution of the Dirac equation for a spin-$\frac{1}{2}$ point particle in a plane-wave field [50, 51]. The latter can be described by the four-vector potential $A^\mu(\varphi)$ in the Lorenz gauge $\partial_\mu A^\mu = 0$, where $\varphi = kx$, with $k^\mu = (\omega, \boldsymbol{k})$, is the wave four-vector ($k^2 = 0$ and $\omega = |\boldsymbol{k}|$) and x^μ the position four-vector. The positron Volkov state wave function is then (for notational simplicity the spin quantum numbers are not

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explicitly indicated)

$$\psi_p(x) = \frac{1}{\sqrt{2\varepsilon_p}}\left(1+\frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kp}\right) v_p e^{iS_p}, \tag{2}$$

where p^μ is the positron four-momentum quantum number, $\varepsilon_p = \sqrt{\boldsymbol{p}^2 + m_e^2}$, and $\mathcal{A}(\varphi) = eA(\varphi)$, whereas S_p is given by

$$S_p = px + \frac{1}{kp}\int^\varphi d\varphi' \left(p\mathcal{A}(\varphi') - \frac{1}{2}\mathcal{A}^2(\varphi')\right), \tag{3}$$

and v_p is the negative-energy constant bi-spinor [51]. The beta-decay 4-point Fermi interaction Hamiltonian is given by [47–49, 53]

$$H_{\text{int}} = \frac{G_F}{\sqrt{2}}\int d^3x \bar{\Psi}_{\text{proton}}(x)\gamma^\mu(g_v + g_a\gamma^5)\Psi_{\text{neutron}}(x) \times \bar{\Psi}_{\text{electron}}(x)\gamma_\mu(1-\gamma^5)\Psi_{\text{neutrino}}(x) + \text{H.C.}, \tag{4}$$

where each operator Ψ denotes the quantum field which contains the operators annihilating the particles and creating the anti-particles indicated as indexes and where the numerical parameters g_v and g_a will be set in the numerical computations to the values $g_v = 1$ and $g_a = -1.262$ [53]. For the proton-transmutation-process in Eq. (1), we need the Hermitian conjugate part of H_{int}. Thus, we also need the wave function of the proton in the external plane-wave field, which, assuming the proton asymptotic four-momentum being $P^\mu = (\varepsilon_P, \boldsymbol{P}) = (\sqrt{m_P^2 + \boldsymbol{P}^2}, \boldsymbol{P})$, is given by

$$\psi_P(x) = \frac{1}{\sqrt{2\varepsilon_P}}\left(1+\frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kP}\right) u_P e^{iS_P}, \tag{5}$$

where

$$S_P = -Px - \frac{1}{kP}\int^\varphi d\varphi' \left(P\mathcal{A}(\varphi') - \frac{1}{2}\mathcal{A}^2(\varphi')\right), \tag{6}$$

and where u_P is the positive-energy constant bi-spinor [51]. The neutron and the neutrino are neutral and therefore we describe them via the free particle wave functions given by

$$\psi_Q(x) = \frac{1}{\sqrt{2\varepsilon_Q}} u_Q e^{-iQx}, \tag{7}$$

$$\psi_q(x) = \frac{1}{\sqrt{2\varepsilon_q}} u_q e^{-iqx}, \tag{8}$$

where $Q^\mu = (\varepsilon_Q, \boldsymbol{Q}) = (\sqrt{m_N^2 + \boldsymbol{Q}^2}, \boldsymbol{Q})$ and $q^\mu = (\varepsilon_q, \boldsymbol{q}) = (\sqrt{m_n^2 + \boldsymbol{q}^2}, \boldsymbol{q})$ denote the four-momenta of the neutron and the neutrino, respectively (note that we are implicitly assuming the neutrino to be a Dirac-like particle even though later the neutrino mass will be neglected).

Using the Volkov state for the proton implies that we are treating it as a point particle and this is acceptable as long as the laser field in the rest frame of the proton has a wavelength significantly longer than the size of the proton, and that the photon energy is much smaller than any potential excitation energy of the proton. Of these two requirements, the latter is the more restrictive one, which corresponds to an energy of 294 MeV for the excitation to the delta-baryon. Assuming a typical value of 1 eV for the laser photon energy, and e.g. a 7 TeV proton, this translates into roughly a 7 keV photon energy in the rest frame of the proton, significantly smaller than the mentioned model restriction.

Under the above assumptions, the transition matrix element is then

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\frac{1}{\sqrt{16\varepsilon_P\varepsilon_Q\varepsilon_p\varepsilon_q}}\int d^4x e^{i(Q+q+p-P)x} Y(\varphi) \times e^{i\int^\varphi d\varphi'\left[\frac{p\mathcal{A}(\varphi')}{kp} - \frac{P\mathcal{A}(\varphi')}{kP} + \frac{1}{2}\mathcal{A}^2(\varphi')\left(\frac{1}{kP}-\frac{1}{kp}\right)\right]}, \tag{9}$$

where we have defined

$$Y(\varphi) = \bar{u}_Q\gamma^\mu(g_v + g_a\gamma^5)\left(1+\frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kP}\right)u_P \times \bar{u}_q\gamma_\mu(1-\gamma^5)\left(1+\frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kp}\right)v_p. \tag{10}$$

Now, anticipating that in a plane wave three light-cone momenta are conserved, it is convenient to write this function in terms of its Fourier transform and so we define

$$\mathcal{Y}(s) = \frac{1}{2\pi}\int d\varphi Y(\varphi) e^{is\varphi} \times e^{i\int^\varphi d\varphi'\left[\frac{p\mathcal{A}(\varphi')}{kp} - \frac{P\mathcal{A}(\varphi')}{kP} + \frac{1}{2}\mathcal{A}^2(\varphi')\left(\frac{1}{kP}-\frac{1}{kp}\right)\right]}, \tag{11}$$

and therefore we can write

$$Y(\varphi)e^{i\int^\varphi d\varphi'\left[\frac{p\mathcal{A}(\varphi')}{kp} - \frac{P\mathcal{A}(\varphi')}{kP} + \frac{1}{2}\mathcal{A}^2(\varphi')\left(\frac{1}{kP}-\frac{1}{kp}\right)\right]} = \int_{-\infty}^{\infty}\mathcal{Y}(s)e^{-is\varphi}ds. \tag{12}$$

By inserting this expression into Eq. (9) and by performing the integration over d^4x we obtain

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\frac{(2\pi)^4}{\sqrt{16\varepsilon_P\varepsilon_Q\varepsilon_p\varepsilon_q}} \times \int ds\delta^4\left(Q+q+p-P-sk\right)\mathcal{Y}(s). \tag{13}$$

At this point the (spin-resolved) transition probability is given by $dP = |\mathcal{M}|^2 d^3\boldsymbol{Q}d^3\boldsymbol{q}d^3\boldsymbol{p}/(2\pi)^9$. After appropriately taking the square of the delta-function, we obtain the probability as

$$dP = \frac{G_F^2}{2}\frac{1}{(2\pi)^4}\frac{1}{16kP}\int ds\frac{d^3\boldsymbol{Q}}{\varepsilon_Q}\frac{d^3\boldsymbol{q}}{\varepsilon_q}\frac{d^3\boldsymbol{p}}{\varepsilon_p} \times \delta^4\left(Q+q+p-P-sk\right)\left|\mathcal{Y}(s)\right|^2, \tag{14}$$

where one may note that each factor is now Lorentz invariant. Now, we turn to the evaluation of the quantity $|\mathcal{Y}(s)|^2$. From Eq. (14), we have that

$$|\mathcal{Y}(s)|^2 = \frac{1}{(2\pi)^2} \iint e^{i\Phi(\varphi,\varphi')} Y(\varphi) Y^\dagger(\varphi') d\varphi d\varphi', \quad (15)$$

where we defined

$$\Phi(\varphi,\varphi') = s(\varphi - \varphi') + \int_{\varphi'}^{\varphi} dx \left[\frac{p\mathcal{A}(x)}{kp} - \frac{P\mathcal{A}(x)}{kP} + \frac{1}{2}\mathcal{A}^2(x) \left(\frac{1}{kP} - \frac{1}{kp} \right) \right]. \quad (16)$$

We therefore have that the probability summed over final spins and averaged over the proton spin is given by

$$dP = \frac{G_F^2}{2} \frac{1}{(2\pi)^6} \frac{1}{32kP} \int ds d\varphi d\varphi' \times e^{i\Phi(\varphi,\varphi')} \sum_{\text{spins}} Y(\varphi) Y^\dagger(\varphi') \times \delta^4 (Q + q + p - P - sk) \frac{d^3\boldsymbol{Q}}{\varepsilon_Q} \frac{d^3\boldsymbol{q}}{\varepsilon_q} \frac{d^3\boldsymbol{p}}{\varepsilon_p}. \quad (17)$$

By applying the usual identities for the products of bispinors when summing over spins, we may write [see Appendix (A) for additional details]

$$\sum_{\text{spins}} Y(\varphi) Y^\dagger(\varphi') = T^{\mu\nu}(\varphi,\varphi') q^\alpha W_{\mu\nu\alpha}(\varphi,\varphi') \quad (18)$$

where the tensors $T^{\mu\nu}(\varphi,\varphi')$ and $W_{\mu\nu\alpha}(\varphi,\varphi')$ are given in terms of traces of gamma matrices, and we need only to keep terms with an even number of gamma matrices, leading to

$$T^{\mu\nu}(\varphi,\varphi') = T_1^{\mu\nu}(\varphi,\varphi') + Q_\alpha T_2^{\mu\nu\alpha}(\varphi,\varphi'), \quad (19)$$

$$T_1^{\mu\nu}(\varphi,\varphi') = m_N m_P \left(g_v^2 - g_a^2\right) \mathrm{Tr} \left[\gamma^\mu \left(1 + \frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kP}\right) \left(1 - \frac{\not{k}\not{\mathcal{A}}(\varphi')}{2kP}\right) \gamma^\nu \right], \quad (20)$$

$$T_2^{\mu\nu\alpha}(\varphi,\varphi') = \mathrm{Tr} \left[\gamma^\alpha \gamma^\mu \left(1 + \frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kP}\right) \not{P} \left(1 - \frac{\not{k}\not{\mathcal{A}}(\varphi')}{2kP}\right) \gamma^\nu \left(g_v^2 + g_a^2 + 2g_v g_a \gamma^5\right) \right], \quad (21)$$

$$W_{\mu\nu\alpha}(\varphi,\varphi') = 2\mathrm{Tr} \left[\gamma_\alpha \gamma_\mu \left(1 + \frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kp}\right) \not{p} \left(1 - \frac{\not{k}\not{\mathcal{A}}(\varphi')}{2kp}\right) \gamma_\nu (1 - \gamma^5) \right]. \quad (22)$$

We may now employ the identities from the Appendix (B) [2] to obtain (for brevity we omit the dependence on φ and φ')

$$\int T^{\mu\nu} q^\alpha W_{\mu\nu\alpha} \delta^4 (Q + q + p - P - sk) \frac{d^3\boldsymbol{q} d^3\boldsymbol{Q}}{\varepsilon_q \varepsilon_Q} = \frac{J}{2l^2}(l^2 - m_N^2) T_1^{\mu\nu} W_{\mu\nu\alpha} l^\alpha + \frac{J}{6l^4} \left[l^2 \left(l^2 + m_N^2\right) - 2m_N^4 \right] l_\alpha l^\beta T_2^{\mu\nu\alpha} W_{\mu\nu\beta} + \frac{J}{12l^2} \left(l^2 - m_N^2\right)^2 T_2^{\mu\nu\alpha} W_{\mu\nu\alpha}, \quad (23)$$

where (setting for simplicity the neutrino mass to zero)

$$J = \theta(s - s_{\min}) \frac{2\pi\sqrt{\left(l^2 - m_N^2\right)^2}}{l^2}, \quad (24)$$

$$l = sk + P - p, \quad (25)$$

$$s_{\min} = \frac{m_N^2 - (P - p)^2}{2k\,(P - p)}, \quad (26)$$

with θ denoting the Heaviside function. The expression of $s = s_{\min}$ can be obtained from the kinematical condition $l^2 > m_N^2$, which follows from the energy-momentum conservation of the delta-function from Eq. (17). Then, by conveniently setting $s = s_{\min} + \rho$, we obtain

$$l^2 = 2k(P - p)\rho + m_N^2. \quad (27)$$

The integrals over φ and φ' in Eq. (17) can be conveniently turned into a double integral over the central phase $\varphi_+ = (\varphi + \varphi')/2$ and over the relative phase $\varphi_- = \varphi - \varphi'$. From now on we realistically assume that the plane wave is sufficiently intense that the classical nonlinearity parameter $\xi = eE/m_e\omega_0 \gg 1$ [3], where E and ω_0 are the electric-field amplitude and the central angular frequency of the laser field, respectively. It is known that, generally speaking, if $\xi \gg 1$ the largest contribution to the integral in φ_- comes from the region $|\varphi_-| \lesssim 1/\xi \ll 1$ [2, 3]. Thus, one can apply the so-called locally constant field approximation (LCFA), where one expands the plane-wave field around $\varphi_- = 0$. The LCFA is discussed in detail in the Appendix (C). We will need to consider both the pre-exponential factor functions and the phase $\Phi(\varphi,\varphi')$ in Eq. (17) and we start from the latter. Within the LCFA, it is appropriate to expand the phase $\Phi(\varphi,\varphi')$ up to cubic terms in φ_- [see Appendix (C)]:

$$\Phi=\tilde{\Phi}+\varphi_-\frac{kP}{2k\,(P-p)\,(kp)}\left(\boldsymbol{p}_\perp-\frac{kp}{kP}\boldsymbol{P}_\perp-\frac{k\,(P-p)}{kP}\left\langle\boldsymbol{\mathcal{A}}_\perp\right\rangle\right)^2, \tag{28}$$

where

$$\left\langle\boldsymbol{\mathcal{A}}_\perp\right\rangle=\frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx, \tag{29}$$

$$\tilde{\Phi}=\rho\varphi_-+\varphi_-\frac{(m_N^2-m_e^2-m_P^2)(kp)(kP)+m_e^2(kP)^2+m_P^2(kp)^2}{2k\,(P-p)\,(kp)(kP)}+\frac{k(P-p)}{2(kP)(kp)}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2\frac{\varphi_-^3}{12}. \tag{30}$$

Here, we have exploited the additional gauge freedom to set the time-component and the space-component parallel to $\boldsymbol{k}$ of the laser four-vector potential, to zero, such that the latter has only non-vanishing transverse components with respect to $\boldsymbol{k}$.

At this point, the quantity $T^{\mu\nu}(\varphi,\varphi')q^\alpha W_{\mu\nu\alpha}(\varphi,\varphi')$ can be evaluated within the LCFA. The computation of the three terms in Eq. (23) and the resulting integrals over $\boldsymbol{p}_\perp$ and φ_- are straightforward but lengthy and we refer to the Appendix (D) for details. Here, we mention that these integrals can be taken analytically. Concerning the integration over $d^2\boldsymbol{p}_\perp$, the phase depends on $\boldsymbol{p}_\perp$ quadratically and the pre-exponential factor contains only powers of $\boldsymbol{p}_\perp$. Thus, this integration can be carried out by using identities for Gaussian integrals in the Appendix (E). Now, after integrating over $\boldsymbol{p}_\perp$, only the reduced phase $\tilde{\Phi}$ remains in the exponent [see Eq. (30)], such that we are left with integrals of the form $\int_{-\infty}^{\infty}\varphi_-^n e^{i(a\varphi_-+b\varphi_-^3)}d\varphi_-$, which can be expressed in terms of modified Bessel functions $K_\alpha(\eta)$ of the second kind [54]. In particular, one can easily show that

$$\int_{-\infty}^{\infty}\varphi_-^n e^{i(a\varphi_-+b\varphi_-^3)}d\varphi_-=c^{n+1}f_n(\eta), \tag{31}$$

where $c=\sqrt{a/(3b)}$, $\eta=2ac/3$ and where

$$f_n(\eta)=\int_{-\infty}^{\infty}z^n e^{i\frac{3}{2}\eta(z+\frac{1}{3}z^3)}dz. \tag{32}$$

In particular we will need

$$if_1(\eta)=-\frac{2}{\sqrt{3}}K_{2/3}(\eta), \tag{33}$$

$$if_{-1}(\eta)=\frac{2}{\sqrt{3}}\int_\eta^\infty K_{1/3}(z)dz, \tag{34}$$

$$f_{-2}(\eta)=\sqrt{3}\eta\left(\int_\eta^\infty K_{1/3}(z)dz-K_{2/3}(\eta)\right). \tag{35}$$

Concerning the convergence of the integrals $f_{-1}(\eta)$ and $f_{-2}(\eta)$, we recall that in taking the Gaussian integrals over $\boldsymbol{p}_\perp$ one implicitly assumes that the coefficient of $\boldsymbol{p}_\perp^2$ in the phase has a infinitesimally small positive imaginary part, which then implies that the variable z in the denominators of the integrands in $f_{-1}(\eta)$ and $f_{-2}(\eta)$ has to be intended to be shifted as $z+i0$ [see the Appendix (E)]. The above results show that the process will be exponentially suppressed when η is large as $K_\alpha(\eta)\sim e^{-\eta}\sqrt{\pi/2\eta}$ for large values of η, which will correspond to relatively low plane-wave field strengths [54]. If we consider the process from the rest frame of the proton around the threshold of $\eta\sim 1$, the particles will be produced as only mildly relativistic, and therefore in the laboratory frame the produced positron will have an energy of the order of $\gamma_P m_e$, where γ_P is the Lorentz factor of the proton. This means that the natural variable to be introduced to describe the positron is

$$\zeta=\frac{m_P}{m_e}\frac{kp}{kP}, \tag{36}$$

which will then be of the order of unity near the threshold. This also means that kp may be neglected compared to kP in our expression for the reaction rate. Within this scheme of approximations, we then obtain from Eq. (30) that

$$\eta=\frac{2}{3}\frac{y^3}{\zeta}\frac{1}{\chi_P}, \tag{37}$$

where

$$y=\sqrt{\frac{l^2-m_e^2-m_P^2}{m_e m_P}\zeta+1+\zeta^2}, \tag{38}$$

$$\chi_P=\frac{e\sqrt{(F^{\mu\nu}P_\nu)^2}}{m_P m_e^2}=\frac{(kP)}{m_P m_e^2}\left|\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right|. \tag{39}$$

Here, χ_P is the ratio of the field strength experienced by the proton in its rest frame and the Schwinger field strength $E_{cr}=m_e^2/e\approx 1.3\times 10^{16}$ V/cm [2, 3]. In order to gain insight into the process, we notice that the quantity η can be written in a physically transparent form by observing that from Eq. (27) the quantity l^2 is at least m_N^2 and that in the region mainly contributing to the transmutation probability we have $\zeta \sim 1$. Thus, by setting $l^2 = m_N^2$ and $\zeta = 1$, we obtain

$$\begin{aligned}\eta &= \frac{2}{3}\left[\frac{m_N^2-(m_P-m_e)^2}{m_e m_P}\right]^{3/2}\frac{1}{\chi_P}\\ &\approx 1.89\left(\frac{m_N+m_e-m_P}{m_e}\right)^{3/2}\frac{1}{\chi_P}\approx\frac{12.5}{\chi_P}.\end{aligned}\tag{40}$$

This equation features the energy gap characterizing the transmutation process as compared to the electron mass, which characterizes the process of electron-positron pair production, and implies that the threshold where the process turns on, i.e., $\eta \sim 1$, is at $\chi_P \sim 13$.

It is sufficient to keep only the leading-order terms and to neglect terms suppressed by the factors m_e/m_P and/or m_e/m_N. Also, we assume the plane wave to be linearly polarized. Under these conditions and by introducing the proton proper time τ via the relation $d\varphi_+ = (kP/m_P)d\tau$, we obtain [see Appendix (D) for additional details]

$$\begin{aligned}\frac{dP}{d\tau d\zeta} &= \frac{G_F^2}{2}\frac{1}{(2\pi)^4}m_e^5 m_P^2\int dz\frac{z^2}{8l^4\zeta^4}\left[1+\zeta^2+\left(\frac{m_N^2-m_e^2-m_P^2}{m_e m_P}\right)\zeta\right]^3\\ &\times\Bigg\{2m_N m_P\left(g_v^2-g_a^2\right)\left[y^2 if_1-\left(1+\zeta^2\right)if_{-1}+\zeta\frac{\chi_P}{y}f_{-2}\right]\\ &+\frac{1}{3}\left(1+\frac{2m_N^2}{l^2}\right)\left(g_v^2+g_a^2\right)\left(2m_P^2+3l^2-m_N^2\right)\\ &\times\left\{-y^2 if_1-\zeta\frac{\chi_P}{y}f_{-2}+if_{-1}\left[1+\left(1+\frac{2m_P^2+l^2-m_N^2}{2m_P^2+3l^2-m_N^2}\frac{l^2-m_N^2}{m_P^2}\right)\zeta^2\right]\right\}\\ &+\frac{l^2-m_N^2}{3}\left[5\left(g_v^2+g_a^2\right)-6g_v g_a\right]\left[if_{-1}\left(1+\zeta^2\right)-y^2 if_1-\zeta\frac{\chi_P}{y}f_{-2}\right]\Bigg\}.\end{aligned}\tag{41}$$

We observe here that we changed variable from ρ to z by introducing

$$z = \zeta\frac{2\rho kP}{(m_N^2-m_e^2-m_P^2)\zeta+m_e m_P(1+\zeta^2)}\tag{42}$$

and therefore eliminating $2\rho k(P-p) \sim 2\rho kP$ in l^2 from Eq. (27) using this expression, we can express l^2 in terms of the independent variable x and the integration variable z.

III. NUMERICAL RESULTS AND DISCUSSION

Below, we report and discuss the results of numerical evaluation of the proton transmutation formula found above in Eq. (41). Concerning the laser pulse, we have chosen the Gaussian pulse form given by

$$A^\mu(\varphi) = a^\mu \sin(\varphi)\, e^{-\frac{\varphi^2}{2\sigma^2}},\tag{43}$$

where $a^\mu = (0, a, 0, 0)$, with $a = E/\omega_0$ and with σ describing the pulse duration. Also, we assume a head-on collision between an ultrarelativistic proton and the laser pulse such that $\chi_P \approx 2\gamma_P E/E_{cr}$.

In Fig. (1) we show a plot of the proton lifetime $\tau_P = (\int d\zeta\, dP/d\tau d\zeta)^{-1}$ in the rest frame of the proton. We point out that this result depends solely on χ_P and therefore that there is no dependence on the laser pulse shape in this figure. The quantity τ_P in Fig. (1) has to be interpreted as the proton lifetime in a constant crossed field of amplitude E. The total proton transmutation probability P in a plane-wave pulse with a given field shape $E(\varphi)$ is obtained by going back to the variable φ_+ and by taking the double integral $P = \int d\zeta d\varphi_+\, dP/d\varphi_+ d\zeta$, with $\chi_P \to \chi_P(\varphi_+) = 2\gamma_P|E(\varphi_+)|/E_{cr}$. As expected,

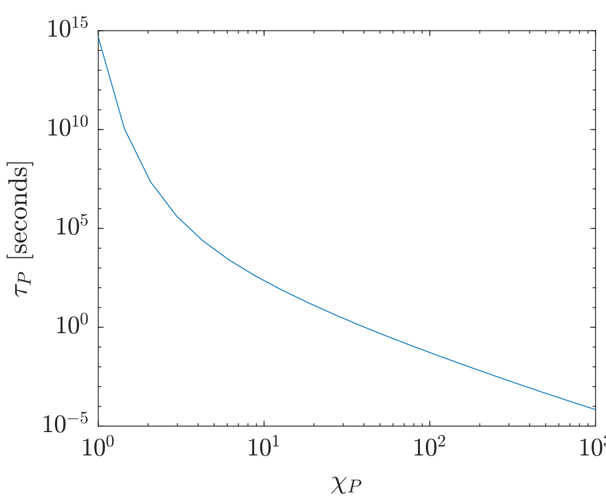

Figure 1. The lifetime τ_P of the proton in its rest frame as a function of the parameter χ_P.

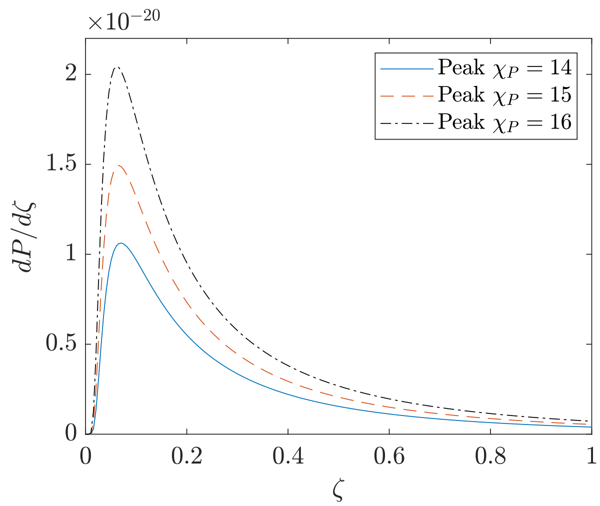

Figure 2. The probability spectrum of the emitted positron as a function of $\zeta = m_P/m_e \times kp/kP$ for a 10-cycle Gaussian pulse [$\sigma = 10$ in Eq. (43)] and for different peak values of χ_P. See the text for the remaining numerical parameters.

the figure shows a rapid decrease of the proton lifetime for increasing values of χ_P. In particular, the lifetime increases rapidly below a certain threshold and above this threshold features a power law dependence scaling roughly as χ_P^{-3}.

In Fig. (2) we show examples of the distribution of the positrons for different peak values of the field strength. For the examples in Fig. (2) we have used $\omega_0 = 1.0$ eV, $\sigma = 10$ and $\varepsilon_P = 7$ TeV as at the LHC. In the cases shown, the spectra show a peak for $\zeta \sim 0.1$, however for larger values of χ_P we have seen that the peak moves towards lower values of ζ. In Table (I) we show the expected results in terms of probability per proton and per collision corresponding to different experimental setups. It is seen that even with future laser facilities brought together with a proton synchrotron such as the LHC, the reaction probability remains small. The physical reason for the probability being so low is that around the threshold where the process is no longer exponentially suppressed, i.e., $\chi_P \sim 10$, the lifetime of the proton becomes comparable to that of a free neutron. More precisely, the proton lifetime at $\chi_P = 10$ is about 235 seconds, which is exceedingly large as compared to the duration of a typical laser pulse on the order of femto- or pico-seconds. Furthermore, in order to to reach high field strengths the laser pulse has been chosen to propagate in the opposite direction as the proton. This implies that the duration of the laser pulse in the rest frame of the proton becomes Lorentz contracted by the Lorentz factor of the proton. In conclusion, for the proton transmutation to be sizable, one would need large field strengths for extended periods of time, i.e., extremely large laser energies. We point out that in Table (I), the probability per collision is the probability per proton times the number N_P of protons in the bunch, i.e., it is assumed that the transverse area of the proton bunch is significantly smaller than the laser pulse focal area.

It is physically interesting to observe the following: The Schwinger field strength contains the mass of the electron and is typically associated with the field strength where production of electron-positron pairs becomes sizable. Therefore, one may rightfully ask why this process, involving protons and neutrons as well, also turns on when the proton experiences a field close to the Schwinger field. This is somewhat a coincidence due to the mass of the neutron and proton differing by about a MeV, i.e., by an amount indeed comparable with twice the electron mass, which corresponds to the energy gap to be overcome in electron-positron pair production. Indeed, Eq. (40) shows that the functions characterizing the transmutation process become sizable at a value of χ_P, which is related to the ratio of the energy gap to be overcome, $(m_N + m_e - m_p)$, to the electron mass.

Suppose that instead of the neutron we had considered producing the neutral delta baryon Δ^0 with a mass of $m_{\Delta^0} = 1232$ MeV. Although the Δ^0 is only about 30% heavier than the neutron with mass $m_N = 939.6$ MeV, the implication for the threshold of the corresponding process would be much more significant. Indeed, if we apply the findings above to this case, we have that $\left[m_{\Delta^0}^2 - (m_P - m_e)^2\right]/m_e m_P = 1332$, such that requiring that $\eta \sim 1$ from Eq. (38) would lead to $\chi_P \sim 3.2 \times 10^4$. This also implies that applying the results obtained in this paper above $\chi_P \approx 10^3$, is not meaningful, as the Δ^0 may be seen as an excitation of the neutron, and therefore the assumption of point like particles in the wave functions is no longer allowed. In addition the emitted positron would also experience a quantum nonlinearity parameter of the order of 10^3, and radiative corrections for the positron interacting with the laser field are expected to become significant [2, 55–66].

Finally, we have also considered the possibility of colliding a proton beam with an XFEL pulse, whose photon energy is typically much larger than in the case of an optical beam. In the case of an XFEL it would be unrealistic to use the above formulas obtained within the LCFA and the opposite regime $\xi \ll 1$ seems more appropriate. Thus, we considered a kinematic situation in which the laser photon energy is high enough that the process is allowed by the absorption of a single photon. In order to obtain an order of magnitude of the resulting transmutations probability, we expanded the probability in Eq. (17) including the leading (quadratic) term in the field and computed the first term in the pre-exponent, corresponding to the second line of Eq. (23). We have found that in the case of the collision of 10 keV photons with 7 TeV protons, the cross-section for the process is on the order of 10^{-7} picobarn. Even assuming optimistic conditions where the field strength is such that $\xi = 1$ and that the pulse contains about 3×10^5 cycles (corresponding to about 120 fs) yields a probability on the order of 10^{-15} for conversion, or roughly 10^{-4} per collision for a bunch containing 10^{11} protons (all passing through the

	HL-LHC Standard laser	HL-LHC exawatt laser	HL-LHC exawatt	FCC exawatt laser	FCC exawatt laser
ξ (peak)	100	2350	7400	2350	7400
I [W/cm^2]	1.85×10^{22}	1.0×10^{25}	1.0×10^{26}	1.0×10^{25}	1.0×10^{26}
ε_P [TeV]	7	7	7	50	50
χ_P (peak)	2.74	32.2	203	230	724
N_P [10^{11}]	2.2	2.2	2.2	1.0	1.0
Prob. per proton	2.2×10^{-26}	9.9×10^{-19}	3.8×10^{-17}	5.8×10^{-17}	1.4×10^{-15}
Prob. per collision	4.8×10^{-15}	2.2×10^{-7}	8.3×10^{-6}	5.8×10^{-6}	1.4×10^{-4}

Table I. Total proton transmutation probabilities for different experimental setups. It is assumed that all protons pass through the center of the laser pulse, i.e., that the proton transverse area is smaller than the laser pulse focal area. A Gaussian laser pulse shape such as that in Eq. (43) is chosen, with $\sigma = 10$.

laser spot).

IV. CONCLUSION

In conclusion, we have presented the formula for the decay rate of a proton into a neutron, a positron, and an electron neutrino in the presence of a strong plane-wave field (proton transmutation). The full Fermi interaction has been employed, meaning that the particles are treated as spin-$\frac{1}{2}$ and that parity-violating effects have been taken into account. We have seen that the process turns on when the proton experiences a field value about 13 times the Schwinger field strength, due to the masses of the neutron and proton differing by about a MeV, i.e., by an amount comparable with the electron mass. We have argued that the composite nature of the neutron and proton can be neglected as long as the external field does not vary too rapidly and for values of the quantum nonlinearity parameter χ_P associated with the proton up to 10^3. We have shown that at $\chi_P = 10^3$ the lifetime of the proton is roughly only 50 microseconds. However, this is still far longer than any realistic strong laser pulse, keeping in mind that this pulse duration should be achieved in the rest frame of the proton. This explains physically why it is challenging to observe the proton transmutation experimentally. Analogous conclusions have been drawn in the case of a collision of a proton beam with an XFEL. However, we have shown that in the case of a strong optical laser field the proton transmutation probability features a non-perturbative dependence on the elementary charge as well as on the laser field strength, which is typical of tunneling-like processes. If, in the future, it becomes possible to drastically increase the number of protons in particle accelerators or the density of laser photons, this mechanism could in principle be an attractive source of anti-neutrino bursts of short duration comparable to the laser pulse duration. One should keep in mind that if the density of laser photons is increased, the limits discussed above on χ_P should not be exceeded. But increasing the laser intensity would allow to decrease γ_P such that the pulse duration in the proton rest frame would not suffer as large a Lorentz contraction.

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Appendix A: Computation of $Y(\varphi)Y^\dagger(\varphi')$

By using the standard properties of the Dirac gamma functions [51], one can write the quantity $Y(\varphi)Y^\dagger(\varphi')$ as

$$
\begin{aligned}
&Y(\varphi)Y^{\dagger}(\varphi')\\
&=\bar{u}_n\gamma^{\mu}(g_v+g_a\gamma^5)\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)u_p\bar{u}_v\gamma_{\mu}(1-\gamma^5)\left(1+\frac{\not{k}\not{A}(\varphi)}{2kp}\right)v\\
&\times\bar{u}_p\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}(g_v+g_a\gamma^5)u_n\bar{v}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kp}\right)\gamma_{\nu}(1-\gamma^5)u_v\\
&=\bar{u}_n\gamma^{\mu}(g_v+g_a\gamma^5)\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)u_p\bar{u}_p\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}(g_v+g_a\gamma^5)u_n\\
&\times\bar{u}_v\gamma_{\mu}(1-\gamma^5)\left(1+\frac{\not{k}\not{A}(\varphi)}{2kp}\right)v\bar{v}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kp}\right)\gamma_{\nu}(1-\gamma^5)u_v\\
&=\mathrm{Tr}\left[\bar{u}_n\gamma^{\mu}(g_v+g_a\gamma^5)\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)u_p\bar{u}_p\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}(g_v+g_a\gamma^5)u_n\right]\\
&\times\mathrm{Tr}\left[\bar{u}_v\gamma_{\mu}(1-\gamma^5)\left(1+\frac{\not{k}\not{A}(\varphi)}{2kp}\right)v\bar{v}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kp}\right)\gamma_{\nu}(1-\gamma^5)u_v\right],
\end{aligned}
\tag{A1}
$$

where u_n and u_p (u_v and v) are the constant bi-spinors corresponding to the neutron and the proton (neutrino and positron), respectively. By summing over the spin of all involve initial and final particles, we obtain

$$
\begin{aligned}
&\sum_{\text{spins}}Y(\varphi)Y^{\dagger}(\varphi')\\
&=\mathrm{Tr}\left[(\not{Q}+m_N)\gamma^{\mu}(g_v+g_a\gamma^5)\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)(\not{P}+m_P)\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}(g_v+g_a\gamma^5)\right]\\
&\times\mathrm{Tr}\left[(\not{q}+m_{\nu})\gamma_{\mu}(1-\gamma^5)\left(1+\frac{\not{k}\not{A}(\varphi)}{2kp}\right)(\not{p}-m_e)\left(1-\frac{\not{k}\not{A}(\varphi')}{2kp}\right)\gamma_{\nu}(1-\gamma^5)\right]\\
&=\mathrm{Tr}\left[(\not{Q}+m_N)\gamma^{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)(g_v+g_a\gamma^5)(\not{P}+m_P)(g_v-g_a\gamma^5)\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}\right]\\
&\times\mathrm{Tr}\left[(\not{q}+m_{\nu})\gamma_{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kp}\right)(1-\gamma^5)(\not{p}-m_e)(1+\gamma^5)\left(1-\frac{\not{k}\not{A}(\varphi')}{2kp}\right)\gamma_{\nu}\right]\\
&=\mathrm{Tr}\left[(\not{Q}+m_N)\gamma^{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)\left\{m_P\left(g_v^2-g_a^2\right)+\not{P}(g_v^2+g_a^2-2g_vg_a\gamma^5)\right\}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}\right]\\
&\times\mathrm{Tr}\left[(\not{q}+m_{\nu})\gamma_{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kp}\right)2(1-\gamma^5)\not{p}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kp}\right)\gamma_{\nu}\right]\\
&=\mathrm{Tr}\left[(\not{Q}+m_N)\gamma^{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)\left\{m_P\left(g_v^2-g_a^2\right)+\not{P}(g_v^2+g_a^2-2g_vg_a\gamma^5)\right\}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}\right]\\
&\times\mathrm{Tr}\left[\not{q}\gamma_{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kp}\right)2(1-\gamma^5)\not{p}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kp}\right)\gamma_{\nu}\right],
\end{aligned}
\tag{A2}
$$

and we set

$$
\begin{aligned}
T^{\mu\nu}(\varphi,\varphi')&=\mathrm{Tr}\left[(\not{Q}+m_N)\gamma^{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)\left[m_P\left(g_v^2-g_a^2\right)+\not{P}(g_v^2+g_a^2-2g_vg_a\gamma^5)\right]\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}\right]\\
&=m_Nm_P\left(g_v^2-g_a^2\right)\mathrm{Tr}\left[\gamma^{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}\right]\\
&+\mathrm{Tr}\left[\not{Q}\gamma^{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kP}\right)\not{P}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kP}\right)\gamma^{\nu}\left(g_v^2+g_a^2+2g_vg_a\gamma^5\right)\right],
\end{aligned}
\tag{A3}
$$

$$
W_{\mu\nu}(\varphi,\varphi')=2\mathrm{Tr}\left[\not{q}\gamma_{\mu}\left(1+\frac{\not{k}\not{A}(\varphi)}{2kp}\right)\not{p}\left(1-\frac{\not{k}\not{A}(\varphi')}{2kp}\right)\gamma_{\nu}(1-\gamma^5)\right].
\tag{A4}
$$

Appendix B: Integrals over the momenta of the neutral particles

Let $l_1 = (\varepsilon_1, \boldsymbol{l}_1) = (\sqrt{m_1^2 + \boldsymbol{l}_1^2}, \boldsymbol{l}_1)$ and $l_2 = (\varepsilon_2, \boldsymbol{l}_2) = (\sqrt{m_2^2 + \boldsymbol{l}_2^2}, \boldsymbol{l}_1)$ be two four-momenta. Let us consider the three integrals

$$J = \int \delta^4(l - l_1 - l_2) \frac{d^3 l_1 d^3 l_2}{\varepsilon_1 \varepsilon_2}, \tag{B1}$$

$$J_\alpha = \int l_{1,\alpha} \delta^4(l - l_1 - l_2) \frac{d^3 l_1 d^3 l_2}{\varepsilon_1 \varepsilon_2}, \tag{B2}$$

$$J_{\alpha\beta} = \int l_{1,\alpha} l_{2,\beta} \delta^4(l - l_1 - l_2) \frac{d^3 l_1 d^3 l_2}{\varepsilon_1 \varepsilon_2}, \tag{B3}$$

where $l = (l^0, \boldsymbol{l})$ is a four-vector and $j = 1, 2$. Due to the four-dimensional delta function, in order these integrals not to vanish, it is required that $l^0 > m_1 + m_2$ and $l^2 > (m_1 + m_2)^2$, which are equivalent to the conditions $l^0 > 0$ and $l^2 > (m_1 + m_2)^2$. Since under a proper Lorentz transformation the integrals J, J_α, and $J_{\alpha\beta}$ are a scalar, a four-vector, and a tensor, respectively, they can be computed by first working in the frame where $\boldsymbol{l} = \boldsymbol{0}$, and it can be shown that (see also Ref. [2])

$$J = \theta(l^0)\theta\left(l^2 - (m_1 + m_2)^2\right) \frac{2\pi\sqrt{\left(l^2 - m_1^2 - m_2^2\right)^2 - 4m_1^2 m_2^2}}{l^2}, \tag{B4}$$

$$J_\alpha = \frac{J}{2l^2} l_\alpha [l^2 + (m_1^2 - m_2^2)], \tag{B5}$$

$$J_{\alpha\beta} = \frac{J}{6l^4} l_\alpha l_\beta \left[l^2 \left(l^2 + m_1^2 + m_2^2\right) - 2\left(m_1^2 - m_2^2\right)^2\right] + \frac{J}{12l^2} g_{\alpha\beta} \left[\left(l^2 - m_1^2 - m_2^2\right)^2 - 4m_1^2 m_2^2\right]. \tag{B6}$$

Appendix C: Computation of $s_{\min}$ and the validity of the LCFA

In this Appendix we will find a useful expression for $s_{\min}$ and conveniently manipulate the phase in the probability. We will here need some identities. We define n^μ as the quantity which in the laboratory frame is given by

$$n^\mu = (1, \boldsymbol{n}) \tag{C1}$$

where $\boldsymbol{n} = \frac{\boldsymbol{k}}{\omega_0}$ is the unit vector along the propagation direction of the plane wave. Then let v be some arbitrary 4-vector and define

$$v_\parallel = \boldsymbol{n} \cdot \boldsymbol{v}, \tag{C2}$$

$$v_+ = v_0 + v_\parallel, \tag{C3}$$

$$v_- = v_0 - v_\parallel = nv. \tag{C4}$$

Then the identity holds

$$2v_+ v_- - \boldsymbol{v}_\perp^2 = \left(v_0 + v_\parallel\right)\left(v_0 - v_\parallel\right) - \boldsymbol{v}_\perp^2 = v_0^2 - v_\parallel^2 - \boldsymbol{v}_\perp^2 = v^2, \tag{C5}$$

such that for the positron four-momentum p, we have that

$$p_+ = \frac{m_e^2 + \boldsymbol{p}_\perp^2}{2np} \tag{C6}$$

and

$$(P - p)^2 = 2(P_+ - p_+)(P_- - p_-) - (\boldsymbol{P}_\perp - \boldsymbol{p}_\perp)^2. \tag{C7}$$

In this way, we can rewrite $s_{\min}$ as

$$\begin{aligned} s_{\min} &= \frac{m_N^2 - (P-p)^2}{2k\,(P-p)} \\ &= \frac{m_N^2}{2k\,(P-p)} + \frac{kP}{2k\,(P-p)\,(kp)} \left(\boldsymbol{p}_\perp^2 - 2\frac{kp}{kP} \boldsymbol{P}_\perp \cdot \boldsymbol{p}_\perp\right) + \frac{m_e^2}{2kp} - \frac{m_P^2 + \boldsymbol{P}_\perp^2}{2kP} + \frac{\boldsymbol{P}_\perp^2}{2k\,(P-p)}. \end{aligned} \tag{C8}$$

Now, we will consider the quantity $\Phi/(\varphi-\varphi')-\rho$:

$$\begin{aligned}
&\frac{\Phi}{\varphi-\varphi'}-\rho\\
&=s_{\min}+\frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi}dx\left[\frac{p\mathcal{A}(x)}{kp}-\frac{P\mathcal{A}(x)}{kP}+\frac{1}{2}\mathcal{A}^2(x)\left(\frac{1}{kP}-\frac{1}{kp}\right)\right]\\
&=\frac{kP}{2k\,(P-p)\,(kp)}\left[\left(\boldsymbol{P}_\perp-\frac{kp}{kP}\boldsymbol{P}_\perp-\frac{1}{\varphi-\varphi'}\frac{k\,(P-p)}{kP}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\right)^2-\left(\frac{kp}{kP}\boldsymbol{P}_\perp+\frac{1}{\varphi-\varphi'}\frac{k\,(P-p)}{kP}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\right)^2\right]\\
&+\frac{m_N^2}{2k\,(P-p)}+\frac{m_e^2}{2kp}-\frac{m_P^2+\boldsymbol{P}_\perp^2}{2kP}+\frac{\boldsymbol{P}_\perp^2}{2k\,(P-p)}+\frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi}dx\left[\frac{\boldsymbol{P}_\perp\boldsymbol{\mathcal{A}}_\perp(x)}{kP}-\frac{1}{2}\boldsymbol{\mathcal{A}}_\perp^2(x)\left(\frac{1}{kP}-\frac{1}{kp}\right)\right]. \qquad \text{(C9)}
\end{aligned}$$

Now, we analyze the terms

$$\begin{aligned}
&\frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi}\frac{\boldsymbol{P}_\perp\boldsymbol{\mathcal{A}}_\perp(x)}{kP}-\frac{1}{2}\boldsymbol{\mathcal{A}}_\perp^2(x)\left(\frac{1}{kP}-\frac{1}{kp}\right)dx-\frac{kP}{2k\,(P-p)\,(kp)}\left(\frac{kp}{kP}\boldsymbol{P}_\perp+\frac{1}{\varphi-\varphi'}\frac{k\,(P-p)}{kP}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\right)^2\\
&=\frac{k(P-p)}{2(kP)(kp)}\left[\frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp^2(x)dx-\left(\frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\right)^2\right]-\frac{1}{2k\,(P-p)}\frac{kp}{kP}\boldsymbol{P}_\perp^2. \qquad \text{(C10)}
\end{aligned}$$

Inserting this expression into the previous equation we obtain

$$\Phi/(\varphi-\varphi')-\rho=\frac{\left(m_N^2-m_e^2-m_P^2\right)(kp)(kP)+m_e^2(kP)^2+m_P^2(kp)^2}{2k\,(P-p)\,(kp)(kP)} \qquad \text{(C11)}$$

$$+\frac{kP}{2k\,(P-p)\,(kp)}\left(\boldsymbol{P}_\perp-\frac{kp}{kP}\boldsymbol{P}_\perp-\frac{1}{\varphi-\varphi'}\frac{k\,(P-p)}{kP}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\right)^2 \qquad \text{(C12)}$$

$$+\left(\frac{k(P-p)}{2(kP)(kp)}\right)\left[\frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp^2(x)dx-\left(\frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\right)^2\right]. \qquad \text{(C13)}$$

Up to this point the calculation has been exact. As we will integrate over $\boldsymbol{p}_\perp$, the line above containing $\boldsymbol{p}_\perp$ will introduce $\boldsymbol{\mathcal{A}}_\perp$ in the front factor as we will perform the substitution

$$\boldsymbol{x}_\perp=\boldsymbol{p}_\perp-\frac{kp}{kP}\boldsymbol{P}_\perp-\frac{1}{\varphi-\varphi'}\frac{k\,(P-p)}{kP}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx, \qquad \text{(C14)}$$

and then integrate with respect to $\boldsymbol{x}_\perp$ instead of $\boldsymbol{p}_\perp$. For this reason, we only need to apply the LCFA to the other terms in the phase. As we have mentioned in the main text, we introduce

$$\varphi_-=\varphi-\varphi', \qquad \text{(C15)}$$

$$\varphi_+=\frac{\varphi+\varphi'}{2}, \qquad \text{(C16)}$$

and expanding the field around φ_+ for small values of $|\varphi_-|$:

$$\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\approx\varphi_-\boldsymbol{\mathcal{A}}_\perp(\varphi_+)+\frac{\varphi_-^3}{24}\frac{d^2\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+^2}, \qquad \text{(C17)}$$

$$\left(\frac{1}{\varphi_-}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\right)^2\approx\boldsymbol{\mathcal{A}}_\perp^2(\varphi_+)+\frac{\varphi_-^2}{12}\boldsymbol{\mathcal{A}}_\perp(\varphi_+)\frac{d^2\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+^2}, \qquad \text{(C18)}$$

$$\frac{1}{\varphi_-}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp^2(x)dx\approx\boldsymbol{\mathcal{A}}_\perp^2(\varphi_+)+\left[\boldsymbol{\mathcal{A}}_\perp(\varphi_+)\frac{d^2\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+^2}+\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2\right]\frac{\varphi_-^2}{12}. \qquad \text{(C19)}$$

Therefore we obtain

$$\begin{aligned}
&\Phi\approx\rho\varphi_-+\varphi_-\frac{\left(m_N^2-m_e^2-m_P^2\right)(kp)(kP)+m_e^2(kP)^2+m_P^2(kp)^2}{2k\,(P-p)\,(kp)(kP)}\\
&+\varphi_-\frac{kP}{2k\,(P-p)\,(kp)}\left(\boldsymbol{P}_\perp-\frac{kp}{kP}\boldsymbol{P}_\perp-\frac{1}{\varphi-\varphi'}\frac{k\,(P-p)}{kP}\int_{\varphi'}^{\varphi}\boldsymbol{\mathcal{A}}_\perp(x)dx\right)^2+\frac{k(P-p)}{2(kP)(kp)}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2\frac{\varphi_-^3}{12}. \qquad \text{(C20)}
\end{aligned}$$

1. Validity of the LCFA

The condition of validity of the LCFA is that the integral over φ_- should be formed over a region where $|\varphi_-|$ is much smaller than unity [2]. By using that

$$\rho = \frac{z}{2\zeta kP}\left[\left(m_N^2 - m_e^2 - m_P^2\right)\zeta + m_e m_P(1+\zeta^2)\right], \tag{C21}$$

we have that

$$\Phi = \frac{\varphi_-}{2kP}\left[(1+z)\left(m_N^2 - m_e^2 - m_P^2 + m_e m_P(\frac{1}{\zeta}+\zeta)\right) + \frac{1}{\zeta}\frac{m_P}{m_e}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2\frac{\varphi_-^2}{12}\right]. \tag{C22}$$

Either the first or the second term dominates and recall that the integral is formed over the region where $\Phi \lesssim 1$. If we assume that the first term is dominant, the condition then becomes ($\zeta \sim 1$)

$$|\varphi_-| = \frac{2kP}{m_N^2 - m_e^2 - m_P^2 + 2m_e m_P} \ll 1. \tag{C23}$$

The parameter describing the laser field strength is given by the so-called classical nonlinearity parameter [2]

$$\xi = \frac{e|a|}{m_e}. \tag{C24}$$

By assuming that $|d\boldsymbol{A}_\perp/d\varphi| \sim |\boldsymbol{A}_\perp(\varphi)|$, we obtain

$$\frac{\chi_P}{\xi} \approx \frac{kP}{m_e m_P}, \tag{C25}$$

such that the above condition becomes

$$\frac{\chi_P}{\xi}\frac{1}{\frac{m_N^2 - m_e^2 - m_P^2}{2m_e m_P} + 1} \ll 1, \tag{C26}$$

or, approximately, $3.5\xi \gg \chi_P$. We consider now the case that the second term in the phase Φ dominates, i.e., that

$$1 \approx \frac{1}{kP}\frac{m_P}{m_e}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2\frac{\varphi_-^3}{24}. \tag{C27}$$

By using Eq. (C25), this may be rewritten as

$$\frac{24\chi_P}{\xi^3} \approx \varphi_-^3, \tag{C28}$$

and therefore $\varphi_-^3 \ll 1$ translates into $\xi^3 \gg 24\chi_P$.

Appendix D: Analytical integrations

In this Appendix we go through the analytical integrations over φ_- and $\boldsymbol{p}_\perp$ of the terms from Eq. (23). For the sake of convenience, we will split up the three lines into three subsections. First, however, it is useful to rewrite the expressions from Eq. (20) to Eq. (22). First, we see that we may rewrite

$$\left(1 + \frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kP}\right)\not{P}\left(1 - \frac{\not{k}\not{\mathcal{A}}(\varphi')}{2kP}\right) = \frac{1}{2kP}\left(m_P^2\not{k} + \not{\Pi}\not{k}\not{\Pi}'\right). \tag{D1}$$

Here we have set

$$\Pi = P - \mathcal{A}(\varphi) + \left(\frac{P\mathcal{A}}{kP} - \frac{\mathcal{A}^2}{2kP}\right)k, \tag{D2}$$

and exploited that $\left(\not{P}-\not{\mathcal{A}}(\varphi)\right)\not{k}=\not{\Pi}\not{k}$. Π' denotes the replacement $\varphi\to\varphi'$. We also need

$$\left(1+\frac{\not{k}\not{\mathcal{A}}(\varphi)}{2kP}\right)\left(1-\frac{\not{k}\not{\mathcal{A}}(\varphi')}{2kP}\right)=1+\frac{\not{k}\left(\not{\mathcal{A}}(\varphi)-\not{\mathcal{A}}(\varphi')\right)}{2kP}. \tag{D3}$$

Similarly, we define

$$\pi=p-\mathcal{A}(\varphi)+\left(\frac{p\mathcal{A}}{kp}-\frac{\mathcal{A}^2}{2kp}\right)k, \tag{D4}$$

and the same identities with the replacement $P\to p$ hold if we also replace $\Pi\to\pi$. Using these identities and carrying out the trace we have that

$$\begin{aligned}T_1^{\mu\nu}&=m_Nm_P\left(g_v^2-g_a^2\right)\mathrm{Tr}\left[\gamma^\mu\left(1+\frac{\not{k}\left(\not{\mathcal{A}}(\varphi)-\not{\mathcal{A}}(\varphi')\right)}{2kP}\right)\gamma^\nu\right]\\&=4m_Nm_P\left(g_v^2-g_a^2\right)\left[g^{\mu\nu}+\frac{k^\mu\left(\mathcal{A}^\nu(\varphi)-\mathcal{A}^\nu(\varphi')\right)-k^\nu\left(\mathcal{A}^\mu(\varphi)-\mathcal{A}^\mu(\varphi')\right)}{2kP}\right].\end{aligned} \tag{D5}$$

Now, we write

$$T_2^{\mu\nu\alpha}=\left(g_v^2+g_a^2\right)T_a^{\mu\nu\alpha}+2g_vg_aT_b^{\mu\nu\alpha} \tag{D6}$$

and then using the above identities we have that

$$T_a^{\mu\nu\alpha}=\frac{1}{2kP}\left[m_P^2\mathrm{Tr}\left(\gamma^\alpha\gamma^\mu\not{k}\gamma^\nu\right)+\mathrm{Tr}\left(\gamma^\alpha\gamma^\mu\not{\Pi}\not{k}\not{\Pi}'\gamma^\nu\right)\right], \tag{D7}$$

$$T_b^{\mu\nu\alpha}=\frac{1}{2kP}\left[m_P^2\mathrm{Tr}\left(\gamma^\alpha\gamma^\mu\not{k}\gamma^\nu\gamma^5\right)+\mathrm{Tr}\left(\gamma^\alpha\gamma^\mu\not{\Pi}\not{k}\not{\Pi}'\gamma^\nu\gamma^5\right)\right]. \tag{D8}$$

At this point we can carry out the traces to obtain

$$\begin{aligned}\frac{kP}{2}T_a^{\mu\nu\alpha}&=(kP)\left[\left(\Pi'^\mu+\Pi^\mu\right)g^{\alpha\nu}+\left(\Pi^\nu+\Pi'^\nu\right)g^{\alpha\mu}-g^{\mu\nu}\left(\Pi^\alpha+\Pi'^\alpha\right)\right]\\&+k^\alpha\left[\Pi'^\nu\Pi^\mu-\Pi'^\mu\Pi^\nu+\left(\Pi\Pi'-m_P^2\right)g^{\mu\nu}\right]\\&+\Pi^\alpha\left(k^\nu\Pi'^\mu-k^\mu\Pi'^\nu\right)+\Pi'^\alpha\left(k^\mu\Pi^\nu-k^\nu\Pi^\mu\right)\\&-\left(k^\nu g^{\alpha\mu}+k^\mu g^{\alpha\nu}\right)\left(\Pi\Pi'-m_P^2\right).\end{aligned} \tag{D9}$$

For the traces involving $\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3$ we will need

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right)=-4i\epsilon^{\mu\nu\rho\sigma}, \tag{D10}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor with convention $\epsilon^{0123}=+1$. Then, by using the identity

$$\gamma^\mu\gamma^\nu\gamma^\rho=\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho}+\gamma^\rho g^{\mu\nu}+i\epsilon^{\sigma\mu\nu\rho}\gamma^5\gamma_\sigma, \tag{D11}$$

we obtain that

$$\begin{aligned}\frac{kP}{2}T_b^{\mu\nu\alpha}&=m_P^2\frac{1}{4}\mathrm{Tr}\left(\gamma^\alpha\gamma^\mu\not{k}\gamma^\nu\gamma^5\right)+\frac{1}{4}\mathrm{Tr}\left(\gamma^\alpha\gamma^\mu\not{\Pi}\not{k}\not{\Pi}'\gamma^\nu\gamma^5\right)\\&=-m_P^2i\epsilon^{\alpha\mu\beta\nu}k_\beta-\frac{1}{4}\mathrm{Tr}\left[\left(\gamma^\nu g^{\mu\alpha}+\gamma^\mu g^{\nu\alpha}-\gamma^\alpha g^{\mu\nu}\right)\not{\Pi}\not{k}\not{\Pi}'\gamma^5\right]-\frac{1}{4}\mathrm{Tr}\left(i\epsilon^{\kappa\nu\alpha\mu}\gamma_\kappa\not{\Pi}\not{k}\not{\Pi}'\right)\\&=i\left(g^{\mu\alpha}\epsilon^{\nu\beta\rho\sigma}+g^{\nu\alpha}\epsilon^{\mu\beta\rho\sigma}-g^{\mu\nu}\epsilon^{\alpha\beta\rho\sigma}\right)\Pi_\beta k_\rho\Pi'_\sigma+i\epsilon^{\mu\nu\alpha\beta}\left[m_P^2k_\beta+(kP)\left(\Pi_\beta+\Pi'_\beta\right)-\left(\Pi\Pi'\right)k_\beta\right].\end{aligned} \tag{D12}$$

At this point it is useful to write down the symmetric and anti-symmetric parts of $T_2^{\mu\nu\alpha}$ and $W_{\mu\nu\alpha}$, with respect to the indexes μ and ν as a symmetric tensor contracted with an anti-symmetric tensor vanishes. We obtain that

$$\begin{aligned}T^{\mu\nu\alpha}_{2,S} &= g^{\mu\nu}\left\{2\left(g_v^2+g_a^2\right)\left[\frac{k^\alpha}{kP}\left(\Pi\Pi'-m_P^2\right)-\left(\Pi^\alpha+\Pi'^\alpha\right)\right]-\frac{4g_vg_a}{kP}i\epsilon^{\alpha\beta\rho\sigma}\Pi_\beta k_\rho\Pi'_\sigma\right\}\\&+g^{\alpha\nu}\left\{2\left(g_v^2+g_a^2\right)\left[\left(\Pi'^\mu+\Pi^\mu\right)-k^\mu\frac{\Pi\Pi'-m_P^2}{kP}\right]+\frac{4g_vg_a}{kP}i\epsilon^{\mu\beta\rho\sigma}\Pi_\beta k_\rho\Pi'_\sigma\right\}\\&+g^{\alpha\mu}\left\{2\left(g_v^2+g_a^2\right)\left[\left(\Pi^\nu+\Pi'^\nu\right)-k^\nu\frac{\Pi\Pi'-m_P^2}{kP}\right]+\frac{4g_vg_a}{kP}i\epsilon^{\nu\beta\rho\sigma}\Pi_\beta k_\rho\Pi'_\sigma\right\}, \qquad \text{(D13)}\\T^{\mu\nu\alpha}_{2,A} &= \frac{2\left(g_v^2+g_a^2\right)}{kP}\left[k^\alpha\left(\Pi'^\nu\Pi^\mu-\Pi'^\mu\Pi^\nu\right)+\Pi^\alpha\left(k^\nu\Pi'^\mu-k^\mu\Pi'^\nu\right)+\Pi'^\alpha\left(k^\mu\Pi^\nu-k^\nu\Pi^\mu\right)\right]\\&+\frac{4g_vg_a}{kP}i\epsilon^{\mu\nu\alpha\beta}\left[(kP)\left(\Pi_\beta+\Pi'_\beta\right)+\left(m_P^2-\Pi\Pi'\right)k_\beta\right]. \qquad \text{(D14)}\end{aligned}$$

We may obtain the same for $W_{\mu\nu\alpha}$ by setting $g_v=1$ and $g_a=-1$ and replace $P\to p$ and $Q\to q$:

$$\begin{aligned}W^S_{\mu\nu\tau} &= g_{\mu\nu}\left\{4\left[\frac{k_\tau}{kp}\left(\pi\pi'-m_e^2\right)-\left(\pi_\tau+\pi'_\tau\right)\right]+\frac{4}{kp}i\epsilon_{\tau\beta\rho\sigma}\pi^\beta k^\rho\pi'^\sigma\right\}\\&+g_{\tau\nu}\left\{4\left[\left(\pi'_\mu+\pi_\mu\right)-k_\mu\frac{\pi\pi'-m_e^2}{kp}\right]-\frac{4}{kp}i\epsilon_{\mu\beta\rho\sigma}\pi^\beta k^\rho\pi'^\sigma\right\}\\&+g_{\tau\mu}\left\{4\left[\left(\pi_\nu+\pi'_\nu\right)-k_\nu\frac{\pi\pi'-m_e^2}{kp}\right]-\frac{4}{kp}i\epsilon_{\nu\beta\rho\sigma}\pi^\beta k^\rho\pi'^\sigma\right\}, \qquad \text{(D15)}\\W^A_{\mu\nu\tau} &= \frac{4}{kp}\left[k_\tau\left(\pi'_\nu\pi_\mu-\pi'_\mu\pi_\nu\right)+\pi_\tau\left(k_\nu\pi'_\mu-k_\mu\pi'_\nu\right)+\pi'_\tau\left(k_\mu\pi_\nu-k_\nu\pi_\mu\right)\right]\\&-\frac{4}{kp}i\epsilon_{\mu\nu\tau\beta}\left[(kp)\left(\pi^\beta+\pi'^\beta\right)+\left(m_e^2-\pi\pi'\right)k^\beta\right]. \qquad \text{(D16)}\end{aligned}$$

1. Line 1

Here, we wish to find

$$\int T_1^{\mu\nu}(\varphi,\varphi')W_{\mu\nu\alpha}(\varphi,\varphi')l^\alpha e^{i\Phi}d\varphi_-d^2\boldsymbol{p}_\perp = \int T_1^{\mu\nu}(\varphi,\varphi')W_{\mu\nu\alpha}(\varphi,\varphi')l^\alpha e^{i\left(\tilde{\Phi}+g\varphi_-x_\perp^2\right)}d\varphi_-d^2\boldsymbol{x}_\perp \qquad \text{(D17)}$$

where we introduced

$$\boldsymbol{x}_\perp = \boldsymbol{p}_\perp-\frac{kp}{kP}\boldsymbol{P}_\perp-\frac{k\left(P-p\right)}{kP}\left\langle\boldsymbol{\mathcal{A}}_\perp\right\rangle, \qquad \text{(D18)}$$

$$g = \frac{kP}{2k\left(P-p\right)(kp)}. \qquad \text{(D19)}$$

By using the expression from the previous appendix we find that

$$\frac{T_1^{\mu\nu}W_{\mu\nu\alpha}l^\alpha}{4m_Nm_P\left(g_v^2-g_a^2\right)} = 8\left[\frac{lk}{kp}\left(\pi\pi'-m_e^2\right)-l\left(\pi+\pi'\right)-\frac{lk}{kP}\frac{\left(\boldsymbol{\mathcal{A}}_\perp-\boldsymbol{\mathcal{A}}'_\perp\right)^2}{2}\right]+8\left(\frac{1}{kp}-\frac{1}{kP}\right)i\epsilon_{\alpha\beta\rho\sigma}l^\alpha\pi^\beta k^\rho\pi'^\sigma, \qquad \text{(D20)}$$

where we employed that

$$\epsilon_{\mu\nu\alpha\beta}l^\alpha\left(\pi^\beta+\pi'^\beta\right)k^\mu\left(\mathcal{A}^\nu-\mathcal{A}'^\nu\right) = \epsilon_{\mu\nu\alpha\beta}l^\alpha\left(\pi^\beta+\pi'^\beta\right)k^\mu\left(\pi'^\nu-\pi^\nu\right) = 2\epsilon_{\mu\nu\alpha\beta}k^\mu\pi'^\nu l^\alpha\pi^\beta, \qquad \text{(D21)}$$

and that the Levi-Civita symbol contracted with the same vector twice vanishes. At this point, we will expand in with respect to φ_- to enforce the LCFA. Each term from Eq. (D20) requires special attention, however the calculation may also be reused later. We have that

$$\pi\pi'-m_e^2 = \Pi\Pi'-m_P^2 = \frac{1}{2}(\boldsymbol{\mathcal{A}}_\perp-\boldsymbol{\mathcal{A}}'_\perp)^2 \approx \frac{\varphi_-^2}{2}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2, \qquad \text{(D22)}$$

and therefore

$$\int \left(\pi\pi' - m_e^2\right) e^{i\left(\tilde{\Phi}+g\varphi_- \boldsymbol{x}_\perp^2\right)} d\varphi_- d^2\boldsymbol{x}_\perp = \int \frac{\varphi_-^2}{2}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2 e^{i\left(\tilde{\Phi}+g\varphi_- \boldsymbol{x}_\perp^2\right)} d\varphi_- d^2\boldsymbol{x}_\perp$$
$$= \int \frac{i\pi}{g}\frac{\varphi_-}{2}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2 e^{i\tilde{\Phi}} d\varphi_-$$
$$= \frac{i\pi}{g} c^2 \frac{f_1}{2}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2. \tag{D23}$$

We remind that

$$c = \sqrt{a/(3b)} = \sqrt{\frac{\rho + \frac{\left(m_N^2 - m_e^2 - m_P^2\right)(kp)(kP) + m_e^2(kP)^2 + m_P^2(kp)^2}{2k(P-p)(kp)(kP)}}{3\frac{k(P-p)}{(kP)(kp)}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2 \frac{1}{24}}} \approx \frac{2}{\left|\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right|}\sqrt{2\rho kp + m_e^2\left(\frac{m_N^2 - m_e^2 - m_P^2}{m_e m_P}\zeta + 1 + \zeta^2\right)}$$
$$= \frac{2m_e y}{\left|\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right|}. \tag{D24}$$

Where we set $k(P-p) \approx kP$. At this level of approximation we also have that

$$\frac{i\pi}{g} c^2 \frac{f_1}{2}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2 = 2\pi(kp)(2m_e y)^2 \frac{if_1}{2}. \tag{D25}$$

This procedure of integrating over $d^2\boldsymbol{p}_\perp$, expanding $\boldsymbol{\mathcal{A}}_\perp$ in φ_-, and integrating over φ_- must be carried out for all the terms.

a. *The term $l\,(\pi+\pi')$*

We have that

$$l\,(\pi+\pi') = (P - p + sk)\left(2p - (\mathcal{A}+\mathcal{A}') + \frac{1}{kp}\left[p(\mathcal{A}+\mathcal{A}') - \frac{\mathcal{A}^2 + \mathcal{A}'^2}{2}\right]k\right)$$
$$= 2Pp\frac{kP}{k(P-p)} - 2m_e^2 + (\boldsymbol{P}_\perp - \boldsymbol{p}_\perp)(\boldsymbol{\mathcal{A}}_\perp + \boldsymbol{\mathcal{A}'}_\perp) + \frac{k(P-p)}{kp}\left[\frac{\boldsymbol{\mathcal{A}}_\perp^2 + \boldsymbol{\mathcal{A}}_\perp'^2}{2} - \boldsymbol{p}_\perp(\boldsymbol{\mathcal{A}}_\perp + \boldsymbol{\mathcal{A}}'_\perp)\right]$$
$$+ 2\rho kp + kp\frac{m_N^2 - m_P^2 - m_e^2}{k\,(P-p)}. \tag{D26}$$

where we used that $s_{\min} = \frac{m_N^2 - (P-p)^2}{2k(P-p)}$. At this point use that

$$Pp = P_+p_- + P_-p_+ - \boldsymbol{P}_\perp\boldsymbol{p}_\perp$$
$$= \left(kp\frac{m_P^2 + \boldsymbol{P}_\perp^2}{2kP} + kP\frac{m_e^2 + \boldsymbol{p}_\perp^2}{2kp} - \boldsymbol{P}_\perp\boldsymbol{p}_\perp\right), \tag{D27}$$

and so we obtain that

$$l\,(\pi+\pi') = \left(kp\frac{m_P^2 + \boldsymbol{P}_\perp^2}{kP} + kP\frac{m_e^2 + \boldsymbol{p}_\perp^2}{kp} - \boldsymbol{P}_\perp\boldsymbol{p}_\perp\right)\frac{kP}{k(P-p)} - 2m_e^2 + (\boldsymbol{P}_\perp - \boldsymbol{p}_\perp)(\boldsymbol{\mathcal{A}}_\perp + \boldsymbol{\mathcal{A}}'_\perp)$$
$$+ \frac{k(P-p)}{kp}\left[\frac{\boldsymbol{\mathcal{A}}_\perp^2 + \boldsymbol{\mathcal{A}}_\perp'^2}{2} - \boldsymbol{p}_\perp(\boldsymbol{\mathcal{A}}_\perp + \boldsymbol{\mathcal{A}}'_\perp)\right] + 2\rho kp + kp\frac{m_N^2 - m_P^2 - m_e^2}{k\,(P-p)}. \tag{D28}$$

Now, we replace $\boldsymbol{p}_\perp = \boldsymbol{x}_\perp + \frac{kp}{kP}\boldsymbol{P}_\perp + \frac{k(P-p)}{kP}\langle \boldsymbol{\mathcal{A}}_\perp \rangle$ to obtain

$$\begin{aligned} l\left(\pi+\pi'\right) &= \left(kp\frac{m_P^2}{kP} + kP\frac{m_e^2}{kp}\right)\frac{kP}{k(P-p)} - 2m_e^2 \\ &\quad + \frac{k(P-p)}{kp}\left[\left(\langle \boldsymbol{\mathcal{A}}_\perp \rangle - \frac{\boldsymbol{\mathcal{A}}_\perp + \boldsymbol{\mathcal{A}}'_\perp}{2}\right)^2 + \frac{(\boldsymbol{\mathcal{A}}_\perp - \boldsymbol{\mathcal{A}}'_\perp)^2}{4}\right] + 2\rho kp + kp\frac{m_N^2 - m_P^2 - m_e^2}{k\left(P-p\right)} \\ &\quad + 2\frac{kP}{kp}\boldsymbol{x}_\perp\left(\langle \boldsymbol{\mathcal{A}}_\perp \rangle - \frac{\boldsymbol{\mathcal{A}}_\perp + \boldsymbol{\mathcal{A}}'_\perp}{2}\right) + \boldsymbol{x}_\perp^2\frac{(kP)^2}{k(P-p)(kp)}. \end{aligned} \tag{D29}$$

Once again approximating $k(P-p) \approx kP$ and by neglecting the difference $\langle \boldsymbol{\mathcal{A}}_\perp \rangle - \left(\boldsymbol{\mathcal{A}}_\perp + \boldsymbol{\mathcal{A}'}_\perp\right)/2$, we obtain

$$l\left(\pi+\pi'\right) \approx \frac{m_e m_P}{\zeta}\left(1+\zeta^2\right) - 2m_e^2 + \frac{1}{\zeta}\frac{m_P}{m_e}\frac{\varphi_-^2}{4}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2 + \zeta m_e^2\left(\frac{l^2 - m_P^2 - m_e^2}{m_e m_P}\right) + \boldsymbol{x}_\perp^2\frac{m_P}{m_e}\frac{1}{\zeta}. \tag{D30}$$

Now using the identities from the Appendix (E), we obtain that

$$\begin{aligned} \int l\left(\pi+\pi'\right)e^{ig\varphi_- \boldsymbol{x}_\perp^2}d^2\boldsymbol{x}_\perp &= \frac{i\pi}{\varphi_- g}\left(\frac{m_e m_P}{\zeta}\left(1+\zeta^2\right) - 2m_e^2 + \frac{1}{\zeta}\frac{m_P}{m_e}\frac{\varphi_-^2}{4}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2 + \zeta m_e^2\left(\frac{l^2 - m_P^2 - m_e^2}{m_e m_P}\right)\right) \\ &\quad - \frac{\pi}{\varphi_-^2 g^2}\frac{m_P}{m_e}\frac{1}{\zeta} \end{aligned} \tag{D31}$$

and therefore

$$\begin{aligned} \int l\left(\pi+\pi'\right)e^{i\left(\tilde{\Phi} + g\varphi_- \boldsymbol{x}_\perp^2\right)}d^2\boldsymbol{x}_\perp d\varphi_- &= \frac{if_{-1}\pi}{g}\left(\frac{m_e m_P}{\zeta}\left(1+\zeta^2\right) - 2m_e^2 + \zeta m_e^2\left(\frac{l^2 - m_P^2 - m_e^2}{m_e m_P}\right)\right) \\ &\quad + \frac{\pi}{g}\frac{1}{\zeta}\frac{m_P}{m_e}c^2\frac{if_1}{4}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2 - \frac{\pi}{g^2}c^{-1}f_{-2}\frac{m_P}{m_e}\frac{1}{\zeta} \\ &\approx if_{-1}2\pi(kP)m_e^2\left(1+\zeta^2\right) + 2\pi(kP)m_e^2 y^2 if_1 - 4\pi(kP)\zeta m_e^2\frac{(kP)\left|\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right|}{2m_e^2 m_P y}f_{-2} \\ &= if_{-1}2\pi(kP)m_e^2\left(1+\zeta^2\right) + 2\pi(kP)m_e^2 y^2 if_1 - 4\pi(kP)\zeta m_e^2\frac{\chi_P}{2y}f_{-2}. \end{aligned} \tag{D32}$$

b. *Term* $\epsilon_{\alpha\beta\rho\sigma}l^\alpha\pi^\beta k^\rho\pi'^\sigma$

By using that repeated four vectors contracted with the Levi-Civita symbol vanishes, we obtain that

$$\begin{aligned} &\epsilon_{\alpha\beta\rho\sigma}l^\alpha\pi^\beta k^\rho\pi^{'\sigma} \\ &= \epsilon_{\alpha\beta\rho\sigma}\left(P^\alpha - p^\alpha\right)\left(p^\beta - A^\beta\right)k^\rho\left(p^\sigma - A'^\sigma\right) \\ &= \epsilon_{\alpha\beta\rho\sigma}P^\alpha\left(A^\sigma - A'^\sigma\right)p^\beta k^\rho \\ &\approx \varphi_-\epsilon_{\alpha\beta\rho\sigma}P^\alpha\frac{dA^\sigma}{d\varphi_+}p^\beta k^\rho \end{aligned} \tag{D33}$$

Now use that

$$g^{\mu\nu} = n^\mu\tilde{n}^\nu + n^\nu\tilde{n}^\mu - e_1^\mu e_1^\nu - e_2^\mu e_2^\nu \tag{D34}$$

with

$$\tilde{n} = \frac{1}{2}\{1, -\boldsymbol{n}\}, \tag{D35}$$

$$e_i = \{0, \boldsymbol{e}_i\} \tag{D36}$$

where $\boldsymbol{e}_i$ and $\boldsymbol{n}$ are unit vectors perpendicular to each other and to $\boldsymbol{n}$. In the setup, we have chosen, the vector potential of the laser is along $\boldsymbol{e}_1$ and we have that

$$\begin{aligned}\varphi_-\epsilon_{\alpha\beta\rho\sigma}P^\alpha\frac{dA^\sigma}{d\varphi_+}p^\beta k^\rho &= \varphi_-\epsilon_{\alpha\beta\rho\sigma}\left[(nP)\tilde{n}^\alpha+(\boldsymbol{P}_\perp\boldsymbol{e}_2)a_2^\alpha\right]\frac{d\mathcal{A}^\sigma}{d\varphi_+}\left[(np)\tilde{n}^\beta+(\boldsymbol{p}_\perp\boldsymbol{e}_2)a_2^\beta\right]k^\rho\\ &= \varphi_-\epsilon_{\alpha\beta\rho\sigma}\left[(nP)(\boldsymbol{p}_\perp\boldsymbol{e}_2)-(\boldsymbol{P}_\perp\boldsymbol{e}_2)(np)\right]\frac{d\mathcal{A}^\sigma}{d\varphi_+}\tilde{n}^\alpha a_2^\beta k^\rho = \varphi_-\epsilon_{\alpha\beta\rho\sigma}\left[(nP)(\boldsymbol{x}_\perp\boldsymbol{e}_2)\right]\frac{d\mathcal{A}^\sigma}{d\varphi_+}\tilde{n}^\alpha a_2^\beta k^\rho,\end{aligned} \tag{D37}$$

where in the last line we put in $\boldsymbol{p}_\perp=\boldsymbol{x}_\perp+\frac{kp}{kP}\boldsymbol{P}_\perp+\frac{k(P-p)}{kP}\langle\boldsymbol{\mathcal{A}}_\perp\rangle$, from the change of variable, which cancels the $\boldsymbol{P}_\perp\boldsymbol{e}_2$ term. The remaining terms are linear in $\boldsymbol{x}_\perp$ and, as may be seen from the identities in the Appendix E, the integral over $\boldsymbol{x}_\perp$ of these terms vanishes.

c. The total contribution from "Line 1" to the probability

By observing that the functions f_n are all dimensionless and of the same order of magnitude, we may neglect terms suppressed by factors of m_e/m_P or m_e/m_N to obtain that

$$\int d^2\boldsymbol{x}_\perp d\varphi_- T_1^{\mu\nu}W_{\mu\nu\alpha}l^\alpha e^{i\left(\tilde{\Phi}+g\varphi_-\boldsymbol{x}_\perp^2\right)}\approx 4m_Nm_P\left(g_v^2-g_a^2\right)2\pi(kP)8\left[m_e^2y^2if_1-if_{-1}m_e^2\left(1+\zeta^2\right)+\zeta m_e^2\frac{\chi_P}{y}f_{-2}\right]. \tag{D38}$$

It is convenient to change variable from ρ to a variable z which should be on the order of unity. This is achieved from looking at the definition of the variable y and recognize that the important size of ρ is when the term containing ρ is of the same size as the other terms. We may write

$$\begin{aligned}y &= \sqrt{\frac{2\rho kP+m_N^2-m_e^2-m_P^2}{m_em_P}\zeta+1+\zeta^2},\\ &= \sqrt{\frac{2\rho kP}{m_em_P}\zeta+\frac{m_N^2-m_e^2-m_P^2}{m_em_P}\zeta+1+\zeta^2}\\ &= \sqrt{\frac{2kP}{m_em_P}\zeta\left[\rho+\frac{m_N^2-m_e^2-m_P^2}{2\zeta kP}\zeta+\frac{m_em_P}{2\zeta kP}\left(1+\zeta^2\right)\right]},\end{aligned} \tag{D39}$$

and therefore we introduce

$$z=\frac{\rho}{\frac{m_N^2-m_e^2-m_P^2}{2\zeta kP}\zeta+\frac{m_em_P}{2\zeta kP}\left(1+\zeta^2\right)}=\zeta\frac{2kP\rho}{\left(m_N^2-m_e^2-m_P^2\right)\zeta+m_em_P\left(1+\zeta^2\right)}. \tag{D40}$$

In this way, the contribution to the probability is

$$dP=\frac{G_F^2}{2}\frac{1}{(2\pi)^6}\frac{1}{32kP}\int d\rho d\varphi_+\frac{J}{2l^2}(l^2-m_N^2)\int T_1^{\mu\nu}W_{\mu\nu\alpha}l^\alpha e^{i\left(\tilde{\Phi}+g\varphi_-x_\perp^2\right)}d^2\boldsymbol{x}_\perp d\varphi_-\frac{dp_\parallel}{\varepsilon_p} \tag{D41}$$

and changing variable from ρ to z, writing $d\varphi_+=kudt=kUd\tau=\frac{kP}{m_P}d\tau$, $dp_\parallel/\varepsilon_p=d(kp)/kp=d\zeta/\zeta$ and

$$\frac{J}{2l^2}(l^2-m_N^2)=\frac{2\pi}{2l^4}(l^2-m_N^2)^2=\frac{2\pi}{2l^4}z^2\left[\frac{\left(m_N^2-m_e^2-m_P^2\right)\zeta+m_em_P\left(1+\zeta^2\right)}{\zeta}\right]^2, \tag{D42}$$

we obtain that

$$\begin{aligned}dP &= \frac{G_F^2}{2}\frac{1}{(2\pi)^6}\frac{1}{32kP}\int\frac{kP}{m_P}d\tau dz\frac{z^2}{2kP}\left[\left(m_N^2-m_e^2-m_P^2\right)\zeta+m_em_P\left(1+\zeta^2\right)\right]^3\frac{d\zeta}{\zeta^4}\\ &\quad\times\frac{2\pi}{2l^4}4m_Nm_P\left(g_v^2-g_a^2\right)2\pi(kP)8\left[m_e^2y^2if_1-if_{-1}m_e^2\left(1+\zeta^2\right)+\zeta m_e^2\frac{\chi_P}{y}f_{-2}\right]\\ &= \frac{G_F^2}{2}\frac{1}{(2\pi)^4}m_e^5m_P^2\int dzd\tau d\zeta\frac{z^2}{\zeta^4}\frac{1}{8l^4}\left[\frac{\left(m_N^2-m_e^2-m_P^2\right)}{m_em_P}\zeta+\left(1+\zeta^2\right)\right]^3\\ &\quad\times 2m_Nm_P\left(g_v^2-g_a^2\right)\left[y^2if_1-if_{-1}\left(1+\zeta^2\right)+\zeta\frac{\chi_P}{y}f_{-2}\right].\end{aligned} \tag{D43}$$

2. Line 2 (symmetric part)

We start by using the identity

$$l_\alpha l^\beta T_2^{\mu\nu\alpha} W_{\mu\nu\beta} = l_\alpha l^\beta T_{2,S}^{\mu\nu\alpha} W_{\mu\nu\beta}^S + l_\alpha l^\beta T_{2,A}^{\mu\nu\alpha} W_{\mu\nu\beta}^A. \tag{D44}$$

Here, we evaluate

$$\int d^2\boldsymbol{x}_\perp d\varphi_- l_\alpha l^\beta T_{2,S}^{\mu\nu\alpha} W_{\mu\nu\beta}^S e^{i\left(\tilde{\Phi}+g\varphi_- \boldsymbol{x}_\perp^2\right)}. \tag{D45}$$

where the labels S and A indicate the symmetric and the anti-symmetric parts of the corresponding tensors. After performing straightforward manipulations reductions we obtain that

$$\begin{aligned}
&l_\alpha l^\beta T_{2,S}^{\mu\nu\alpha} W_{\mu\nu\beta}^S \\
&= 16\left\{\left(g_v^2+g_a^2\right)\left[\frac{lk}{kP}\left(\Pi\Pi'-m_P^2\right)-l\left(\Pi+\Pi'\right)\right]-\frac{2g_v g_a}{kP} i\epsilon^{\alpha\beta\rho\sigma} l_\alpha \Pi_\beta k_\rho \Pi'_\sigma\right\} \\
&\times\left\{\left[\frac{lk}{kp}\left(\pi\pi'-m_e^2\right)-l\left(\pi+\pi'\right)\right]+\frac{1}{kp} i\epsilon_{\tau\theta\phi\kappa} l^\tau \pi^\theta k^\phi \pi'^\kappa\right\} \\
&+16l^2\left(g_v^2+g_a^2\right)\left[\left(\Pi'+\Pi\right)\cdot\left(\pi'+\pi\right)-2kP\frac{\pi\pi'-m_e^2}{kp}-2kp\frac{\Pi\Pi'-m_P^2}{kP}\right] \\
&+16l^2\left[\frac{2g_v g_a}{kP} i\epsilon^{\mu\beta\rho\sigma}\left(\pi'_\mu+\pi_\mu\right)\Pi_\beta k_\rho \Pi'_\sigma-\frac{1}{kp} i\epsilon_{\mu\theta\phi\kappa}\left(\Pi'^\mu+\Pi^\mu\right)\pi^\theta k^\phi \pi'^\kappa\right].
\end{aligned} \tag{D46}$$

In this expression, we encounter some terms of the same type as before and also some new ones. The terms of the type $\epsilon^{\mu\beta\rho\sigma}\left(\pi'_\mu+\pi_\mu\right)\Pi_\beta k_\rho \Pi'_\sigma$ vanish under our assumption of linear polarization.

a. *Term $l\left(\Pi+\Pi'\right)$*

We have that

$$l\left(\Pi+\Pi'\right) = \left(P-p+sk\right)\left(2P-\left(\mathcal{A}+\mathcal{A}'\right)+\frac{1}{kP}\left[\frac{\boldsymbol{\mathcal{A}}_\perp^2+\boldsymbol{\mathcal{A}}_\perp'^2}{2}-\boldsymbol{P}_\perp(\boldsymbol{\mathcal{A}}_\perp+\mathcal{A}'_\perp)\right]k\right), \tag{D47}$$

which, after performing the same kind of reduction as for $l\left(\pi+\pi'\right)$, leads to

$$\begin{aligned}
l\left(\Pi+\Pi'\right) = {}& 2m_P^2+\left(kp\frac{m_P^2}{kP}+kP\frac{m_e^2}{kp}\right)\frac{kp}{k\left(P-p\right)}+\frac{k(P-p)}{kP}\frac{\left(\boldsymbol{\mathcal{A}}_\perp-\boldsymbol{\mathcal{A}}'_\perp\right)^2}{4}+2\rho kP \\
&+kP\frac{m_N^2-m_P^2-m_e^2}{k\left(P-p\right)}+\boldsymbol{x}_\perp^2\frac{kP}{k\left(P-p\right)}.
\end{aligned} \tag{D48}$$

b. *The total contribution from the symmetric part of "Line 2"*

We note that several terms vanish when performing the integration over $d^2\boldsymbol{x}_\perp$ in this contribution. We have that

$$\begin{aligned}
&\int d^2\boldsymbol{x}_\perp l_\alpha l^\beta T_{2,S}^{\mu\nu\alpha} W_{\mu\nu\beta}^S \\
&= \int d^2\boldsymbol{x}_\perp 16\left(g_v^2+g_a^2\right)\left[\frac{lk}{kP}\left(\Pi\Pi'-m_P^2\right)-l\left(\Pi+\Pi'\right)\right]\left[\frac{lk}{kp}\left(\pi\pi'-m_e^2\right)-l\left(\pi+\pi'\right)\right] \\
&+32\frac{kP}{kp} g_v g_a (\boldsymbol{x}_\perp \boldsymbol{e}_2)^2\left(\boldsymbol{\mathcal{A}}_\perp-\boldsymbol{\mathcal{A}}'_\perp\right)^2 \\
&+16l^2\left(g_v^2+g_a^2\right)\left[\left(\Pi'+\Pi\right)\left(\pi'+\pi\right)-2kP\frac{\pi\pi'-m_e^2}{kp}-2kp\frac{\Pi\Pi'-m_P^2}{kP}\right].
\end{aligned} \tag{D49}$$

Now using that in the LCFA and to leading order in m_e/m_P

$$l\left(\pi+\pi'\right)\approx\left(kp\frac{m_P^2}{kP}+kP\frac{m_e^2}{kp}\right)+\frac{kP}{kp}\frac{\varphi^2}{4}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2+(l^2-m_N^2)\frac{kp}{kP}+\boldsymbol{x}_\perp^2\frac{kP}{kp}, \tag{D50}$$

$$l\left(\Pi+\Pi'\right)\approx 2m_P^2+\frac{\varphi^2}{4}\left(\frac{d\boldsymbol{\mathcal{A}}_\perp}{d\varphi_+}\right)^2+(l^2-m_N^2)+\boldsymbol{x}_\perp^2, \tag{D51}$$

$$\left(\Pi+\Pi'\right)\left(\pi+\pi'\right)\approx 2kP\frac{m_e^2+\boldsymbol{x}_\perp^2}{kp}+2kp\frac{m_P^2}{kP}+2\left(\frac{kP}{kp}+\frac{kp}{kP}\right)\frac{(\boldsymbol{\mathcal{A}}_\perp-\boldsymbol{\mathcal{A}}'_\perp)^2}{4} \tag{D52}$$

we obtain that to leading order

$$\begin{aligned}&\int d^2\boldsymbol{x}_\perp d\varphi_- l_\alpha l^\beta T_{2,S}^{\mu\nu\alpha}W^S_{\mu\nu\beta}\\&=16\left(g_v^2+g_a^2\right)2\pi(kP)\left(2m_P^2+3l^2-m_N^2\right)\\&\times\left\{-m_e^2y^2if_1-m_e^2\zeta\frac{\chi_P}{y}f_{-2}+m_e^2if_{-1}\left[1+\left(1+\frac{2m_P^2+l^2-m_N^2}{2m_P^2+3l^2-m_N^2}\frac{l^2-m_N^2}{m_P^2}\right)\zeta^2\right]\right\}\end{aligned} \tag{D53}$$

3. Line 2 (anti-symmetric part)

By proceeding analogously as above, we obtain that

$$\begin{aligned}&l_\alpha l^\tau T_{2,A}^{\mu\nu\alpha}W^A_{\mu\nu\tau}\\&=\frac{16\left(g_v^2+g_a^2\right)}{(kP)(kp)}\left\{(lk)^2\left[\left(\Pi'\pi'\right)\left(\Pi\pi\right)-\left(\Pi'\pi\right)\left(\Pi\pi'\right)\right]\right.\\&+(lk)(kP)\left[(l\pi)\pi'-(l\pi')\pi\right]\cdot\left(\Pi-\Pi'\right)\\&+(lk)(kp)\left[(l\Pi)\Pi'-(l\Pi')\Pi\right]\cdot\left(\pi-\pi'\right)\\&\left.-(kp)(kP)\left(l\Pi-l\Pi'\right)\left(l\pi-l\pi'\right)\right\}\\&-\frac{16\left(g_v^2+g_a^2\right)}{kP}\left\{\left[(lk)\,\Pi'^\nu\Pi^\mu+(l\Pi)\,k^\nu\Pi'^\mu+(l\Pi')\,k^\mu\Pi^\nu\right]\left(\pi^\beta+\pi'^\beta\right)+\frac{lk}{kp}\Pi'^\nu\Pi^\mu\left(m_e^2-\pi\pi'\right)k^\beta\right\}i\epsilon_{\mu\nu\alpha\beta}l^\alpha\\&+\frac{32g_vg_a}{kp}\left\{\left[(lk)\,\pi'_\nu\pi_\mu+(l\pi)\,k_\nu\pi'_\mu+(l\pi')\,k_\mu\pi_\nu\right]\left(\Pi_\beta+\Pi'_\beta\right)+\frac{lk}{kP}\pi'_\nu\pi_\mu\left(m_P^2-\Pi\Pi'\right)k_\beta\right\}i\epsilon^{\mu\nu\alpha\beta}l_\alpha\\&+\frac{32g_vg_a}{(kP)(kp)}\left\{l^2\left[(kP)(kp)\left(\pi+\pi'\right)\cdot\left(\Pi+\Pi'\right)+2(kP)^2\left(m_e^2-\pi\pi'\right)+2(kp)^2\left(m_P^2-\Pi\Pi'\right)\right]\right.\\&\left.-\left[(kP)l\cdot\left(\Pi+\Pi'\right)+(lk)\left(m_P^2-\Pi\Pi'\right)\right]\left[(kp)l\cdot\left(\pi+\pi'\right)+(lk)\left(m_e^2-\pi\pi'\right)\right]\right\}\end{aligned} \tag{D54}$$

This whole part, however, only contributes with terms suppressed by at least m_e/m_P.

4. Line 3

After performing reductions from the initial expression, one obtains that

$$\begin{aligned}T_{2,S}^{\mu\nu\alpha}W^S_{\mu\nu\alpha}&=80\left(g_v^2+g_a^2\right)\left[\left(\Pi'+\Pi\right)\cdot\left(\pi'+\pi\right)-2kp\frac{\Pi\Pi'-m_P^2}{kP}-2kP\frac{\pi\pi'-m_e^2}{kp}\right]\\&+\frac{160g_vg_a}{kP}i\epsilon^{\mu\beta\rho\sigma}\left(\pi'_\mu+\pi_\mu\right)\Pi_\beta k_\rho\Pi'_\sigma-\frac{80\left(g_v^2+g_a^2\right)}{kp}i\epsilon_{\mu\theta\phi\kappa}\left(\Pi'^\mu+\Pi^\mu\right)\pi^\theta k^\phi\pi'^\kappa\\&+160g_vg_a\left(\Pi'-\Pi\right)\left(\pi'-\pi\right).\end{aligned} \tag{D55}$$

By using the identity

$$\left(\Pi-\Pi'\right)\cdot\left(\pi'-\pi\right)=\left(\boldsymbol{\mathcal{A}}'_\perp-\boldsymbol{\mathcal{A}}_\perp\right)^2, \tag{D56}$$

it is seen that this contribution contains only terms which have already been analyzed. For the anti-symmetric part, we obtain after reduction that

$$\begin{aligned}T^{\mu\nu\alpha}_{2,A}W^{A}_{\mu\nu\alpha} &= 48\left(g_v^2+g_a^2\right)\left(\Pi-\Pi'\right)\cdot\left(\pi'-\pi\right)\\ &-\frac{48\left(g_v^2+g_a^2\right)}{kP}i\epsilon_{\mu\nu\alpha\beta}\left(\pi^\beta+\pi'^\beta\right)k^\alpha\Pi'^\nu\Pi^\mu\\ &+\frac{96g_vg_a}{kp}i\epsilon^{\mu\nu\alpha\beta}\left(\Pi_\beta+\Pi'_\beta\right)k_\alpha\pi'_\nu\pi_\mu\\ &-96g_vg_a\left[\left(\Pi+\Pi'\right)\cdot\left(\pi+\pi'\right)+\frac{2kP}{kp}\left(m_e^2-\pi\pi'\right)+\frac{2kp}{kP}\left(m_P^2-\Pi\Pi'\right)\right],\end{aligned} \tag{D57}$$

which also contains only terms which we have already treated.

Appendix E: Gaussian integrals

Here we report some identities for Gaussian integrals, which are needed to perform the integration over the transverse momentum of the positron:

$$\int_{-\infty}^{\infty}e^{ia\boldsymbol{x}_\perp^2}d^2\boldsymbol{x}_\perp=\int_0^\infty e^{iar^2}2\pi r dr=\pi\int_0^\infty e^{iar^2}dr^2=\frac{i\pi}{a}, \tag{E1}$$

$$\int_{-\infty}^{\infty}\boldsymbol{c}\cdot\boldsymbol{x}_\perp e^{ia\boldsymbol{x}_\perp^2}d^2\boldsymbol{x}_\perp=0, \tag{E2}$$

$$\int_{-\infty}^{\infty}\boldsymbol{x}_\perp^2e^{ia\boldsymbol{x}_\perp^2}d^2\boldsymbol{x}_\perp=\int_0^\infty r^2e^{iar^2}2\pi r dr=\pi\int_0^\infty xe^{iax}dx=-\frac{\pi}{a^2}, \tag{E3}$$

$$\int_{-\infty}^{\infty}\boldsymbol{x}_\perp^4e^{ia\boldsymbol{x}_\perp^2}d^2\boldsymbol{x}_\perp=\int_0^\infty r^4e^{iar^2}2\pi r dr=\pi\int_0^\infty x^2e^{iax}dx=-\frac{2\pi i}{a^3}, \tag{E4}$$

$$\int_{-\infty}^{\infty}\left(\boldsymbol{c}\cdot\boldsymbol{x}_\perp\right)\left(\boldsymbol{d}\cdot\boldsymbol{x}_\perp\right)e^{ia\boldsymbol{x}_\perp^2}d^2\boldsymbol{x}_\perp=-\frac{\pi}{2a^2}\boldsymbol{c}\cdot\boldsymbol{d}. \tag{E5}$$

Note that in the derivations it is implicitly assumed that $\mathrm{Im}(a)>0$ for the integrals to converge.

[1] S. N. Ahmed, A. E. Anthony, and E. W. Beier et al. (SNO Collaboration), Phys. Rev. Lett. **92**, 102004 (2004).
[2] V. I. Ritus, J. Sov. Laser Res. **6**, 497 (1985).
[3] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, Rev. Mod. Phys. **84**, 1177 (2012).
[4] H. R. Reiss, J. Math. Phys. (N.Y.) **3**, 59 (1962).
[5] A. I. Nikishov and V. I. Ritus, Sov. Phys. JETP **19**, 529 (1964).
[6] N. B. Narozhny and M. S. Fofanov, J. Exp. Theor. Phys. **90**, 415 (2000).
[7] S. P. Roshchupkin, Phys. At. Nucl. **64**, 243 (2001).
[8] H. R. Reiss, Eur. Phys. J. D **55**, 365 (2009).
[9] T. Heinzl, A. Ilderton, and M. Marklund, Phys. Lett. B **692**, 250 (2010).
[10] T.-O. Müller and C. Müller, Phys. Lett. B **696**, 201 (2011).
[11] A. I. Titov, H. Takabe, B. Kämpfer, and A. Hosaka, Phys. Rev. Lett. **108**, 240406 (2012).
[12] T. Nousch, D. Seipt, B. Kämpfer, and A. Titov, Phys. Lett. B **715**, 246 (2012).
[13] K. Krajewska, C. Müller, and J. Z. Kamiński, Phys. Rev. A **87**, 062107 (2013).
[14] M. J. A. Jansen and C. Müller, Phys. Rev. A **88**, 052125 (2013).
[15] S. Augustin and C. Müller, Phys. Lett. B **737**, 114 (2014).
[16] S. Meuren, K. Z. Hatsagortsyan, C. H. Keitel, and A. Di Piazza, Phys. Rev. D **91**, 013009 (2015).
[17] S. Meuren, C. H. Keitel, and A. Di Piazza, Phys. Rev. D **93**, 085028 (2016).
[18] M. J. A. Jansen, J. Z. Kamiński, K. Krajewska, and C. Müller, Phys. Rev. D **94**, 013010 (2016).
[19] A. Di Piazza, Phys. Rev. Lett. **117**, 213201 (2016).
[20] A. I. Titov, H. Takabe, B. Kämpfer, and A. Hosaka, Phys. Rev. Lett. **108**, 240406 (2012).
[21] J. Esberg, K. Kirsebom, H. Knudsen, H. D. Thomsen, E. Uggerhøj, U. I. Uggerhøj, P. Sona, A. Mangiarotti, T. J. Ketel, A. Dizdar, M. M. Dalton, S. Ballestrero, and S. H. Connell (CERN NA63), Phys. Rev. D **82**, 072002 (2010).
[22] F. Mackenroth and A. Di Piazza, Phys. Rev. D **98**, 116002 (2018).

[23] V. Dinu and G. Torgrimsson, Phys. Rev. D **97**, 036021 (2018).
[24] B. King and A. M. Fedotov, Phys. Rev. D **98**, 016005 (2018).
[25] A. Ilderton, Phys. Rev. Lett. **106**, 020404 (2011).
[26] R. Moore, M. Parker, A. Baurichter, K. Kirsebom, R. Medenwaldt, U. Mikkelsen, S. Møller, E. Uggerhøj, T. Worm, N. Doble, K. Elsener, S. Ballestrero, P. Sona, V. Strakhovenko, C. Biino, and Z. Vilakazi, Nucl. Instrum. Methods Phys. Res., B **119**, 149 (1996).
[27] V. Baier, V. Katkov, and V. Strakhovenko, Phys. Lett. A **104**, 231 (1984).
[28] J. Kimball, N. Cue, L. Roth, and B. Marsh, Phys. Rev. Lett. **50**, 950 (1983).
[29] T. N. Wistisen, Phys. Rev. D **101**, 076017 (2020).
[30] F. Sauter, Z. Phys. **69**, 742 (1931).
[31] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
[32] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[33] E. Brezin and C. Itzykson, Phys. Rev. D **2**, 1191 (1970).
[34] V. S. Popov, JETP Lett. **13**, 185 (1971).
[35] N. B. Narozhny, S. S. Bulanov, V. D. Mur, and V. S. Popov, J. Exp. Theor. Phys. **102**, 9 (2006).
[36] R. Schützhold, H. Gies, and G. Dunne, Phys. Rev. Lett. **101**, 130404 (2008).
[37] G. V. Dunne, H. Gies, and R. Schützhold, Phys. Rev. D **80**, 111301(R) (2009).
[38] A. Di Piazza, E. Lötstedt, A. I. Milstein, and C. H. Keitel, Phys. Rev. Lett. **103**, 170403 (2009).
[39] S. S. Bulanov, V. D. Mur, N. B. Narozhny, J. Nees, and V. S. Popov, Phys. Rev. Lett. **104**, 220404 (2010).
[40] A. Di Piazza, Phys. Rev. D **70**, 053013 (2004).
[41] C. Kohlfürst, M. Mitter, G. von Winckel, F. Hebenstreit, and R. Alkofer, Phys. Rev. D **88**, 045028 (2013).
[42] V. Ginzburg and G. Zharkov, JETP **20**, 1525 (1965).
[43] D. A. T. Vanzella and G. E. A. Matsas, Phys. Rev. Lett. **87**, 151301 (2001).
[44] R. Müller, Phys. Rev. D **56**, 953 (1997).
[45] M. Blasone, G. Lambiase, G. G. Luciano, and L. Petruzziello, The European Physical Journal C **80**, 1 (2020).
[46] G. E. A. Matsas and D. A. T. Vanzella, Phys. Rev. D **59**, 094004 (1999).
[47] E. Fermi, Z. Phys. **88**, 161 (1934).
[48] E. C. G. Sudarshan and R. E. Marshak, Phys. Rev. **109**, 1860 (1958).
[49] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[50] D. Volkov, Z. Phys **94**, 250 (1935).
[51] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum electrodynamics* (Butterworth-Heinemann, Oxford, 2008).
[52] E. K. Akhmedov, Phys. At. Nucl. **74**, 1299 (2011).
[53] R. N. Cahn and G. Goldhaber, *The experimental foundations of particle physics* (Cambridge University Press, 2009).
[54] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, 2010).
[55] V. I. Ritus, Sov. Phys. JETP **30**, 1181 (1970).
[56] N. B. Narozhny, Phys. Rev. D **20**, 1313 (1979).
[57] N. B. Narozhny, Phys. Rev. D **21**, 1176 (1980).
[58] D. Morozov, N. Narozhnyi, and V. Ritus, Sov. Phys. JETP **53**, 1103 (1981).
[59] A. Fedotov, Journal of Physics: Conference Series **826**, 012027 (2017).
[60] T. Podszus and A. Di Piazza, Phys. Rev. D **99**, 076004 (2019).
[61] A. Ilderton, Phys. Rev. D **99**, 085002 (2019).
[62] C. Baumann, E. N. Nerush, A. Pukhov, and I. Y. Kostyukov, Sci. Rep. **9**, 9407 (2019).
[63] T. G. Blackburn, A. Ilderton, M. Marklund, and C. P. Ridgers, New J. Phys. **21**, 053040 (2019).
[64] V. Yakimenko, S. Meuren, F. Del Gaudio, C. Baumann, A. M. Fedotov, F. Fiuza, T. Grismayer, M. J. Hogan, A. Pukhov, L. O. Silva, and G. White, Phys. Rev. Lett. **122**, 190404 (2019).
[65] A. Di Piazza, T. N. Wistisen, M. Tamburini, and U. I. Uggerhøj, Phys. Rev. Lett. **124**, 044801 (2020).
[66] A. Di Piazza and M. A. Lopez-Lopez, Phys. Rev. D **102**, 076018 (2020).