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Rational homotopy equivalences and singular chains

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Bousfield and Kan's $\mathbb{Q}$–completion and fiberwise $\mathbb{Q}$–completion of spaces lead to two different approaches to the rational homotopy theory of nonsimply connected spaces. In the first approach, a map is a weak equivalence if it induces an isomorphism on rational homology. In the second, a map of path-connected pointed spaces is a weak equivalence if it induces an isomorphism between fundamental groups and higher rationalized homotopy groups; we call these maps π_1–rational homotopy equivalences. We compare these two notions and show that π_1–rational homotopy equivalences correspond to maps that induce Ω–quasi-isomorphisms on the rational singular chains, ie maps that induce a quasi-isomorphism after applying the cobar functor to the dg coassociative coalgebra of rational singular chains. This implies that both notions of rational homotopy equivalence can be deduced from the rational singular chains by using different algebraic notions of weak equivalences: quasi-isomorphisms and Ω–quasi-isomorphisms. We further show that, in the second approach, there are no dg coalgebra models of the chains that are both strictly cocommutative and coassociative.

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1 Introduction

One of the questions that gave birth to rational homotopy theory is the *commutative cochains problem*, which, given a commutative ring $\boldsymbol{k}$, asks whether there exists a commutative differential graded (dg) associative $\boldsymbol{k}$–algebra functorially associated to any topological space that is weakly equivalent to the dg associative algebra of singular $\boldsymbol{k}$–cochains on the space with the cup product; see Sullivan [20] and Quillen [15]. Here we study a coalgebra version of this problem, which requires a careful consideration of what it means for two coalgebras to be weakly equivalent and for two possibly nonsimply connected spaces to be rationally homotopy equivalent, as we now explain.

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For technical reasons we use a pointed version of the normalized singular chains and cochains (Definition 12). The *pointed normalized singular chains* on a pointed space (X,b), denoted by $C_*(X,b,\boldsymbol{k})$, form a connected coaugmented dg coassociative $\boldsymbol{k}$–coalgebra with the Alexander–Whitney diagonal approximation as coproduct. The linear dual of $C_*(X,b,\boldsymbol{k})$, denoted by $C^*(X,b,\boldsymbol{k})$, is the connected augmented dg associative $\boldsymbol{k}$–algebra of pointed normalized singular cochains with the cup product. The Alexander–Whitney coproduct on $C_*(X,b;\boldsymbol{k})$ is not strictly cocommutative, but its cocommutativity holds up to an infinite coherent family of homotopies. This algebraic structure may be described using the language of operads; namely, the dg coassociative coalgebra structure on $C_*(X,b;\boldsymbol{k})$ extends to an E_∞*–coalgebra* structure. This dualizes to an E_∞*–algebra* structure on $C^*(X,b;\boldsymbol{k})$ extending the cup product.

Denote by Top_* the category of pointed path-connected topological spaces and by $\mathrm{CDGA}_{\boldsymbol{k}}$ the category of augmented commutative dg associative $\boldsymbol{k}$–algebras. A pointed version of the commutative cochains problem is given by the following question:

Question 1 *Is there a functor* $\mathcal{A}\colon \mathrm{Top}_* \to \mathrm{CDGA}_{\boldsymbol{k}}$ *such that for any* $(X,b)\in \mathrm{Top}_*$, $\mathcal{A}(X,b)$ *can be connected by a zigzag of quasi-isomorphisms of augmented dg associative* $\boldsymbol{k}$*–algebras to* $C^*(X,b;\boldsymbol{k})$?

Steenrod operations are obstructions for the existence of a functor $\mathcal{A}$ when $\boldsymbol{k}=\mathbb{Z}$ or a field of nonzero characteristic. Sullivan and Quillen showed via different approaches that when $\boldsymbol{k}=\mathbb{Q}$ such a functor $\mathcal{A}$ exists and, furthermore, the quasi-isomorphism type of the rational commutative dg algebra $\mathcal{A}(X,b)$ determines the rational homotopy type of X when X is a simply connected space of finite type.

In [4; 5], Bousfield and Kan describe two possible rational completions for general (not necessarily nilpotent) spaces both leading to different extensions of the classical rational homotopy theory of Sullivan and Quillen. The first one, known as the $\mathbb{Q}$*–completion* of a space, naturally associates to any space X another space $\mathbb{Q}_\infty X$ together with a map $\rho\colon X\to \mathbb{Q}_\infty X$. When X is a nilpotent space, $\mathbb{Q}_\infty X$ has the Maltsev completion of $\pi_1(X,b)$ as fundamental group and the rationalized higher homotopy groups of X as higher homotopy groups.

We call a continuous map $f\colon X\to Y$ a $\mathbb{Q}_\infty$*–homotopy equivalence* if $\mathbb{Q}_\infty f\colon \mathbb{Q}_\infty X\to \mathbb{Q}_\infty Y$ is a weak homotopy equivalence. This is the notion of weak equivalence in the extension of rational homotopy theory of Buijs, Félix, Murillo and Tanré [6]. The $\mathbb{Q}$–completion construction satisfies the following properties:

Proposition 2 [4] *Let $f\colon (X,b)\to(Y,c)$ be a continuous map of pointed path-connected spaces. Then the following are equivalent*:

(1) *The map $f\colon (X,b)\to(Y,c)$ is a $\mathbb{Q}_\infty$–homotopy equivalence*, *ie the induced map between $\mathbb{Q}$–completions $\mathbb{Q}_\infty f\colon \mathbb{Q}_\infty X\to\mathbb{Q}_\infty Y$ is a weak homotopy equivalence.*

(2) *The induced map $C_*(f;\mathbb{Q})\colon C_*(X,b;\mathbb{Q})\to C_*(Y,c;\mathbb{Q})$ is a quasi-isomorphism.*

Furthermore, *if X and Y are both nilpotent spaces then* (1) *and* (2) *are equivalent to*:

(3) *The map $f\colon (X,b)\to(Y,c)$ induces an isomorphism between Maltsev completions of the fundamental groups and an isomorphism of higher rationalized homotopy groups.*

The second completion functor, known as the *fiberwise $\mathbb{Q}$–completion*, associates to any space X another space $\mathbb{Q}^*_\infty X$, having the same fundamental group as X and higher homotopy groups isomorphic to the rationalized higher homotopy groups of X. A continuous map $f\colon X\to Y$ is a *π_1–rational homotopy equivalence* if $\mathbb{Q}^*_\infty f\colon \mathbb{Q}^*_\infty X\to \mathbb{Q}^*_\infty Y$ is a weak homotopy equivalence. This is the notion of weak equivalence in the extension of rational homotopy theory of Gómez-Tato, Halperin and Tanré [10].

In Section 3, we prove that π_1–rational homotopy equivalences are detected by the dg coassociative coalgebra of pointed normalized singular chains with rational coefficients. Then, in Section 4, we study a coalgebra version of Question 1 which fits with π_1–rational homotopy equivalences. More precisely, our first result is the following theorem:

Theorem 3 *Let $f\colon (X,b)\to(Y,c)$ be a continuous map of pointed path-connected spaces. Then the following are equivalent*:

(1) *The map $f\colon (X,b)\to(Y,c)$ is a π_1–rational homotopy equivalence*, *ie the induced map between fiberwise $\mathbb{Q}$–completions $\mathbb{Q}^*_\infty f\colon \mathbb{Q}^*_\infty X\to\mathbb{Q}^*_\infty Y$ is a weak homotopy equivalence.*

(2) *The induced maps $\pi_1(f)\colon \pi_1(X,b)\to\pi_1(Y,c)$ and*

$$\pi_n(f)\otimes\mathbb{Q}\colon \pi_n(X,b)\otimes\mathbb{Q}\to\pi_n(Y,c)\otimes\mathbb{Q}$$

for $n\geq 2$ are isomorphisms.

(3) *The induced map* $C_*(f;\mathbb{Q})\colon C_*(X,b;\mathbb{Q})\to C_*(Y,c;\mathbb{Q})$ *is an* Ω*–quasi-isomorphism, where an* Ω*–quasi-isomorphism is a map that induces an isomorphism after applying the cobar construction* (see Section 2.1).

(4) *The induced map* $\pi_1(f)\colon \pi_1(X,b)\to\pi_1(Y,c)$ *is an isomorphism and for every* $\mathbb{Q}$*–representation* A *of* $\pi_1(Y,c)$ *the induced map on the homology with local coefficients* $H_*(f)\colon H_*(X;f^*A)\to H_*(Y;A)$ *is an isomorphism.*

The above theorem says that the notion of π_1–rational homotopy equivalence is in fact a rational notion, ie it can be described in terms of maps that preserve algebraic structures on rational vector spaces. The proof of the equivalence between (2) and (3) of Theorem 3 uses a recent extension of a classical theorem of Adams, proven in [18] by the first and third authors, which says that for any pointed *path-connected* space (X,b) there is a natural quasi-isomorphism of dg algebras

$$\theta\colon \Omega C_*(X,b;\mathbb{Q})\simeq C_*^{\square}(\Omega_b X;\mathbb{Q}),$$

where $C_*^{\square}(\Omega_b X;\mathbb{Q})$ is the dg algebra of rational cubical singular chains on the (Moore) based loop space of X at b. The proof of Theorem 3 also uses the fact that θ is a quasi-isomorphism of dg bialgebras for natural bialgebra structures on $\Omega C_*(X,b;\mathbb{Q})$ and $C_*^{\square}(\Omega_b X;\mathbb{Q})$.

Quillen proved that associated to any simply connected space X there is a rational cocommutative dg coassociative coalgebra which is quasi-isomorphic to the rational chains on X. Under the light of Theorem 3 we may now ask a stronger question, namely, if for any path-connected (X,b) the dg coalgebra $C_*(X,b;\mathbb{Q})$ may be strictified into a rational cocommutative dg coassociative coalgebra which still detects the fundamental group (or at least the fundamental group algebra) and the higher rational homotopy groups. More precisely, denoting by $\mathrm{CDGC}_{\boldsymbol{k}}$ the category of coaugmented conilpotent dg cocommutative coassociative $\boldsymbol{k}$–coalgebras we ask the following question, which we call the *cocommutative chains problem*:

Question 4 *Is there a functor* $\mathcal{C}\colon \mathrm{Top}_*\to\mathrm{CDGC}_{\boldsymbol{k}}$ *such that for any* $(X,b)\in\mathrm{Top}_*$, $\mathcal{C}(X,b)$ *can be connected by a zigzag of* Ω*–quasi-isomorphisms of coaugmented conilpotent dg coassociative coalgebras to* $C_*(X,b;\boldsymbol{k})$?

Steenrod operations are also an obstruction for the existence of $\mathcal{C}$ when $\boldsymbol{k}=\mathbb{Z}$ or a field of nonzero characteristic. In Section 4, we prove that there is no such functor $\mathcal{C}$ even when $\boldsymbol{k}$ is a field of characteristic zero.

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2 Algebraic preliminaries

In this section we recall the algebraic constructions and results that will be used in the proofs of our main theorems in Sections 3 and 4. Let $\boldsymbol{k}$ be a commutative ring. We assume familiarity with the notions of differential graded (dg) $\boldsymbol{k}$–algebras and $\boldsymbol{k}$–coalgebras. The phrases "dg algebra" and "dg coalgebra" will mean "unital augmented differential graded associative $\boldsymbol{k}$–algebra" and "counital coaugmented conilpotent differential graded coassociative $\boldsymbol{k}$–coalgebra", respectively. A graded $\boldsymbol{k}$–(co)algebra V is *connected* if $V_n = 0$ for all $n < 0$ and $V_0 \cong \boldsymbol{k}$. We also assume familiarity with bialgebras and Hopf algebras; in this paper, by "bialgebra" we will mean a unital augmented associative counital coaugmented coassociative bialgebra. A Hopf algebra is a bialgebra that admits an antipode. Antipodes are unique when they exist. We refer to [13] for further background.

2.1 The cobar construction

Let $\mathrm{DGA}_{\boldsymbol{k}}$ and $\mathrm{DGC}_{\boldsymbol{k}}$ denote the categories of dg algebras and dg coalgebras, respectively. Recall the definition of the *cobar* functor

$$\Omega\colon \mathrm{DGC}_{\boldsymbol{k}} \to \mathrm{DGA}_{\boldsymbol{k}}.$$

Given a dg coalgebra C define a dg algebra

$$\Omega C := (T(s^{-1}\overline{C}), D),$$

where $\overline{C}$ is the cokernel of the coaugmentation $\boldsymbol{k} \to C$, s^{-1} is the shift functor which lowers degree by 1, $T(s^{-1}\overline{C}) = \boldsymbol{k} \oplus \bigoplus_{i=1}^{\infty} (s^{-1}\overline{C})^{\otimes i}$ is the unital tensor algebra, and the differential D is defined as follows. Let $\partial\colon C \to C$ and $\Delta\colon C \to C \otimes C$ be the differential and coproduct of C. Now extend the linear map

$$-s^{-1} \circ \partial \circ s^{+1} + (s^{-1} \otimes s^{-1}) \circ \Delta \circ s^{+1} \colon s^{-1}\overline{C} \to T(s^{-1}\overline{C})$$

as a derivation to obtain $D\colon T(s^{-1}\overline{C}) \to T(s^{-1}\overline{C})$. The coassociativity of Δ, the compatibility of ∂ and Δ, and the fact that $\partial^2 = 0$ together imply that $D^2 = 0$.

A *quasi-isomorphism* of dg $\boldsymbol{k}$–modules is a chain map which induces an isomorphism in homology. A map of dg (co)algebras is said to be a quasi-isomorphism if the underlying map of dg $\boldsymbol{k}$–modules is. The following stronger notion of weak equivalence between dg coalgebras will play a fundamental role in this article:

Definition 5 A map of dg coalgebras $f\colon C \to C'$ is called an Ω*–quasi-isomorphism* if $\Omega f\colon \Omega C \to \Omega C'$ is a quasi-isomorphism of dg algebras.

Remark 6 Any Ω–quasi-isomorphism is a quasi-isomorphism, but not vice versa; an example of this fact can be found in Proposition 2.4.3 of [13]. Another example is given by considering the simplicial set S which has exactly one vertex, one nondegenerate 1–simplex, and all the higher simplices are degenerate. Since $|S|$ is homotopy equivalent to the circle S^1, the dg coalgebras of simplicial chains $C_*^{\Delta}(S;\boldsymbol{k})$ and singular chains $C_*(S^1;\boldsymbol{k})$ are quasi-isomorphic. However, $H_0(\Omega C_*^{\Delta}(S))$ is isomorphic to the polynomial algebra $\boldsymbol{k}[x]$, while $H_0(\Omega C_*(S^1;\boldsymbol{k}))$ is isomorphic to $\boldsymbol{k}[x,x^{-1}]$.

A quasi-isomorphism between simply connected dg coalgebras (namely, nonnegatively graded dg coalgebras C such that $C_0 \cong \boldsymbol{k}$ and $C_1 = 0$) is an Ω–quasi-isomorphism, as discussed in Section 2.4 of [13].

2.2 Lie algebras and related constructions

For the rest of the section we work over a field $\boldsymbol{k}$ of characteristic zero. We review some classical constructions and results which will be used in Sections 3 and 4. We refer to [15] or [9] for further details.

Given a dg vector space V, denote by SV the *symmetric algebra* generated by V. The dg commutative algebra SV is defined as the quotient of the tensor algebra TV by the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$ for $x, y \in V$. The unital dg associative algebra structure on TV induces a unital commutative dg associative algebra structure on SV. Moreover, SV is a commutative cocommutative dg bialgebra when equipped with coproduct $\Delta\colon SV \to SV \otimes SV$ given by extending $\Delta(x) = x \otimes 1 + 1 \otimes x$ as an algebra map and counit $S(V) \to \boldsymbol{k}$ induced by the projection $TV \to T^0V = \boldsymbol{k}$.

Given a dg Lie algebra L denote by UL the *universal enveloping algebra* of L. The dg associative algebra UL is defined as the quotient of TL by the ideal generated by elements of the form $x\otimes y-(-1)^{|x||y|}y\otimes x-[x,y]$ for $x,y\in L$. Moreover, UL is a cocommutative dg bialgebra when equipped with the coproduct also determined by the formula $\Delta(x)=x\otimes 1+1\otimes x$ and counit induced by the projection $TV\to T^0V=\boldsymbol{k}$.

The universal enveloping algebra construction defines a functor from dg Lie algebras to cocommutative dg bialgebras which commutes with homology.

Theorem 7 [15, Appendix B, Proposition 2.1] *For any dg Lie algebra L there is a natural isomorphism*

$$UH_*(L)\cong H_*(UL)$$

of graded cocommutative bialgebras.

The Poincaré–Birkhoff–Witt (PBW) theorem relates S and U.

Theorem 8 (Poincaré–Birkhoff–Witt [15, Appendix B, Theorem 2.3]) *For any dg Lie algebra L there is an isomorphism of cocommutative dg coalgebras*

$$e\colon SL\xrightarrow{\cong} UL.$$

Since the symmetric algebra functor preserves quasi-isomorphisms we immediately obtain the following corollary:

Corollary 9 *If $f\colon L\to L'$ is a quasi-isomorphism of dg Lie algebras, then $Uf\colon UL\to UL'$ is a quasi-isomorphism as well.*

In [15], Quillen introduced a functor

$$\mathcal{L}\colon \mathrm{CDGC}_{\boldsymbol{k}}\to \mathrm{DGL}_{\boldsymbol{k}},$$

where $\mathrm{DGL}_{\boldsymbol{k}}$ denotes the category of dg Lie algebras, defined as follows. Given $C\in\mathrm{CDGC}_{\boldsymbol{k}}$ with differential $\partial\colon C\to C$, the dg Lie algebra $\mathcal{L}C$ has as underlying graded Lie algebra the free graded Lie algebra $(L(s^{-1}\overline{C}),[\cdot,\cdot])$ generated by $s^{-1}\overline{C}$, the desuspension of the coaugmentation ideal of C. The differential $d_{\mathcal{L}C}$ is determined by

$$d_{\mathcal{L}C}(\tau x)=-\tau\partial x-\frac{1}{2}\sum_{(x)}(-1)^{|x'|}[\tau x',\tau x''],$$

where $\tau\colon C\to L(s^{-1}\overline{C})$ is the canonical map given by $\tau x=s^{-1}\overline{x}$ and $\Delta(x)=\sum_{(x)}x'\otimes x''$ denotes the coproduct of C.

The functors Ω and $\mathcal{L}$ are related via U as follows:

Lemma 10 [15, page 290] *For any cocommutative dg coalgebra C there is a natural isomorphism of augmented dg associative algebras*

$$\Omega C \cong U\mathcal{L}C.$$

We finish this section by recalling the Milnor–Moore theorem in topology.

Theorem 11 (Milnor–Moore [9, Theorem 21.5]) *Let (X, b) be a simply connected pointed path-connected space. There is an isomorphism of graded Hopf algebras*

$$U(\pi_*(\Omega_b X) \otimes \boldsymbol{k}) \cong H_*(\Omega_b X; \boldsymbol{k}),$$

where the graded Lie algebra structure on $\pi_*(\Omega_b X) \otimes \boldsymbol{k}$ is given by the Whitehead bracket.

3 Coalgebras and nonsimply connected rational homotopy theory

In this section we define two notions of weak equivalences between spaces for two different approaches to extending rational homotopy theory to nonsimply connected spaces. These are based on two rational completions for spaces described in [4]. We explain the sense in which the rational dg coalgebra of pointed normalized singular chains detects each of these weak equivalences. All topological spaces will be assumed to be semilocally simply connected and locally path-connected, so that universal covers exist. Denote by Top_* the category of pointed path-connected spaces. We first define the version of singular chains on pointed spaces that will be used.

Definition 12 For any $(X, b) \in \mathrm{Top}_*$ and any commutative ring $\boldsymbol{k}$, the coaugmented connected dg $\boldsymbol{k}$–coalgebra of *pointed normalized singular chains* $(C_*(X, b; \boldsymbol{k}), \partial, \Delta)$ is defined as follows. The underlying graded $\boldsymbol{k}$–module is obtained by considering the graded submodule of the ordinary singular chains generated by those continuous maps $\sigma\colon \Delta^n \to X$ that send the vertices of the n–simplex Δ^n to $b \in X$, and then modding out by degenerate simplices. The usual boundary operator for singular chains induces a differential $\partial\colon C_*(X, b; \boldsymbol{k}) \to C_{*-1}(X, b; \boldsymbol{k})$ and the Alexander–Whitney diagonal approximation map induces a compatible coassociative coproduct

$$\Delta\colon C_*(X, b; \boldsymbol{k}) \to C_*(X, b; \boldsymbol{k}) \otimes C_*(X, b; \boldsymbol{k}).$$

Note that $C_0(X, b; \boldsymbol{k}) \cong \boldsymbol{k}$, so the counit and coaugmentation are canonically defined. This construction defines a functor

$$C_*\colon \mathrm{Top}_* \to \mathrm{DGC}^0_{\boldsymbol{k}},$$

where $\mathrm{DGC}^0_{\boldsymbol{k}}$ denotes the full subcategory of $\mathrm{DGC}_{\boldsymbol{k}}$ whose objects are connected dg coalgebras.

In [4], Bousfield and Kan define the $\mathbb{Q}$*–completion* of a space X as

$$\mathbb{Q}_\infty X = |\overline{\boldsymbol{W}}\, \mathbb{Q}_\infty(\boldsymbol{G}\,\mathrm{Sing}(X, b))|,$$

where $\mathrm{Sing}(X, b)$ is the subsimiplicial set of $\mathrm{Sing}(X)$ consisting of those singular simplices $\Delta^n \to X$ that send the vertices of Δ^n to b, $\boldsymbol{G}\colon \mathrm{sSet} \to \mathrm{sGrp}$ is the Kan loop group functor from simplicial sets to simplicial groups, $\mathbb{Q}_\infty G$ denotes the dimension-wise (algebraic) $\mathbb{Q}$–completion of any simplicial group G, $\overline{\boldsymbol{W}}\colon \mathrm{sGrp} \to \mathrm{sSet}$ is the classifying space functor, and $|\cdot|$ denotes geometric realization. Note we have used a slightly different notation from that in [4].

Definition 13 A continuous map $f\colon X \to Y$ between path-connected spaces is a $\mathbb{Q}_\infty$*–homotopy equivalence* if $\mathbb{Q}_\infty f\colon \mathbb{Q}_\infty X \to \mathbb{Q}_\infty Y$, the induced map on the $\mathbb{Q}$–completions, is a weak homotopy equivalence.

The rational singular chains are able to detect $\mathbb{Q}_\infty$–homotopy equivalences in the following sense:

Proposition 14 [4] *Let $f\colon (X, b) \to (Y, c)$ be a continuous map of pointed path-connected spaces. Then the following are equivalent:*

(1) *The map $f\colon (X, b) \to (Y, c)$ is a $\mathbb{Q}_\infty$–homotopy equivalence, ie the induced map between $\mathbb{Q}$–completions $\mathbb{Q}_\infty f\colon \mathbb{Q}_\infty X \to \mathbb{Q}_\infty Y$ is a weak homotopy equivalence.*

(2) *The induced map $C_*(f; \mathbb{Q})\colon C_*(X, b; \mathbb{Q}) \to C_*(Y, c; \mathbb{Q})$ is a quasi-isomorphism.*

Furthermore, if X and Y are both nilpotent spaces, then (1) *and* (2) *are equivalent to:*

(3) *$f\colon (X, b) \to (Y, c)$ induces an isomorphism between Maltsev completions of the fundamental groups and an isomorphism of higher rationalized homotopy groups.*

Proof The equivalence between (1) and (2) is exactly [4, 5.2]. If X is a nilpotent space then, by combining the results of [4], we see that $\rho\colon X \to \mathbb{Q}_\infty X$ induces an isomorphism on the Maltsev completions of the fundamental groups and an isomorphism $\pi_n(\rho)\otimes\mathbb{Q}\colon \pi_n(X)\otimes\mathbb{Q} \cong \pi_n(\mathbb{Q}_\infty X)$ for all $n \geq 2$. This is because nilpotent spaces are $\mathbb{Q}$–good and the $\mathbb{Q}$–completion of a nilpotent group is given by the Maltsev completion. It therefore follows that if X and Y are both nilpotent then (3) is equivalent to (1). □

In [4], Bousfield and Kan also define a second possibility of completion for nonsimply connected spaces, called *fiberwise $\mathbb{Q}$–completion*. This is done by fiberwise $\mathbb{Q}$–completing the fibration $\widetilde{X} \to X \to B\pi_1(X,b)$, where $\widetilde{X}$ denotes the universal cover of X and $B\pi_1(X,b)$ is the classifying space of $\pi_1(X,b)$. The fiberwise $\mathbb{Q}$–completion uses the algebraic *relative $\mathbb{Q}$–completion* for a short exact sequence of groups. This results in a fibration

$$\mathbb{Q}_\infty\widetilde{X} \to \mathbb{Q}^*_\infty X \to B\pi_1(X,b),$$

naturally associated to any X, whose fiber is the ordinary $\mathbb{Q}$–completion of $\widetilde{X}$. Thus, to any space X we may functorially associate a new space $\mathbb{Q}^*_\infty X$ which has the same fundamental group as X and whose higher homotopy groups are the rationalized homotopy groups of X.

Definition 15 A continuous map $f\colon X \to Y$ between path-connected spaces is a *π_1–rational homotopy equivalence* if $\mathbb{Q}^*_\infty f\colon \mathbb{Q}^*_\infty X \to \mathbb{Q}^*_\infty Y$, the induced map on the fiberwise $\mathbb{Q}$–completions, is a weak homotopy equivalence.

Our next result gives three alternative characterizations of π_1–rational homotopy equivalences in terms of the coalgebras of pointed normalized singular chains, homotopy groups and homology with local coefficients.

Theorem 16 *Let $f\colon (X,b) \to (Y,c)$ be a continuous map of pointed path-connected spaces. Then the following are equivalent*:

(1) *The map $f\colon (X,b) \to (Y,c)$ is a π_1–rational homotopy equivalence*, *ie the induced map between fiberwise $\mathbb{Q}$–completions $\mathbb{Q}^*_\infty f\colon \mathbb{Q}^*_\infty X \to \mathbb{Q}^*_\infty Y$ is a weak homotopy equivalence.*

(2) *The induced maps $\pi_1(f)\colon \pi_1(X,b) \to \pi_1(Y,c)$ and*

$$\pi_n(f)\otimes\mathbb{Q}\colon \pi_n(X,b)\otimes\mathbb{Q} \to \pi_n(Y,c)\otimes\mathbb{Q}$$

for $n \geq 2$ are isomorphisms.

(3) *The induced map* $C_*(f;\mathbb{Q})\colon C_*(X,b;\mathbb{Q})\to C_*(Y,c;\mathbb{Q})$ *is an* Ω*–quasi-isomorphism.*

(4) *The induced map* $\pi_1(f)\colon \pi_1(X,b)\to\pi_1(Y,c)$ *is an isomorphism and for every* $\mathbb{Q}$*–representation* A *of* $\pi_1(Y,c)$ *the induced map on the homology with local coefficients* $H_*(f)\colon H_*(X;f^*A)\to H_*(Y;A)$ *is an isomorphism.*

In the above theorem, by a $\mathbb{Q}$*–representation* of a group G we mean a $\mathbb{Q}$–vector space A together with a left $\mathbb{Q}[G]$–module structure. If $f\colon G'\to G$ is a group homomorphism and A is a $\mathbb{Q}$–representation of G, then f^*A denotes the $\mathbb{Q}$–representation of G' given by the pullback action of $\mathbb{Q}[G']$ on A via f.

Remark 17 A direct observation that follows from Proposition 14 and Theorem 16 is that both notions of rational homotopy equivalence can be deduced from the pointed normalized singular chains with rational coefficients. This might come as a surprise, since this means that the singular chains with rational chains are capable of capturing the highly nonrational fundamental group in the case of π_1–rational homotopy equivalences and the Maltsev completion of the fundamental group in the case of $\mathbb{Q}_\infty$–homotopy equivalences. This can be interpreted as saying that both approaches to nonsimply connected rational homotopy theory are in fact rational, ie can be deduced from algebraic structure on a rational chain complex. We would further like to point out that both notions of rational homotopy equivalence coincide for simply connected spaces.

Before we prove Theorem 16 we first recall two results proven in [18; 19] regarding an extension of the classical Adams' cobar theorem [1]. For completeness, we will sketch a proof of the following theorem and for a detailed proof we refer the reader to [19] or [16]. Let $\Omega_b X$ denote the topological monoid of Moore loops in X based at $b\in X$.

Theorem 18 *Let* (X,b) *be a pointed path-connected space. Then*:

(1) *There is a natural quasi-isomorphism of dg algebras*

$$\theta\colon \Omega C_*(X,b;\mathbb{Q})\to C_*^{\square}(\Omega_b X;\mathbb{Q}),$$

where $C_*^{\square}(\Omega_b X;\mathbb{Q})$ *denotes the normalized singular cubical chains on* $\Omega_b X$ *with rational coefficients.*

(2) *There is a natural coassociative coproduct*

$$\nabla\colon \Omega C_*(X,b;\mathbb{Q})\to \Omega C_*(X,b;\mathbb{Q})\otimes\Omega C_*(X,b;\mathbb{Q})$$

making $\Omega C_*(X,b;\mathbb{Q})$ a dg bialgebra such that θ becomes a quasi-isomorphism of dg bialgebras, when $C_*^{\square}(\Omega_b X;\mathbb{Q})$ is equipped with the natural diagonal approximation coproduct for cubical chains.

Sketch of proof (1) The fact that there exists a quasi-isomorphism of dg algebras $\theta\colon \Omega C_*(X,b;\mathbb{Q}) \to C_*^{\square}(\Omega_b X;\mathbb{Q})$ is an extension of a classical theorem of Adams proven in [18] by observing that $\Omega C_*(X,b;\mathbb{Q})$ is naturally isomorphic as a dg algebra to the chains on a monoidal cubical set with connections (denoted by $\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))$ in [18; 19]), whose geometric realization is naturally homotopy equivalent to the based loop space. This natural homotopy equivalence induces a quasi-isomorphism from the dg algebra of chains on $\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))$ to $C_*^{\square}(\Omega_b X;\mathbb{Q})$.

(2) The chain complex of normalized cubical chains on any cubical set (with or without connections) has a natural coassociative coproduct approximating the diagonal map. The quasi-isomorphism from the cubical chains on $\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))$ to $C_*^{\square}(\Omega_b X;\mathbb{Q})$ preserves coproducts. To construct ∇, we transfer the coproduct of the cubical chains on $\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))$ to $\Omega C_*(X,b;\mathbb{Q})$ via the isomorphism between them. □

Remark 19 The construction of the coproduct on $\Omega C_*(X,b;\mathbb{Q})$ builds upon an idea originally described in [2]. It is related to the E_∞–coalgebra structure on $C_*(X,b;\mathbb{Q})$ as follows. It is shown in [3; 14] that $C_*(X,b;\mathbb{Q})$ has a natural structure of a coalgebra over the *surjection operad*, usually denoted by $\mathcal{X}$, extending the dg coassociative coalgebra structure given by the Alexander–Whitney diagonal approximation. The surjection operad $\mathcal{X}$ is a particular model for the E_∞–operad. It is explained in [12] that for any connected $\mathcal{X}$–coalgebra $\boldsymbol{C}$ with underlying dg coassociative coalgebra C, the structure maps of the $\mathcal{X}$–coalgebra structure corresponding to the E_2 portion of the operad induce a dg bialgebra structure on ΩC. In the case of $C_*(X,b;\mathbb{Q})$, the coproduct of the dg bialgebra structure on $\Omega C_*(X,b;\mathbb{Q})$ coincides with the coproduct ∇ outlined in the proof of Theorem 18. More details may be found in [19, Theorems 2 and 3].

Remark 20 Two other extensions of Adams' cobar theorem to the nonsimply connected case may be found in [8; 11]. These approaches add formal inverses for the 1–simplices in different ways. In [8], an extension of Adams' construction is described for nonsimply connected CW–complexes by adding homotopy inverses for all 1–simplices together with formal homotopies at the level of the cellular chains. In [11], a different extension is described for any simplicial set by adding formal (strict) inverses for all 1–simplices after applying the cobar construction. In the approach of the first and third authors in [18], no inverses need be added since the construction is preformed at the level of the Kan complex of singular chains, which already contains inverses up to homotopy.

Since there is an isomorphism of bialgebras $H_0(\Omega_b X;\mathbb{Q}) \cong \mathbb{Q}[\pi_1(X,b)]$ and any group algebra is a Hopf algebra (a bialgebra which has an antipode map) whose grouplike elements are the underlying group, we immediately obtain the following corollary:

Corollary 21 *The map $\theta\colon \Omega C_*(X,b;\mathbb{Q}) \to C^{\square}_*(\Omega_b X;\mathbb{Q})$ induces an isomorphism of bialgebras $H_0(\Omega C_*(X,b;\mathbb{Q})) \cong \mathbb{Q}[\pi_1(X,b)]$. In particular, there exists an antipode on the bialgebra $H_0(\Omega C_*(X,b;\mathbb{Q}))$, which makes it a Hopf algebra. The isomorphism class of the fundamental group of X may be recovered functorially as the group of grouplike elements of $H_0(\Omega C_*(X,b;\mathbb{Q}))$.*

We now use the above results to prove Theorem 16. An integral version of the equivalence between (2), (3) and (4) of Theorem 16 was proven in [17].

Proof of Theorem 16 The equivalence between (1) and (2) follows directly from the fact that $\mathbb{Q}^*_\infty X$ is the total space of a fibration $\mathbb{Q}_\infty \widetilde{X} \to \mathbb{Q}^*_\infty X \to B\pi_1(X,b)$, since then we can use the long exact sequence for a fibration together with the natural isomorphisms $\pi_n(\mathbb{Q}_\infty \widetilde{X}) \cong \pi_n(X) \otimes \mathbb{Q}$.

The proof that (3) implies (4) is similar to the proof of [17, Theorem 12], as we now explain. If $C_*(f;\mathbb{Q})\colon C_*(X,b;\mathbb{Q}) \to C_*(Y,c;\mathbb{Q})$ is an Ω–quasi-isomorphism, then f induces an isomorphism of Hopf algebras

$$H_0(\Omega(f))\colon H_0(\Omega C_*(X,b;\mathbb{Q})) \xrightarrow{\cong} H_0(\Omega C_*(Y,c;\mathbb{Q}))$$

and, by Corollary 21, after applying the grouplike elements functor to the isomorphism $H_0(\Omega(f))$ we obtain that $\pi_1(f)\colon \pi_1(X,b) \to \pi_1(Y,c)$ is an isomorphism. It follows from [17, Proposition 10] that if $C_*(f)\colon C_*(X,b;\mathbb{Q}) \to C_*(Y,c;\mathbb{Q})$ is an Ω–quasi-isomorphism, then the induced map on homology with coefficients in any rational local system is an isomorphism.

To prove (4) implies (2), we consider the pointed universal covers of the pointed spaces (X,b) and (Y,c), which we denote by $(\widetilde{X},[b])$ and $(\widetilde{Y},[c])$, respectively. By a standard lifting argument we get an induced map $\widetilde{f}\colon (\widetilde{X},[b]) \to (\widetilde{Y},[c])$, which is unique up to homotopy. As explained in Section 5.2 of [7], the rational homology of the universal cover of X can be computed by the homology of X with local coefficients in the fundamental group algebra, ie $H_*(\widetilde{X};\mathbb{Q}) \cong H_*\big(X;\mathbb{Q}[\pi_1(X,b)]\big)$, where $\mathbb{Q}[\pi_1(X,b)]$ is the representation of $\pi_1(X,b)$ through the left multiplication of $\pi_1(X,b)$ on itself. By assumption, this implies that $\tilde{f}$ induces an isomorphism on the rational homology of the universal covers. Consequently, since the universal covers are simply connected, Whitehead's theorem yields that $\pi_n(\tilde{f})\otimes\mathbb{Q}\colon \pi_n(\widetilde{X},[b])\otimes\mathbb{Q}\to\pi_n(\widetilde{Y},[c])\otimes\mathbb{Q}$ are isomorphisms for all $n\geq 2$. It then follows from the long exact sequence in homotopy that $\pi_n(f)\otimes\mathbb{Q}\colon \pi_n(X,b)\otimes\mathbb{Q}\to\pi_n(Y,c)\otimes\mathbb{Q}$ are isomorphisms for all $n\geq 2$ as well.

We now show (2) implies (3). Suppose the maps $\pi_1(f)\colon\pi_1(X,b)\to\pi_1(Y,c)$ and $\pi_n(f)\otimes\mathbb{Q}\colon\pi_n(X,b)\otimes\mathbb{Q}\to\pi_n(Y,c)\otimes\mathbb{Q}$ for $n\geq 2$ are isomorphisms. By Theorem 18, this is equivalent to showing that f induces an isomorphism on homology for the corresponding loop spaces. Again we will look at the universal covers to deduce the results. In particular, the lift $\tilde{f}$ induces an isomorphism

$$\pi_n(\tilde{f})\otimes\mathbb{Q}\colon \pi_n(\widetilde{X})\otimes\mathbb{Q}\xrightarrow{\cong}\pi_n(\widetilde{Y})\otimes\mathbb{Q}$$

for each $n\geq 1$. Therefore,

$$\pi_{n-1}(\Omega(\tilde{f}))\otimes\mathbb{Q}\colon \pi_{n-1}(\Omega_{[b]}\widetilde{X})\otimes\mathbb{Q}\xrightarrow{\cong}\pi_{n-1}(\Omega_{[c]}\widetilde{Y})\otimes\mathbb{Q}$$

are isomorphisms for each $n\geq 1$. This means that the map $\pi_n(\Omega(\tilde{f}))\otimes\mathbb{Q}$ is an isomorphism of Lie algebras and, applying the universal enveloping algebra functor to each of these maps, by the Milnor–Moore theorem (Theorem 11), we get an isomorphism

$$H_*(\Omega\tilde{f})\colon H_*(\Omega_{[b]}\widetilde{X};\mathbb{Q})\xrightarrow{\cong}H_*(\Omega_{[c]}\widetilde{Y};\mathbb{Q}).$$

Recall that there is a homotopy equivalence $\Omega_b X\simeq\Omega_{[b]}\widetilde{X}\times\pi_1(X,b)^{\mathrm{disc}}$, where $\pi_1(X,b)^{\mathrm{disc}}$ denotes $\pi_1(X,b)$ considered as a discrete space, inducing a natural isomorphism of graded vector spaces

$$H_*(\Omega_b X;\mathbb{Q})\cong H_*(\Omega_{[b]}\widetilde{X};\mathbb{Q})\otimes H_*(\pi_1(X,b)^{\mathrm{disc}};\mathbb{Q}).$$

Similarly,

$$H_*(\Omega_c Y;\mathbb{Q})\cong H_*(\Omega_{[c]}\widetilde{Y};\mathbb{Q})\otimes H_*(\pi_1(Y,c)^{\mathrm{disc}};\mathbb{Q}).$$

The latter two isomorphisms together with the fact that $H_*(\Omega\tilde{f})$ and $\pi_1(f)$ are isomorphisms imply that $H_*(\Omega(f))\colon H_*(\Omega_b X;\mathbb{Q})\to H_*(\Omega_c Y;\mathbb{Q})$ is an isomorphism as well. Finally, by Theorem 18, it follows that $\Omega C_*(f)\colon\Omega C_*(X,b;\mathbb{Q})\to\Omega C_*(Y,c;\mathbb{Q})$ is a quasi-isomorphism, as desired. □

4 The cocommutative chains problem

The goal of this section is to give a negative answer to Question 4. If we restrict our attention to simply connected spaces and coalgebras with quasi-isomorphisms as weak equivalences, then Quillen's rational homotopy theory provides a positive answer to the cocommutative chains problem: naturally associated to any simply connected space there is a dg Lie algebra whose Chevalley–Eilenberg complex is a cocommutative dg coassociative coalgebra quasi-isomorphic to the rational chains on the space [15]. However, if we consider the problem for all path-connected spaces and for coalgebras with Ω–quasi-isomorphisms as weak equivalences, as asked in Question 4, the answer turns out to be negative.

Suppose there were a cocommutative dg coalgebra $\mathcal{C}(X,b)$ such that $\Omega\mathcal{C}(X,b)$ is quasi-isomorphic as a dg algebra to $\Omega C_*(X,b;\boldsymbol{k})$. By Corollary 21, $H_0(\Omega C_*(X,b;\boldsymbol{k}))\cong \boldsymbol{k}[\pi_1(X,b)]$, so for a hypothetical strictly cocommutative model $\mathcal{C}(X,b)$ we would also have an isomorphism of algebras $H_0(\Omega\mathcal{C}(X,b))\cong \boldsymbol{k}[\pi_1(X,b)]$. In the proof of the upcoming theorem we show that, if C is a cocommutative dg coalgebra, $H_0(\Omega C)$ is always isomorphic as a vector space to a polynomial algebra. Therefore $H_0(\Omega C)$ is always infinite-dimensional, which means it cannot be isomorphic to the group algebra of a finite group.

Recall $\mathrm{CDGC}_{\boldsymbol{k}}$ denotes the category of cocommutative dg $\boldsymbol{k}$–coalgebras and Top_* the category of pointed path-connected spaces.

Theorem 22 *Let $\boldsymbol{k}$ be a field of characteristic zero. There is no functor*

$$\mathcal{C}\colon \mathrm{Top}_* \to \mathrm{CDGC}_{\boldsymbol{k}}$$

such that for any $(X,b)\in \mathrm{Top}_*$, $\mathcal{C}(X,b)$ is Ω–quasi-isomorphic to $C_*(X,b;\boldsymbol{k})$ as dg coassociative coalgebras.

Proof Suppose there exists such a functor $\mathcal{C}$ and let $(X,b)\in\mathrm{Top}_*$. Lemma 10 says that there is an isomorphism of dg associative algebras

$$\Omega\mathcal{C}(X,b)\cong U\mathcal{L}\mathcal{C}(X,b),$$

so the homologies are isomorphic as algebras:

$$H_*(\Omega\mathcal{C}(X,b))\cong H_*(U\mathcal{L}\mathcal{C}(X,b)).$$

By Theorem 7, we have an isomorphism of associative algebras

$$H_*(U\mathcal{L}\mathcal{C}(X,b)) \cong UH_*(\mathcal{L}\mathcal{C}(X,b)).$$

The PBW theorem (Theorem 8) gives an isomorphism of graded vector spaces,

$$UH_*(\mathcal{L}\mathcal{C}(X,b)) \cong SH_*(\mathcal{L}\mathcal{C}(X,b)).$$

In particular,

$$H_0(\Omega\mathcal{C}(X,b)) \cong SH_0(\mathcal{L}\mathcal{C}(X,b)).$$

By Corollary 21, our assumption yields an isomorphism of algebras

$$\boldsymbol{k}[\pi_1(X,b)] \cong H_0(\Omega C_*(X,b;\boldsymbol{k})) \cong H_0(\Omega\mathcal{C}(X,b)) \cong SH_0(\mathcal{L}\mathcal{C}(X,b))$$

for all pointed path-connected spaces (X,b). Note that $SH_0(\mathcal{L}\mathcal{C}(X,b))$ is either one- or infinite-dimensional as a vector space. Since $\pi_1(X,b)$ can be any arbitrary group and, in particular, a finite group has finite-dimensional group algebra, we obtain a contradiction. □

Remark 23 We would like to point out that a similar argument also implies that there is no functor

$$\mathcal{C}\colon \mathrm{Top}_* \to C_{\infty,\boldsymbol{k}},$$

where $C_{\infty,\boldsymbol{k}}$ denotes the category of C_∞–coalgebras over $\boldsymbol{k}$. A C_∞–coalgebra is a cocommutative coalgebra whose coassociativity is relaxed up to homotopy, ie it is a coalgebra whose binary coproduct is strictly cocommutative, but only coassociative up to a sequence of coherent higher homotopies. The difference with an E_∞–coalgebra is that in an E_∞–coalgebra both the cocommutativity and coassociativity are relaxed up to homotopy, while in a C_∞–coalgebra only the coassociativity is relaxed up to homotopy (see Sections 13.1.9 and 13.1.10 of [13] for more details). Since there is a morphism of operads $A_\infty \to C_\infty$, we can also use the cobar construction for A_∞–coalgebras for C_∞–coalgebras. This cobar construction is defined in a similar way as for cocommutative algebra, ie it is defined as the tensor algebra generated by the coaugmentation ideal, with a differential coming from the coalgebra structure. For a C_∞–coalgebra C, this cobar construction again factors as $\Omega C \cong U\mathcal{L}C$, where $\mathcal{L}$ now denotes the C_∞ analog of $\mathcal{L}$ and U is the universal enveloping algebra. By the PBW theorem, this algebra will always be the symmetric algebra on some vector space V and is therefore either one-dimensional or infinite-dimensional and can never model finite fundamental groups, therefore showing that there is no functor $\mathcal{C}$ from connected spaces to C_∞–coalgebras such that $\mathcal{C}(X,b)$ is Ω–quasi-isomorphic to $C_*(X,b;\boldsymbol{k})$ as A_∞–algebras.

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