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The mass shift and the anomalous magnetic moment of the electron in an intense plane wave field

Master thesis

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The mass shift and the anomalous magnetic moment of the electron in an intense plane wave field:

The value of the anomalous magnetic moment of the electron is the most accurately verified prediction of Quantum Electrodynamics (QED). Upcoming QED experiments on the interaction of electrons with intense laser fields open up the way of checking whether this high degree of agreement for the value of the electron anomalous magnetic moment persists in intense background fields. The possibility of experimentally verifying the expression for the anomalous magnetic moment of the electron in intense laser fields calls for computing radiative corrections beyond the leading-order result, that already includes the background field exactly. While in vacuum QED the anomalous magnetic moment is extracted from the vertex diagram for which the external photon provides the magnetic field interacting with the electron, in the strong field case the mass operator is used, where the magnetic field of the plane wave is exploited instead. Hence, in the thesis, the renormalized momentum space mass operator for an off-shell electron in the presence of an arbitrary plane-wave background is computed in light-cone coordinates, which have the advantage of making transparent the conserved quantities. Sandwiching between on-shell electron states, a new representation, more compact than the one known from the literature [VS75], is obtained. Solving the Schwinger-Dyson equation for the electron, in which the determined mass operator is inserted, the electron mass shift in an arbitrary plane wave is obtained. The expression for the electron mass shift generalizes the already known expressions from the literature [VS71; Rit70] for the constant crossed field case. The spin-dependent part of the electron mass shift is related to the anomalous magnetic moment of the electron in the plane wave. In the locally constant field approximation, the anomalous magnetic moment of the electron is extracted and reduces to Schwinger's famous result when the background is removed. However, due to the non-local dependence of the electron mass shift on the field, it is not generally possible to define a local expression of the electron anomalous magnetic moment in an arbitrary plane.

Contents

Units, notation and considerations

Units

Except in the introduction, Heaviside and natural units are used, $\epsilon_0 = \hbar = c = 1$.

Spacetime coordinates

The thesis assumes we are working on a Minkowski flat spacetime, where

- Einstein summation convention is always implicitly assumed for all types of repeated indices which appear only on one side of an equation;
- in Minkowski (canonical) coordinates
  - the four-dimensional space-time indices are denoted by lowercase Greek letters $(\mu, \nu, \cdots)$, taking the values $0, 1, 2, 3$ and three-dimensional space indices by lowercase Latin letters $(i, j, \cdots)$, taking the values $1, 2, 3$
  - the three-dimensional vectors are denoted by bold symbols, i.e.

    $$\boldsymbol{a} = (a^1, a^2, a^3)$$

  - the metric $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+, -, -, -)$ is used to raise and lower indices
  - the indefinite bilinear form evaluated on 4-vectors can be expressed as

    $$(xy) \equiv x \cdot y = \eta_{\mu\nu} x^\mu y^\nu = (x^0)(y^0) - (\boldsymbol{x}^\perp \cdot \boldsymbol{y}^\perp) \implies x^2 = (x^0)^2 - (\boldsymbol{x}^\perp)^2$$

  - the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ is conveniently chosen with $\epsilon^{0123} = 1$
  - the Feynman slashed notation is used in the introduction and the first chapter, with the hat denoting operators. Starting from Chapter 3, the hat is used to denote

    $$\hat{A} \equiv \gamma^\mu A_\mu,$$

    while operators no longer have a hat (the operator character is stated explicitly in the text instead).
- in light-cone coordinates (LCC)
  - the four-dimensional space-time indices are denoted by $(\mu' = +, -, \perp = \{1, 2\})$
  - the vector components are denoted by $x^{\mu'}_{\mathrm{LCC}} = \{x^+, x^-, \boldsymbol{x}^\perp = (x^{\perp,1}, x^{\perp,2})\}$ with the subscript LCC ommited when clear

- the basis of the four-dimensional four-vector space is taken as

$$\left\{n^\mu = (1, \boldsymbol{n}), \tilde{n}^\mu = (1, -\boldsymbol{n})/2, a_1^\mu = (0, \boldsymbol{a}_1^\perp), a_2^\mu = (0, \boldsymbol{a}_2^\perp)\right\},$$
$$\boldsymbol{n}^2 = 1, \quad \boldsymbol{a}_i \boldsymbol{a}_j = \delta_{ij}, \quad \boldsymbol{n} = \boldsymbol{a}_1 \times \boldsymbol{a}_2.$$

  where $n^2 = \tilde{n}^2 = 0$ are timelike vectors, chosen such that $(n\tilde{n}) = 1$.
- the basis vectors a_j are transversal $(na_j) = -\boldsymbol{n} \cdot \boldsymbol{a}_j = 0$ and similarly, $(\tilde{n}a_j) = 0$ $(j = 1, 2)$, to the spatial propagation direction $\boldsymbol{n}$
- the transversal basis vectors a_1^μ and a_2^μ are orthonormal with respect to the 3-vector inner product

$$(a_j a_{j'}) = -\boldsymbol{a}_j \cdot \boldsymbol{a}_{j'} = -\delta_{jj'} \qquad (j, j' = 1, 2) \tag{0.1}$$

- the completeness relation for the basis is

$$\eta^{\mu\nu} = n^\mu \tilde{n}^\nu + \tilde{n}^\mu n^\nu - a_1^\mu a_1^\nu - a_2^\mu a_2^\nu \tag{0.2}$$

- the vector components are related to Minkowski vector components by

$$x^- = (\tilde{n}x) = \frac{1}{2}\left(x^0 + x^\parallel\right), \quad x^+ = (nx) = \left(x^0 - x^\parallel\right),$$
$$\boldsymbol{x}^\perp\,|_{\mathrm{LCC}} = \boldsymbol{x}^\perp\big|_{\mathrm{canonical}} = -\left((xa_1), (xa_2)\right) = (\boldsymbol{x} \cdot \boldsymbol{a}_1, \boldsymbol{x} \cdot \boldsymbol{a}_2)$$

  where the transversal vectors are denoted by $x^\perp = (x^1, x^2)$ and where $x^\parallel = \boldsymbol{n} \cdot \boldsymbol{x}$ is the 3-vector projection along the propagation direction, with $\boldsymbol{n} = \boldsymbol{k}/\omega$ the unit 3-vector in the direction of propagation, and $x^\perp = \boldsymbol{x}\boldsymbol{a}_1^\perp + \boldsymbol{x}\boldsymbol{a}_2^\perp$ with $x^\perp \cdot \boldsymbol{n} = 0$ the perpendicular projection, where $\boldsymbol{a}_1^\perp$ and $\boldsymbol{a}_2^\perp$ are unit vectors that span the perpendicular plane. It follows that $\boldsymbol{k}^\perp = 0$ and $\boldsymbol{k}^\parallel = \omega\boldsymbol{n}$.
- the metric is off-diagonal in the $(+, -)$ subspace

$$\eta_{+-} = \eta_{-+} = 1, \quad \eta_{++} = \eta_{--} = 0$$

  and also does not couple the $(+, -)$ subspace to the $\perp = (\mathrm{I}, \mathrm{II})$ subspace

$$\eta_{+\perp} = \eta_{-\perp} = 0, \quad \perp = \mathrm{I}, \mathrm{II}$$

- the indefinite bilinear product can be expressed as

$$a \cdot b = a^+ b^- + a^- b^+ - \boldsymbol{a}^\perp \boldsymbol{b}^\perp \implies a^2 = 2a^+ a^- - \left(\boldsymbol{a}^\perp\right)^2$$

- the Levi-Civita symbol $\epsilon^{+-12} = -1$

Dirac spinor conventions

The Dirac conjugated spinor is denoted by $\bar{\psi} \equiv \psi^\dagger \gamma^0$.
The Dirac conjugated matrix is denoted by $\bar{\Gamma} \equiv \gamma^0 \Gamma^\dagger \gamma^0$.
Dirac spinor indices are denoted by lowercase Latin letters $(a, b, \cdots)$, taking the values $1, 2, 3, 4$ and are suppressed $(\bar{u}_{p'} \gamma^\mu u_p = (u_{p'})^*_a \gamma^0_{ab} \gamma^\mu_{bc} (u_p)_c)$.
The Lorentz transformations generator in the spinor representation is $\sigma^{\mu\nu} \equiv [\gamma^\mu, \gamma^\nu]/2$

Field conventions

- The four-vector potential of the plane wave is $A^\mu(\phi)$.
- The field strength tensor for a plane wave is $F^{\mu\nu}(\phi) = \partial^\mu A^\nu(\phi) - \partial^\nu A^\mu(\phi) = n^\mu A'^\nu(\phi) - n^\nu A'^\mu(\phi)$, while the dual field strength tensor is $\tilde{F}^{\mu\nu}(\phi) = \frac{\varepsilon^{\mu\nu\lambda\rho}}{2} F_{\lambda\rho}(\phi)$.
- The charged multiplied quantities $\mathcal{A}^\mu(\phi) = eA^\mu(\phi)$, $\mathcal{F}^{\mu\nu}(\phi) = eF^{\mu\nu}(\phi)$ and $\tilde{\mathcal{F}}^{\mu\nu}(\phi) = e\tilde{F}^{\mu\nu}(\phi)$ come in handy.
- The Lorentz gauge, fully fixed by $A^0 = 0$ is employed throughout the thesis.

Momentum operators

The eigenstate basis of the position operator X^μ (and the momentum $P^\mu = i\partial^\mu$, respectively) is $|x\rangle$ ($|p\rangle$) such that

$$\begin{aligned} X^\mu |x\rangle &= x^\mu |x\rangle, \\ P^\mu |p\rangle &= p^\mu |p\rangle, \end{aligned} \tag{0.3}$$

normalized according to $\langle x \mid y\rangle = \delta^{(4)}(x-y) \left[\langle p \mid q\rangle = (2\pi)^4 \delta^{(4)}(p-q)\right]$.

In light cone-coordinates, the momentum operators read

$$\boldsymbol{P}_\perp = -i(\boldsymbol{a}_1 \cdot \boldsymbol{\nabla}, \boldsymbol{a}_2 \cdot \boldsymbol{\nabla}),$$
$$P_\tau = -i\partial_\tau = -i(nP) = -i(\partial_t + \partial_{x^\parallel}), P_\phi = -i\partial_\phi = -(\tilde{n}P) = -\left(i\partial_t - i\partial_{x^\parallel}\right)/2,$$

and satisfy $P_\phi|p\rangle = -p_+|p\rangle, P_\tau|p\rangle = -p_-|p\rangle$ and $\boldsymbol{P}_\perp|p\rangle = \boldsymbol{p}_\perp|p\rangle$.

The following commutation relations, consistent with $[X^\mu, P^\nu] = -i\eta^{\mu\nu}$, hold

$$[\Phi, P_\phi] = [T, P_\tau] = i, \quad [X_{\perp,j}, P_{\perp,k}] = i\delta_{jk} \qquad (j, k = 1, 2), \tag{0.4}$$

where T is the operator corresponding to the light-cone time τ.

Integral measures

When the integral is in configuration space or in momentum space, the following shorthand for the integral measure is used to compactify the formulas

$$\int_x \equiv \int d^4x, \quad \int_p \equiv \int \frac{d^4p}{(2\pi)^4}. \tag{0.5}$$

When the integral is over some other variable, with integration limits, it is stated explicitly when a similar shorthand is used.

1 Introduction

The development of the quantum field theoretic framework, initiated by Dirac's 1927 seminal paper "The Quantum Theory of the Emission and Absorption of Radiation" and reaching a firm ground with the 1932 review article by Fermi [Fer32] that taught generations of physicists [Sch02], has proven to be a stepping stone in the formulation of fundamental modern-day theories.

Of particular physical importance, the Standard Model (SM) provides an incredibly accurate description of the microscopic phenomena, albeit neglecting gravitational effects, of justifiably less importance at energies currently accesible in colliders. The physical consistency of the model, without which the model wouldn't be able to account for the particle masses, was confirmed by the discovery of the Higgs boson at the LHC [ATL12; CMS12]. However, in what concerns its mathematical consistency, the SM still lacks a solid mathematical foundation, which relies on solving the Yang-Mills problem [JW00].

Despite the long line of success in terms of experimental agreement, there are still issues, other than including gravity, that aren't accounted for by the Standard Model, like explaining neutrino mass or, equivalently, their oscillations, or finding an explanation for the finely-tuned cancellation occurring in the so-called electroweak hierarchy problem. All this points to the necessity of an extension or a high-energy (UV) completion, which renders the SM a low-energy ("effective") theory embedded in a more complete "grand unified theory" at scales of $\sim 10^{16}$ GeV, where the strong and electroweak force become comparable, in turn embedded into the hypothetical "theory of everything" that should comprise a description of gravity up to the Planck scale ($\sim 10^{19}$ GeV).

1.1 Precision tests of the SM and QED

Quantum Electrodynamics, the unbroken part of the spontaneously broken electroweak theory developed by Glashow, Salam and Weinberg in the 1970s, embedded in the Standard Model, provides at the perturbative level predictions experimentally confirmed within an incredibly high degree of precision.

Particularly, there is a 10 significant digits agreement between the theoretical QED prediction and the experimentally determined value for the electromagnetic fine structure constant

$$\begin{aligned} \alpha_{\text{exp}}^{-1} &= 137.035999037(91) \qquad \text{[Bou+11]}, \\ \alpha_{\text{theo}}^{-1} &= 137.035999084(51) \qquad \text{[HFG08]}. \end{aligned} \tag{1.1}$$

The most stringent experimental tests on QED come from the measurement of the anomalous magnetic moment of either a free [HFG08; HHG11] or a bound [Stu+11] electron. For the free electron, there is a remarkable agreement of 12 significant digits for the gyromagnetic factor of the electron

$$\begin{aligned} g_{\text{exp}}/2 &= 1.00115965218073(28) \qquad \text{[HFG08]}, \\ g_{\text{theo}}/2 &= 1.00115965218178(77) \qquad \text{[Aoy+12]}. \end{aligned} \tag{1.2}$$

Including the hadronic and weak contributions, the value of the magnetic moment $\mu_\pm$ is also the most precise prediction of the SM, expressed as [Gab+19; HFG08]

$$-\frac{\mu_-}{\mu_B} = \frac{\mu_+}{\mu_B} = 1 + C_2\left(\frac{\alpha}{\pi}\right) + C_4\left(\frac{\alpha}{\pi}\right)^2 + C_6\left(\frac{\alpha}{\pi}\right)^3 + C_8\left(\frac{\alpha}{\pi}\right)^4 + \ldots + a_{\text{hadronic}} + a_{\text{weak}}, \tag{1.3}$$

where the positron magnetic moment μ_+ is the same with opposite sign as that of the electron μ_-, the 1 is the leading contribution of the tree level Dirac theory [Dir28], and the C_n terms represent the QED series expansion in powers of α/π.

The 1-loop coefficient $C_2 = 0.5$ is Schwinger's famouos leading order (LO) radiative correction from 1948 [Sch48], representing the first successful application of covariant renormalization theory.

The coefficients up to C_8 have also been calculated analytically [Lap17]. The two remaining terms a_{hadronic} and a_{weak} are the hadronic and weak interaction contributions, and have been estimated [Jeg19; Gab+19] to be subdominant to the QED corrections up to 5-loops for the electron.

It is useful to define

$$a_\ell \equiv \frac{g_\ell - 2}{2}, \qquad (\ell = e, \mu, \tau), \tag{1.4}$$

in which case, Schwinger's QED prediction[1] becomes

$$a_\ell^{\text{QED(LO)}} = \frac{\alpha}{2\pi}. \tag{1.5}$$

Motivated by the high degree of agreement between theory and experiment in the vacuum case, where QED carries the dominant contribution [Gab+19], the present thesis calculates the electron mass shift and magnetic moment correction in the strong field of a plane wave, at one-loop. But first, we need to see what considerations are implied by the strong field approximation.

[1] calculated by Schwinger for the electron, which has the same QED value by QED lepton universality

1.2 QED in strong fields

An active area of research is the study of phenomena in the presence of strong background fields, where perturbative techniques become intractable, as the interaction with a large number of background photons needs to be considered. However, there are profiles for the background fields, like that of a plane wave (as in this thesis), or of a homogeneous field in space and time, or that of the Coulomb field [BGJ75] [BG14], where the interaction with the field can be solved for exactly. This holds when the background field is treated classically, as it is the case for quantum fields of high numbers of particles, i.e. approaching coherent states [Gla63], that are not significantly altered (depleted or enhanched) during an interaction. Mathematically, the vacuum expectation value (VEV) of the background photon field operator will now be nonzero, as opposed to vacuum QED where the VEV vanishes, while the expectation value of the background field operator between coherent states will satisfy classical Maxwell dynamics. The remaining quantized radiation field, describing the interaction, is treated perturbatively[2]. This semiperturbative approach that includes the contribution of the background field exactly is called the *Furry picture* [Fur51; Moo09].

SFQED typical scales

An important scale that arises in QED in the presence of a strong field is set by the electric and magnetic field strengths at which the QED vacuum state $|\Omega\rangle$ is significantly altered [Sau31; Sch51]. Since the classical picture can be applied for the background field, this can be qualitatively described by considering an electric field E_0 applied to the virtual particles that arise for short periods $\tau_C \sim \hbar/mc^2 \sim 10^{-21}$ s as quantum fluctuations of the vacuum. These are always present, even in the absence of the external field. However, the effect of the field is to polarize the vacuum by separating the particle-antiparticle pairs and when it gets strong enough such that the separation between an electron and the corresponding positron is over a distance of a Compton wavelength $\lambda_C = \hbar c/mc^2 = 3.86 \times 10^{-13}$ m, the virtual particles are produced on-shell (in the so-called Schwinger mechanism) and the vacuum becomes unstable. The value at which this happens is called the (Sauter-Schwinger) critical field $E_{cr} = m^2c^3/\hbar|e| \simeq 1.32 \times 10^{16}$ V/cm and, hence, corresponds to a field that accelerates the electron to an energy comparable to its rest energy mc^2 over a distance of the (reduced) Compton wavelength λ_C of the electron, the inherent QED length scale. One may also define the critical magnetic field strength $B_{\rm cr} = m^2c^2/|e|\hbar \simeq 4.41\times10^9$ T as the value at which the interaction energy of a Bohr magneton $\mu_B = |e|\hbar/2m$ with the external magnetic field B_0 is of the order of the electron rest energy mc^2. Notice that, informally, as magnetic fields cannot do work on the virtual particles, critical magnetic fields cannot induce spontaneous pair creation. Such critical electric field strengths can be seen ocurring naturally in electrons

[2] Unless we are in the strongly coupled regime of QED, where perturbation theory breaks down [Rit85]

bound to highly charged ions [Stu+11], while critical magnetic fields are found in magnetars [TZW15]. The associated critical field intensity $I_{\text{cr}} = 2.32 \times 10^{29}$ W/cm^2 is, however, at the present time, still far from being attained in the laboratory.

1.3 Current experimental status

With the advent of new laser technologies such as chirped pulse amplification [SM85] and parametric chirped pulse amplification [PSY86], new experiments dedicated to addressing the high-intensity frontier [Di +12] pave the way to test novel features of nonlinear QED effects in the *strong-field regime*, where the interaction with the laser field needs to be treated *non-perturbatively*. For example, facilities like ELI-NP [Ur+15] or Vulcan [Dan+04] undergo experiments that can deliver laser pulses in the petawatt (PW) range, reaching intensities of $10^{23} - 10^{24}$ W/cm^2 and 10^{21} W/cm^2 respectively. While terrestrial facilities are still a long way from reaching the critical intensity of $I_{\text{cr}} = c\epsilon_0 E_{\text{cr}}^2 \simeq 2.32 \times 10^{29}\ W/cm^2$ in the laboratory frame where the vacuum itself becomes unstable with respect to electron positron pair production, planned experiments at DESY [Abr+21] and SLAC [Meu+20] probe the critical electric field E_{cr} in the rest frame of the electron, where an enhanched value of the field $E^* = \gamma E$ is seen, boosted by the Lorentz γ factor with respect to the laboratory frame.

Although most of the current SFQED experiments rely on lasers, there are also mentionable laserless experiments, like the proposed E332 experiment at the FACET-II facility at SLAC [Cor+20], which employs a collimated and high-current electron beam directed onto a series of aluminum foils, spaced 10 μm apart. The incident electron beam undergoes a focusing effect in the near-field coherent transition radiation and exits together with a generated dense $\gamma-$ray pulse.

The extremely high-energy regime ($\chi \gg 1$) of SFQED is still entirely untested and poorly understood theoretically.

1.4 Outline of the thesis

Chapter 2 briefly introduces the necessary notions to understand the area of strong-field QED (and particularly the thesis) in the adopted conventions. Specifically, covariant quantization of the photon field is presented and the background field method is introduced by defining coherent states and explaining the Furry picture. Since light-cone coordinates make the computation of the mass operator transparent, they are also defined. Afterwards, the solution of the classical Lorentz equation for a electron in a plane wave is presented, after which the quantum (Volkov) solution is discussed. The chapter concludes with the Feynman rules for strong field QED.

Chapter 3 provides a detailed calculation of the mass operator for an off-shell electron in a plane wave background field, starting from the Feynman rules stated in the previous chapter. The calculation is general as it does not assume either state (incoming or outgoing) to be on-shell. However, in order to extract the mass shift of the electron in the plane-wave background, the external electron states are put on-shell.

In Chapter 4, the mass shift in a linearly-polarized plane-wave background, with an electron having the spin quantization axis align with the magnetic field of the plane wave in its rest frame, is calculated, after which, the anomalous magnetic moment correction for the electron is extracted, in the locally constant field approximation (LCFA).

In Chapter 5 the concluding remarks are outlined.

2

Strong field QED (SFQED)

This chapter briefly introduces the notation and the necessary notions in understanding the area of strong field QED, by reviewing the basic concepts of classical electrodynamics [Lan75; Jac98] and quantum electrodynamics [BPL12; Cla06; Ryd08], discussed in most textbooks.

The limits of validity of the strong field approximation, as mentioned in the introduction, are explained here by introducing coherent states for the background photon field, such that its expectation value over those states can be treated classically. The field on top of the background is quantized and treated perturbatively. Afterwards, Dirac equation in a classical plane-wave field is solved exactly, leading to the so-called Volkov solutions [BPL12]. Light-cone coordinates are introduced, as they make the calculations simpler and transparent, which can be seen from the fact that the electromagnetic 4-potential depends nontrivially only on the phase ϕ [BPL12; Bra19], the light cone time.

2.1 Quantization of the electromagnetic field

The free electromagnetic field is described by [Jac98; Lan75]

$$\mathcal{L}_{\text{em}} = -\frac{1}{4e^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}, \tag{2.1}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ is the field strength tensor and e is the electron charge. This leads to the equation of motion [Jac98]

$$\Box \mathcal{A}^\mu - \partial^\mu \partial_\nu \mathcal{A}^\nu = 0. \tag{2.2}$$

In the Lorentz gauge $\partial_\mu \mathcal{A}^\mu = 0$[1], such that the equation of motion

$$\Box \mathcal{A}^\mu = 0 \tag{2.3}$$

[1] Note that the Lorentz condition doesn't completely fix the gauge but still allows for a residual (longitudinal/scalar) gauge freedom.

The canonical momentum π^0 from

$$\pi^\mu \equiv \frac{\partial \mathcal{L}_{\text{em}}}{\partial \dot{A}_\mu} = -\frac{1}{2e^2}\frac{\partial \mathcal{F}_{\rho\sigma}}{\partial \dot{A}_\mu}\mathcal{F}^{\rho\sigma} = -\frac{1}{e^2}\mathcal{F}^{\rho\sigma} \underbrace{\frac{\partial}{\partial \dot{A}_\mu}\left(\partial_\rho \mathcal{A}_\sigma\right)}_{=e\delta_{\rho 0}\eta^\mu_\sigma} = -F^{0\mu} = F^{\mu 0}$$
$$= \begin{cases} 0, & \mu = 0, \\ -\dot{A}^i + \partial^i A^0 = E^i, & \mu = i = 1,2,3, \end{cases} \tag{2.4}$$

shows that A^0 is a non-propagating mode [Mic19]. Therefore, the canonical quantization procedure given by

$$\left[\hat{A}_\mu(t,\boldsymbol{x}), \hat{\pi}_\nu(t,\boldsymbol{x}')\right] = i\eta_{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{2.5}$$

cannot be imposed [Cla06] (the hats are used here to denote operators and not the Feynman slashed notation), which can be easily seen from the 00-component.

One way to solve this problem is to add a gauge fixing term as a Lagrange multiplier and obtain a modified gauge-fixed Lagrangian density [Ryd08]

$$\mathcal{L}^{(\xi)}_{\text{em}} = -\frac{1}{4e^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{\xi}{2e^2}\left(\partial_\mu \mathcal{A}^\mu\right)^2, \tag{2.6}$$

under the Lorentz gauge-fixing Lagrange constraint $\partial_\mu \mathcal{A}^\mu = 0$ and with the gauge-fixing parameter ξ[2].

$$\begin{aligned} \frac{e^2\partial \mathcal{L}^{(\xi)}_{\text{em}}}{\partial(\partial_\mu \mathcal{A}_\nu)} &= -\partial^\mu \mathcal{A}^\nu + \partial^\nu \mathcal{A}^\mu - \xi\eta^{\mu\nu}\partial_\rho \mathcal{A}^\rho, \\ \frac{e^2\partial \mathcal{L}^{(\xi)}_{\text{em}}}{\partial \mathcal{A}_\nu} &= 0. \end{aligned} \tag{2.7}$$

From the Euler-Lagrange equations, the following equations of motion follow

$$\Box \mathcal{A}^\mu - (1-\xi)\partial^\mu\partial_\nu \mathcal{A}^\nu = 0. \tag{2.8}$$

For example, in the Feynman gauge ($\xi = 1$) we get the canonical momenta[3]

$$\begin{aligned} \pi^\mu \equiv \frac{\partial \mathcal{L}^{(\xi=1)}_{\text{em}}}{\partial \dot{A}_\mu} &= -\partial^0 A^\mu + \partial^\mu A^0 - \eta^{\mu 0}\partial_\lambda A^\lambda \\ &= \begin{cases} -\dot{A}^0 + \dot{A}^0 - \eta^{00}\partial_\lambda A^\lambda = -\partial_\lambda A^\lambda, & \mu = 0, \\ -\dot{A}^i + \partial^i A^0, & \mu = i = 1,2,3. \end{cases} \end{aligned} \tag{2.9}$$

Although this might seem the same as what we had before, with $\pi^0 = 0$, we know that the gauge fixing condition $\partial_\lambda A^\lambda = 0$ holds at the classical level and need not

[2] The choice $\xi = 1$ is called the Feynman gauge.
[3] Hats denoting operators are not written when the equations hold at the classical level.

hold at the quantum level as well. In fact, it cannot hold at the quantum level, as an operator condition, i.e. $\partial_\mu \hat{A}^\mu = \hat{0}$, since

$$\left[\hat{A}^\mu(x), \hat{A}^\nu\left(x'\right)\right] = iD^{\mu\nu}\left(x - x'\right) \implies \left[\partial_\mu A^\mu(x), A^\nu\left(x'\right)\right] = i\partial_\mu D^{\mu\nu}\left(x - x'\right), \quad (2.10)$$

where $D^{\mu\nu}(x - x')$ is the photon propagator. Notice that the RHS of 2.10 is always different from 0. Furthermore, the equal-time commutation relation (ECTR), evaluated at 00

$$\left[\hat{A}^\mu(t, \boldsymbol{x}), \hat{\pi}^\nu(t, \boldsymbol{x}')\right] = i\eta^{\mu\nu}\delta^3(\boldsymbol{x}-\boldsymbol{x}') \implies \left[\hat{A}^0(t, \boldsymbol{x}), \hat{\pi}^0(t, \boldsymbol{x}')\right] = i\delta^3(\boldsymbol{x}-\boldsymbol{x}'), \quad (2.11)$$

cannot be imposed either, again, because if $\hat{\pi}^0 = -\partial_\mu \hat{A}^\mu = \hat{0}$, the LHS is zero whereas the RHS is non-zero. Hence, at the quantum (operator) level

$$\begin{aligned} &A^\mu \to \hat{A}^\mu; \\ &\pi^0 \to \hat{\pi}^0 = -\partial_\lambda \hat{A}^\lambda \neq \hat{0}. \end{aligned} \quad (2.12)$$

One might still try to get around this by integrating by parts $\mathcal{L}_{\text{em}}^{(\xi=1)}$ and dropping the boundary term to obtain the Fermi Lagrangian

$$\mathcal{L}_{\text{em}}^{\text{Fermi}} = -\frac{1}{2e^2}\left(\partial^\mu \mathcal{A}^\nu\right)\left(\partial_\mu \mathcal{A}_\nu\right), \quad (2.13)$$

in which case $\pi^\mu = -\dot{A}^\mu(x)$, such that in the 4-divergence of the commutation relation, i.e.

$$\left[\partial_\mu \hat{A}^\mu(t, \boldsymbol{x}), \dot{\hat{A}}^\nu\left(t, \boldsymbol{x}'\right)\right] + \left[\hat{A}^\mu(t, \boldsymbol{x}), \ddot{\hat{A}}^\nu\left(t, \boldsymbol{x}'\right)\right] = -i\partial^\nu \delta^3\left(\boldsymbol{x} - \boldsymbol{x}'\right), \quad (2.14)$$

the first term again clearly vanishes, while the second term, by using the equations of motions, is equal to $\left[\hat{A}^\mu(t, \boldsymbol{x}), \boldsymbol{\nabla}'^2 \hat{A}^\nu\left(t, \boldsymbol{x}'\right)\right]$, which vanishes.

However, we can relax the gauge-fixing condition [Gup50; Ble50] to hold only on the physical states

$$\left(\partial_\mu \hat{A}^\mu\right)|\psi_{phys}\rangle = |\varnothing\rangle, \quad (2.15)$$

since the expectation values between physical states lead to measurable quantities, and not the operators themselves. The Lorentz condition imposed on the states, together with the equal-time commutation relations (ECTR) [Ryd08]

$$\begin{aligned} &\left[\hat{A}^\mu(t, \boldsymbol{x}), \hat{\pi}^\nu(t, \boldsymbol{x}')\right] = i\eta^{\mu\nu}\delta^3(\boldsymbol{x} - \boldsymbol{x}'); \\ &\left[\hat{A}^\mu(t, \boldsymbol{x}), \hat{A}^\nu(t, \boldsymbol{x}')\right] = \hat{0}; \\ &\left[\hat{\pi}^\mu(t, \boldsymbol{x}), \hat{\pi}^\nu(t, \boldsymbol{x}')\right] = \hat{0}, \end{aligned} \quad (2.16)$$

are called *Gupta-Bleuer covariant quantization.* The mode expansion is similar to that of four real Klein-Gordon scalar fields [Meu15]

$$\hat{\mathcal{A}}^\mu(x) = \hat{\mathcal{A}}^\mu_+(x)+\hat{\mathcal{A}}^\mu_-(x), \quad \hat{\mathcal{A}}^\mu_+(x) = \sum_{r=0,1,2,3}\int \frac{d^3q}{(2\pi)^3}\frac{e}{\sqrt{2\omega_{\boldsymbol{q}}}}\hat{a}_r(\boldsymbol{q})e^{-iqx}\epsilon^\mu_r(\boldsymbol{q}), \tag{2.17}$$

where $\hat{\mathcal{A}}^\mu_-(x) = \hat{\mathcal{A}}^{\dagger\mu}_+(x)$, $\hat{\mathcal{A}}^\mu_+(x)$ are the negative (positive) frequency modes, $\omega_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2}$ is the energy of mode $\boldsymbol{q}$ and $\epsilon^\mu_{\boldsymbol{q},r}$ are the polarization vector [Sch08]:

$$\begin{aligned} \epsilon^\mu_1, \epsilon^\mu_2 &\text{ transversal polarization },\\ \epsilon^\mu_3 &\text{ longitudinal polarization },\\ \epsilon^\mu_0 &\text{ scalar or timelike polarization }. \end{aligned}$$

For each mode **q** they satisfy the orthogonality and completeness relations [Sch08; Meu15]

$$\begin{aligned} \epsilon_r(\boldsymbol{q})\epsilon_s(\boldsymbol{q}) \equiv \epsilon_{r\mu}(\boldsymbol{q})\epsilon^\mu_s(\boldsymbol{q}) = -\zeta_r\delta_{rs}, \quad r,s=0,1,2,3\\ \sum_r \zeta_r e^\mu_r(\boldsymbol{q})e^{*\nu}_r(\boldsymbol{q}) = -\eta^{\mu\nu}, \end{aligned} \tag{2.18}$$

where $\zeta_r = -\eta^{rr} = (-1,+1,+1,+1)$.

It is sometimes useful to align the polarization vectors [Sch08] as

$$\begin{aligned} \epsilon^\mu_0(\boldsymbol{q}) &= n^\mu \equiv (1,0,0,0)\\ \epsilon^\mu_r(\boldsymbol{q}) &= (0, \boldsymbol{\epsilon}_r(\boldsymbol{q})) \quad r=1,2,3, \end{aligned} \tag{2.19}$$

where $\boldsymbol{\epsilon}_{r=1,2}(\boldsymbol{q})$ are orthonormal vectors, also orthogonal to $\boldsymbol{q}$, while

$$\boldsymbol{\epsilon}_3(\boldsymbol{q}) = \boldsymbol{q}/|\boldsymbol{q}|, \tag{2.20}$$

such that

$$\begin{aligned} \mathbf{n}\cdot\boldsymbol{\epsilon}_r(\boldsymbol{q}) = 0, \qquad r=1,2\\ \boldsymbol{\epsilon}_r(\boldsymbol{q})\boldsymbol{\epsilon}_s(\boldsymbol{q}) = \delta_{rs}, \quad r,s=1,2,3. \end{aligned} \tag{2.21}$$

Alternatively, from the mode expansion we see that

$$\partial_\mu\hat{A}^\mu_+(x)\left|\psi_{phys}\right\rangle = \left|\varnothing\right\rangle \implies L(\boldsymbol{q})\left|\psi_{phys}\right\rangle \equiv (\hat{a}_3(\boldsymbol{q})-\hat{a}_0(\boldsymbol{q}))\left|\psi_{phys}\right\rangle = \left|\varnothing\right\rangle, \tag{2.22}$$

suffices for 2.15 to be valid. This also shows that the longitudinal and timelike components come "in pairs" and consequently, they can always be gauge-shifted away to remain only with purely transversal states (e.g. in 2.17, we can take $r = 1,2$). One-photon states are created using the operator

$$\left|\boldsymbol{q}r\right\rangle = \hat{a}^\dagger_r(\boldsymbol{q})\left|\Omega\right\rangle, \tag{2.23}$$

where the photon creation and annihilation operators satisfy the ECTR[4]

$$\begin{aligned}\left[\hat{a}_r(\boldsymbol{q}), \hat{a}_s^\dagger(\boldsymbol{q}')\right] &= \zeta_r \delta_{rs}(2\pi)^3\delta^{(3)}(\boldsymbol{q}-\boldsymbol{q}'),\\ \left[\hat{a}_r(\boldsymbol{q}), \hat{a}_s(\boldsymbol{q}')\right] &= \left[\hat{a}_r^\dagger(\boldsymbol{q}), \hat{a}_s^\dagger(\boldsymbol{q}')\right] = 0.\end{aligned} \tag{2.24}$$

We see that the vacuum state $|\Omega\rangle$ is physical, i.e.

$$L(\boldsymbol{q})\,|\Omega\rangle \equiv (\hat{a}_3(\boldsymbol{q}) - \hat{a}_0(\boldsymbol{q}))\,|\Omega\rangle = |\varnothing\rangle\,, \tag{2.25}$$

whereas from the relation

$$\langle \boldsymbol{q}r|\boldsymbol{q}r\rangle = \langle\Omega|\,\hat{a}_r(\boldsymbol{q})\hat{a}_r^\dagger(\boldsymbol{q})\,|\Omega\rangle = \zeta_r\,\langle\Omega|\Omega\rangle = \zeta_r, \tag{2.26}$$

we see that $r = 0$ states have a negative value ($\xi_0 = -1$) of the nondgenerate sesquilinear form $\langle.|.\rangle$, but they do not belong however to the physical Hilbert space of states

$$\hat{L}(\boldsymbol{q})\,|\boldsymbol{q}0\rangle = \hat{L}(\boldsymbol{q})\hat{a}_0^\dagger(\boldsymbol{q})\,|\Omega\rangle = (\hat{a}_0(\boldsymbol{q}) - \hat{a}_3(\boldsymbol{q}))\,\hat{a}_0^\dagger(\boldsymbol{q})\,|\Omega\rangle = -(2\pi)^3\delta^{(3)}(0)\,|\Omega\rangle \neq |\varnothing\rangle\,. \tag{2.27}$$

Although the two transversal polarizations are physical,

$$\hat{L}(\boldsymbol{q})\,|\boldsymbol{q}r\rangle = \hat{L}(\boldsymbol{q})\hat{a}_r^\dagger(\boldsymbol{q})\,|\Omega\rangle = |\varnothing\rangle\,, \qquad r = 1, 2, \tag{2.28}$$

the combination $|\boldsymbol{q}0 - \boldsymbol{q}2\rangle \equiv \left(\hat{a}_0^\dagger(\boldsymbol{q}) - \hat{a}_3^\dagger(\boldsymbol{q})\right)|\Omega\rangle$ also satisfies

$$\hat{L}(\boldsymbol{q})\,|\boldsymbol{q}0 - \boldsymbol{q}2\rangle = |\varnothing\rangle\,, \tag{2.29}$$

and furthermore, it has zero norm

$$\langle \boldsymbol{q}0 - \boldsymbol{q}2|\boldsymbol{q}0 - \boldsymbol{q}2\rangle = 0. \tag{2.30}$$

This might seem as a third polarization, but it is actually the residual gauge freedom of the Lorentz gauge, and hence, unphysical. Therefore the truly physical space state mods out the 0-norm states, i.e. is the quotient space $\mathcal{H}_{\text{phys}}/\mathcal{H}_0$ [Sch19].

In summary, we have quantized the electromagnetic field $\mathcal{A}^\mu \to \hat{\mathcal{A}}^\mu$ by contructing the physical Hilbert space for a photon, i.e. for the 1-particle states, which can be generalized to multiparticle states to obtain the photon Fock space.

2.2 Background field method and coherent states

Following [Gla63; MKP13; Meu15; Sei12], the photon coherent state is defined as

$$|A\rangle_{\text{coherent}} = \hat{D}\,|\Omega\rangle\,, \tag{2.31}$$

[4] compatible with the field ECTR 2.16

where $|\Omega\rangle$ is the photon Fock space vacuum ($\langle\Omega|\Omega\rangle = 1$ and $\langle A|A\rangle = 1$) and $\hat{D}$ is the unitary displacement operator, defined by [Sei12]

$$\hat{D} = \exp\left[\sum_{r=0,1,2,3} \int \frac{d^3q}{(2\pi)^3} \left(a_{\boldsymbol{q},\sigma}\hat{a}_r^\dagger(\boldsymbol{q}) - a_r^*(\boldsymbol{q})\hat{a}_r(\boldsymbol{q})\right)\right]. \tag{2.32}$$

The state $|A\rangle$ produced by the displacement operator is the eigenstate of the quantum operator [Sei12], i.e.

$$\hat{\mathcal{A}}_+^\mu(x)\,|A\rangle = \mathcal{A}_+^\mu(x)\,|A\rangle\,, \tag{2.33}$$

such that the expectation value

$$\mathcal{A}^\mu \equiv \langle A|\,\hat{\mathcal{A}}^\mu\,|A\rangle\,, \tag{2.34}$$

satisfies the classical equations of motion, which justifies calling coherent states "classical-like" states.

Equivalently, a single-mode coherent state $|A;\boldsymbol{q},r\rangle$ with momentum $\boldsymbol{q}$ and polarization r is defined as the eigenstate of the annihilation operator $\hat{a}_r(\boldsymbol{q})$ [Sei12], i.e.

$$\hat{a}_r(\boldsymbol{q})\,|A;\boldsymbol{q}r\rangle = a_r(\boldsymbol{q})\,|A;\boldsymbol{q}r\rangle\,, \tag{2.35}$$

with $\hat{a}_r(\boldsymbol{q})$ and $\hat{a}_s^\dagger(\boldsymbol{q})$ satisfying the usual commutation relations 2.24. Under the Lorentz condition ($\partial_\mu\mathcal{A}^\mu = 0$), this leads to a similar mode decomposition (a solution to $\Box\mathcal{A}^\mu = 0$)

$$\mathcal{A}^\mu(x) = \mathcal{A}_+^\mu(x) + \mathcal{A}_+^{*\mu}(x), \quad \mathcal{A}_+^\mu(x) = \sum_{r=0,1,2,3} \int \frac{d^3q}{(2\pi)^3} \frac{e}{\sqrt{2\omega_{\boldsymbol{q}}}} a_r(\boldsymbol{q}) e^{-iqx} \epsilon_r^\mu(\boldsymbol{q}). \tag{2.36}$$

where $a_r(\boldsymbol{q})$ can be also thought of as Fourier expansion coefficients. Similar to the quantum case, the longitudinal and scalar modes are purely gauge, and in 2.36 we can also restrict the sum over just the transversal components $r = 1, 2$ amounting to a complete gauge fixing, but this would break Lorentz invariance [Sch19]. The remaining quantized radiation $\hat{\mathcal{A}}^\mu_{\text{radiation}}$ describes the quantum fluctuations around the classical value from 2.34 [Meu15].

We can see that coherent states are, loosely speaking, highly-occupied states [Kai18]. From 2.31, by representing the single-mode coherent state $|A;\boldsymbol{q},r\rangle$ in 2.35 in terms of multiparticle states $|n_r\boldsymbol{q}\rangle$

$$|A;\boldsymbol{q},r\rangle = \mathrm{e}^{-\frac{|a_r(\boldsymbol{q})|^2}{2}} \sum_{n=0}^{\infty} \frac{a_r(\boldsymbol{q})^n}{\sqrt{n!}}\,|n_r(\boldsymbol{q})\rangle\,, \tag{2.37}$$

and calculate the probability of a coherent state to be found in a particular n-particle number Fock-state

$$\langle n|A;\boldsymbol{q},r\rangle^2 = \mathrm{e}^{-(a_r(\boldsymbol{q}))^2} \frac{|a_r(\boldsymbol{q})|^{2n}}{n!} = \mathrm{e}^{-\langle n_r(\boldsymbol{q})\rangle} \frac{\langle n_r(\boldsymbol{q})\rangle}{n!}, \tag{2.38}$$

we discover that it follows a Poisson distribution with expected value

$$\langle A; \boldsymbol{q}, r| \, \hat{n}_r(\boldsymbol{q}) \, |A; \boldsymbol{q}, r\rangle = |a_r(\boldsymbol{q})|^2 = \langle n_r(\boldsymbol{q})\rangle, \tag{2.39}$$

where $\hat{n}_r(\boldsymbol{q}) \equiv \hat{a}^\dagger_r(\boldsymbol{q})\hat{a}_r(\boldsymbol{q})$ is the single-mode number operator, and dispersion

$$|a_r(\boldsymbol{q})| = \sqrt{\langle n_r(\boldsymbol{q})\rangle}, \tag{2.40}$$

such that it provides an intuitive understanding of the high-occupation number limit, where states look approximately coherent.

Using the shift properties of the displacement operator, it can be shown that [MKP13; Meu15]

$$\left.\begin{aligned} \hat{D}^{-1}\hat{a}_r(\boldsymbol{q})\hat{D} = \hat{a}_r(\boldsymbol{q}) + a_r(\boldsymbol{q}) \\ \hat{D}^{-1}\hat{a}^\dagger_r(\boldsymbol{q})\hat{D} = \hat{a}^\dagger_r(\boldsymbol{q}) + a^*_r(\boldsymbol{q}) \end{aligned}\right\} \implies \hat{D}^{-1}\hat{\mathcal{A}}^\mu\hat{D} = \hat{\mathcal{A}}^\mu + \mathcal{A}^\mu. \tag{2.41}$$

Under the assumption that the plane-wave background looks classical[5], then it can be described by a coherent state $|A\rangle$. When incoming $|i; A\rangle$ and outgoing plane waves $|f; A\rangle$ are not significantly altered in the process of interaction with the quantum system (such that the same coherent state $|A\rangle$ is valid for both), the matrix elements describing the process have the form [Sei12]

$$\langle f; A| \, \hat{S}[\hat{\mathcal{A}}] \, |i; A\rangle \equiv \langle A| \, \hat{S}[\hat{\mathcal{A}}] \, |A\rangle = \langle\Omega| \, \hat{D}^{-1}\hat{S}[\hat{\mathcal{A}}]D \, |\Omega\rangle = \langle\Omega| \, \hat{S}[\hat{\mathcal{A}}+\mathcal{A}] \, |\Omega\rangle \,, \tag{2.42}$$

where $\hat{S}[\hat{\mathcal{A}}]$ is the S-matrix operator. This shows that the vacuum expectation value of $\hat{S}[\hat{\mathcal{A}} + \mathcal{A}]$ gives the same result as computing between the same coherent states the S-matrix evaluated on the photon field $\hat{S}[\hat{\mathcal{A}}]$.

Therefore, the *strong field approximation* assumes the conditions are satisfied such that one can work with vacuum expectation values using the shifted field

$$\hat{\mathcal{A}}^\mu_{\text{initial}} \to \hat{\mathcal{A}}^\mu_{\text{shifted}} = \hat{\mathcal{A}}^\mu_{\text{radiative}} + \mathcal{A}^\mu_{\text{background}}, \tag{2.43}$$

instead of the initial photon field $\hat{\mathcal{A}}^\mu_{\text{initial}}$ sandwiched between coherent states $|A\rangle$.

Unlike QED where $\langle\Omega| \, \hat{\mathcal{A}}^\mu_{\text{initial}} \, |\Omega\rangle = 0$, we now have

$$\langle\Omega| \, \hat{\mathcal{A}}^\mu_{\text{shifted}} \, |\Omega\rangle = \mathcal{A}^\mu_{\text{background}}, \qquad \langle\Omega| \, \hat{\mathcal{A}}^\mu_{\text{radiative}} \, |\Omega\rangle = 0, \tag{2.44}$$

while the radiative part $\hat{\mathcal{A}}^\mu_{\text{radiation}}$ represents quantum fluctuations around this VEV.

2.3 Classical plane-wave background fields

A plane wave background field, described by the charge-multiplied electromagnetic 4-potential $\mathcal{A}^\mu = eA^\mu$ (where A^μ is the 4-vector potential), can only depend on the wave phase $\phi = (nx)$ [BPL12], i.e. $\mathcal{A}^\mu = \mathcal{A}^\mu(\phi)$, where $n^2 = 0$ is the timelike vector that determines the wave direction of propagation $n^\mu = (1, \boldsymbol{n})$.

[5] in the sense mentioned in 2.34

2.3.1 Light-cone coordinates

Considering the background plane-wave phase $\phi = (nx)$ dependence, the timelike vector n^μ is a natural choice as a basis vector. However, in order to uniquely describe any 4-vector in this four-dimensional vector space, three more basis vectors are required, $\tilde{n}^\mu$ and a_i^μ $(i = 1, 2)$, where $\tilde{n}^\mu$ is another timelike vector $(\tilde{n}^2 = 0)$ chosen such that $n \cdot \tilde{n} = 1$ and a_i^μ orthogonal to n $(n \cdot a_i = 0)$ and $\tilde{n}$ $(\tilde{n} \cdot a_i = 0)$ and orthonormal to one another $(a_i \cdot a_j = -\delta_{i,j}$, for $i, j = 1, 2)$. Hence, the following conditions define the light-cone basis $\{n^\mu, \tilde{n}^\mu, a_1^\mu, a_2^\mu\}$ of the Minkowski vector space [MKP13; Meu15; Bra19]

$$n^2 = \tilde{n}^2 = (na_j) = (\tilde{n}a_i) = 0, \quad (n\tilde{n}) = 1, \quad (a_i a_j) = -\delta_{ij}, \quad i, j = 1, 2 \tag{2.45}$$

with an orientation given by $\Omega = \epsilon_{\mu\nu\rho\sigma} n^\mu \tilde{n}^\nu a_1^\rho a_2^\sigma = 1$, where $\Omega^2 = 1$ [MKP13; Meu15]. Since for an on-shell photon $n^2 = 0$ and also $\tilde{n}^2 = 0$, these coordinates are justifiably called *light-cone coordinates*. In what follows, the following choice is adopted

$$\begin{gathered} \left\{n^\mu = (1, \boldsymbol{n}), \tilde{n}^\mu = (1, -\boldsymbol{n})/2, a_1^\mu = (0, \boldsymbol{a}_1^\perp), a_2^\mu = (0, \boldsymbol{a}_2^\perp)\right\}, \\ \boldsymbol{n}^2 = 1, \quad \boldsymbol{a}_i \boldsymbol{a}_j = \delta_{ij}, \quad \boldsymbol{n} = \boldsymbol{a}_1 \times \boldsymbol{a}_2. \end{gathered} \tag{2.46}$$

In this light-cone basis 2.45, under the choice 2.46, the light-cone vector components $x_{LCC}^{\mu'} = \{x^+, x^-, \boldsymbol{x}^\perp\}$, $(\mu' = +, -, \perp = \{1, 2\})$ are related to Minkowski (canonical) vector components $x_{\text{canonical}}^\mu = \{x^0, x^1, x^2, x^3\}$, $(\mu = 0, 1, 2, 3)$ by

$$\begin{aligned} x^- = (\tilde{n}x) = \frac{1}{2}\left(x^0 + x^\parallel\right), \quad x^+ = (nx) = \left(x^0 - x^\parallel\right), \\ \boldsymbol{x}^\perp\,|_{\text{LCC}} = \boldsymbol{x}^\perp\big|_{\text{canonical}} = -\left((xa_1), (xa_2)\right) = (\boldsymbol{x} \cdot \boldsymbol{a}_1, \boldsymbol{x} \cdot \boldsymbol{a}_2), \end{aligned} \tag{2.47}$$

where the transversal vectors are denoted by $x^\perp = (x^1, x^2)$ and where $x^\parallel = \boldsymbol{n} \cdot \boldsymbol{x}$ is the 3-vector projection along the propagation direction, with $\boldsymbol{n} = \boldsymbol{k}/\omega$ the unit 3-vector in the direction of propagation, and $x^\perp = \boldsymbol{x}\boldsymbol{a}_1^\perp + \boldsymbol{x}\boldsymbol{a}_2^\perp$ with $x^\perp \cdot \boldsymbol{n} = 0$ the perpendicular projection, where $\boldsymbol{a}_1^\perp$ and $\boldsymbol{a}_2^\perp$ are unit vectors that span the perpendicular plane. It follows that $\boldsymbol{k}^\perp = 0$ and $\boldsymbol{k}^\parallel = \omega\boldsymbol{n}$. The relation between vector and covector components takes the form

$$x^+ = x_-, \quad x^- = x_+, \quad x_\perp = -x^\perp, \tag{2.48}$$

which can be easily seen by computing the metric in the new coordinates

$$\eta_{\mu\nu}^{\text{LCC}} = \frac{\partial x^\alpha}{\partial x_{\text{LCC}}^\mu} \frac{\partial x^\beta}{\partial x_{\text{LCC}}^\nu} \eta_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\text{LCC}}^{\mu\nu}, \tag{2.49}$$

which shows that the metric is off-diagonal in the $(+, -)$ subspace

$$\eta_{+-} = \eta_{-+} = 1, \quad \eta_{++} = \eta_{--} = 0,$$

and also does not couple the $(+,-)$ subspace to the $\perp = (1,2)$ subspace

$$\eta_{+\perp} = \eta_{-\perp} = 0, \quad \perp = 1,2.$$

The light-cone metric has the following decomposition [MKP13; Meu15]

$$\eta^{\mu\nu} = n^\mu \tilde{n}^\nu + \tilde{n}^\mu n^\nu - a_1^\mu a_1^\nu - a_2^\mu a_2^\nu, \tag{2.50}$$

The indefinite bilinear product can be expressed as

$$x \cdot y = x^+ y^- + x^- y^+ - \boldsymbol{x}^\perp \boldsymbol{y}^\perp \implies x^2 = 2x^+ x^- - \left(\boldsymbol{x}^\perp\right)^2.$$

2.3.2 Plane-wave solutions of Maxwell's equations

For a plane-wave, the field strength tensor $\mathcal{F}^{\mu\nu} = \mathcal{F}^{\mu\nu}(\phi) \equiv eF^{\mu\nu}(\phi)$ [BPL12] expressed uniquely via the 4-potential $\mathcal{A}^\mu(\phi)$ by

$$\mathcal{F}^{\mu\nu}(\phi) = \partial^\mu \mathcal{A}^\nu(\phi) - \partial^\nu \mathcal{A}^\mu(\phi) = n^\mu \mathcal{A}'^\nu(\phi) - n^\nu \mathcal{A}'^\mu(\phi), \tag{2.51}$$

satisfies the homogeneous Maxwell's equations [Lan75]

$$\partial_\mu \mathcal{F}^{\mu\nu}(\phi) = n_\mu \mathcal{F}'^{\mu\nu}(\phi) = 0, \quad \partial^\mu \mathcal{F}^*_{\mu\nu}(\phi) = n^\mu \mathcal{F}'^*_{\mu\nu}(\phi) = 0. \tag{2.52}$$

Because $\mathcal{F}^{\mu\nu}(\phi)$ is antisymmetric, it can have at most 6 linearly independent components. Neglecting any constant (ϕ-independent) parts, from the first of Maxwell's equations it follows that all the coefficients proportional to $\tilde{n}^\mu$ must vanish (since $n \cdot \tilde{n} = 1$), whereas the second Maxwell's equations it follows that also the coefficient for the combination $a_1^\mu a_2^\nu - a_2^\mu a_1^\nu$ must vanish (as $n^\mu \epsilon_{\mu\nu\rho\sigma}(a_1^\rho a_2^\sigma - a_2^\rho a_1^\sigma) \neq 0$) [Bra19]. The general solution has then two independent coefficients [MKP13; Meu15; Bra19]

$$F^{\mu\nu}(\phi) = \sum_{i=1,2} f_i^{\mu\nu} \psi_i'(\phi) \tag{2.53}$$

where [MKP13; Meu15]

$$f_i^{\mu\nu} = n^\mu a_i^\nu - n^\nu a_i^\mu, \quad f_{i\rho}^\mu f_j^{\rho\nu} = -\delta_{ij} a_i^2 n^\mu n^\nu, \quad n_\mu f_i^{\mu\nu} = 0, \tag{2.54}$$

and a_i^μ, $i = 1,2$ define the field amplitudes in the two polarizarion directions ($n \cdot a_i = 0$, $a_1 \cdot a_2 = 0$) [Bra19]. The functions $\psi_i(\phi)$ describe just the shape of the field and not the amplitude ($|\psi_i(\phi)|, |\psi_i'(\phi)| \simeq 1$), a shape of finite extent, i.e. vanishing towards infinity ($\psi_i(\pm\infty) = \psi_i'(\pm\infty) = 0$) [Meu15]. The Lorentz gauge fixing condition is

$$\partial \cdot A(\phi) = 0 \implies n \cdot A'(\phi) = 0 \text{ or } A'_-(\phi) = 0, \tag{2.55}$$

and choosing the wave-packets such that the fields vanish at $\pm\infty$, there can be no constant component, i.e.

$$n \cdot A(\phi) = 0 \text{ or } A_-(\phi) = 0. \tag{2.56}$$

The solution of 2.51 in the Lorentz-gauge 2.55 for the four-potential is [Meu15]

$$A^\mu(\phi) = \sum_{i=1,2} a_i^\mu \psi_i(\phi). \tag{2.57}$$

2.3.3 Motion of an electron in a plane-wave background

The Lorentz invariant action for a free charged particle in an electromagnetic field is [Lan75]

$$
\begin{aligned}
S &= \int_a^b \left(-m ds - eA \cdot dx\right) \\
&= \int_{t_1}^{t_2} \left(-\frac{m}{\gamma} - eA^0 \frac{dx_0}{dt} - eA^i \frac{dx_i}{dt}\right) dt \\
&= \int_{\tau_1}^{\tau_2} \left(-m\sqrt{\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}} - eA_\mu \frac{dx^\mu}{d\tau}\right) d\tau.
\end{aligned}
\tag{2.58}
$$

That is, using $u^\mu = dx^\mu/d\tau$, the Lagrangian can be written in a manifestly covariant form, when integrated over the proper time to obtain the action

$$
S[x(\tau)] = \int_{\tau_1}^{\tau_2} L_{\text{covariant}}(x,u,\tau) d\tau, \quad L_{\text{covariant}}(x,u,\tau) = -m\sqrt{u \cdot u} - eA \cdot u. \tag{2.59}
$$

This approach makes transparent obtaining the covariant Lorentz force equation. By using the Euler-Lagrange equations we get for the equations of motion [Jac98]

$$
\left.
\begin{aligned}
P_\mu \equiv \frac{\partial L}{\partial u^\mu} = m u_\mu + eA_\mu = p_\mu + eA_\mu \\
\partial_\mu L = q\partial_\mu \left(u^\lambda A_\lambda\right) = e u^\lambda \partial_\mu A_\lambda
\end{aligned}
\right\}
\implies \frac{d}{d\tau}\left(m u_\mu + eA_\mu\right) = e u^\lambda \partial_\mu A_\lambda. \tag{2.60}
$$

Noticing that $\frac{A_\lambda}{d\tau} = u_\lambda A^\lambda$, we get the covariant Lorentz force equation [Lan75; Di +12] (for a plane-wave background $F^{\mu\nu} = F^{\mu\nu}(\phi)$)

$$
\frac{dp^\mu}{d\tau} = eF^{\mu\nu}(\phi) u_\nu. \tag{2.61}
$$

Alternatively, we can work noncovariantly with the ordinary Lagrangian, integrated over the time coordinate [Jac98; Lan75]

$$
\left.
\begin{aligned}
L = -\frac{m}{\gamma} - eA^0 + e\boldsymbol{A} \cdot \boldsymbol{v} \implies \boldsymbol{P} = \frac{\partial L}{\partial \boldsymbol{v}} = \boldsymbol{p} + e\boldsymbol{A} \\
H = \boldsymbol{v} \cdot \frac{\partial L}{\partial \boldsymbol{v}} - L = \frac{m}{\gamma} + eA^0
\end{aligned}
\right\}
\implies P^\mu = p^\mu + eA^\mu. \tag{2.62}
$$

$$
p^2 = m^2 \implies \left(H - eA^0\right)^2 = m^2 + (\boldsymbol{P} - e\boldsymbol{A})^2 \implies H = \sqrt{m^2 + (\boldsymbol{P} - e\boldsymbol{A})^2} + eA^0 \tag{2.63}
$$

The Hamilton-Jacobi equation (HJE) can be easily obtained from the on-shell relation, by using the canonical 4-momentum obtained either from the relativistic or from the non-relativistic Lagrangian, and is [Lan75]

$$
\left.
\begin{aligned}
P_\mu = -\partial_\mu S = p_\mu + \mathcal{A}_\mu \\
p^2 = m^2
\end{aligned}
\right\}
\implies \boxed{\left(\partial_\mu S + \mathcal{A}_\mu\right)\left(\partial^\mu S + \mathcal{A}^\mu\right) = m^2}. \tag{2.64}
$$

For solving the HJE for a plane wave background, the ansatz is taken [BPL12]

$$S = -p_0 \cdot x + F(\phi) \tag{2.65}$$

where p_0^μ is a constant 4-vector satisfying $p_0^2 = m^2$ and $S = -p_0 \cdot x$ is the solution for the HJE for a free particle with a momentum $p_0^\mu = -\partial_\mu S$, i.e. $(\partial_\mu S)^2 = m^2$.

Plugging in the ansatz

$$[-p_0 + n \cdot F'(\phi) + \mathcal{A}(\phi)]^2 = m^2 \implies \mathcal{A}^2(\phi) - 2(p_0 \cdot n)F'(\phi) - 2(p_0 \cdot \mathcal{A}) = 0. \tag{2.66}$$

Using $(p_0)_- = (p_0 n) = {p_0}^0 - {p_0}^\parallel$, we have

$$F'(\phi) = \frac{1}{2(p_0)_-}\mathcal{A}^2(\phi) - \frac{1}{(p_0)_-}(p_0 \cdot \mathcal{A}(\phi)). \tag{2.67}$$

Integrating 2.67, we get

$$F(\phi) = \frac{1}{2(p_0)_-}\int_{-\infty}^{\phi} \mathcal{A}^2(\phi')d\phi' - \frac{1}{(p_0)_-}\int_{-\infty}^{\phi} (p_0 \cdot \mathcal{A}(\phi'))\, d\phi', \tag{2.68}$$

such that the action is [Lan75]

$$\boxed{S = -p_0 \cdot x + \frac{1}{2(p_0)_-}\int_{-\infty}^{\phi} \mathcal{A}^2(\phi')d\phi' - \frac{1}{(p_0)_-}\int_{-\infty}^{\phi} (p_0 \cdot \mathcal{A}(\phi'))\, d\phi'}. \tag{2.69}$$

Having computed the action S under the Lorentz condition, we can retrieve

$$\boxed{\begin{aligned} p_\mu(\phi) &= -\partial_\mu S(\phi) - \mathcal{A}_\mu(\phi) \\ &= (p_0)_\mu - \mathcal{A}_\mu(\phi) + n_\mu\left(\frac{(p_0 \cdot \mathcal{A}(\phi))}{(p_0)_-} - \frac{\mathcal{A}^2(\phi)}{2(p_0)_-}\right). \end{aligned}} \tag{2.70}$$

Writing the same relation for a reference phase ϕ_0 and subtracting, we get the value of the momentum at phase ϕ in terms of the value at reference phase ϕ_0,

$$p_\mu(\phi) = p_\mu(\phi_0) - (\mathcal{A}_\mu(\phi) - \mathcal{A}_\mu(\phi_0)) + n_\mu\left(\frac{[p_0 \cdot (\mathcal{A}(\phi) - A(\phi_0))]}{(p_0)_-} - \frac{\left(\mathcal{A}^2(\phi) - \mathcal{A}^2(\phi_0)\right)}{2(p_0)_-}\right). \tag{2.71}$$

Assuming that the wave packets are such that $\mathcal{A}^\mu(\phi_0) = 0$ (for example for $\phi_0 = -\infty$) and also denoting $p_\mu(\phi_0) =: p_\mu$, the final solution becomes [Lan75], by denoting $\pi^\mu(\phi) \equiv p^\mu(\phi)$

$$\pi^\mu(\phi) = p^\mu - \mathcal{A}^\mu(\phi) + n^\mu\left(\frac{p \cdot \mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right), \tag{2.72}$$

which is known as the dressed momentum. Notice that in deriving the HJE equation, we started from the on-shell condition $\pi(\phi) \cdot \pi(\phi) = m^2$.

Choosing the direction of propagation along $\boldsymbol{n}$, with $\boldsymbol{n}^2 = 1$, and completely fixing the Lorentz gauge ($A_-(\phi) = 0$) by choosing $A^\mu(\phi) = (0, \boldsymbol{A}(\phi))$, with $\boldsymbol{A}(\phi)$ transversal ($\boldsymbol{A}(\phi) \cdot \boldsymbol{n} = 0$), the dressed momentum $p^\mu(\phi) = (\varepsilon(\phi), \boldsymbol{p}(\phi))$ at the arbitrary phase ϕ from 2.72, given the initial condition $p^\mu(\phi_0) = p_0^\mu = (\varepsilon_0, \boldsymbol{p}_0)$, becomes [Lan75] [Di +12]

$$\begin{aligned}
\boldsymbol{p}^\perp(\phi) &= \boldsymbol{p}_0^\perp - e\left(\boldsymbol{A}^\perp(\phi) - \boldsymbol{A}^\perp(\phi_0)\right), \\
p^\parallel(\phi) &= p_0^\parallel - \frac{e}{(p_0)_-}\boldsymbol{p}_0^\perp\left(\boldsymbol{A}^\perp(\phi) - \boldsymbol{A}^\perp(\phi_0)\right) + \frac{e^2}{2(p_0)_-}\left(\boldsymbol{A}^2(\phi) - \boldsymbol{A}^2(\phi_0)\right), \\
\varepsilon(\phi) &= \varepsilon(\phi_0) - \frac{e}{(p_0)_-}\boldsymbol{p_0}^\perp\left(\boldsymbol{A}^\perp(\phi) - \boldsymbol{A}^\perp(\phi_0)\right) + \frac{e^2}{2(p_0)_-}\left(\boldsymbol{A}^2(\phi) - \boldsymbol{A}^2(\phi_0)\right).
\end{aligned} \tag{2.73}$$

Using that p_- is conserved, as it is the conjugate to the cyclic variable $t+x^\parallel$, we have that $p^\parallel(\phi) = \varepsilon(\phi) - p_-$. Inserting this in the on-shell condition $\varepsilon(\phi) + p^\parallel(\phi) = \left[\boldsymbol{p}_\perp^2(\phi) + m^2\right]/p_-$ we have that

$$\varepsilon(\phi) = \frac{p_-}{2} + \frac{\left[\boldsymbol{p}_\perp^2(\phi) + m^2\right]}{2p_-}, \tag{2.74}$$

which shows that $\varepsilon(\phi) \geq m$ (or $\varepsilon(\phi) \leq m$) for $p_- > 0$ ($p_- < 0$, respectively), and hence, classicaly there are no bound states [Di +12].

2.3.4 SFQED parameters

In classical electrodynamics, Maxwell's equations are linear and do not allow for self interactions [Lan75]. However, in QED photons do manifest self-interactions at loop level such that when the vacuum is subjected to an external field, it becomes birefringent [HE36].

High intensities and energies require taking these nonlinear effects into account. In order to do this, *Lorentz and gauge invariant parameters* that characterize the non-linearities in the strong-field regime need to be constructed.

Classical non-linearity parameter

At the classical level, the non-linear dynamics of an electron in the field $A^\mu(\phi)$ is characterized by the *classical non-linearity parameter* ξ, [Di +12]

$$\xi = \sqrt{\xi_1^2 + \xi_2^2}, \quad \hat{\xi}_i = \xi_i/\xi, \quad \xi_i = \frac{|e|\sqrt{-a_i^2}}{m} = \frac{|e|E_i}{m\omega} = \frac{|e|\lambda E_i}{2\pi m}, \tag{2.75}$$

where ξ_i (and E_i) represents the classical non-linearity parameter (electric field, respectively) for polarization a_i^μ, m is the mass of the charged particle being accelerated, λ (and ω) is the wavelength (angular frequency, respectively) of the plane-wave. It can be understood as the work performed by the laser field on the electron in one laser wavelength in units of the electron rest mass, which shows that $\xi \geq 1$ determines the relativistic regime [Di +12]. Notice however that although work is performed on the charge, according to the Lawson-Woodward theorem [ESL09] a plane wave field can't provide a net acceleration to an ultrarelativistic charged particle in vacuum.

The classical non-linearity parameter can also be interpreted quantum mechanically as the number of photons absorbed over a Compton wavelength $\lambda_C = 1/m$ in the interaction with the quantum system, as it can be seen from

$$\xi = \frac{|e|\lambda_C E}{\omega}. \tag{2.76}$$

Alternatively, this shows that for $\xi \geq 1$ the interaction involves multiple photons and enters the nonlinear regime. Therefore, this regime describes multiphoton effects.

In the weakly relativistic regime $\xi \leq 1$, the probability for an electron to interact with n laser photons scales as ξ^{2n}, meaning that the leading-order corrections are supressed and the interaction with the laser field can be treated perturbatively [Bra19].

Quantum non-linearity parameter

One can characterize the quantum effects like photon recoil or pair production by defining the *quantum non-linearity parameter* [Rit72; Di +12; Meu15]

$$\chi = \sqrt{\chi_1^2 + \chi_2^2}, \quad \hat{\chi}_i = \chi_i/\chi, \quad \chi_i = \frac{|e|\sqrt{q f_i^2 q}}{m^3} = \eta \xi_i, \quad \eta = \frac{\sqrt{(kq)^2}}{m^2}. \tag{2.77}$$

Since ξ_i and f_i^2 are gauge and Lorentz invariant, χ_i is as well.

Field strength invariants

Other parameters that characterize the strength of the electromagnetic field can be introduced, like the scalar and pseudoscalar field strength invariants [Sei12; Di +12], defined for an electromagnetic field strength $F^{\mu\nu}(x) = (\boldsymbol{E}(x), \boldsymbol{B}(x))$ by

$$\begin{aligned} \mathscr{F}(x) &= \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) = \frac{1}{2}\left(B^2(x) - E^2(x)\right), \\ \mathscr{G}(x) &= \frac{1}{4} F^*_{\mu\nu}(x) F^{\mu\nu}(x) = -\boldsymbol{B}(x)\boldsymbol{E}(x), \end{aligned} \tag{2.78}$$

where $F^*_{\mu\nu}(x) = \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}(x)/2$ is the dual field strength tensor. However, for the plane-wave case or the crossed field case, both vanish identically, i.e. $\mathscr{F} = \mathscr{G} \equiv 0$, which means that the vacuum remains stable in the presence of a plane-wave background [Sch51; Meu15].

2.4 Quantization in the presence of a background plane-wave

Starting from the QED Lagrangian density [Mic19; Sch19]

$$\mathcal{L}_{\text{QED}} = \hat{\bar{\psi}}(i\partial\!\!\!/ - m)\hat{\psi} - \frac{1}{4e^2}(\hat{\mathcal{F}} \cdot \hat{\mathcal{F}})_{\text{initial}} - \bar{\psi}\gamma_\mu\psi\hat{\mathcal{A}}^\mu_{\text{initial}}, \tag{2.79}$$

where $\mathcal{F}^{\mu\nu} = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu$ the field strength tensor and ψ is the electron spinor[6], we can obtain the (SFQED) Lagrangian density in an external plane-wave field by the shift 2.43 of the photon field, i.e. $\hat{\mathcal{A}}^\mu_{\text{initial}} \to \hat{\mathcal{A}}^\mu_{\text{shifted}} = \hat{\mathcal{A}}^\mu + \mathcal{A}^\mu$ (not writing the subscripts any more not to clutter the equations).

Letting j^μ be an external current coupling to the full field, the SFQED Lagrangian becomes

$$\mathcal{L}_{\text{SFQED}} = \bar{\psi}\left[\gamma^\mu\left(i\partial_\mu - \hat{\mathcal{A}}_\mu - \mathcal{A}_\mu\right) - m\right]\psi - \frac{1}{4e^2}\left(\hat{\mathcal{F}} + \mathcal{F}\right)^2 + j^\mu\left(\hat{\mathcal{A}}_\mu + \mathcal{A}_\mu\right), \tag{2.80}$$

The background field and four-current are given functions fulfilling Maxwells equations $\partial_\mu \mathcal{F}^{\mu\nu} = ej^\nu$ and this is why we can drop the pure background terms to get

$$\mathcal{L}_{\text{SFQED}} = \bar{\psi}\left[\gamma^\mu\left(i\partial_\mu - \hat{\mathcal{A}}_\mu - \mathcal{A}_\mu\right) - m\right]\psi - j^\mu \hat{A}_\mu - \frac{1}{4e^2}\hat{\mathcal{F}}_{\mu\nu}\hat{\mathcal{F}}^{\mu\nu} - \frac{1}{2e^2}\hat{\mathcal{F}}_{\mu\nu}\mathcal{F}^{\mu\nu}. \tag{2.81}$$

By integrating by parts the term

$$\begin{aligned}\hat{\mathcal{F}}^{\mu\nu}\mathcal{F}_{\mu\nu} &= \left(\partial^\mu \hat{\mathcal{A}}^\nu - \partial^\nu \hat{\mathcal{A}}^\mu\right)\mathcal{F}_{\mu\nu} \\ &= \partial^\mu\left(\hat{\mathcal{A}}^\nu \mathcal{F}_{\mu\nu}\right) - \hat{\mathcal{A}}^\nu\left(\partial^\mu \mathcal{F}_{\mu\nu}\right) - \partial^\nu\left(\hat{\mathcal{A}}^\mu \mathcal{F}_{\mu\nu}\right) + \hat{\mathcal{A}}^\mu\left(\partial^\nu \mathcal{F}_{\mu\nu}\right) \\ &= \partial^\mu\left(\hat{\mathcal{A}}^\nu \mathcal{F}_{\mu\nu}\right) - \partial^\nu\left(\hat{A}^\mu \mathcal{F}_{\mu\nu}\right) - 2e\hat{\mathcal{A}}^\nu j_\nu.\end{aligned} \tag{2.82}$$

we obtain the Lagrangian density in strong-field QED as

$$\mathcal{L}_{\text{SFQED}} = \bar{\psi}(i\gamma\cdot\partial - m)\psi - \frac{1}{4e^2}\hat{\mathcal{F}}_{\mu\nu}\hat{\mathcal{F}}^{\mu\nu} - \bar{\psi}\gamma^\mu\psi\left(\mathcal{A}_\mu + \hat{\mathcal{A}}_\mu\right) = \mathcal{L}_{\text{QED}} + \mathcal{L}_{int,B} \tag{2.83}$$

The effect of the external field is to give rise to an additional vertex corresponding to the interaction term [Meu15; Sei12]

$$\mathcal{L}_{int,B} = -\bar{\psi}\gamma^\mu\psi\mathcal{A}_\mu \tag{2.84}$$

which, in the case where we can solve for the interaction of the electron with the plane-wave field exactly[7], can be included as

$$\mathcal{L}_{\text{Volkov}} = \bar{\psi}(\gamma^\mu(i\partial_\mu - \mathcal{A}_\mu) - m)\psi \implies \mathcal{L}_{\text{SFQED}} = \mathcal{L}_{\text{Volkov}} + \mathcal{L}_{\text{QED,int}} + \mathcal{L}_{\text{em}} \tag{2.85}$$

where

$$\begin{aligned}\mathcal{L}_{\text{QED,int}} &= -\bar{\psi}\gamma_\mu\psi\hat{\mathcal{A}}^\mu, \\ \mathcal{L}_{\text{em}} &= -\frac{1}{4e^2}\hat{\mathcal{F}}_{\mu\nu}\hat{\mathcal{F}}^{\mu\nu},\end{aligned} \tag{2.86}$$

By using the Volkov Lagrangian to find the equations of motion and solve for the spinor in the presence of the background field, the effect of the plane-wave background is then

[6] Although ψ are still operator-valued distributions, the hat over ψ is not written for notational convenience, not to interfere with the bar.

[7] For more such cases see [BGJ75; BG14].

already included nonperturbatively [Sei12]. Therefore, if we want to add a gauge fixing term $\mathcal{L}_\xi$[8], we can do so only for the radiative part, i.e.

$$\mathcal{L}^{(\xi)}_{\text{SFQED}} = \mathcal{L}_{\text{Volkov}} + \mathcal{L}_{\text{QED,int}} + \mathcal{L}_{\text{em}} + \mathcal{L}_\xi, \qquad \mathcal{L}_\xi = -\frac{\xi}{2}(\partial \cdot \hat{\mathcal{A}})^2. \tag{2.87}$$

Then, when computing matrix elements, we can use the following S-matrix operator [VN 74; Mit75; Sei12] in the Furry picture [Fur51]

$$\hat{S}[\hat{\mathcal{A}}] = \mathcal{T}\exp\left\{-i\int d^4x \mathcal{H}_{\text{int}}(x)\right\} = \mathcal{T}\exp\left\{-i\int d^4x : \hat{\bar{\psi}}^{(V)}(x)\gamma^\mu \hat{\mathcal{A}}_\mu(x)\hat{\psi}^{(V)}(x) :\right\}, \tag{2.88}$$

where $\hat{\psi}^{(V)}$ is the Volkov solution of the Dirac equation in the plane wave background field, while $\mathcal{T}$ is the time ordering operator and : denotes normal ordering.

The Volkov spinor similarly admits a mode decomposition into positive and negative energy parts [MKP13; Sei12]

$$\hat{\psi}^{(V)}(x) = \sum_\alpha \psi^{(+)}_\alpha(x)\hat{c}_\alpha + \psi^{(-)}_\alpha \hat{d}^\dagger_\alpha. \tag{2.89}$$

2.4.1 Volkov solution

The Volkov Lagrangian density leads to the Dirac equation in the presence of an external background field [BPL12]

$$(i\not{\partial} - \not{\mathcal{A}}(\phi) - m)\psi^{(V)}_\alpha(x) = 0, \tag{2.90}$$

where α denotes the set of quantum numbers identifying the state of the particle, and $\phi = (nx)$ is the phase of the plane-wave background.

The solution was first derived by [Vol35] and was introduced in the context of SFQED in the review by [Mit75]. A clear exposition of the derivation can be found in [BPL12]. First, by applying the $(i\not{\partial} - \not{\mathcal{A}}(\phi) + m)$ to 2.90, we get [BPL12]

$$\left[-\partial^2 - 2i(\mathcal{A}\cdot\partial) + \mathcal{A}^2 - m^2 - i\not{n}\not{\mathcal{A}}'\right]\psi_\alpha(x) = 0. \tag{2.91}$$

We seek a solution [BPL12] to 2.91 of the form

$$\psi_{p,\sigma}(x) = e^{-ipx}F(\phi), \tag{2.92}$$

for on-shell electron of momentum p ($p^2 = m^2$) and spin σ[9].

Inserting the ansatz 2.92 and using

$$\partial^\mu F(\phi) = n^\mu F'(\phi), \quad \partial^2 F(\phi) = n^2 F''(\phi) = 0, \tag{2.93}$$

we get the equation for $F(\phi)$

$$2i(np)F' + \left[-2(p\cdot\mathcal{A}) + \mathcal{A}^2 - i\not{n}\not{\mathcal{A}}'\right]F = 0, \tag{2.94}$$

[8] ξ here doesn't represent the classical non-linearity parameter, but the gauge-fixing parameter.
[9] Notice that if $p^2 \neq 0$ initially, we can add δp, i.e. $p + \delta p = p'$, such that $(p')^2 = m^2$ and absorb the remaining phase $\exp(i\delta p x)$ into $F(\phi)$. Then, the form 2.92 continues to be valid.

which is solved by [BPL12]

$$F(\phi) = \exp\left\{-i\int_0^{\phi}\left[\frac{(p\cdot\mathcal{A})}{(np)} - \frac{\mathcal{A}^2}{2(np)}\right]d\phi + \frac{\not{n}\not{\mathcal{A}}}{2(np)}\right\}u_\sigma(p), \tag{2.95}$$

where $u_\sigma(p)$ (and $v_\sigma(p)$) are the free momentum space Dirac spinor solutions, i.e.

$$(\not{p} - m)u_\sigma(p) = 0, \quad (\not{p} + m)v_\sigma(p) = 0, \tag{2.96}$$

satisfying the normalization conditions (for $p^\mu = (\varepsilon_{\boldsymbol{p}}, \boldsymbol{p})$) [Sch19; Mic19]

$$\begin{aligned} \bar{u}_\sigma(p)u_{\sigma'}(p) &= 2m\delta_{\sigma\sigma'}, \quad \bar{v}_\sigma(p)v_{\sigma'}(p) = -2m\delta_{\sigma\sigma'}, \\ u_\sigma^\dagger(p)u_{\sigma'}(p) &= 2\varepsilon_{\boldsymbol{p}}\delta_{\sigma\sigma'}, \quad v_\sigma^\dagger(p)v_{\sigma'}(p) = 2\varepsilon_{\boldsymbol{p}}\delta_{\sigma\sigma'}. \end{aligned} \tag{2.97}$$

Taylor expanding 2.95 and noticing that the terms in $(\not{n}\not{\mathcal{A}})^2$ vanish, we get for the positive energy Volkov states (for the particles)

$$U_\sigma(p,x) \equiv \Psi_\sigma^{(V)}(p,x) = E(p,x)u_\sigma(p), \tag{2.98}$$

where $E(p,x)$ are the Ritus matrices, and $\bar{E}_p(x) = \gamma^0 E_p^\dagger(x)\gamma^0$ the Dirac-conjugate matrix

$$E(p,x) = \left[\mathbf{1} + \frac{\not{n}\not{\mathcal{A}}(\phi)}{2(np)}\right]e^{iS_p(x)}, \quad \bar{E}(p,x) = \left[\mathbf{1} + \frac{\not{\mathcal{A}}(\phi)\not{n}}{2(np)}\right]e^{-iS_p(x)}, \tag{2.99}$$

and the phase (which is also the classical phase from 2.69 [Lan75]) is

$$S_p(x) = -px - \int_{-\infty}^{\phi} d\phi' \left[\frac{p\mathcal{A}(\phi')}{np} - \frac{\mathcal{A}^2(\phi')}{2(np)}\right], \tag{2.100}$$

which shows that, the Volkov states are in a sense semi-classical.

Noticing that in the limit of a vanishing external field at $\phi = \pm\infty$, i.e. $\mathcal{A}_\mu(\pm\infty) = 0$, the Ritus matrices $E(p,x) \to e^{-ipx}$ such that

$$\Psi_\sigma^{(V)}(p,x) \to \psi_\sigma(p,x) = e^{-ipx}u_\sigma(p), \tag{2.101}$$

the free spinor solutions are recovered [Meu15].

The negative energy solutions (corresponding to the antiparticles), are obtained from the replacements $p \to -p$ and $\sigma \to -\sigma$ [Di +12; BPL12]. Inserting $v_\sigma(p) \equiv u_{-\sigma}(-p)$

$$V_\sigma(p,x) \equiv \Psi_{-\sigma}^{(V)}(-p,x) = E(-p,x)v_\sigma(p). \tag{2.102}$$

By imposing the canonical fermionic anticommutation relations [Sch19; Mic19], we get the quantized Volkov solution (as in 2.88)

$$\hat{\psi}^{(V)}(x) = \sum_{\sigma=+,-}\int \frac{d^3p}{(2\pi)^3\sqrt{2\varepsilon_{\boldsymbol{p}}}}\left[\hat{c}_{p,\sigma}U_\sigma(\boldsymbol{p},x) + \hat{d}_{p,\sigma}^\dagger V_\sigma(\boldsymbol{p},x)\right]. \tag{2.103}$$

The Volkov state from 2.98 is an eigenfunction of the momentum operators[10] [BPL12; Pia18]

$$\boldsymbol{P}_\perp = -i(\boldsymbol{a}_1\cdot\boldsymbol{\nabla}, \boldsymbol{a}_2\cdot\boldsymbol{\nabla}), P_\tau = -i\partial_\tau = -i(nP) = -i(\partial_t + \partial_{x_\parallel}), (\gamma\Pi)^2 \tag{2.104}$$

[10] Check Section 2.3.1 for a definition of light-cone coordinates.

with eigenvalues $\boldsymbol{p}_\perp, -p_- = -(p_0 - p_3)$ and p^2 (not necessarily on-shell), which comes from the same property shared by the Ritus matrices [Rit72]. Since the operators in 2.104 commute with the Volkov Hamiltonian (the Legendre transform of the Volkov Lagrangian), the corresponding eigenvalues are conserved [BPL12].

It is easy to check that the Ritus matrices satisfy

$$\bar{E}(p,x)E(p,x) = E(p,x)\bar{E}(p,x) = \mathbf{1} \tag{2.105}$$

and that they convert momentum operators into momentum variables [Rit72; Meu15; Pia18] (even when analytically continued off-shell $p^2 \neq m^2$)[11],

$$\left[i\not{\partial}_x - \mathcal{A}(\phi)\right] E(p,x) = E(p,x)\not{p}, \quad -i\partial_x^\mu \bar{E}(p,x)\gamma_\mu - \bar{E}(p,x)\mathcal{A}(\phi) = \not{p}\bar{E}(p,x), \tag{2.106}$$

where the derivative acts only on the Ritus matrix and its Dirac-conjugate.

The Ritus matrices also form a complete set [Pia18; Meu15]

$$\int \frac{d^4p}{(2\pi)^4} E(p,x)\bar{E}(p,x') = \delta^4\left(x - x'\right), \quad \int d^4x \bar{E}(p',x)E(p,x) = (2\pi)^4\delta^4\left(p' - p\right) \tag{2.107}$$

2.4.2 Volkov propagator

The electron Green's function in an external plane-wave background field is [Pia18]

$$\left\{\gamma^\mu\left[i\partial_\mu - \mathcal{A}_\mu(\phi)\right] - m\right\} G\left(x,x'\right) = \delta^{(4)}\left(x - x'\right). \tag{2.108}$$

In order to get the dressed Feynman propagator of the Volkov field $\hat{\Psi}^{(V)}(x)$

$$G_F(x,x') \equiv -i\left\langle\Omega\right| \mathcal{T}\left[\hat{\Psi}^{(V)}(x)\hat{\bar{\Psi}}^{(V)}\left(x'\right)\right]\left|\Omega\right\rangle, \tag{2.109}$$

[$\mathcal{T}$ is the time ordering operator] we specify boundary conditions that are equivalent to shifting the mass $m \to m - i0$.

Plugging in 2.109 the anticommutation relations and the mode expansion of the Volkov field operator from 2.103 [Pia18]

$$\begin{aligned} G\left(x,x'\right) = &- i\theta\left(x^0 - x'^0\right)\sum_\sigma \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\varepsilon_{\boldsymbol{p}}} U_\sigma(\boldsymbol{p},x)\bar{U}_\sigma\left(\boldsymbol{p},x'\right) \\ &+ i\theta\left(x'^0 - x^0\right)\sum_\sigma \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\varepsilon_{\boldsymbol{p}}} V_\sigma(\boldsymbol{p},x)\bar{V}_\sigma\left(\boldsymbol{p},x'\right), \end{aligned} \tag{2.110}$$

can be put in the Volkov form [Vol35; Rit72]

$$\hat{G}\left(x,x'\right) = \int \frac{d^4p}{(2\pi)^4} E(p,x)\frac{\not{p} + m}{p^2 - m^2 + i0}\bar{E}\left(p,x'\right). \tag{2.111}$$

Within the operator technique, the operator $\hat{G}$[12], corresponding to the Green's function $G(x,x') = \langle x|\hat{G}|x'\rangle$ is defined as

$$\hat{G} = \frac{1}{\not{\Pi} - m + i0} = (\not{\Pi} + m)\frac{1}{\not{\Pi}^2 - m^2 + i0} = \frac{1}{\not{\Pi}^2 - m^2 + i0}(\not{\Pi} + m), \tag{2.112}$$

[11] For a proof of see Appendix A.1.
[12] dropping the F subscript indicating the Feynman propagator.

where $\Pi^\mu = P^\mu - eA^\mu(\Phi)$.

Employing the Schwinger parametrization expansion for the exponential (Appendix C.2)

$$\begin{aligned}\frac{1}{\not{\Pi}^2 - m^2 + i0} &= (-i)\int_0^\infty ds e^{is\left(\not{\Pi}^2 - m^2\right)} \\ &= (-i)\int_0^\infty ds e^{is\left\{2P_\tau P_\phi - [\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi)]^2 - m^2 - ie\hat{n}\hat{A}'(\Phi)\right\}},\end{aligned} \tag{2.113}$$

where in the second line was used that

$$\begin{aligned}\not{\Pi}^2 - m^2 &= [P - eA(\Phi)]^2 - m^2 - \frac{ie}{2}\sigma^{\mu\nu}F_{\mu\nu}(\Phi) \\ &= 2P_\tau P_\phi - [\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi)]^2 - m^2 - ie\not{n}\not{A}'(\Phi)\end{aligned} \tag{2.114}$$

where $\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/2$, $F_{\mu\nu}(\Phi) = \partial_\mu A_\nu(\Phi) - \partial_\nu A_\mu(\Phi) = n_\mu A'_\nu(\Phi) - n_\nu A'_\mu(\Phi)$, with the prime indicating the derivative with respect to the operator Φ corresponding to the variable ϕ, and the light-cone momentum operators $P_\tau = -i\partial_\tau = -i(nP)$ and $P_\phi = -i\partial_\phi = -(\tilde{n}P)$[13].

Following [Pia18], we can write the exponential from the integral 2.113

$$e^{is\left\{2P_\tau P_\phi - [\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi)]^2 - m^2 - ie\not{n}\not{A}'(\Phi)\right\}} = L(\boldsymbol{s})e^{2isP_\tau P_\phi} \tag{2.115}$$

and noticing that $L(s)$ satisfies the differential equation [Pia18]

$$\begin{aligned}\frac{dL}{ds} &= -iLe^{2isP_\tau P_\phi}\left\{[\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi)]^2 + m^2 + ie\not{n}\not{A}'(\Phi)\right\}e^{-2isP_\tau P_\phi} \\ &= -iL\left\{[\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi + 2sP_\tau)]^2 + m^2 + ie\not{n}\not{A}'(\Phi + 2sP_\tau)\right\}\end{aligned} \tag{2.116}$$

where in the second line the shift formula $\exp(2isP_\phi P_\tau) f(\Phi)\exp(-2isP_\phi P_\tau) = f(\Phi + 2sP_\tau)$ for an arbitrary function $f(\Phi)$ was used. The solution of this differential equation with initial condition $L(0) = 1$ is [Pia18]

$$L(s) = e^{-i\int_0^s ds'\left\{[\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi + 2s'P_\tau)]^2 + m^2 + ie\not{n}\not{A}'(\Phi + 2s'P_\tau)\right\}}. \tag{2.117}$$

Then, plugging in 2.113, Taylor expanding and using that $\hat{n}\hat{A}(\Phi)\not{n}\not{A}(\Phi') = 0$,

$$\begin{aligned}\frac{1}{\not{\Pi}^2 - m^2 + i0} &= (-i)\int_0^\infty ds e^{-im^2 s}\left\{1 + \frac{e}{2P_T}\not{n}\left[\not{A}(\Phi + 2sP_T) - \not{A}(\Phi)\right]\right\} \\ &\quad\times e^{-i\int_0^s ds'[\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi + 2s'P_T)]^2} e^{2isP_T P_\phi},\end{aligned} \tag{2.118}$$

such that $\hat{G}$ can be written in the form [Pia18; PL20]

$$\begin{aligned}\hat{G} =&(-i)\int_0^\infty ds e^{-im^2 s}\left\{1 + \frac{e}{2P_\tau}\not{n}\left[\not{A}(\Phi + 2sP_\tau) - \not{A}(\Phi)\right]\right\} \\ &\times e^{-i\int_0^s ds'[\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi + 2s'P_\tau)]^2} e^{2isP_\tau P_\phi}(\not{\Pi} + m),\end{aligned} \tag{2.119}$$

[13] See the list of notations and conventions

or, equivalently [Pia18],

$$\hat{G} =(-i)(\not{\Pi} + m) \times \int_0^\infty ds e^{-im^2 s} e^{2isP_\tau P_\phi} e^{-i\int_0^s ds' [\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi - 2s'P_\tau)]^2} \times \left\{1 - \frac{e}{2P_\tau}\not{n}\left[\not{A}(\Phi - 2sP_\tau) - \not{A}(\Phi)\right]\right\}. \tag{2.120}$$

2.4.3 Effective (dressed) momentum space vertex

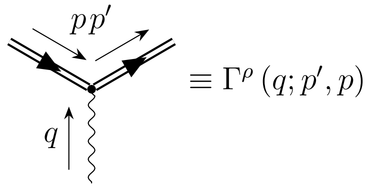

Figure 2.1: Effective strong field QED vertex in momentum space, corresponding to an incoming electron with momentum p, an exchanged photon with momentum q, and an outgoing electron with momentum p'.

In momentum space, the free QED vertex $-ie\gamma^\mu$ gets modified to the effective vertex function $\Gamma^\rho(q; p', p)$ (see Figure 2.1), becoming "dressed" by the Ritus matrices $E(p, x)$ and acquiring the following form [Rit72; Mit75; Pia18; Meu15]

$$\Gamma^\rho(q; p', p) = -ie \int d^4x e^{-iqx} \bar{E}(p', x)\gamma^\rho E(p, x). \tag{2.121}$$

2.4.4 Photon propagator

The photon propagator in the Feynman gauge $-iD_{\mu\nu}(x - y)$ is given by

$$-iD_{\mu\nu}(x - y) = -i \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{\eta_{\mu\nu}}{q^2 - \lambda^2 + i0} \tag{2.122}$$

where λ^2 is the square of a fictitious photon mass, introduced to avoid infrared divergences.

3

One-loop electron mass operator in a plane-wave background field

In vacuum QED the magnetic moment of the electron is extracted from the vertex diagram, as in Figure 3.1, by constraining the form of the vertex function using Ward identities [Mic19]. However, in strong field QED, the magnetic field of the intense plane wave background field can be used instead, leading to the mass operator diagram as the one relevant for extracting the magnetic moment. This can be seen from the Pauli interaction term in 2.114, coming from the Volkov propagator.

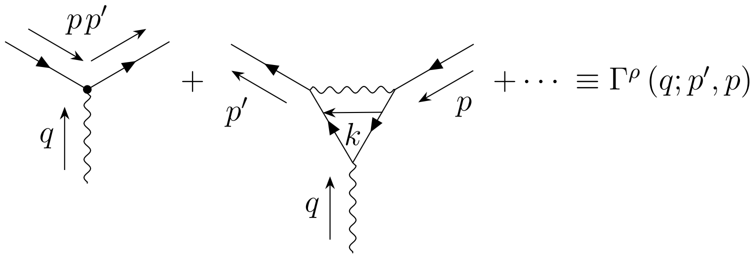

Figure 3.1: Vertex function in vacuum QED for an electron with incoming momentum p and outgoing momentum p', with the magnetic field provided by the external virtual photon of momentum q. The tree-level and 1-loop radiative corrections are shown.

3.1 The mass operator for the off-shell electron

Using the configuration space Feynman rules for strong field QED (i.e. in this case replacing the external Dirac spinors with Volkov states and the propagator with the Volkov propagator), the matrix amplitude for the one-loop diagram 3.2 contributing to the mass operator for an incoming electron with momentum p and spin σ, which exits with momentum p' and spin σ', is given by

$$\begin{aligned}-i\mathcal{M}_{\sigma'\sigma}(p',p) &= \int d^4x d^4y\ (-iD_{\mu\nu}(x-y))\bar{U}_{\sigma'}(p',y)(-ie\gamma^{\nu})(iG(y,x))(-ie\gamma^{\mu})U_{\sigma}(p,x)\\ &= -e^2\int d^4x d^4y\ D_{\mu\nu}(x-y)\bar{U}_{\sigma'}(p',y)\gamma^{\nu}G(y,x)\gamma^{\mu}U_{\sigma}(p,x),\end{aligned}$$

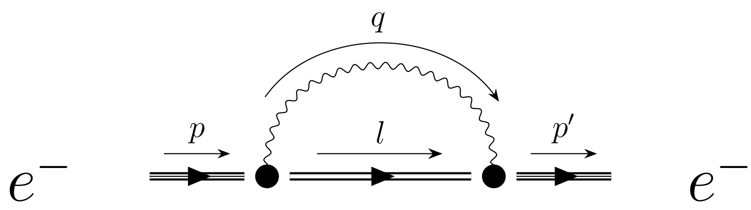

Figure 3.2: One-loop mass operator for an electron in an external field. The electron has incoming momentum p, outgoing momentum p' (it can exchange momentum with the background field), the electron Volkov propagator corresponds to the momentum l and the exchanged photon has loop momentum q.

$$\tag{3.1}$$

where $D_{\mu\nu}$ is the photon propagator, and $U_\sigma(p,x)$ (see Equation 2.98) and $\bar{U}_\sigma(p,x)$ is the Volkov spinor and its Dirac conjugate, respectively.

Equivalently, the matrix element can be obtained from the Feynman rules for strong field QED in momentum space [MMF]

$$-i\mathcal{M}_{\sigma'\sigma}(p',p) = \bar{u}_{\sigma'}(p') \underbrace{\left[\int \frac{d^4l}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\Gamma^\mu\left(q;p',l\right)\left(iG(l)\right)\Gamma^\nu(-q;l,p)\left(-iD_{\mu\nu}(q)\right)\right]}_{\equiv -iM(p',p)} u_\sigma(p), \tag{3.2}$$

where $\Gamma^\rho\left(q;p',p\right)$ is the effective vertex function (see Equation 2.121). The advantage of obtaining the mass operator $-iM(p',p)$ in momentum space, as in Equation 3.2, comes from the fact that it can be readily evaluated by sandwiching between free-electron spinors and putting the spinors on-shell. However, although using the momentum space SFQED Feynman rules leads to the same result, the configuration space computation is provided next.

Writing out the Volkov state $U_\sigma(p,x) = \langle x|p\sigma\rangle$[1] and the explicit expression for the photon propagator and the electron propagator, we get in configuration space, in operatorial form

$$\begin{aligned}
-i\mathcal{M}_{\sigma'\sigma}(p',p) &= -e^2 \int d^4x d^4y \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x-y)}}{q^2-\lambda^2+i0}\left\langle p'\sigma'|y\right\rangle \gamma^0\gamma_\mu \left\langle y|G|x\right\rangle \gamma^\mu \left\langle x|p\sigma\right\rangle \\
&= -e^2 \int_x\int_y\int_q \frac{1}{q^2-\lambda^2+i0}\langle p'\sigma'|\gamma^0\gamma_\mu|y\rangle\langle y|e^{iqX}G\gamma^\mu e^{-iqX}|x\rangle\langle x|p\sigma\rangle \\
&= -e^2 \int_q \frac{1}{q^2-\lambda^2+i0}\langle p'\sigma'|\gamma^0\gamma_\mu e^{iqX}Ge^{-iqX}\gamma^\mu|ps\rangle,
\end{aligned} \tag{3.3}$$

where in the last line the integral over x and y was carried out by using the completeness relation, and the shorthand notation for the integral measure was used (which will be used throughout the rest of the computation to simplify the typesetting).

[1] the Volkov state $|p\sigma\rangle$ should not be confused with the momentum operator eigenstate.

Shifting the propagator $e^{iqX}G(P)e^{-iqX} = G(P+q)$ from expression 2.119

$$e^{iqX}Ge^{-iqX} = (-i)\int_0^\infty dse^{-im^2s}\left\{1+\frac{e}{2(P+q)_T}\hat{n}\left[\hat{A}\left(\Phi+2s(P+q)_\tau\right)-\hat{A}(\Phi)\right]\right\}$$
$$\times\, e^{-i\int_0^s ds'\left[(\boldsymbol{P}+\boldsymbol{q})_\perp - e\boldsymbol{A}_\perp(\Phi+2s'(P+q)_\tau)\right]^2} e^{2is(P+q)_\tau(P+q)_\phi}(\hat{\Pi}+q+m), \tag{3.4}$$

where $q_\tau = -q_-$ is the eigenvalue of the P_τ operator acting on a state $|q\rangle$ ($P_\tau|q\rangle = -q_-|q\rangle$), and similarly $q_\phi = -q_+$. Noting that for an off-shell incoming particle, the operator acting on $|p\sigma\rangle$, (using $\gamma^\mu\hat{q} = 2q^\mu - \hat{q}\gamma^\mu$) can be written as

$$\left(\hat{\Pi}+\hat{q}+m\right)\gamma^\mu = \left(2\left(\Pi+q\right)^\mu - \gamma^\mu\left(\hat{\Pi}+\hat{q}-m\right)\right) = \left(2\Pi^\mu - \gamma^\mu\left(\hat{\Pi}-m\right)+\hat{q}\gamma^\mu\right), \tag{3.5}$$

the matrix element, writing the $\Pi(\Phi)$-dependence explicitly, becomes

$$\begin{aligned}
-i\mathcal{M}_{\sigma'\sigma}(p',p) &= ie^2\int_q\int_s \frac{e^{-im^2s}}{q^2-\lambda^2+i0}\langle p'\sigma'|\gamma^0\gamma_\mu \\
&\quad\times\left\{1+\frac{e}{2(P+q)_\tau}\hat{n}\left[\hat{A}\left(\Phi+2s(P+q)_\tau\right)-\hat{A}(\Phi)\right]\right\} \\
&\quad\times e^{-i\int_0^s ds'\left[(\boldsymbol{P}+\boldsymbol{q})_\perp - e\boldsymbol{A}_\perp(\Phi+2s'(P+q)_\tau)\right]^2} e^{2is(P+q)_\tau(P+q)_\phi} \\
&\quad\times\left[2\Pi^\mu(\Phi)-\gamma^\mu\left(\hat{\Pi}(\Phi)-m\right)+\hat{q}\gamma^\mu\right]|p\sigma\rangle \\
&= ie^2\int_q\int_s\int_x \frac{e^{-im^2s}}{q^2-\lambda^2+i0}\bar{U}_{\sigma'}(p',x)\gamma_\mu\langle x| \\
&\quad\times\left\{1+\frac{e}{2(P+q)_\tau}\hat{n}\left[\hat{A}\left(\Phi+2s(P+q)_\tau\right)-\hat{A}(\Phi)\right]\right\} \\
&\quad\times e^{-i\int_0^s ds'\left[(\boldsymbol{P}+\boldsymbol{q})_\perp - e\boldsymbol{A}_\perp(\Phi+2s'(P+q)_\tau)\right]^2} e^{2is(P+q)_\tau(P+q)_\phi} \\
&\quad\times\left[2\Pi^\mu(\Phi)-\gamma^\mu\left(\hat{\Pi}(\Phi)-m\right)+\hat{q}\gamma^\mu\right]|p\sigma\rangle,
\end{aligned} \tag{3.6}$$

where in the second step a resolution of the identity over x was inserted to the left to reconstruct the Volkov states.

Writing the state $\langle x| = \langle\phi,\tau,\boldsymbol{x}_\perp|$ and replacing $q_\tau = -q_-$ and $q_\phi = -q_+$

$$\begin{aligned}
-i\mathcal{M}_{\sigma'\sigma}(p',p) &= ie^2\int_q\int_s\int_x \frac{e^{-im^2s}}{q^2-\lambda^2+i0}\bar{U}_{\sigma'}(p',x)\gamma_\mu\langle\phi,\tau,\boldsymbol{x}_\perp| \\
&\quad\times\left\{1+\frac{e}{2(P_\tau-q_-)}\hat{n}\left[\hat{A}\left(\phi+2s(P_\tau-q_-)\right)-\hat{A}(\phi)\right]\right\} \\
&\quad\times e^{-i\int_0^s ds'\left[(\boldsymbol{P}+\boldsymbol{q})_\perp - e\boldsymbol{A}_\perp(\phi+2s'(P_\tau-q_-))\right]^2} e^{2is(P_\tau-q_-)(P_\phi-q_+)} \\
&\quad\times\left[2\Pi^\mu(\Phi)-\gamma^\mu\left(\hat{\Pi}(\Phi)-m\right)+\hat{q}\gamma^\mu\right]|p\sigma\rangle.
\end{aligned} \tag{3.7}$$

Acting with Φ on it (to the left), i.e. $\langle\phi,\tau,\boldsymbol{x}_\perp|\Phi = \langle\phi,\tau,\boldsymbol{x}_\perp|\phi$, while at the same time acting on the $|p\sigma\rangle$ Volkov state with the projectors, i.e. by using $P_\tau|p\sigma\rangle = -p_-|p\sigma\rangle$, $\boldsymbol{P}_\perp|p\sigma\rangle = \boldsymbol{p}_\perp|p\sigma\rangle$ (notice however that the Volkov ket state $|p\sigma\rangle$ is not an eigenstate of P_ϕ, from 2.104), the matrix element is

$$
\begin{aligned}
-i\mathcal{M}_{\sigma'\sigma}(p',p) = ie^2 \int_q\int_s\int_x & \frac{e^{-im^2 s}}{q^2-\lambda^2+i0}\bar{U}_{\sigma'}(p',x)\gamma_\mu \\
& \times \left\{1 - \frac{e}{2(p_-+q_-)}\hat{n}\left[\hat{A}\left(\phi - 2s(p_-+q_-)\right) - \hat{A}(\phi)\right]\right\} \\
& \times e^{-i\int_0^s ds' \left[(\boldsymbol{p}+\boldsymbol{q})_\perp - e\boldsymbol{A}_\perp(\phi - 2s'(p_-+q_-))\right]^2} \\
& \times \langle \phi, \tau, \boldsymbol{x}_\perp | e^{-2is(p_-+q_-)(P_\phi - q_+)}\left[2\Pi^\mu(\Phi) - \gamma^\mu\left(\hat{\Pi}(\Phi) - m\right) + \hat{q}\gamma^\mu\right] |p\sigma\rangle,
\end{aligned} \tag{3.8}
$$

where the bra state $\langle x| = \langle \phi, \tau, \boldsymbol{x}_\perp|$ was also passed through the numbers.

Inserting the identity ($\mathbb{1} = e^{2is(p_-+q_-)P_\phi}e^{-2is(p_-+q_-)P_\phi}$) to the left of $|ps\rangle$ in order to use the shift formula, the matrix element is

$$
\begin{aligned}
& -i\mathcal{M}_{\sigma'\sigma}(p',p) \\
& = ie^2 \int_q\int_s\int_x \frac{e^{-im^2 s}}{q^2-\lambda^2+i0}\bar{U}_{\sigma'}(p',x)\gamma_\mu \\
& \times \left\{1 - \frac{e}{2(p_-+q_-)}\hat{n}\left[\hat{A}\left(\phi + 2s(-p_-+q_T)\right) - \hat{A}(\phi)\right]\right\} \\
& \times e^{-i\int_0^s ds' \left[(\boldsymbol{p}+\boldsymbol{q})_\perp - e\boldsymbol{A}_\perp(\phi - 2s'(p_-+q_-))\right]^2} \\
& \times \langle \phi, \tau, \boldsymbol{x}_\perp | e^{2is(p_-+q_-)q_+} \underbrace{e^{-2is(p_-+q_-)P_\phi}\left[\hat{\Pi}(\phi) + \hat{q} + m\right]e^{2is(p_-+q_-)P_\phi}}_{\text{use shift formula}} e^{-2is(p_-+q_-)P_\phi}\gamma^\mu |p\sigma\rangle,
\end{aligned} \tag{3.9}
$$

where in the last line we can use $\exp(isP_\phi)\, f(\phi)\exp(-isP_\phi) = f(\phi+s)$ for

$$
e^{-2is(p_-+q_-)P_\phi}\left[\hat{\Pi}(\phi) + \hat{q} + m\right]e^{2is(p_-+q_-)P_\phi} = \hat{\Pi}\left(\phi - 2s(p_-+q_-)\right) + \hat{q} + m, \tag{3.10}
$$

Pulling the bra $\langle \phi, \tau, \boldsymbol{x}_\perp|$ through to form the Volkov state $U_\sigma(p,x)$, adding an exponential $e^{-2is(-p_--q_-)(-i\partial_\phi)}1 = 1$ acting on the Volkov state from the right and again using the ϕ-shift formula, the matrix element becomes

$$
\begin{aligned}
-i\mathcal{M}_{\sigma'\sigma}(p',p) = ie^2 \int_q\int_s\int_x & \frac{e^{-im^2 s}}{q^2-\lambda^2+i0}\bar{U}_{\sigma'}(p',x)\gamma_\mu \\
& \times \left\{1 - \frac{e}{2(p_-+q_-)}\hat{n}\left[\hat{A}\left(\phi - 2s(p_-+q_-)\right) - \hat{A}(\phi)\right]\right\} \\
& \times e^{-i\int_0^s ds' \left[(\boldsymbol{p}+\boldsymbol{q})_\perp - e\boldsymbol{A}_\perp(\phi - 2s'(p_-+q_-))\right]^2} \times e^{2is(p_-+q_-)q_+} \\
& \times \left[2\Pi^\mu(\phi - 2s(p_-+q_-)) - \gamma^\mu\left(\hat{\Pi}(\phi - 2s(p_-+q_-)) - m\right) + \hat{q}\gamma^\mu\right] \\
& \times U_\sigma\left(p, \phi - 2s(p_-+q_-), \tau, \boldsymbol{x}_\perp\right).
\end{aligned} \tag{3.11}
$$

Denoting $\phi_s \equiv \phi - 2s(p_- + q_-)$ and writing in terms of $\mathcal{A}(\phi) \equiv eA(\phi)$, using Corrolary A.1 and the ϕ-evolved state from Theorem A.2 to get

$$\Pi^\mu(\phi_s)U_\sigma(p,\phi_s,\tau,\boldsymbol{x}_\perp) = \left[\pi_p^\mu(\phi_s) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi_s)}{2p_-}n^\mu\right] M(\phi_s,\phi)U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp), \quad (3.12)$$

where $\pi_p^\mu(\phi)$ is the dressed momentum from 2.72 with $\lim_{\phi\to\pm\infty}\pi_p^\mu(\phi) = p^\mu$, the matrix element can be expressed as

$$\begin{aligned}
-i\mathcal{M}_{\sigma'\sigma}(p',p) = ie^2 &\int_q\int_s\int_x \frac{e^{-im^2 s}}{q^2-\lambda^2+i0}\bar{U}_{\sigma'}(p',x)\gamma_\mu \\
&\times \left\{1 - \frac{1}{2(p_-+q_-)}\hat{n}\left[\hat{\mathcal{A}}\left(\phi_s\right) - \hat{\mathcal{A}}(\phi)\right]\right\} \\
&\times e^{-i\int_0^s ds'\left[(\boldsymbol{p}+\boldsymbol{q})_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2} \times e^{2is(p_-+q_-)q_+} \\
&\times \left[2\left(\pi_p^\mu(\phi_s) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi_s)}{2p_-}n^\mu\right) - \gamma^\mu\left(\hat{\Pi}(\phi_s) - m\right) + \hat{q}\gamma^\mu\right] \\
&\times U_\sigma\left(p,\phi_s,\tau,\boldsymbol{x}_\perp\right)
\end{aligned} \quad (3.13)$$

Temporarily writing the matrix element as in 3.2, i.e.

$$-i\mathcal{M}_{\sigma'\sigma}(p',p) = \bar{u}_{\sigma'}(p')(-iM(p',p))u_\sigma(p), \quad (3.14)$$

after which the sandwich between free Dirac spinors will be taken, we get that

$$\begin{aligned}
-iM(p',p) = ie^2 &\int_q\int_s\int_x \frac{e^{-im^2 s}}{q^2-\lambda^2+i0}\bar{E}(p',x)\gamma_\mu \\
&\times \left\{1 - \frac{1}{2(p_-+q_-)}\hat{n}\left[\hat{\mathcal{A}}\left(\phi_s\right) - \hat{\mathcal{A}}(\phi)\right]\right\} \\
&\times e^{-i\int_0^s ds'\left[(\boldsymbol{p}+\boldsymbol{q})_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2} \times e^{2is(p_-+q_-)q_+} \\
&\times \left[2\left(\pi_p^\mu(\phi_s) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi_s)}{2p_-}n^\mu\right) - \gamma^\mu\left(\hat{\Pi}(\phi_s) - m\right) + \hat{q}\gamma^\mu\right] \\
&\times E\left(p,\phi_s,\tau,\boldsymbol{x}_\perp\right)
\end{aligned} \quad (3.15)$$

Using Theorem A.1 and the Schwinger parametrization (see Appendix C.2) for the photon propagator, i.e.

$$\frac{1}{q^2-\lambda^2+i0} = -i\int_0^\infty du e^{iu(q^2-\lambda^2)}, \quad (3.16)$$

the matrix element becomes

$$\begin{aligned}
-iM(p',p) = ie^2 &\int_q\int_s\int_x\int_0^\infty du e^{iu\left(q^2-\lambda^2\right)} e^{-im^2 s}\bar{E}(p',x)\gamma_\mu \\
&\times \left\{1 - \frac{1}{2(p_-+q_-)}\hat{n}\left[\hat{\mathcal{A}}\left(\phi_s\right) - \hat{\mathcal{A}}(\phi)\right]\right\} \\
&\times e^{-i\int_0^s ds'\left[(\boldsymbol{p}+\boldsymbol{q})_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2} \times e^{2is(p_-+q_-)q_+} \\
&\times \left\langle\left[2\left(\pi_p^\mu(\phi_s) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi_s)}{2p_-}n^\mu\right) + \hat{q}\gamma^\mu\right] M(\phi_s,\phi)E\left(p,x\right) + \gamma^\mu M(\phi_s,\phi)E(p,x)(m-\hat{p})\right\rangle
\end{aligned} \quad (3.17)$$

Inserting $q^2 = 2q_+q_- - (\boldsymbol{q}_\perp)^2$ to separate the q_+ exponentials in order to integrate and expanding the $M(\phi_s, \phi)$ from Equation A.8

$$
\begin{aligned}
-iM(p',p) = e^2 \int_q \int_s \int_x \int_u & e^{-im^2 s} e^{-iu\left[(\boldsymbol{q}_\perp)^2+\lambda^2\right]} e^{2i[sp_- + (u+s)q_-]q_+} \\
& \times \bar{E}(p',x) \times e^{-i\int_0^s ds' \left[(\boldsymbol{p}+\boldsymbol{q})_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2} e^{-i\int_\phi^{\phi_s} d\tilde{\phi} \left(p_+ + \frac{p\cdot\mathcal{A}(\tilde{\phi})}{p_-} - \frac{\mathcal{A}^2(\tilde{\phi})}{2p_-}\right)} \\
& \times \gamma_\mu \left\{ 1 - \frac{1}{2(p_- + q_-)} \hat{n} \left[\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right] \right\} \\
& \times \left\langle \left[2\left(\pi_p^\mu(\phi_s) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi_s)}{2p_-} n^\mu \right) + \hat{q}\gamma^\mu \right] \left[1 + \frac{1}{2p_-}\hat{n}\left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right) \right] E(p,x) \right. \\
& \left. + \gamma^\mu \left[1 + \frac{1}{2p_-}\hat{n}\left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right) \right] E(p,x)\,(m - \hat{p}) \right\rangle .
\end{aligned}
\tag{3.18}
$$

Expanding the brackets using the identities $\gamma_\mu\gamma^\mu = 4\mathbb{1}$, $\gamma_\mu\hat{q}\gamma^\mu = -2\hat{q}$, and $\gamma_\mu\hat{n}\hat{\mathcal{A}}\gamma^\mu = 4(n\cdot\mathcal{A}) = 0$, where $n\cdot\mathcal{A} = 0$ is the Lorentz gauge condition

$$
\begin{aligned}
&\gamma_\mu \left[1 - \frac{1}{2(p_- + q_-)}\hat{n}\left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right) \right] \left[2\left(\pi_p^\mu(\phi_s) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi_s)}{2p_-} n^\mu \right) + \hat{q}\gamma^\mu \right] = \\
&= 2\hat{\pi}_p(\phi_s) - 2\hat{q} - \frac{1}{p_- + q_-}\hat{\pi}_p(\phi_s)\hat{n}\left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right) - \frac{1}{2(p_- + q_-)}\gamma_\mu\hat{n}\left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right)\hat{q}\gamma^\mu \\
&= 2\left(\hat{\pi}_p(\phi_s) - \hat{q}\right) - \frac{1}{p_- + q_-}\hat{\pi}_p(\phi_s)\hat{n}\left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right) + \frac{1}{p_- + q_-}\hat{q}\left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right)\hat{n} \\
&= 2\left(\hat{\pi}_p(\phi_s) - \hat{q}\right) - \frac{1}{p_- + q_-}\left(\hat{\pi}_p(\phi_s) + \hat{q}\right)\hat{n}\left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right),
\end{aligned}
\tag{3.19}
$$

and changing variables $\tilde{\phi} = \phi - 2s'(p_- + q_-) \mapsto s'$ into the integral, with s' going from 0 to s, the exponential becomes

$$
-i\int_\phi^{\phi_s} d\tilde{\phi}\left(p_+ + \frac{p\cdot\mathcal{A}(\tilde{\phi})}{p_-} - \frac{\mathcal{A}^2(\tilde{\phi})}{2p_-}\right) = 2i(p_- + q_-)\int_0^s ds'\left(p_+ + \frac{p\cdot\mathcal{A}(\phi_{s'})}{p_-} - \frac{\mathcal{A}^2(\phi_{s'})}{2p_-}\right) \tag{3.20}
$$

the matrix element acquires the following form

$$
\begin{aligned}
&- iM(p',p) \\
&= e^2 \int_q \int_s \int_x \int_u e^{-im^2 s} e^{-iu\left[(\boldsymbol{q}_\perp)^2+\lambda^2\right]} e^{2i[sp_- + (u+s)q_-]q_+} \\
&\times \bar{E}(p',x) \\
&\times e^{\left\{ 2i(p_- + q_-)\int_0^s ds'\left(p_+ + \frac{p\cdot\mathcal{A}(\phi_{s'})}{p_-} - \frac{\mathcal{A}^2(\phi_{s'})}{2p_-}\right) - i\int_0^s ds'\left[(\boldsymbol{p}+\boldsymbol{q})_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 \right\}} \\
&\times \left\langle \left[2\left(\hat{\pi}_p(\phi_s) - \hat{q}\right) - \frac{1}{p_- + q_-}\left(\hat{\pi}_p(\phi_s) + \hat{q}\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s) \right] \left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s) \right] E(p,x) \right. \\
&\left. + 4\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s) \right] E(p,x)\,(m - \hat{p}) \right\rangle \\
&\times e^{2i[sp_- + (u+s)q_-]q_+},
\end{aligned}
\tag{3.21}
$$

where it was denoted $\boxed{\Delta\mathcal{A}(\phi_s) \equiv \mathcal{A}(\phi_s) - \mathcal{A}(\phi)}$.

Decomposing 3.21 on the bispinor basis, i.e $\hat{q} = \gamma^+ q_+ + \gamma^- q_- + \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp$

$$
\begin{aligned}
-iM(p', p) = e^2 \int_q \int_s \int_x \int_u & e^{-im^2 s} e^{-iu\left[(\boldsymbol{q}_\perp)^2 + \lambda^2\right]} \\
& \times \bar{E}(p', x) \\
& \times e^{\left\{2i(p_- + q_-)\int_0^s ds' \left(p_+ + \frac{p\cdot\mathcal{A}(\phi_{s'})}{p_-} - \frac{\mathcal{A}^2(\phi_{s'})}{2p_-}\right) - i\int_0^s ds' \left[(\boldsymbol{p}+\boldsymbol{q})_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2\right\}} \\
& \times \Big\langle \left[2\left(\hat{\pi}_p(\phi_s) - \gamma^+ q_+ - \gamma^- q_- - \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp\right)\right. \\
& \left. - \frac{1}{p_- + q_-}\left(\hat{\pi}_p(\phi_s) + \gamma^+ q_+ + \gamma^- q_- + \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp\right) \hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\
& \left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] E(p, x) + 4\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] E(p, x)(m - \hat{p}) \Big\rangle \\
& \times e^{2i[sp_- + (u+s)q_-]q_+}.
\end{aligned}
\tag{3.22}
$$

We can bring down q_+ by taking the ∂_u derivative, i.e.

$$
\begin{aligned}
q_+ e^{2i[sp_- + (u+s)q_-]q_+} &= \frac{-i}{2q_-}\partial_u e^{2i[sp_- + (u+s)q_-]q_+}, \\
&= \frac{-i}{2(u+s)}\partial_{q_-} e^{2i[sp_- + (u+s)q_-]q_+},
\end{aligned}
\tag{3.23}
$$

or alternatively, by taking the ∂_{q_-} derivative, as in the second line.

Taking the ∂_{q_-} derivative, we obtain

$$
\begin{aligned}
-iM(p', p) = e^2 \int_q \int_s \int_x \int_u & e^{-im^2 s} e^{-iu\left[(\boldsymbol{q}_\perp)^2 + \lambda^2\right]} \\
& \times \bar{E}(p', x) \\
& \times e^{\left\{2i(p_- + q_-)\int_0^s ds' \left(p_+ + \frac{p\cdot\mathcal{A}(\phi_{s'})}{p_-} - \frac{\mathcal{A}^2(\phi_{s'})}{2p_-}\right) - i\int_0^s ds' \left[(\boldsymbol{p}+\boldsymbol{q})_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2\right\}} \\
& \times \left\langle \left[2\left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)}\gamma^+ \partial_{q_-} - \gamma^- q_- - \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp\right)\right.\right. \\
& \left. - \frac{1}{p_- + q_-}\left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)}\gamma^+ \partial_{q_-} + \gamma^- q_- + \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp\right) \hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\
& \left. \times \left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] E(p, x) + 4\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] E(p, x)(m - \hat{p}) \right\rangle \\
& \times e^{2i[sp_- + (u+s)q_-]q_+}.
\end{aligned}
\tag{3.24}
$$

Since the q_- derivative will act on the phase, it is useful to put the phase in a more transparent form.

The phase can be written as

$$
\begin{aligned}
&2i\left(p_-+q_-\right)\int_0^s ds'\left(p_++\frac{p\cdot\mathcal{A}(\phi_{s'})}{p_-}-\frac{\mathcal{A}^2\left(\phi_{s'}\right)}{2p_-}\right)-i\int_0^s ds'\left[\left(\boldsymbol{p}+\boldsymbol{q}\right)_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2\\
&=2is\left(p_-+q_-\right)p_++2i\frac{p_-+q_-}{p_-}\int_0^s ds'\left(p\cdot\mathcal{A}(\phi_{s'})-\frac{\mathcal{A}^2\left(\phi_{s'}\right)}{2}\right)-i\int_0^s ds'\left[\left(\boldsymbol{p}+\boldsymbol{q}\right)_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2\\
&=2is\left(p_-+q_-\right)p_++2i\left(1+\frac{q_-}{p_-}\right)\int_0^s ds'\left(-\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)+\frac{\boldsymbol{\mathcal{A}}_\perp^2\left(\phi_{s'}\right)}{2}\right)-i\int_0^s ds'\left[\left(\boldsymbol{p}+\boldsymbol{q}\right)_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2\\
&=2is\left(p_-+q_-\right)p_+-2i\left(1+\frac{q_-}{p_-}\right)\int_0^s ds'\left(\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)-\frac{\boldsymbol{\mathcal{A}}_\perp^2\left(\phi_{s'}\right)}{2}\right)-i\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2-is\left(\boldsymbol{q}_\perp\right)^2\\
&-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]\\
&=2is\left(p_-+q_-\right)p_+-i\left(1+\frac{q_-}{p_-}\right)\int_0^s ds'\left(2\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)-\boldsymbol{\mathcal{A}}_\perp^2\left(\phi_{s'}\right)\right)-i\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2-is\left(\boldsymbol{q}_\perp\right)^2\\
&-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]\\
&=2is\left(p_-+q_-\right)p_+-i\left(1+\frac{q_-}{p_-}\right)\int_0^s ds'\left(2\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)-\boldsymbol{\mathcal{A}}_\perp^2\left(\phi_{s'}\right)\right)-is(\boldsymbol{p}_\perp)^2-i\int_0^s ds'\left[\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2\\
&+2i\boldsymbol{p}_\perp\int_0^s ds'\left[\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]-is\left(\boldsymbol{q}_\perp\right)^2-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right].
\end{aligned}
\tag{3.25}
$$

Continuing the manipulation of the phase exponential

$$
\begin{aligned}
&2i\left(p_-+q_-\right)\int_0^s ds'\left(p_++\frac{p\cdot\mathcal{A}(\phi_{s'})}{p_-}-\frac{\mathcal{A}^2\left(\phi_{s'}\right)}{2p_-}\right)-i\int_0^s ds'\left[\left(\boldsymbol{p}+\boldsymbol{q}\right)_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2\\
&=2is\left(p_-+q_-\right)p_+-i\frac{q_-}{p_-}\int_0^s ds'\left(2\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)-\boldsymbol{\mathcal{A}}_\perp^2\left(\phi_{s'}\right)\right)-is(\boldsymbol{p}_\perp)^2-is\left(\boldsymbol{q}_\perp\right)^2\\
&-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]\\
&=2is\left(p_-+q_-\right)p_+-i\frac{q_-}{p_-}\int_0^s ds'\left(2\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)-\boldsymbol{\mathcal{A}}_\perp^2\left(\phi_{s'}\right)\right)-2isp_-\frac{\left(\boldsymbol{p}_\perp\right)^2}{2p_-}-is\left(\boldsymbol{q}_\perp\right)^2\\
&-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]\\
&=2is\left(p_-+q_-\right)p_+-i\frac{q_-}{p_-}\int_0^s ds'\left(2\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)-\boldsymbol{\mathcal{A}}_\perp^2\left(\phi_{s'}\right)\right)-2isp_-\frac{\left(\boldsymbol{p}_\perp\right)^2}{2p_-}-2is\frac{q_-}{2p_-}\left(\boldsymbol{p}_\perp\right)^2+2is\frac{q_-}{2p_-}\left(\boldsymbol{p}_\perp\right)^2\\
&-is\left(\boldsymbol{q}_\perp\right)^2-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]\\
&=2is\left(p_-+q_-\right)p_+-i\frac{q_-}{p_-}\int_0^s ds'\left(2\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)-\left(\boldsymbol{p}_\perp\right)^2-\boldsymbol{\mathcal{A}}_\perp^2(\phi_{s'})\right)-2is(p_-+q_-)\frac{\left(\boldsymbol{p}_\perp\right)^2}{2p_-}-is\left(\boldsymbol{q}_\perp\right)^2\\
&-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]\\
&=2is\left(p_-+q_-\right)\left(p_+-\frac{\left(\boldsymbol{p}_\perp\right)^2}{2p_-}\right)+i\frac{q_-}{p_-}\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2-is\left(\boldsymbol{q}_\perp\right)^2-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]\\
&=is\left(1+\frac{q_-}{p_-}\right)p^2+i\frac{q_-}{p_-}\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2-2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]-is\left(\boldsymbol{q}_\perp\right)^2,
\end{aligned}
\tag{3.26}
$$

where in the last line $p^2=2p_+p_--\left(\boldsymbol{p}_\perp\right)^2 \implies p^2/2p_-=p_+-\left(\boldsymbol{p}_\perp\right)^2/2p_-$ was used.

Integrating the exponential $e^{2i[sp_-+(u+s)q_-]q_+}$ over $\frac{dq_+}{2\pi}$ to give $\delta\left(2\left[sp_-+(u+s)q_-\right]\right)$

and inserting the phase from 3.26

$$
\begin{aligned}
-iM(p',p) = \frac{e^2}{2}\int_{q/q_+}\int_s\int_x\int_u & e^{-im^2 s}e^{-iu\left[(\boldsymbol{q}_\perp)^2+\lambda^2\right]} \\
& \times \bar{E}(p',x) \\
& \times e^{\left\{is\left(1+\frac{q_-}{p_-}\right)p^2+i\frac{q_-}{p_-}\int_0^s ds'[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp(\phi_{s'})]^2-2i\boldsymbol{q}_\perp\int_0^s ds'[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp(\phi_{s'})]-is(\boldsymbol{q}_\perp)^2\right\}} \\
& \times \left\langle\left[2\left(\hat{\pi}_p(\phi_s)+\frac{i}{2(u+s)}\gamma^+\partial_{q_-}-\gamma^- q_- -\boldsymbol{\gamma}^\perp\boldsymbol{q}_\perp\right)\right.\right. \\
& -\frac{1}{p_-+q_-}\left(\hat{\pi}_p(\phi_s)-\frac{i}{2(u+s)}\gamma^+\partial_{q_-}+\gamma^- q_- +\boldsymbol{\gamma}^\perp\boldsymbol{q}_\perp\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\Big] \\
& \times\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right)+4\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right)\left(m-\hat{p}\right)\Big\rangle \\
& \times\delta(sp_-+(u+s)q_-),
\end{aligned}
\tag{3.27}
$$

where $\delta\left(2\left[sp_-+(u+s)q_-\right]\right)=\frac{1}{2}\delta\left(sp_-+(u+s)q_-\right)$ was used to give an overall factor of 1/2 (and $\int_{q/q_+}$ denotes the 4-momentum integral without the integral $\int\frac{dq_+}{2\pi}$).

When g has a real root at x_0, we can use $\delta(g(x))=\frac{\delta(x-x_0)}{|g'(x_0)|}$, such that in our case

$$
\delta\left(sp_-+(u+s)q_-\right)=\frac{1}{|u+s|}\delta\left(q_--\left(\frac{-sp_-}{(u+s)}\right)\right)=\frac{1}{(u+s)}\delta\left(q_--\left(\frac{-sp_-}{(u+s)}\right)\right)\quad u,s\geq 0,
\tag{3.28}
$$

and inserting this into the matrix element

$$
\begin{aligned}
-iM(p',p) = \frac{e^2}{2}\int_{q/q_+}\int_s\int_x\int_u & \frac{1}{u+s}e^{-im^2 s}e^{-iu\left[(\boldsymbol{q}_\perp)^2+\lambda^2\right]} \\
& \times \bar{E}(p',x) \\
& \times e^{\left\{is\left(1+\frac{q_-}{p_-}\right)p^2+i\frac{q_-}{p_-}\int_0^s ds'[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp(\phi_{s'})]^2-2i\boldsymbol{q}_\perp\int_0^s ds'[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{A}}_\perp(\phi_{s'})]-is(\boldsymbol{q}_\perp)^2\right\}} \\
& \times \left\langle\left[2\left(\hat{\pi}_p(\phi_s)+\frac{i}{2(u+s)}\gamma^+\partial_{q_-}-\gamma^- q_- -\boldsymbol{\gamma}^\perp\boldsymbol{q}_\perp\right)\right.\right. \\
& -\frac{1}{p_-+q_-}\left(\hat{\pi}_p(\phi_s)-\frac{i}{2(u+s)}\gamma^+\partial_{q_-}+\gamma^- q_- +\boldsymbol{\gamma}^\perp\boldsymbol{q}_\perp\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\Big] \\
& \times\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right)+4\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right)\left(m-\hat{p}\right)\Big\rangle \\
& \times\delta\left(q_--\left(\frac{-sp_-}{(u+s)}\right)\right).
\end{aligned}
\tag{3.29}
$$

Using that for an arbitrary function $f(x)$ and for a real constant a,

$$
\int_{-\infty}^{\infty}\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-a)\right]f(x)dx=-\int_{-\infty}^{\infty}\delta(x-a)f'(x)dx,
\tag{3.30}
$$

the matrix element reduces to

$$
\begin{aligned}
-iM(p',p) = \frac{e^2}{2} \int_{q/q_+} \int_s \int_x \int_u & \frac{1}{u+s} e^{-im^2 s} e^{-iu\left[(\boldsymbol{q}_\perp)^2+\lambda^2\right]} \\
& \times \bar{E}(p',x) \\
& \times e^{\left\{is\left(1+\frac{q_-}{p_-}\right)p^2 + i\frac{q_-}{p_-}\int_0^s ds' [\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})]^2 - 2i\boldsymbol{q}_\perp \int_0^s ds' [\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})] - is(\boldsymbol{q}_\perp)^2\right\}} \\
& \times \left\langle \left[2\left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)}\gamma^+ \overleftarrow{\partial}_{q_-} - \gamma^- q_- - \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp \right) \right.\right. \\
& - \left.\frac{1}{p_-+q_-}\left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)}\gamma^+ \overleftarrow{\partial}_{q_-} + \gamma^- q_- + \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp \right) \hat{n}\widehat{\Delta\mathcal{A}}(\phi_s) \right] \\
& \times \left.\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] E(p,x) + 4\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] E(p,x)(m-\hat{p}) \right\rangle \\
& \times \delta\left(q_- - \left(\frac{-sp_-}{(u+s)}\right)\right),
\end{aligned}
\tag{3.31}
$$

where the arrow over the derivative indicates that the derivative acts on the left.

Taking into account that $\phi_s \equiv \phi - 2s(p_- + q_-)$ and that

$$
\left.\begin{aligned}
\partial_{q_-} &= \frac{d\phi_s}{dq_-}\partial_{\phi_s} = -2s\partial_{\phi_s} \\
\partial_{\phi_s} &= -\frac{1}{2(p_-+q_-)}\partial_s
\end{aligned}\right\} \implies \partial_{q_-} = \frac{s}{p_-+q_-}\partial_s. \tag{3.32}
$$

Denoting $\mathcal{A}'(\phi_s) \equiv \partial_{\phi_s}\mathcal{A}(\phi_s)$ the derivative with respect to the argument, the phase derivative can be written as

$$
\begin{aligned}
&\frac{\partial_{q_-} \exp\left\{is\left(1+\frac{q_-}{p_-}\right)p^2 + i\frac{q_-}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 - 2i\boldsymbol{q}_\perp \int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right] - is(\boldsymbol{q}_\perp)^2\right\}}{\exp\{\cdots\}} \equiv \Diamond \\
&= \left[is\frac{p^2}{p_-} + \frac{i}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 + 2i\frac{q_-}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]\left(-\boldsymbol{\mathcal{A}}'_\perp\right)(\phi_{s'})\left(-2s'\right) \right. \\
&\left. +2i\boldsymbol{q}_\perp \int_0^s ds' \boldsymbol{\mathcal{A}}'_\perp(\phi_{s'})\left(-2s'\right) \right] \\
&= i\left[s\frac{p^2}{p_-} + \frac{1}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 + 4\frac{q_-}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]\boldsymbol{\mathcal{A}}'_\perp(\phi_{s'})s' - 4\boldsymbol{q}_\perp \int_0^s ds' \boldsymbol{\mathcal{A}}'_\perp(\phi_{s'})s' \right] \\
&= i\left[s\frac{p^2}{p_-} + \frac{1}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 + 4\frac{1}{-2(p_-+q_-)}\frac{q_-}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]s'\partial_{s'}\boldsymbol{\mathcal{A}}_\perp(\phi_{s'}) \right. \\
&\left. -4\frac{1}{-2(p_-+q_-)}\boldsymbol{q}_\perp \int_0^s ds' \partial_{s'}\boldsymbol{\mathcal{A}}_\perp(\phi_{s'})s' \right] \\
&= i\left[s\frac{p^2}{p_-} + \frac{1}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 - \frac{2}{p_-+q_-}\frac{q_-}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]s'\partial_{s'}\boldsymbol{\mathcal{A}}_\perp(\phi_{s'}) \right. \\
&\left. +\frac{2}{p_-+q_-}\boldsymbol{q}_\perp \int_0^s ds' \partial_{s'}\boldsymbol{\mathcal{A}}_\perp(\phi_{s'})s' \right] \\
&= i\left[s\frac{p^2}{p_-} + \frac{1}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 + \frac{1}{p_-+q_-}\frac{q_-}{p_-}\int_0^s ds' s'\partial_{s'}\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 \right. \\
&\left. +\frac{2}{p_-+q_-}\boldsymbol{q}_\perp \int_0^s ds' \partial_{s'}\boldsymbol{\mathcal{A}}_\perp(\phi_{s'})s' \right] \\
&= i\left[s\frac{p^2}{p_-} + \frac{1}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 + \frac{1}{p_-+q_-}\frac{q_-}{p_-}s\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_s)\right]^2 - \frac{1}{p_-+q_-}\frac{q_-}{p_-}\int_0^s ds' \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})\right]^2 \right. \\
&\left. +\frac{2s}{p_-+q_-}\boldsymbol{q}_\perp \cdot \boldsymbol{\mathcal{A}}_\perp(\phi_s) - \frac{2}{p_-+q_-}\boldsymbol{q}_\perp \cdot \int_0^s ds' \boldsymbol{\mathcal{A}}_\perp(\phi_{s'}) \right],
\end{aligned}
$$

$$(3.33)$$

where the dots in the denominator of the first line indicate the argument of the same exponential as in the numerator, and where the quantity $\Diamond$ was defined.

Combininig the second and fourth term from expression 3.33

$$\Diamond = i\left[s\frac{p^2}{p_-} + \frac{1}{p_- + q_-}\int_0^s ds'\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2 + \frac{1}{p_- + q_-}\frac{q_-}{p_-}s\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right]^2 + \frac{2s}{p_- + q_-}\boldsymbol{q}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right. \\ \left. - \frac{2}{p_- + q_-}\boldsymbol{q}_\perp\cdot\int_0^s ds'\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right], \tag{3.34}$$

where the $\partial_{\phi'_s}$ derivative was changed into a $\partial_{s'}$ using Equation 3.32 and was integrated by parts. Taking the q_- derivative, and applying the product rule to the $1\ (p_- + q_-)$ term

$$\begin{aligned} -iM(p', p) = {} & \frac{e^2}{2}\int_{q/q_+}\int_s\int_x\int_u \frac{1}{u+s}e^{-im^2 s}e^{-iu\left[(\boldsymbol{q}_\perp)^2+\lambda^2\right]} \\ & \times \bar{E}(p', x) \\ & \times e^{\left\{is\left(1+\frac{q_-}{p_-}\right)p^2 + i\frac{q_-}{p_-}\int_0^s ds'[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})]^2 - 2i\boldsymbol{q}_\perp\int_0^s ds'[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{s'})] - is(\boldsymbol{q}_\perp)^2\right\}} \\ & \times \left\langle\left[2\left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)}\gamma^+\Diamond - \gamma^- q_- - \boldsymbol{\gamma}^\perp\boldsymbol{q}_\perp\right)\right.\right. \\ & - \left.\frac{1}{p_- + q_-}\left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)}\gamma^+\left(\Diamond - \frac{1}{p_- + q_-}\right) + \gamma^- q_- + \boldsymbol{\gamma}^\perp\boldsymbol{q}_\perp\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\ & \times \left.\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p, x\right) + 4\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p, x\right)\left(m - \hat{p}\right)\right\rangle \\ & \times \delta\left(q_- - \left(\frac{-sp_-}{(u+s)}\right)\right). \end{aligned} \tag{3.35}$$

The (q_-) - integration over the delta function (the additional $1/2\pi$ from the integration measure remains) fixes q_-, i.e.

$$q_- = -\frac{sp_-}{u+s} \implies \frac{1}{p_- + q_-} = \frac{u+s}{up_-}, \quad \frac{q_-}{p_-} = -\frac{s}{u+s}, \tag{3.36}$$

such that the derivative term $\Diamond$ from the factor, from Equation 3.34, becomes

$$
\begin{aligned}
&\Diamond\Big|_{q_-=-sp_-/(u+s)} \\
&= i\left[s\frac{p^2}{p_-} + \frac{u+s}{up_-}\int_0^s ds'\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2 - \frac{s^2}{up_-}\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right]^2 + \frac{2(u+s)s}{up_-}\boldsymbol{q}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right. \\
&\left.-\frac{2(u+s)}{up_-}\boldsymbol{q}_\perp\cdot\int_0^s ds'\boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right] \\
&= i\left[s\frac{p^2}{p_-} + \frac{(u+s)s}{up_-}\int_0^1 dy\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2 - \frac{s^2}{up_-}\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right]^2 + \frac{2(u+s)s}{up_-}\boldsymbol{q}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right. \\
&\left.-\frac{2(u+s)s}{up_-}\boldsymbol{q}_\perp\cdot\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right] \\
&= i\left[s\frac{p^2}{p_-} + \frac{(u+s)s}{up_-}\int_0^1 dy\left[\left(\boldsymbol{p}_\perp\right)^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right) + \left(\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2\right] - \frac{s^2}{up_-}\left[\left(\boldsymbol{p}_\perp\right)^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right) + \left(\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right)^2\right]\right. \\
&\left.+\frac{2(u+s)s}{up_-}\boldsymbol{q}_\perp\cdot\left[\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right) - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]\right] \\
&= i\left[s\frac{2p_+p_- - \left(\boldsymbol{p}_\perp\right)^2}{p_-} + s\frac{\left(\boldsymbol{p}_\perp\right)^2}{p_-} + \frac{(u+s)s}{up_-}\int_0^1 dy\left[\left(\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]\right. \\
&\left.-\frac{s^2}{up_-}\left[\left(\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right)^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right] + \frac{2(u+s)s}{up_-}\boldsymbol{q}_\perp\cdot\left[\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right) - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]\right] \\
&= i\left[2sp_+ + \frac{(u+s)s}{up_-}\int_0^1 dy\left[\left(\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right] - \frac{s^2}{up_-}\left[\left(\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right)^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right)\right]\right. \\
&\left.+\frac{2(u+s)s}{up_-}\boldsymbol{q}_\perp\cdot\left[\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right) - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]\right],
\end{aligned}
\tag{3.37}
$$

where the q_- subscript indicates that the q_- variable is fixed.

Then, the phase becomes, after a change of variables $s' = sy \to y$

$$
\begin{aligned}
&is\left(1+\frac{q_-}{p_-}\right)p^2 + i\frac{q_-}{p_-}\int_0^s ds'\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right]^2 - 2i\boldsymbol{q}_\perp\int_0^s ds'\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{s'}\right)\right] - is\left(\boldsymbol{q}_\perp\right)^2 \\
&= i\left[\frac{su}{u+s}p^2 - \frac{s^2}{u+s}\int_0^1 dy\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2 - 2s\boldsymbol{q}_\perp\int_0^1 dy\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right] - s\left(\boldsymbol{q}_\perp\right)^2\right],
\end{aligned}
\tag{3.38}
$$

such that the matrix element is

$$
\begin{aligned}
-iM(p',p) = \frac{e^2}{4\pi}&\int_{q/\{q_+,q_-\}}\int_s\int_x\int_u \frac{1}{u+s}e^{-im^2s}e^{-iu\left[\left(\boldsymbol{q}_\perp\right)^2+\lambda^2\right]} \\
&\times \bar{E}(p',x) \\
&\times e^{\left\{i\left[\frac{su}{u+s}p^2 - \frac{s^2}{u+s}\int_0^1 dy\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2 - 2s\boldsymbol{q}_\perp\int_0^1 dy\left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right] - s\left(\boldsymbol{q}_\perp\right)^2\right]\right\}} \\
&\times \left\langle\left[2\left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)}\gamma^+\Diamond\Big|_{q_-} + \frac{s}{u+s}\gamma^-p_- - \boldsymbol{\gamma}^\perp\boldsymbol{q}_\perp\right)\right.\right. \\
&\left.-\frac{u+s}{up_-}\left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)}\gamma^+\left(\Diamond\Big|_{q_-} - \frac{u+s}{up_-}\right) - \frac{s}{u+s}\gamma^-p_- + \boldsymbol{\gamma}^\perp\boldsymbol{q}_\perp\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\
&\left.\times\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right) + 4\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right)\left(m-\hat{p}\right)\right\rangle,
\end{aligned}
\tag{3.39}
$$

where the subscript $q/\{q_+,q_-\}$ in the measure stands for an integral over just $\boldsymbol{q}_\perp$.

Writing the Gaussian part at the end, combining the rest of the exponentials and using that $\gamma^+\hat{n} = \hat{n}^2 = n^2 = 0$ to get rid of the term $(u+s)/up_-$, we get

$$\begin{aligned}
-iM(p',p) = \frac{e^2}{4\pi} \int_{q/\{q_+,q_-\}} \int_s \int_x \int_u \frac{1}{u+s} \\
\times \exp\left\{ i \left[\frac{su}{u+s} p^2 - m^2 s - u\lambda^2 - \frac{s^2}{u+s} \int_0^1 dy \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right]^2 \right] \right\} \\
\times \bar{E}(p',x) \\
\times \left\langle \left[2 \left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)} \gamma^+ \Diamond \Big|_{q_-} + \frac{s}{u+s} \gamma^- p_- - \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp \right) \right.\right. \\
\left. - \frac{u+s}{up_-} \left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)} \gamma^+ \Diamond \Big|_{q_-} - \frac{s}{u+s} \gamma^- p_- + \boldsymbol{\gamma}^\perp \boldsymbol{q}_\perp \right) \hat{n} \widehat{\Delta\mathcal{A}}(\phi_s) \right] \\
\times \left. \left[1 + \frac{1}{2p_-} \hat{n} \widehat{\Delta\mathcal{A}}(\phi_s) \right] E(p,x) + 4 \left[1 + \frac{1}{2p_-} \hat{n} \widehat{\Delta\mathcal{A}}(\phi_s) \right] E(p,x)(m - \hat{p}) \right\rangle \\
\times \exp\left\{ -i(u+s)(\boldsymbol{q}_\perp)^2 - 2is\boldsymbol{q}_\perp \int_0^1 dy \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right] \right\}.
\end{aligned} \tag{3.40}$$

Denoting

$$\boxed{a \equiv 2i(u+s),\ \boldsymbol{b} \equiv -2is \left[\boldsymbol{p}_\perp - \int_0^1 dy \boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) \right] = -2is\boldsymbol{\mathcal{P}}_\perp(p),} \tag{3.41}$$

the Gaussian exponential becomes

$$\begin{aligned}
\exp\left\{ -i(u+s)(\boldsymbol{q}_\perp)^2 - 2is\boldsymbol{q}_\perp \left[\boldsymbol{p}_\perp - \int_0^1 dy \boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) \right] \right\} &= \exp\left\{ -\frac{a}{2} (\boldsymbol{q}_\perp)^2 + \boldsymbol{b}\boldsymbol{q}_\perp \right\} \\
&= \exp\left\{ -\frac{a}{2} (\boldsymbol{q}_\perp - a^{-1}\boldsymbol{b})^2 + \frac{\boldsymbol{b}^2}{2a} \right\},
\end{aligned} \tag{3.42}$$

which needs to be integrated over all values of $\boldsymbol{q}_\perp$, but first $\boldsymbol{q}_\perp$ must be expressed as $\boldsymbol{\partial_b}$ applied to the Gaussian exponential, leading to

$$\begin{aligned}
-iM(p',p) = \frac{e^2}{4\pi} \int_{q/\{q_+,q_-\}} \int_s \int_x \int_u \frac{1}{u+s} \\
\times \exp\left\{ i \left[\frac{su}{u+s} p^2 - m^2 s - u\lambda^2 - \frac{s^2}{u+s} \int_0^1 dy \left[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right]^2 \right] \right\} \\
\times \bar{E}(p',x) \\
\times \left\langle \left[2 \left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)} \gamma^+ \Diamond \Big|_{q_-} (\boldsymbol{q}_\perp = \boldsymbol{\partial_b}) + \frac{s}{u+s} \gamma^- p_- - \boldsymbol{\gamma}^\perp \boldsymbol{\partial_b} \right) \right.\right. \\
\left. - \frac{u+s}{up_-} \left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)} \gamma^+ \left(\Diamond \Big|_{q_-} (\boldsymbol{q}_\perp = \boldsymbol{\partial_b}) \right) - \frac{s}{u+s} \gamma^- p_- + \boldsymbol{\gamma}^\perp \boldsymbol{\partial_b} \right) \hat{n} \widehat{\Delta\mathcal{A}}(\phi_s) \right] \\
\times \left. \left[1 + \frac{1}{2p_-} \hat{n} \widehat{\Delta\mathcal{A}}(\phi_s) \right] E(p,x) + 4 \left[1 + \frac{1}{2p_-} \hat{n} \widehat{\Delta\mathcal{A}}(\phi_s) \right] E(p,x)(m - \hat{p}) \right\rangle \\
\times \exp\left\{ -\frac{a}{2} (\boldsymbol{q}_\perp - a^{-1}\boldsymbol{b})^2 + \frac{\boldsymbol{b}^2}{2a} \right\}.
\end{aligned} \tag{3.43}$$

Pulling the $\boldsymbol{q}_\perp$ integral through and shifting the integral over all real values $\boldsymbol{q} \to \boldsymbol{q}'_\perp = \boldsymbol{q}_\perp - a^{-1}\boldsymbol{b}$ and dropping the prime

$$\begin{aligned}
-iM(p',p) = &\frac{e^2}{4\pi}\int_s\int_x\int_u \frac{1}{u+s} \times \exp\left\{i\left[\frac{su}{u+s}p^2 - m^2 s - u\lambda^2 - \frac{s^2}{u+s}\int_0^1 dy\,[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})]^2\right]\right\} \\
&\times \bar{E}(p',x) \\
&\times \left\langle\left[2\left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)}\gamma^+\Diamond\Big|_{q_-}(\boldsymbol{q}_\perp = \boldsymbol{\partial_b}) + \frac{s}{u+s}\gamma^- p_- - \boldsymbol{\gamma}^\perp\boldsymbol{\partial_b}\right)\right.\right. \\
&- \left.\frac{u+s}{up_-}\left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)}\gamma^+\left(\Diamond\Big|_{q_-}(\boldsymbol{q}_\perp = \boldsymbol{\partial_b})\right) - \frac{s}{u+s}\gamma^- p_- + \boldsymbol{\gamma}^\perp\boldsymbol{\partial_b}\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\
&\times \left.\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E(p,x) + 4\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\mathcal{A}}(\phi_s)\right]E(p,x)(m-\hat{p})\right\rangle \\
&\times \int\frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2}\exp\left\{-\frac{a}{2}(\boldsymbol{q}_\perp)^2 + \frac{\boldsymbol{b}^2}{2a}\right\}.
\end{aligned} \tag{3.44}$$

Carrying out the 2-dimensional Gaussian integral

$$\begin{aligned}
-iM(p',p) = &\frac{e^2}{4\pi}\int_s\int_x\int_u \frac{1}{u+s} \times \exp\left\{i\left[\frac{su}{u+s}p^2 - m^2 s - u\lambda^2 - \frac{s^2}{u+s}\int_0^1 dy\,[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})]^2\right]\right\} \\
&\times \bar{E}(p',x) \\
&\times \left\langle\left[2\left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)}\gamma^+\Diamond\Big|_{q_-}(\boldsymbol{q}_\perp = \boldsymbol{\partial_b}) + \frac{s}{u+s}\gamma^- p_- - \boldsymbol{\gamma}^\perp\boldsymbol{\partial_b}\right)\right.\right. \\
&- \left.\frac{u+s}{up_-}\left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)}\gamma^+\left(\Diamond\Big|_{q_-}(\boldsymbol{q}_\perp = \boldsymbol{\partial_b}) - \frac{u+s}{up_-}\right) - \frac{s}{u+s}\gamma^- p_- + \boldsymbol{\gamma}^\perp\boldsymbol{\partial_b}\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\
&\times \left.\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]U_s(p,x) + 4\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E(p,x)(m-\hat{p})\right\rangle \\
&\times \frac{1}{(2\pi)^2}\left[\frac{2\pi}{a}\exp\left\{\frac{\boldsymbol{b}^2}{2a}\right\}\right].
\end{aligned} \tag{3.45}$$

Taking the $\boldsymbol{\partial_b}$ derivatives, rearranging the $2\pi/a$ factor and combining the phases

$$\begin{aligned}
-iM(p',p) = &\frac{e^2}{4\pi}\frac{1}{2\pi}\int_s\int_x\int_u \frac{1}{a}\frac{1}{u+s} \times \exp\left\{i\left[\frac{su}{u+s}p^2 - m^2 s - u\lambda^2 - \frac{s^2}{u+s}\int_0^1 dy\,[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})]^2\right] + \frac{\boldsymbol{b}^2}{2a}\right\} \\
&\times \bar{E}(p',x) \\
&\times \left\langle\left[2\left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)}\gamma^+\Diamond\Big|_{q_-}\left(\boldsymbol{q}_\perp = \frac{\boldsymbol{b}}{a}\right) + \frac{s}{u+s}\gamma^- p_- - \boldsymbol{\gamma}^\perp\frac{\boldsymbol{b}}{a}\right)\right.\right. \\
&- \left.\frac{u+s}{up_-}\left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)}\gamma^+\left(\Diamond\Big|_{q_-}\left(\boldsymbol{q}_\perp = \frac{\boldsymbol{b}}{a}\right)\right) - \frac{s}{u+s}\gamma^- p_- + \boldsymbol{\gamma}^\perp\frac{\boldsymbol{b}}{a}\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\
&\times \left.\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E(p,x) + 4\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E(p,x)(m-\hat{p})\right\rangle.
\end{aligned} \tag{3.46}$$

Reinserting a, $\boldsymbol{b}$ from 3.41 and simplifying the factor of 2

$$
\begin{aligned}
& - iM(p', p) \\
& = \frac{e^2}{4\pi}\frac{1}{2\pi}\int_s\int_x\int_u \frac{1}{i(u+s)^2} \\
& \times \exp\left\{ i\left[\frac{su}{u+s}p^2 - m^2 s - u\lambda^2 - \frac{s^2}{u+s}\int_0^1 dy\,[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})]^2 + \frac{s^2}{u+s}(\boldsymbol{\mathcal{P}}_\perp(p))^2\right]\right\} \\
& \times \bar{E}(p', x) \\
& \times \left\langle \left[\left(\hat{\pi}_p(\phi_s) - \frac{i}{2(u+s)}\gamma^+ \Diamond\Big|_{q_-}\left(\boldsymbol{q}_\perp = -\frac{s}{u+s}\boldsymbol{\mathcal{P}}_\perp(p)\right) + \frac{s}{u+s}\gamma^- p_- + \frac{s}{u+s}\boldsymbol{\gamma}^\perp \boldsymbol{\mathcal{P}}_\perp(p)\right)\right.\right. \\
& - \left.\frac{u+s}{2up_-}\left(\hat{\pi}_p(\phi_s) + \frac{i}{2(u+s)}\gamma^+\left(\Diamond\Big|_{q_-}\left(\boldsymbol{q}_\perp = -\frac{s}{u+s}\boldsymbol{\mathcal{P}}_\perp(p)\right)\right) - \frac{s}{u+s}\gamma^- p_- - \frac{s}{u+s}\boldsymbol{\gamma}^\perp \boldsymbol{\mathcal{P}}_\perp(p)\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\
& \times \left.\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] E(p, x) + 2\left[1 + \frac{1}{2p_-}\hat{n}\widehat{\mathcal{A}}(\phi_s)\right] E(p, x)(m - \hat{p})\right\rangle,
\end{aligned}
\tag{3.47}
$$

where now, in addition to q_- being fixed by the delta function integration to $q_- = -sp_-/(u+s)$, the perpendicular component $\boldsymbol{q}_\perp$ is set to $\boldsymbol{q}_\perp = -s\boldsymbol{\mathcal{P}}_\perp(p)/(u+s)$, which will shortly be denoted by $\Diamond|_{q_-, \boldsymbol{q}_\perp}$, remembering the values.

Denoting

$$
\boxed{
\begin{aligned}
\Delta & \equiv \left[\int_0^1 dy\,(\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys}))^2 - \left(\boldsymbol{p}_\perp - \int_0^1 dy \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right] \\
& \equiv \left[\int_0^1 dy\,(\boldsymbol{\mathcal{A}}_\perp)^2(\phi_{ys}) - \left(\int_0^1 dy \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right],
\end{aligned}
}
\tag{3.48}
$$

the phase becomes

$$
\begin{aligned}
& \frac{su}{u+s}p^2 - m^2 s - u\lambda^2 - \frac{s^2}{u+s}\int_0^1 dy\,[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{A}}_\perp(\phi_{ys})]^2 + \frac{s^2}{u+s}(\boldsymbol{\mathcal{P}}_\perp(p))^2 \\
& = \frac{su}{u+s}\left(p^2 - m^2\right) - \frac{s^2}{u+s}\left[m^2 + \Delta\right] - u\lambda^2
\end{aligned}
\tag{3.49}
$$

and the factor can be written as

$$
\begin{aligned}
-i\Diamond\Big|_{q_-,\boldsymbol{q}_\perp} &= \Big[2sp_+ + \frac{(u+s)s}{up_-}\int_0^1 dy\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}))^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right] - \frac{s^2}{up_-}\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_s))^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp(\phi_s)\right] \\
&+ \frac{2(u+s)s}{up_-}\boldsymbol{q}_\perp\cdot\left[\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right]\Big]\Big|_{\boldsymbol{q}_\perp=-\frac{s}{u+s}\boldsymbol{\mathcal{P}}_\perp(p)} \\
&= \Big[2sp_+ + \frac{(u+s)s}{up_-}\int_0^1 dy\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}))^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right] - \frac{s^2}{up_-}\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_s))^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp(\phi_s)\right] \\
&- \frac{2s^2}{up_-}\left[\boldsymbol{p}_\perp - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(ys)\right]\cdot\left[\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right]\Big] \\
&= \Big[2sp_+ + \frac{(u+s)s}{up_-}\int_0^1 dy\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}))^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right] - \frac{s^2}{up_-}\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_s))^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp(\phi_s)\right] \\
&- \frac{2s^2}{up_-}\left[\boldsymbol{p}_\perp - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(ys)\right]\cdot\left[\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right]\Big] \\
&= \Big[2sp_+ + \frac{(u+s)s}{up_-}\int_0^1 dy\,(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}))^2 - \frac{2(u+s)s}{up_-}\boldsymbol{p}_\perp\cdot\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) \\
&- \frac{s^2}{up_-}\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_s))^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp(\phi_s)\right] \\
&- \frac{2s^2}{up_-}\left[\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \boldsymbol{\mathcal{A}}_\perp(\phi_s)\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) - \boldsymbol{p}_\perp\cdot\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) + \left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right]\Big] \\
&= \Big[2s\left(p_+ - \frac{1}{p_-}\boldsymbol{p}_\perp\cdot\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right) + \frac{(u+s)s}{up_-}\int_0^1 dy\,(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}))^2 \\
&- \frac{s^2}{up_-}\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_s))^2 - 2\boldsymbol{p}_\perp\boldsymbol{\mathcal{A}}_\perp(\phi_s)\right] \\
&- \frac{2s^2}{up_-}\left[\boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \boldsymbol{\mathcal{A}}_\perp(\phi_s)\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) + \left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right]\Big].
\end{aligned}
\tag{3.50}
$$

> **Observation 3.1.1** Introducing $\Delta\mathcal{A}^\mu(\phi_s) \equiv \mathcal{A}^\mu(\phi_s) - \mathcal{A}^\mu(\phi)$, 3.48 becomes
>
> $$\boxed{\Delta \equiv \left[\int_0^1 dy\,(\boldsymbol{\mathcal{A}}_\perp)^2(\phi_{ys}) - \left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right] = \left[\int_0^1 dy\,(\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}))^2 - \left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right]} \tag{3.51}$$

Defining the integrated momentum

$$\boxed{\mathcal{P}^\mu(p) \equiv p^\mu - \int_0^1 dy\mathcal{A}^\mu(\phi_{ys}) + \frac{n^\mu}{p_-}\left(p\cdot\int_0^1 dy\mathcal{A}(\phi_{ys})\right) - \frac{n^\mu}{2p_-}\left(\int_0^1 dy\mathcal{A}(\phi_{ys})\right)^2,} \tag{3.52}$$

and using the relations $n^+ = n_-, \boldsymbol{n}_\perp = 0, n^+ = n^2 = 0, n^- = \tilde{n}n = 1,\quad A^+ = A_- = n\cdot A \overset{Lorentz}{=} 0,\quad A^0 = A^3 = 0 \implies A_+ = A^- = 0$, the decomposition on the bispinor basis of 3.52 is

$$
\begin{aligned}
\hat{\mathcal{P}}(p) &= \gamma^+\mathcal{P}_+(p) + \gamma^-\mathcal{P}_-(p) + \boldsymbol{\gamma}^\perp\boldsymbol{\mathcal{P}}_\perp(p) \\
&= \gamma^+\left[p_+ + \frac{1}{p_-}\left(p\cdot\int_0^1 dy\mathcal{A}(\phi_{ys})\right) - \frac{1}{2p_-}\left(\int_0^1 dy\mathcal{A}(\phi_{ys})\right)^2\right] + \gamma^-p_- + \boldsymbol{\gamma}^\perp\left(\boldsymbol{p}_\perp - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right) \\
&= \gamma^+\left[p_+ - \frac{1}{p_-}\left(\boldsymbol{p}_\perp\cdot\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right) + \frac{1}{2p_-}\left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right] + \gamma^-p_- + \boldsymbol{\gamma}^\perp\left(\boldsymbol{p}_\perp - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right),
\end{aligned}
$$

(3.53)

which shows that the projection $\boldsymbol{\mathcal{P}}_\perp(p)$ obtained in 3.53 is consistent with the previous definition from 3.41.

$$
\begin{aligned}
-i\Diamond\Big|_{q_-,\boldsymbol{q}_\perp} &= \Bigg[2s\left(p_+ - \frac{1}{p_-}\boldsymbol{p}_\perp\cdot\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right) + \frac{(u+s)s}{up_-}\int_0^1 dy\left(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2 \\
&\quad - \frac{s^2}{up_-}\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_s))^2 - 2\boldsymbol{\mathcal{A}}_\perp(\phi_s)\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) + 2\left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right]\Bigg] \\
&= \Bigg[2s\left(p_+ - \frac{1}{p_-}\boldsymbol{p}_\perp\cdot\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) + \frac{1}{2p_-}\left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right) - \frac{us}{up_-}\left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2 \\
&\quad + \frac{(u+s)s}{up_-}\int_0^1 dy\left(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2 \\
&\quad - \frac{s^2}{up_-}\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_s))^2 - 2\boldsymbol{\mathcal{A}}_\perp(\phi_s)\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) + 2\left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right]\Bigg] \\
&= \Bigg[2s\left(p_+ - \frac{1}{p_-}\boldsymbol{p}_\perp\cdot\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) + \frac{1}{2p_-}\left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right) \\
&\quad - \frac{(u+s)s}{up_-}\left[\left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2 - \int_0^1 dy\left(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right] \\
&\quad - \frac{s^2}{up_-}\left[(\boldsymbol{\mathcal{A}}_\perp(\phi_s))^2 - 2\boldsymbol{\mathcal{A}}_\perp(\phi_s)\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) + \left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right]\Bigg] \\
&= \Bigg[2s\mathcal{P}_+(p) - \frac{(u+s)s}{up_-}\left[\left(\int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2 - \int_0^1 dy\left(\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right] \\
&\quad - \frac{s^2}{up_-}\left(\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\Bigg],
\end{aligned} \tag{3.54}
$$

where $\mathcal{P}_+$ from expression 3.53 was inserted.

Inserting Δ from the definition 3.48 and noticing that

$$
\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}) = \Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_{ys}), \tag{3.55}
$$

we get

$$
-i\Diamond\Big|_{q_-,\boldsymbol{q}_\perp} = \left[2s\mathcal{P}_+(p) - \frac{(u+s)s}{up_-}\Delta - \frac{s^2}{up_-}\left(\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right)^2\right]. \tag{3.56}
$$

Therefore, we can write

$$
\boxed{-\frac{i}{2(u+s)}\Diamond\Big|_{q_-,\boldsymbol{q}_\perp} = \frac{s}{u+s}\left[\mathcal{P}_+(p) + \mathscr{A}(\phi_s)\right],} \tag{3.57}
$$

where temporarily the quantity $\mathscr{A}(\phi_s)$ is introduced by

$$
\boxed{\mathscr{A}(\phi_s) \equiv \frac{u+s}{2up_-}\Delta - \frac{s}{2up_-}\left[\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_{ys})\right]^2.} \tag{3.58}
$$

Simplifying the i, and plugging in the matrix element the expression from 3.57

$$\begin{aligned}
M(p',p) = &\frac{e^2}{4\pi}\frac{1}{2\pi}\int_s\int_x\int_u \frac{1}{(u+s)^2} \times \exp\left\{i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}\left[m^2+\Delta\right] - u\lambda^2\right]\right\} \\
&\times \bar{E}(p',x) \\
&\times \left\{\left[\left(\hat{\pi}_p(\phi_s) + \frac{s}{u+s}\gamma^+\mathcal{P}_+(p) + \frac{s}{u+s}\gamma^+\mathscr{A}(\phi_s) + \frac{s}{u+s}\gamma^- p_- + \frac{s}{u+s}\boldsymbol{\gamma}^\perp\boldsymbol{\mathcal{P}}_\perp(p)\right)\right.\right. \\
&- \left.\frac{u+s}{2up_-}\left(\hat{\pi}_p(\phi_s) - \frac{s}{u+s}\gamma^+\mathcal{P}_+(p) + \frac{s}{u+s}\gamma^+\mathscr{A}(\phi_s) - \frac{s}{u+s}\gamma^- p_- - \frac{s}{u+s}\boldsymbol{\gamma}^\perp\boldsymbol{\mathcal{P}}_\perp(p)\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] \\
&\times \left.\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right) + 2\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right)\left(m-\hat{p}\right)\right\},
\end{aligned} \tag{3.59}$$

and using the bispinor expansion of $\hat{\mathcal{P}}(p)$ from equation 3.53, inserting $\alpha = e^2/4\pi$

$$\begin{aligned}
M(p',p) = &\frac{\alpha}{2\pi}\int_s\int_x\int_u \frac{1}{(u+s)^2} \times \exp\left\{i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}\left[m^2+\Delta\right] - u\lambda^2\right]\right\} \\
&\times E(p',x) \\
&\times \left\{\left[\left(\hat{\pi}_p(\phi_s) + \frac{s}{u+s}\hat{\mathcal{P}}(p) + \frac{s}{u+s}\mathscr{A}(\phi_s)\hat{n}\right) - \frac{u+s}{2up_-}\left(\hat{\pi}_p(\phi_s) - \frac{s}{u+s}\hat{\mathcal{P}}(p)\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]\right. \\
&\times \left.\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right) + 2\left[1+\frac{1}{2p_-}\hat{n}\hat{\mathcal{A}}(\phi_s)\right]E\left(p,x\right)\left(m-\hat{p}\right)\right\}.
\end{aligned} \tag{3.60}$$

Factoring $\hat{\pi}_p(\phi_s)$ and $\hat{\mathcal{P}}(p)$

$$\begin{aligned}
M(p',p) = &\frac{\alpha}{2\pi}\int_s\int_x\int_u \frac{1}{(u+s)^2} \times \exp\left\{i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}\left(m^2+\Delta\right) - u\lambda^2\right]\right\} \\
&\times \bar{E}(p',x) \\
&\times \left\{\left[\hat{\pi}_p(\phi_s)\left(1-\frac{u+s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right) + \frac{s}{u+s}\hat{\mathcal{P}}(p)\left(1+\frac{s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right) + \frac{s}{u+s}\mathscr{A}(\phi_s)\hat{n}\right]\right. \\
&\times \left.\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right]E\left(p,x\right) + 2\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]E\left(p,x\right)\left(m-\hat{p}\right)\right\}.
\end{aligned} \tag{3.61}$$

Expanding the brackets

$$\begin{aligned}
&\left[1-\frac{u+s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] = 1 + \frac{1}{2p_-}\left(1-\frac{u+s}{u}\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s) = 1 - \frac{s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s) \\
&\left[1+\frac{s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)\right] = 1 + \frac{1}{2p_-}\left(1+\frac{s}{2u}\right)\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s) = 1 + \frac{2u+s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s),
\end{aligned} \tag{3.62}$$

and inserting in the matrix element

$$\begin{aligned}
M(p',p) = &\frac{\alpha}{2\pi}\int_s\int_x\int_u \frac{1}{(u+s)^2} \times \exp\left\{i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}\left(m^2+\Delta\right) - u\lambda^2\right]\right\} \\
&\times \bar{E}(p',x) \\
&\times \left\{\left[\hat{\pi}_p(\phi_s)\left(1-\frac{s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right) + \frac{s}{u+s}\hat{\mathcal{P}}(p)\left(1+\frac{2u+s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right) + \frac{s}{u+s}\mathscr{A}(\phi_s)\hat{n}\right]\right. \\
&\times \left.E\left(p,x\right) + 2\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]E\left(p,x\right)\left(m-\hat{p}\right)\right\}.
\end{aligned} \tag{3.63}$$

3.2 Sandwiching between the Ritus matrices

The sandwich between the Ritus matrices of a matrix Γ can be taken using

$$\bar{E}(p',x)\Gamma E(p,x) = \exp\{-i\Delta S(x)\}\bar{E}(p,x)\Gamma E(p,x) \tag{3.64}$$

where, under the conservation laws $p'_- = p_-$ and $\boldsymbol{p'}_\perp = \boldsymbol{p}_\perp$, the phase difference is

$$
\begin{aligned}
\Delta S(x) &\equiv S_{p'}(x) - S_p(x) \qquad (p'_- = p_-) \\
&= -(p'-p)\cdot x - \int_{-\infty}^{\phi} d\varphi \frac{(p'-p)\cdot \mathcal{A}(\varphi))}{p_-} \qquad (\boldsymbol{p'}_\perp = \boldsymbol{p}_\perp, p'_- = p_-) \\
&= -(p'_+ - p_+)x_- - \int_{-\infty}^{\phi} d\varphi \frac{(p'_+ - p_+)\cdot \mathcal{A}_-(\varphi))}{p_-} \qquad (\mathcal{A}_- = 0, x_- = (nx) = \phi) \\
&= -(p'_+ - p_+)\phi \implies \boxed{\Delta S(x) = -(p'_+ - p_+)\phi}.
\end{aligned} \tag{3.65}
$$

Specifically, in our case we have for $\hat{\pi}_p(\phi_s)\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]$

$$
\begin{aligned}
&\bar{E}(p,x)\hat{\pi}_p(\phi_s)\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]E(p,x) \\
&= \bar{E}(p,x)\hat{\pi}_p(\phi_s)E(p,x)\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right] \\
&= \left[\hat{\pi}_p(\phi_s) - \left(\pi_p(\phi_s)\cdot\mathcal{A}(\phi)\right)\frac{\hat{n}}{p_-} + \underbrace{\left(\pi_p(\phi_s)\cdot n\right)}_{p_-}\left(\frac{\hat{\mathcal{A}}(\phi)}{p_-} - \frac{\hat{n}\mathcal{A}^2(\phi)}{2(p_-)^2}\right)\right]\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right] \\
&= \left[\hat{\pi}_p(\phi_s) + \hat{\mathcal{A}}(\phi) - \left(\pi_p(\phi_s)\cdot\mathcal{A}(\phi)\right)\frac{\hat{n}}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\hat{n}\right]\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right] \\
&= \left[\hat{p} - \hat{\mathcal{A}}(\phi_s) + \frac{p\cdot\mathcal{A}(\phi_s)}{p_-}\hat{n} - \frac{\mathcal{A}^2(\phi_s)}{2p_-}\hat{n} + \hat{\mathcal{A}}(\phi) - \left[\left(p - \mathcal{A}\left(\phi_s\right)\right)\cdot\mathcal{A}(\phi)\right]\frac{\hat{n}}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\hat{n}\right]\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right] \\
&= \left[\hat{p} - \left(\hat{\mathcal{A}}(\phi_s) - \hat{\mathcal{A}}(\phi)\right) + \frac{p\cdot\left(\mathcal{A}(\phi_s) - \mathcal{A}(\phi)\right)}{p_-}\hat{n} - \frac{\mathcal{A}^2(\phi_s) + \mathcal{A}^2(\phi) - 2\left(\mathcal{A}\left(\phi_s\right)\cdot\mathcal{A}\left(\phi\right)\right)}{2p_-}\hat{n}\right]\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right] \\
&= \left[\hat{p} - \widehat{\Delta\mathcal{A}}(\phi_s) + \frac{p\cdot\Delta\mathcal{A}(\phi_s)}{p_-}\hat{n} - \frac{\left(\Delta\mathcal{A}\right)^2\left(\phi_s\right)}{2p_-}\hat{n}\right]\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right] \\
&= \hat{\pi}_{\Delta\mathcal{A}}(\phi_s)\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]
\end{aligned} \tag{3.66}
$$

and similarly for the other quantity $\hat{\mathcal{P}}(p)\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]$

$$
\bar{E}(p,x)\hat{\mathcal{P}}(p)\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]E(p,x) = \hat{\mathcal{P}}_{\Delta\mathcal{A}}(\phi_s)\left[1 + a\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right], \tag{3.67}
$$

where the following two momenta were introduced

$$
\boxed{
\begin{aligned}
&\pi^\lambda_{p,\Delta\mathcal{A}}\left(\phi_s\right) \equiv p^\lambda - \Delta\mathcal{A}^\lambda\left(\phi_s\right) + \frac{n^\lambda}{p_-}\left(p\cdot\Delta\mathcal{A}\left(\phi_s\right)\right) - \frac{n^\lambda}{2p_-}\left[\Delta\mathcal{A}\left(\phi_s\right)\right]^2 \\
&\mathcal{P}^\lambda_{\Delta\mathcal{A}(\phi_s)}(p) \equiv p^\lambda - \int_0^1 dy\Delta\mathcal{A}^\lambda\left(\phi_{ys}\right) + \frac{n^\lambda}{p_-}\left(p\cdot\int_0^1 dy\Delta\mathcal{A}\left(\phi_{ys}\right)\right) - \frac{n^\lambda}{2p_-}\left(\int_0^1 dy\Delta\mathcal{A}\left(\phi_{ys}\right)\right)^2
\end{aligned}
} \tag{3.68}
$$

Carrying out the integrals over the variables $\boldsymbol{x}_\perp$ and τ, which provide delta functions enforcing the conservation laws $\boldsymbol{p}_\perp = \boldsymbol{p}'_\perp$ and $p_- = p'_-$,

$$\begin{aligned} M(p',p) &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp\right)\delta\left(p'_- - p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+ - p_+\right)\phi\right\}\int_s\int_u \frac{1}{(u+s)^2} \\ &\times \exp\left\{i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}\left(m^2+\Delta\right) - u\lambda^2\right]\right\} \\ &\times \left\{\hat{\pi}_{p,\Delta\mathcal{A}}(\phi_s)\left(1 - \frac{s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right)\right. \\ &\left. + \frac{s}{u+s}\hat{\mathcal{P}}_{\Delta\mathcal{A}(\phi_s)}(p)\left(1+\frac{2u+s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right) + \frac{s}{u+s}\mathscr{A}(\phi_s)\hat{n} + 2\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right](m-\hat{p})\right\}. \end{aligned} \tag{3.69}$$

Inserting the full form for Δ from Observation 3.1.1 and $\mathscr{A}(\phi_s)$, given by

$$\boxed{\mathscr{A}(\phi_s) = \frac{u+s}{2up_-}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2\right] - \frac{s}{2up_-}\left[\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2,} \tag{3.70}$$

the matrix element becomes

$$\boxed{\begin{aligned} &M(p',p) \\ &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp\right)\delta\left(p'_- - p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+ - p_+\right)\phi\right\}\int_s\int_u \frac{1}{(u+s)^2} \\ &\times \exp\left\{i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}\left(m^2+\Delta\right) - u\lambda^2\right]\right\} \\ &\times \left\{\hat{\pi}_{p,\Delta\mathcal{A}}(\phi_s)\left[1 - \frac{s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right] + \frac{s}{u+s}\hat{\mathcal{P}}_{\Delta\mathcal{A}(\phi_s)}(p)\left[1+\frac{2u+s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]\right. \\ &+\frac{s}{u+s}\left\langle\frac{u+s}{2up_-}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2\right] - \frac{s}{2up_-}\left[\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2\right\rangle\hat{n} \\ &\left.+2\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right](m-\hat{p})\right\}. \end{aligned}} \tag{3.71}$$

3.3 Mass operator renormalization

The mass operator we have found diverges in the same way as the mass operator of a spinor particle when there is no external field. By subtracting and adding the vacuum contribution, the plane wave mass operator can be written as

$$M = (M - M(\mathcal{A}=0)) + M(\mathcal{A}=0), \tag{3.72}$$

such that we have the regular part and the vacuum contribution, which is divergent.

Renormalizing only the vacuum part, i.e. $M(\mathcal{A}=0) \to M_R(\mathcal{A}=0)$, we get the renormalized plane wave mass operator, now regular

$$M_R = (M - M(\mathcal{A}=0)) + M_R(\mathcal{A}=0). \tag{3.73}$$

Evaluating the matrix element in vacuum

$$\begin{aligned} M(\mathcal{A}=0) &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp\right)\delta\left(p'_- - p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+ - p_+\right)\phi\right\}\int_s\int_u \frac{1}{(u+s)^2} \\ &\times \exp\left\{i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}m^2 - u\lambda^2\right]\right\} \\ &\times \left\{-\frac{u}{u+s}\hat{p} + 2m\right\}. \end{aligned} \tag{3.74}$$

The mass operator renormalization of vacuum QED is standard, and is carried out by employing the condition [VS75; VS71; Kai18; Mic19]

$$M_R\left(A_\mu=0\right)=M\left(A_\mu=0\right)-M\left(A_\mu=0\right)\Big|_{\hat{p}=m}-(\hat{p}-m)\,\frac{\partial M\left(A_\mu=0\right)}{\partial\hat{p}}\Big|_{\hat{p}=m} \tag{3.75}$$

at the on-shell renormalization point $\hat{p}=m$, $p^2=m^2$.

The vacuum matrix element 3.74, evaluated on-shell is then

$$\begin{aligned} M\left(\mathcal{A}=0\right)\Big|_{\hat{p}=m} &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\\ &\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2-u\lambda^2\right]\right\}\\ &\times\left\{\frac{u+2s}{u+s}m\right\} \end{aligned} \tag{3.76}$$

For calculating the derivative in the vacuum case

$$\begin{aligned} \frac{\partial M\left(\mathcal{A}=0\right)}{\partial\hat{p}} &= -(2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\\ &\times\exp\left\{i\left[\frac{su}{u+s}\left(p^2-m^2\right)-\frac{s^2}{u+s}m^2-u\lambda^2\right]\right\}\times\left\{\frac{u}{u+s}+i\frac{2su}{u+s}\hat{p}\left[\frac{u}{u+s}\hat{p}-2m\right]\right\} \end{aligned} \tag{3.77}$$

Evaluating the derivative 3.77 on-shell

$$\begin{aligned} \frac{\partial M\left(\mathcal{A}=0\right)}{\partial\hat{p}}\Bigg|_{\hat{p}=m} &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\\ &\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2-u\lambda^2\right]\right\}\\ &\times\left\{-\frac{u}{u+s}-i\frac{2su}{u+s}m\left[-\frac{u+2s}{u+s}m\right]\right\}\qquad\left(=\left\{-\frac{u}{u+s}\left[1-2i\frac{u+2s}{u+s}m^2s\right]\right\}\right). \end{aligned} \tag{3.78}$$

The renormalized vacuum mass operator becomes

$$\boxed{\begin{aligned} M_R\left(\mathcal{A}=0\right) &= -(2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\\ &\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2-u\lambda^2\right]\right\}\\ &\times\left\{\exp\left[\frac{isu}{u+s}\left(p^2-m^2\right)\right]\times\left(\frac{u}{u+s}\hat{p}-2m\right)+\frac{u+2s}{u+s}m-\frac{u}{u+s}\left[1-2i\frac{u+2s}{u+s}m^2s\right](\hat{p}-m)\right\}. \end{aligned}} \tag{3.79}$$

Then, the difference $M_R\left(\mathcal{A}=0\right)-M\left(\mathcal{A}=0\right)$ is

$$\boxed{\begin{aligned} M_R\left(\mathcal{A}=0\right)-M\left(\mathcal{A}=0\right) &= -(2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\\ &\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2-u\lambda^2\right]\right\}\\ &\times\left\{\frac{u+2s}{u+s}m-\frac{u}{u+s}\left[1-2i\frac{u+2s}{u+s}m^2s\right](\hat{p}-m)\right\}. \end{aligned}} \tag{3.80}$$

The renormalized plane wave mass operator becomes, from condition 3.73

$$\boxed{\begin{aligned}&M_R(p',p)\\&=(2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2-u\lambda^2\right]\right\}\\&\times\left\{\exp i\left[\frac{su}{u+s}\left(p^2-m^2\right)-\frac{s^2}{u+s}\Delta\right]\right.\\&\times\left[\hat{\pi}_{p,\Delta\mathcal{A}}(\phi_s)\left[1-\frac{s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]+\frac{s}{u+s}\hat{\mathcal{P}}_{\Delta\mathcal{A}(\phi_s)}(p)\left[1+\frac{2u+s}{2up_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right]\right.\\&+\frac{s\hat{n}}{2(u+s)}\left\langle\frac{u+s}{up_-}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2\right]-\frac{s}{up_-}\left[\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s)-\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2\right\rangle\\&\left.\left.+2\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right](m-\hat{p})\right]-\frac{u+2s}{u+s}m+\frac{u}{u+s}\left(1-2i\frac{u+2s}{u+s}m^2s\right)(\hat{p}-m)\right\}.\end{aligned}}$$

(3.81)

which is a more compact form, but equivalent, to the one in [VS75], having the advantage that it can be easily put on-shell and that in light-cone coordinates the conserved quantities are manifest through the delta functions appearing in the mass operator.

Decomposing the product $\gamma^\alpha\gamma^\beta\gamma^\delta$ on the gamma matrices basis $\{\mathbf{1},\gamma^5,i\gamma^\mu\gamma^5,i\sigma^{\mu\nu}\}$

$$\gamma^\alpha\gamma^\beta\gamma^\delta=g^{\beta\delta}\gamma^\alpha-g^{\alpha\delta}\gamma^\beta+\gamma^{\alpha\beta}\gamma^\delta+\varepsilon^{\alpha\beta\delta}{}_\mu i\gamma^5\gamma^\mu,\tag{3.82}$$

the triple product can be decomposed as

$$\hat{a}\hat{b}\hat{c}=\hat{a}(b\cdot c)-\hat{b}(a\cdot c)+\hat{c}(a\cdot b)+i\varepsilon_{\mu\alpha\beta\delta}\gamma^5\gamma^\mu a^\alpha b^\beta c^\delta,\tag{3.83}$$

Particulary, $\hat{\pi}(\phi_s)\hat{n}\hat{\mathcal{A}}$ and $\hat{\Pi}(p)\hat{n}\hat{\mathcal{A}}$ as

$$\begin{aligned}\hat{\pi}\hat{n}\hat{\mathcal{A}}&=\hat{\pi}(n\cdot\mathcal{A})-\hat{n}(\pi\cdot\mathcal{A})+\hat{\mathcal{A}}(\pi\cdot n)+i\gamma^5\varepsilon_{\mu\alpha\beta\delta}\gamma^\mu\pi^\alpha n^\beta\mathcal{A}^\delta\\&=-\hat{n}(\pi\cdot\mathcal{A})+\hat{\mathcal{A}}\pi_-+i\gamma^5\gamma^\mu\frac{1}{2}\varepsilon_{\mu\alpha\beta\delta}\pi^\alpha\left(n^\beta\mathcal{A}^\delta+n^\beta\mathcal{A}^\delta\right)\\&=-\hat{n}(\pi\cdot\mathcal{A})+\hat{\mathcal{A}}\pi_-+i\gamma^5\gamma^\mu\left[\frac{1}{2}\varepsilon_{\mu\alpha\beta\delta}\left(n^\beta\mathcal{A}^\delta-n^\delta\mathcal{A}^\beta\right)\right]\pi^\alpha\\&=-\hat{n}(\pi\cdot\mathcal{A})+\hat{\mathcal{A}}\pi_-+i\gamma^5\gamma^\mu\left[\frac{1}{2}\varepsilon_{\mu\alpha\beta\delta}\mathscr{F}^{\beta\delta}\right]\pi^\alpha=-\hat{n}(\pi\cdot\mathcal{A})+\hat{\mathcal{A}}\pi_-+i\gamma^5\gamma^\mu\mathscr{F}^*_{\mu\alpha}\pi^\alpha\\&=\boxed{-\hat{n}(\pi\cdot\mathcal{A})+\hat{\mathcal{A}}\pi_-+i\gamma^5\left(\gamma\mathscr{F}^*\pi\right)}\\&=-\gamma^\alpha(n_\alpha\pi^\beta\mathcal{A}_\beta-\mathcal{A}_\alpha n_\beta\pi^\beta)+i\gamma^5\left(\gamma\mathscr{F}^*\pi\right)=-\gamma^\alpha\pi^\beta(n_\alpha\mathcal{A}_\beta-\mathcal{A}_\alpha n_\beta)+i\gamma^5\left(\gamma\mathscr{F}^*\pi\right)\\&=-\gamma^\alpha\mathcal{F}_{\alpha\beta}\pi^\beta+i\gamma^5\left(\gamma\mathscr{F}^*\pi\right)=\boxed{i\gamma^5\left(\gamma\mathscr{F}^*\pi\right)-\left(\gamma\mathscr{F}\pi\right)},\end{aligned}\tag{3.84}$$

where was introduced

$$\boxed{\mathscr{F}_{\alpha\beta}(\phi)\equiv n_\alpha\mathcal{A}_\beta(\phi)-\mathcal{A}_\alpha n_\beta(\phi),\quad \mathscr{F}^*_{\alpha\beta}(\phi)\equiv\frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\mathscr{F}^{\mu\nu}(\phi)}\tag{3.85}$$

Then, using the three product decomposition from 3.84, we have

$$\hat{\pi}\hat{n}\widehat{\Delta\mathcal{A}}(\phi_s)=i\gamma^5\left(\gamma\Delta\mathscr{F}^*(\phi_s)\pi\right)-\left(\gamma\Delta\mathscr{F}(\phi_s)\pi\right),\tag{3.86}$$

where

$$\boxed{\begin{aligned}&\Delta\mathscr{F}^{\mu\nu}(\phi_s) \equiv \mathscr{F}^{\mu\nu}(\phi_s) - \mathscr{F}^{\mu\nu}(\phi) = n^\mu(\mathcal{A}(\phi_s) - \mathcal{A}(\phi))^\nu - n^\nu(\mathcal{A}(\phi_s) - \mathcal{A}(\phi)^\mu,\\ &\Delta\mathscr{F}^*_{\alpha\beta}(\phi_s) \equiv \varepsilon_{\alpha\beta\mu\nu}\Delta\mathscr{F}^{\mu\nu}(\phi_s)/2,\end{aligned}} \quad (3.87)$$

were defined. Inserting this into the matrix element

$$\begin{aligned}&M_R(p',p)\\ &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp\right)\delta\left(p'_- - p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+ - p_+\right)\phi\right\}\int_s\int_u \frac{1}{(u+s)^2}\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2 - u\lambda^2\right]\right\}\\ &\times\left\{\exp i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}\Delta\right]\right.\\ &\times\left[\left(\hat{\pi}_{p,\Delta\mathcal{A}}(\phi_s) + \frac{s}{u+s}\hat{\mathcal{P}}_{\Delta\mathcal{A}(\phi_s)}(p)\right) - \frac{s}{2up_-}\left(\hat{\pi}_{p,\Delta\mathcal{A}}(\phi_s) - \frac{2u+s}{u+s}\hat{\mathcal{P}}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right.\\ &+\frac{s\hat{n}}{2(u+s)}\left\langle\frac{u+s}{up_-}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2\right] - \frac{s}{up_-}\left[\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2\right\rangle\\ &\left.\left.+2\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right](m-\hat{p})\right] - \frac{u+2s}{u+s}m + \frac{u}{u+s}\left[1-2i\frac{u+2s}{u+s}m^2 s\right](\hat{p}-m)\right\}\end{aligned} \quad (3.88)$$

Denoting $\boxed{\mathscr{P}(p,\Delta\mathcal{A}(\phi_s),s,u) \equiv \pi_{p,\Delta\mathcal{A}}(\phi_s) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)}$

$$\boxed{\begin{aligned}&M_R(p',p)\\ &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp\right)\delta\left(p'_- - p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+ - p_+\right)\phi\right\}\int_s\int_u \frac{1}{(u+s)^2}\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2 - u\lambda^2\right]\right\}\\ &\times\left\{\exp i\left[\frac{su}{u+s}\left(p^2-m^2\right) - \frac{s^2}{u+s}\Delta\right]\right.\\ &\times\left[\left(\hat{\pi}_{p,\Delta\mathcal{A}}(\phi_s) + \frac{s}{u+s}\hat{\mathcal{P}}_{\Delta\mathcal{A}(\phi_s)}(p)\right) - \frac{s}{2up_-}\left[i\gamma^5\left(\gamma\Delta\mathscr{F}^*(\phi_s)\mathscr{P}(p,\Delta\mathcal{A}(\phi_s),s,u)\right) - \left(\gamma\Delta\mathscr{F}(\phi_s)\mathscr{P}(p,\Delta\mathcal{A}(\phi_s),s,u)\right)\right]\right.\\ &+\frac{s\hat{n}}{2(u+s)}\left\langle\frac{u+s}{up_-}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2\right] - \frac{s}{up_-}\left[\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2\right\rangle\\ &\left.\left.+2\left[1+\frac{1}{2p_-}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right](m-\hat{p})\right] - \frac{u+2s}{u+s}m + \frac{u}{u+s}\left[1-2i\frac{u+2s}{u+s}m^2 s\right](\hat{p}-m)\right\}\end{aligned}} \quad (3.89)$$

Expression 3.89 reduces in the constant crossed field case (where $\boldsymbol{A}(\phi) = -\boldsymbol{E}\phi$) to the corresponding expression from [Rit70; Rit85].

3.4 On-shell renormalized mass operator

Sandwiching between the free Dirac spinors of the same momenta (the incoming momentum equals the outgoing momentum, i.e. $p = p'$), using the normalization relation 2.97 and the relations

$$\bar{u}_{\sigma'}(p)\gamma^\mu u_\sigma(p) = 2p^\mu\delta_{\sigma'\sigma} \implies \begin{cases}\bar{u}_{\sigma'}(p)\hat{n}u_\sigma(p) = 2(n\cdot p)\delta_{\sigma'\sigma} = 2p_-\delta_{\sigma'\sigma}\\ \bar{u}_{\sigma'}(p)\hat{\mathcal{A}}u_\sigma(p) = 2(p\cdot\mathcal{A})\delta_{\sigma'\sigma}\end{cases}$$

and denoting

$$\boxed{\xi^{\mu}_{\sigma'\sigma} \equiv -\bar{u}_{\sigma'}(p)\gamma^5\gamma^{\mu}u_{\sigma}(p)} \tag{3.90}$$

the matrix element becomes

$$\begin{aligned}
&\mathcal{M}_{R,\sigma'\sigma}\left(p',p\right)\\
&= (2\pi)^3\delta\left(\boldsymbol{p}'_{\perp}-\boldsymbol{p}_{\perp}\right)\delta\left(p'_{-}-p_{-}\right)\frac{\alpha}{2\pi}\int_{\phi}\exp\left\{i\left(p'_{+}-p_{+}\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2-u\lambda^2\right]\right\}\\
&\times\left\{\exp i\left[\frac{su}{u+s}\left(p^2-m^2\right)-\frac{s^2}{u+s}\Delta\right]\right.\\
&\times\left[2p\cdot\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)+\frac{s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\delta_{\sigma'\sigma}\right.\\
&+\frac{s}{2up_{-}}2\left[im\left(\xi_{\sigma'\sigma}\Delta\mathscr{F}^*(\phi_s)\mathscr{P}(p,\Delta\mathcal{A}(\phi_s),s,u)\right)+\delta_{\sigma\sigma'}\left(p\Delta\mathscr{F}(\phi_s)\mathscr{P}(p,\Delta\mathcal{A}(\phi_s),s,u)\right)\right]\\
&+\frac{s}{u+s}\left\langle\frac{u+s}{2up_{-}}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_{\perp}\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_{\perp}\left(\phi_{ys}\right)\right)^2\right]-\frac{s}{2up_{-}}\left[\Delta\boldsymbol{\mathcal{A}}_{\perp}(\phi_s)-\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_{\perp}\left(\phi_{ys}\right)\right]^2\right\rangle 2p_{-}\delta_{\sigma\sigma'}\\
&+2\bar{u}_{\sigma'}(p)\left[1+\frac{1}{2p_{-}}\hat{n}\widehat{\Delta\mathcal{A}}\left(\phi_s\right)\right](m-\hat{p})\,u_{\sigma}(p)\Big]\\
&\left.-\frac{u+2s}{u+s}2m^2\delta_{\sigma'\sigma}+\frac{u}{u+s}\left[1-2i\frac{u+2s}{u+s}m^2s\right]\left(2p^2-2m^2\right)\delta_{\sigma'\sigma}\right\}
\end{aligned} \tag{3.91}$$

Evaluating the matrix element **on shell** ($\hat{p}u_p = mu_p$ with $p^2 = m^2$) leads to

$$\begin{aligned}
&\mathcal{M}_{R,\sigma'\sigma}\left(p',p\right)\\
&= (2\pi)^3\delta\left(\boldsymbol{p}'_{\perp}-\boldsymbol{p}_{\perp}\right)\delta\left(p'_{-}-p_{-}\right)\frac{\alpha}{2\pi}\int_{\phi}\exp\left\{i\left(p'_{+}-p_{+}\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\times\exp\left\{i\left[-\frac{s^2}{u+s}m^2-u\lambda^2\right]\right\}\\
&\times\left\{\exp\left[-i\frac{s^2}{u+s}\Delta\right]\right.\\
&\times\left[2p\cdot\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)+\frac{s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\delta_{\sigma\sigma'}\right.\\
&+\frac{s}{up_{-}}\left[im\left(\xi_{\sigma'\sigma}\Delta\mathscr{F}^*(\phi_s)\mathscr{P}(p,\Delta\mathcal{A}(\phi_s),s,u)\right)+\delta_{\sigma\sigma'}\left(p\Delta\mathscr{F}(\phi_s)\mathscr{P}(p,\Delta\mathcal{A}(\phi_s),s,u)\right)\right]\\
&+\frac{s}{u(u+s)}\left\langle(u+s)\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_{\perp}\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_{\perp}\left(\phi_{ys}\right)\right)^2\right]-s\left[\Delta\boldsymbol{\mathcal{A}}_{\perp}(\phi_s)-\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_{\perp}\left(\phi_{ys}\right)\right]^2\right\rangle\delta_{\sigma'\sigma}\Big]\\
&\left.-\frac{u+2s}{u+s}2m^2\delta_{\sigma'\sigma}\right\}
\end{aligned} \tag{3.92}$$

Changing $m^2s = s'$ and $m^2u = u' \to u$, then dropping the primes $s' \to s$ and

$u' \to u$ and dropping the fictitious photon mass λ^2

$$\begin{aligned}
\mathcal{M}_{R,\sigma'\sigma}(p',p) &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp\right)\delta\left(p'_- - p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+ - p_+\right)\phi\right\}\int_s\int_u \frac{1}{(u+s)^2}\times\exp\left\{i\left[-\frac{s^2}{u+s}\right]\right\} \\
&\times\left\{\exp\left[-i\frac{1}{m^2}\frac{s^2}{u+s}\Delta\right]\right. \\
&\times\left[2p\cdot\left(\pi_{p,\Delta\mathcal{A}}\left(\phi_{s/m^2}\right) + \frac{s}{u+s}\mathcal{P}_{\Delta\mathcal{A}\left(\phi_{s/m^2}\right)}(p)\right)\delta_{\sigma'\sigma}\right. \\
&+\frac{s}{up_-}\left[im\left(\xi_{\sigma'\sigma}\Delta\mathscr{F}^*\left(\phi_{s/m^2}\right)\left(\pi_{p,\Delta\mathcal{A}}\left(\phi_{s/m^2}\right) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}\left(\phi_{s/m^2}\right)}(p)\right)\right)\right. \\
&\left.+\delta_{\sigma'\sigma}\left(p\Delta\mathscr{F}(\phi_{s/m^2})\left(\pi_{p,\Delta\mathcal{A}}\left(\phi_{s/m^2}\right) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}\left(\phi_{s/m^2}\right)}(p)\right)\right)\right] \\
&+\left\langle\frac{s}{u}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys/m^2}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys/m^2}\right)\right)^2\right]\right. \\
&\left.\left.-\frac{s^2}{u(u+s)}\left[\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_{s/m^2}) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys/m^2}\right)\right]^2\right\rangle\delta_{\sigma'\sigma}\right] \\
&\left.-\frac{u+2s}{u+s}2m^2\delta_{\sigma'\sigma}\right\}.
\end{aligned} \tag{3.93}$$

Changing notation $\phi_{s/m^2} \equiv \phi - 2\frac{s}{m^2}\frac{up_-}{u+s} \mapsto \phi_s \equiv \phi - 2\frac{s}{m^2}\frac{up_-}{u+s}$

$$\boxed{\begin{aligned}
&\mathcal{M}_{R,\sigma'\sigma}(p',p) \\
&= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp\right)\delta\left(p'_- - p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+ - p_+\right)\phi\right\}\int_s\int_u \frac{1}{(u+s)^2}\times\exp\left\{i\left[-\frac{s^2}{u+s}\right]\right\} \\
&\times\left\{\exp\left[-i\frac{1}{m^2}\frac{s^2}{u+s}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2\right]\right]\right. \\
&\times\left[2p\cdot\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) + \frac{s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\delta_{\sigma'\sigma}\right. \\
&+\frac{s}{up_-}\left[im\left(\xi_{\sigma'\sigma}\Delta\mathscr{F}^*(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right)\right. \\
&\left.+\delta_{\sigma'\sigma}\left(p\Delta\mathscr{F}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right)\right] \\
&\left.+\left\langle\frac{s}{u}\left[\int_0^1 dy\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)^2\right] - \frac{s^2}{u(u+s)}\left[\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right]^2\right\rangle\delta_{\sigma'\sigma}\right] \\
&\left.-\frac{u+2s}{u+s}2m^2\delta_{\sigma'\sigma}\right\}.
\end{aligned}} \tag{3.94}$$

Defining now the dimensionless quantities corresponding to 3.85

$$\begin{aligned}
\boldsymbol{\xi}_\perp(\phi) \equiv \mathcal{A}_\perp(\phi)/m, &\quad \Delta\boldsymbol{\xi}_\perp(\phi_s) \equiv \boldsymbol{\xi}_\perp(\phi_s) - \boldsymbol{\xi}_\perp(\phi) \\
\zeta^\mu_{\sigma'\sigma} &= -\bar{u}_{\sigma'}(p)\gamma^5\gamma^\mu u_\sigma(p)/2m \equiv \xi^\mu_{\sigma'\sigma}/2m \\
\tilde{\xi}^{\mu\nu}(\phi) \equiv \mathscr{F}^{*\mu\nu}(\phi)/m, &\quad \Delta\tilde{\xi}^{\mu\nu}(\phi_s) \equiv \Delta\tilde{\xi}^{\mu\nu}(\phi_s) - \tilde{\xi}^{\mu\nu}(\phi)
\end{aligned} \tag{3.95}$$

where $\zeta^{\mu}_{\sigma'\sigma}$ is the covariant spin-density matrix [Sei+18], the matrix element is

$$
\begin{aligned}
&\mathcal{M}_{R,\sigma'\sigma}(p',p)\\
&= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp - \boldsymbol{p}_\perp\right)\delta\left(p'_- - p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+ - p_+\right)\phi\right\}\int_s\int_u \frac{1}{(u+s)^2}\times\exp\left\{i\left[-\frac{s^2}{u+s}\right]\right\}\\
&\times\left\{\exp\left[-i\frac{s^2}{u+s}\left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]\right]\right.\\
&\times\left[2p\cdot\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) + \frac{s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\delta_{\sigma'\sigma}\right.\\
&+i\frac{s}{u}\frac{1}{p_-}\left[m^2\left(\zeta_{\sigma'\sigma}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right)\right.\\
&\left.\left.-i\delta_{\sigma'\sigma}\left(p\Delta\mathscr{F}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right)\right]\right]\\
&+m^2\left\langle\frac{s}{u}\left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right] - \frac{s^2}{u(u+s)}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s) - \int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right]^2\right\rangle\delta_{\sigma'\sigma}\Bigg]\\
&\left.-\frac{u+2s}{u+s}2m^2\delta_{\sigma'\sigma}\right\}.
\end{aligned}
\tag{3.96}
$$

The following simplifications can be made

$$
\begin{aligned}
&p^\mu\Delta\mathscr{F}_{\mu\nu}\left(\phi_s\right)\left(\pi_{p,\Delta\mathcal{A}}\left(\phi_s\right) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)^\nu\\
&= p^\mu\Delta\left[n_\mu\mathcal{A}_\nu(\phi_s) - n_\nu\mathcal{A}_\mu(\phi_s)\right]\left(\pi_{p,\Delta\mathcal{A}}\left(\phi_s\right) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)^\nu\\
&= p^\mu\Delta\left[n_\mu\mathcal{A}_\nu(\phi_s) - n_\nu\mathcal{A}_\mu(\phi_s)\right]\left[-\Delta\mathcal{A}^\nu\left(\phi_s\right) - \frac{2u+s}{u+s}\left(-\int_0^1 dy\Delta\mathcal{A}^\nu\left(\phi_{ys}\right)\right)\right]\\
&= p^\mu\Delta\left[n_\mu\mathcal{A}_\nu(\phi_s) - n_\nu\mathcal{A}_\mu(\phi_s)\right]\left[-\Delta\mathcal{A}^\nu\left(\phi_s\right) + \frac{2u+s}{u+s}\int_0^1 dy\Delta\mathcal{A}^\nu\left(\phi_{ys}\right)\right]\\
&= p^\mu\Delta\left[n_\mu\mathcal{A}_\nu(\phi_s)\right]\left[-\Delta\mathcal{A}^\nu\left(\phi_s\right) + \frac{2u+s}{u+s}\int_0^1 dy\Delta\mathcal{A}^\nu\left(\phi_{ys}\right)\right]\\
&= p_-\Delta\mathcal{A}_\nu(\phi_s)\left(-\Delta\mathcal{A}^\nu\left(\phi_s\right) + \frac{2u+s}{u+s}\int_0^1 dy\Delta\mathcal{A}^\nu\left(\phi_{ys}\right)\right) \qquad \left(\text{Lorentz gauge and } A^0 = 0\right)\\
&= p_-\Delta\boldsymbol{\mathcal{A}}_\perp(\phi_s)\cdot\left(\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_s\right) - \frac{2u+s}{u+s}\int_0^1 dy\Delta\boldsymbol{\mathcal{A}}_\perp\left(\phi_{ys}\right)\right)\\
&= m^2 p_-\Delta\boldsymbol{\xi}_\perp(\phi_s)\cdot\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_s\right) - \frac{2u+s}{u+s}\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right),
\end{aligned}
\tag{3.97}
$$

$$
\begin{aligned}
&p\cdot\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) + \frac{s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\\
&= p^2\left(1 + \frac{s}{u+s}\right) - p\cdot\left(\Delta\mathcal{A}\left(\phi_s\right) + \frac{s}{u+s}\int_0^1 dy\Delta\mathcal{A}\left(\phi_{ys}\right)\right) + p\cdot\left(\Delta\mathcal{A}\left(\phi_s\right) + \frac{s}{u+s}\int_0^1 dy\Delta\mathcal{A}\left(\phi_{ys}\right)\right)\\
&-\frac{1}{2}\left(\left[\Delta\mathcal{A}\left(\phi_s\right)\right]^2 + \frac{s}{u+s}\left(\int_0^1 dy\Delta\mathcal{A}\left(\phi_{ys}\right)\right)^2\right)\\
&= \frac{u+2s}{u+s}m^2 - \frac{1}{2}\left(\left[\Delta\mathcal{A}\left(\phi_s\right)\right]^2 + \frac{s}{u+s}\left(\int_0^1 dy\Delta\mathcal{A}\left(\phi_{ys}\right)\right)^2\right)\\
&\implies 2p\cdot\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) + \frac{s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right) = \frac{u+2s}{u+s}2m^2 + m^2\left(\left[\Delta\boldsymbol{\xi}_\perp\left(\phi_s\right)\right]^2 + \frac{s}{u+s}\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right),
\end{aligned}
\tag{3.98}
$$

where in the last lines the notation from 3.95 was inserted.

Using these simplifications, dividing by m^2, the matrix element becomes

$$\begin{aligned}
\frac{\mathcal{M}_{R,\sigma'\sigma}(p',p)}{m^2} &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\times\exp\left\{-i\frac{s^2}{u+s}\right\}\\
&\times\left\{\exp\left[-i\frac{s^2}{u+s}\left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]\right]\right.\\
&\times\left[\frac{u+2s}{u+s}2+\left(\left[\Delta\boldsymbol{\xi}_\perp\left(\phi_s\right)\right]^2+\frac{s}{u+s}\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right)\delta_{\sigma'\sigma}\right.\\
&+i\frac{s}{u}\frac{1}{p_-}\left(\zeta_{\sigma'\sigma}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)-\frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right)\\
&+\frac{s}{u}\delta_{\sigma'\sigma}\Delta\boldsymbol{\xi}_\perp(\phi_s)\cdot\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_s\right)-\frac{2u+s}{u+s}\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)\\
&+\left.\left\langle\frac{s}{u}\left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]-\frac{s^2}{u(u+s)}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)-\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right]^2\right\rangle\delta_{\sigma'\sigma}\right]\\
&\left.-\frac{u+2s}{u+s}2\delta_{\sigma'\sigma}\right\}.
\end{aligned} \tag{3.99}$$

After simple algebraic manipulations, by collecting terms, the matrix element is

$$\begin{aligned}
\frac{\mathcal{M}_{R,\sigma'\sigma}(p',p)}{m^2} &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\times\exp\left\{-i\frac{s^2}{u+s}\right\}\\
&\times\left\{\exp\left[-i\frac{s^2}{u+s}\left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]\right]\right.\\
&\times\left[\left(\frac{u+2s}{u+s}2+\frac{u+2s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp\left(\phi_s\right)\right]^2-2\frac{s}{u}\frac{s}{u+s}\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right)\delta_{\sigma'\sigma}\right.\\
&+i\frac{s}{u}\frac{1}{p_-}\left(\zeta_{\sigma'\sigma}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)-\frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right)\\
&+\left.\left.\left\langle\frac{s}{u}\left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]-\frac{s}{u}\frac{2u-s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right]\right\rangle\delta_{\sigma\sigma'}\right]-\frac{u+2s}{u+s}2\delta_{\sigma'\sigma}\right\}.
\end{aligned} \tag{3.100}$$

Collecting the $(u+2s)/(u+s)$ terms

$$\boxed{\begin{aligned}
\frac{\mathcal{M}_{R,\sigma'\sigma}(p',p)}{m^2} &= (2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\frac{\alpha}{2\pi}\int_\phi \exp\left\{i\left(p'_+-p_+\right)\phi\right\}\int_s\int_u\frac{1}{(u+s)^2}\\
&\times\left\{2\frac{u+2s}{u+s}\delta_{\sigma'\sigma}\left[\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]\right]-\exp\left\{-i\frac{s^2}{u+s}\right\}\right]\right.\\
&+\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]\right]\\
&\times\left[\left(\frac{u+2s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp\left(\phi_s\right)\right]^2-2\frac{s}{u}\frac{s}{u+s}\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right)\delta_{\sigma'\sigma}\right.\\
&+i\frac{s}{u}\frac{1}{p_-}\left(\zeta_{\sigma'\sigma}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)-\frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right)\\
&+\left.\left.\left\langle\frac{s}{u}\left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]-\frac{s}{u}\frac{2u-s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right]\right\rangle\delta_{\sigma'\sigma}\right]\right\}.
\end{aligned}} \tag{3.101}$$

Therefore, the matrix element can $M_R\left(p',p\right)$ can be cast in the form

$$M_R\left(p',p\right)=(2\pi)^3\delta\left(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp\right)\delta\left(p'_--p_-\right)\int d\phi e^{i\left(p'_+-p_+\right)\phi}\mathcal{M}_R\left(p',p,\phi\right) \tag{3.102}$$

and defining $\mathcal{M}_{R,\sigma\sigma'}(p',p,\phi)$ as

$$\mathcal{M}_{R,\sigma'\sigma}(p',p,\phi) \equiv \bar{u}_{\sigma'}(p)\mathcal{M}_R(p',p,\phi)\,u_\sigma(p)/2m \tag{3.103}$$

where the spin orientations along the spin quantization axis can take the values $\sigma' = \pm 1$ and $\sigma = \pm 1$, we see that

$$\begin{aligned}
\frac{\mathcal{M}_{R,\sigma'\sigma}(p',p,\phi)}{m} &= \frac{\alpha}{4\pi}\int_s\int_u \frac{1}{(u+s)^2} \\
&\times\left\{2\frac{u+2s}{u+s}\delta_{\sigma'\sigma}\left[\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\right]\right]-\exp\left\{-i\frac{s^2}{u+s}\right\}\right]\right. \\
&+\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\right]\right] \\
&\times\left[\left(\frac{u+2s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\right]^2-2\frac{s}{u}\frac{s}{u+s}\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\right)\delta_{\sigma'\sigma}\right. \\
&+i\frac{s}{u}\frac{1}{p_-}\left(\zeta_{\sigma'\sigma}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)-\frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right) \\
&\left.\left.+\left\langle\frac{s}{u}\left[\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2\right]-\frac{s}{u}\frac{2u-s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right]\right\rangle\delta_{\sigma'\sigma}\right]\right\},
\end{aligned} \tag{3.104}$$

which will be used next in obtaining the mass shift using the Schwinger-Dyson equation. In spin space, writing the components

$$\begin{aligned}
\frac{\mathcal{M}_{R,\pm\pm}(p',p,\phi)}{m} &= \frac{\alpha}{4\pi}\int_s\int_u \frac{1}{(u+s)^2} \\
&\times\left\{2\frac{u+2s}{u+s}\left[\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\right]\right]-\exp\left\{-i\frac{s^2}{u+s}\right\}\right]\right. \\
&+\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\right]\right] \\
&\times\left[\left(\frac{u+2s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\right]^2-2\frac{s}{u}\frac{s}{u+s}\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\right)\right. \\
&+i\frac{s}{u}\frac{1}{p_-}\left(\zeta_{\pm\pm}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)-\frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right) \\
&\left.\left.+\left\langle\frac{s}{u}\left[\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2\right]-\frac{s}{u}\frac{2u-s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right]\right\rangle\right]\right\},
\end{aligned} \tag{3.105}$$

and also for the off-diagonal terms in spin space

$$\begin{aligned}
\frac{\mathcal{M}_{R,\pm\mp}(p',p,\phi)}{m} &= i\frac{\alpha}{4\pi}\int_s\int_u \frac{1}{(u+s)^2} \\
&\times\left\{\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\right]\right]\right. \\
&\left.\times\left[\frac{s}{u}\frac{1}{p_-}\left(\zeta_{\pm\mp}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)-\frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right)\right]\right\},
\end{aligned} \tag{3.106}$$

4 Mass shift and anomalous magnetic moment

4.1 The mass shift of the electron

The Schwinger-Dyson equation in terms of the renormalized quantities is

$$\{\gamma^\mu [i\partial_\mu - \mathcal{A}_\mu(\phi)] - m\} \Psi(x) = \int d^4y M_R(x,y)\Psi(y) \tag{4.1}$$

where $\Psi(x)$ is the spinor of mass m in the plane-wave background $\mathcal{A}^\mu(\phi) = eA^\mu(\phi)$.

In order to identify the mass shift, the ansatz for the self-energy corrected Volkov state can be taken of the form

$$\Psi(x) = f_\sigma(p,\phi)\, E(p,x)\, u_\sigma(p), \tag{4.2}$$

where $f_\sigma(p,\phi)$ is the correction factor that includes a correction $\delta f_\sigma(p,\phi)$ of the order of the mass operator, which is $\mathcal{O}(\alpha)$, i.e.

$$f_\sigma(p,\phi) = 1 + \delta f_\sigma(p,\phi), \quad \delta f_\sigma(p,\phi) = \mathcal{O}(\alpha). \tag{4.3}$$

Similarly, as it will be shown below, the ansatz in 4.2 can be viewed as a Volkov state where the Ritus matrix gets mass shifted (of $\delta m_\sigma(p,\phi)$)

$$\Psi(x) = E(p,x)\,|_{(m+\delta m_\sigma(p,\phi))} u_\sigma(p), \tag{4.4}$$

which will help relate $f_\sigma(p,\phi)$ to the mass shift. To determine the mass shift, notice that to the first-order in α

$$f_\sigma(p,\phi) \approx e^{\int_0^\phi d\phi' \delta f'_\sigma(p,\phi')}, \tag{4.5}$$

where the prime over δf_σ indicates the derivative with respect to the light-cone time ϕ. To see that the shift $\delta f'_\sigma(p,\phi)$ can be expressed in terms of the mass shift $\delta m_\sigma(p,\phi)$, notice (on-shell $p_+ = (m^2 + \boldsymbol{p}_\perp^2)/2p_-$)

$$\begin{aligned}
\Psi(x) &= f_\sigma(p,\phi)\, E(p,x)\, u_\sigma(p) \\
&= e^{\int_0^\phi d\phi' \delta f'_\sigma(p,\phi')} \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] e^{i\left\{-p_+\phi - p_- T + \boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp - \int_0^\phi d\varphi \left[\frac{(p\cdot\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]\right\}} u_\sigma(p) \\
&= \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] e^{i\left\{-i\int_0^\phi d\phi' \delta f'_\sigma(p,\phi') - \int_0^\phi d\phi' \left(m^2 + \boldsymbol{p}_\perp^2\right)/2p_- - p_- T + \boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp - \int_0^\phi d\varphi\left[\frac{(p\cdot\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]\right\}} u_\sigma(p).
\end{aligned} \tag{4.6}$$

Using the two equivalent forms from 4.2 and 4.4, that is $f_\sigma(p,\phi)\,E(p,x)\,u_\sigma(p) = E(p,x)\,|_{(m+\delta m_\sigma(p,\phi))} u_\sigma(p)$ and that the interaction with the plane wave (from the self-energy diagram) conserves the momenta p_- and $\boldsymbol{p}_\perp$, such that the change can be only in the p_+ component, i.e.

$$\begin{aligned}&\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] e^{i\left\{-i\int_0^\phi d\phi'\delta f'_\sigma(p,\phi') - \int_0^\phi d\phi'\left(m^2+\boldsymbol{p}_\perp^2\right)/2p_- - p_-\tau + \boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp - \int_0^\phi d\varphi\left[\frac{(p\cdot\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]\right\}} u_\sigma(p)\\ &= \left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] e^{i\left\{-\int_0^\phi d\phi'\left((m+\delta m_\sigma(p,\phi'))^2+\boldsymbol{p}_\perp^2\right)/2p_- - p_-\tau + \boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp - \int_0^\phi d\varphi\left[\frac{(p\cdot\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]\right\}} u_\sigma(p)\end{aligned} \tag{4.7}$$

it can be identified that

$$\begin{aligned} i\int_0^\phi d\phi'\delta f'_\sigma(p,\phi') + m^2\phi/2p_- &= \int_0^\phi d\phi'\left(m+\delta m_\sigma(p,\phi')\right)^2\phi/2p_-\\ &\simeq m^2\phi/2p_- + \int_0^\phi d\phi' 2m\delta m_\sigma(p,\phi')\phi/2p_-\end{aligned} \tag{4.8}$$

such that if $\delta f'_\sigma(p,\phi) \simeq -im\delta m_\sigma(p,\phi)/p_-$, then

$$\boxed{f_\sigma(p,\phi) \approx e^{-i\frac{m}{p_-}\int_0^\phi d\phi'\delta m_\sigma(p,\phi')}} \tag{4.9}$$

which is valid under the assumption $(m/p_-)\left|\int_0^\phi d\phi'\delta m_\sigma(p,\phi')\right| \ll 1$.

Equivalently, the change in p_+ can be written as

$$\boxed{-i\delta p_{+,\sigma}(p,\phi') = -im\delta m_\sigma(p,\phi')/p_-} \tag{4.10}$$

where $p_+ = \left(m^2+\boldsymbol{p}_\perp^2\right)/2p_- \mapsto p_+ + \delta p_{+,\sigma}(p,\phi) = \left((m+\delta m_\sigma(p,\phi))^2+\boldsymbol{p}_\perp^2\right)/2p_-$, while the other momentum components are unchanged.

4.2 The Schwinger-Dyson equation

Summing over the two spin components to get the unpolarized state, the substitution of the ansatz for the mass shifted Volkov state 4.2, of the form

$$\Psi(x) = E(p,x)e^{-i\frac{m}{p_-}\int_0^\phi d\phi'\delta m_\sigma(p,\phi')}u_\sigma(p) \tag{4.11}$$

into the Schwinger-Dyson equation 4.1, leads to

$$\begin{aligned}&\left\{\gamma^\mu\left[i\partial_\mu - \mathcal{A}_\mu(\phi)\right] - m\right\}E(p,x)e^{-i\frac{m}{p_-}\int_0^\phi d\phi'\delta m_\sigma(p,\phi')}u_\sigma(p)\\ &= \int d^4y M_R(x,y)E(p,y)e^{-i\frac{m}{p_-}\int_0^\phi d\phi'\delta m_\sigma(p,\phi')}u_\sigma(p).\end{aligned} \tag{4.12}$$

Knowing that

$$M_R(x,y) = \int \frac{d^4l}{(2\pi)^4}\frac{d^4l'}{(2\pi)^4}E(l,x)M_R(l,l')\bar{E}(l',y), \tag{4.13}$$

and dropping the $\delta f_\sigma = \mathcal{O}(\alpha)$ as $M_R(l,l')$ is already of order α, the equation is

$$\begin{aligned}&\left\{\gamma^\mu\left[i\partial_\mu - \mathcal{A}_\mu(\phi)\right] - m\right\} E(p,x) e^{-i\frac{m}{p_-}\int_0^\phi d\phi' \delta m_\sigma(p,\phi')} u_\sigma(p) \\ &= \int_y \int_l \int_{l'} E(l,x) M_R(l,l') \bar{E}(l',y) E(p,y) u_\sigma(p),\end{aligned} \tag{4.14}$$

where the shorthand notation for the integral (not writing explicitly the measure) has been used.

Using the completeness relation 2.107 of the Ritus matrices to get a momentum space $(2\pi)^4\delta^4(l'-p)$ and integrating over l', the Schwinger-Dyson equation becomes

$$\left\{\gamma^\mu\left[i\partial_\mu - \mathcal{A}_\mu(\phi)\right] - m\right\} E(p,x) e^{-i\frac{m}{p_-}\int_0^\phi d\phi' \delta m_\sigma(p,\phi')} u_\sigma(p) = \int_l E(l,x) M_R(l,p) u_\sigma(p). \tag{4.15}$$

Inserting on the RHS the form from 3.102, while on the LHS using that $\gamma^\mu\left[i\partial_\mu - \mathcal{A}_\mu(\phi)\right] E(p,x) = E(p,x)\left[\hat{p} + \gamma^\mu i\partial_\mu\right]$ (see Appendix A.1 for the derivation) and that $\partial_\mu g(\phi) = n_\mu \partial_\phi g(\phi)$

$$\begin{aligned}&\left\{E(p,x)\left[(\hat{p} - m) + i\hat{n}\partial_\phi\right]\right\} e^{-i\frac{m}{p_-}\int_0^\phi d\phi' \delta m_\sigma(p,\phi')} u_\sigma(p) \\ &= \int_l E(l,x)(2\pi)^3\delta^2\left(\boldsymbol{p}_\perp - \boldsymbol{l}_\perp\right)\delta\left(p_- - l_-\right) \int d\phi e^{-i(p_+ - l_+)\phi} \mathcal{M}_R\left(l,p,\phi\right) u_\sigma(p).\end{aligned} \tag{4.16}$$

Using the on-shell relation $(\hat{p} - m)\, u_\sigma(p) = 0$[1] on the LHS

$$\begin{aligned}&\left[E(p,x) i\hat{n}\partial_\phi\right] e^{-i\frac{m}{p_-}\int_0^\phi d\phi' \delta m_\sigma(p,\phi')} u_\sigma(p) \\ &= \int_l E(l,x)(2\pi)^3\delta^2\left(\boldsymbol{p}_\perp - \boldsymbol{l}_\perp\right)\delta\left(p_- - l_-\right) \int d\phi e^{-i(p_+ - l_+)\phi} \mathcal{M}_R\left(l,p,\phi\right) u_\sigma(p).\end{aligned} \tag{4.17}$$

Applying the ϕ-derivative to the exponential on the LHS and keeping only the $\mathcal{O}(\delta m_\sigma(p,\phi))$ terms, the equation becomes

$$\begin{aligned}&\left[E(p,x)\hat{n}\frac{m}{p_-}\delta m_\sigma\left(p,\phi\right)\right] u_\sigma(p) \\ &= \int_l E(l,x)(2\pi)^3\delta^2\left(\boldsymbol{p}_\perp - \boldsymbol{l}_\perp\right)\delta\left(p_- - l_-\right) \int d\phi e^{-i(p_+ - l_+)\phi} \mathcal{M}_R\left(l,p,\phi\right) u_\sigma(p).\end{aligned} \tag{4.18}$$

Projecting by $\bar{E}(p,x)$ from the left (using $\bar{E}(p,x)E(p,x) = 1$) and also that $\delta^2\left(\boldsymbol{p}_\perp - \boldsymbol{l}_\perp\right)\delta\left(p_- - l_-\right) E(l,x) = \delta^2\left(\boldsymbol{p}_\perp - \boldsymbol{l}_\perp\right)\delta\left(p_- - l_-\right) E(p,x) e^{i(p_+ - l_+)\phi_x}$, the Schwinger-Dyson equation is

$$\begin{aligned}&\left[\hat{n}\frac{m}{p_-}\delta m_\sigma\left(p,\phi\right)\right] u_\sigma(p) \\ &= \int_l e^{i(p_+ - l_+)\phi_x}(2\pi)^3\delta^2\left(\boldsymbol{p}_\perp - \boldsymbol{l}_\perp\right)\delta\left(p_- - l_-\right) \int d\phi e^{-i(p_+ - l_+)\phi} \mathcal{M}_R\left(l,p,\phi\right) u_\sigma(p).\end{aligned} \tag{4.19}$$

Integrating over $\boldsymbol{l}_\perp$ and l_-, only the l_+ integral remains

$$\left[\hat{n}\frac{m}{p_-}\delta m_\sigma\left(p,\phi\right)\right] u_\sigma(p) = \int_{l_+} e^{i(p_+ - l_+)\phi_x} \int d\phi e^{-i(p_+ - l_+)\phi} \mathcal{M}_R\left(l,p,\phi\right)|_{\boldsymbol{l}_\perp = \boldsymbol{p}_\perp, l_- = p_-} u_\sigma(p). \tag{4.20}$$

[1] See 2.96

Since $\mathcal{M}_R(l,p,\phi)$ does not depend on l_+ as found previously, we can pull it outside the l_+ integral and change the order

$$\hat{n}\frac{m}{p_-}\delta m_\sigma(p,\phi)\,u_\sigma(p) = \int d\phi \mathcal{M}_R(p,p,\phi)\int_{l_+} e^{i(p_+-l_+)(\phi_x-\phi)}u_\sigma(p). \tag{4.21}$$

Using that $\int \frac{dl_+}{(2\pi)}e^{-il_+(\phi_x-\phi)} = \delta(\phi_x-\phi)$

$$\hat{n}\frac{m}{p_-}\delta m_\sigma(p,\phi)\,u_\sigma(p) = \int d\phi \mathcal{M}_R(p,p,\phi)\,e^{ip_+(\phi_x-\phi)}\delta(\phi_x-\phi)\,u_\sigma(p). \tag{4.22}$$

Integrating over the ϕ and dropping the x from ϕ_x

$$\hat{n}\frac{m}{p_-}\delta m_\sigma(p,\phi)\,u_\sigma(p) = \mathcal{M}_R(p,p,\phi)\,u_\sigma(p). \tag{4.23}$$

Projecting by multiplying by $\bar{u}_{\sigma'}(p)$ on the LHS

$$\frac{m}{p_-}\delta m_\sigma(p,\phi)\,\bar{u}_{\sigma'}(p)\hat{n}u_\sigma(p) = \bar{u}_{\sigma'}(p)\mathcal{M}_R(p,p,\phi)\,u_\sigma(p). \tag{4.24}$$

Using $\bar{u}_{\sigma'}(p)\hat{n}u_\sigma(p) = 2p_-\delta_{\sigma'\sigma}$

$$\frac{m}{p_-}\delta m_\sigma(p,\phi)\,2p_-\delta_{\sigma'\sigma} - \bar{u}_{\sigma'}(p)\mathcal{M}_R(p,p,\phi)\,u_\sigma(p) = 0. \tag{4.25}$$

Diving by $2m$ and plugging in the expression from 3.103

$$\delta m_\sigma(p,\phi)\,\delta_{\sigma'\sigma} - \mathcal{M}_{R,\sigma'\sigma}(p,p,\phi) = 0. \tag{4.26}$$

Therefore, the mass shift comes from the diagonal part of the matrix element

$$\boxed{\delta m_\sigma(p,\phi) - \mathcal{M}_{R,\sigma\sigma}(p,p,\phi) = 0.} \tag{4.27}$$

Plugging in $\mathcal{M}_{R,\sigma\sigma}(p,p,\phi)$ from 3.105, the electron mass shift is

$$\boxed{\begin{aligned}\frac{\delta m_\sigma(p,\phi)}{m} &= \frac{\alpha}{4\pi}\int_s\int_u \frac{1}{(u+s)^2} \\ &\times \Bigg\{2\frac{u+2s}{u+s}\Bigg[\exp\Bigg[-i\frac{s^2}{u+s}\Bigg[1+\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2 - \left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\Bigg]\Bigg] - \exp\left\{-i\frac{s^2}{u+s}\right\}\Bigg] \\ &+ \exp\Bigg[-i\frac{s^2}{u+s}\Bigg[1+\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2 - \left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\Bigg]\Bigg] \\ &\times \Bigg[\frac{u+2s}{u+s}(\Delta\boldsymbol{\xi}_\perp(\phi_s))^2 - 2\frac{s}{u}\frac{s}{u+s}\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2 \\ &+ i\frac{s}{u}\frac{1}{p_-}\left[\zeta_{\sigma\sigma}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s) - \frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right] \\ &+ \frac{s}{u}\left[\int_0^1 dy\,(\Delta\boldsymbol{\xi}_\perp(\phi_{ys}))^2\right] - \frac{s}{u}\frac{2u-s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right]\Bigg]\Bigg\}.\end{aligned}} \tag{4.28}$$

4.3 Electron mass shift in a linearly polarizated plane-wave

Expression 4.28 simplifies if a linearly polarized plane-wave background is used,

$$A^\mu(\phi) = A_0 a^\mu \psi(\phi), \tag{4.29}$$

with the amplitude $A_0 < 0$ and profile shape $\psi(\phi)$, under the gauge fixing condition $a^0 = 0$, which completely fixes the Lorentz gauge, and where $\boldsymbol{a}$ (normalizated to $\boldsymbol{a}^2 = 1$) defines the direction of oscillation.

For example, for a monochromatic plane wave of angular frequency ω_0 and peark electric field E_0, the amplitude is $A_0 = -E_0/\omega_0$ and a choice $\psi(\phi) = \cos(\omega_0\phi)$, while for a constant crossed field case $A_0 = -E_0/\omega_0$ and $\psi(\phi) = \omega_0\phi$.

The field strength tensor $F^{\mu\nu}(\phi)$ and its dual $\tilde{F}^{\mu\nu}(\phi)$ (see 2.53) are

$$F^{\mu\nu}(\phi) = A_0^{\mu\nu}\psi'(\phi) \quad \tilde{F}^{\mu\nu}(\phi) = \tilde{A}_0^{\mu\nu}\psi'(\phi), \tag{4.30}$$

with the corresponding amplitudes given by

$$A_0^{\mu\nu} = A_0\left(n^\mu a^\nu - n^\nu a^\mu\right), \qquad \tilde{A}_0^{\mu\nu} = (1/2)\varepsilon^{\mu\nu\lambda\rho}A_{0,\lambda\rho}. \tag{4.31}$$

Using 3.85 similar but dimensionless quantities are defined as

$$\xi^{\mu\nu}(\phi) = \mathscr{F}^{\mu\nu}(\phi)/m, \qquad \tilde{\xi}^{\mu\nu}(\phi) = \mathscr{F}^{*\mu\nu}(\phi)/m, \tag{4.32}$$

having the corresponding differences

$$\Delta\xi^{\mu\nu}(\phi_s) = \left[\xi^{\mu\nu}(\phi_s) - \xi^{\mu\nu}(\phi)\right]/m, \quad \Delta\tilde{\xi}^{\mu\nu}(\phi_s) = \left[\tilde{\xi}^{\mu\nu}(\phi_s) - \tilde{\xi}^{\mu\nu}(\phi)\right]/m. \tag{4.33}$$

Defining the spin 4-pseudovector (see Appendix A) [PP21]

$$\zeta^\mu = -\sigma\bar{u}_\sigma(p)\gamma^5\gamma^\mu u_\sigma(p)/2m = \sigma\zeta^\mu_{\sigma\sigma} \tag{4.34}$$

where $\sigma = \pm 1$ is the quantum number describing the spin degree of freedom of the spinor $u_\sigma(p)$ along the chosen spin quantization axis. Choosing the spin quantization axis parallel to the magnetic field axis in the electron rest frame, it can be written as (Appendix B.2)

$$\zeta^\mu = -\tilde{A}_0^{\mu\nu}p_\nu/\left(p_- A_0\right). \tag{4.35}$$

Defining the 4-vector

$$\tilde{\chi}^\mu(\phi) \equiv \tilde{F}^{\mu\nu}(\phi)p_\nu/mE_{\text{cr}} = \tilde{\mathcal{F}}^{\mu\nu}(\phi)p_\nu/meE_{\text{cr}} = \left(\tilde{\xi}'(\phi)p\right)^\mu/eE_{\text{cr}}, \tag{4.36}$$

the local quantum nonlinearity parameter can be written as

$$\chi(\phi) = -p_- A_0\psi'(\phi)/mE_{cr} = -(\zeta\tilde{\chi}(\phi)) \overset{E_{\text{cr}}=m^2/|e|}{=} -p_- eA_0\psi'(\phi)/m^3. \tag{4.37}$$

Then, the spin-dependent factor from 4.28 can be manipulated as

$$
\begin{aligned}
& i\frac{s}{u}\frac{1}{p_-}\left[\zeta_{\sigma\sigma}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)-\frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right] \\
&= i\sigma\frac{s}{u}\frac{1}{p_-}\left[\zeta\Delta\tilde{\xi}(\phi_s)\left(-\frac{u}{u+s}p-\Delta\mathcal{A}\left(\phi_s\right)+\frac{2u+s}{u+s}\int_0^1 dy\Delta\mathcal{A}\left(\phi_{ys}\right)\right)\right] \\
&= i\sigma\frac{s}{u}\frac{1}{p_-}\left[-\frac{u}{u+s}\zeta\Delta\tilde{\xi}(\phi_s)p\right] \\
&= i\sigma\frac{s}{u}\frac{1}{p_-}\left[-\frac{u}{u+s}\zeta\int_{\phi}^{\phi_s} d\tilde{\phi}\left[\tilde{\xi}'\left(\tilde{\phi}\right)\right]p\right] \\
&= i\sigma\frac{s}{u}\frac{eE_{\text{cr}}}{p_-}\left[-\frac{u}{u+s}\zeta\int_{\phi}^{\phi_s} d\tilde{\phi}\tilde{\chi}\left(\tilde{\phi}\right)\right] \\
&= -i\sigma\frac{s}{u}\frac{eE_{\text{cr}}}{p_-}\left[-\frac{u}{u+s}\int_{\phi}^{\phi_s} d\tilde{\phi}\chi\left(\tilde{\phi}\right)\right] \\
&= i\sigma\frac{s}{u+s}\frac{eE_{\text{cr}}}{p_-}\int_{\phi}^{\phi_s} d\tilde{\phi}\chi(\tilde{\phi}),
\end{aligned}
\tag{4.38}
$$

where in the third line $\Delta\tilde{\xi}(\phi_s)\cdot\Delta\mathcal{A}(\phi_s)=0$ and $\Delta\tilde{\xi}(\phi_s)\cdot n=0$ were used.

Changing $\tilde{\phi}\equiv\phi_{ys}=\phi-2uysp_-/m^2(u+s)\mapsto y(\tilde{\phi})=(\phi-\tilde{\phi})\frac{m^2(u+s)}{2usp_-}$

$$
\begin{aligned}
& i\frac{s}{u}\frac{1}{p_-}\left[\zeta_{\sigma\sigma}\Delta\tilde{\xi}(\phi_s)\left(\pi_{p,\Delta\mathcal{A}}(\phi_s)-\frac{2u+s}{u+s}\mathcal{P}_{\Delta\mathcal{A}(\phi_s)}(p)\right)\right] \\
&= i\sigma\frac{s}{u+s}\frac{eE_{\text{cr}}}{p_-}\frac{2usp_-}{m^2(u+s)}\int_0^1 dy\chi(\phi_{ys}) \qquad (E_{\text{cr}}=m^2/e) \\
&= 2i\sigma\frac{us^2}{(u+s)^2}\int_0^1 dy\chi(\phi_{ys}),
\end{aligned}
\tag{4.39}
$$

such that the matrix element becomes

$$
\boxed{
\begin{aligned}
\frac{\delta m_\sigma\left(p,\phi\right)}{m} &= \frac{\alpha}{4\pi}\int_s\int_u\frac{1}{(u+s)^2} \\
&\times\left\{2\frac{u+2s}{u+s}\left[\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]\right]-\exp\left\{-i\frac{s^2}{u+s}\right\}\right]\right. \\
&+\exp\left[-i\frac{s^2}{u+s}\left[1+\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2-\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]\right] \\
&\times\left[\frac{u+2s}{u+s}\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_s\right)\right)^2-2\frac{s}{u}\frac{s}{u+s}\left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right. \\
&+2i\sigma\frac{us^2}{(u+s)^2}\int_0^1 dy\chi(\phi_{ys}) \\
&\left.\left.+\frac{s}{u}\left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right)^2\right]-\frac{s}{u}\frac{2u-s}{u+s}\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\int_0^1 dy\Delta\boldsymbol{\xi}_\perp\left(\phi_{ys}\right)\right]\right]\right\}.
\end{aligned}
}
\tag{4.40}
$$

Due to the fact that the expression in 4.40 is non-local, a local expression for the anomalous magnetic moment of the electron can't be generally obtained. However, employing the locally constant field approximation allows the extraction of such a local expression.

4.3.1 Locally constant field approximation (LCFA)

In the locally constant field approximation the factor is expanded to order $\mathcal{O}\left((\phi_s-\phi)^2\right)$ and the phase to $\mathcal{O}\left((\phi_s-\phi)^3\right)$ and assume that the external field oscillates slowly (such that space-time variations of the field can be neglected) compared to the timescale of the process considered (emission and absorbtion of a photon), that is

$$\frac{\omega p_-}{m^2} \ll 1. \tag{4.41}$$

The phase difference is given by

$$\phi_s = \phi - 2usp_-/m^2(u+s) \implies \phi_s - \phi = -\frac{2sup_-}{(u+s)m^2}. \tag{4.42}$$

From 4.37, $\chi^2(\phi)$ is calculated as

$$\chi(\phi) = -p_-\mathcal{A}_0\psi'(\phi)/m^3 \implies \chi^2(\phi) = (p_-)^2\left(\mathcal{A}_0\psi'(\phi)\right)^2/m^6 = (p_-)^2\left(\boldsymbol{\xi}'_\perp(\phi)\right)^2/m^4, \tag{4.43}$$

and the Taylor expansion of $\chi(\phi_{ys})$ gives

$$\begin{aligned}
\chi(\phi_{ys}) &\simeq \chi(\phi) + (\phi_{ys}-\phi)\,\chi'(\phi) \\
&\simeq \chi(\phi) - \frac{2syup_-}{(u+s)m^2}\chi'(\phi) \\
\implies \int_0^1 dy\chi(\phi_{ys}) &\simeq \chi(\phi) - \frac{2sup_-}{(u+s)m^2}\chi'(\phi)\int_0^1 dyy \simeq \chi(\phi) - \frac{1}{2}\frac{2sup_-}{(u+s)m^2}\chi'(\phi) \simeq \chi(\phi),
\end{aligned} \tag{4.44}$$

where in the last step (of the integral) the LCFA relation from 4.41 was employed.
The following Taylor expansions follow

$$\begin{aligned}
\Delta\boldsymbol{\xi}_\perp(\phi_s) &= \boldsymbol{\xi}_\perp(\phi_s) - \boldsymbol{\xi}_\perp(\phi) \\
&= \boldsymbol{\xi}_\perp(\phi) + (\phi_s-\phi)\frac{d}{d\phi_s}\boldsymbol{\xi}_\perp(\phi_s)\Big|_\phi + (\phi_s-\phi)^2\frac{d^2}{d\phi_s^2}\boldsymbol{\xi}_\perp(\phi_s)\Big|_\phi + \mathcal{O}\left((\phi_s-\phi)^3\right) - \boldsymbol{\xi}_\perp(\phi) \\
&= (\phi_s-\phi)\frac{d}{d\phi_s}\boldsymbol{\xi}_\perp(\phi_s)\Big|_\phi + (\phi_s-\phi)^2\frac{d^2}{d\phi_s^2}\boldsymbol{\xi}_\perp(\phi_s)\Big|_\phi + \mathcal{O}\left((\phi_s-\phi)^3\right) \\
&=: (\phi_s-\phi)\,\boldsymbol{\xi}'_\perp(\phi_s)\Big|_\phi + (\phi_s-\phi)^2\,\boldsymbol{\xi}''_\perp(\phi_s)\Big|_\phi + \mathcal{O}\left((\phi_s-\phi)^3\right) \\
&= -\frac{2sup_-}{(u+s)m^2}\boldsymbol{\xi}'_\perp(\phi) + \left(-\frac{2sup_-}{(u+s)m^2}\right)^2\boldsymbol{\xi}''_\perp(\phi) + \mathcal{O}\left((\phi_s-\phi)^3\right),
\end{aligned} \tag{4.45}$$

$$\begin{aligned}
\left[\Delta\boldsymbol{\xi}_\perp(\phi_s)\right]^2 &= \left[\boldsymbol{\xi}_\perp(\phi_s) - \boldsymbol{\xi}_\perp(\phi)\right]^2 \\
&=: \left[(\phi_s-\phi)\,\boldsymbol{\xi}'_\perp(\phi_s)\Big|_\phi\right]^2 + \mathcal{O}\left((\phi_s-\phi)^3\right) \\
&= \left(-\frac{2sup_-}{(u+s)m^2}\right)^2\left[\boldsymbol{\xi}'_\perp(\phi)\right]^2 + \mathcal{O}\left((\phi_s-\phi)^3\right) \\
&\simeq \left(\frac{2su}{u+s}\right)^2\left[\frac{p_-}{m^2}\boldsymbol{\xi}'_\perp(\phi)\right]^2 = \left(\frac{2su}{u+s}\right)^2\chi^2(\phi),
\end{aligned} \tag{4.46}$$

$$\left[\int_0^1 dy \Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right]^2 \simeq \left[\int_0^1 dy\left(-\frac{2syup_-}{(u+s)m^2}\right)\boldsymbol{\xi}'_\perp(\phi)\right]^2$$
$$\simeq \left[\left(-\frac{2sup_-}{(u+s)m^2}\right)\boldsymbol{\xi}'_\perp(\phi)\int_0^1 dyy\right]^2 \simeq \left(\frac{2su}{u+s}\right)^2\left(\frac{p_-}{m^2}\boldsymbol{\xi}'_\perp(\phi)\right)^2\left[\int_0^1 dyy\right]^2$$
$$\simeq \frac{1}{4}\left(\frac{2su}{u+s}\right)^2\chi^2(\phi), \tag{4.47}$$

$$\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2 \simeq \int_0^1 dy\left[\left(-\frac{2syup_-}{(u+s)m^2}\right)\boldsymbol{\xi}'_\perp(\phi)\right]^2$$
$$\simeq \left[\left(-\frac{2su}{u+s}\right)\left(\frac{p_-}{m^2}\boldsymbol{\xi}'_\perp(\phi)\right)\right]^2\int_0^1 dyy^2 \simeq \frac{1}{3}\left(\frac{2su}{u+s}\right)^2\chi^2(\phi), \tag{4.48}$$

$$\Delta\boldsymbol{\xi}_\perp(\phi_s)\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys}) \simeq -\frac{2sup_-}{(u+s)m^2}\boldsymbol{\xi}'_\perp(\phi)\int_0^1 dy\left[-\frac{2syup_-}{(u+s)m^2}\right]\boldsymbol{\xi}'_\perp(\phi)$$
$$\simeq \left[\frac{2su}{u+s}\frac{p_-}{m^2}\boldsymbol{\xi}'_\perp(\phi)\right]^2\int_0^1 dyy \tag{4.49}$$
$$\simeq \frac{1}{2}\left(\frac{2su}{u+s}\right)^2\chi^2(\phi),$$

where in the last steps of 4.45 - 4.49, 4.43 was inserted.

Introducing

$$\boxed{\Delta_\xi = \left[\int_0^1 dy\left(\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2 - \left(\int_0^1 dy\Delta\boldsymbol{\xi}_\perp(\phi_{ys})\right)^2\right] = \Delta/m^2,} \tag{4.50}$$

leads to the expansion

$$
\begin{aligned}
\Delta_\xi &\simeq \int_0^1 dy \left(\Delta \boldsymbol{\xi}_\perp \left(\phi_{ys}\right)\right)^2 - \left(\int_0^1 dy \Delta \boldsymbol{\xi}_\perp \left(\phi_{ys}\right)\right)^2 \\
&\simeq \int_0^1 dy \left(\left(\phi_{ys} - \phi\right) \boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 - \left(\int_0^1 dy \left(\phi_{ys} - \phi\right) \boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 \\
&\simeq \left(\boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 \int_0^1 dy \left(\phi_{ys} - \phi\right)^2 - \left(\boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 \left(\int_0^1 dy \left(\phi_{ys} - \phi\right)\right)^2 \\
&\simeq \left(\boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 \int_0^1 dy \left(-\frac{2syup_-}{(u+s)m^2}\right)^2 - \left(\boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 \left(-\frac{2sup_-}{(u+s)m^2} \int_0^1 dyy\right)^2 \\
&\simeq \left(\boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 \left(\frac{2sup_-}{(u+s)m^2}\right)^2 \int_0^1 dyy^2 - \left(\boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 \left(\frac{2sup_-}{(u+s)m^2}\right)^2 \left(\int_0^1 dyy\right)^2 \\
&\simeq \left(\boldsymbol{\xi}'_\perp \left(\phi_{ys}\right)\Big|_\phi\right)^2 \left(\frac{2sup_-}{(u+s)m^2}\right)^2 \left[\frac{1}{3} - \frac{1}{4}\right] \\
&\simeq \frac{1}{3} \left(\frac{sup_-}{(u+s)m^2}\right)^2 \left(\boldsymbol{\xi}'_\perp (\phi)\right)^2 \\
&\simeq \frac{1}{3} \left(\frac{su}{u+s}\right)^2 \left(\frac{p_-}{m^2} \boldsymbol{\xi}'_\perp (\phi)\right)^2 \\
&\simeq \frac{1}{3} \left(\frac{su}{u+s}\right)^2 \chi^2(\phi).
\end{aligned}
\tag{4.51}
$$

Inserting these expansions, the mass shift in the LCFA becomes

$$
\begin{aligned}
\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{4\pi} \int_s \int_u \frac{1}{(u+s)^2} \\
&\times \left\{ 2\frac{u+2s}{u+s} \left[\exp\left[-i\frac{s^2}{u+s} \left[1 + \frac{1}{3}\left(\frac{su}{u+s}\right)^2 \chi^2(\phi) \right] \right] - \exp\left\{ -i\frac{s^2}{u+s} \right\} \right] \right. \\
&+ \exp\left[-i\frac{s^2}{u+s} \left[1 + \frac{1}{3}\left(\frac{su}{u+s}\right)^2 \chi^2(\phi) \right] \right] \\
&\times \left[\frac{u+2s}{u+s} \left(\frac{2su}{u+s}\right)^2 \chi^2(\phi) - 2\frac{s}{u}\frac{s}{u+s}\frac{1}{4}\left(\frac{2su}{u+s}\right)^2 \chi^2(\phi) + 2i\sigma\frac{us^2}{(u+s)^2}\chi(\phi) \right. \\
&\left.\left. + \frac{s}{u}\frac{1}{3}\left(\frac{2su}{u+s}\right)^2 \chi^2(\phi) - \frac{s}{u}\frac{2u-s}{u+s}\frac{1}{2}\left(\frac{2su}{u+s}\right)^2 \chi^2(\phi) \right] \right\}.
\end{aligned}
\tag{4.52}
$$

Factoring $(2su/(u+s))^2$, the mass shift becomes

$$
\begin{aligned}
\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{4\pi} \int_s \int_u \frac{1}{(u+s)^2} \\
&\times \left\{ 2\frac{u+2s}{u+s} \left[\exp\left[-i\frac{s^2}{u+s} \left[1 + \frac{1}{3}\left(\frac{su}{u+s}\right)^2 \chi^2(\phi) \right] \right] - \exp\left\{ -i\frac{s^2}{u+s} \right\} \right] \right. \\
&+ \exp\left[-i\frac{s^2}{u+s} \left[1 + \frac{1}{3}\left(\frac{su}{u+s}\right)^2 \chi^2(\phi) \right] \right] \\
&\left. \times \left(\frac{2su}{u+s}\right)^2 \left[\frac{u+2s}{u+s}\chi^2(\phi) - 2\frac{s}{u}\frac{s}{u+s}\frac{1}{4}\chi^2(\phi) + \frac{s}{u}\frac{1}{3}\chi^2(\phi) - \frac{s}{u}\frac{2u-s}{u+s}\frac{1}{2}\chi^2(\phi) + i\sigma\frac{1}{2u}\chi(\phi) \right] \right\}.
\end{aligned}
\tag{4.53}
$$

Introducing $s = uv \implies \int ds = u \int dv$

$$\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} = \frac{\alpha}{4\pi}\int_u u \int_v \frac{1}{(u+uv)^2}$$
$$\times \left\{ 2\frac{u+2uv}{u+uv}\left[\exp\left[-i\frac{(uv)^2}{u+uv}\left[1+\frac{1}{3}\left(\frac{uvu}{u+uv}\right)^2\chi^2(\phi)\right]\right] - \exp\left\{-i\frac{(uv)^2}{u+uv}\right\}\right]\right.$$
$$+ \exp\left[-i\frac{(uv)^2}{u+uv}\left[1+\frac{1}{3}\left(\frac{uvu}{u+uv}\right)^2\chi^2(\phi)\right]\right]$$
$$\left.\times\left(\frac{2uvu}{u+uv}\right)^2\left[\frac{u+2uv}{u+uv}\chi^2(\phi) - 2\frac{uv}{u}\frac{uv}{u+uv}\frac{1}{4}\chi^2(\phi) + \frac{uv}{u}\frac{1}{3}\chi^2(\phi) - \frac{uv}{u}\frac{2u-uv}{u+uv}\frac{1}{2}\chi^2(\phi) + i\sigma\frac{1}{2u}\chi(\phi)\right]\right\}. \tag{4.54}$$

Simplifying

$$\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} = \frac{\alpha}{4\pi}\int_v\int_u \frac{1}{u(1+v)^2}$$
$$\times \left\{ 2\frac{1+2v}{1+v}\left[\exp\left[-i\frac{uv^2}{1+v}\left[1+\frac{1}{3}\left(\frac{uv}{1+v}\right)^2\chi^2(\phi)\right]\right] - \exp\left\{-i\frac{uv^2}{1+v}\right\}\right]\right.$$
$$+ \exp\left[-i\frac{uv^2}{1+v}\left[1+\frac{1}{3}\left(\frac{uv}{1+v}\right)^2\chi^2(\phi)\right]\right]$$
$$\left.\times\left(\frac{2uv}{1+v}\right)^2\left[\frac{1+2v}{1+v}\chi^2(\phi) - \frac{v^2}{1+v}\frac{1}{2}\chi^2(\phi) + v\frac{1}{3}\chi^2(\phi) - v\frac{2-v}{1+v}\frac{1}{2}\chi^2(\phi) + i\sigma\frac{1}{2u}\chi(\phi)\right]\right\}. \tag{4.55}$$

Changing variables $u \to u' = (v^2u)/(1+v) \implies (1+v)du'/v^2 = du$

$$\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} = \frac{\alpha}{4\pi}\int_v \frac{1+v}{v^2}\int_{u'} \frac{v^2}{u'(1+v)^3}$$
$$\times \left\{ 2\frac{1+2v}{1+v}\left[\exp\left[-iu'\left[1+\frac{1}{3}\left(\frac{u'}{v}\right)^2\chi^2(\phi)\right]\right] - \exp\left\{-iu'\right\}\right] +\right.$$
$$+ \exp\left[-iu'\left[1+\frac{1}{3}\left(\frac{u'}{v}\right)^2\chi^2(\phi)\right]\right]$$
$$\left.\times\left(\frac{2u'}{v}\right)^2\left[\frac{1+2v}{1+v}\chi^2(\phi) - \frac{v^2}{1+v}\frac{1}{2}\chi^2(\phi) + v\frac{1}{3}\chi^2(\phi) - v\frac{2-v}{1+v}\frac{1}{2}\chi^2(\phi) + i\sigma\frac{v^2}{2(1+v)u'}\chi(\phi)\right]\right\}. \tag{4.56}$$

Dropping the primes and simplifying

$$\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} = \frac{\alpha}{4\pi}\int_u\int_v \frac{1}{u(1+v)^2}$$
$$\times \left\{ 2\frac{1+2v}{1+v}\left[\exp\left[-iu\left[1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right]\right] - \exp\left\{-iu\right\}\right] +\right.$$
$$+ \exp\left[-iu\left[1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right]\right]$$
$$\left.\times\left(\frac{2u}{v}\right)^2\left[\frac{1+2v}{1+v}\chi^2(\phi) - \frac{v^2}{1+v}\frac{1}{2}\chi^2(\phi) + v\frac{1}{3}\chi^2(\phi) - v\frac{2-v}{1+v}\frac{1}{2}\chi^2(\phi) + i\sigma\frac{v^2}{2(1+v)u}\chi(\phi)\right]\right\}. \tag{4.57}$$

Factoring in u/v^2 and simplifying

$$\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} = \frac{\alpha}{2\pi}\int_u\int_v \frac{1}{u(1+v)^2} \times \left\{ \frac{1+2v}{1+v}\left[\exp\left[-iu\left[1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right]\right] - \exp\left\{-iu\right\}\right]\right.$$
$$+ u\exp\left[-iu\left[1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right]\right]$$
$$\left.\times\left[2\frac{1+2v}{1+v}\frac{\chi^2(\phi)}{v^2}u - \frac{v^2}{1+v}\frac{\chi^2(\phi)}{v^2}u + 2\frac{u}{v}\frac{1}{3}\chi^2(\phi) - \frac{u}{v}\frac{2-v}{1+v}\chi^2(\phi) + i\sigma\frac{1}{1+v}\chi(\phi)\right]\right\}. \tag{4.58}$$

Factoring $\chi^2(\phi)u/v^2$

$$\boxed{\begin{aligned}\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{2\pi}\int_u\int_v \frac{1}{u(1+v)^2}\times\left\{\frac{1+2v}{1+v}\left[\exp\left[-iu\left[1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right]\right]-\exp\{-iu\}\right]\right.\\ &+u\exp\left[-iu\left(1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right)\right]\\ &\left.\times\left[\left(2\frac{1+2v}{1+v}-\frac{v^2}{1+v}+\frac{2v}{3}-\frac{(2-v)v}{1+v}\right)\frac{\chi^2(\phi)}{v^2}u+i\sigma\frac{1}{1+v}\chi(\phi)\right]\right\}.\end{aligned}} \tag{4.59}$$

Using the identity[2]

$$\int_0^\infty \frac{dudv}{u(1+v)^2}\frac{1+2v}{1+v}\left\{e^{-iu\left[1+\frac{1}{3}\frac{\chi^2(\phi)}{v^2}u^2\right]}-e^{-iu}\right\} = -\frac{\chi^2(\phi)}{3}\int_0^\infty\frac{dudv}{(1+v)^2}e^{-iu\left[1+\frac{1}{3}\frac{\chi^2(\phi)}{v^2}u^2\right]}\frac{1+v-3v^2}{1+v}\frac{u}{v^2}, \tag{4.60}$$

and factoring out $1/(1+v)$

$$\begin{aligned}\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{2\pi}\int_u\int_v\frac{1}{(1+v)^3}\\ &\times\left\{\exp\left[-iu\left(1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right)\right]\right.\\ &\left.\times\left[\left(-\frac{1+v-3v^2}{3}+2(1+2v)-v^2+\frac{2v(1+v)}{3}-(2-v)v\right)\frac{\chi^2(\phi)}{v^2}u+i\sigma\chi(\phi)\right]\right\},\end{aligned} \tag{4.61}$$

which after simple algebraic manipulations becomes

$$\boxed{\begin{aligned}\frac{\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{2\pi}\int_u\int_v\frac{1}{(1+v)^3}\\ &\times\left\{\exp\left[-iu\left(1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right)\right]\times\left[\left(\frac{5v^2+7v+5}{3}\right)\frac{\chi^2(\phi)}{v^2}u+i\sigma\chi(\phi)\right]\right\}.\end{aligned}} \tag{4.62}$$

The mass shift can be split into a spin dependent and a spin independent part

$$\delta m_\sigma^{(\mathrm{LCFA})}(p,\phi) = \delta m_{\sigma,\zeta=0}^{(\mathrm{LCFA})}(p,\phi)+\delta m_{\sigma,\zeta}^{(\mathrm{LCFA})}(p,\phi), \tag{4.63}$$

with the identification

$$\begin{aligned}\frac{\delta m_{\sigma,\zeta=0}^{(\mathrm{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{2\pi}\int_0^\infty du\int_0^\infty dv\frac{1}{(1+v)^3}\\ &\times\left\{\exp\left[-iu\left(1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right)\right]\times\left[\left(\frac{5v^2+7v+5}{3}\right)\frac{\chi^2(\phi)}{v^2}u\right]\right\}, \\ \frac{\delta m_{\sigma,\zeta}^{(\mathrm{LCFA})}(p,\phi)}{m} &= i\sigma\chi(\phi)\frac{\alpha}{2\pi}\int_0^\infty du\int_0^\infty dv\frac{1}{(1+v)^3}\times\exp\left[-iu\left(1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right)\right].\end{aligned} \tag{4.64}$$

[2] For a proof, see Appendix C.1.

Observation 4.3.1 Substituting $u \to \tau = u/z \implies du = zd\tau$ [a], with $z \equiv v/\chi(\phi)$,

$$\boxed{\begin{aligned}\frac{\delta m_\sigma^{(\text{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{2\pi}\int_0^\infty dv \frac{z}{(1+v)^3} \\ &\times \int_0^\infty d\tau \left\{ \exp\left[-iz\tau\left(1+\frac{1}{3}\tau^2\right)\right] \times \left[\left(\frac{5v^2+7v+5}{3}\right)\frac{\tau}{z} + i\sigma\chi(\phi)\right]\right\}\end{aligned}}. \tag{4.65}$$

$$\begin{aligned}\frac{\delta m_{\sigma,\zeta=0}^{(\text{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{2\pi}\int_0^\infty dv \frac{1}{(1+v)^3}\left(\frac{5v^2+7v+5}{3}\right) \times \int_0^\infty d\tau\tau \left\{ \exp\left[-i\frac{v}{\chi(\phi)}\left(\tau+\frac{1}{3}\tau^3\right)\right]\right\} \\ \frac{\delta m_{\sigma,\zeta}^{(\text{LCFA})}(p,\phi)}{m} &= i\sigma\frac{\alpha}{2\pi}\int_0^\infty dv \frac{v}{(1+v)^3}\int_0^\infty d\tau \left\{ \exp\left[-i\frac{v}{\chi(\phi)}\left(\tau+\frac{1}{3}\tau^3\right)\right]\right\}\end{aligned} \tag{4.66}$$

This result for the electron mass shift in a plane wave background, within the LCFA, generalizes the constant crossed field result from [VS71], with the quantum nonlinearity parameter depending now on the phase, i.e. $\chi = \chi(\phi)$.

[a] The τ here does not represent the light-cone time.

Observation 4.3.2 Denoting $t \equiv u\left(\frac{\chi(\phi)}{v}\right)^{\frac{2}{3}} = u/z$, $z \equiv \left(\frac{\chi(\phi)}{v}\right)^{-\frac{2}{3}}$, $du = zdt$

$$\boxed{\begin{aligned}\frac{\delta m_\sigma^{(\text{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{2\pi}\int_v \frac{z}{(1+v)^3} \\ &\times \int_t \left\{ \exp\left[-i\left(zt+\frac{1}{3}t^3\right)\right] \times \left[\left(\frac{5v^2+7v+5}{3}\right)\frac{t}{z^2} + i\sigma\chi(\phi)\right]\right\}\end{aligned}} \tag{4.67}$$

Writing the mass shifts in the form

$$\begin{aligned}\frac{\delta m_{\sigma,\zeta=0}^{(\text{LCFA})}(p,\phi)}{m} &= \frac{\alpha}{2\pi}\int_v \frac{1}{z(1+v)^3} \\ &\times \int_t \left\{ \exp\left[-i\left(zt+\frac{1}{3}t^3\right)\right] \times \left[\left(\frac{5v^2+7v+5}{3}\right)t\right]\right\} \\ \frac{\delta m_{\sigma,\zeta}^{(\text{LCFA})}(p,\phi)}{m} &= i\sigma\chi(\phi)\frac{\alpha}{2\pi}\int_v \frac{z}{(1+v)^3}\int_t \left\{ \exp\left[-i\left(zt+\frac{1}{3}t^3\right)\right]\right\}\end{aligned} \tag{4.68}$$

makes transparent the comparison with the constant crossed field result from [Rit70], and generalizes it by letting the quantum nonlinearity parameter depend on the phase, i.e. $\chi = \chi(\phi)$.

Within the LCFA, the spin dependent expression for the mass shift from 4.64 is now local (unlike the expression from 4.40), such that it can be used to extract a local expression for the anomalous magnetic moment of the electron.

4.3.2 Electron anomalous magnetic moment in a plane-wave

In the rest frame of the electron, the real part of the mass shift $\delta m_{\sigma,\zeta}^{(\mathrm{LCFA})}(p,\phi)$ can be interpreted as the interaction energy $-\delta\boldsymbol{\mu}^{(\mathrm{LCFA})}\cdot\boldsymbol{B}_0(\phi)$ of the change in anomalous magnetic moment with the magnetic field $\boldsymbol{B}_0(\phi)$ [Rit72; VS71]. This provides the condition

$$\mathrm{Re}\,\delta m_{\sigma,\zeta}^{(\mathrm{LCFA})}(p,\phi) = -\delta\boldsymbol{\mu}^{(\mathrm{LCFA})}\cdot\boldsymbol{B}_0(\phi), \tag{4.69}$$

where $\delta\boldsymbol{\mu}^{(\mathrm{LCFA})}$ is the shift of the anomalous magnetic moment of the electron in the LCFA and $\boldsymbol{B}_0(\phi)$ is the magnetic field of the plane wave in the rest frame of the electron.

Then, knowing that we can relate the shift in the magnetic moment $\delta\boldsymbol{\mu}^{(\mathrm{LCFA})}$ to the shift in the gyromagnetic factor of the electron $\delta g^{(\mathrm{LCFA})} = g^{(\mathrm{LCFA})} - 2$, in the LCFA, by the relation

$$-\delta\boldsymbol{\mu}^{(\mathrm{LCFA})} = \delta g^{(\mathrm{LCFA})}\mu_B(\sigma/2)\boldsymbol{\zeta}, \tag{4.70}$$

where, in the rest frame $\zeta^\mu_{\mathrm{rest}} = (0,\boldsymbol{\zeta})$, $\mu_B = |e|/2m$ is the Bohr magneton and $\delta g^{(\mathrm{LCFA})}$ is the change in the electron gyromagnetic factor (in the linearly polarized plane wave background), we have that

$$\frac{\delta g^{(\mathrm{LCFA})}}{2} = \frac{m}{\sigma\mu_B\,(\boldsymbol{\zeta}\cdot\boldsymbol{B}_0)}\,\mathrm{Re}\,\frac{\delta m_{\sigma,\zeta}^{(\mathrm{LCFA})}(p,\phi)}{m}. \tag{4.71}$$

In order to compute 4.71, first notice that in the rest frame, where $p^\mu = (m,\mathbf{0})$, $\zeta^\mu = (0,\boldsymbol{\zeta})$, and $\tilde{F}^{i0}(\phi) = -B_0^i(\phi))$

$$\left.\begin{aligned} \chi(\phi) &= -\tfrac{2\mu_B}{m}E_{\mathrm{cr}}\zeta_\mu\tilde{\chi}^\mu(\phi) \\ \tilde{\chi}^\mu(\phi) &= \tilde{F}^{\mu\nu}(\phi)p_\nu/mE_{\mathrm{cr}} \end{aligned}\right\} \implies \chi_{\mathrm{rest}}(\phi) = \frac{2\mu_B}{m}\boldsymbol{\zeta}\cdot\boldsymbol{B}_0(\phi). \tag{4.72}$$

A straightforward computation gives then, plugging 4.62 and 4.72 in 4.71, the value of the electron gyromagnetic factor, as follows

$$\begin{aligned} \frac{\delta g^{(\mathrm{LCFA})}}{2} &= \frac{m}{\sigma\mu_B\,(\boldsymbol{\zeta}\cdot\boldsymbol{B}_0(\phi))}\frac{\alpha}{2\pi}\,\mathrm{Re}\int_u\int_v\frac{1}{(1+v)^3}\left\{\exp\left[-iu\left(1+\frac{1}{3}\left(\frac{u}{v}\right)^2\chi^2(\phi)\right)\right][i\sigma\chi(\phi)]\right\} \\ &= \frac{m}{\mu_B\,(\boldsymbol{\zeta}\cdot\boldsymbol{B}_0(\phi))}\frac{\alpha}{2\pi}\,\mathrm{Re}\int_u\int_v\frac{1}{(1+v)^3}e^{-iu\left[1+\frac{1}{3}\frac{\chi^2(\phi)}{v^2}u^2\right]}\left[i\frac{2\mu_B}{m}\boldsymbol{\zeta}\cdot\boldsymbol{B}_0(\phi)\right] \\ &= -\frac{\alpha}{\pi}\,\mathrm{Im}\int_0^\infty\frac{dudv}{(1+v)^3}e^{-iu\left[1+\frac{1}{3}\frac{\chi^2(\phi)}{v^2}u^2\right]}. \end{aligned} \tag{4.73}$$

where $\mathrm{Re}\,iz = -\,\mathrm{Im}\,z$ for $z\in\mathbb{C}$ was used in the last line.

Recalling the $i0^+ = \lim_{\epsilon\to0^+} i\epsilon$ pole prescription from Equation 3.16 to evaluate the integral $\int_0^\infty du e^{-i(1+i\varepsilon)u} = -i(1+i\varepsilon)^{-1} \to -i$, the gyromagnetic factor from Equation 4.73 reproduces Schwinger's result $\delta g_0/2 = \alpha/2\pi$ when the background field is removed, i.e. $\chi(\phi)\to 0$.

5 Conclusions

In conclusion, the mass operator in the presence of an arbitrary plane-wave background field was obtained for an off-shell electron.

Putting the external spinor states from the mass operator on-shell and solving the Schwinger-Dyson equation, the mass shift of the electron was determined. The obtained expression for the electron mass shift simplified by specializing to the case of a linearly polarized plane-wave background and choosing the spin quantization axis along the direction of the magnetic field (of the plane-wave) in the electron rest frame. However, the electron mass shift featured a nonlocal dependence on the plane-wave field, which prevented a convenient description of the spin-dependent part in terms of an electron anomalous magnetic moment.

To obtain a local expression, the locally constant field approximation was employed, which allowed the extraction of the anomalous magnetic moment. As a consistency check, removing the background field led to Schwinger's famous result. The electron mass shift in the locally constant field approximation generalized the expressions for the constant crossed field case previously studied in the literature.

Appendix

A Volkov states

Theorem A.1 Denoting the Ritus state $|E(p,x)\rangle \equiv E(p,x)|\Omega\rangle$, where $E(p,x)$ is the Ritus matrix, the following holds [Rit70]

$$\hat{\Pi}(\phi)|E(p,x)\rangle = \hat{\pi}_p(\phi)|E(p,x)\rangle = |E(p,x)\rangle\hat{p} \tag{A.1}$$

where $\Pi^\mu(\phi) = i\partial^\mu - \mathcal{A}^\mu(\phi)$[a]

[a] the derivative acts only on $E(p,x)$, but it can be generalized to act on the product $E(p,x)f(x)$.

Proof:

$$S_p(x) = -(px) - \int_{-\infty}^{\phi} d\varphi \left[\frac{p\cdot\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right] \implies \partial_\mu S_p(x) = -p_\mu - n_\mu\left(\frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right)$$

$$\begin{aligned}
\hat{\Pi}(\phi)|E(p,x)\rangle &= \gamma^\mu\left(i\partial_\mu - \mathcal{A}_\mu(\Phi)\right)\left\{\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]e^{iS_p(x)}|\Omega\rangle\right\} \\
&= \left\{i\gamma^\mu\partial_\mu\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]e^{iS_p(x)} + \gamma^\mu\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]i\partial_\mu e^{iS_p(x)} - \hat{\mathcal{A}}(\phi)E(p,x)\right\}|\Omega\rangle \\
&= \left\{i\gamma^\mu\frac{\hat{n}n_\mu\hat{\mathcal{A}}'(\phi)}{2p_-}e^{iS_p(x)} + \gamma^\mu\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]i(i\partial_\mu S_p(x))e^{iS_p(x)} - \hat{\mathcal{A}}(\phi)E(p,x)\right\}|\Omega\rangle \\
&= \left\{-\gamma^\mu(\partial_\mu S_p(x))\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]e^{iS_p(x)} - \hat{\mathcal{A}}(\phi)E(p,x)\right\}|\Omega\rangle \\
&= \left\{-\gamma^\mu\left[-p_\mu - n_\mu\left(\frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right)\right] - \hat{\mathcal{A}}(\phi)\right\}E(p,x)|\Omega\rangle \\
&= \left\{\hat{p} - \hat{\mathcal{A}}(\phi) + \hat{n}\left(\frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right)\right\}E(p,x)|\Omega\rangle = \hat{\pi}_p(\phi)|E(p,x)\rangle
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\hat{\Pi}(\phi)|E(p,x)\rangle &= \gamma^\mu\left(i\partial_\mu - \mathcal{A}_\mu(\Phi)\right)\left\{\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]e^{iS_p(x)}|\Omega\rangle\right\} \\
&= \left\{i\gamma^\mu\frac{\hat{n}n_\mu\hat{\mathcal{A}}'(\phi)}{2p_-}e^{iS_p(x)} + \gamma^\mu\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]i(i\partial_\mu S_p(x))e^{iS_p(x)} - \hat{\mathcal{A}}(\phi)\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]e^{iS_p(x)}\right\}|\Omega\rangle \\
&= \left\{-\gamma^\mu(\partial_\mu S_p(x))\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]e^{iS_p(x)} - \left[\mathbf{1} - \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]e^{iS_p(x)}\hat{\mathcal{A}}(\phi)\right\}|\Omega\rangle \\
&= e^{iS_p(x)}\left\{-\gamma^\mu\left[-p_\mu - n_\mu\left(\frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right)\right]\left[\mathbf{1} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] - \hat{\mathcal{A}}(\phi) + \frac{\hat{n}\mathcal{A}^2(\phi)}{2p_-}\right\}|\Omega\rangle \\
&= e^{iS_p(x)}\left\{\hat{p} + \hat{n}\left(\frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right) + \frac{\hat{p}\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-} - \hat{\mathcal{A}}(\phi) + \frac{\hat{n}\mathcal{A}^2(\phi)}{2p_-}\right\}|\Omega\rangle \\
&= e^{iS_p(x)}\left\{\hat{p} - \hat{\mathcal{A}}(\phi) + \hat{n}\frac{p\cdot\mathcal{A}(\phi)}{p_-} + \frac{(2p_- - \hat{n}\hat{p})\,\hat{\mathcal{A}}(\phi)}{2p_-}\right\}|\Omega\rangle \\
&= e^{iS_p(x)}\left\{\hat{p} + \hat{n}\frac{p\cdot\mathcal{A}(\phi)}{p_-} - \hat{n}\frac{\left(2p\cdot\mathcal{A}(\phi) - \hat{\mathcal{A}}(\phi)\hat{p}\right)}{2p_-}\right\}|\Omega\rangle \\
&= e^{iS_p(x)}\left\{\hat{p} + \frac{\hat{n}\hat{\mathcal{A}}(\phi)\hat{p}}{2p_-}\right\}|\Omega\rangle = |E(p,x)\rangle\hat{p}
\end{aligned}$$

(A.3)

Theorem A.2 The state $U_\sigma(p,x)$ is governed by the ϕ-evolution equation

$$\boxed{i\partial_\phi U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp) = K(\phi)U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp)} \tag{A.4}$$

where the matrix $K(\phi)$ is given by

$$\boxed{K(\phi) \equiv p_+ + \frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-} + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-}} \tag{A.5}$$

The solution at some $\phi=\phi_F$, for the initial condition

$$U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp) = U_\sigma(p,\phi_I,\tau,\boldsymbol{x}_\perp) \tag{A.6}$$

is given by

$$\boxed{U_\sigma(p,\phi_F,\tau,\boldsymbol{x}_\perp) = M(\phi_F,\phi_I)U_\sigma(p,\phi_I,\tau,\boldsymbol{x}_\perp)} \tag{A.7}$$

with the ϕ-evolution matrix

$$\boxed{\begin{aligned} M(\phi_F,\phi_I) &\equiv \exp\left\{-i\int_{\phi_I}^{\phi_F} d\phi K(\phi)\right\} \\ &= \exp\left\{-i\int_{\phi_I}^{\phi_F} d\phi\left[p_+ + \frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-} + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-}\right]\right\} \qquad \left(\hat{n}\hat{\mathcal{A}}(\phi)\hat{n}\hat{\mathcal{A}}(\phi')=0\right) \\ &= \exp\left[-i\int_{\phi_I}^{\phi_F} d\phi\left(p_+ + \frac{p\cdot\mathcal{A}(\phi))}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right)\right]\left[1+\frac{1}{2p_-}\hat{n}\left(\hat{\mathcal{A}}(\phi_F)-\hat{\mathcal{A}}(\phi_I)\right)\right] \end{aligned}} \tag{A.8}$$

Proof:

$$\begin{aligned} \partial_\phi U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp) = &\left\{\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-} + \left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right](-i)\left(p_+ + \frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right)\right\} \\ &\times \exp\left\{i\left[-(p_+\phi + p_-\tau - \boldsymbol{p}_\perp\boldsymbol{x}_\perp) - \int_{-\infty}^{\phi} d\varphi\left[\frac{p\cdot\mathcal{A}(\varphi)}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]\right]\right\}u_\sigma(p) \\ &= \left\{\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-}\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] - \cancel{\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-}\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}} - i\left(p_+ + \frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]\right\} \\ &\times \exp\left\{i\left[-(p_+\phi + p_-\tau - \boldsymbol{p}_\perp\boldsymbol{x}_\perp) - \int_{-\infty}^{\phi} d\varphi\left(\frac{p\cdot\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right)\right]\right\}u_\sigma(p) \\ &= \left\{\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-} - i\left(p_+ + \frac{p\cdot\mathcal{A}(\phi))}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right)\right\}\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] \\ &\times \exp\left\{i\left\{-(p_+\phi + p_-\tau - \boldsymbol{p}_\perp\boldsymbol{x}_\perp) - \int_{-\infty}^{\phi} d\varphi\left[\frac{p\cdot\mathcal{A}(\varphi)}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]\right\}\right\}u_\sigma(p) \\ &= (-i)\underbrace{\left\{\left(p_+ + \frac{p\cdot\mathcal{A}(\phi))}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-}\right) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-}\right\}}_{\equiv K(\phi)} U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp) \end{aligned} \tag{A.9}$$

Theorem A.3 Denoting $\boxed{\mathcal{A}(\phi) \equiv eA(\phi)}$ with $\boxed{\phi = n \cdot x}$, it holds that [Rit85]

$$\Pi^\mu(\phi) U_\sigma(p,x) = \left[\pi_p^\mu(\phi) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-} n^\mu\right] U_\sigma(p,x) \tag{A.10}$$

Alternatively

$$\Pi^\mu(\phi) E(p,x) = \left[\pi_p^\mu(\phi) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-} n^\mu\right] E(p,x) \implies \hat{\Pi}(\phi) E(p,x) = \hat{\pi}_p(\phi) E(p,x) \tag{A.11}$$

where $\pi_p^\mu(\phi)$ is the classical kinetic four-momentum of an electron in the plane wave $\mathcal{A}(\phi)$, with $\lim_{\phi\to\pm\infty} \pi_p^\mu(\phi) = p^\mu$, given by

$$\boxed{\pi_p^\mu(\phi) \equiv p^\mu - \mathcal{A}^\mu(\phi) + \frac{p \cdot \mathcal{A}(\phi)}{p_-} n^\mu - \frac{\mathcal{A}^2(\phi)}{2p_-} n^\mu} \tag{A.12}$$

Proof: Decomposing P^μ into the light-cone basis, i.e.

$$\begin{aligned} P^\mu &= \eta^{\mu\nu} P_\nu = (n^\mu \tilde{n}^\nu + \tilde{n}^\mu n^\nu - a_1^\mu a_1^\nu - a_2^\mu a_2^\nu) P_\nu \\ &= n^\mu (\tilde{n} \cdot P) + \tilde{n}^\mu (n \cdot P) - a_1^\mu (a_1 \cdot P) - a_2^\mu (a_2 \cdot P) \\ &= n^\mu (-P_\phi) + \tilde{n}^\mu (-P_\tau) - a_1^\mu (-P_{\perp,1}) - a_2^\mu (-P_{\perp,2}) \\ &= -n^\mu P_\phi - \tilde{n}^\mu P_\tau + a_1^\mu P_{\perp,1} + a_2^\mu P_{\perp,2} \\ &= -n^\mu (-i\partial_\phi) - \tilde{n}^\mu P_\tau + a_1^\mu P_{\perp,1} + a_2^\mu P_{\perp,1} \\ &= n^\mu (i\partial_\phi) - \tilde{n}^\mu P_\tau + a_1^\mu P_{\perp,1} + a_2^\mu P_{\perp,2} \end{aligned} \tag{A.13}$$

Using that the Volkov state is an eigenstate of the momentum operator along τ and $\perp$, i.e. $P_{\perp,(\tau)} |p\sigma\rangle = (-) p_{\perp,(\tau)} |p\sigma\rangle$

$$P^\mu |p\sigma\rangle = \left[-n^\mu P_\phi + \tilde{n}^\mu p_- + a_1^\mu p_{\perp,1} + a_2^\mu p_{\perp,2}\right] |p\sigma\rangle \tag{A.14}$$

Representing the momentum operator on the Volkov states, i.e.

$$\begin{aligned} \langle \phi, \tau, \boldsymbol{x}_\perp | P^\mu | p\sigma\rangle &= \left[n^\mu (i\partial_\phi) + \tilde{n}^\mu p_- + a_1^\mu p_{\perp,1} + a_2^\mu p_{\perp,2}\right] \langle \phi, \tau, \boldsymbol{x}_\perp | p\sigma\rangle \\ &= \left[n^\mu (i\partial_\phi) + \tilde{n}^\mu p_- + a_1^\mu p_{\perp,1} + a_2^\mu p_{\perp,2}\right] U_\sigma(p,x) \end{aligned} \tag{A.15}$$

$$
\begin{aligned}
\Pi^\mu(\phi)U_\sigma(p,x) &= \left(P^\mu - \mathcal{A}^\mu(\phi)\right)U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp)\\
&= \left[n^\mu(i\partial_\phi) + \tilde{n}^\mu p_- + a_1^\mu p_{\perp,1} + a_2^\mu p_{\perp,2} - \mathcal{A}^\mu(\phi)\right]U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp)\\
&= \left[n^\mu K(\phi) + \tilde{n}^\mu p_- + a_1^\mu p_{\perp,1} + a_2^\mu p_{\perp,2} - \mathcal{A}^\mu(\phi)\right]U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp)\\
&= \left[n^\mu\left(p_+ + \frac{p\cdot\mathcal{A}(\phi)}{p_-} - \frac{\mathcal{A}^2(\phi)}{2p_-} + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-}\right) + \tilde{n}^\mu p_- + a_1^\mu p_{\perp,1} + a_2^\mu p_{\perp,2} - \mathcal{A}^\mu(\phi)\right]U_\sigma(p,\phi,\tau,\boldsymbol{x}_\perp)\\
&= \left[\underbrace{\underbrace{\left(n^\mu p_+ + \tilde{n}^\mu p_- + a_1^\mu p_{\perp,1} + a_2^\mu p_{\perp,2}\right)}_{p^\mu} - \mathcal{A}^\mu(\phi) + \frac{p\cdot\mathcal{A}(\phi)}{p_-}n^\mu - \frac{\mathcal{A}^2(\phi)}{2p_-}n^\mu}_{\pi_p^\mu(\phi)} + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-}n^\mu\right]U_\sigma(p,x)\\
&= \left[\pi_p^\mu(\phi) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2p_-}n^\mu\right]U_\sigma(p,x)
\end{aligned}
\tag{A.16}
$$

where in the third line A.4 was used and in the second last line it was used that the momentum p^μ is written in light-cone coordinates as

$$
\begin{aligned}
p^\mu &= \eta^{\mu\nu}p_\nu = (n^\mu\tilde{n}^\nu + \tilde{n}^\mu n^\nu - a_1^\mu a_1^\nu - a_2^\mu a_2^\nu)p_\nu\\
&= n^\mu(\tilde{n}\cdot p) + \tilde{n}^\mu(n\cdot p) - a_1^\mu(a_1\cdot p) - a_2^\mu(a_2\cdot p)\\
&= n^\mu(\tilde{n}\cdot p) + \tilde{n}^\mu(n\cdot p) - a_1^\mu(-\boldsymbol{a_1}\cdot\boldsymbol{p}) - a_2^\mu(-\boldsymbol{a_2}\cdot\boldsymbol{p}) \quad (a_k^0 = 0)\\
&= n^\mu p_+ + \tilde{n}^\mu p_- + a_1^\mu p_{\perp,1} + a_2^\mu p_{\perp,2} \quad (p_{\perp,k} = \boldsymbol{a}_k\cdot\boldsymbol{p})
\end{aligned}
\tag{A.17}
$$

Corollary A.1

$$
\Pi^\mu(\phi+c)U_\sigma(p,\phi+c,\tau,\boldsymbol{x}_\perp) = \left[\pi_p^\mu(\phi+c) + i\frac{\hat{n}\hat{\mathcal{A}}'(\phi+c)}{2p_-}n^\mu\right]M(\phi+c,\phi)U_\sigma(p,x)
\tag{A.18}
$$

$$
\overset{\hat{n}\hat{n}=0}{\Longrightarrow} \hat{\Pi}(\phi+c)U_\sigma(p,\phi+c,\tau,\boldsymbol{x}_\perp) = \hat{\pi}_p(\phi+c)M(\phi+c,\phi)U_\sigma(p,x)
\tag{A.19}
$$

Proof: The proof follows trivially from

$$
\begin{aligned}
i\partial_\phi U_\sigma\left(p,\phi+c,\tau,\boldsymbol{x}_\perp\right) &= i\frac{\partial(\phi+c)}{\partial\phi}\frac{\partial}{\partial(\phi+c)}U_\sigma\left(p,\phi+c,\tau,\boldsymbol{x}_\perp\right)\\
&= i\partial_{\phi+c}U_\sigma\left(p,\phi+c,\tau,\boldsymbol{x}_\perp\right) = K(\phi+c)U_\sigma\left(p,\phi+c,\tau,\boldsymbol{x}_\perp\right)
\end{aligned}
\tag{A.20}
$$

Hence,

$$
i\partial_\phi U_s\left(p,\phi+c,\tau,\boldsymbol{x}_\perp\right) = K(\phi+c)U_s\left(p,\phi+c,\tau,\boldsymbol{x}_\perp\right) = K(\phi+c)M(\phi+c,\phi)U_s\left(p,\phi,\tau,\boldsymbol{x}_\perp\right)
\tag{A.21}
$$

B Spin 4-pseudovector

Measurable quantities depend on the spinors only through their density matrices [Meu15; BPL12]

$$\rho_\sigma^{(u)} = u_\sigma(\boldsymbol{p})\bar{u}_\sigma(\boldsymbol{p}), \quad \rho_\sigma^{(v)} = v_\sigma(\boldsymbol{p})\bar{v}_\sigma(\boldsymbol{p}) \tag{B.22}$$

where $u_\sigma(\boldsymbol{p})$ $(v_\sigma(\boldsymbol{p}))$[1] denotes the electron (positron) with spin quantum number $\sigma = \pm 1$ denoting the spin orientation along the direction $\boldsymbol{\zeta}$[2].

Decomposing the density matrix on the spinor space matrix basis $\{\mathbf{1}, \gamma^5, \gamma^\mu, i\gamma^\mu\gamma^5, i\sigma^{\mu\nu}\}$ leads to [Meu15]

$$\begin{aligned} \rho_\sigma^{(u)} &= \frac{1}{2}(\hat{p}+m)\left(1+\sigma\gamma^5\hat{\zeta}(\boldsymbol{p})\right) = \frac{1}{2}(\hat{p}+m)\left(1+\sigma S(\boldsymbol{p})\right) \\ \rho_\sigma^{(v)} &= \frac{1}{2}(\hat{p}-m)\left(1+\sigma\gamma^5\hat{\zeta}(\boldsymbol{p})\right) = \frac{1}{2}(\hat{p}-m)\left(1+\sigma S(\boldsymbol{p})\right) \end{aligned} \tag{B.23}$$

where $\zeta^\mu(\boldsymbol{p})$ is the spin 4-pseudovector defined as [BPL12]

$$\begin{aligned} \zeta^\mu(\boldsymbol{p}) &\equiv -\frac{\sigma}{2m}\,\mathbf{tr}\left(\rho_\sigma^{(u)}\gamma^5\gamma^\mu\right) \\ \zeta^\mu(\boldsymbol{p}) &\equiv \frac{\sigma}{2m}\,\mathbf{tr}\left(\rho_\sigma^{(v)}\gamma^5\gamma^\mu\right) \end{aligned} \tag{B.24}$$

that in the rest frame $(p^\mu = (m, \mathbf{0}))$ $\zeta^\mu(\mathbf{0}) = (0, \boldsymbol{\zeta})$ is purely spacelike, where $\boldsymbol{\zeta}$ defines the spin quantization axis and is normalized to $\boldsymbol{\zeta}^2 = 1$ for a pure state [BPL12], such that

$$\zeta^2(\boldsymbol{p}) = -1, \quad \zeta^\mu(\boldsymbol{p})p_\mu = 0, \tag{B.25}$$

and $S(\boldsymbol{p}) = \gamma^5\hat{\zeta}(\boldsymbol{p})$ is the spin operator satisfying [Sei+18]

$$\begin{aligned} S(\boldsymbol{p})u_\sigma(\boldsymbol{p}) &= \sigma u_\sigma(\boldsymbol{p}), \\ S(\boldsymbol{p})v_\sigma(\boldsymbol{p}) &= \sigma v_\sigma(\boldsymbol{p}). \end{aligned} \tag{B.26}$$

Alternatively, it is easy to see that B.24 can be written as

$$\begin{aligned} \zeta^\mu(\boldsymbol{p}) &= -\frac{\sigma}{2m}\bar{u}_\sigma(p)\gamma^5\gamma^\mu u_\sigma(p), \\ \zeta^\mu(\boldsymbol{p}) &= \frac{\sigma}{2m}\bar{v}_\sigma(p)\gamma^5\gamma^\mu v_\sigma(p). \end{aligned} \tag{B.27}$$

In the standard Pauli-Dirac representation [Mic19; Sch19][3]

$$u_\sigma(\boldsymbol{p}) = \begin{pmatrix} \sqrt{\epsilon+m}\,\omega_\sigma \\ \sqrt{\epsilon-m}(\hat{\boldsymbol{p}}\sigma)\omega_\sigma \end{pmatrix}, \quad v_\sigma(\boldsymbol{p}) = \begin{pmatrix} \sqrt{\epsilon-m}(\hat{\boldsymbol{p}}\boldsymbol{\sigma})\omega'_\sigma \\ \sqrt{\epsilon+m}\,\omega'_\sigma \end{pmatrix} \tag{B.28}$$

[1] For an on-shell spinor, it is enough to specify the spatial momentum $\boldsymbol{p}$ rather than the full momentum p.

[2] A more correct notation for the electron spinor would be $u_{\boldsymbol{\zeta}\sigma}(\boldsymbol{p})$

[3] Similarly, the momentum space spinor should be more correctly denoted by $\omega_{\boldsymbol{\zeta}\sigma}$, by specifying the spin quantization axis.

where $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$, ω_σ is a two-component spinor normalized as $\omega^\dagger_{\sigma'}\omega_\sigma = \delta_{\sigma'\sigma}$, the spin 4-pseudovector at momentum $\boldsymbol{p}$ can be computed as [Meu15; Sei+18]

$$\zeta^\mu(\boldsymbol{p}) = \left(\frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}}{m}, \boldsymbol{\zeta} + \frac{\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{\zeta})}{m(\epsilon+m)}\right) = \Lambda^\mu_\nu(\boldsymbol{p})\zeta^\nu(\boldsymbol{0}), \tag{B.29}$$

where $\Lambda^\mu_\nu(\boldsymbol{p})$ is the Lorentz boost matrix to momentum $\boldsymbol{p}$ and $\zeta^\mu(\boldsymbol{0}) = (0, \boldsymbol{\zeta})$ is the rest frame spin vector, $\boldsymbol{\zeta} = \sigma\omega^\dagger_\sigma\boldsymbol{\sigma}\omega_\sigma$ with $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ the Pauli matrices.

B.1 Rest frame spin relations

It is easily seen that in the rest frame, the Dirac spinors reduce to [Mic19]

$$u_\sigma(\boldsymbol{0}) = \sqrt{2m}\begin{pmatrix}\omega_\sigma\\0\end{pmatrix}, \quad v_\sigma(\boldsymbol{0}) = \sqrt{2m}\begin{pmatrix}0\\\omega'_\sigma\end{pmatrix} \tag{B.30}$$

such that the density matrices, taking $\boldsymbol{p} = 0$ and $\zeta^\mu = (0, \boldsymbol{\zeta})$ in B.23, reduce accordingly to [BPL12]

$$\begin{aligned}\rho^{(u)}_{\text{rest},\sigma} &= \frac{m}{2}\left(1+\gamma^0\right)\left(1+\sigma\gamma^5\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}\right),\\ \rho^{(v)}_{\text{rest},\sigma} &= -\frac{m}{2}\left(1-\gamma^0\right)\left(1+\sigma\gamma^5\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}\right),\end{aligned} \tag{B.31}$$

which can be written in two component form as

$$\begin{aligned}\omega_\sigma\omega^\dagger_\sigma &= \frac{1}{2}\left(1+\sigma\boldsymbol{\sigma}\cdot\boldsymbol{\zeta}\right),\\ \omega'_\sigma\omega'^\dagger_\sigma &= \frac{1}{2}\left(1-\sigma\boldsymbol{\sigma}\cdot\boldsymbol{\zeta}\right).\end{aligned} \tag{B.32}$$

The spin operator reads now $S(\boldsymbol{0}) = -\gamma^5\boldsymbol{\gamma}\cdot\boldsymbol{\zeta} =: \gamma^0\boldsymbol{\Sigma}\cdot\boldsymbol{\zeta}$ [$\boldsymbol{\Sigma} = \gamma^5\gamma^0\boldsymbol{\gamma}$], such that the eigenvalue equations B.26 in the rest frame are

$$\begin{aligned}\gamma^0\boldsymbol{\Sigma}\cdot\boldsymbol{\zeta}u_\sigma(\boldsymbol{0}) = \sigma u_\sigma(\boldsymbol{0}), &\quad \boldsymbol{\Sigma}\cdot\boldsymbol{\zeta}u_\sigma(\boldsymbol{0}) = \sigma u_\sigma(\boldsymbol{0}),\\ \gamma^0\boldsymbol{\Sigma}\cdot\boldsymbol{\zeta}v_\sigma(\boldsymbol{0}) = \sigma v_\sigma(\boldsymbol{0}), &\quad \boldsymbol{\Sigma}\cdot\boldsymbol{\zeta}v_\sigma(\boldsymbol{0}) = -\sigma v_\sigma(\boldsymbol{0}),\end{aligned} \tag{B.33}$$

or in two component form [Sei+18]

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\omega_\sigma &= \sigma\omega_\sigma,\\ (\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\omega'_\sigma &= -\sigma\omega'_\sigma.\end{aligned} \tag{B.34}$$

The motivation for the definition B.24 comes from noticing that in a pure electron state $\psi_e(x)$ of spin σ, the mean value of the spin is given by the quantity[4] [BPL12]

$$\begin{aligned}\mathcal{S}(t) = \left\langle \frac{\boldsymbol{\Sigma}}{2} \right\rangle &= \frac{1}{2}\int d^3\boldsymbol{x}\psi_e^\dagger(x)\boldsymbol{\Sigma}\psi_e(x) \\ &= \frac{1}{4m}u_\sigma^\dagger(\mathbf{0})\boldsymbol{\Sigma}u_\sigma(\mathbf{0}) \\ &= \frac{1}{4m}\bar{u}_\sigma(\mathbf{0})\gamma^0\boldsymbol{\Sigma}u_\sigma(\mathbf{0}) \\ &= \frac{1}{4m}\,\mathbf{tr}\left(\rho_\sigma^{(u)}\gamma^0\boldsymbol{\Sigma}(\boldsymbol{p})\right) \\ &= -\frac{1}{4m}\,\mathbf{tr}\left(\rho_\sigma^{(u)}\gamma^5\boldsymbol{\gamma}\right) = \frac{\sigma}{2}\boldsymbol{\zeta}.\end{aligned} \tag{B.35}$$

equivalent to the corresponding expectation value over two-component spinors

$$\mathcal{S}(t) = \left\langle \frac{\boldsymbol{\sigma}}{2} \right\rangle = \frac{1}{2}\omega_\sigma^\dagger\boldsymbol{\sigma}\omega_\sigma = \frac{\sigma}{2}\boldsymbol{\zeta}. \tag{B.36}$$

B.2 Canonical spin quantization axis

In the case of a linearly polarized plane wave background $F^{\mu\nu} = A_0^{\mu\nu}\psi'(\phi)$ (see 2.53, 4.30), the direction of oscillation of the magnetic field does not change and hence it can be chosen as a quantization axis for the spin 4-pseudovector $\zeta^\mu(\mathbf{0})$ of the electron in its rest frame [Meu15]. Therefore, at an arbitrary momentum $\boldsymbol{p}$ the spin 4-pseudovector can be defined as [PP21]

$$\zeta^\mu(\boldsymbol{p}) = -\tilde{A}_0^{\mu\nu}p_\nu/\left(p_-A_0\right), \quad (p\zeta) = 0, \quad (n\zeta) = 0. \tag{B.37}$$

Inserting $\tilde{A}_0^{0i} = (A_0/2)\varepsilon^{0i\alpha\beta}\left(n_\alpha a_\beta - n_\beta a_\alpha\right) = -\boldsymbol{B}_0^i$, it can be seen that in the rest frame (where $p_- = m$, $p^\mu = (m, \mathbf{0})$), the spin 4-pseudovector becomes [Sei+18]

$$\zeta^\mu(\mathbf{0}) = \left(0, \frac{\boldsymbol{B}_0}{|\boldsymbol{B}_0|}\right) \tag{B.38}$$

which shows that the choice B.37 is consistent, i.e. the spin quantization axis $\boldsymbol{\zeta}$ (with $\boldsymbol{\zeta}^2 = 1$) is really parallel to the magnetic field axis in the electron rest frame.

C Useful identities

Observation C.1

$$\int_0^\infty \frac{dudv}{u(1+v)^2}\frac{1+2v}{1+v}\left\{e^{-iu\left[1+\frac{1}{3}\frac{\chi^2(\phi)}{v^2}u^2\right]} - e^{-iu}\right\} = -\frac{\chi^2(\phi)}{3}\int_0^\infty \frac{dudv}{(1+v)^2}e^{-iu\left[1+\frac{1}{3}\frac{\chi^2(\phi)}{v^2}u^2\right]}\frac{1+v-3v^2}{1+v}\frac{u}{v^2} \tag{C.39}$$

[4] A more rigorous proof would use the mode expansion of the spinor and then integrate over a narrow region $\Delta\boldsymbol{p}$ centered at 0 in 3-momentum space.

Proof:

$$
\begin{aligned}
&\int_0^\infty \frac{du dv}{u(1+v)^2}\frac{1+2v}{1+v}\left\{e^{-iu\left[1+\frac{1}{3}\frac{\chi^2(\phi)}{v^2}u^2\right]} - e^{-iu}\right\}\\
&= \int_0^\infty \frac{dv}{(1+v)^2}\int_0^\infty \frac{du}{u}\left(2-\frac{1}{1+v}\right)e^{-iu}\left\{e^{-i\left[\frac{1}{3}\frac{\chi^2(\phi)}{v^2}u^3\right]}-1\right\}\\
&= \int_0^\infty \frac{dv}{(1+v)^2}\int_0^\infty \frac{du}{u}\left(2-\frac{1}{1+v}\right)e^{-i\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}u}\left(e^{-i\frac{u^3}{3}}-1\right)\\
&= \int_0^\infty \frac{dv}{(1+v)^2}\left(2-\frac{1}{1+v}\right)\int_0^\infty du\left(e^{-i\frac{u^3}{3}}-1\right) i\int_{\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}}^\infty dz e^{-iuz}\\
&= \int_0^\infty \frac{dv}{(1+v)^2}\left(2-\frac{1}{1+v}\right)\int_0^\infty du\left(e^{-i\frac{u^3}{3}}-1\right) i\frac{d}{du}\int_{\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}}^\infty \frac{dz}{-iz}e^{-iuz}\\
&\overset{PI}{=} \int_0^\infty \frac{dv}{(1+v)^2}\left(2-\frac{1}{1+v}\right)\int_0^\infty du\,\left(-iu^2\right)e^{-i\frac{u^3}{3}}\int_{\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}}^\infty \frac{dz}{z}e^{-iuz}\\
&= \int_0^\infty dv\frac{d}{dv}\left[-\frac{2}{(1+v)}+2+\frac{1}{2}\frac{1}{(1+v)^2}-\frac{1}{2}\right]\int_0^\infty du\,\left(-iu^2\right)e^{-i\frac{u^3}{3}}\int_{\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}}^\infty \frac{dz}{z}e^{-iuz}\\
&= \int_0^\infty dv(-1)\left[-\frac{2}{(1+v)}+2+\frac{1}{2}\frac{1}{(1+v)^2}-\frac{1}{2}\right]\int_0^\infty du\,\left(-iu^2\right)e^{-i\frac{u^3}{3}}(-1)\frac{e^{-i\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}u}}{\left(\frac{v^2}{\chi^2(\phi)}\right)^{\frac{1}{3}}}\frac{2}{3}\left(\frac{1}{\chi(\phi)}\right)^{\frac{2}{3}}\frac{1}{v^{\frac{1}{3}}}\\
&= -\frac{2i}{3}\int_0^\infty \frac{dv}{v}\left[\frac{2v}{(1+v)}+\frac{1-1-2v-v^2}{2(1+v)^2}\right]\int_0^\infty du u^2 e^{-i\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}u-i\frac{u^3}{3}}\\
&= -\frac{2i}{3}\int_0^\infty dv\left[\frac{2}{(1+v)}-\frac{2+v}{2(1+v)^2}\right]\int_0^\infty du u i\frac{d}{d\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}}e^{-i\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}u}e^{-i\frac{u^3}{3}}\\
&= \frac{2}{3}\int_0^\infty dv\left[\frac{2}{(1+v)}-\frac{2+v}{2(1+v)^2}\right]\int_0^\infty du u\frac{1}{\frac{2}{3}\frac{1}{v^{\frac{1}{3}}}\frac{1}{\chi(\phi)^{\frac{2}{3}}}}\frac{d}{dv}e^{\left[-i\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}u-i\frac{u^3}{3}\right]}\\
&= \chi(\phi)^{\frac{2}{3}}\int_0^\infty dv v^{\frac{1}{3}}\left[\frac{2}{(1+v)}-\frac{2+v}{2(1+v)^2}\right]\int_0^\infty du u\frac{d}{dv}e^{\left[-i\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}u-i\frac{u^3}{3}\right]}\\
&\overset{PI}{=} -\chi(\phi)^{\frac{2}{3}}\int_0^\infty dv\left\{\frac{1}{3}\frac{1}{v^{\frac{2}{3}}}\left[\frac{2}{(1+v)}-\frac{2+v}{2(1+v)^2}\right]+v^{\frac{1}{3}}\left[-\frac{2}{(1+v)^2}+\frac{1}{2}\frac{1}{(1+v)^2}+\frac{1}{(1+v)^3}\right]\right\}\\
&\times\int_0^\infty du u e^{\left[-i\left(\frac{v}{\chi(\phi)}\right)^{\frac{2}{3}}u-i\frac{u^3}{3}\right]} \qquad \left(\text{now } u\mapsto\frac{v^{1/3}}{(\chi(\phi))^{1/3}}u\right)\\
&= -\int_0^\infty dv\left\{\frac{1}{3}\left[\frac{2}{(1+v)}-\frac{2+v}{2(1+v)^2}\right]+v\left[-\frac{3}{2}\frac{1}{(1+v)^2}+\frac{1}{(1+v)^3}\right]\right\}\int_0^\infty du u e^{\left[-i\left(\frac{v}{\chi(\phi)}\right)\left(u+\frac{u^3}{3}\right)\right]}\\
&= -\int_0^\infty dv\left\{\frac{1}{3}\frac{4(1+v)-(2+v)}{2(1+v)^2}+\frac{v}{2(1+v)^3}\left[2-3(1+v)\right]\right\}\int_0^\infty du u e^{\left[-i\left(\frac{v}{\chi(\phi)}\right)\left(u+\frac{u^3}{3}\right)\right]}\\
&= -\int_0^\infty dv\left\{\frac{1}{3}\frac{2+3v}{2(1+v)^2}-\frac{v(1+3v)}{2(1+v)^3}\right\}\int_0^\infty du u e^{\left[-i\left(\frac{v}{\chi(\phi)}\right)\left(u+\frac{u^3}{3}\right)\right]}\\
&= -\int_0^\infty dv\frac{(2+3v)(1+v)-v(1+3v)3}{6(1+v)^3}\int_0^\infty du u e^{\left[-i\left(\frac{v}{\chi(\phi)}\right)\left(u+\frac{u^3}{3}\right)\right]}\\
&= -\int_0^\infty dv\frac{1+v-3v^2}{3(1+v)^3}\int_0^\infty du u e^{\left[-i\left(\frac{v}{\chi(\phi)}\right)\left(u+\frac{u^3}{3}\right)\right]} \qquad \left(u\to\frac{\chi(\phi)}{v}u\right)\\
&= -\frac{\chi^2(\phi)}{3}\int_0^\infty dv\frac{1+v-3v^2}{3(1+v)^3}\int_0^\infty du e^{-iu\left[u+\left(\frac{\chi(\phi)}{v}\right)^{\frac{2}{3}}\frac{u^2}{3}\right]}\frac{u}{v^2}
\end{aligned}
\tag{C.40}
$$

Observation C.2 (Schwinger parametrization) In terms of the Schwinger **proper time** s, the following integral identity holds when $\text{Im}(A) > 0$ [Sch19]

$$\frac{1}{A} = -i \int_0^\infty ds e^{isA} \quad (\text{Im}(A) > 0) \tag{C.41}$$

Publications

The results presented in detail in the thesis are published in:

On the electron mass shift in an intense plane wave
A. Di Piazza and T. Pătuleanu
arXiv:2106.13720 [hep-ph]
[Submitted on 25 Jun 2021]
[Submitted also to Physical Review D]

List of Figures

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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