Invent. math. (2021) 225:395–426
https://doi.org/10.1007/s00222-021-01030-8

Helly meets Garside and Artin

Jingyin Huang[1] · **Damian Osajda**[2,3]

Received: 7 June 2019 / Accepted: 7 January 2021 / Published online: 15 February 2021

Abstract A graph is Helly if every family of pairwise intersecting combinatorial balls has a nonempty intersection. We show that weak Garside groups of finite type and FC-type Artin groups are Helly, that is, they act geometrically on Helly graphs. In particular, such groups act geometrically on spaces with a convex geodesic bicombing, equipping them with a nonpositive-curvature-like structure. That structure has many properties of a CAT(0) structure and, additionally, it has a combinatorial flavor implying biautomaticity. As immediate consequences we obtain new results for FC-type Artin groups (in particular braid groups and spherical Artin groups) and weak Garside groups, including e.g. fundamental groups of the complements of complexified finite simplicial arrangements of hyperplanes, braid groups of well-generated complex reflection groups, and one-relator groups with non-trivial center. Among the results are: biautomaticity, existence of EZ and Tits boundaries, the Farrell–Jones conjecture, the coarse Baum–Connes conjecture, and a description of higher

✉ Damian Osajda
dosaj@math.uni.wroc.pl

Jingyin Huang
huang.929@osu.edu

[1] Department of Mathematics, The Ohio State University, 100 Math Tower, 231 W 18th Ave, Columbus, OH 43210, USA

[2] Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland

[3] Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-656 Warsaw, Poland

order homological and homotopical Dehn functions. As a means of proving the Helly property we introduce and use the notion of a (generalized) cell Helly complex.

Mathematics Subject Classification 20F65 · 20F36 · 20F67

Contents

1 Introduction

1.1 Main result

In this article we design geometric models for Artin groups and Garside groups, which are two generalizations of braid groups.

Among the vast literature on these two classes of groups, we are particularly inspired by the results concerning the Garside structure, rooted in [30,42, 52], and the small cancellation structure for certain Artin groups, exhibited in [4,5,78]. These two aspects actually have interesting interactions, yielding connections with injective metric spaces and Helly graphs.

For geodesic metric spaces, being *injective* (or, in other words, *hyperconvex* or *having the binary intersection property*) can be characterized as follows: any collection of pairwise intersecting closed metric balls has a nonempty intersection. The graph-theoretic analogues of injective metric spaces are called *Helly graphs* (Definition 3.1). Groups acting on injective metric spaces were studied in [45,69], leading to a number of results parallel to the CAT(0) setting [43,44,72]. The article [31] initiates the studies of *Helly groups*, that is, groups acting properly and cocompactly by graph automorphisms (i.e., *geometrically*) on Helly graphs.

Theorem *Weak Garside groups of finite type and Artin groups of type FC are Helly.*

The Main Theorem should be seen as providing geometric models for groups in question with a nonpositive-curvature-like structure. Consequences of this theorem are explained in Sect. 1.2.

See Sects. 4.1 and 2.2 for the definitions of weak Garside groups and Artin groups. Examples of (weak) Garside groups of finite type include:

(1) fundamental groups of the complements of complexified finite simplicial arrangements of hyperplanes [42];
(2) spherical Artin groups [30,42];
(3) braid groups of well-generated complex reflection groups [14];
(4) structure groups of non-degenerate, involutive and braided set-theoretical solutions of the quantum Yang–Baxter equation [36];
(5) one-relator groups with non-trivial center and tree products of cyclic groups [76].

It is known that Garside groups are biautomatic [35,41]. This suggests the non-positively curved (NPC) geometry of these groups. However, it is natural to ask for other notions of NPC for Garside groups in order to understand problems of coarse or continuous nature and to provide more convexity than the Garside normal form does.

A few classes of Artin groups are known to be CAT(0) [8,11,24,25,29,33,54,55]. This leads to the speculation that all Artin groups should be NPC. However, due to the difficulty of verifying the CAT(0) condition in higher dimensional situations, it is not even known whether all braid groups are CAT(0), though some evidences are presented in [46]. In particular, it is widely open whether Artin groups of type FC are CAT(0).

The Main Theorem applies to a large class of Artin groups and all finite type weak Garside groups (in particular all braid groups). Though the formulation is combinatorial, it has a continuous counterpart which is very similar to CAT(0). All Helly groups act geometrically on injective metric spaces [31], which in particular have convex geodesic bicombings. The convexity one obtains here is much stronger than biautomaticity and is weaker than CAT(0). See Sect. 1.2 for details.

On the other hand the Main Theorem equips the groups with a robust combinatorial structure—the one of Helly graphs. By a result from [31] it implies e.g. biautomaticity—an important algorithmic property not implied by CAT(0) (see [70]). While allowing to deduce ‘CAT(0)-like’ behavior as in the previous paragraph, the property of being Helly can be checked by local combinatorial conditions via a local-to-global characterization from [32], see Theorem 3.2 below.

The Main Theorem also provides a higher dimensional generalization of an old result of Pride [78] saying that 2-dimensional Artin groups of type FC are C(4)-T(4). The notion of cell Helly introduced in Sect. 3 serves as a higher dimensional counterpart of the C(4)-T(4) small cancellation condition in the sense of Lemma 3.8. We prove that the universal covers of the Salvetti complexes of higher dimensional Artin groups of type FC are cell Helly.

Let us note that 2-dimensional Artin groups of type FC are known to satisfy several different versions of NPC: small cancellation [78], CAT(0) [24], systolic [57,58], quadric [59], and Helly.

1.2 Consequences of the Main Theorem

In the Corollary below we list immediate consequences of being Helly, for groups as in the Main Theorem. In [31] it is shown that the class of Helly groups is closed under operations of free products with amalgamations over finite subgroups, graph products (in particular, direct products), quotients by finite normal subgroups, and (as an immediate consequence of the definition) taking finite index subgroups. Moreover, it is shown there that e.g. (Gromov) hyperbolic groups, CAT(0) cubical groups, uniform lattices in buildings of type $\widetilde{C}_n$, and graphical C(4)-T(4) small cancellation groups are all Helly. Therefore, the results listed below in the Corollary apply to various combinations of all those classes of groups.

Except for item (6) below (biautomaticity) all other results (1)–(5) are rather direct consequences of well known facts concerning injective metric spaces—see explanations in Sect. 1.3 below, and [31] for more details. Biautomaticity of Helly groups is one of the main results of the paper [31].

Corollary *Suppose that a group G is either a weak Garside group of finite type or an Artin group of type FC. Then the following hold.*

(1) G acts geometrically on an injective metric space X_G, and hence on a metric space with a convex geodesic bicombing. Moreover, X_G is a piecewise ℓ^∞-polyhedron complex.
(2) G admits an EZ-boundary and a Tits boundary.
(3) The Farrell–Jones conjecture with finite wreath products holds for G.
(4) The coarse Baum–Connes conjecture holds for G.
(5) The k-th order homological Dehn function and homotopical Dehn function of G are Euclidean, i.e. $f(l) \preceq l^{\frac{k+1}{k}}$.
(6) G is biautomatic.

The recent work of Kleiner and Lang [66] provides many new insights into the asymptotic geometry of injective metric spaces. Together with (1) of the above corollary, it opens up the possibility of studying Gromov's program of quasi-isometric classification and rigidity for higher dimensional Artin groups which are not right-angled. This actually serves as one of our motivations to look for good geometric models for such groups. Previous works in this direction concern mainly the right-angled or the 2-dimensional case [20,26,56,62,63,71].

We also present an alternative proof of the fact that the Salvetti complex for Artin groups of type FC is aspherical (see Theorem 5.8), which implies the $K(\pi, 1)$-conjecture for such groups. This has been previously established by Charney and Davis [34] by showing contractibility of the associated Deligne complex. Our proof does not rely on the Deligne complex.

1.3 Details and comments on items in corollary

(1) *Injective metric spaces* also known as *hyperconvex metric spaces* or *absolute retracts* (in the category of metric spaces with 1-Lipschitz maps) form an important class of metric spaces rediscovered several times in history [3,47,64] and studied thoroughly. One important feature of injective metric spaces of finite dimension is that they admit a unique convex, consistent, reversible geodesic bicombing [44, Theorems 1.1&1.2]. Recall that a *geodesic bicombing* on a metric space (X, d) is a map

$$\sigma \colon X \times X \times [0, 1] \to X,$$

such that for every pair $(x, y) \in X \times X$ the function $\sigma_{xy} := \sigma(x, y, \cdot)$ is a constant speed geodesic from x to y. We call σ *convex* if the function $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex for all $x, y, x', y' \in X$. The bicombing σ is *consistent* if $\sigma_{pq}(\lambda) = \sigma_{xy}((1-\lambda)s + \lambda t)$, for all $x, y \in X$, $0 \leq s \leq t \leq 1$, $p := \sigma_{xy}(s)$, $q := \sigma_{xy}(t)$, and $\lambda \in [0, 1]$. It is called *reversible* if $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all $x, y \in X$ and $t \in [0, 1]$.

It is shown in [31] that if a group G acts geometrically on a Helly graph Γ then it acts geometrically on an *injective hull* of $V(\Gamma)$ (set of vertices of Γ equipped with a path metric)—the smallest injective metric space containing isometrically $V(\Gamma)$. The proof is a straightforward consequence of the fact that the injective hull can be defined as a space of real-valued functions, the so-called *metric forms* [47,64,69], and similarly the smallest Helly graph containing isometrically a given graph—the *Helly hull*—can be defined analogously using integer metric forms (see e.g. [65, Section II.3]). It follows that G acts on a space with a convex geodesic bicombing. Another example of a space with convex geodesic bicombing is a CAT(0) space with a combing consisting of all (unit speed) geodesics. As noted in Sect. 1.1 only in very particular cases it is possible to construct a CAT(0) space acted upon geometrically by an Artin group. We establish the existence of a slightly weaker structure in much greater generality.

Holt [60] proves falsification by fellow traveller for Garside groups. It follows then from [69] that such groups act properly (not necessarily cocompactly) on injective metric spaces. Garside groups have Garside normal forms, which can be thought of as a combinatorial combing (not in the sense above). However, even coarsely this combing is not convex—see some further discussion in (4) below.

The properties considered in (2)–(5) below are immediate consequences of acting geometrically on a space with a convex geodesic bicombing.

(2) Similarly to a CAT(0) space, an injective metric space (or, more generally, a space with a convex geodesic bicombing) has a boundary at infinity, with two different topologies: the cone topology and the Tits topology [44,66]. The former gives rise to an *EZ-boundary* (cf. [15,49]) of G [44]. The topology of an EZ-boundary reflects some algebraic properties of the group, and its existence implies e.g. the Novikov conjecture. The Tits topology is finer and provides subtle quasi-isometric invariants of G [66]. It is observed in [31] that Helly groups admit EZ-boundaries.

(3) For a discrete group G the *Farrell–Jones conjecture* asserts that the K-theoretic (or L-theoretic) *assembly map*, that is a homomorphism from a homology group of the classifying space $E_{\mathcal{VCY}}(G)$ to a K-group (or L-group) of a group ring, is an isomorphism (see e.g. [23,51,67,79] for details). We say that G satisfies the *Farrell–Jones conjecture with finite wreath products* if for any finite group F the wreath product $G \wr F$ satisfies the Farrell–Jones conjecture. It is proved in [67] that groups acting on spaces with a convex geodesic bicombing satisfy the Farrell–Jones conjecture with finite wreath products.

The Farrell–Jones conjecture implies, for example, the Novikov Conjecture, and the Borel Conjecture in high dimensions. Farrell–Roushon [51] established the Farrell–Jones conjecture for braid groups, Bartels–Lück [23] and Wegner [83] proved it for CAT(0) groups (see Sect. 1.1 for examples of CAT(0) Artin groups). Recently, Roushon [79] established the Farrell–Jones conjecture for some (but not all) spherical and Euclidean Artin groups. His proof relies on observing that some Artin groups fit into appropriate exact sequences. After completion of the first version of the current paper Brück-Kielak-Wu [22] presented another proof of the Farrell–Jones conjecture with finite wreath products for a subclass of the class of FC type Artin groups—for even Artin groups of type FC.

(4) For a metric space X the *coarse assembly map* is a homomorphism from the coarse K-homology of X to the K-theory of the Roe-algebra of X. The space X satisfies the *coarse Baum–Connes conjecture* if the coarse assembly map is an isomorphism. A finitely generated group Γ satisfies the coarse Baum–Connes conjecture if the conjecture holds for Γ seen as a metric space with the

word metric given by a finite generating set. Equivalently, the conjecture holds for Γ if a metric space (equivalently: every metric space) acted geometrically upon by Γ satisfies the conjecture. As observed in [31], it follows from the work of Fukaya–Oguni [50] that Helly groups satisfy the coarse Baum–Connes conjecture.

For CAT(0) groups the coarse Baum–Connes conjecture holds by [61]. Groups with finite asymptotic dimension satisfy the coarse Baum–Connes conjecture. It is shown in [16] that braid groups as well as spherical Artin groups A_n, C_n, and Artin groups of affine type $\widetilde{A}_n$, $\widetilde{C}_n$ have finite asymptotic dimension.

For proving the coarse Baum–Connes conjecture in [50] a coarse version of a convex bicombing is used. Essential there is that two geodesics in the combing with common endpoints 'converge' at least linearly. This is not true for the usual combing in Garside groups. For example, such a combing for $\mathbb{Z}^2$ between points (n, n) and $(0, 0)$ follows roughly the diagonal. Similarly the combing line between $(n + k, n)$ and $(0, 0)$ follows roughly a diagonal for $n \gg k$. Hence, for large n, the two lines stay at roughly constant distance for a long time—this behavior is different from what is required.

(5) Higher order homological (resp. homotopical) Dehn functions measure the difficulty of filling higher dimensional cycles (resp. spheres) by chains (resp. balls), see [1,10] for backgrounds. The homological case of (5) is a consequence of (1) and [84], see the discussion in [1, Section 2] for more details. The homotopical case of (5) follows from (1) and [10]. The Garside case of (5) has been known before as it can be deduced from the biautomaticity and [10,84]. The Artin case of (5) is new.

(6) Biautomaticity is an important algorithmic property of a group. It implies, among others, that the Dehn function is at most quadratic, and that the Word Problem and the Conjugacy Problem are solvable; see e.g. [48]. It is proved in [31] that all Helly groups are biautomatic.

Biautomaticity of all FC-type Artin groups is new (and so is the solution of the Conjugacy Problem and bounding the Dehn function for these Artin groups). Altobelli [2] showed that FC-type Artin groups are asynchronously automatic, and hence have solvable Word Problem. Biautomaticity for few classes of Artin groups was shown before by: Pride together with Gersten-Short (triangle-free Artin groups) [53,78], Charney [35] (spherical), Peifer [75] (extra-large type), Brady-McCammond [24] (three-generator large-type Artin groups and some generalizations), Huang–Osajda [58] (almost large-type). Weak Garside groups of finite type are biautomatic by [35,41].

1.4 Sketch of the proof of the main result

Here we only discuss the special case of braid groups. Each braid group is the fundamental group of its Salvetti complex X. The universal cover $\widetilde{X}$ admits a natural cellulation by Coxeter cells. The key point is that the combinatorics of how these Coxeter cells in $\widetilde{X}$ intersect each other follows a special pattern which is a reminiscent of $CAT(0)$ cube complexes (but much more general). We formulate such pattern as the definition of cell Helly complex, see Definition 3.5 and Remark 3.7. By the local-to-global characterization of Helly graphs established in [32], the proof of the main theorem reduces to showing that $\widetilde{X}$ is cell Helly, and showing that the cell Hellyness can be further reduced to a calculation using Garside theory.

1.5 Organization of the paper

In Sect. 2, we provide some background on Coxeter groups and Artin groups. In Sect. 3, we discuss Helly graphs, introduce the notion of cell Helly complex, and present some natural examples. In Sect. 4, we prove the Garside case of the Main Theorem and in Sect. 5 we deal with the Artin group case.

2 Preliminaries

All graphs in this article are *simplicial*, that is, they do not contain loops or multiple edges. For a connected graph we equip the set of its vertices with a path metric: the distance of two vertices is the length (the number of edges) of the shortest edge-path between those vertices. A *full subgraph* of a graph Γ is a subgraph Γ' such that two vertices are adjacent in Γ' if and only if they are adjacent in Γ.

2.1 Lattices

A standard reference for the lattice theory is [18]. Let $(P, \leq)$ be a partially ordered set (poset), and let L be an arbitrary subset. An element $u \in P$ is an *upper bound* of L if $s \leq u$ for each $s \in L$. An upper bound u of L is its *least upper bound*, denoted by $\bigvee L$, if $u \leq x$ for each upper bound x of L. Similarly, $v \in P$ is a *lower bound* of L if $v \leq s$ for each $s \in L$. A lower bound v of L is its *greatest lower bound*, denoted by $\bigwedge L$, if $x \leq v$ for each lower bound x of L. Given $a, b \in P$, the *interval* between a and b, denoted $[a, b]$, is the collection of all elements $x \in P$ satisfying $a \leq x \leq b$.

A poset $(P, \leq)$ is a *lattice* if each two-element subset $\{a, b\} \subset P$ has a least upper bound—*join*, denoted by $a \vee b$—and a greatest lower bound—*meet*, denoted by $a \wedge b$.

Lemma 2.1 [18, Theorem IV.8.15] *Suppose* $(P, \leq)$ *is a lattice. Then*

(1) if two intervals $[a, b]$ *and* $[c, d]$ *of* P *have nonempty intersection, then* $[a, b] \cap [c, d] = [a \vee c, b \wedge d]$*;*
(2) if elements of a finite collection $\{[a_i, b_i]\}_{i=1}^n$ *of intervals in* P *pairwise intersect, then the intersection of all members in the collection is a nonempty interval* $\bigcap_{i=1}^n [a_i, b_1] = [\bigvee\{a_i\}, \bigwedge\{b_i\}]$.

2.2 Artin groups and Coxeter groups

Let Γ be a finite simple graph with each of its edges labeled by an integer ≥ 2. Let $V\Gamma$ be the vertex set of Γ. The *Artin group with defining graph* Γ, denoted A_Γ, is given by the following presentation:

$$\langle s_i \in V\Gamma \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \text{ for each } s_i \text{ and } s_j \text{ spanning an edge labeled by } m_{ij} \rangle.$$

The *Artin monoid with defining graph* Γ, denoted A_Γ^+ is the monoid with the same presentation. The natural map $A_\Gamma^+ \to A_\Gamma$ is injective [73].

The *Coxeter group with defining graph* Γ, denoted W_Γ, is given by the following presentation:

$$\langle s_i \in V\Gamma \mid s_i^2 = 1 \text{ for any } s_i \in V\Gamma,\ (s_i s_j)^{m_{ij}} = 1 \text{ for each } s_i \text{ and } s_j \text{ spanning an edge labeled by } m_{ij} \rangle.$$

Note that there is a quotient map $A_\Gamma \to W_\Gamma$ whose kernel is normally generated by the squares of generators of A_Γ.

Theorem 2.2 *Let* Γ_1 *and* Γ_2 *be full subgraphs of* Γ *with the induced edge labelings. Then*

(1) the natural homomorphisms $A_{\Gamma_1} \to A_\Gamma$ *and* $W_{\Gamma_1} \to W_\Gamma$ *are injective;*
(2) $A_{\Gamma_1} \cap A_{\Gamma_2} = A_{\Gamma_1 \cap \Gamma_2}$ *and* $W_{\Gamma_1} \cap W_{\Gamma_2} = W_{\Gamma_1 \cap \Gamma_2}$.

The Coxeter case of this theorem is standard, see e.g. [28] or [39], the Artin case of this theorem is proved in [81].

A *standard parabolic subgroup* of an Artin group or a Coxeter group is a subgroup generated by a subset of its generating set.

Theorem 2.3 [38] *Let $\Gamma' \subset \Gamma$ be a full subgraph. Let $X' \to X$ be the embedding from the Cayley graph of $A_{\Gamma'}$ (with respect to the standard presentation) to the Cayley graph of A_Γ induced by the monomorphism $A_{\Gamma'} \to A_\Gamma$. Then X' is convex in X, i.e. any geodesic of X between two vertices of X' is contained in X'.*

Given $x, y \in A_\Gamma$, we define $x \preccurlyeq y$ (resp. $y \succcurlyeq x$) if $y = xz$ (resp. $y = zx$) for some z in the Artin monoid A_Γ^+. By considering the length homomorphism $A_\Gamma \to \mathbb{Z}$ which sends each generator of A_Γ to 1, one deduces that $\preccurlyeq$ and $\succcurlyeq$ are both partial orders on A_Γ. They are called the *prefix order* and the *suffix order*, respectively. The prefix order is invariant under the left action of A_Γ on itself.

Now we recall the *weak order* on a Coxeter group W_Γ. Each element $w \in W_\Gamma$ can be expressed as a word in the free monoid on the generating set of W_Γ. One shortest such expression (with respect to the word norm) is called a *reduced expression* of w. Given $x, y \in W_\Gamma$, define $x \preccurlyeq y$ (resp. $y \succcurlyeq x$), if some reduced expression of y has its prefix (resp. suffix) being a reduced expression of x. Note that $\preccurlyeq$ and $\succcurlyeq$ give two partial orders on W_Γ, called the *right weak order* and the *left weak order*, respectively.

The defining graph Γ is *spherical* if W_Γ is a finite group, in this case, we also say the Artin group A_Γ is *spherical*. An Artin group is of *type FC* if any complete subgraph $\Gamma' \subset \Gamma$ is spherical.

Theorem 2.4 *Suppose Γ is spherical. Then*

(1) $(W_\Gamma, \preccurlyeq)$ and $(W_\Gamma, \succcurlyeq)$ are both lattices;
(2) $(A_\Gamma, \preccurlyeq)$ and $(A_\Gamma, \succcurlyeq)$ are both lattices.

The first assertion is standard, see e.g. [6]. The second assertion follows from [30,42].

2.3 Davis complexes and oriented Coxeter cells

By a cell, we always mean a closed cell unless otherwise specified.

Definition 2.5 (*Davis complex*) Given a Coxeter group W_Γ, let $\mathcal{P}$ be the poset of left cosets of spherical standard parabolic subgroups in W_Γ (with respect to inclusion) and let Δ_Γ be the geometric realization of this poset (i.e. Δ_Γ is a simplicial complex whose simplices correspond to chains in $\mathcal{P}$). Now we modify the cell structure on Δ_Γ to define a new complex D_Γ, called the *Davis complex*. The cells in D_Γ are full subcomplexes of Δ_Γ spanned by a given vertex v and all other vertices which are $\leq v$ (note that vertices of Δ_Γ correspond to elements in $\mathcal{P}$, hence inherit the partial order).

Suppose W_Γ is finite with n generators. Then there is a canonical faithful orthogonal action of W_Γ on the Euclidean space $\mathbb{E}^n$. Take a point in $\mathbb{E}^n$ with trivial stabilizer, then the convex hull of the orbit of this point under the W_Γ action (with its natural cell structure) is isomorphic to D_Γ. In such case, we call D_Γ a *Coxeter cell*.

In general, the 1-skeleton of D_Γ is the unoriented Cayley graph of W_Γ (i.e. we start with the usual Cayley graph and identify the double edges arising from s_i^2 as single edges), and D_Γ can be constructed from the unoriented Cayley graph by filling Coxeter cells in a natural way. Each edge of D_Γ is labeled by a generator of W_Γ. We endow $D_\Gamma^{(1)}$ with the path metric.

We can identify the unoriented Cayley graph of W_Γ with the Hasse diagram of W_Γ with respect to right weak order. Thus we can orient each edge of D_Γ from its smaller vertex to its larger vertex. When W_Γ is finite, D_Γ with such edge orientation is called an *oriented Coxeter cell*. Note that there is a unique *source vertex* (resp. *sink vertex*) of D_Γ corresponding to the identity element (resp. longest element) of W_Γ, where each edge containing this vertex is oriented away from (resp. oriented towards) this vertex. The source and sink vertices are called *tips* of D_Γ.

The following fact is standard, see e.g. [28].

Theorem 2.6 *Let $\mathcal{R}$ be a left coset of a standard parabolic subgroup, and let $v \in W_\Gamma$ be an arbitrary element. Then the following hold.*

(1) $\mathcal{R}$ is convex in the sense that for two vertices $v_1, v_2 \in \mathcal{R}$, the vertex set of any geodesic in $D_\Gamma^{(1)}$ joining v_1 and v_2 is inside $\mathcal{R}$.
(2) There is a unique vertex $u \in \mathcal{R}$ such that $d(v, u) = d(v, \mathcal{R})$, where d denotes the path metric on the 1-skeleton of D_Γ.
(3) Pick $v_1 \in \mathcal{R}$. Then $d(v, v_1) = d(v, u) + d(u, v_1)$.

Theorem 2.6 implies that the face of an oriented Coxeter cell C, with its edge labeling and orientation from C, can be identified with a lower dimensional Coxeter cell in a way which preserves edge labeling and orientation.

The following lemma is well-known. For example, it follows from [9, Lemma 3.4.9 and Corollary 5.5.16]. From another perspective, Theorem 2.6 (2)&(3) says that left cosets are the so-called *gated subgraphs* of $D_\Gamma^{(1)}$. It is a well-known fact (see e.g. [82, Proposition I.5.12(2)]) that any family of gated subgraphs has the finite Helly property. In fact, the proof provided below (for the convenience of the reader), is the proof of this feature for families of gated subgraphs.

Lemma 2.7 *Let $\{R_i\}_{i=1}^n$ be a finite collection of left cosets of standard parabolic subgroups of W_Γ. Then $\cap_{i=1}^n R_i \neq \emptyset$ given $\{R_i\}_{i=1}^n$ pairwise intersect.*

Proof We first look at the case $n = 3$. Pick $x \in R_2 \cap R_3$. Let y be the unique point in R_1 closest to x (Theorem 2.6 (2)). For $i = 2, 3$, let y_i be the unique point in $R_1 \cap R_i$ closest to x. By Theorem 2.6 (3), $d(x, y_2) = d(x, y) + d(y, y_2)$. Thus $y \in R_2$ by the convexity of R_2 (Theorem 2.6 (1)). Thus $y = y_2$. Similarly, $y = y_3$. Hence $y \in \cap_{i=1}^{3} R_i$. For $n > 3$, we proceed by the induction on n. For $2 \leq i \leq n$, let $\tau_i = R_1 \cap R_i$. Then each τ_i is a left coset of a standard subgroup by Theorem 2.2. Moreover, $\{\tau_i\}_{i=2}^{n}$ pairwise intersect by the $n = 3$ case. Thus $\cap_{i=2}^{n} \tau_i \neq \emptyset$ by induction, and the lemma follows. □

2.4 Salvetti complexes

Definition 2.8 (*Salvetti complex*) Given an Artin group S_Γ, we define a cell complex, called the *Salvetti complex* and denoted by S_Γ, as follows. The 1-skeleton of S_Γ is a wedge sum of circles, one circle for each generator. We label each circle by its associated generators and choose an orientation for each circle. Suppose the $(n-1)$-skeleton of S_Γ has been defined. Now for each spherical subgraph $\Gamma' \subset \Gamma$ with n vertices, we attach an oriented Coxeter cell $D_{\Gamma'}$ with the attaching map being a cellular map whose restriction to each face of the boundary of $D_{\Gamma'}$ coincides with the attaching map of a lower dimensional oriented Coxeter cell (when $n = 2$, we require the attaching map preserves the orientation and the labeling of edges).

The 2-skeleton of S_Γ is the presentation complex of A_Γ. Hence $\pi_1(S_\Gamma) = A_\Gamma$. Let X_Γ be the universal cover of S_Γ. The 1-skeleton of X_Γ is the Cayley graph of A_Γ, and it is endowed with the path metric with each edge having length $= 1$. We pull back labeling and orientation of edges in S_Γ to X_Γ. A *positive path* in $X^{(1)}_\Gamma$ from a vertex $x \in X_\Gamma$ to another vertex y is an edge path such that one reads of a word in the Artin monoid by following this path. By considering the length homomorphism $A_\Gamma \to \mathbb{Z}$, we obtain that each positive path is geodesic.

We identify $X^{(0)}_\Gamma$ with elements in A_Γ. Hence $X^{(0)}_\Gamma$ inherits the prefix order $\preccurlyeq$ and the suffix order $\succcurlyeq$. Then $x \preccurlyeq y$ if and only if there is a positive path in $X^{(1)}_\Gamma$ from x to y.

The quotient homomorphism $A_\Gamma \to W_\Gamma$ induces a graph morphism $X^{(1)}_\Gamma \to D^{(1)}_\Gamma$ which preserves labeling of edges. Take a cell of S_Γ represented by its attaching map $q : D_{\Gamma'} \to S_\Gamma$ with $D_{\Gamma'}$ being an oriented Coxeter cell. Let $\tilde{q} : D_{\Gamma'} \to X_\Gamma$ be a lift of q. Note that $\tilde{q}$ respects orientation and labeling of edges. It follows from definition that $D^{(1)}_{\Gamma'} \to X^{(1)}_\Gamma \to D^{(1)}_\Gamma$ is an embedding, hence $D^{(1)}_{\Gamma'} \to X^{(1)}_\Gamma$ is an embedding. By induction on dimension, we have that $\tilde{q}$ is an embedding. Hence each cell in X_Γ is embedded. Given a vertex $x \in X_\Gamma$, there is a one to one correspondence between cells of X_Γ whose source vertex is x and spherical subgraphs of Γ. Then $X_\Gamma^{(1)} \to D_\Gamma^{(1)}$ extends naturally to a cellular map $\pi : X_\Gamma \to D_\Gamma$ which is a homeomorphism on each closed cell.

Lemma 2.9 *Let $q : C \to X_\Gamma$ be the attaching map from an oriented Coxeter cell C to X_Γ. Let a and b be the source vertex and the sink vertex of C. Let $\preccurlyeq_C$ be the right weak order on $C^{(0)}$. Then the following hold (we use d to denote the path metric on the 1-skeleton).*

(1) the map $(C^{(1)}, d) \to (X_\Gamma^{(1)}, d)$ is an isometric embedding;
(2) for any two vertices $x, y \in C$, $x \preccurlyeq_C y$ if and only if $q(x) \preccurlyeq q(y)$, moreover, $[q(x), q(y)]_\preccurlyeq = q([a, b]_{\preccurlyeq_C})$ where $[q(x), q(y)]_\preccurlyeq$ denotes the interval between two elements with respect to the order $\preccurlyeq$;
(3) for two cells C_1 and C_2 in X_Γ, if $C_1^{(0)} \subset C_2^{(0)}$, then there exists a cell $C_3 \subset C_2$ such that $C_3^{(0)} = C_1^{(0)}$.

Proof Assertion (1) follows from the fact that the map $X_\Gamma^{(1)} \to D_\Gamma^{(1)}$ is 1-Lipschitz. Now we prove (2). The 'only if' direction follows from the fact that q preserves orientation of edges. Now assume $q(x) \preccurlyeq q(y)$. Since $q(a) \preccurlyeq q(x) \preccurlyeq q(y) \preccurlyeq q(b)$, there is a positive path ω from $q(a)$ to $q(b)$ passing through $q(x)$ and $q(y)$. Let $\bar{C}^{(1)}$, $\bar{\omega}$, $\bar{a}$ and $\bar{b}$ be the images of $q(C^{(1)})$, ω, $q(a)$ and $q(b)$ under $X_\Gamma^{(1)} \to D_\Gamma^{(1)}$. Then $d(a, b) = d(\bar{a}, \bar{b})$, which also equals to the length of ω (recall each positive path is geodesic). Thus $length(\omega) = length(\bar{\omega})$. By Theorem 2.6, $\bar{\omega} \subset \bar{C}^{(1)}$. Then $\bar{\omega}$ (hence ω) corresponds to the longest word in the finite Coxeter group associated C. Thus $\omega \subset q(C)$. It follows that $x \preccurlyeq y$. The proof of the moreover statement in (2) is similar.

For (3), there is a positive path ω from the source of C_1 to the sink of C_1 corresponding to the longest word of some finite Coxeter group. The proof of (2) implies that $\omega \subset C_2$. Then ω determines $C_3 \subset C_2$ as required. □

Lemma 2.10 *Let C_1 and C_2 be two cells of X_Γ. Then the following are equivalent:*

(1) $C_1 = C_2$;
(2) $C_1^{(0)} = C_2^{(0)}$;
(3) C_1 and C_2 have the same sink.

Proof (1) $\Rightarrow$ (2) is trivial. (3) $\Rightarrow$ (1) follows from the definition of cells. Now we prove (2) $\Rightarrow$ (3). First assume C_1 and C_2 are two edges. By considering the label preserving map $X_\Gamma^{(1)} \to D_\Gamma^{(1)}$, we obtain that C_1 and C_2 have the same label. On the other hand, Theorem 2.2 (1) implies that map from the circle associated with a generator in S_Γ to S_Γ is π_1-injective. Thus $C_1 = C_2$. Now we look at the general case. By Lemma 2.9 (3), two vertices are adjacent in C_1 if and only if they are adjacent in C_2. Then (2) $\Rightarrow$ (3) follows from the 1-dimensional case by considering the orientation of edges on the cells. □

A *full subcomplex* $K \subset X_\Gamma$ is a subcomplex such that if a cell of X_Γ has the vertex set contained in K, then the cell is contained in K. It is clear that intersection of two full subcomplexes is also a full subcomplex. The following is a consequence of Lemma 2.9 (2) and Lemma 2.10.

Lemma 2.11 *Any cell in X_Γ is a full subcomplex.*

Take a full subgraph $\Gamma' \subset \Gamma$, then there is an embedding $S_{\Gamma'} \to S_\Gamma$ which is π_1-injective (Theorem 2.2 (1)). A *standard subcomplex of type* Γ' of X_Γ is a connected component of the preimage of $S_{\Gamma'}$ under the covering map $X_\Gamma \to S_\Gamma$. Each such standard subcomplex can be naturally identified with the universal cover $X_{\Gamma'}$ of $S_{\Gamma'}$. A standard subcomplex is *spherical* if its type is spherical. We define the *type* of a cell in X_Γ to be the type of the smallest standard subcomplex that contains this cell.

Lemma 2.12 *Any standard subcomplex of X_Γ is a full subcomplex.*

Proof Let $X \subset X_\Gamma$ be a standard subcomplex of type Γ'. Let $C \subset X_\Gamma$ be a cell such that $C^{(0)} \subset X$. By considering the map $X_\Gamma^{(1)} \to D_\Gamma^{(1)}$, we obtain that each edge of C is labeled by a vertex of Γ'. Thus C is of type Γ'' for $\Gamma'' \subset \Gamma'$. Hence C is mapped to $S_{\Gamma''}$ under $X_\Gamma \to S_\Gamma$. Since $C \cup X$ is connected, $C \subset X$. Then the lemma follows. □

Lemma 2.13 *Let $X_1, X_2 \subset X_\Gamma$ be standard subcomplexes of type Γ_1, Γ_2.*

(1) Suppose $X_1 \cap X_2 \neq \emptyset$. Then $X_1 \cap X_2$ is a standard subcomplex of type $\Gamma_1 \cap \Gamma_2$.
(2) Let $C \subset X_\Gamma$ be a cell of type Γ_1 such that $C \cap X_2 \neq \emptyset$. Then $C \cap X_2$ is a cell of type $\Gamma_1 \cap \Gamma_2$.
(3) For elements $a, b \in X_1^{(0)}$, the interval between a and b with respect to $(X_\Gamma^{(0)}, \preccurlyeq)$ is contained in $X_1^{(0)}$.
(4) We identify $X_1^{(0)}$ with A_{Γ_1}, hence $X_1^{(0)}$ inherits a prefix order $\preccurlyeq_{\Gamma_1}$ from A_{Γ_1}. Then $\preccurlyeq_{\Gamma_1}$ coincides with the restriction of $(A_\Gamma, \preccurlyeq)$ to $X_1^{(0)}$.

Proof By Theorem 2.2, there exists a standard subcomplex $X_0 \subset X_\Gamma$ such that $X_0^{(0)} = X_1^{(0)} \cap X_2^{(0)}$. Then (1) follows from Lemma 2.12. Now we prove (2). Let X_1 be the standard subcomplex of type Γ_1 containing C. Then $C \cap X_2$ is contained in $X_1 \cap X_2$, which is a standard subcomplex of type $\Gamma_1 \cap \Gamma_2$. This together with Lemma 2.12 imply that $C \cap X_2$ is a disjoint union of faces of C of type $\Gamma_1 \cap \Gamma_2$. It remains to show $C \cap X_2$ is connected. Suppose the contrary holds. Pick a shortest geodesic edge path $\omega \subset C$ (with respect to the path metric on $C^{(1)}$) traveling from a vertex in one component of $C \cap X_2$ to another component. Then ω is also a geodesic path in $X_\Gamma^{(1)}$ by Lemma 2.9. Since two end points of ω are in X_2, we have $\omega \subset X_2$ by Theorem 2.3, which yields a contradiction. Assertions (3) and (4) follow from Theorem 2.3. □

3 Helly graphs and cell-Helly complexes

See the beginning of Sect. 2 for our convention on graphs. A *clique* of a graph is a complete subgraph. A *maximal clique* is a clique not properly contained in another clique. Each graph is endowed with the path metric such that each edge has length 1. A *ball* centered at x with radius r in a graph Γ is the full subgraph spanned by all vertices at distance $\leq r$ from x. As for Helly graphs we follow usually the notation from [31,32].

Definition 3.1 A family $\mathcal{F}$ of subsets of a set satisfies the (*finite*) *Helly property* if for any (finite) subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of pairwise intersecting subsets the intersection $\bigcap \mathcal{F}'$ is nonempty.

A graph is (*finitely*) *clique Helly* if the family of all maximal cliques, as subsets of the vertex sets of the graph, satisfies the (finite) Helly property.

A graph is (*finitely*) *Helly* if the family of all balls, as subsets of the vertex sets of the graph, satisfies the (finite) Helly property.

The notion of clique Helly is local and the notion of Helly is global. Our main tool for proving Hellyness of some graphs is the following local-to-global characterization.

The *clique complex* of a graph Γ is the flag completion of Γ. The *triangle complex* of Γ is the 2-skeleton of the clique complex, that is, the 2-dimensional simplicial complex obtained by spanning 2-simplices (that is, triangles) on all 3-cycles (that is, 3-cliques) of Γ.

Theorem 3.2 *(i)* ([32, Theorem 3.5]) *Let Γ be a simplicial graph. Suppose that Γ is finitely clique Helly, and that the clique complex of Γ is simply-connected. Then Γ is finitely Helly.*

(ii) ([77, Proposition 3.1.2]) *If in addition Γ does not have infinite cliques, then Γ is Helly.*

Parallel to this theorem, a local-to-global result for injective metric spaces is obtained in [72].

Lemma 3.3 *Let X be a set and let $\{X_i\}_{i\in I}$ be a family of subsets of X. Define a simplicial graph Y whose vertex set is X, and two points are adjacent if there exists i such that they are contained in X_i. Suppose that*

(1) there are no infinite cliques in Y;
(2) $\{X_i\}_{i\in I}$ has finite Helly property, i.e. any finite family of pairwise intersecting elements in $\{X_i\}_{i\in I}$ has nonempty intersection;
(3) for any triple $X_{i_1}, X_{i_2}, X_{i_3}$ of pairwise intersecting subsets in $\{X_i\}_{i\in I}$, there exists $i_0 \in I$ such that

$$(X_{i_1} \cap X_{i_2}) \cup (X_{i_2} \cap X_{i_3}) \cup (X_{i_3} \cap X_{i_1}) \subseteq X_{i_0}.$$

Then Y is finitely clique Helly.

Proof Let $S \subset X$ be a subset. We claim if S is the vertex set of a clique in Y, then there exists $i \in I$ such that $S \subset X_i$. We prove it by induction on $|S|$. Note that $|S| < \infty$ by assumption (1), and the case $|S| = 2$ is trivial. Suppose $|S| = n \geq 3$. Let $\{x_1, x_2, x_3\}$ be three pairwise distinct vertices in S and let $S_i = S \backslash \{x_i\}$. It follows from induction assumption that there exists X_i such that $S_i \subset X_i$. Note that $\{X_{i_1}, X_{i_2}, X_{i_3}\}$ pairwise intersect. Thus by assumption (3), there exists $X_{i_0} \in \{X_i\}_{i \in I}$ such that $S \subset (X_{i_1} \cap X_{i_2}) \cup (X_{i_2} \cap X_{i_3}) \cup (X_{i_3} \cap X_{i_1}) \subset X_{i_0}$. Hence the claim follows. The claim implies that if S is the vertex set of a maximal clique in Y, then $S = X_i$ for some $i \in I$. Now the lemma follows from assumption (2). □

The lemma above justifies the following definition that will be our main tool for showing that some groups act nicely on Helly graphs. A subcomplex of X is *finite* it has only finitely many cells.

Definition 3.4 Let X be a combinatorial complex (cf. [17, Chapter I.8.A]) and let $\{X_i\}_{i \in I}$ be a family of finite full subcomplexes covering X. We call $(X, \{X_i\})$ (or simply X, when the family $\{X_i\}$ is evident) a *generalized cell Helly* complex if the following conditions are satisfied:

(1) each intersection $X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_k}$ is either empty or connected and simply connected (in particular, all members of $\{X_i\}$ are connected and simply connected);
(2) the family $\{X_i\}_{i \in I}$ has finite Helly property, i.e. any finite family of pairwise intersecting elements in $\{X_i\}_{i \in I}$ has nonempty intersection;
(3) for any triple $X_{i_1}, X_{i_2}, X_{i_3}$ of pairwise intersecting subcomplexes in $\{X_i\}_{i \in I}$, there exists $i_0 \in I$ such that

$$(X_{i_1} \cap X_{i_2}) \cup (X_{i_2} \cap X_{i_3}) \cup (X_{i_3} \cap X_{i_1}) \subseteq X_{i_0}.$$

The subcomplexes X_i are called *generalized cells* of X.

In this article we will only deal with the following special case of generalized cell Helly complex.

Definition 3.5 Suppose X is a combinatorial complex such that each of its closed cell is embedded and each closed cell is a full subcomplex of X. Let $\{X_i\}_{i \in I}$ be the collection of all closed cells in X. We say X is *cell Helly* if $(X, \{X_i\})$ satisfies Definition 3.4.

A natural class of cell Helly complexes is described in the lemma below, whose proof is left to the reader.

Lemma 3.6 *Suppose X is a simply-connected cube complex such that each cube in X is embedded and any intersection of two cubes of X is a sub-cube. Then X is cell Helly if and only if X is a $CAT(0)$ cube complex.*

Remark 3.7 It can be seen here that the condition (3) from Definition 3.4 plays a role of Gromov's flagness condition. Intuitively, it tells that all the corners (of possible positive curvature) are "filled" (and hence curvature around them is nonpositive).

Pride [78] showed that 2-dimensional Artin groups of type FC are C(4)-T(4) small cancellation groups. The following lemma together with Theorem 5.8 can be seen as an extension of Pride's result to higher dimensions. We refer to [59,78] for the definition of C(4)-T(4) small cancellation conditions. Here, we assume that cells are determined by their attaching maps (this corresponds to the usual small cancellation theory, for groups without torsion). In particular, by [59, Proposition 3.5] it means that the intersection of two cells is either empty or a vertex or an interval.

Lemma 3.8 *Suppose X is a 2-dimensional simply-connected combinatorial complex such that each closed cell is embedded. Then X is cell Helly if and only if X satisfies the C(4)-T(4) small cancellation conditions.*

Proof If X is a C(4)-T(4) complex then, by [59, Proposition 3.8], for any pairwise intersecting cells C_0, C_1, C_2 we have $C_i \cap C_j \subseteq C_k$, for $\{i, j, k\} = \{0, 1, 2\}$. It follows that X is cell Helly.

Now suppose X is cell Helly. Since nonempty intersections of cells are connected and simply connected, such intersections must be vertices or intervals.

We check the C(4) condition. First suppose there is a cell C with the boundary covered by two intervals $C \cap C_0$ and $C \cap C_1$ being intersections with cells C_0, C_1. Since, by Definition 3.4(1), the intersection $C_0 \cap C_1$ has to be a nontrivial interval. But then the union $(C \cap C_0) \cup (C_0 \cap C_1) \cup (C_1 \cap C)$ is a graph being a union of three intervals along their endpoints, contradicting Definition 3.4(3). Now, suppose there are cells C, C_0, C_1, C_2 such that $C \cap C_i \neq \emptyset$ and $C_i \cap C_{i+1(\mathrm{mod}\,3)} \neq \emptyset$. It follows that there is a common intersection $C \cap C_0 \cap C_1 \cap C_2$. This brings us to the previous case.

For the T(4) condition suppose there are cells C_0, C_1, C_2 with $C_i \cap C_{i+1(\mathrm{mod}\,3)}$ being a nontrivial interval containing a vertex v, for all i. Then $(C_0 \cap C_1) \cup (C_1 \cap C_2) \cup (C_2 \cap C_0)$ cannot be contained in a single cell, contradicting Definition 3.4(3). □

Definition 3.4 generalizes Definition 3.5 in the same way as the graphical small cancellation theory generalizes the classical small cancellation theory, another example to consider is X being a C(4)-T(4) graphical small cancellation complex. Defining $\{X_i\}$ as the family of simplicial cones over relators, we obtain a generalized cell Helly complex (proved in [31]).

Later we will see more examples of cell Helly complexes whose cells are zonotopes (Corollary 4.11) or Coxeter cells (Theorem 5.8).

For a pair $(X, \{X_i\}_{i\in I})$ as above, we define its *thickening* Y to be the following simplicial complex. The 0-skeleton of Y is $X^{(0)}$, and two vertices of $Y^{(0)}$ are adjacent if they are contained in the same generalized cell of X. Then Y is defined to be the flag complex of $Y^{(1)}$. We call $(X, \{X_i\}_{i\in I})$ *locally bounded* if there does not exist an infinite strictly increasing sequence of vertex sets $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots$ such that for every j there exists i_j with $A_j \subseteq X_{i_j}$.

Theorem 3.9 *The thickening of a locally bounded generalized cell Helly complex is clique Helly. The thickening of a simply connected locally bounded generalized cell Helly complex is Helly.*

Proof Let $(X, \{X_i\}_{i\in I})$ be a locally bounded generalized cell Helly complex and let Y be its thickening. It follows from the proof of Lemma 3.3 that the vertex set of each finite clique of Y is contained in some X_{i_j} (this does not use assumption (1) of Lemma 3.3). This together with local boundedness imply that there are no infinite cliques in Y. The first statement follows from Lemma 3.3. By the Nerve Theorem [19, Theorem 6] applied to the covering of X by generalized cells $\{X_i\}$, and the covering of Y by simplices corresponding to generalized cells we get that X and Y are have the same connectivity and isomorphic fundamental groups. If X is simply connected then so is Y and hence, by Theorem 3.2, the thickening is Helly. □

We say that a group G *acts* on a generalized cell Helly complex $(X, \{X_i\}_{i\in I})$ if it acts by automorphisms on X and preserves the family $\{X_i\}_{i\in I}$. For such an action clearly G acts by automorphisms on the thickening of $(X, \{X_i\}_{i\in I})$. Moreover, we have the following.

Proposition 3.10 *If a group G acts geometrically on a simply connected locally bounded generalized cell Helly complex $(X, \{X_i\}_{i\in I})$ then G acts geometrically on the Helly graph being the 1-skeleton $Y^{(1)}$ of the thickening Y. In particular, G is Helly.*

Proof We first show that each vertex v in $Y^{(1)}$ is adjacent to finitely many other vertices. Suppose v is adjacent to infinitely many different vertices $\{v_i\}_{i=1}^{\infty}$. Take X_i which contains both v and v_i. As X is uniformly locally finite, we can assume $\lim_{i\to\infty} d_{X^{(1)}}(v, v_i) = \infty$ where $d_{X^{(1)}}$ denotes the path metric on $X^{(1)}$. Let P_i be a shortest edge path in X_i connecting v and v_i. Then P_i subconverges to an infinite path as $i \to \infty$, which contradicts the local boundedness assumption.

Since the G-action on X is cocompact, the quotient $Y^{(0)}/G$ of the vertex set is finite. Since $(X, \{X_i\})$ is locally bounded there are no infinite cliques (see the proof of Theorem 3.9) and hence the quotient $Y^{(1)}/G$ is finite.

To prove properness it remains to show that, for the G-action on $Y^{(1)}$, stabilizers of cliques are finite. Such stabilizers correspond to stabilizers of finite vertex sets for the G-action on X, which are finite by the properness of the G-action on X. $\square$

We close the section with a simple but useful observation, that will allow us to reprove contractibility of the universal cover of the Salvetti complex of an FC-type Artin group in Sect. 5.

Proposition 3.11 *Let $(X, \{X_i\}_{i\in I})$ be a simply connected locally bounded generalized cell Helly complex such that the intersection of any collection of generalized cells is either empty or contractible. Then X is contractible.*

Proof By Borsuk's Nerve Theorem [27] applied to the covering of X by generalized cells $\{X_i\}$, and the covering of Y by simplices corresponding to generalized cells, we have that X is homotopically equivalent to its thickening. The latter is contractible as a Helly complex by Theorem 3.9. $\square$

4 Weak Garside groups act on Helly graphs

4.1 Basic properties of Garside categories

We follow the treatment of Garside categories in [13,68].

Let $\mathcal{C}$ be a small category. One may think of $\mathcal{C}$ as of an oriented graph, whose vertices are objects in $\mathcal{C}$ and oriented edges are morphisms of $\mathcal{C}$. Arrows in $\mathcal{C}$ compose like paths: $x \stackrel{f}{\to} y \stackrel{g}{\to} z$ is composed into $x \stackrel{fg}{\to} z$. For objects $x, y \in \mathcal{C}$, let $\mathcal{C}_{x\to}$ denote the collection of morphisms whose source object is x. Similarly we define $\mathcal{C}_{\to y}$ and $\mathcal{C}_{x\to y}$.

For two morphisms f and g, we define $f \preccurlyeq g$ if there exists a morphism h such that $g = fh$. Define $g \succcurlyeq f$ if there exists a morphism h such that $g = hf$. Then $(\mathcal{C}_{x\to}, \preccurlyeq)$ and $(\mathcal{C}_{\to y}, \succcurlyeq)$ are posets. A nontrivial morphism f which cannot be factorized into two nontrivial factors is an *atom*.

The category $\mathcal{C}$ is *cancellative* if, whenever a relation $afb = agb$ holds between composed morphisms, it implies $f = g$. $\mathcal{C}$ is *homogeneous* if there exists a length function l from the set of $\mathcal{C}$-morphisms to $\mathbb{Z}_{\geq 0}$ such that $l(fg) = l(f) + l(g)$ and $(l(f) = 0) \Leftrightarrow (f$ is a unit).

We consider the triple $(\mathcal{C}, \mathcal{C} \stackrel{\phi}{\to} \mathcal{C}, 1_{\mathcal{C}} \stackrel{\Delta}{\Rightarrow} \phi)$ where ϕ is an automorphism of $\mathcal{C}$ and Δ is a natural transformation from the identity function to ϕ. For an object $x \in \mathcal{C}$, Δ gives morphisms $x \stackrel{\Delta(x)}{\longrightarrow} \phi(x)$ and $\phi^{-1}(x) \stackrel{\Delta(\phi^{-1}(x))}{\longrightarrow} x$. We denote the first morphism by Δ_x and the second morphism by Δ^x. A morphism $x \stackrel{f}{\to} y$ is *simple* if there exists a morphism $y \stackrel{f^*}{\to} \phi(x)$ such that $ff^* = \Delta_x$. When $\mathcal{C}$ is cancellative, such f^* is unique.

Definition 4.1 A *homogeneous categorical Garside structure* is a triple $(\mathcal{C}, \mathcal{C} \xrightarrow{\phi} \mathcal{C}, 1_{\mathcal{C}} \overset{\Delta}{\Rightarrow} \phi)$ such that:

(1) ϕ is an automorphism of $\mathcal{C}$ and Δ is a natural transformation from the identity function to ϕ;
(2) $\mathcal{C}$ is homogeneous and cancellative;
(3) all atoms of $\mathcal{C}$ are simple;
(4) for any object x, $\mathcal{C}_{x\to}$ and $\mathcal{C}_{\to x}$ are lattices.

It has *finite type* if the collection of simple morphisms of $\mathcal{C}$ is finite.

Definition 4.2 A *Garside category* is a category $\mathcal{C}$ that can be equipped with ϕ and Δ to obtain a homogeneous categorical Garside structure. A *Garside groupoid* is the enveloping groupoid of a Garside category. See [40, Section 3.1] for a detailed definition of enveloping groupoid. Informally speaking, it is a groupoid obtained by adding formal inverses to all morphisms in a Garside category.

Let x be an object in a groupoid $\mathcal{G}$. The *isotropy* group $\mathcal{G}_x$ at x is the group of morphisms from x to itself. A *weak Garside group* is a group isomorphic to the isotropy group of an object in a Garside groupoid. A weak Garside group has *finite type* if its associated Garside category has finite type.

A *Garside monoid* is a Garside category with a single object and a *Garside group* is a Garside groupoid with a single object.

It follows from [40, Proposition 3.11] that if $\mathcal{G}$ is a Garside groupoid, then the natural functor $\mathcal{C} \to \mathcal{G}$ is injective. Moreover, by [40, lemma 3.13], the triple $(\mathcal{C}, \mathcal{C} \xrightarrow{\phi} \mathcal{C}, 1_{\mathcal{C}} \overset{\Delta}{\Rightarrow} \phi)$ naturally extends to $(\mathcal{G}, \mathcal{G} \xrightarrow{\phi} \mathcal{G}, 1_{\mathcal{G}} \overset{\Delta}{\Rightarrow} \phi)$ (we denote the extensions of ϕ and Δ by ϕ and Δ as well).

Define $\mathcal{G}_{x\to}$ and $\mathcal{G}_{\to x}$ in a similar way to $\mathcal{C}_{x\to}$ and $\mathcal{C}_{\to x}$. For two morphisms x and y of $\mathcal{G}$, define $x \preccurlyeq y$ if there exists a morphism z of $\mathcal{C}$ such that $y = xz$. Define $y \succcurlyeq x$ analogously. The following Lemma 4.3 is well-known for inclusion from a Garside monoid to its associated Garside group. The more general case concerning Garside category can be proved similarly.

Lemma 4.3 *For each object x of $\mathcal{G}$, $\mathcal{G}_{x\to}$ and $\mathcal{G}_{\to x}$ are lattices. Moreover, the natural inclusions $\mathcal{C}_{x\to} \to \mathcal{G}_{x\to}$ and $\mathcal{C}_{\to x} \to \mathcal{G}_{\to x}$ are homomorphisms between lattices.*

For morphisms $f_1, f_2 \in \mathcal{G}_{x\to}$ (resp. $\mathcal{G}_{\to x}$), we use $f_1 \wedge_p f_2$ (resp. $f_1 \wedge_s f_2$) to denote the greatest lower bound of f_1 and f_2 with respect to $\preccurlyeq$ (resp. $\succcurlyeq$), where p (resp. s) stands for prefix (resp. suffix) order. Similarly, we define $f_1 \vee_p f_2$ and $f_1 \vee_s f_2$.

Lemma 4.4 *Let $\mathcal{G}$ be a Garside groupoid and let $f : x \to y$ be a morphism of $\mathcal{G}$. Then*

(1) the map $(\mathcal{G}_{y\rightarrow}, \preccurlyeq) \rightarrow (\mathcal{G}_{x\rightarrow}, \preccurlyeq)$ *mapping* $h \in \mathcal{G}_y$ *to* fh *is a lattice monomorphism;*

(2) the bijection $(\mathcal{G}_{x\rightarrow}, \preccurlyeq) \rightarrow (\mathcal{G}_{\rightarrow x}, \succcurlyeq)$ *mapping* $h \in \mathcal{G}_x$ *to* h^{-1} *is order-revering, i.e.* $h_1 \preccurlyeq h_2$ *if and only if* $h_1^{-1} \succcurlyeq h_2^{-1}$*;*

(3) choose elements $\{a_i\}_{i=1}^n \subset \mathcal{G}_{\rightarrow x}$*, then* $a_1^{-1} \vee_p a_2^{-1} \vee_p \cdots \vee_p a_n^{-1} = (a_1 \wedge_s a_2 \wedge_s \cdots \wedge_s a_n)^{-1}$ *and* $a_1^{-1} \wedge_p a_2^{-1} \wedge_p \cdots \wedge_p a_n^{-1} = (a_1 \vee_s a_2 \vee_s \cdots \vee_s a_n)^{-1}$.

Proof Assertion (1) follows from the definition. To see (2), note that if $h_1 \preccurlyeq h_2$, then there exists a morphism k of $\mathcal{C}$ such that $h_2 = h_1 k$. Thus $h_1^{-1} = k h_2^{-1}$. Hence $h_1^{-1} \succcurlyeq h_2^{-1}$. The other direction is similar. (3) follows from (2). □

Lemma 4.5 *Let* $\mathcal{G}$ *be a Garside groupoid. Let* $\{a_i\}_{i=1}^n$ *and* c *be morphisms in* $\mathcal{G}$*. Let* $\{b_i\}_{i=1}^n$ *be morphisms in* $\mathcal{C}$*.*

(1) Suppose $a_i b_i = c$ *for* $1 \leq i \leq n$*. Let* $a = a_1 \vee_p a_2 \vee_p \cdots \vee_p a_n$ *and* $b = b_1 \wedge_s b_2 \wedge_s \cdots \wedge_s b_n$*. Then* $ab = c$*.*

(2) Suppose $a_i b_i = c$ *for* $1 \leq i \leq n$*. Let* $a = a_1 \wedge_p a_2 \wedge_p \cdots \wedge_p a_n$ *and* $b = b_1 \vee_s b_2 \vee_s \cdots \vee_s b_n$*. Then* $ab = c$*.*

(3) $(cb_1) \wedge_p (cb_2) \wedge_p \cdots \wedge_p (cb_n) = c(b_1 \wedge_p \cdots \wedge_p b_n)$*.*

(4) $(cb_1) \vee_p (cb_2) \vee_p \cdots \vee_p (cb_n) = c(b_1 \vee_p \cdots \vee_p b_n)$*.*

Proof For (1), note that all the a_i have the same source object and all the b_i have the same target object. Thus a and b are well-defined. It follows from Lemma 4.4 (1) that $c^{-1}a = (c^{-1}a_1) \vee_p (c^{-1}a_2) \vee_p \cdots \vee_p (c^{-1}a_n) = b_1^{-1} \vee_p b_2^{-1} \vee_p \cdots \vee_p b_n^{-1}$. By Lemma 4.4 (3), $b_1^{-1} \vee_p \cdots \vee_p b_n^{-1} = (b_1 \wedge_s b_2 \wedge_s \cdots \wedge_s b_n)^{-1} = b^{-1}$. Thus (1) follows. Assertion (2) can be proved similarly. (3) and (4) follow from Lemma 4.4 (1). □

For a morphism f of $\mathcal{G}$, denote the source object and the target object of f by $s(f)$ and $t(f)$.

Lemma 4.6 *Suppose* f *is a morphism of* $\mathcal{G}$*. Then* $f \preccurlyeq \Delta_{s(f)}$ *if and only if* $\Delta^{t(f)} \succcurlyeq f$*.*

Proof If $f \preccurlyeq \Delta_{s(f)}$, then $\Delta_{s(f)} = fg$ for g in $\mathcal{C}$. Let h be the unique morphism such that $g = \phi(h)$. By the definition of ϕ, we have that h is in $\mathcal{C}$. Moreover, h fits into the following commuting diagram:

$$
\begin{tikzcd}
\phi^{-1}(t(f)) \arrow[r, "h"] \arrow[d, "\Delta^{t(f)}"] & s(f) \arrow[d, "\Delta_{s(f)}"] \\
t(f) = s(g) \arrow[r, "g"] & t(g) = t(\Delta_{s(f)})
\end{tikzcd}
$$

Thus $h\Delta_{s(f)} = \Delta^{t(f)} g$, which implies that $h(fg) = \Delta^{t(f)} g$. Hence $hf = \Delta^{t(f)}$ and $\Delta^{t(f)} \succcurlyeq f$. The other direction is similar. □

Lemma 4.7 *Let x be an object of $\mathcal{G}$. Let Φ_1 be the collection of simple morphisms of $\mathcal{G}$ with target object x and let Φ_2 be the collection of simple morphisms with source object x. For $a \in \Phi_1$, we define a^* to be the morphism such that $aa^* = \Delta_{s(a)}$. Then*

(1) the map $a \to a^*$ gives a bijection $\Phi_1 \to \Phi_2$;
(2) for $a, b \in \Phi_1$, we have $b \succcurlyeq a$ if and only if $b^* \preccurlyeq a^*$;
(3) for $a, b \in \Phi_1$, we have $(a \wedge_s b)^* = a^* \vee_p b^*$ and $(a \vee_s b)^* = a^* \wedge_p b^*$.

Proof For (1), since $\Delta^{t(a^*)} = \Delta_{s(a)} \succcurlyeq a^*$, we have $a^* \preccurlyeq \Delta_{s(a^*)}$ by Lemma 4.6. Thus $a^* \in \Phi_2$. The map is clearly injective and surjectivity follows from Lemma 4.6. For (2), $b \succcurlyeq a$ implies $b = ka$ for a morphism k of $\mathcal{C}$. From $\Delta_{s(b)} = bb^* = kab^*$, we have that $\Delta^{t(b^*)} = \Delta_{s(b)} \succcurlyeq ab^*$, hence $ab^* \preccurlyeq \Delta_{s(a)}$ by Lemma 4.6. Then there is a morphism h of $\mathcal{C}$ such that $ab^*h = \Delta_{s(a)} = aa^*$, hence $b^*h = a^*$ and $b^* \preccurlyeq a^*$. The other direction is similar. For (3), note that elements in Φ_1 (resp. Φ_2) are closed under $\wedge_s$ and $\vee_s$ (resp. $\wedge_p$ and $\vee_p$), so Φ_1 and Φ_2 are lattices. Now assertion (3) follows from (2) and (1). □

4.2 Garside groups are Helly

Let $\mathcal{G}$ be a Garside groupoid and let x be an object of $\mathcal{G}$. A *cell* of $\mathcal{G}_{x\to}$ is an interval of the form $[f, f\Delta_{t(f)}]$ with respect to the order $\preccurlyeq$.

Lemma 4.8 *Let C_1, C_2 and C_3 be pairwise intersecting cells of $\mathcal{G}_{x\to}$. Then there is a cell C of $\mathcal{G}_{x\to}$ such that $D = (C_1 \cap C_2) \cup (C_2 \cap C_3) \cup (C_3 \cap C_1)$ is contained in C.*

Proof Since each C_i is an interval in $(\mathcal{G}_{x\to}, \preccurlyeq)$, we write $C_i = [f_i, g_i]$. Let $C_{ij} = C_i \cap C_j$. Then each C_{ij} is also an interval (Lemma 2.1 (1)). We write $C_{ij} = [f_{ij}, g_{ij}]$. Then $f_{ij} = f_i \vee_p f_j$ and $g_{ij} = g_i \wedge_p g_j$. Choose a morphism h in $\cap_{i=1}^3 C_i$ (Lemma 2.1 (2)). Since $f_i \preccurlyeq h \preccurlyeq g_i$, there are morphisms w_i and w_i^* in $\mathcal{C}$ such that $h = f_i w_i$ and $g_i = hw_i^*$. Note that $w_i w_i^* = \Delta_{s(f_i)}$. Moreover, by Lemma 4.5 (1) and (3), we have

$$f_{ij} \cdot (w_i \wedge_s w_j) = h \text{ and } h \cdot (w_i^* \wedge_p w_j^*) = g_{ij}. \tag{1}$$

Let f (resp. g) be the greatest lower bound (resp. least upper bound) over all elements in D with respect to the prefix order $\preccurlyeq$. Since D is a union of three intervals, we have $f = f_{12} \wedge_p f_{23} \wedge_p f_{31}$ and $g = g_{12} \vee_p g_{23} \vee_p g_{31}$. Define $a = (w_1 \wedge_s w_2) \vee_s (w_2 \wedge_s w_3) \vee_s (w_3 \wedge_s w_1)$ and $b = (w_1^* \wedge_p w_2^*) \vee_p (w_2^* \wedge_p w_3^*) \vee_p (w_3^* \wedge_p w_1^*)$. Then by the formula (1) above and Lemma 4.5 (2) and (4), we have $fa = h$ and $hb = g$.

We claim $ab \preccurlyeq \Delta_{s(a)}$. This claim implies that the cell $[f, f\Delta_{s(a)}]$ contains the interval $[f, g]$, hence it contains D. Thus the lemma follows from the claim.

Now we prove the claim. Since $w_i \preccurlyeq \Delta_{s(w_i)}$ for $1 \le i \le 3$, by Lemma 4.6, $\Delta_{t(w_i)} \succcurlyeq w_i$. Then $\Delta_{t(w_i)} \succcurlyeq a$. By Lemma 4.6 again, $a \preccurlyeq \Delta_{s(a)}$. Let a^* be a morphism in $\mathcal{C}$ such that $aa^* = \Delta_{s(a)}$. It suffices to show $b \preccurlyeq a^*$. By repeatedly applying Lemma 4.7 (3), we have $a^* = (w_1 \wedge_s w_2)^* \wedge_p (w_2 \wedge_s w_3)^* \wedge_p (w_3 \wedge_s w_1)^* = (w_1^* \vee_p w_2^*) \wedge_p (w_2^* \vee_p w_3^*) \wedge_p (w_3^* \vee_p w_1^*)$. Note that b is the least upper bound of three terms, however, each of these terms is $\preccurlyeq$ every term of a^*, so each of these terms is $\preccurlyeq a^*$, hence $b \preccurlyeq a^*$. □

Theorem 4.9 *Let G be a weak Garside group of finite type associated with a Garside groupoid $\mathcal{G}$. Then G acts geometrically on a Helly graph.*

Proof Let x be an object of $\mathcal{G}$. Suppose $G = \mathcal{G}_x$. Let Y be a simplicial graph whose vertex set is $\mathcal{G}_{x\to}$ and two vertices are adjacent if they are in the same cell of $\mathcal{G}_{x\to}$. The left action $\mathcal{G}_x \curvearrowright \mathcal{G}_{x\to}$ preserves the collection of cells, hence induces an action $\mathcal{G}_x \curvearrowright Y$.

Note that Y is uniformly locally finite. Indeed, given $f \in \mathcal{G}_{x\to}$, g is adjacent to f if and only if $g = fa^{-1}b$ for some simple morphisms a and b of $\mathcal{C}$. Since the collection of simple morphisms is finite, there are finitely many possibilities for such g. Thus Y satisfies assumption (1) of Lemma 3.3 with $\{X_i\}_{i\in I}$ being cells of $\mathcal{G}_{x\to}$. Assumption (2) of Lemma 3.3 follows from Lemma 2.1 (2), as each cell is an interval in the lattice $\mathcal{G}_{x\to}$. Assumption (3) of Lemma 3.3 follows from Lemma 4.8. Thus Y is finitely clique Helly by Lemma 3.3. By Theorem 3.2, it remains to show that the flag complex $F(Y)$ of Y is simply connected.

Let Z be the cover graph of the poset $(\mathcal{G}_{x\to}, \preccurlyeq)$, i.e. two vertices f and h are adjacent if $f = hk$ or $h = fk$ for some atom k. Since all atoms are simple, Z is a subgraph of Y. As every morphism in $\mathcal{C}$ can be decomposed into a product of finitely many atoms by Definition 4.1 (2), each edge of Y is homotopic rel its end points in $F(Y)$ to an edge path in Z. Let $\omega \subset Y$ be an edge loop. We may assume $\omega \subset Z$ up to homotopy in $F(Y)$. Let $f \in \mathcal{G}_{x\to}$ be a common lower bound for the collection of vertices of ω. We denote consecutive vertices of ω by $\{ff_i\}_{i=1}^n$ where each f_i is a morphism of $\mathcal{C}$. We define the complexity $\tau(\omega)$ of ω to be $\max_{1\le i\le n}\{l(f_i)\}$, where l is the length function from Sect. 4.1. Suppose $l(f_{i_0}) = \tau(\omega)$. Then, by definition of l, we have $l(f_{i_0-1}) < l(f_{i_0})$ and $l(f_{i_0+1}) < l(f_{i_0})$. Thus $f_{i_0} = f_{i_0-1}a_1$ and $f_{i_0} = f_{i_0+1}a_2$ for atoms a_1 and a_2 of $\mathcal{C}$. Let $f'_{i_0} = f_{i_0-1} \wedge_p f_{i_0+1}$. Let ω_1 (resp. ω_2) be the edge path in Z from f'_{i_0} to f_{i_0-1} (resp. f_{i_0+1}) corresponding to a decomposition of $(f'_{i_0})^{-1}f_{i_0-1}$ (resp. $(f'_{i_0})^{-1}f_{i_0+1}$) into atoms (ω_i might be a point). We replace the sub-path $\overline{f_{i_0-1}f_{i_0}} \cup \overline{f_{i_0+1}f_{i_0}}$ of ω by $\omega_1 \cup \omega_2$ to obtain a new edge loop $\omega' \subset Z$. Clearly $\tau(\omega') < \tau(\omega)$. Moreover, since $f'_{i_0}(a_1 \vee_s a_2) = f_{i_0}$ (Lemma 4.5) and $a_1 \vee_s a_2$ is simple, by Lemma 4.6 $\overline{f_{i_0-1}f_{i_0}} \cup \overline{f_{i_0+1}f_{i_0}} \cup \omega_1 \cup \omega_2$ is contained in a simplex of $F(Y)$. Thus ω and ω' are homotopic in $F(Y)$. By repeating this process, we can homotop ω to a constant path, which finishes the proof. □

Remark 4.10 The simple connectedness of $F(Y)$ above be can alternatively deduced from [13, Corollary 7.6], where Bessis considers a graph Γ on $\mathcal{G}_{x\rightarrow}$ such that two vertices are adjacent if they differ by a simple morphism in $\mathcal{C}$. It is proved that the flag complex $F(\Gamma)$ is contractible. As Γ is a proper subgraph of Y, it is not hard to deduce the simply-connectedness of $F(Y)$. The definition of Γ goes back to work of Bestvina [15] in the case of spherical Artin groups, and Charney-Meier-Whittlesey [37] in the case of Garside group.

For a braid group G, Theorem 4.9 gives rise to many different Helly graphs (hence injective metric spaces) upon which G acts geometrically. One can choose different Garside structures on G: either we use the interval $[1, \Delta^k]$ as a cell, or we use the dual Garside structure [12,21]. One can also apply some combinatorial operations to a Helly graph—e.g. taking the Rips complex or the face complex—in order to obtain a new Helly graph (see [31]).

Corollary 4.11 *Let $\mathcal{A}$ be a finite simplicial central arrangement of hyperplanes in a finite dimensional real vector space V. Let* $\mathrm{Sal}(\mathcal{A})$ *be its Salvetti complex and let* $\widetilde{\mathrm{Sal}}(\mathcal{A})$ *be the universal cover of* $\mathrm{Sal}(\mathcal{A})$. *Then* $\widetilde{\mathrm{Sal}}(\mathcal{A})$ *is cell Helly.*

Proof We refer to [74,80] for relevant definitions and background. Let $M(\mathcal{A}) = V_{\mathbb{C}} - \left(\bigcup_{H\in\mathcal{A}} H_{\mathbb{C}}\right)$ and $G = \pi_1(M(\mathcal{A}))$. Then G is a weak Garside group [42]. G can be identified as vertices of $\widetilde{\mathrm{Sal}}(\mathcal{A})$, and a cell of G in the sense of Lemma 4.8 can be identified with the vertex set of a top dimensional cell in $\widetilde{\mathrm{Sal}}(\mathcal{A})$, moreover, the vertex set of each cell of $\widetilde{\mathrm{Sal}}(\mathcal{A})$ can be identified with an interval in G. Thus Definition 3.4 (2) holds. Similarly to Lemma 2.11, one can prove that any closed cell in $\widetilde{\mathrm{Sal}}(\mathcal{A})$ is a full subcomplex. Hence any intersection of cells is a full subcomplex spanned by vertices in an interval of one cell. Therefore, Definition 3.4 (3) holds. Moreover, the fact that cells are full together with Lemma 4.8 imply that the collection of top dimensional cells in $\widetilde{\mathrm{Sal}}(\mathcal{A})$ satisfies Definition 3.4 (3). Therefore, the same holds for the collection of cells in $\widetilde{\mathrm{Sal}}(\mathcal{A})$ as every cell is contained in a top dimensional cell. □

5 Artin groups and Helly graphs

5.1 Helly property for cells in X_Γ

Recall that $X^{(0)}_\Gamma$ is endowed with the prefix order $\preccurlyeq$. For a cell $C \in X_\Gamma$, we use $\preccurlyeq_C$ to denote the right weak order on $C^{(0)}$ (we view C as an oriented Coxeter cell). And for a standard subcomplex $X_1 \subset X_\Gamma$ of type Γ_1, we use $\preccurlyeq_{\Gamma_1}$ to denote the prefix order on $X^{(0)}_1$ coming from A_{Γ_1}.

Lemma 5.1 *Let $C_1, C_2 \subset X_\Gamma$ be two cells with non-empty intersection. Then the vertex set of $C_1 \cap C_2$ is an interval in $(X_\Gamma^{(0)}, \preccurlyeq)$.*

Proof For $i = 1, 2$, suppose C_i is of type Γ_i and let X_i be the standard subcomplex of type Γ_i containing C_i. Let $X_0 = X_1 \cap X_2$, which is a standard subcomplex of type $\Gamma_0 = \Gamma_1 \cap \Gamma_2$. Let $D_i = C_i \cap X_0$. Then D_i is a cell by Lemma 2.13. Moreover, $D_1^{(0)}$ and $D_2^{(0)}$ are intervals in $(X_0^{(0)}, \preccurlyeq_{\Gamma_0})$ by Lemma 2.9. As $(X_0^{(0)}, \preccurlyeq_{\Gamma_0})$ is a lattice (Theorem 2.4), $(C_1 \cap C_2)^{(0)} = (D_1 \cap D_2)^{(0)}$ is an interval in $(X_0^{(0)}, \preccurlyeq_{\Gamma_0})$ (Lemma 2.1), hence it is also an interval in $(X_\Gamma^{(0)}, \preccurlyeq)$ by Lemma 2.13 (3) and (4). □

Lemma 5.2 *Let C be an oriented Coxeter cell. Let K be the full subcomplex of C spanned by vertices in an interval of $C^{(0)}$. Then K is contractible.*

Proof Let $[x, y]$ be an interval of $C^{(0)}$. By [6, Proposition 3.1.6], we can assume x is the source of C. Let $\ell : C^{(0)} \to \mathbb{Z}$ be the function measuring the combinatorial distance from x to a given point in $C^{(0)}$. We extend ℓ to $C \to \mathbb{R}$ such that ℓ is affine when restricted to each cell. Note that ℓ is non-constant on each cell with dimension > 0. Then $\ell|_K$ is a Morse function in the sense of [7, Definition 2.2]. Moreover, it follows from [28, Chapter IV, Exercise 22] that the descending link ([7, Definition 2.4]) of each vertex for $\ell|_K$ is a simplex, hence contractible. Thus K is contractible by [7, Lemma 2.5]. □

Lemma 5.3 *Suppose Γ is spherical. Let C_1 and C_2 be two cells in X_Γ, and let X_0 be a standard subcomplex of X_Γ of type Γ_0. Suppose that C_1, C_2, and X_0 pairwise intersect. Then their triple intersection is nonempty.*

Proof For $i = 1, 2$, let v_i be a vertex of $C_i \cap X_0$. Since $(X^{(0)}, \preccurlyeq_{\Gamma_0})$ is a lattice by Theorem 2.4, we can find $a, b \in X^{(0)}$ such that $a \preccurlyeq_{\Gamma_0} v_i \preccurlyeq_{\Gamma_0} b$ for $i = 1, 2$. Let I be the interval between a and b with respect to the prefix order on $X_\Gamma^{(0)}$. Then $I \subset X_0$ by Lemma 2.13 (3). Note that C_1, C_2 and I pairwise intersect by construction. Since $C_i^{(0)}$ is an interval in $(X_\Gamma^{(0)}, \preccurlyeq)$ for $i = 1, 2$ (Lemma 2.9 (2)), and $(X_\Gamma^{(0)}, \preccurlyeq)$ is a lattice (recall that Γ is spherical), we have $C_1 \cap C_2 \cap I \neq \emptyset$ by Lemma 2.1 (2). Since $I \subset X_0$, the lemma follows. □

Lemma 5.4 *Let C_1, C_2, C_3 be three cells in X_Γ. Suppose they pairwise intersect. Then $C_1 \cap C_2 \cap C_3 \neq \emptyset$.*

Proof Let Γ_i be the type of C_i and let X_i be the standard subcomplex of type Γ_i containing C_i. Let $\Gamma_0 = \cap_{i=1}^{3} \Gamma_i$.

Case 1: $\Gamma_1 = \Gamma_2 = \Gamma_3$. Then $X_1 = X_2 = X_3$. Since each $C_i^{(0)}$ is an interval in $(X_1^{(0)}, \preccurlyeq_{\Gamma_1})$ (Lemma 2.9) and $(X_1^{(0)}, \preccurlyeq_{\Gamma_1})$ is a lattice (Theorem 2.4), the lemma follows from Lemma 2.1.

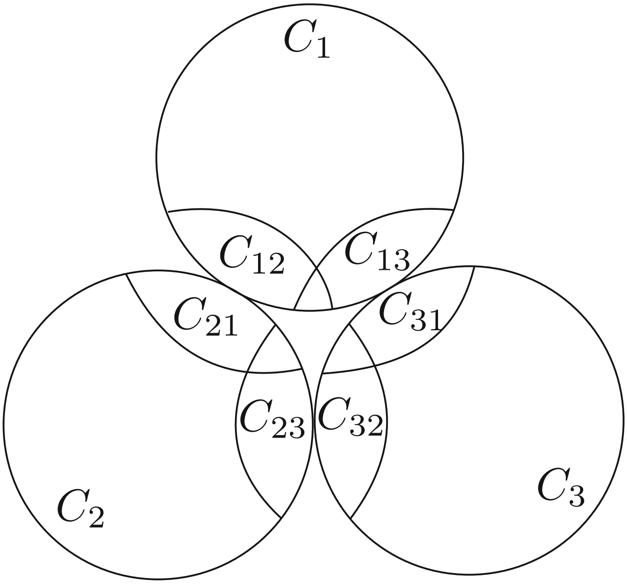

Fig. 1 Lemma 5.4, Case 3

Case 2: only two of the Γ_i *are equal*. We suppose without loss of generality that $\Gamma_1 = \Gamma_2 \neq \Gamma_3$. Let $C_3' = C_3 \cap X_1$. Then C_3' is a cell by Lemma 2.13. Moreover, C_1, C_2 and C_3' pairwise intersect. As in the previous case, we have $C_1 \cap C_2 \cap C_3' \neq \emptyset$, hence the lemma follows.

Case 3: the Γ_i *are pairwise distinct*. For $1 \leq i \neq j \leq 3$, let $C_{ij} = C_i \cap X_j$ (see Fig. 1). We claim that $C_{12} \cap C_{13}$ is a cell of type Γ_0 (when $\Gamma_0 = \emptyset$, the intersection is a vertex). Consider the cellular map $\pi : X_\Gamma \to D_\Gamma$ defined just before Lemma 2.9. Let $\bar{C}_i$ be the image of C_i. Note that C_{12} is a cell of type $\Gamma_1 \cap \Gamma_2$ (Lemma 2.13), and $\bar{C}_1 \cap \bar{C}_2$ is a cell of type $\Gamma_1 \cap \Gamma_2$ (Theorem 2.2), hence $\pi(C_{12}) = \bar{C}_1 \cap \bar{C}_2$. Similarly, $\pi(C_{13}) = \bar{C}_1 \cap \bar{C}_3$. Since $\{\bar{C}_i\}_{i=1}^3$ pairwise intersect, their triple intersection is nonempty by Lemma 2.7. Thus $\pi(C_{12}) \cap \pi(C_{13}) \neq \emptyset$. Hence $\pi(C_{12}) \cap \pi(C_{13})$ is a cell of type Γ_0. Now, the claim follows from the fact that π restricted to C_1 is an embedding.

For $i = 1, 2, 3$, let $\tau_i = C_{i,i+1} \cap C_{i,i+2}$ (indices are taken mod 3). By the previous paragraph, each τ_i is a cell of type Γ_0. Let $X_0 = \cap_{i=1}^3 X_i$. Then $\tau_i \subset X_0$ for $1 \leq i \leq 3$. In particular X_0 is nonempty, hence is a standard subcomplex of type Γ_0 by Lemma 2.13 (1). Now we claim $\cap_{i=1}^3 \tau_i \neq \emptyset$. Note that the lemma follows from this claim since $\tau_i \subset C_i$ for $1 \leq i \leq 3$.

Since each $\tau_i^{(0)}$ is an interval in the lattice $(X_0^{(0)}, \preccurlyeq_{\Gamma_0})$, it suffices to show that τ_i pairwise intersect. We only prove $\tau_1 \cap \tau_2 \neq \emptyset$, as the other pairs are similar. Now, consider the triple $\{C_{12}, C_{21}, X_0\}$. Note that $\tau_1 \subset C_{12}$, $\tau_2 \subset C_{21}$ and $\tau_i \subset X_0$ for $i = 1, 2$. Therefore $\tau_1 \cap \tau_2 \subset C_{12} \cap C_{21} \cap X_0$. On the other hand, since C_{12} is a cell of type $\Gamma_1 \cap \Gamma_2$ and $C_{12} \cap X_0 \supset \tau_1 \neq \emptyset$, we have that $C_{12} \cap X_0$ is a cell of type Γ_0 from Lemma 2.13, hence $C_{12} \cap X_0 = \tau_1$. Similarly $C_{21} \cap X_0 = \tau_2$. Thus actually $\tau_1 \cap \tau_2 = C_{12} \cap C_{21} \cap X_0$. It remains to show that $C_{12} \cap C_{21} \cap X_0 \neq \emptyset$ This follows from Lemma 5.3, since C_{12}, C_{21}, X_0 are contained in $X_1 \cap X_2$, which is a spherical standard subcomplex. □

Proposition 5.5 *Let $\{C_i\}_{i=1}^n$ be a finite collection of cells in X_Γ. Suppose they pairwise intersect. Then $\cap_{i=1}^n C_i$ is a nonempty contractible subcomplex of X_Γ.*

Proof For $2 \leq i \leq n$, let $\tau_i = C_1 \cap C_i$. Then $\tau_i^{(0)}$ is an interval in $(X_\Gamma^{(0)}, \preccurlyeq)$, hence it is also an interval in $(C_1^{(0)}, \preccurlyeq_C)$ by Lemma 2.9 (2). Moreover, it follows from Lemma 5.4 that $\{\tau_i\}_{i=2}^n$ pairwise intersect. Since $(C_1^{(0)}, \preccurlyeq_C)$ is a lattice (Theorem 2.4 (1)), we have that $\cap_{i=2}^n \tau_i^{(0)}$ is a nonempty interval in $(C_1^{(0)}, \preccurlyeq_C)$ by Lemma 2.1, hence $\cap_{i=1}^n C_i \neq \emptyset$. Since each C_i is a full subcomplex of X_Γ (Lemma 2.11), $\cap_{i=1}^n C_i$ is full in X_Γ (hence is full in C_1). Now, contractibility of $\cap_{i=1}^n C_i$ follows from Lemma 5.2. □

5.2 Artin groups of type FC act on Helly graphs

Lemma 5.6 *Suppose Γ is spherical. Let C_1, C_2 and C_3 be pairwise intersecting cells in X_Γ. Then there is a cell $C \subset X_\Gamma$ such that $(C_1 \cap C_2) \cup (C_2 \cap C_3) \cup (C_3 \cap C_1) \subseteq C$.*

Proof It suffices to consider the special case when each C_i is a maximal cell of X_Γ. Consider the cell C of type Γ in X_Γ with the source vertex being the identity. Let Δ be the sink vertex of C. Then by [30,42], A_Γ is a Garside group with A_Γ^+ being its Garside monoid and Δ being its Garside element (the ϕ in Definition 4.1 corresponds to conjugating by Δ). Moreover, cells in the sense of Lemma 4.8 correspond to vertices of maximal cells of X_Γ. Thus the lemma follows from Lemmas 4.8 and 2.11. □

Proposition 5.7 *Suppose that Γ is of type FC. Let C_1, C_2 and C_3 be maximal pairwise intersecting cells in X_Γ. Then there is a maximal cell $C \subset X_\Gamma$ such that $D = (C_1 \cap C_2) \cup (C_2 \cap C_3) \cup (C_3 \cap C_1) \subseteq C$.*

Proof For $1 \leq i \leq 3$, suppose C_i is of type Γ_i and let X_i be the standard spherical subcomplex of type Γ_i. If $X_1 = X_2 = X_3$, then the proposition follows from Lemma 5.6.

Suppose two of the X_i are the same. We assume without loss of generality that $X_1 = X_2 \neq X_3$. Let $C_3' = C_3 \cap X_1$. Then $C_i \cap C_3 = C_i \cap C_3'$ for $i = 1, 2$. Hence $D \subset (C_1 \cap C_2) \cup (C_2 \cap C_3') \cup (C_3' \cap C_1)$. By Lemma 2.13 (2), C_3' is a cell in X_1. Then we are done by Lemma 5.6.

It remains to consider the case when X_1, X_2, and X_3 are mutually distinct. By Proposition 5.5, there is a vertex p in $\cap_{i=1}^{3} C_i$. Let $C_{ij} = C_i \cap C_j$ and $X_{ij} = X_i \cap X_j$. Then X_{ij} is a spherical standard subcomplex of type $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$ by Lemma 2.13 (1) (but $C_{ij} \subset X_{ij}$ might not be a cell in general). Since Γ is of type FC, there is a spherical full subgraph $\Gamma_0 \subset \Gamma$ such that Γ_0 contains $\Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{31}$. Let X_0 be the spherical standard subcomplex of X_Γ of type Γ_0 that contains p. Then for $1 \leq i \neq j \leq 3$, we have $p \in C_i \cap C_j \subset X_{ij} \subset X_0$. It follows that $D \subset (C'_1 \cap C'_2) \cup (C'_2 \cap C'_3) \cup (C'_3 \cap C'_1)$ where $C'_i = C_i \cap X_0$. Since each C'_i is a cell (Lemma 2.13 (2)) and X_0 is spherical, the proposition follows from Lemma 5.6. □

Theorem 5.8 *Suppose Γ is of type FC. Then $(X_\Gamma, \{C_i\})$ is a locally bounded cell Helly complex and, consequently, A_Γ acts geometrically on a Helly graph being the 1-skeleton of the thickening of $(X_\Gamma, \{C_i\})$. Moreover, $X(\Gamma)$ is contractible.*

Proof The conditions (1) and (2) of Definition 3.4 are satisfied by Proposition 5.5. The condition (3) holds by Proposition 5.7. Since X_Γ is simply connected, the theorem holds by Theorem 3.9 and Proposition 3.10. Contractibility follows from Propositions 5.5 and 3.11. □

Acknowledgements J.H. thanks the Max-Planck Institute of Mathematics (Bonn), where part of this work was done, for its hospitality. J.H. also thanks R. Charney for helpful discussions on Garside groups. We would like to thank V. Chepoi, T. Fukaya, and U. Lang for comments on earlier versions of the paper. Part of the work on the paper was carried out while D.O. was visiting McGill University. We would like to thank the Department of Mathematics and Statistics of McGill University for its hospitality during that stay. D.O. was partially supported by (Polish) Narodowe Centrum Nauki, UMO-2017/25/B/ST1/01335. We thank the organizers of the conference "Nonpositively Curved Groups in the Mediterranean", Nahsholim 2017, for inviting us. A part of the work on the paper was carried out there. This work was partially supported by the Grant 346300 for IMPAN from the Simons Foundation and the matching 2015–2019 Polish MNiSW fund.

References

1. Abrams, A., Brady, N., Dani, P., Duchin, M., Young, R.: Pushing fillings in right-angled Artin groups. J. Lond. Math. Soc. (2) **87**(3), 663–688 (2013)

2. Altobelli, J.A.: The word problem for Artin groups of FC type. J. Pure Appl. Algebra **129**(1), 1–22 (1998)
3. Aronszajn, N., Panitchpakdi, P.: Extension of uniformly continuous transformations and hyperconvex metric spaces. Pac. J. Math. **6**, 405–439 (1956)
4. Appel, K.I.: On Artin groups and Coxeter groups of large type. Contributions to group theory, pp. 50–78 (1984)
5. Appel, K.I., Schupp, P.E.: Artin groups and infinite Coxeter groups. Invent. Math. **72**(2), 201–220 (1983)
6. Björner, A., Brenti, F.: Combinatorics of Coxeter Groups. Graduate Texts in Mathematics, vol. 231. Springer, New York (2005)
7. Bestvina, M., Brady, N.: Morse theory and finiteness properties of groups. Invent. Math. **129**(3), 445–470 (1997)
8. Brady, N., Crisp, J.: Two-dimensional Artin groups with CAT(0) dimension three. In: Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000), pp. 185–214 (2002)
9. Buekenhout, F., Cohen, A.M..: Diagram geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 57, Springer, Heidelberg. Related to classical groups and buildings (2013)
10. Behrstock, J., Druţu, C.: Combinatorial higher dimensional isoperimetry and divergence. J. Topol. Anal. **11**(03), 499–534 (2019)
11. Bell, R.W.: Three-dimensional FC Artin groups are CAT(0). Geom. Dedic. **113**, 21–53 (2005)
12. Bessis, D.: The dual braid monoid. Ann. Sci. École Norm. Sup. (4) **36**(5), 647–683 (2003)
13. Bessis, D.: Garside categories, periodic loops and cyclic sets. arXiv:math/0610778 (2006)
14. Bessis, D.: Finite complex reflection arrangements are $K(\pi, 1)$. Ann. Math. (2) **181**(3), 809–904 (2015)
15. Bestvina, M.: Non-positively curved aspects of Artin groups of finite type. Geom. Topol. **3**, 269–302 (1999)
16. Bell, G.C., Fujiwara, K.: The asymptotic dimension of a curve graph is finite. J. Lond. Math. Soc. (2) **77**(1), 33–50 (2008)
17. Bridson, M.R., Haefliger, A.: Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319. Springer, Berlin (1999)
18. Birkhoff, G.: Lattice Theory, 3rd edn. American Mathematical Society Colloquium Publications, vol. XXV. American Mathematical Society, Providence (1967)
19. Björner, A.: Nerves, fibers and homotopy groups. J. Combin. Theory Ser. A **102**(1), 88–93 (2003)
20. Behrstock, J.A., Januszkiewicz, T., Neumann, W.D.: Quasi-isometric classification of some high dimensional right-angled Artin groups. Groups Geom. Dyn. **4**(4), 681–692 (2010)
21. Birman, J., Ko, K.H., Lee, S.J.: A new approach to the word and conjugacy problems in the braid groups. Adv. Math. **139**(2), 322–353 (1998)
22. Brück, B., Kielak, D., Wu, X.: The farrell–jones conjecture for normally poly-free groups. Proc. Am. Math. Soc. https://doi.org/10.1090/proc/15357
23. Bartels, A., Lück, W.: The Borel conjecture for hyperbolic and CAT(0)-groups. Ann. Math. (2) **175**(2), 631–689 (2012)
24. Brady, T., McCammond, J.P.: Three-generator Artin groups of large type are biautomatic. J. Pure Appl. Algebra **151**(1), 1–9 (2000)
25. Brady, T., McCammond, J.P.: Braids, posets and orthoschemes. Algebr. Geom. Topol. **10**(4), 2277–2314 (2010)
26. Behrstock, J.A., Neumann, W.D.: Quasi-isometric classification of graph manifold groups. Duke Math. J. **141**(2), 217–240 (2008)

27. Borsuk, K.: On the imbedding of systems of compacta in simplicial complexes. Fund. Math. **35**, 217–234 (1948)
28. Bourbaki, N.: Lie Groups and Lie Algebras. Chapters 4–6, Elements of Mathematics (Berlin). Springer, Berlin (2002). Translated from the 1968 French original by Andrew Pressley
29. Brady, T.: Artin groups of finite type with three generators. Mich. Math. J. **47**(2), 313–324 (2000)
30. Brieskorn, E., Saito, K.: Artin–Gruppen und Coxeter–Gruppen. Invent. Math. **17**, 245–271 (1972)
31. Chalopin, J., Chepoi, V., Genevois, A., Hirai, H., Osajda, D.: Helly groups. arXiv:2002.06895 (2020)
32. Chalopin, J., Chepoi, V., Hirai, H., Osajda, D.: Weakly modular graphs and nonpositive curvature. Mem. Am. Math. Soc. **268**(1309), 159 (2020)
33. Charney, R., Davis, M.W.: Finite $K(\pi, 1)s$ for Artin groups, Prospects in topology (Princeton, NJ, 1994), 1995, pp. 110–124 (1995)
34. Charney, R., Davis, M.W.: The $K(\pi, 1)$-problem for hyperplane complements associated to infinite re ection groups. J. Am. Math. Soc. **8**(3), 597–627 (1995)
35. Charney, R.: Artin groups of finite type are biautomatic. Math. Ann. **292**(4), 671–683 (1992)
36. Chouraqui, F.: Garside groups and Yang–Baxter equation. Commun. Algebra **38**(12), 4441–4460 (2010)
37. Charney, R., Meier, J., Whittlesey, K.: Bestvina's normal form complex and the homology of Garside groups. Geom. Dedic. **105**, 171–188 (2004)
38. Charney, R., Paris, L.: Convexity of parabolic subgroups in Artin groups. Bull. Lond. Math. Soc. **46**(6), 1248–1255 (2014)
39. Davis, M.W.: The Geometry and Topology of Coxeter Groups. London Mathematical Society Monographs Series, vol. 32. Princeton University Press, Princeton (2008)
40. Dehornoy, P., Digne, F., Godelle, E., Krammer, D., Michel, J.: Foundations of Garside theory, EMS Tracts in Mathematics, vol. 22. European Mathematical Society (EMS), Zürich (2015). Author name on title page: Daan Kramer
41. Dehornoy, P.: Groupes de Garside. Ann. Sci. École Norm. Sup. (4) **35**(2), 267–306 (2002)
42. Deligne, P.: Les immeubles des groupes de tresses généralisés. Invent. Math. **17**, 273–302 (1972)
43. Descombes, D.: Asymptotic rank of spaces with bicombings. Math. Z. **284**(3–4), 947–960 (2016)
44. Descombes, D., Lang, U.: Convex geodesic bicombings and hyperbolicity. Geom. Dedic. **177**, 367–384 (2015)
45. Descombes, D., Lang, U.: Flats in spaces with convex geodesic bicombings. Anal. Geom. Metr. Spaces **4**(1), 68–84 (2016)
46. Dougherty, M., McCammond, J., Witzel, S.: Boundary braids. arXiv:1811.05603 (2018)
47. Dress, A.W.M.: Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces. Adv. Math. **53**(3), 321–402 (1984)
48. Epstein, D.B.A., Cannon, J.W., Holt, D.F., Levy, S.V.F., Paterson, M.S., Thurston, W.P.: Word Processing in Groups. Jones and Bartlett Publishers, Boston (1992)
49. Farrell, F.T., Lafont, J.-F.: EZ-structures and topological applications. Comment. Math. Helv. **80**(1), 103–121 (2005)
50. Fukaya, T., Oguni, S.: A coarse Cartan–Hadamard theorem with application to the coarse Baum–Connes conjecture. J. Topol. Anal. **12**(3), 857–895 (2020)
51. Farrell, F.T., Roushon, S.K.: The Whitehead groups of braid groups vanish. Int. Math. Res. Not. **10**, 515–526 (2000)

52. Garside, F.A.: The braid group and other groups. Q. J. Math. Oxf. Ser. (2) **20**, 235–254 (1969)
53. Gersten, S.M., Short, H.B.: Small cancellation theory and automatic groups. Invent. Math. **102**(2), 305–334 (1990)
54. Haettel, T.: XXL type Artin groups are CAT(0) and acylindrically hyperbolic. arXiv:1905.11032v3 (2019)
55. Haettel, T., Kielak, D., Schwer, P.: The 6-strand braid group is CAT(0). Geom. Dedic. **182**, 263–286 (2016)
56. Huang, J., Osajda, D.: Quasi-Euclidean tilings over 2-dimensional Artin groups and their applications. Mem. Am. Math. Soc. (2017). arXiv: 1711.00122 (to appear)
57. Huang, J., Osajda, D.: Metric systolicity and two-dimensional Artin groups. Math. Ann. **374**(3–4), 1311–1352 (2019)
58. Huang, J., Osajda, D.: Large-type Artin groups are systolic. Proc. Lond. Math. Soc. **120**(1), 95–123 (2020)
59. Hoda, N.: Quadric complexes. Mich. Math. J. **69**(2), 241–271 (2020)
60. Holt, D.F.: Garside groups have the falsification by fellow-traveller property. Groups Geom. Dyn. **4**(4), 777–784 (2010)
61. Higson, N., Roe, J.: On the Coarse Baum–Connes Conjecture. Novikov Conjectures, Index Theorems and Rigidity, vol. 2 (Oberwolfach, 1993), pp. 227–254 (1995)
62. Huang, J.: Quasi-isometric classification of right-angled Artin groups II: several infinite out cases. arXiv:1603.02372 (2016)
63. Huang, J.: Quasi-isometric classification of right-angled Artin groups, I: the finite out case. Geom. Topol. **21**(6), 3467–3537 (2017)
64. Isbell, J.R.: Six theorems about injective metric spaces. Comment. Math. Helv. **39**, 65–76 (1964)
65. Jawhari, E.M., Pouzet, M., Misane, D.: Retracts: graphs and ordered sets from the metric point of view. Combinatorics and ordered sets (Arcata, Calif., 1985), pp. 175–226 (1986)
66. Kleiner, B., Lang, U.: Higher rank hyperbolicity. Invent. Math. **221**(2), 597–664 (2020)
67. Kasprowski, D., Rüping, H.: The Farrell–Jones conjecture for hyperbolic and CAT(0)-groups revisited. J. Topol. Anal. **9**(4), 551–569 (2017)
68. Krammer, D.: A class of Garside groupoid structures on the pure braid group. Trans. Am. Math. Soc. **360**(8), 4029–4061 (2008)
69. Lang, U.: Injective hulls of certain discrete metric spaces and groups. J. Topol. Anal. **5**(3), 297–331 (2013)
70. Leary, I., Minasyan, A.: Commensurating HNN-extensions: non-positive curvature and biautomaticity. Geom. Topol. arXiv:1907.03515 (to appear)
71. Margolis, A.: Quasi-isometry classification of right-angled Artin groups that split over cyclic subgroups. Groups Geom. Dyn. **14**(4), 1351–1417 (2020)
72. Miesch, B.: The Cartan–Hadamard theorem for metric spaces with local geodesic bicombings. Enseign. Math. **63**(1–2), 233–247 (2017)
73. Paris, L.: Artin monoids inject in their groups. Comment. Math. Helv. **77**(3), 609–637 (2002)
74. Paris, L.: Universal cover of Salvetti's complex and topology of simplicial arrangements of hyperplanes. Trans. Am. Math. Soc. **340**(1), 149–178 (1993)
75. Peifer, D.: Artin groups of extra-large type are biautomatic. J. Pure Appl. Algebra **110**(1), 15–56 (1996)
76. Picantin, M.: Tree products of cyclic groups and HNN extensions. arXiv:1306.5724 (2013)
77. Polat, N.: Convexity and fixed-point properties in Helly graphs. Discrete Math. **229**(1–3), 197–211 (2001). Combinatorics, graph theory, algorithms and applications
78. Pride, S.J.: On Tits' conjecture and other questions concerning Artin and generalized Artin groups. Invent. Math. **86**(2), 347–356 (1986)

79. Roushon, S.K.: A certain structure of Artin groups and the isomorphism conjecture. Can. J. Math. (2020). https://doi.org/10.4153/S0008414X2000036X
80. Salvetti, M.: Topology of the complement of real hyperplanes in C^N. Invent. Math. **88**(3), 603–618 (1987)
81. van der Lek, H.: The homotopy type of complex hyperplane complements. PhD thesis, Katholieke Universiteit te Nijmegen (1983)
82. van de Vel, M.L.J.: Theory of Convex Structures. North-Holland Mathematical Library, vol. 50. North-Holland Publishing Co., Amsterdam (1993)
83. Wegner, C.: The K-theoretic Farrell–Jones conjecture for CAT(0)-groups. Proc. Am. Math. Soc. **140**(3), 779–793 (2012)
84. Wenger, S.: Isoperimetric inequalities of Euclidean type in metric spaces. Geom. Funct. Anal. **15**(2), 534–554 (2005)