

Model Reduction of Parametric Differential-Algebraic Systems by Balanced Truncation

J. Przybilla* M. Voigt†

*Max Planck Institute for Dynamics of Complex Technical Systems, Sandtorstraße 1, 39106 Magdeburg, Germany.

Email: przybilla@mpi-magdeburg.mpg.de, ORCID: 0000-0002-8703-8735

†UniDistance Suisse, Überlandstrasse 12, 3900 Brig, Switzerland.

Email: matthias.voigt@fernuni.ch, ORCID: 0000-0001-8491-1861

Abstract: In this article, we deduce a procedure to apply balanced truncation to parameter-dependent differential-algebraic systems. For that, we solve projected Lyapunov equations to compute the Gramians that are required for the truncation procedure. This process would lead to high computational costs if we perform it for a large number of parameters. Hence, we combine this approach with the reduced basis method that determines a reduced representation of the Lyapunov equation solutions for the parameters of interest. Residual-based error estimators are then used to evaluate the quality of the approximations. To apply the error estimators, a uniformly strictly dissipative state-space realization of the system is needed. We demonstrate how this property can be enforced by suitable state-space transformations. We illustrate the effectiveness of our approach on several models from fluid dynamics and mechanics. We further consider an application of the method in the context of damping optimization.

Keywords: model order reduction, reduced basis method, balanced truncation, error estimation, parameter-dependency, differential-algebraic equations

AMS subject classifications: 41A20, 65F45, 65F55, 65L80, 70G60, 76D05, 93A15

Novelty statement: We propose a new method that reduces parametric differential-algebraic equations by combining projection methods for differential-algebraic equations and the reduced basis methods for Lyapunov equations. To apply the reduced basis method new error estimators are invented. The performance of our new method is illustrated for several examples.

1 Introduction

In the modeling of various industrial processes one often obtains systems of differential-algebraic equations (DAEs) that are of high dimensions. Typical examples are electrical circuits, thermal and diffusion processes, multibody systems, and certain discretized partial differential equations [8, 9]. Often, these systems are further dependent on parameters that can describe physical properties such as material constants and which are often subject to optimization. The resulting parameter-dependent differential-

algebraic systems have the form

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\mu)z(t) &= \mathcal{A}(\mu)z(t) + \mathcal{B}(\mu)u(t), \\ y(t) &= \mathcal{C}(\mu)z(t), \end{aligned} \quad (1)$$

where $\mathcal{E}(\mu)$, $\mathcal{A}(\mu) \in \mathbb{R}^{N,N}$, $\mathcal{B}(\mu) \in \mathbb{R}^{N,m}$ and $\mathcal{C}(\mu) \in \mathbb{R}^{p,N}$ are dependent on parameters $\mu \in \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^d$. The input, the state and the output are described by $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^N$ and $y(t) \in \mathbb{R}^p$. The matrix $\mathcal{E}(\mu)$ can be a singular. However, throughout this paper, the pencil $s\mathcal{E}(\mu) - \mathcal{A}(\mu)$ is assumed to be regular for all $\mu \in \mathcal{D}$, that is $\det(s\mathcal{E}(\mu) - \mathcal{A}(\mu))$ is not the zero polynomial for all $\mu \in \mathcal{D}$. The model (1) is also referred to as the full-order model (FOM). Often, one also considers the input/output mapping of the system in the frequency domain. This relation is typically expressed by the *transfer function* $\mathcal{G}(\mu, s) := \mathcal{C}(\mu)(s\mathcal{E}(\mu) - \mathcal{A}(\mu))^{-1}\mathcal{B}(\mu)$.

Because of the high state-space dimension N of (1) it is useful to apply reduction methods to extract the essential information of the system and its solution. More precisely, we want to determine a reduced-order model (ROM)

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_R(\mu)z_R(t) &= \mathcal{A}_R(\mu)z_R(t) + \mathcal{B}_R(\mu)u(t), \\ y_R(t) &= \mathcal{C}_R(\mu)z_R(t), \end{aligned}$$

with $\mathcal{E}_R(\mu)$, $\mathcal{A}_R(\mu) \in \mathbb{R}^{r,r}$, $\mathcal{B}_R(\mu) \in \mathbb{R}^{r,m}$ and $\mathcal{C}_R \in \mathbb{R}^{p,r}$, $r \ll N$. The reduced state and output are $z_R(t) \in \mathbb{R}^r$ and $y_R(t) \in \mathbb{R}^p$. The aim is to determine the surrogate model in such a way that the output of the ROM well-approximates the output of the FOM for all admissible inputs $u(\cdot)$ and all parameters $\mu \in \mathcal{D}$. In terms of the transfer function this means that the error $\|\mathcal{G}(\mu, \cdot) - \mathcal{G}_R(\mu, \cdot)\|$ is sufficiently small in an appropriate norm for all $\mu \in \mathcal{D}$, where $\mathcal{G}_R(\mu, s)$ denotes the transfer function of the ROM.

For parameter-independent problems, i. e.,

$$\mathcal{E}(\mu) \equiv \mathcal{E}, \quad \mathcal{A}(\mu) \equiv \mathcal{A}, \quad \mathcal{B}(\mu) \equiv \mathcal{B}, \quad \mathcal{C}(\mu) \equiv \mathcal{C}, \quad \mathcal{G}(\mu, s) \equiv \mathcal{G}(s),$$

there are several classes of model order reduction techniques. Examples are singular value based approaches like balanced truncation [5, 25, 38] and Hankel norm approximations [14]. Additionally, there are Krylov subspace based methods such as the iterative rational Krylov algorithm (IRKA) [5, 12, 15, 16] and moment matching as well as data driven methods such as the Loewner framework [23]. Corresponding methods for parameter-dependent systems are introduced in [3, 4, 13].

In this article we consider balanced truncation which is one of the most popular reduction techniques. This is mainly due to the guaranteed asymptotic stability of the ROM, the existence of an error bound, and appealing numerical techniques for the involved Lyapunov equations [6, 10, 31, 33].

Within balanced truncation, certain Lyapunov equations corresponding to the original system need to be solved. The solutions of these equations which are called Gramians. They describe the input-to-state and state-to-output behavior and are used to construct projection matrices for the reduction. The multiplication of the system matrices with these projection matrices then results in a reduced-order model.

All the above-mentioned methods focus on the case $\mathcal{E} = I_N$ and are, however, not directly applicable in the DAE case. Even if the problem is not parameter-dependent, there are several challenges that one has to face (here we assume for simplicity, that the problem is parameter-independent):

- a) Since the matrix \mathcal{E} is typically singular, the transfer function $\mathcal{G}(s)$ is possibly improper, i. e., it may have a polynomial part which is unbounded for growing values of $|s|$. If the reduced transfer function $\mathcal{G}_R(s)$ does not match this polynomial part, then the error transfer function $\mathcal{G}(s) - \mathcal{G}_R(s)$

is not an element of the rational Hardy spaces $\mathcal{RH}_\infty^{p \times m}$ or $\mathcal{RH}_2^{p \times m}$ (see, e. g. [40] for a definition) and the output error cannot be bounded by the input norm. Thus, a model reduction scheme for DAEs must preserve the polynomial part of its transfer function. This is addressed in [16, 24, 35].

- b) In balanced truncation, one has to solve two large-scale Lyapunov equations. In the DAE case, one has to consider a generalization of these – so-called *projected* Lyapunov equations. There, the Lyapunov equations are projected to the subspace of the DAE in which the dynamics evolves. This involves certain projectors which cannot be explicitly formed since this would destroy the sparsity structure of the coefficient matrices. However, for DAE systems of special structure, the projectors are of a particular form which can be exploited from the numerical point of view. More precisely, in the solution algorithms for Lyapunov equations, the projectors can be applied implicitly without forming them. For details, we refer to [7, 18, 30, 37].

If we want to reduce a parameter-dependent DAE by balanced truncation, we would have to solve the Lyapunov equations for each individual parameter of interest which is computationally prohibitive. To avoid these computational costs, in this paper we make use of the reduced basis method which is a well-established method to reduce parameter-dependent partial differential equations [19, 28]. This method has also been applied to parametric standard Lyapunov equations in [33]. There, the parametric Lyapunov equations are only solved for few sampling parameters. Then, based on these solutions, a reduced subspace in which the Lyapunov equation solutions approximately live, is constructed. The latter steps form the computationally expensive *offline phase*. After that, using the reduced basis representation, the Lyapunov equations can be solved much more efficiently for all $\mu \in \mathcal{D}$ in the *online phase*. A crucial question in the offline phase is the choice of the sample parameters. Usually, a grid of test parameters is selected. For these, the error is quantified using a posteriori error estimators. Then new samples are taken at those parameters at which the error estimator delivers the largest error.

In this paper we generalize the reduced basis balanced truncation method of [33] to differential-algebraic systems with a focus on structured systems from certain application areas. The main problems we solve in this paper are the following:

- a) We derive error estimators for parameter-dependent projected Lyapunov equations. This requires the given system to be uniformly strictly dissipative. Since this condition is not always satisfied, we discuss transformations to achieve this.
- b) We apply this approach to several application problems and show its benefits in optimization problems for mechanical systems.

This paper is organized as follows. In Section 2, three model problems are introduced that motivate the method presented in this paper. In Section 3, we review projection techniques to eliminate the algebraic equations in (1). Afterwards, in Section 4, we consider model order reduction by balanced truncation for projected systems. We further address the numerical challenges that arise in computing the solutions of the required Lyapunov equations. Since solving Lyapunov equations for every requested parameter leads to high computational costs, in Section 5 the reduced basis method is presented, which was first applied to standard Lyapunov equations in [33]. We derive two error estimators for our method, one of them is motivated by the estimator from [33], the other one is an adaption of the estimator presented in [31]. Finally, in Section 6, we evaluate the method of this paper by applying it to our model problems from Section 2.

2 Model problems

2.1 Problem 1: Stokes-like DAEs

The first example is the system

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} E(\mu) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} &= \begin{bmatrix} A(\mu) & G(\mu) \\ G(\mu)^\top & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B(\mu) \\ 0 \end{bmatrix} u(t), \\ y(t) &= [C(\mu) \quad 0] \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \end{aligned} \quad (2)$$

which arises, i. e., if we discretize the incompressible Navier-Stokes equation. The parameter-independent version is presented in [24, 35]. The system matrices are dependent on a parameter $\mu \in \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^d$ is the parameter domain. For fluid-flow problems, the matrices $E(\mu), A(\mu) \in \mathbb{R}^{n,n}$ represent the masses and the discretized Laplace operator. Naturally, it holds that $E(\mu) = E(\mu)^\top > 0$ and $A(\mu) = A(\mu)^\top < 0$ for all $\mu \in \mathcal{D}$. The discrete gradient is given by $G(\mu) \in \mathbb{R}^{n,q}$ which we assume to be of full rank for all $\mu \in \mathcal{D}$. The matrices $B(\mu) \in \mathbb{R}^{n,m}$ and $C(\mu) \in \mathbb{R}^{p,n}$ are the input and the output matrices, respectively. The state of the system at time t is given by $x(t) \in \mathbb{R}^n$ and $\lambda(t) \in \mathbb{R}^q$. The vectors $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the input and output of the system.

We assume that $E(\cdot)$ and $A(\cdot)$ are *affine in the parameter* μ , i. e.,

$$E(\mu) = \sum_{k=1}^{n_E} \Theta_k^E(\mu) E_k \quad \text{and} \quad A(\mu) = \sum_{k=1}^{n_A} \Theta_k^A(\mu) A_k,$$

where $\Theta_k^E(\cdot), \Theta_k^A(\cdot)$ are *non-negative* parameter-dependent functions and E_k, A_k parameter-independent, positive and negative semi-definite matrices. Moreover, for reasons of computational efficiency, we always assume that $n_E, n_A \ll n$.

2.2 Problem 2: Mechanical systems

The second system we consider is of the form

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} I_{n_x} & 0 & 0 \\ 0 & M(\mu) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda(t) \end{bmatrix} &= \begin{bmatrix} 0 & I_{n_x} & 0 \\ -K(\mu) & -D(\mu) & G(\mu) \\ G(\mu)^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_x(\mu) \\ 0 \end{bmatrix} u(t), \\ y(t) &= [C_x(\mu) \quad 0 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda(t) \end{bmatrix}, \end{aligned} \quad (3)$$

which results from a linearization of the spring-mass-damper model presented in [24]. The mass, damping, and stiffness matrices $M(\mu), D(\mu), K(\mu) \in \mathbb{R}^{n_x, n_x}$ are assumed to be symmetric and positive definite for all $\mu \in \mathcal{D}$. The matrices $B_x(\mu) \in \mathbb{R}^{n_x, m}$ and $C_x(\mu) \in \mathbb{R}^{p, n_x}$ are the input and the output matrices. The matrix $G(\mu) \in \mathbb{R}^{n_x, q}$ reflects algebraic constraints on the system and is of full rank. In this example, the state includes the displacement $x_1(t) \in \mathbb{R}^{n_x}$, the velocity $x_2(t) \in \mathbb{R}^{n_x}$, and $\lambda(t) \in \mathbb{R}^q$.

For convenience, we also define

$$\begin{aligned} E(\mu) &:= \begin{bmatrix} I_{n_x} & 0 \\ 0 & M(\mu) \end{bmatrix}, \quad A(\mu) := \begin{bmatrix} 0 & I_{n_x} \\ -K(\mu) & -D(\mu) \end{bmatrix}, \quad B(\mu) := \begin{bmatrix} 0 \\ B_x(\mu) \end{bmatrix}, \\ C(\mu) &:= [C_x(\mu) \quad 0], \quad n := 2n_x. \end{aligned} \quad (4)$$

Then with this notation, we can write the mechanical system similarly as in (2) with the difference

that the off-diagonal blocks of the state matrix are not the transposes of each other.

Similarly to the first model problem, we assume that $M(\cdot)$, $D(\cdot)$, and $K(\cdot)$ can be written as

$$M(\mu) = \sum_{k=1}^{n_M} \Theta_k^M(\mu) M_k, \quad D(\mu) = \sum_{k=1}^{n_D} \Theta_k^D(\mu) D_k \quad \text{and} \quad K(\mu) = \sum_{k=1}^{n_K} \Theta_k^K(\mu) K_k,$$

where $\Theta_k^M(\cdot)$, $\Theta_k^D(\cdot)$, $\Theta_k^K(\cdot)$ are *non-negative* parameter-dependent functions and M_k, D_k, K_k parameter-independent symmetric and positive semi-definite matrices with $n_M, n_D, n_K \ll n$.

2.3 Problem 3: Optimization problems

We consider an optimization problem which is discussed in [4, 39]. We have a mechanical system

$$\begin{aligned} \frac{d}{dt} \underbrace{\begin{bmatrix} I_{n_x} & 0 \\ 0 & M \end{bmatrix}}{=:E} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & I_{n_x} \\ -K & -D(\mu) \end{bmatrix}}{=:A(\mu)} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ F \end{bmatrix}}{=:B} u(t), \\ y(t) &= \underbrace{\begin{bmatrix} C_x & 0 \end{bmatrix}}{=:C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned} \tag{5}$$

The matrices $M, K \in \mathbb{R}^{n_x, n_x}$ (both symmetric and positive definite), $F \in \mathbb{R}^{n_x, m}$, and $C_x \in \mathbb{R}^{p, n_x}$ are the mass, stiffness, input, and output matrices, respectively. The damping matrix is given by $D(\mu) := D_{\text{int}} + B_x H(\mu) B_x^T$ where $D_{\text{int}} \in \mathbb{R}^{n_x, n_x}$ is the internal damping and the expression $B_x H(\mu) B_x^T$ describes the external dampers. There $B_x \in \mathbb{R}^{n_x, \ell}$ contains the external dampers' positions and $H(\mu) := \text{diag}(h_1(\mu), \dots, h_\ell(\mu)) \in \mathbb{R}^{\ell, \ell}$ is a diagonal matrix containing the dampers' viscosities $h_1(\mu), \dots, h_\ell(\mu) \geq 0$ for all $\mu \in \mathcal{D}$. The vectors $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, the primary excitation input, and the output at time t . The aim is to choose $h_1(\mu), \dots, h_\ell(\mu)$ such that the external input $Fu(\cdot)$ affects the output $y(\cdot)$ as little as possible. Therefore, we wish to minimize the response energy which is given by

$$J(\mu) = \text{tr}(CP(\mu)C^T), \tag{6}$$

where $P(\mu)$ solves the Lyapunov equation

$$A(\mu)P(\mu)E^T + EP(\mu)A(\mu)^T = -BB^T. \tag{7}$$

We will use reduction techniques, presented in Section 5, to efficiently solve the parametric Lyapunov equation (7) and subsequently solve the reduced optimization problem.

3 Elimination of algebraic equations

In this section we briefly describe the projection technique for eliminating the algebraic constraints in DAEs of the forms (2) and (3) for a fixed parameter $\mu \in \mathcal{D}$. Therefore, for simplicity of presentation, we omit μ in this section. The details of this can be found in [18, 30, 35]. These projections integrate the state that results from the algebraic equations into the differential equations of the system, such that the algebraic equations are eliminated. These considerations are useful for deriving a projection technique for the corresponding Lyapunov equations. First, we deduce the projections for Stokes-like systems.

As presented in [18], we can build a projection matrix

$$\Pi := I_n - G (G^T E^{-1} G)^{-1} G^T E^{-1}.$$

With the help of this projector we obtain the projected system

$$\begin{aligned} \frac{d}{dt} E_\Pi x(t) &= A_\Pi x(t) + B_\Pi u(t), \\ y(t) &= C_\Pi x(t), \end{aligned} \tag{8}$$

with $E_\Pi := \Pi E \Pi^T$, $A_\Pi := \Pi A \Pi^T$, $B_\Pi := \Pi B$, $C_\Pi := C \Pi^T$. The state $\lambda(t)$ can be constructed by

$$\lambda(t) = -(G^T E^{-1} G)^{-1} G^T E^{-1} A x(t) - (G^T E^{-1} G)^{-1} G^T E^{-1} B u(t). \tag{9}$$

Note that (2) (for a fixed parameter) has exactly the same solutions as (8) together with (9) and $x(t) = \Pi^T \lambda(t)$.

Since the matrix pencil $sE_\Pi - A_\Pi$ is singular, we use a factorization $\Pi = \Phi_l \Phi_r^T$, with $\Phi_l, \Phi_r \in \mathbb{R}^{n,k}$, $\Phi_l^T \Phi_r = I_k$ such that we obtain the system

$$\begin{aligned} \frac{d}{dt} E_\Phi \tilde{x}(t) &= A_\Phi \tilde{x}(t) + B_\Phi u(t), \\ y(t) &= C_\Phi \tilde{x}(t), \end{aligned} \tag{10}$$

with $\tilde{x}(t) := \Phi_l^T x(t)$, $E_\Phi := \Phi_r^T E \Phi_r$, $A_\Phi := \Phi_r^T A \Phi_r$, $B_\Phi := \Phi_r^T B$, $C_\Phi := C \Phi_r$. From system (10) we can derive the solutions of (2) (for a fixed parameter) and (8). Now the matrix pencil $sE_\Phi - A_\Phi$ is regular.

To eliminate the algebraic equations in the case of mechanical systems (3), one can proceed similarly as above. The main difference is the choice of the projection matrix which now becomes $\Pi := \text{diag}(\hat{\Pi}, \hat{\Pi})$ with

$$\hat{\Pi} := I_{n_x} - G (G^T M^{-1} G)^{-1} G^T M^{-1}.$$

By projecting with this Π , we obtain systems of the form (8) and (10), see also [30].

4 Model reduction of differential-algebraic systems

The aim of this section is to reduce the projected systems presented in the previous section. When doing so, we can simultaneously reduce the original differential-algebraic equations from Section 2. In this section we will still focus entirely on the fixed-parameter case.

We utilize balanced truncation modified for projected systems which is presented in Subsection 4.1. Afterwards, in Subsection 4.2, we compute the projected Gramians that are needed for balanced truncation by adapting the ADI method to projected Lyapunov equations.

4.1 Balanced truncation

The aim of this subsection is to find a reduced system

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_r z_r(t) &= \mathcal{A}_r z_r(t) + \mathcal{B}_r u(t), \\ y_r(t) &= \mathcal{C}_r z_r(t), \end{aligned} \tag{11}$$

with $\mathcal{E}_r, \mathcal{A}_r \in \mathbb{R}^{r,r}$, $\mathcal{B}_r \in \mathbb{R}^{r,m}$, $\mathcal{C}_r \in \mathbb{R}^{p,r}$, $r \ll N$ which approximates the input/output behavior of the original system (2) (for a fixed parameter) and (8).

We consider the controllability Gramian P_Π and the observability Gramian Q_Π of the projected system (8). We obtain them by solving the projected Lyapunov equations

$$E_\Pi P_\Pi A_\Pi^\top + A_\Pi P_\Pi E_\Pi^\top = -B_\Pi B_\Pi^\top, \quad \Pi^\top P_\Pi \Pi = P_\Pi, \quad (12a)$$

$$E_\Pi^\top Q_\Pi A_\Pi + A_\Pi^\top Q_\Pi E_\Pi = -C_\Pi^\top C_\Pi, \quad \Pi^\top Q_\Pi \Pi = Q_\Pi. \quad (12b)$$

These Gramians P_Π and Q_Π correspond to the proper controllability and observability Gramians as introduced by Stykel [34].

We summarize solution techniques for projected Lyapunov equations in Subsection 4.2. Since P_Π and Q_Π are positive semidefinite, there exist factorizations

$$P_\Pi = R_\Pi R_\Pi^\top, \quad Q_\Pi = S_\Pi S_\Pi^\top$$

with factors $R_\Pi \in \mathbb{R}^{N,N}$, $S_\Pi \in \mathbb{R}^{N,N}$. We consider the singular value decomposition

$$S_\Pi^\top E_\Pi R_\Pi = U \Sigma V^\top = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix},$$

where the matrix Σ is a diagonal matrix with decreasing nonnegative entries that are called Hankel singular values. They are an indicator for the observability and reachability of certain states of the system. With the matrix Σ_1 which contains the r largest Hankel singular values, we construct the left and right projection matrices as

$$W := S_\Pi U_1 \Sigma_1^{-\frac{1}{2}} \quad \text{and} \quad T := R_\Pi V_1 \Sigma_1^{-\frac{1}{2}}.$$

Then we obtain the reduced system (11) by setting

$$\mathcal{E}_r := W^\top E_\Pi T = I_r, \quad \mathcal{A}_r := W^\top A_\Pi T, \quad \mathcal{B}_r := W^\top B_\Pi, \quad \mathcal{C}_r := C_\Pi T.$$

Remark 1. The projection matrices fulfill $W = \Pi^\top W$ and $T = \Pi^\top T$ and hence, we also have

$$\mathcal{E}_r := W^\top E T, \quad \mathcal{A}_r := W^\top A T, \quad \mathcal{B}_r := W^\top B, \quad \mathcal{C}_r := C T.$$

As in [1, Theorem 7.9] and [5, Theorem 6.4], if $\sigma_r > \sigma_{r+1}$, then the ROM is asymptotically stable and one can estimate the output error of the reduced system by

$$\begin{aligned} \|y - y_r\|_{L^2([0,\infty),\mathbb{R}^p)} &\leq \|\mathcal{G} - \mathcal{G}_r\|_\infty \|u\|_{L^2([0,\infty),\mathbb{R}^m)}, \\ &\leq \left(2 \sum_{j=r+1}^n \sigma_j \right) \|u\|_{L^2([0,\infty),\mathbb{R}^m)}, \end{aligned} \quad (14)$$

where $\mathcal{G}(s) := \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$ and $\mathcal{G}_r(s) := \mathcal{C}_r(s\mathcal{E}_r - \mathcal{A}_r)^{-1}\mathcal{B}_r$ are the *transfer functions* of the original and the reduced system. The \mathcal{H}_∞ -norm is defined as $\|\mathcal{G}\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(\mathcal{G}(i\omega))$.

4.2 Solving projected Lyapunov equations

The aim of this section is to solve the projected Lyapunov equations (12) in order to compute the Gramians of system (8). For standard systems with $\mathcal{E} = I_N$, there are several methods to solve the corresponding Lyapunov equations. If small-scale matrices are considered, the Bartels-Stewart

algorithm [2] or Hammarling's method [17] are used. These methods however, are inefficient in the case of large matrices. In typical applications of practical relevance, the solution of a Lyapunov equation is often of very low numerical rank. So it is desired to compute low-rank factors of these solutions directly. State-of-the-art methods are the ADI method [21, 27], the sign function method [5] or Krylov subspace methods [32]. In the literature there exist also several extensions to projected Lyapunov equations such as [18, 37].

In this section, we utilize the ADI method for projected Lyapunov equations presented in [18] to compute a *low-rank factor* $R_\Pi \in \mathbb{R}^{n,k}$ with $n \gg k$ such that $P_\Pi \approx R_\Pi R_\Pi^\top$ approximately solves (12a). Analogously, we obtain the low-rank factors S_Π of the solution Q_Π for the solution of (12b).

We define the matrices

$$\begin{aligned} \mathcal{S}_\Pi(p) &:= \Phi_r (E_\Phi + pA_\Phi)^{-1} \Phi_r^\top, \\ \mathcal{T}_\Pi(p) &:= E_\Pi - \bar{p}A_\Pi, \\ \mathcal{C}_\Pi(p) &:= \mathcal{S}_\Pi(p)\mathcal{T}_\Pi(p), \end{aligned} \tag{15}$$

for a shift parameter $p \in \mathbb{C}^-$. As proven in [18, Lemma 5.5], the solution P_Π of (12a) can be approximated by $P_\Pi \approx Z_k Z_k^\mathbb{H}$ with

$$Z_k := \alpha(p) [\mathcal{S}_\Pi(p)B_\Pi \quad \mathcal{C}_\Pi(p)\mathcal{S}_\Pi(p)B_\Pi \quad \dots \quad \mathcal{C}_\Pi(p)^{k-1}\mathcal{S}_\Pi(p)B_\Pi] \tag{16}$$

and $\alpha(p) := \sqrt{-2\operatorname{Re}(p)}$. In order to improve the convergence of the method, typically a different shift is chosen in each iteration. If the coefficient matrices defining the Lyapunov equation are real, then the iteration matrices Z_k can also be chosen to be real, if a double shift with p and \bar{p} is carried out. For the details on this procedure and the choice of shifts we refer to [21].

Another problem in practice is the construction of the matrices in (15). Since Π and Φ_r are generally dense matrices, also the matrices in (15) are dense. Thus, the projectors should only be applied implicitly without constructing them explicitly at any point in the algorithm. Hence, we use [18, Lemma 5.4 and 5.3] to determine the entries of Z_k for the case of a Stokes-like system without the explicit computations of projection matrices as follows. The first lemma provides that $U := \mathcal{S}_\Pi(p)B_\Pi$ solves

$$\begin{bmatrix} E + pA & G \\ G^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ \Lambda \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

which can be used to compute the first m columns of Z_k . The second lemma shows that for a matrix H with $H = \Pi^\top H$, the equation

$$\begin{bmatrix} E + pA & G \\ G^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ \Lambda \end{bmatrix} = \begin{bmatrix} (E - \bar{p}A)H \\ 0 \end{bmatrix}$$

has the solution $U = \mathcal{C}_\Pi(p)H$. That way, we obtain the remaining columns of Z_k .

5 Reduced basis method

In this section, we return to the problem of reducing parameter-dependent systems. Since the solution of the Lyapunov equations is the most expensive part of balanced truncation, we aim to limit the number of Lyapunov equation solves. To achieve this, the reduced basis method presented by Son and Stykel in [33] can be applied.

The main idea consists of finding a reduced representation of the solutions of the Lyapunov equations

(12a) of the form

$$P_{\Pi}(\mu) \approx Z(\mu)Z(\mu)^{\text{H}} \quad \text{with} \quad Z(\mu) = V(\mu)\tilde{Z}(\mu) \quad \forall \mu \in \mathcal{D}, \quad (17)$$

where $\Pi(\mu)$ denotes the projector for the parameter value μ and $V(\mu) \in \mathbb{R}^{n, r_{\mathbf{V}}(\mu)}$ has orthonormal columns with $\Pi(\mu)^{\text{T}}V(\mu) = V(\mu)$. The low-rank factor $\tilde{Z}(\mu)$ solves the reduced Lyapunov equation

$$\begin{aligned} & V(\mu)^{\text{T}}E_{\Pi}(\mu)V(\mu)\tilde{Z}(\mu)\tilde{Z}(\mu)^{\text{H}}V(\mu)^{\text{T}}A_{\Pi}(\mu)^{\text{T}}V(\mu) \\ & + V(\mu)^{\text{T}}A_{\Pi}(\mu)V(\mu)\tilde{Z}(\mu)\tilde{Z}(\mu)^{\text{H}}V(\mu)^{\text{T}}E_{\Pi}(\mu)^{\text{T}}V(\mu) = -V(\mu)^{\text{T}}B_{\Pi}(\mu)B_{\Pi}(\mu)^{\text{T}}V(\mu), \end{aligned} \quad (18)$$

where $E_{\Pi}(\mu) := \Pi(\mu)E(\mu)\Pi(\mu)^{\text{T}}$, $A_{\Pi}(\mu) := \Pi(\mu)A(\mu)\Pi(\mu)^{\text{T}}$, and $B_{\Pi}(\mu) := \Pi(\mu)B(\mu)$.

The equation $\Pi(\mu)^{\text{T}}V(\mu) = V(\mu)$ replaces the condition

$$\Pi(\mu)^{\text{T}}(V(\mu)\tilde{Z}(\mu))(V(\mu)\tilde{Z}(\mu))^{\text{H}}\Pi(\mu) = (V(\mu)\tilde{Z}(\mu))(V(\mu)\tilde{Z}(\mu))^{\text{H}}.$$

For the case of a Stokes-like system (2), this condition is further equivalent to the property $G(\mu)^{\text{T}}V(\mu) = 0$, i. e., all columns of $V(\mu)$ lie in the kernel of $G(\mu)^{\text{T}}$.

In practice, we aim to determine a matrix $\mathbf{V} \in \mathbb{R}^{n, r_{\mathbf{V}}}$ with $n \gg r_{\mathbf{V}}$ that contains the information of $V(\mu)$ for all $\mu \in \mathcal{D}$ globally. When we have determined such a \mathbf{V} , we set

$$V(\mu) = \text{orth}(\Pi(\mu)^{\text{T}}\mathbf{V}), \quad (19)$$

where $\text{orth}(\Pi(\mu)^{\text{T}}\mathbf{V})$ denotes the orthonormalized columns of $\Pi(\mu)^{\text{T}}\mathbf{V}$. If the matrix \mathbf{V} has been found, then the reduced-order model for a particular parameter $\mu \in \mathcal{D}$ can be determined very efficiently in the so-called *online phase*, where one simply solves (18) with (17) and (19) to determine the two Gramians and then projects the system as in Subsection 4.1.

The computation of the *reduced basis* \mathbf{V} is done beforehand in the *offline phase*. There, we solve the full-order Lyapunov equation only for those parameters, where their solutions are currently approximated worst to enrich the current reduced basis. A posteriori error estimators are employed on a test parameter set to find these points efficiently.

We determine \mathbf{V} by considering a test-parameter set $\mathcal{D}_{\text{Test}} \subset \mathcal{D}$. We begin by computing a low-rank factor $Z(\mu_1)$ in $\mu_1 \in \mathcal{D}_{\text{Test}}$ such that $P_{\Pi}(\mu_1) = Z(\mu_1)Z(\mu_1)^{\text{H}}$ solves the projected Lyapunov equation

$$\begin{aligned} & E_{\Pi}(\mu)P_{\Pi}(\mu)A_{\Pi}(\mu)^{\text{T}} + A_{\Pi}(\mu)P_{\Pi}(\mu)E_{\Pi}(\mu)^{\text{T}} = -B_{\Pi}(\mu)B_{\Pi}(\mu)^{\text{T}}, \\ & \Pi(\mu)^{\text{T}}P_{\Pi}(\mu)\Pi(\mu) = P_{\Pi}(\mu) \end{aligned} \quad (20)$$

for $\mu = \mu_1$.

The first reduced basis is then given by $\mathbf{V} := \text{orth}(Z(\mu_1))$. Next we determine the test parameter $\mu_2 \in \mathcal{D}_{\text{Test}}$ for which the solution $P_{\Pi}(\mu_2)$ of the Lyapunov equation (12a) is approximated worst by using (17), (18), and (19). For that, one of the error estimators $\Delta_{\mathbf{V}}(\mu)$ presented in Subsection 5.1 is utilized. For this parameter μ_2 , we solve the projected Lyapunov equation (20) with $\mu = \mu_2$ to obtain the low-rank factor $Z(\mu_2)$. Then we define the new reduced basis by setting $\mathbf{V} := \text{orth}([\mathbf{V}, Z(\mu_2)])$. This procedure is repeated until the error estimator $\Delta_{\mathbf{V}}$ is smaller than a prescribed tolerance for every test parameter in $\mathcal{D}_{\text{Test}}$. This method results in Algorithm 1.

Algorithm 1 Reduced basis method for projected Lyapunov equations (offline phase)**Input:** Dynamical system (1), test parameter set $\mathcal{D}_{\text{Test}}$, tolerance Tol.**Output:** Reduced basis matrix \mathbf{V}

- 1: Choose any $\mu_1 \in \mathcal{D}_{\text{Test}}$.
- 2: Solve the projected Lyapunov equation (20) for $\mu = \mu_1$ to obtain $Z(\mu_1)$.
- 3: Set $\mathcal{M} := \{\mu_1\}$.
- 4: Set $\mathbf{V} := \text{orth}(Z(\mu_1))$.
- 5: Set $\hat{\mu} := \arg \max_{\mu \in \mathcal{D}_{\text{Test}} \setminus \mathcal{M}} \Delta_V(\mu)$.
- 6: Set $\Delta_V^{\max} := \Delta_V(\hat{\mu})$.
- 7: **while** $\Delta_V^{\max} > \text{Tol}$ **do**
- 8: Solve the projected Lyapunov equation (20) for $\mu = \hat{\mu}$ to obtain $Z(\hat{\mu})$.
- 9: Set $\mathcal{M} := \mathcal{M} \cup \{\hat{\mu}\}$.
- 10: Set $\mathbf{V} := \text{orth}([\mathbf{V}, Z(\hat{\mu})])$.
- 11: $\hat{\mu} := \arg \max_{\mu \in \mathcal{D}_{\text{Test}} \setminus \mathcal{M}} \Delta_V(\mu)$
- 12: Set $\Delta_V^{\max} := \Delta_V(\hat{\mu})$.
- 13: **end while**

5.1 Error estimation

We want to estimate the error $\mathcal{E}_{\Pi}(\mu) := P_{\Pi}(\mu) - Z(\mu)Z(\mu)^{\text{H}}$ with the help of the residual

$$\mathcal{R}_{\Pi}(\mu) := A_{\Pi}(\mu)Z(\mu)Z(\mu)^{\text{H}}E_{\Pi}(\mu)^{\text{T}} + E_{\Pi}(\mu)Z(\mu)Z(\mu)^{\text{H}}A_{\Pi}(\mu)^{\text{T}} + B_{\Pi}(\mu)B_{\Pi}(\mu)^{\text{T}}.$$

As in [33], we consider the linear system

$$L_{\Pi}(\mu)x_{\Pi}(\mu) = b_{\Pi}(\mu), \quad x_{\Pi}(\mu) = \mathbf{\Pi}(\mu)^{\text{T}}x_{\Pi}(\mu),$$

with

$$\begin{aligned} L_{\Pi}(\mu) &:= -A_{\Pi}(\mu) \otimes E_{\Pi}(\mu) - E_{\Pi}(\mu) \otimes A_{\Pi}(\mu), \\ x_{\Pi}(\mu) &:= \text{vec}(P_{\Pi}(\mu)), \quad b_{\Pi}(\mu) := \text{vec}(B_{\Pi}(\mu)B_{\Pi}(\mu)^{\text{T}}), \quad \mathbf{\Pi}(\mu) = \Pi(\mu) \otimes \Pi(\mu), \end{aligned}$$

which is equivalent to the projected Lyapunov equation (20). The operator \otimes denotes the *Kronecker product* and vec the *vectorization operator* that stacks the columns of the matrix on top of one another. Additionally, for the decomposition $\Pi(\mu) = \Phi_1(\mu)\Phi_r(\mu)^{\text{T}}$ with $\Phi_1(\mu)^{\text{T}}\Phi_r(\mu) = I_k$ and with $E_{\Phi}(\mu) := \Phi_r(\mu)^{\text{T}}E(\mu)\Phi_r(\mu)$ and $A_{\Phi}(\mu) := \Phi_r(\mu)^{\text{T}}A(\mu)\Phi_r(\mu)$ we define

$$\Phi_r(\mu) := \Phi_r(\mu) \otimes \Phi_r(\mu), \quad \Phi_1(\mu) := \Phi_1(\mu) \otimes \Phi_1(\mu),$$

as well as

$$L_{\Phi}(\mu) := -A_{\Phi}(\mu) \otimes E_{\Phi}(\mu) - E_{\Phi}(\mu) \otimes A_{\Phi}(\mu).$$

Then for $\hat{x}_{\Pi}(\mu) := \text{vec}(Z(\mu)Z(\mu)^{\text{H}})$ it holds that

$$\begin{aligned} \|\mathcal{E}_{\Pi}(\mu)\|_{\text{F}} &= \|\hat{x}_{\Pi}(\mu) - x_{\Pi}(\mu)\|_2 \\ &= \|\mathbf{\Pi}(\mu)^{\text{T}}(\hat{x}_{\Pi}(\mu) - x_{\Pi}(\mu))\|_2 \\ &= \|\Phi_r(\mu)\Phi_1(\mu)^{\text{T}}(\hat{x}_{\Pi}(\mu) - x_{\Pi}(\mu))\|_2 \\ &= \|\Phi_r(\mu)L_{\Phi}(\mu)^{-1}L_{\Phi}(\mu)\Phi_1(\mu)^{\text{T}}(\hat{x}_{\Pi}(\mu) - x_{\Pi}(\mu))\|_2 \\ &= \|\Phi_r(\mu)L_{\Phi}(\mu)^{-1}\Phi_r(\mu)^{\text{T}}\Phi_1(\mu)L_{\Phi}(\mu)\Phi_1(\mu)^{\text{T}}(\hat{x}_{\Pi}(\mu) - x_{\Pi}(\mu))\|_2 \\ &= \|\Phi_r(\mu)L_{\Phi}(\mu)^{-1}\Phi_r(\mu)^{\text{T}}(L_{\Pi}(\mu)\hat{x}_{\Pi}(\mu) - b_{\Pi}(\mu))\|_2. \end{aligned} \tag{21}$$

We want to estimate the error $\|\mathcal{E}_\Pi(\mu)\|_F$ for which we will impose an additional assumption on the matrix pencils $sE(\mu) - A(\mu)$ in (2) and (4).

Definition 5.1. The family $\{sE(\mu) - A(\mu) \in \mathbb{R}[s]^{n,n} \mid \mu \in \mathcal{D}\}$ of matrix pencils is called *uniformly strictly dissipative*, if

$$E(\mu) = E(\mu)^T > 0 \quad \text{and} \quad A^S(\mu) := \frac{1}{2}(A(\mu) + A(\mu)^T) < 0 \quad \forall \mu \in \mathcal{D}.$$

The uniform strict dissipativity provides the solvability of the reduced Lyapunov equation (18) for all $\mu \in \mathcal{D}$. Additionally, it is used to derive an error bound. The pencil family $\{sE(\mu) - A(\mu) \mid \mu \in \mathcal{D}\}$ in example (2) is naturally uniformly strictly dissipative. On the other hand, the pencil family $\{sE(\mu) - A(\mu) \mid \mu \in \mathcal{D}\}$ in (4) is not uniformly strictly dissipative. However, we can enforce this condition by a certain transformation of the system, see Subsection 5.2 for the details.

Now we assume w.l.o.g. that the columns of $\Phi_r(\mu)$ and hence those of $\Phi_r(\mu)$ are orthonormal. Moreover, we define

$$L(\mu) := -A(\mu) \otimes E(\mu) - E(\mu) \otimes A(\mu).$$

Then Corollary 3.1.5 from [20] and the eigenvalue properties of the Kronecker product, given in [20, 22] (see also [33]), lead to the estimate

$$\begin{aligned} \sigma_{\min}(L_\Phi(\mu)) &= \sigma_{\min}(\Phi_r(\mu)^T L(\mu) \Phi_r(\mu)) \geq \lambda_{\min}\left(\frac{1}{2}\Phi_r(\mu)^T (L(\mu) + L(\mu)^T) \Phi_r(\mu)\right) \\ &\geq \lambda_{\min}\left(\frac{1}{2}(L(\mu) + L(\mu)^T)\right) = 2\lambda_{\min}(E(\mu))\lambda_{\min}(-A^S(\mu)) \\ &\geq 2 \min_{k=1,\dots,n_E} \frac{\Theta_k^E(\mu)}{\Theta_k^E(\bar{\mu})} \lambda_{\min}(E(\bar{\mu})) \min_{k=1,\dots,n_A} \frac{\Theta_k^A(\mu)}{\Theta_k^A(\bar{\mu})} \lambda_{\min}(-A^S(\bar{\mu})) =: \alpha(\mu), \end{aligned}$$

where $\bar{\mu}$ is an arbitrary fixed parameter in \mathcal{D} .

We compute the eigenvalues of $E(\bar{\mu})$ and $A^S(\bar{\mu})$ once to find a lower bound $\alpha(\mu)$ of the minimal singular value $\sigma_{\min}(L_\Phi(\mu))$ for every parameter $\mu \in \mathcal{D}$.

Applying this result to equation (21) provides the error estimate

$$\begin{aligned} \|\mathcal{E}_\Pi(\mu)\|_F &\leq \sigma_{\max}(L_\Phi(\mu)^{-1}) \cdot \|L_\Pi(\mu)\hat{x}_\Pi(\mu) - b_\Pi(\mu)\|_2 \\ &= \frac{\|L_\Pi(\mu)\hat{x}_\Pi(\mu) - b_\Pi(\mu)\|_2}{\sigma_{\min}(L_\Phi(\mu))} \\ &\leq \frac{\|\mathcal{R}_\Pi(\mu)\|_F}{\alpha(\mu)} =: \Delta_V^{(1)}(\mu). \end{aligned} \tag{22}$$

To improve the estimator (22), we consider the error bound presented in [31]. We can determine the exact error $\mathcal{E}_\Pi(\mu)$ by solving the error equation

$$\begin{aligned} A_\Pi(\mu)\mathcal{E}_\Pi(\mu)E_\Pi(\mu)^T + E_\Pi(\mu)\mathcal{E}_\Pi(\mu)A_\Pi(\mu)^T \\ = A_\Pi(\mu)Z(\mu)Z(\mu)^H E_\Pi(\mu)^T + E_\Pi(\mu)Z(\mu)Z(\mu)^H A_\Pi(\mu)^T + B_\Pi(\mu)B_\Pi(\mu)^T, \end{aligned} \tag{23}$$

with $\mathcal{E}_\Pi(\mu) = \Pi(\mu)^T \mathcal{E}_\Pi(\mu) \Pi(\mu)$ and where $Z(\mu) = V(\mu)\tilde{Z}(\mu)$ is our approximate solution factor of the original Lyapunov equation.

Assuming that $\hat{\mathcal{E}}_\Pi(\mu) \approx \mathcal{E}_\Pi(\mu)$ is an error estimate that is not necessarily an upper bound, an error

bound $\Delta_V^{(2)}(\mu)$ is derived by

$$\begin{aligned} \|\mathcal{E}_\Pi(\mu)\|_F &= \|\mathcal{E}_\Pi(\mu) + \widehat{\mathcal{E}}_\Pi(\mu) - \widehat{\mathcal{E}}_\Pi(\mu)\|_F \\ &\leq \|\widehat{\mathcal{E}}_\Pi(\mu)\|_F + \|\mathcal{E}_\Pi(\mu) - \widehat{\mathcal{E}}_\Pi(\mu)\|_F \\ &\leq \|\widehat{\mathcal{E}}_\Pi(\mu)\|_F + \frac{\|\widehat{\mathcal{R}}_\Pi(\mu)\|_F}{\alpha(\mu)} := \Delta_V^{(2)}(\mu), \end{aligned} \quad (24)$$

where the residual $\widehat{\mathcal{R}}_\Pi(\mu)$ is defined as

$$\begin{aligned} \widehat{\mathcal{R}}_\Pi(\mu) &:= A_\Pi(\mu) \left(Z(\mu)Z(\mu)^H + \widehat{\mathcal{E}}_\Pi(\mu) \right) E_\Pi(\mu)^T \\ &\quad + E_\Pi(\mu) \left(Z(\mu)Z(\mu)^H + \widehat{\mathcal{E}}_\Pi(\mu) \right) A_\Pi(\mu)^T + B_\Pi(\mu)B_\Pi(\mu)^T. \end{aligned}$$

It remains to obtain an error estimate $\widehat{\mathcal{E}}_\Pi(\mu)$ for all $\mu \in \mathcal{D}_{\text{Test}}$. To do so, we solve the error equation (23) approximately by modifying the projected ADI iteration from Section 4.2 and the reduced basis method in Algorithm 1.

The right-hand side of the error equation (23) can be written as a product of $B_l(\mu)$ and $B_r(\mu)^H$ where

$$\begin{aligned} B_l(\mu) &:= [A_\Pi(\mu)Z(\mu) \quad E_\Pi(\mu)Z(\mu) \quad B_\Pi(\mu)], \\ B_r(\mu) &:= [E_\Pi(\mu)Z(\mu) \quad A_\Pi(\mu)Z(\mu) \quad B_\Pi(\mu)]. \end{aligned}$$

For simplicity we consider the modified ADI iteration for the parameter-independent case. Analogously to the derivation in [18], it can be shown that the iteration

$$\begin{aligned} \mathcal{E}_k &= Z_l^k(Z_r^k)^H = \mathcal{C}_\Pi(p)\mathcal{E}_{k-1}\mathcal{C}_\Pi(p)^H - 2\text{Re}(p)\mathcal{S}_\Pi(p)B_lB_r^T\mathcal{S}_\Pi(p)^H \\ &= \mathcal{C}_\Pi(p)Z_l^{k-1}(Z_r^{k-1})^H\mathcal{C}_\Pi(p)^H - 2\text{Re}(p)\mathcal{S}_\Pi(p)B_lB_r^T\mathcal{S}_\Pi(p)^H \end{aligned}$$

converges to the solution \mathcal{E}_Π of (23), where $\mathcal{S}_\Pi(p)$ and $\mathcal{C}_\Pi(p)$ are defined as in (15) for a shift-parameter $p \in \mathbb{C}^-$. The factors Z_l^k and Z_r^k can be written as

$$\begin{aligned} Z_l^k &= \alpha(p) [\mathcal{S}_\Pi(p)B_l \quad \mathcal{C}_\Pi(p)\mathcal{S}_\Pi(p)B_l \quad \dots \quad \mathcal{C}_\Pi(p)^{k-1}\mathcal{S}_\Pi(p)B_l], \\ Z_r^k &= \alpha(p) [\mathcal{S}_\Pi(p)B_r \quad \mathcal{C}_\Pi(p)\mathcal{S}_\Pi(p)B_r \quad \dots \quad \mathcal{C}_\Pi(p)^{k-1}\mathcal{S}_\Pi(p)B_r] \end{aligned}$$

with $\alpha(p) = \sqrt{-2\text{Re}(p)}$. Consequently, one can modify the projected ADI iteration from Section 4.2 in such a way, that we multiply in every iteration step the last columns of the current factors Z_l^k, Z_r^k by $\mathcal{C}_\Pi(p)$ and concatenate the product with the current factors to obtain Z_l^{k+1} and Z_r^{k+1} . As in Section 4.2, we can derive the ADI iteration for a single shift. The multi-shift version can be formulated similarly.

We include the modified projected ADI iteration into the reduced basis method presented in Algorithm 1 by solving the error equation (23) in Step 2 and 8 to obtain the two factors $Z_l(\mu), Z_r(\mu)$ with $Z_l(\mu)Z_r(\mu)^H \approx \mathcal{E}_\Pi(\mu)$. The orthonormal basis computation in Step 4 and 10 is replaced by $\mathbf{V} := \text{orth}([\mathbf{V}, Z_l(\mu)])$. Note that here, the columns of $Z_l(\mu)$ and $Z_r(\mu)$ span the same subspace. As error estimate we use $\Delta_V^{(1)}$. After we have determined the orthonormal basis, we solve equation (23) on the corresponding subspace to get an approximate error $\widehat{\mathcal{E}}_\Pi(\mu)$, that we use for the error bound $\Delta_V^{(2)}(\mu)$.

5.2 Enforcing uniform strict dissipativity

In this section we demonstrate how to achieve uniform strict dissipativity for our model problems. Recall that this is needed in order to derive the error estimators from the previous subsection. First of all, the pencil family $\{sE(\mu) - A(\mu) \mid \mu \in \mathcal{D}\}$ for the Stokes-like problem is uniformly strictly dissipative by its structure. However, this is not the case for the mechanical systems in Subsections 2.2 and 2.3. Therefore, in order to apply our theory, we transform these systems to make them uniformly strictly dissipative. We only discuss this transformation for the system (3) from Subsection 2.2 since for the other one it is analogous.

Here we apply a transformation presented in [11, 26]. We observe that by adding certain productive zeros and assuming $\gamma(\mu) \neq 0$, (3) is equivalent to the system

$$\begin{aligned} \frac{d}{dt}K(\mu)x(t) + \gamma(\mu)\frac{d^2}{dt^2}M(\mu)x(t) &= -\gamma(\mu)K(\mu)x(t) + \frac{d}{dt}K(\mu)x(t) \\ &\quad - \gamma(\mu)\frac{d}{dt}D(\mu)x(t) + \gamma(\mu)G(\mu)\lambda(t) + \gamma(\mu)B_x u(t), \\ \gamma(\mu)\frac{d}{dt}M(\mu)x(t) + \frac{d^2}{dt^2}M(\mu)x(t) &= -K(\mu)x(t) + \gamma(\mu)\frac{d}{dt}M(\mu)x(t) \\ &\quad - \frac{d}{dt}D(\mu)x(t) + G(\mu)\lambda(t) + B_x(\mu)u(t), \\ 0 &= G(\mu)^T x(t), \\ 0 &= \frac{d}{dt}G(\mu)^T x(t), \end{aligned}$$

where $x(t)$ is equal to $x_1(t)$ and $\frac{d}{dt}x(t)$ equal to $x_2(t)$ from system (3). The last equation is a direct consequence from the equation above and provides no extra restrictions. These equations can be written as first order system. By defining the matrices

$$\begin{aligned} E(\mu) &:= \begin{bmatrix} K(\mu) & \gamma(\mu)M(\mu) \\ \gamma(\mu)M(\mu) & M(\mu) \end{bmatrix}, \quad A(\mu) := \begin{bmatrix} -\gamma(\mu)K(\mu) & K(\mu) - \gamma(\mu)D(\mu) \\ -K(\mu) & -D(\mu) + \gamma(\mu)M(\mu) \end{bmatrix}, \\ B(\mu) &:= \begin{bmatrix} \gamma(\mu)B_x(\mu) \\ B_x(\mu) \end{bmatrix}, \end{aligned}$$

and replacing $G(\mu)$ by $\text{diag}(G(\mu), G(\mu))$, we generate a system in the form (2) such that we can apply all methods for this kind of system in the following.

As shown in [11, 26] the transformed system has a uniformly strictly dissipative pencil family $\{sE(\mu) - A(\mu) \mid \mu \in \mathcal{D}\}$, if we choose $\gamma(\mu) > 0$ such that

$$\gamma(\mu) \left(\lambda_{\max} \left(M(\mu) + \frac{1}{4} D(\mu)K(\mu)^{-1}D(\mu) \right) \right) \leq \lambda_{\min}(D(\mu)) \quad \forall \mu \in \mathcal{D}. \quad (25)$$

Because of the symmetry and positive definiteness of $D(\mu)K(\mu)^{-1}D(\mu)$ and the submultiplicativity of the norm we obtain

$$\begin{aligned} \lambda_{\max}(D(\mu)K(\mu)^{-1}D(\mu)) &= \|D(\mu)K(\mu)^{-1}D(\mu)\|_2 \\ &\leq \|D(\mu)\|_2^2 \cdot \|K(\mu)^{-1}\|_2 = \lambda_{\max}(D(\mu))^2 \cdot \lambda_{\max}(K(\mu)^{-1}). \end{aligned}$$

By Weyl's lemma [20, 22], we also have

$$\lambda_{\max} \left(M(\mu) + \frac{1}{4} D(\mu)K(\mu)^{-1}D(\mu) \right) \leq \lambda_{\max}(M(\mu)) + \frac{1}{4} \lambda_{\max}(D(\mu)K(\mu)^{-1}D(\mu)).$$

Therefore, condition (25) is satisfied, if we choose $\gamma(\mu)$ such that

$$\gamma(\mu) \left(\lambda_{\max}(M(\mu)) + \frac{1}{4} \lambda_{\max}(D(\mu))^2 \lambda_{\max}(K(\mu)^{-1}) \right) \leq \lambda_{\min}(D(\mu)) \quad \forall \mu \in \mathcal{D}.$$

This can be achieved by the choice

$$\gamma(\mu) \leq \frac{\lambda_{\min}(D(\mu))}{\lambda_{\max}(M(\mu)) + \frac{1}{4} \lambda_{\max}(D(\mu))^2 \lambda_{\max}(K(\mu)^{-1})}. \quad (26)$$

To avoid computing the eigenvalues of $M(\mu)$, $D(\mu)$, $K(\mu)$ for every parameter μ , we further use the estimates

$$\lambda_{\max}(D(\mu)) \leq \sum_{k=1}^{n_D} \Theta_k^D(\mu) \lambda_{\max}(D_k) \quad \text{and} \quad \lambda_{\min}(D(\mu)) \geq \sum_{k=1}^{n_D} \Theta_k^D(\mu) \lambda_{\min}(D_k),$$

that are a consequence of Weyl's lemma. Similar estimates are also valid for $M(\mu)$ and $K(\mu)$.

Assume now that there exist $k_D \in \{1, \dots, n_D\}$ and $k_K \in \{1, \dots, n_K\}$ such that D_{k_D} and K_{k_K} are *positive definite* and that $\Theta_{k_D}^D(\mu) > 0$ and $\Theta_{k_K}^K(\mu) > 0$ for all $\mu \in \mathcal{D}$. Then we can further estimate $\gamma(\mu)$ as

$$\gamma(\mu) \leq \frac{\sum_{k=1}^{n_D} \Theta_k^D(\mu) \lambda_{\min}(D_k)}{\sum_{k=1}^{n_M} \Theta_k^M(\mu) \lambda_{\max}(M_k) + \frac{1}{4} (\sum_{k=1}^{n_D} \Theta_k^D(\mu) \lambda_{\max}(D_k))^2 / (\sum_{k=1}^{n_K} \Theta_k^K(\mu) \lambda_{\min}(K_k))},$$

while ensuring that we can choose $\gamma(\mu) > 0$ for all $\mu \in \mathcal{D}$.

The benefit of the above estimate is that the extremal eigenvalues of the matrices M_k , D_k , and K_k have to be computed only once in the beginning and otherwise, its evaluation is cheap, since $\Theta_k^M(\cdot)$, $\Theta_k^D(\cdot)$, $\Theta_k^K(\cdot)$ are scalar-valued functions.

6 Numerical results

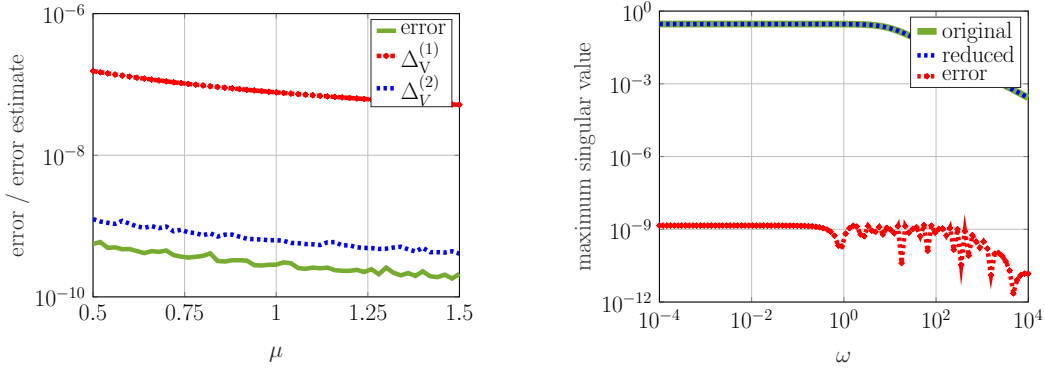
In this section, we present the numerical results for the two differential-algebraic systems and the optimization problem from Section 2. The computations have been done on a computer with 2 Intel Xeon Silver 4110 CPUs running at 2.1 GHz and equipped with 192 GB total main memory. The experiments use MATLAB[®]R2017b (9.3.0.713579) and examples and methods from M-M.E.S.S.-2.0. [29].

6.1 Problem 1: Stokes equation

We consider a system that arises, if we consider the creeping flow in capillaries or porous media. It has the following structure

$$\begin{aligned} \frac{d}{dt} v(\zeta, t) &= \mu \Delta v(\zeta, t) - \nabla p(\zeta, t) + f(\zeta, t), \\ 0 &= \operatorname{div}(v(\zeta, t)), \end{aligned} \quad (27)$$

with appropriate initial and boundary conditions. The position in the domain $\Omega \subset \mathbb{R}^d$ is described by $\zeta \in \Omega$ and $t \geq 0$ is the time. For simplicity, we use a classical solution concept and assume that the external force $f : \Omega \times [0, \infty) \rightarrow \mathbb{R}^d$ is continuous and that the velocities $v : \Omega \times [0, \infty) \rightarrow \mathbb{R}^d$ and pressures $p : \Omega \times [0, \infty) \rightarrow \mathbb{R}^d$ satisfy the necessary smoothness conditions. The parameter $\mu \in \mathcal{D}$



(a) Error and error estimates of the approximated Gramians. For evaluating the error, the Lyapunov equations are also solved with the residual tolerance 10^{-15} to obtain an accurate estimate of the exact solution.

(b) Sigma plots of the original, reduced, and error transfer functions for the parameter $\mu = 0.65$.

Figure 1: Results for the reduction of system (2).

represents the dynamic viscosity. We discretize system (27) by finite differences as shown in [24, 36] and add an output equation. Then we obtain a discretized system of the form (2), where only the matrix $A(\mu) = \mu A$ for $A \in \mathbb{R}^{n \times n}$ depends on the parameter μ .

Our example matrices E, A, G, B, C are created by the M-M.E.S.S. function `stokes_FVM` and we choose the dimensions $n = 3280, q = 1681$, and $m, p = 1$ and the parameter set $\mathcal{D} = [\frac{1}{2}, \frac{3}{2}]$. The reduced basis method from Section 5 produces projection space dimensions 24 and 24 for the projected Lyapunov equations. The errors and their estimates are presented in Figure 1a for the controllability Lyapunov equation. The error is evaluated by

$$\text{error} \approx \left\| P_{\Pi}^{\text{acc}}(\mu) - Z(\mu)Z(\mu)^H \right\|_{\text{F}},$$

where $P_{\Pi}^{\text{acc}}(\mu)$ denotes an accurate approximation of the exact solution $P_{\Pi}(\mu)$.

We can observe, that after the first iteration step, we obtain an error which is smaller than the tolerance 10^{-4} . Additionally, we see that the error estimator $\Delta_V^{(2)}$ provides a sharp bound of the actual error while $\Delta_V^{(1)}$ leads to more conservative error bounds.

Afterwards, we apply balanced truncation to obtain the reduced system of dimension $r = 10$. We demonstrate the quality of the reduced system by evaluating the error of the transfer functions. We show the sigma plot of the original and reduced transfer function as well as the error for the parameter $\mu = 0.65$ in Figure 1b. We observe that the error is smaller than 10^{-4} in the entire frequency band $[10^{-4}, 10^4]$.

6.2 Problem 2: Mechanical system

As an example for system (3) we consider a constrained mass-spring-damper system which is depicted in Figure 2 and taken from [24]. This system is composed of masses $m_i, i = 1, \dots, g$ which are connected with each other by springs with spring constants k_i and dampers with damping constants $d_i, i = 1, \dots, g - 1$. Moreover, each mass is connected to the ground by a spring with spring constant κ_i and a damper with damping constant δ_i for $i = 1, \dots, g$. An algebraic constraint is given by the condition that there is a rigid connection between the masses m_1 and m_g . Our input force acts on the

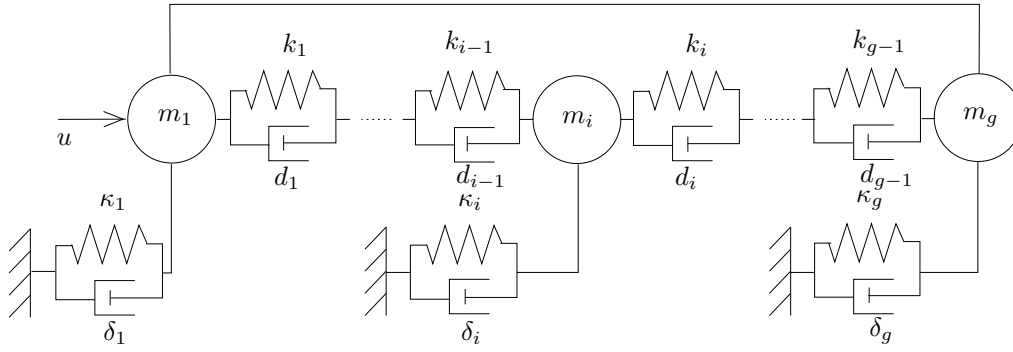


Figure 2: Example 2 – Constrained mass-damper-spring system

first mass m_1 only. As output we take the position of the first mass. This leads to a system of the form (3) with the matrices

$$M = \text{diag}(m_1, \dots, m_g),$$

$$D = \begin{bmatrix} d_1 + \delta_1 & -d_1 & & & \\ -d_1 & d_1 + d_2 + \delta_2 & -d_2 & & \\ & & \ddots & \ddots & \\ & -d_{g-2} & & d_{g-2} + d_{g-1} + \delta_{g-1} & -d_{g-1} \\ & & & -d_{g-1} & d_{g-1} + \delta_g \end{bmatrix},$$

$$K = \begin{bmatrix} k_1 + \kappa_1 & -k_1 & & & \\ -k_1 & k_1 + k_2 + \kappa_2 & -k_2 & & \\ & & \ddots & \ddots & \\ & -k_{g-2} & & k_{g-2} + k_{g-1} + \kappa_{g-1} & -k_{g-1} \\ & & & -k_{g-1} & k_{g-1} + \kappa_g \end{bmatrix},$$

$$G = [1, 0, \dots, 0, -1]^T, \quad B = [1, 0, \dots, 0]^T, \quad C = [1, 0, \dots, 0].$$

The dimension n coincides with the dimension g in Figure 2. The matrices are generated by the M-M.E.S.S. function `msd_ind3`. We choose $g = 6000$ as well as

$$m_1 = \dots = m_g = 1, \quad k_1 = \dots = k_{g-1} = 1.5, \quad d_1 = \dots = d_{g-1} = 0.7, \\ \kappa_1 = \dots = \kappa_g = 2, \quad \delta_1 = \dots = \delta_g = 0.9.$$

Assume that we want to reduce the model for various damper configurations for δ_i . So assume that

$$D(\mu) := D + \mu I_g,$$

where $\mu \in \mathcal{D} = [0, 1]$.

We evaluate the error for the reduced basis method, where we consider especially the controllability Gramian. After the first iteration, the error for the Gramian is already below 10^{-4} with a projection space of dimension 22 in the entire parameter domain \mathcal{D} . The results are presented in Figure 3a. Again, we see that the error estimator $\Delta_V^{(2)}$ approximates the error \mathcal{E}_Π well, especially compared to the estimation by $\Delta_V^{(1)}$. For the observability Gramian, we obtain a projection space dimension of 22 and so we only have to solve small Lyapunov equations of dimensions 22 and 22 in the online phase, respectively.

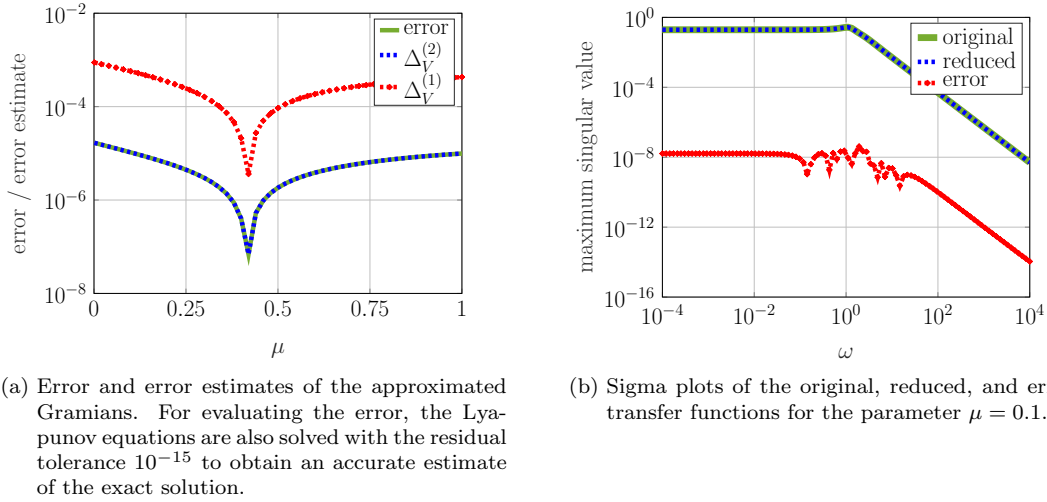


Figure 3: Results for the reduction of system (3).

The reduction results for the parameter $\mu = 0.1$ are shown in Figure 3b, where we depict the original, reduced, and error transfer function. The reduced system has the dimension $r = 10$. We observe that the \mathcal{H}_∞ -norm error of the reduced transfer function is smaller than 10^{-4} . Therefore, the procedure of this article works well for this example, too.

6.3 Problem 3: Optimization problem

Now we revisit the optimization problem detailed in Subsection 2.3. As in the second example the matrices are generated by the M-M.E.S.S. function `msd.ind3`. As internal damping model we choose the Rayleigh damping

$$D_{\text{int}} = \beta M + \zeta K,$$

with $\beta = 0.02$ and $\zeta = 0.01$ in our example.

Recall that our aim is to choose $H(\mu)$ such that the external input $Fu(\cdot)$ affects the output $y(\cdot)$ as little as possible. Therefore, we want to minimize the system response which is given by

$$J(\mu) := \int_{-\infty}^{\infty} \text{tr}(\mathcal{G}(\mu, i\omega)^H \mathcal{G}(\mu, i\omega)) d\omega,$$

where the function $\mathcal{G}(\mu, s)$ is the parameter-dependent transfer function of the system (5). The function $J(\cdot)$ can be expressed as (6) and (7).

The MATLAB function `fminbnd` is used to find the minimizer of $J(\cdot)$ in the parameter set $\mathcal{D} = [0, 100]$. As described in (6) and (7), a Lyapunov equation has to be solved in every iteration step. In [4] this problem is solved by the dominant pole algorithm and in [39] by rational interpolation. In this article, on the other hand, we apply the reduced basis method, presented in Section 5, to accelerate the minimization process. In the reduced basis method we use the tolerance 10^{-3} . Then we can solve the corresponding reduced Lyapunov equations in each step of the minimization process.

The resulting minimizers and times are presented in Table 1. Both optimization methods lead to the same result. Even though, the reduction in the offline phase takes 28.9 seconds, the online phase

Table 1: Results and times for the optimization

system	dimension	optimization time	minimizer
original system	12000	118.0 sec	100
reduced system	131	14.2 sec	100

provides an acceleration and the overall runtime is still smaller compared to the approach without reduction.

7 Conclusion

This paper has addressed the model reduction of parameter-dependent differential-algebraic systems by balanced truncation. To apply the balancing procedure we have utilized certain projections to eliminate the algebraic constraints. This has enabled us to compute the necessary Gramians efficiently by solving projected Lyapunov equations.

To handle the parameter-dependency, we have applied the reduced basis method, which is split into the *offline phase* and the *online phase*. In the *offline phase*, we have computed the basis of the subspace which approximates the solution space of the Lyapunov equation. To evaluate the quality of this subspace we have derived two error estimators. Afterwards, in the *online phase*, we have solved a reduced Lyapunov equation to obtain an approximation of the Gramian for a parameter value of interest efficiently. Therefore, a balanced truncation for this parameter value can also be carried out very fast.

This method has been illustrated by numerical examples of index two and three. In particular, we were able to reduce the associated Lyapunov equations to very small dimensions. We have evaluated our error estimators, where the second one almost estimates the error exactly. Further, we have considered an optimization problem where applying the reduced basis method has accelerated the minimization of the system response significantly.

Code Availability

The code and data that has been used to generate the numerical results of this work are freely available under the DOI 10.5281/zenodo.5145752 under the 3-clause BSD license and is authored by Jennifer Przybilla and Matthias Voigt.

References

- [1] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*, volume 6 of *Adv. Des. Control*. SIAM Publications, Philadelphia, PA, 2005. doi:10.1137/1.9780898718713.
- [2] R. H. Bartels and G. W. Stewart. Solution of the matrix equation $AX + XB = C$: Algorithm 432. *Comm. ACM*, 15:820–826, 1972. doi:10.1145/361573.361582.
- [3] U. Baur, P. Benner, and L. Feng. Model order reduction for linear and nonlinear systems: A system-theoretic perspective. *Arch. Comput. Methods Eng.*, 21(4):331–358, 2014. doi:10.1007/s11831-014-9111-2.

-
- [4] P. Benner, P. Kürschner, Z. Tomljanović, and N. Truhar. Semi-active damping optimization of vibrational systems using the parametric dominant pole algorithm. *Z. Angew. Math. Mech.*, 96(5):604–619, 2016. doi:10.1002/zamm.201400158.
- [5] P. Benner, M. Ohlberger, A. Cohen, and K. Willcox, editors. *Model Reduction and Approximation: Theory and Algorithms*. Computational Science & Engineering, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2017. doi:10.1137/1.9781611974829.
- [6] P. Benner and J. Saak. Numerical solution of large and sparse continuous time algebraic matrix Riccati and Lyapunov equations: a state of the art survey. *GAMM Mitteilungen*, 36(1):32–52, August 2013. doi:10.1002/gamm.201310003.
- [7] P. Benner, J. Saak, and M. M. Uddin. Balancing based model reduction for structured index-2 unstable descriptor systems with application to flow control. *Numer. Algebra Control Optim.*, 6(1):1–20, March 2016. doi:10.3934/naco.2016.6.1.
- [8] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1995.
- [9] S. L. Campbell. *Singular Systems of Differential Equations*, volume 40 of *Research Notes in Mathematics*. Pitman Advanced Publishing Program, London, 1980.
- [10] V. Druskin and V. Simoncini. Adaptive rational Krylov subspaces for large-scale dynamical systems. *Systems Control Lett.*, 60(8):546–560, 2011. doi:10.1016/j.sysconle.2011.04.013.
- [11] R. Eid, R. Castañé-Selga, H. Panzer, T. Wolf, and B. Lohmann. Stability-preserving parametric model reduction by matrix interpolation. *Mathematical and Computer Modelling of Dynamical Systems*, 17(4):319–335, 2011. doi:10.1080/13873954.2011.547671.
- [12] G. Flagg, C. Beattie, and S. Gugercin. Convergence of the iterative rational Krylov algorithm. *Systems Control Lett.*, 61(6):688–691, 2012. doi:10.1016/j.sysconle.2012.03.005.
- [13] M. Geuss, H. Panzer, and B. Lohmann. On parametric model order reduction by matrix interpolation. *Proceedings of the 12th European Control Conference*, pages 3433–3438, 2013. doi:10.23919/ECC.2013.6669829.
- [14] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L^∞ norms. *Internat. J. Control*, 39:1115–1193, 1984.
- [15] S. Gugercin, A. C. Antoulas, and C. Beattie. \mathcal{H}_2 model reduction for large-scale linear dynamical systems. *SIAM J. Matrix Anal. Appl.*, 30(2):609–638, 2008. doi:10.1137/060666123.
- [16] S. Gugercin, T. Stykel, and S. Wyatt. Model reduction of descriptor systems by interpolatory projection methods. *SIAM J. Sci. Comput.*, 35(5):B1010–B1033, 2013. doi:10.1137/130906635.
- [17] S. J. Hammarling. Numerical solution of the stable, non-negative definite Lyapunov equation. *IMA J. Numer. Anal.*, 2:303–323, 1982.
- [18] M. Heinkenschloss, D. C. Sorensen, and K. Sun. Balanced truncation model reduction for a class of descriptor systems with application to the Oseen equations. *SIAM J. Sci. Comput.*, 30(2):1038–1063, 2008. doi:10.1137/070681910.
- [19] J. S. Hesthaven, G. Rozza, and B. Stamm. *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. SpringerBriefs in Mathematics. Springer, Cham, 2016. doi:10.1007/978-3-319-22470-1.
- [20] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.

-
- [21] Patrick Kürschner. *Efficient Low-Rank Solution of Large-Scale Matrix Equations*. Dissertation, Otto-von-Guericke-Universität, Magdeburg, Germany, April 2016. URL: <http://hdl.handle.net/11858/00-001M-0000-0029-CE18-2>.
- [22] P. Lancaster and M. Tismenetsky. *The Theory of Matrices*. Academic Press, Orlando, 2nd edition, 1985.
- [23] A. J. Mayo and A. C. Antoulas. A framework for the solution of the generalized realization problem. *Linear Algebra Appl.*, 425(2–3):634–662, 2007. Special Issue in honor of P. A. Fuhrmann, Edited by A. C. Antoulas, U. Helmke, J. Rosenthal, V. Vinnikov, and E. Zerz. doi:10.1016/j.laa.2007.03.008.
- [24] V. Mehrmann and T. Stykel. Balanced truncation model reduction for large-scale systems in descriptor form. In P. Benner, V. Mehrmann, and D. C. Sorensen, editors, *Dimension Reduction of Large-Scale Systems*, volume 45 of *Lect. Notes Comput. Sci. Eng.*, pages 83–115. Springer-Verlag, Berlin/Heidelberg, Germany, 2005. doi:10.1007/3-540-27909-1_3.
- [25] B. C. Moore. Principal component analysis in linear systems: controllability, observability, and model reduction. *IEEE Trans. Autom. Control*, AC-26(1):17–32, 1981. doi:10.1109/TAC.1981.1102568.
- [26] H. Panzer, T. Wolf, and B. Lohmann. A strictly dissipative state space representation of second order systems. *at-Automatisierungstechnik*, 60(7):392–397, 2012. doi:10.1524/auto.2012.1015.
- [27] T. Penzl. A cyclic low rank Smith method for large sparse Lyapunov equations. *SIAM J. Sci. Comput.*, 21(4):1401–1418, 2000. doi:10.1137/S1064827598347666.
- [28] A. Quarteroni, A. Manzoni, and F. Negri. *Reduced Basis Methods for Partial Differential Equations*, volume 92 of *La Matematica per il 3+2*. Springer International Publishing, 2016. ISBN: 978-3-319-15430-5.
- [29] J. Saak, M. Köhler, and P. Benner. M-M.E.S.S.-2.0 – the Matrix Equations Sparse Solvers library, August 2019. see also: <https://www.mpi-magdeburg.mpg.de/projects/mess>. doi:10.5281/zenodo.3368844.
- [30] J. Saak and M. Voigt. Model reduction of constrained mechanical systems in M-M.E.S.S. *IFAC-PapersOnLine 9th Vienna International Conference on Mathematical Modelling MATHMOD 2018, Vienna, Austria, 21–23 February 2018*, 51(2):661–666, 2018. doi:10.1016/j.ifacol.2018.03.112.
- [31] A. Schmidt, D. Wittwar, and B. Haasdonk. Rigorous and effective a-posteriori error bounds for nonlinear problems – Application to RB methods. *Advances in Computational Mathematics*, 46:32, 2020. doi:10.1007/s10444-020-09741-x.
- [32] V. Simoncini and V. Druskin. Convergence analysis of projection methods for the numerical solution of large Lyapunov equations. *SIAM J. Numer. Anal.*, 47(2):828–843, 2009. doi:10.1137/070699378.
- [33] N. T. Son and T. Stykel. Solving parameter-dependent Lyapunov equations using the reduced basis method with application to parametric model order reduction. *SIAM J. Matrix Anal. Appl.*, 38(2):478–504, 2017. doi:10.1137/15M1027097.
- [34] T. Stykel. *Analysis and Numerical Solution of Generalized Lyapunov Equations*. Dissertation, TU Berlin, 2002. URL: http://webdoc.sub.gwdg.de/ebook/e/2003/tu-berlin/stykel_tatjana.pdf.
- [35] T. Stykel. Gramian-based model reduction for descriptor systems. *Math. Control Signals Systems*, 16(4):297–319, 2004. doi:10.1007/s00498-004-0141-4.

-
- [36] T. Stykel. Balanced truncation model reduction for semidiscretized Stokes equation. *Linear Algebra Appl.*, 415(2–3):262–289, 2006. doi:10.1016/j.laa.2004.01.015.
- [37] T. Stykel. Low-rank iterative methods for projected generalized Lyapunov equations. *Electron. Trans. Numer. Anal.*, 30:187–202, 2008. URL: <http://etna.mcs.kent.edu/vol.30.2008/pp187-202.dir/pp187-202.pdf>.
- [38] M. S. Tombs and I. Postlethwaite. Truncated balanced realization of a stable non-minimal state-space system. *Internat. J. Control*, 46(4):1319–1330, 1987.
- [39] Z. Tomljanović, C. Beattie, and S. Gugercin. Damping optimization of parameter dependent mechanical systems by rational interpolation. *Advances in Computational Mathematics*, 44(6):1797–1820, 2018. doi:10.1007/s10444-018-9605-9.
- [40] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, NJ, 1996.