

ON EGYPTIAN FRACTIONS OF LENGTH 3

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ABSTRACT. Let a, n be positive integers that are relatively prime. We say that a/n can be represented as an Egyptian fraction of length k if there exist positive integers m_1, \dots, m_k such that $\frac{a}{n} = \frac{1}{m_1} + \dots + \frac{1}{m_k}$. Let $A_k(n)$ be the number of solutions a to this equation. In this article, we give a formula for $A_2(p)$ and a parametrization for Egyptian fractions of length 3, which allows us to give bounds to $A_3(n)$, to $f_a(n) = \#\{(m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\}$, and finally to $F(n) = \#\{(a, m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\}$.

1. INTRODUCTION

Historical background. The most ancient mathematical texts are mostly related to computations involving proportions, fractions, and inverses of integers (sometimes connected with problems related to geometry). Many traces of these mathematics are found in Sumerian or Babylonian clay tablets covering a period of several millennia¹.

For Egyptian mathematics, many papyri present computations involving sums of unit fractions (fractions of the form $1/n$) and sometimes also the fraction $2/3$; see e.g. the Rhind Mathematical Papyrus. This document, estimated from 1550 BCE,

2020 *Mathematics Subject Classification.* 11D68, 11D45, 13A05, 01A16.

Key words and phrases. Diophantine equations; Egyptian fractions; divisibility; history of mathematics; Erdős–Straus conjecture; asymptotic estimate.

This paper was written during visits of Florian Luca in 2018 and 2019 to University Roma Tre, to the Université Paris Nord, as well as during the workshop *Number Theory in the Americas* in Casa Matemática Oaxaca where also Carlos Alexis Gómez and Enrique Treviño participated. These authors thank these institutions for their hospitality. In addition, Florian Luca was supported in part by grant CPRR160325161141 and an A-rated scientist award from the NRF of South Africa and by grant no. 17-02804S of the Czech Granting Agency. Francesco Pappalardi was supported in part by the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (GNSAGA) from the Istituto Nazionale di Alta Matematica Francesco Severi (INdAM).

¹In case the reader may have the chance to visit the corresponding museums, let us mention e.g. the Sumerian tablets from Shuruppak (Istanbul Museum, dated circa 2500 BCE), the Babylonian tablets VAT 6505, 7535, 7621, 8512 (Berlin Museum), Plimpton 322 (Columbia University), 015-189 (Hermitage Museum), YBC 4675 (Yale University), AO 64456, AO 17264, and AO 6555 (the Esagil tablet, Louvre Museum, dated 229 BCE).

is a copy by the scribe Ahmes of older documents. It gives a list of decompositions of $2/n$ into unit fractions; such decompositions are also found in the Lahun Mathematical Papyri (UC 32159 and UC 32160, conserved at the University College London), which are dated circa 1800 BCE; see [27].

As traditional, we call an *Egyptian fraction decomposition* (or, in short, an Egyptian fraction) any rational number a/n , seen as a sum of unit fractions (obviously, all rational numbers possess such a decomposition!). It is often said that Egyptian fractions were related to parts of the Eye of Horus (an ancient Egyptian symbol of protection and royal power). However, this esoteric hypothesis (made popular via the seminal work of the Egyptologist Gardiner) is nowadays refuted [37].

As narrated in his survey [20], Ron Graham once asked André Weil what he thought to be the reason that led Egyptians to use this numerical system. André Weil answered jokingly “It is easy to explain. They took a wrong turn!”. However, it is fair to say that, though it is not the most efficient system, it possesses interesting algorithmic aspects and has several applications: for a modern overview of the use of fractions in Egyptian mathematics, see [36].

Babylonian and Greek mathematics were later further developed by Arabic and Indian mathematicians. One book which played an important role in the transmission of Arabic mathematics to Europe is the *Liber Abaci* of Fibonacci, in 1202 (see [33] for a translation into English). This book focuses mostly on the use of fractions and on the *modus Indorum* (the method of the Indians), i.e., the Hindu-Arabic numeral base 10 system that we all use nowadays. He shows how to use these two concepts to solve many problems, often related to trading/financial computations. With respect to fractions, he presents several methods to get Egyptian fraction decompositions, like e.g.

$$\frac{97}{100} = \frac{1}{50} + \frac{1}{5} + \frac{1}{4} + \frac{1}{2}.$$

Alternatively, a greedy method (nowadays called Fibonacci’s greedy algorithm for Egyptian fractions) gives

$$\frac{97}{100} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{86} + \frac{1}{25800}.$$

Similar decompositions were later also considered by Lambert [29] and Sylvester [42]. Sylvester’s attention for this topic was in fact due to the father of the history of mathematics discipline, Moritz Cantor, who mentions (a few years after the translation of the Rhind papyrus) these Egyptian mathematics in the first volume of his monumental 4000-page *Vorlesungen über die Geschichte der Mathematik* [11].

Modern times. Later, in the midst of the twentieth century, Erdős attracted mathematicians’ attention to this topic, by proving or formulating puzzling conjectures related to Egyptian fraction decompositions, and also by establishing nice links with number theory. Egyptian fractions were the subject of the third published article of Erdős (the sum of unit fractions with denominators in arithmetic progression is not an integer, [17]) and of his last (posthumous) published article

with Graham and Butler (all integers are sums of distinct unit fractions with denominators involving 3 distinct prime factors, [10]). Erdős also popularized some conjectures, analysed densities related to these fractions [18, 16, 40], and considered the minimal number of unit fractions needed to express a rational [7, 8, 6].

There are still many unsolved problems regarding Egyptian fractions; see e.g. [21, Section D.11] for a survey. We can end by mentioning further applications or links with total parallel resistance ($\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \dots$), trees and Huffman codes [23], Diophantine equations [1, 43], Engel expansion [12], continued fractions and Farey series [5, 22], products of Abelian groups [2], combinatorial number theory [19, 20, 34], and many asymptotic analyses [28, 9, 13, 15, 26, 24, 25, 14, 30].

Our result. Our article analyzes the Egyptian fraction Diophantine equation

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}, \quad \text{where } a, n, m_1, m_2, m_3 \in \mathbb{N}. \tag{1.1}$$

The famous Erdős–Straus conjecture asserts that, for $a = 4$, there is always a solution to this equation (for any $n > 1$); see [18] for the origin of this conjecture and see [31, Chapter 30.1] for some nontrivial progress on it. A lesser-known conjecture due to Sierpiński asserts that, for $a = 5$, there is always a solution [41], and we additionally conjecture that this is also the case for $a = 6$ and $a = 7$ (for $n \geq a/3$). In fact, a conjecture of Schinzel [39] asserts that any positive integer a is a solution of Equation (1.1) for n large enough (e.g., it seems that $8/n$ is a sum of 3 unit fractions for $n > 241$). For sure, for each n , there is a finite number of integers a which can be solution: the structure of the equation constrains a to be between 1 and $3n$. For fixed n , we set

$$A_k(n) := \#\{a : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \dots + \frac{1}{m_k}\}.$$

It is shown in [14] that $A_2(n) \ll n^\varepsilon$ and that, for $k \geq 3$, one has

$$A_k(n) \ll n^{\alpha_k + \varepsilon}, \quad \text{where } \alpha_k = 1 - 2/(3^{k-2} + 1).$$

In particular, $A_3(n) \ll n^{1/2 + \varepsilon}$. Here and in what follows, all implied constants in the Vinogradov symbol depend on a parameter $\varepsilon > 0$ which can be taken arbitrarily small. In this article, we give a different proof of $A_3(n) \ll n^{1/2 + \varepsilon}$ and we get the following explicit inequality:

Theorem 1.1. *Introducing $h(n) := C/\log \log n$ (for some constant $C \approx 1.066$ given in Lemma 4.1 in Section 4), one has for $n \geq 57000$:*

$$A_3(n) \leq 10n^{\frac{1}{2} + \frac{13}{4}h(n)} \log n.$$

In order to prove this result in Section 4, we give in Lemma 3.1 of Section 3 a parametrization of the solutions to (1.1). Furthermore, thanks to this parametrization lemma, in Section 5 we prove bounds on

$$f_a(n) := \#\left\{ (m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}$$

and

$$F(n) := \# \left\{ (a, m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}.$$

Note that $f_a(n)$ counts the number of representations of a/n as an Egyptian fraction of length 3, while $F(n)$ counts all possible Egyptian fractions of length 3 with denominator n . We also include a formula and numerical tables in Section 2.

2. A FORMULA AND SOME NUMERICS

In this section, we give the first few values of our main sequences and some closed-form formulas. In addition to the sequences $A_k(n)$ which count the integers a which are solutions of the Egyptian fraction Diophantine equation

$$\frac{a}{n} = \sum_{i=1}^k \frac{1}{m_i} \quad \text{for some positive integers } m_1, \dots, m_k, \quad (2.1)$$

we shall also make use of some auxiliary sequences, $A_k^*(n)$, which consist of the number of integers a which are solutions of Equation (2.1), with the additional constraint that a is coprime to n .

The sequences $A_k(n)$ and $A_k^*(n)$ are easily computed via an exhaustive search. Some values can be more directly computed via the following closed-form formulas.

Proposition 2.1. *For any fixed prime p , the number of fractions a/p and the number of irreducible fractions a/p which can be written as a sum of two unit fractions are given by*

$$A_2(p) = 2 + d(p+1) \quad \text{and} \quad A_2^*(p) = d(p+1),$$

where $d(n) = \sum_{d|n} 1$ denotes as usual the number of divisors of n .

Proof. First, if a is any divisor of $n+1$, say $a = (n+1)/f$, then one has the decomposition $a/n = 1/(nf) + 1/f$. Let us now prove that all the decompositions are of this type, whenever $n = p$ is prime and $\gcd(a, n) = 1$. Equation (2.1) can be rewritten as $am_1m_2 = n(m_1+m_2)$. As $\gcd(a, n) = 1$, this is forcing $n \mid m_1m_2$. This gives that m_1 or m_2 is a multiple of p . Without loss of generality, say $m_1 = pf$. Thus one has $am_1m_2 = n(pf+m_2)$, i.e., $afm_2 = pf+m_2$, which implies $f \mid m_2$. Setting $m_2 = fg$ and simplifying, one gets $afg = (p+g)$, so $g \mid p$. As p is prime, either one has $g = 1$, which leads to $af = p+1$ (and thus a is any divisor of $p+1$), or one has $g = p$, which leads to $afp = 2p$ (and thus $a = 1$ or $a = 2$). Altogether, this gives $d(p+1)$ possible values for a , all actually leading to a legitimate Egyptian fraction decomposition of a/p . This proves $A_2^*(p) = d(p+1)$.

Now, consider Equation (2.1) with $n = p$ (where p is prime) and $\gcd(a, p) \neq 1$. This gives exactly two additional decompositions: $\frac{a}{p} = \frac{1}{2} + \frac{1}{2}$ (for $a = p$) and $\frac{a}{p} = \frac{1}{1} + \frac{1}{1}$ (for $a = 2p$). Thus, one has $A_2(p) = 2 + A_2^*(p) = 2 + d(p+1)$. \square

Unfortunately, there is no such simple formula for composite n . The obstruction comes from the fact that the factors of n spread between m_1 and m_2 (like in the above proof) and this leads to a more intricate disjunction of cases too cumbersome to be captured by a simple formula.

n	$A_2(n)$	$A_3(n)$	n	$A_2(n)$	$A_3(n)$	n	$A_2(n)$	$A_3(n)$	n	$A_2(n)$	$A_3(n)$
1	2	3	26	15	36	51	20	58	76	30	84
2	4	6	27	18	41	52	27	68	77	25	77
3	5	8	28	23	49	53	10	36	78	39	101
4	7	11	29	10	27	54	35	82	79	12	46
5	6	11	30	29	58	55	24	66	80	49	118
6	10	16	31	8	28	56	36	85	81	28	81
7	6	13	32	23	51	57	21	62	82	18	62
8	11	19	33	18	44	58	18	54	83	14	52
9	10	19	34	17	42	59	14	41	84	60	139
10	12	22	35	20	49	60	51	109	85	22	79
11	8	16	36	34	69	61	6	33	86	19	65
12	17	29	37	6	28	62	18	57	87	25	79
13	6	18	38	17	45	63	33	86	88	39	106
14	13	26	39	20	51	64	32	82	89	14	49
15	14	29	40	33	71	65	22	69	90	58	138
16	16	31	41	10	31	66	36	89	91	20	80
17	8	21	42	34	74	67	8	40	92	29	89
18	20	38	43	8	32	68	30	80	93	21	77
19	8	22	44	25	61	69	25	71	94	21	70
20	21	41	45	28	69	70	39	98	95	24	83
21	17	37	46	17	48	71	14	44	96	59	143
22	14	32	47	12	36	72	54	121	97	8	47
23	10	25	48	41	87	73	6	38	98	32	98
24	27	51	49	14	48	74	17	59	99	36	107
25	12	33	50	27	67	75	33	91	100	48	128

TABLE 1. Number $A_k(n)$ of integers a which are solutions of the Egyptian fraction Diophantine equation $\frac{a}{n} = \frac{1}{m_1} + \dots + \frac{1}{m_k}$, for $k = 2, 3$ and $n = 1, \dots, 100$. The sequences $A_2(n)$ and $A_3(n)$ are OEIS A308219 and OEIS A308221 in the On-Line Encyclopedia of Integer Sequences.

n	$A_2^*(n)$	$A_3^*(n)$	n	$A_2^*(n)$	$A_3^*(n)$	n	$A_2^*(n)$	$A_3^*(n)$	n	$A_2^*(n)$	$A_3^*(n)$
1	2	3	26	7	15	51	9	32	76	10	34
2	2	3	27	8	22	52	9	27	77	13	51
3	3	5	28	7	18	53	8	33	78	7	27
4	3	5	29	8	24	54	7	22	79	10	43
5	4	8	30	6	13	55	12	42	80	11	35
6	3	5	31	6	25	56	9	28	81	10	40
7	4	10	32	7	20	57	10	35	82	6	28
8	4	8	33	7	23	58	6	24	83	12	49
9	5	11	34	7	18	59	12	38	84	12	34
10	4	8	35	10	28	60	9	24	85	10	50
11	6	13	36	7	18	61	4	30	86	9	30
12	4	8	37	4	25	62	8	26	87	12	47
13	4	15	38	7	20	63	11	38	88	10	37
14	5	10	39	11	28	64	9	31	89	12	46
15	5	13	40	8	22	65	12	43	90	10	29
16	5	12	41	8	28	66	9	24	91	10	52
17	6	18	42	7	19	67	6	37	92	9	36
18	5	11	43	6	29	68	10	33	93	10	44
19	6	19	44	8	24	69	12	41	94	7	31
20	6	14	45	9	29	70	8	28	95	12	53
21	8	19	46	5	20	71	12	41	96	11	36
22	4	13	47	10	33	72	10	30	97	6	44
23	8	22	48	9	24	73	4	35	98	11	37
24	6	14	49	8	35	74	9	28	99	13	52
25	6	22	50	9	23	75	13	40	100	12	42

TABLE 2. Number $A_k^*(n)$ of integers a which are solutions of the Egyptian fraction Diophantine equation $\frac{a}{n} = \frac{1}{m_1} + \dots + \frac{1}{m_k}$ (with a coprime to n), for $k = 2, 3$ and $n = 1, \dots, 100$. The sequences $A_2^*(n)$ and $A_3^*(n)$ are OEIS A308220 and OEIS A308415 in the On-Line Encyclopedia of Integer Sequences.

3. A PARAMETRIZATION LEMMA

The proof of Theorem 1.1 is based on the following lemma which characterizes the solutions of Equation (3.1) below for $k = 3$. A similar (but simpler) characterization for $k = 2$ appears as Lemma 1 in [14] or in [35, 4, 3]; see also [38] for another existence criterion when $a = 4$.

Lemma 3.1 (Parametrization lemma). *Consider an Egyptian fraction decomposition of the irreducible fraction a/n :*

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \dots + \frac{1}{m_k} \quad (\text{with } \gcd(a, n) = 1 \text{ and } k = 3)^2. \tag{3.1}$$

Then there exist $2k$ integers $D_1, \dots, D_k, v_1, \dots, v_k$ with

- (i) $\text{lcm}[D_1, \dots, D_k] \mid n$ and $\gcd(D_1, \dots, D_k) = 1$;
 - (ii) $av_1 \cdots v_k \mid \sum_{j=1}^k D_j v_j$ and $\gcd(v_i, D_j v_j) = 1$ when $i \neq j$,
- and the denominators of the Egyptian fractions are given by

$$m_i = \frac{n \sum_{j=1}^k D_j v_j}{a D_i v_i} \quad \text{for } i = 1, \dots, k. \tag{3.2}$$

Conversely, if conditions (i)–(ii) are fulfilled, then the m_i 's defined via (3.2) are integers, and denominators of k unit fractions summing to a/n .

Remark 3.2. This decomposition may not be unique. For example, both

$$\begin{aligned} (D_1, D_2, D_3, v_1, v_2, v_3) &= (3, 1, 1, 3, 2, 1) \quad \text{and} \\ (D_1, D_2, D_3, v_1, v_2, v_3) &= (9, 1, 1, 1, 2, 1) \end{aligned}$$

correspond to the decomposition $\frac{2}{27} = \frac{1}{18} + \frac{1}{81} + \frac{1}{162}$.

Remark 3.3. It may be tempting to state the very same lemma for $k > 3$. However, this does not work: indeed, already for $k = 4$ one may have denominators m_i such that there are no tuples of D_k 's, v_k 's satisfying (i) and (ii). This is e.g. the case for

$$\frac{1}{13} = \frac{1}{14} + \frac{1}{364} + \frac{1}{365} + \frac{1}{132860}.$$

Only the converse direction works for any k : if the tuples of D_k 's and v_k 's do exist, then they give a decomposition.

Proof of Lemma 3.1. Let $g = \gcd(m_1, m_2, m_3)$. Write $m_i = gm'_i$ for $i = 1, 2, 3$. So, $\gcd(m'_1, m'_2, m'_3) = 1$. We get

$$\frac{ag}{n} = \frac{1}{m'_1} + \frac{1}{m'_2} + \frac{1}{m'_3}.$$

Further, the left-hand side fraction gets irreducible by simplifying it via the factorizations

$$g = \gcd(g, n)g' \quad \text{and} \quad n = \gcd(g, n)n',$$

²Though Lemma 3.1 holds verbatim for $k = 3$ only, we state it with the parameter k as we also discuss variations of this lemma for different values of k .

so one obtains

$$\frac{ag'}{n'} = \frac{1}{m'_1} + \frac{1}{m'_2} + \frac{1}{m'_3}. \tag{3.3}$$

Put

$$P = \prod_{p|m'_1m'_2m'_3} p.$$

Note that no prime factor p of P divides all three of m'_1, m'_2, m'_3 . Split them as follows:

- Q is the largest divisor of P formed with primes p that divide just one of the m'_1, m'_2, m'_3 .
- R is the largest divisor of P formed with primes p which divide two of m'_1, m'_2, m'_3 , say m'_i and m'_j but³ $\nu_p(m'_i) \neq \nu_p(m'_j)$.
- $S = P/(QR)$ (i.e., the product of the remaining primes, those having the same valuation in two of the m_i 's).

For $i = 1, 2, 3$, write

$$m'_i = q_i r_i s_i, \tag{3.4}$$

where q_i is formed only of primes from Q , r_i is formed of primes from R , and s_i is formed of primes from S . We show that

$$q_1 q_2 q_3 \operatorname{lcm}[r_1, r_2, r_3] \mid n'; \tag{3.5}$$

$$s_1 = u_2 u_3, \quad s_2 = u_1 u_3, \quad s_3 = u_1 u_2 \quad \text{for some integers } u_1, u_2, u_3. \tag{3.6}$$

Now, rewrite (3.3) as

$$\frac{ag'}{n'} = \frac{1}{q_1 r_1 s_1} + \frac{m'_2 + m'_3}{m'_2 m'_3} = \frac{m'_2 m'_3 + q_1 r_1 s_1 (m'_2 + m'_3)}{q_1 r_1 s_1 m'_2 m'_3}.$$

On the right-hand side, q_1 is coprime to $m'_2 m'_3 + q_1 r_1 s_1 (m'_2 + m'_3)$ (because by the definition of Q , q_1 is coprime to $m'_2 m'_3$). So, it must be the case that $q_1 \mid n'$, as ag'/n' is irreducible. Similarly, q_2, q_3 divide n' and since any two of the q_i 's are mutually coprime, it follows that $q_1 q_2 q_3 \mid n'$. Consider next r_1 . It is formed by primes from R , so for each prime factor p of r_1 there exists $i \in \{2, 3\}$ such that $p \mid r_i$. Say $i = 2$, then we introduce $\alpha_1 := \nu_p(r_1)$ and $\alpha_2 := \nu_p(r_2)$, with $\alpha_2 > \alpha_1$ (these two assumptions, $i = 2$ and $\alpha_i > \alpha_1$, cause no loss of generality to this proof: the other cases would be handled similarly). Now, writing $m'_1 = p^{\alpha_1} m''_1$, $m'_2 = p^{\alpha_2} m''_2$, we have

$$\frac{ag'}{n'} = \frac{1}{p^{\alpha_1} m''_1} + \frac{1}{p^{\alpha_2} m''_2} + \frac{1}{m'_3} = \frac{m'_3 m''_2 p^{\alpha_2 - \alpha_1} + m''_1 m'_3 + p^{\alpha_2} m''_2 m''_1}{p^{\alpha_2} m''_1 m''_2 m'_3}.$$

On the right, p^{α_2} is coprime to the numerator $m'_3 m''_2 p^{\alpha_2 - \alpha_1} + m''_1 m'_3 + p^{\alpha_2} m''_2 m''_1$. Thus, $p^{\alpha_2} \mid n'$. Note that $p^{\alpha_2} = \operatorname{lcm}[p^{\alpha_1}, p^{\alpha_2}]$. Proceeding one prime at a time for the primes dividing r_1, r_2, r_3 , we get to the conclusion that $\operatorname{lcm}[r_1, r_2, r_3] \mid n'$. Since $q_1 q_2 q_3$ and $\operatorname{lcm}[r_1, r_2, r_3]$ have no prime factor in common, and since $q_1 q_2 q_3 \mid n'$, this proves Formula (3.5).

³We use the classical notation $\nu_p(m)$ for the exponent of p in the factorization of m .

Now, Formula (3.6) is a simple linear algebra problem. Namely, for $i_1 \in \{1, 2, 3\}$, let i_2, i_3 be such that $\{1, 2, 3\} = \{i_1, i_2, i_3\}$ and write

$$s_{i_1} = s_{i_1}^{(i_2)} s_{i_1}^{(i_3)}, \quad \text{where } s_{i_1}^{(i_2)} = \prod_{p|\gcd(s_{i_1}, s_{i_2})} p^{\nu_p(s_{i_1})}.$$

The condition that $\nu_p(s_{i_1}) = \nu_p(s_{i_2})$ if $p \mid \gcd(s_{i_1}, s_{i_2})$ shows that $s_{i_1}^{(i_2)} = s_{i_2}^{(i_1)}$ for all $i_1 \neq i_2$. This gives Formula (3.6).

Now, rewrite (3.3) using (3.4) and (3.5): putting $n'' = n' / (q_1 q_2 q_3 \text{lcm}[r_1, r_2, r_3])$, we get

$$\frac{ag'}{n''} = \frac{q_2 q_3 \frac{\text{lcm}[r_1, r_2, r_3]}{r_1} u_1 + q_1 q_3 \frac{\text{lcm}[r_1, r_2, r_3]}{r_2} u_2 + q_1 q_2 \frac{\text{lcm}[r_1, r_2, r_3]}{r_3} u_3}{u_1 u_2 u_3}.$$

It is clear that u_1, u_2, u_3 are mutually coprime since any common prime factor of two of them will divide all three of m'_1, m'_2, m'_3 . So, write $u_i = d_i u'_i$, where d_i is the largest factor of u_i whose prime factors divide n'' and u_i , and where u'_i is coprime to n'' . Similarly, write $n'' = d'_1 d'_2 d'_3 n'''$, where d'_i is the largest factor of n'' whose prime factors divide d_i . We then get

$$\frac{ag'}{n'''} \prod_{j=1}^3 \left(\frac{d_j}{d'_j} \right) = \frac{q_2 q_3 \frac{\text{lcm}[r_1, r_2, r_3]}{r_1} u_1 + q_1 q_3 \frac{\text{lcm}[r_1, r_2, r_3]}{r_2} u_2 + q_1 q_2 \frac{\text{lcm}[r_1, r_2, r_3]}{r_3} u_3}{u'_1 u'_2 u'_3}. \tag{3.7}$$

On the right, $u'_1 u'_2 u'_3$ is divisible only by primes coprime to n'' so $u'_1 u'_2 u'_3$ divides the numerator

$$q_2 q_3 \frac{\text{lcm}[r_1, r_2, r_3]}{r_1} u_1 + q_1 q_3 \frac{\text{lcm}[r_1, r_2, r_3]}{r_2} u_2 + q_1 q_2 \frac{\text{lcm}[r_1, r_2, r_3]}{r_3} u_3.$$

So, the left-hand side of (3.7) is an integer. This shows that $d'_i \mid d_i$ for $i = 1, 2, 3$ and $n''' = 1$ (since the four quantities d_i/d'_i for $i = 1, 2, 3$ and n''' are rational numbers supported on mutually disjoint sets of prime factors of n'' and ag' is coprime to n''). Thus, in fact $n'' = d'_1 d'_2 d'_3$ and we can write $u_i = d'_i v_i$, where $v_i = (d_i/d'_i) u'_i$. Hence, we get

$$ag' = \frac{q_2 q_3 \frac{\text{lcm}[r_1, r_2, r_3]}{r_1} d'_1 v_1 + q_1 q_3 \frac{\text{lcm}[r_1, r_2, r_3]}{r_2} d'_2 v_2 + q_1 q_2 \frac{\text{lcm}[r_1, r_2, r_3]}{r_3} d'_3 v_3}{v_1 v_2 v_3}. \tag{3.8}$$

Putting (for $i \in \{1, 2, 3\}$)

$$D_i := \frac{q_1 \cdots q_3}{q_i} \frac{\text{lcm}[r_1, r_2, r_3]}{r_i} d'_i,$$

we have that each D_i is a divisor of $n' = n / \gcd(g, n)$, so

$$\text{lcm}[D_1, D_2, D_3] \mid q_1 q_2 q_3 \text{lcm}[r_1, r_2, r_3] d'_1 d'_2 d'_3 = n' d'_1 d'_2 d'_3 = n' n'' = n,$$

which is part of condition (i) of our parametrization lemma (Lemma 3.1). The second part of condition (i) is now easy. Indeed, $\gcd(D_1, D_2, D_3)$ cannot be divisible by primes from either Q or R , and d'_i is coprime to d'_j (since d'_i and d'_j are supported on primes dividing d_i and d_j which are divisors of u_i and u_j , respectively), which

shows that indeed $\gcd(D_1, D_2, D_3) = 1$. Rewriting Equation (3.8) in terms of these D_i 's gives

$$av_1v_2v_3 \mid D_1v_1 + D_2v_2 + D_3v_3,$$

which is the first part of condition (ii) of our parametrization lemma (Lemma 3.1). The second part is also clear since v_i is a divisor of u_i , which is coprime to u_j for any $j \neq i$ with $\{i, j\} \subset \{1, 2, 3\}$. The converse direction is obvious: if one has the divisibility conditions (i)–(ii), it is clear that the m_i 's defined via (3.2) are integers and satisfy Equation (3.1). \square

4. AN EXPLICIT BOUND ON $A_3(n)$

To prove explicit results, we will use the following lemma from [32].

Lemma 4.1 (Nicolas and Robin, [32]). *Let $d(n)$ be the number of divisors of n and let $h(n) := C/\log \log n$, where $C := \frac{2 \log(48) \log(\log(6983776800))}{\log(6983776800)} \approx 1.066$. Then*

$$d(n) \leq n^{h(n)}.$$

Because of this lemma, we will use $h(n) = C/\log \log n$ for the rest of the paper.

Proof of Theorem 1.1. Consider

$$\mathcal{A}_3^*(n) = \left\{ a : \gcd(a, n) = 1, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\} \quad \text{and} \quad A_3^*(n) = \#\mathcal{A}_3^*(n).$$

From the parametrization lemma (Lemma 3.1), if $a \in \mathcal{A}_3^*(n)$, there exist integers $D_1, D_2, D_3, v_1, v_2, v_3$ satisfying $D_i \mid n$, $v_1v_2v_3 \mid D_1v_1 + D_2v_2 + D_3v_3$, and

$$a \mid \frac{D_1v_1 + D_2v_2 + D_3v_3}{v_1v_2v_3}.$$

Let A be such that $Av_1v_2v_3 = D_1v_1 + D_2v_2 + D_3v_3$. Then $a \mid A$.

First suppose $A \leq n^{1/2+\alpha}$. Then $A_3^*(n)$ is bounded above by

$$\begin{aligned} \sum_{A \leq n^{1/2+\alpha}} d(A) &\leq n^{1/2+\alpha} \log(n^{1/2+\alpha}) + n^{1/2+\alpha} \\ &= \left(\frac{1}{2} + \alpha\right) n^{1/2+\alpha} \log n + n^{1/2+\alpha}. \end{aligned}$$

Now, suppose $A > n^{1/2+\alpha}$. Fix D_1, D_2, D_3 as divisors of n . There are $d(n)^3 \leq n^{3h(n)}$ ways of doing this. Suppose $v_1, v_2 \leq v_3$; one then has

$$Av_1v_2v_3 = D_1v_1 + D_2v_2 + D_3v_3 \leq (D_1 + D_2 + D_3)v_3 \leq 3nv_3.$$

Therefore $m := v_1v_2 \leq 3n^{1/2-\alpha}$. Once v_1, v_2 are chosen, there are at most $d(D_1v_1 + D_2v_2)$ choices of v_3 (since $v_1v_2v_3 \mid D_1v_1 + D_2v_2 + D_3v_3$). We have

$$D_1v_1 + D_2v_2 \leq 2nv_3 \leq 6n^{3/2-\alpha}.$$

Therefore

$$d(D_1v_1 + D_2v_2) \leq 6^{h(n)} n^{3h(n)/2-\alpha h(n)}.$$

We can thus bound the contribution of the a 's appearing when $A > n^{1/2+\alpha}$ by

$$3n^{3h(n)}6^{h(n)}n^{3h(n)/2-\alpha h(n)} \sum_{m \leq 3n^{1/2-\alpha}} d(m).$$

Given that

$$\begin{aligned} \sum_{m \leq 3n^{1/2-\alpha}} d(m) &\leq 3n^{\frac{1}{2}-\alpha} \log(3n^{\frac{1}{2}-\alpha}) + 3n^{\frac{1}{2}-\alpha} \\ &= 3n^{\frac{1}{2}-\alpha} \left(\frac{1}{2} - \alpha \right) \log n + 3n^{\frac{1}{2}-\alpha} \log(3e) \end{aligned}$$

and that $6^{h(n)} \leq \frac{20}{9}$ for $n \geq 57000$, we get

$$\begin{aligned} A_3^*(n) &\leq 10n^{\frac{1}{2}+\frac{9}{2}h(n)-\alpha-\alpha h(n)} \log n + 20n^{\frac{1}{2}+\frac{9}{2}h(n)-\alpha-\alpha h(n)} \log(3e) \\ &\quad - 20n^{\frac{1}{2}+\frac{9}{2}h(n)-\alpha-\alpha h(n)} \log n + \left(\frac{1}{2} + \alpha \right) n^{\frac{1}{2}+\alpha} + n^{\frac{1}{2}+\alpha}. \end{aligned}$$

Choose $\alpha = \frac{9}{4+2h(n)}h(n) \leq \frac{9}{4}h(n)$. We then have

$$\begin{aligned} A_3^*(n) &\leq n^{\frac{1}{2}+\frac{9}{4}h(n)} \left(10 \log n + 20 \log(3e) - 20\alpha \log n + \frac{1}{2} + \alpha + 1 \right) \\ &\leq 10n^{\frac{1}{2}+\frac{9}{4}h(n)} \log n. \end{aligned}$$

For the last inequality we use that, for $n \geq 20$, $h(n) \leq 1$, so

$$\alpha \geq \frac{9}{2+2h(n)}h(n) \geq \frac{9}{4}h(n), \quad \text{and} \quad \alpha \leq \frac{9}{2}h(n) \leq \frac{9}{2}.$$

For $n > e$, $\log n / \log \log n \geq e$; therefore

$$20\alpha \log n \geq 45h(n) \log n > 45e \cdot C > 20 \log(3e) + \frac{3}{2} + \frac{9}{2}.$$

Hence, for $n \geq 57000$,

$$A_3^*(n) \leq 10n^{\frac{1}{2}+\frac{9}{4}h(n)} \log n.$$

This gives the bound in Theorem 1.1:

$$A_3(n) = \sum_{d|n} A_3^*(d) \leq 10n^{\frac{1}{2}+\frac{13}{4}h(n)} \log n. \quad \square$$

Corollary 4.2. For $n \geq 10^{10^{23}}$,

$$A_3(n) < \frac{1}{100}n^{\frac{1}{2}+\frac{1}{15}}.$$

Proof. When $n \geq 10^{10^{23}}$,

$$\frac{13}{4}h(n) + \frac{\log \log n}{\log n} + \frac{\log 1000}{\log n} < \frac{1}{15}.$$

Therefore, using Theorem 1.1, we get

$$A_3(n) \leq 10n^{\frac{1}{2}+\frac{13}{4}h(n)} \log n = \frac{1}{100}n^{\frac{1}{2}+\frac{13}{4}h(n)+\frac{\log \log n}{\log n}+\frac{\log 1000}{\log n}} < \frac{1}{100}n^{\frac{1}{2}+\frac{1}{15}}. \quad \square$$

5. ON THE NUMBER OF LENGTH 3 REPRESENTATIONS

In this section we study

$$f_a(n) = \# \left\{ (m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\},$$

and

$$F(n) = \# \left\{ (a, m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}.$$

Theorem 5.1.

$$f_a(n) \leq n^\varepsilon \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right). \tag{5.1}$$

Remark 5.2. Choosing $\rho = 1/3 + (2/3)(\log a / \log n)$ to balance between the two estimates, we get that

$$f_a(n) \ll \frac{n^{2/3+\varepsilon}}{a^{2/3}}. \tag{5.2}$$

In particular, $f_a(n) \ll n^{2/3+\varepsilon}$ uniformly in a . It is interesting to compare this bound with the one obtained for $a = 4$ by Elsholtz and Tao in [15, Proposition 1.7]: For primes p , they get as $p \rightarrow \infty$

$$f_4(p) \ll p^{3/5+o(1)}.$$

Now, if one just counts the triplets D_1, D_2, D_3 satisfying the condition (i) in the parametrization lemma 3.1, it is possible to adapt their reasoning to obtain a similar bound for any a , when p is replaced by a composite integer n . Our bound (5.2) is slightly worse (it gives the exponent $2/3 + \varepsilon$ instead of $3/5 + \varepsilon$) but it works for fixed or bounded a and it allows us to get better exponents for larger a (when $a \sim n^c$ with $1/10 < c < 1$).

Proof of Theorem 5.1. We use the parametrization lemma (Lemma 3.1). Indeed, there are divisors D_1, D_2, D_3 of n such that

$$a \mid (D_1v_1 + D_2v_2 + D_3v_3)/(v_1v_2v_3).$$

Fix D_1, D_2, D_3 . They can be fixed in at most $d(n)^3 \ll n^\varepsilon$ ways. Assume $v_1 \leq v_2 \leq v_3$ (this just possibly creates a factor 6 for the number of solutions). Furthermore, we define the integers A and b by $A := ab = (D_1v_1 + D_2v_2 + D_3v_3)/(v_1v_2v_3)$. Note that

$$\frac{a}{n} = \frac{1}{b(n/D_1)v_2v_3} + \frac{1}{b(n/D_2)v_1v_3} + \frac{1}{b(n/D_3)v_1v_2} := \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}$$

and that

$$Av_1v_2v_3 = D_1v_1 + D_2v_2 + D_3v_3 \leq 3nv_3.$$

Now, let ρ be a parameter to be fixed later. First, let us assume that $A > n^\rho$. Then $v_1v_2 \leq 3n^{1-\rho}$, so the pair (v_1, v_2) can be chosen in at most $n^{1-\rho+\varepsilon}$ ways. Having chosen (v_1, v_2) , v_3 is a divisor of $D_1v_1 + D_2v_2$, so it can be chosen in at most n^ε ways, and after that everything is determined, so A is unique. Note that such A might not end up being divisible by the number a we started with so not all such

solutions will contribute to $f_a(n)$. This gives the second part of the right-hand side inequality in the statement of the theorem. So, we may assume that $ab \leq n^\rho$, that is,

$$b \leq \frac{n^\rho}{a},$$

and then we have $v_1 \leq 3(n/ab)^{1/2}$. Fix v_1 ; it can be fixed in at most $3(n/ab)^{1/2}$ ways. Now put $A_1 := Av_1 = (ab)v_1$, $B_1 := D_1v_1$, and note that they are fixed. Further,

$$A_1v_2v_3 = B_1 + D_2v_2 + D_3v_3$$

and the only variables are v_2, v_3 . The above can be rewritten as

$$A_1v_2v_3 - D_2v_2 - D_3v_3 + D_3D_2/A_1 = B_1 + (D_2D_3/A_1),$$

or equivalently as

$$(A_1v_3 - D_2)(A_1v_2 - D_3) = A_1B_1 + D_2D_3.$$

It thus follows that $A_1v_2 - D_3$ can be chosen in $d(A_1B_1 + D_2D_3) \ll n^\varepsilon$ ways and then v_3 is uniquely determined. We thus get that for fixed b, v_1, D_1, D_2, D_3 there are n^ε possibilities for (v_2, v_3) . Summing over v_1 , it follows that there are $\ll n^\varepsilon(n/ab)^{1/2}$ possibilities. Summing over $b \leq n^\rho/a$, we get a count of

$$\frac{n^{1/2+\varepsilon}}{a^{1/2}} \sum_{b \leq 3n^\rho/a} \frac{1}{b^{1/2}} \ll \frac{n^{1/2+2\varepsilon}}{a^{1/2}} \int_1^{3n^\rho/a} \frac{dt}{t^{1/2}} \ll \frac{n^{1/2+\rho/2+2\varepsilon}}{a}.$$

Thus,

$$f_a(n) \leq n^\varepsilon \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right),$$

which is (5.1). □

Theorem 5.3. *Let $F(n) = \sum_a f_a(n)$ be the count of all (a, m_1, m_2, m_3) such that $\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}$. We have $F(n) \ll n^{5/6+\varepsilon}$.*

Proof. Let $\varepsilon > 0$. Note that

$$\sum_{a \leq n^\alpha} f_a(n) \leq n^{2/3+\alpha+\varepsilon}. \tag{5.3}$$

This estimate follows from using $\rho = 1/3$ in

$$f_a(n) \leq n^\varepsilon \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right) \leq n^\varepsilon \left(n^{1/2+\rho/2} + n^{1-\rho} \right).$$

Now note that

$$\sum_{n^\alpha \leq a \leq n^\beta} f_a(n) \leq n^{2/3+\frac{3\beta-2\alpha}{3}+\varepsilon}. \tag{5.4}$$

This follows from using $\rho = \frac{2\alpha+1}{3}$ in

$$f_a(n) \leq n^\varepsilon \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right) \leq n^\varepsilon \left(n^{1/2+\rho/2-\alpha} + n^{1-\rho} \right).$$

Finally note that

$$\sum_{a \geq n^\gamma} f_a(n) \leq n^{1/2 + \frac{2-2\gamma}{3} + \varepsilon}. \tag{5.5}$$

This follows from using $\rho = \frac{2\gamma+1}{3}$ and $A_3(n) \ll n^{1/2+\varepsilon}$ in

$$f_a(n) \leq n^\varepsilon \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right) \leq n^\varepsilon \left(n^{1/2+\rho/2-\gamma} + n^{1-\rho} \right).$$

Now let $x_1 = 1/6$ and $x_k = \frac{1}{6} + \frac{2}{3}x_{k-1}$ for $k \geq 2$. Then $x_k = \frac{1}{2} - \frac{(2/3)^k}{2}$. Let i be fixed such that $(2/3)^i < \varepsilon$. Consider the intervals

$$[1, n^{x_1}], [n^{x_1}, n^{x_2}], \dots, [n^{x_{i-1}}, n^{x_i}], [n^{x_i}, n^{1/2}], [n^{1/2}, \infty).$$

From (5.3), (5.4), and (5.5), we have that

$$\sum_{a \in I} f_a(n) \ll n^{5/6+\varepsilon},$$

for any interval $I \neq [n^{x_i}, n^{1/2}]$. Using (5.4) and our choice of i , we get

$$\sum_{n^{x_i} \leq a \leq n^{1/2}} f_a(n) \ll n^{5/6 + \frac{4}{3}\varepsilon}.$$

Therefore

$$F(n) \leq (i + 1)n^{5/6+\varepsilon} + n^{5/6 + \frac{4}{3}\varepsilon} \leq n^{5/6+2\varepsilon}. \quad \square$$

Problem 5.4. The first values for which $F(n) < n$ are $F(8821) = 8590$, $F(11161) = 10270$, and $F(11941) = 10120$. It is an open problem to find the largest n such that $F(n) > n$. We can however show that such an n is smaller than $10^{10^{23}}$.

Theorem 5.5. For $n \geq 10^{10^{23}}$, $F(n) < \frac{1}{10}n$.

The proof of this theorem requires the explicit upper bound for $A_3(n)$ from Corollary 4.2. It also requires the following explicit version of Theorem 5.1:

Theorem 5.6. Let $1/3 \leq \rho$ and $n \geq 11000$. Then

$$f_a(n) \leq 6n^{5h(n)} \left(6\sqrt{2} \frac{n^{1/2+\rho/2}}{a} 10^{h(n)} + \frac{3}{2} n^{1-\rho} \log n 6^{h(n)} \right).$$

Proof. We revisit the proof of Theorem 5.1 with more technical bounds. First, the assumption that $v_1 \leq v_2 \leq v_3$ just introduces (at most) a factor 6 for the number of solutions. Then, D_1, D_2, D_3 can be fixed in at most $d(n)^3 \leq n^{3h(n)}$ ways (this last bound follows from the Nicolas–Robin result, see Lemma 4.1).

Now let us assume that $A > n^\rho$ in

$$Av_1v_2v_3 = D_1v_1 + D_2v_2 + D_3v_3 \leq 3nv_3.$$

Then $v_1v_2 \leq 3n^{1-\rho}$, so the pair (v_1, v_2) can be chosen in at most

$$\begin{aligned} \sum_{v_1 \leq \sqrt{3n^{1-\rho}}} \sum_{v_2 \leq 3n^{1-\rho}/v_1} 1 &\leq \sum_{v_1 \leq \sqrt{3n^{1-\rho}}} \frac{3n^{1-\rho}}{v_1} \\ &\leq 3n^{1-\rho} \log(\sqrt{3}n^{1/2-\rho/2}) + 3n^{1-\rho} \\ &\leq \frac{3}{2}n^{1-\rho} \log n - \frac{3\rho}{2}n^{1-\rho} \log n + \frac{3}{2}n^{1-\rho} \log 3 + 3n^{1-\rho} \\ &\leq \frac{3}{2}n^{1-\rho} \log n \end{aligned}$$

ways. The last step of the inequality follows from using that $\rho \geq 1/3$ and $n \geq 11000$.

Having chosen (v_1, v_2) , v_3 is a divisor of $D_1v_1 + D_2v_2$, so it can be chosen in $d(D_1v_1 + D_2v_2)$ ways, and after that everything is determined, so A is unique. Now

$$D_1v_1 + D_2v_2 \leq (D_1 + D_2)v_2 \leq (2n)(3n) = 6n^2,$$

so $d(D_1v_1 + D_2v_2) \leq (6n^2)^{h(n)}$. This gives the second part of the right-hand side inequality in the statement of the theorem (after factoring out an $n^{2h(n)}$).

For the first part, we may assume that $ab \leq n^\rho$, and then we have $v_1 \leq 3(n/ab)^{1/2}$. Since

$$A_1B_1 + D_2D_3 = abD_1v_1^2 + D_2D_3 \leq abn \left(9\frac{n}{ab}\right) + n^2 = 10n^2,$$

we get

$$d(A_1B_1 + D_2D_3) \leq (10n^2)^{h(n)}.$$

We have thus bounded the number of possibilities for (v_2, v_3) given fixed b, v_1, D_1, D_2, D_3 . To finish our estimate we use that $v_1 \leq 3(n/ab)^{1/2}$ and that

$$\sum_{b \leq n^\rho/a} \frac{1}{b^{1/2}} \leq \int_1^{\frac{n^\rho}{a}+1} \frac{1}{t^{1/2}} dt \leq \int_1^{2\frac{n^\rho}{a}} \frac{1}{t^{1/2}} dt \leq 2\sqrt{2} \frac{n^{\rho/2}}{a^{1/2}}. \quad \square$$

Corollary 5.7. *If $n \geq 10^{10^{23}}$, then*

$$f_a(n) < \frac{1}{100}n^{\frac{1}{10}} \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right).$$

Proof. For $n \geq 10^6$, $6\sqrt{2} \cdot 10^{h(n)} \leq 2 \log n$, and for $n \geq 10^{334}$, $\frac{3}{2}6^{h(n)} \leq 2$. Therefore

$$f_a(n) \leq 12(\log n)n^{5h(n)} \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right).$$

But we have, for $n \geq 10^{10^{23}}$,

$$12(\log n)n^{5h(n)} = \frac{1}{100}n^{5h(n) + \frac{\log \log n}{\log n} + \frac{\log 1200}{\log n}} < \frac{1}{100}n^{1/10}. \quad \square$$

We are now ready to prove Theorem 5.5.

Proof of Theorem 5.5. Let $n \geq 10^{10^{23}}$. The proof will be similar to the proof of Theorem 5.3. Applying Corollary 5.7 with $\rho = 1/3$ yields

$$\sum_{a \leq n^{7/30}} f_a(n) \leq \frac{2}{100} n^{\frac{2}{3} + \frac{7}{30} + \frac{1}{10}} = \frac{2}{100} n. \quad (5.6)$$

Now applying $\rho = 22/45$ to Corollary 5.7, we get

$$\sum_{n^{7/30} \leq a \leq n^{7/18}} f_a(n) \leq \frac{2}{100} n^{\frac{23}{45} + \frac{7}{18} + \frac{1}{10}} = \frac{2}{100} n. \quad (5.7)$$

Applying $\rho = 89/135$ to Corollary 5.7 yields

$$\sum_{n^{7/18} \leq a \leq n^{1/2}} f_a(n) \leq \frac{2}{100} n^{\frac{46}{135} + \frac{1}{2} + \frac{1}{10}} = \frac{2}{100} n^{\frac{127}{135}} < \frac{2}{100} n. \quad (5.8)$$

Applying $\rho = \frac{2}{3}$ to Corollary 5.7 and using Corollary 4.2 yields

$$\sum_{a \geq n^{1/2}} f_a(n) < \frac{2}{10000} n^{\frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15}} = \frac{2}{10000} n. \quad (5.9)$$

The proof follows from combining (5.6), (5.7), (5.8), and (5.9). \square

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Received: October 21, 2019

Accepted: December 2, 2019