Surface charges in Chern-Simons gravity with $T\bar{T}$ deformation

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Abstract

The $T\bar{T}$ deformed 2D CFTs correspond to AdS_3 gravity with Dirichlet boundary condition at finite cutoff or equivalently a mixed boundary condition at spatial infinity. Starting from the Bañados geometry, we obtained the $T\bar{T}$ deformed Chern-Simons gauge fields which are parametrized by two classes of independent charges. With help of the mixed boundary condition, the residual gauge symmetries of deformed gauge fields and the associated surface charges can be obtained respectively. These surface charges satisfy a non-linear deformed Virasoro algebra, which is obtained differently in the literature. Finally, the time-independent charges can be systematically constructed from these surface charges and they satisfy the field-dependent Virasoro algebra.

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Contents

1 Introduction

The $T\bar{T}$ deformed 2D CFTs attract a lot of interests because of its integrability and holographic duality [1–3]. The action of $T\bar{T}$ deformation is defined by a flow equation along with the determinant of the stress tensor, which is also called $T\bar{T}$ operator. Although the deformation is irrelevant and the $T\bar{T}$ flow takes a theory from IR to UV, many physical quantities can be exactly computed in terms of the un-deformed quantities, such us the Lagrangian [4], finite-size spectrum [2], and partition function [5, 6]. The $T\bar{T}$ deformed theory also has many equivalent descriptions, such as the $T\bar{T}$ deformation can be treated as the original theory dressed by the JT gravity [7–9], the deformation is equivalent to a random coordinate transformation infinitesimally [10], and it also refers to the non-critical string theory [11, 12]. The $T\bar{T}$ deformation can also be obtained through a field-dependent change of coordinates [13, 14].

In the holographic aspects, it was proposed that the $T\bar{T}$ deformed CFT corresponds to the AdS_3 gravity with Dirichlet boundary condition at finite radial [3, 15], and the cutoff radius is related to the deformation parameter. The finite-size spectrum turned out to be the quasi-local energy of the BTZ black hole at finite radius, and the $T\bar{T}$ flow equation coincides with the Hamilton-Jacobi equation governing the radial evolution of the classical gravity action in AdS_3 [3, 16, 17]. It was known that the Dirichlet boundary conditions at finite radius correspond to mixed boundary conditions at infinity[18–20]. Instead of working with a Dirichlet boundary condition on a cutoff surface, the other way to think about the holographic description is imposing a mixed boundary condition at the asymptotic AdS_3 boundary [21]. This mixed boundary condition leads to a deformed bulk solution, which is constructed through a field-dependent coordinate transformation [21]. The boundary dynamics of AdS_3 with the mixed boundary condition can be described by the $T\bar{T}$ deformed coadjoint orbit of the Virasoro group [22, 23]. Many holographic features of the $T\bar{T}$ deformed CFT have been explored [24–35]. For more, see the recent review of the $T\bar{T}$ deformation [36].

It is shown that the $T\bar{T}$ deformation preserves integrability [1, 37]. Alternatively, it is interesting to investigate chaotic behavior under the $T\bar{T}$ deformation. The out of time ordered correlation function(OTOC) has been widely used to capture the chaotic or integrability. As the fundamental observables, as well as the basics components of OTOC, the correlation functions for $T\bar{T}$ deformation are also studied [38–44]. Recently, The $T\bar{T}$ deformations have been considered in other models including integrable lattice models and non-relativistic integrable field theories [45–50].

The deformed 2D CFTs still have infinitely many charges. The charge algebra would play a crucial role in extracting the exact results in the $T\bar{T}$ deformation. These charges have been explored on both the field side and gravity side. The calculation from the boundary field side shows that some additional winding terms in Poisson brackets are not fixed due to certain ambiguities of the field-dependent coordinates [51, 52]. On the gravity side, the charge algebra was obtained by considering 3D gravity with Dirichlet boundary conditions on a finite boundary [21, 53]. In AdS_3/CFT_2, the Chern-Simons formalism is a powerful tool to study the boundary dynamics and asymptotic symmetries [54–57]. Even more, it can be naturally generalized to higher spin gravity [58–60]. In the present work, we prefer to use the Chern-Simons form to study the asymptotic symmetries of AdS_3 with mixed boundary conditions. An analogy to the Bañados geometry, we rewrite the deformed AdS_3 solution into Chern-Simons gauge fields, which are also parametrized by two independent functions. After imposing the mixed boundary condition, we find the residual gauge symmetries and associated surface charges in Chern-Simons theory. The charge algebra is also obtained by using the covariant phase space method [53]. The resulting algebra turns out to be a non-linear deformation of the Virasoro algebra, which coincides with the result in [53]. Further, we systematically construct the time-independent charges which satisfy the same charge algebra given in [21].

This paper is organized as following: Section 2 is a review of the global symmetries in Chern-Simons theory. In section 3, we rewrite the deformed solutions of AdS_3 in the Chern-Simons form. The $T\bar{T}$ deformed gauge field can be parametrized by the two classes of independent deformed charges. In section 4, we obtain a set of residual gauge transformations which keep the deformed gauge connections asymptotically invariant. The residual gauge transformations generate a set of surface charges and the corresponding charge algebra is also obtained. We comment on the surface charges and the charge algebra in section 5. Conclusions and discussions are given in section 6. Some calculation details are presented in the appendices.

2 Review of surface charges in Chern-Simons theory

This section is to review some well-known facts about Chern-Simons theory following the Refs [61, 62]. We start from the Chern-Simons theory defined on a manifold with topology $M = \mathbb{R} \times \Sigma$, whose action is

$$I(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{2.1}$$

In the Hamiltonian form, the action can be expressed as

$$I(A) = \frac{k}{4\pi} \int_{\mathbb{R}} dt \int_{\Sigma} d^2x \varepsilon^{ij} g_{ab} (\dot{A}_i^a A_j^b + A_t^a F_{ij}^b) + B, \tag{2.2}$$

where the g_{ab} is the Cartan-Killing metric of the gauge group. The B is a boundary term that depends on the imposed boundary condition. The boundary term plays a crucial role in the charges and the symmetry algebra [63]. As a consequence, we may get the different charges and symmetries by imposing various boundary conditions in Chern-Simons theory.

From the Hamiltonian form, we learn that the A_i^a are the dynamics fields and A_t^a is the Lagrange multiplier. Varying with respect to A_i^a, one can get the equation of motion

$$F_{ti} = \partial_t A_i^a - \partial_i A_t^a + f_{bc}^a A_t^b A_i^c = 0. \tag{2.3}$$

The Lagrange multiplier gives the constraint

$$G_a \equiv \frac{k}{4\pi} g_{ac} \varepsilon^{ij} F_{ij}^c = 0. \tag{2.4}$$

The canonical momentum of the dynamical fields A_i^a are A_j^b, which satisfy the basic Poisson bracket

$$\{A_i^a(x), A_j^b(y)\} = \frac{2\pi}{k} g^{ab} \varepsilon_{ij} \delta(x-y). \tag{2.5}$$

The Poisson bracket for any two functions G, H can be defined as

$$\{G(A_i^a), H(A_j^b)\} = \frac{2\pi}{k} \int dx^2 \varepsilon_{ij} g^{ab} \frac{\delta G}{\delta A_i^a} \frac{\delta H}{\delta A_j^b}. \tag{2.6}$$

Therefore, one can find the constraints satisfy the Poisson algebra

$$\{G_a(x), G_b(y)\} = f_{ab}^c G_c(x) \delta(x-y), \tag{2.7}$$

which implies this is the first class constraints.

Moreover, we should also consider the smeared generator which is defined as

$$G(\eta) = \int_\Sigma G_a \eta^a + Q(\eta), \quad Q(\eta) = -\frac{k}{2\pi} \int_{\partial\Sigma} \eta_a A^a. \tag{2.8}$$

The supplemented term $Q(\eta)$ is to make the smeared generator differentiable [61]. In general, the parameter η is the set of gauge transformations that preserve the imposed boundary conditions. The Poisson bracket of the smeared generators is

$$\{G(\eta), G(\lambda)\} = G([\eta, \lambda]) + C(\eta, \lambda), \quad C(\eta, \lambda) = \frac{k}{2\pi} \int_{\partial\Sigma} \eta_a d\lambda^a,$$

where C is the central charge term. As a consequence, the Poisson bracket of the smeared generator is a central extension of the algebra of the gauge generator (2.7). The central extension comes from the surface term $Q(\eta)$ in the definition of the smeared generator. It is worth noting that the smeared generator does not vanish when the constraints $G_a = 0$ are imposed. The transformations generated by $G(\eta)$ are not the trivial gauge transformations. The residual gauge generators can transform from one physical state to another one [55]. The asymptotic symmetry, also called global symmetry, is defined as the quotient of the group of residual gauge transformations modulo the group of the trivial gauge transformations. This is the origin of infinitely many boundary degrees of freedom in Chern-Simons theory.

After disentangling the constraints (2.4), the $Q(\eta)$ define the surface charges of the Chern-Simons theory. It turns out the surface charges satisfy the same Poisson bracket algebra

$$\{Q(\eta), Q(\lambda)\} = Q([\eta, \lambda]) + C(\eta, \lambda). \tag{2.9}$$

Furthermore, the variation of a function in phase space can be generate by the surface charges

$$\delta_\lambda F = \{Q(\lambda), F\} \tag{2.10}$$

Given a certain boundary condition, we can find the gauge transformations preserving the boundary condition. The corresponding charges induced by the boundary condition can also be obtained. This technique was widely used in AdS_3 with various boundary conditions [55, 64–70]. In this paper, we would like to apply this approach to study the surface charges of the Chern-Simons gravity theory with the mixed boundary condition for $T\bar{T}$ deformation [21].

3 Chern-Simons formalism and $T\bar{T}$ deformation

The AdS_3 gravity can be formulated as $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory [71]. The action can be written as the sum of the left-moving part and right-moving part

$$S(A, \bar{A}) = I(A) - I(\bar{A}), \quad \text{with} \quad k = \frac{1}{4G}, \tag{3.1}$$

where the gauge fields are the combination of vielbein and spin connection

$$A^a = \omega^a + e^a, \quad \bar{A}^a = \omega^a - e^a. \tag{3.2}$$

The equations of motion are

$$dA + A \wedge A = 0, \quad d\bar{A} + \bar{A} \wedge \bar{A} = 0. \tag{3.3}$$

which agree with first order gravitational field equations. Given an AdS_3 solution, we have an equivalent description in the Chern-Simons formalism.

In particular, the Bañados geometry [55] in Fefferman-Graham gauge is following

$$ds^2 = \frac{dr^2}{r^2} + r^2 \left(dz d\bar{z} + \frac{1}{r^2}\mathcal{L}(z)dz^2 + \frac{1}{r^2}\bar{\mathcal{L}}(\bar{z})d\bar{z}^2 + \frac{1}{r^4}\mathcal{L}(z)\bar{\mathcal{L}}(\bar{z})dzd\bar{z} \right), \tag{3.4}$$

where the $\mathcal{L}(z)$ and $\bar{\mathcal{L}}(\bar{z})$ are arbitrary holomorphic and antiholomorphic functions, respectively. For the case of BTZ black hole, the parameters are constants associated with the mass and angular momentum of the black hole

$$\mathcal{L} = \frac{M+J}{2}, \quad \bar{\mathcal{L}} = \frac{M-J}{2}. \tag{3.5}$$

Up to the Lorentz rotation, the corresponding Chern-Simons gauge connections can be fixed as

$$\tilde{A} = \frac{dr}{r}L_0 + rdzL_{-1} + \frac{1}{r}\mathcal{L}dzL_1, \tag{3.6}$$

$$\tilde{\bar{A}} = -\frac{dr}{r}L_0 - \frac{1}{r}\bar{\mathcal{L}}d\bar{z}L_{-1} - rd\bar{z}L_1. \tag{3.7}$$

where the L_{-1}, L_0, L_1 are the generators of $SL(2,\mathbb{R})$. In this paper, we use the following generators of $SL(2,\mathbb{R})$

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, L_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.8}$$

with the commutation relations

$$[L_{-1}, L_0] = L_{-1}, \quad [L_{-1}, L_1] = -2L_0, \quad [L_0, L_1] = L_1. \tag{3.9}$$

The non-zero components of Cartan-Killing metric are

$$\mathrm{Tr}\,(L_{-1}L_1) = \mathrm{Tr}\,(L_1 L_{-1}) = 1, \quad \mathrm{Tr}\,(L_0 L_0) = \frac{1}{2}. \tag{3.10}$$

In this paper, we will use the $(\tilde{A}, \tilde{\bar{A}})$ denote the original gauge fields and $(A, \bar{A})$ denote the deformed gauge fields. Following [72], the r-dependent of the gauge fields can be eliminated through a gauge transformation

$$\tilde{A} = b(d + \tilde{a})b^{-1}, \quad \tilde{\bar{A}} = b(d + \tilde{\bar{a}})b^{-1}, \quad b = e^{\ln r L_0}. \tag{3.11}$$

The induced gauge fields take the form

$$\tilde{a} = (L_{-1} + \mathcal{L}L_1)dz, \quad \tilde{\bar{a}} = (\bar{\mathcal{L}}L_{-1} + L_1)d\bar{z}, \tag{3.12}$$

which can be treated as the gauge connection defined on the boundary. For the Bañados geometry, the residual gauge symmetry generates the Virasoro algebra [55].

The $T\bar{T}$ deformed CFT correspond to the AdS_3 gravity with a mixed boundary condition. The deformed AdS_3 solution can also be constructed from original one via a field-dependent coordinate transformation [21]. For the Bañados geometry, the field-dependent coordinate transformation reads

$$dz = \frac{1}{1 - \mu^2 \mathcal{L}_\mu \bar{\mathcal{L}}_\mu}(dw - \mu\bar{\mathcal{L}}_\mu d\bar{w}), \quad d\bar{z} = \frac{1}{1 - \mu^2 \mathcal{L}_\mu \bar{\mathcal{L}}_\mu}(d\bar{w} - \mu\mathcal{L}_\mu dw), \tag{3.13}$$

where $\mathcal{L}_\mu \equiv \mathcal{L}(z(\mu, w, \bar{w})), \bar{\mathcal{L}}_\mu \equiv \mathcal{L}(z(\mu, w, \bar{w}))$, and μ is the deformation parameter. Then, we can obtain the deformed the gauge fields

$$A = \frac{1}{r}L_0 dr + \frac{1}{1 - \mu^2 \mathcal{L}_\mu \bar{\mathcal{L}}_\mu}\left(rL_{-1} + \frac{1}{r}\mathcal{L}_\mu L_1\right)(dw - \mu\bar{\mathcal{L}}_\mu d\bar{w}), \tag{3.14}$$

$$\bar{A} = -\frac{1}{r}L_0 dr - \frac{1}{1 - \mu^2 \mathcal{L}_\mu \bar{\mathcal{L}}_\mu}\left(\frac{1}{r}\bar{\mathcal{L}}_\mu L_{-1} + rL_1\right)(d\bar{w} - \mu\mathcal{L}_\mu dw). \tag{3.15}$$

The Bañados geometry is parametrized by the holomorphic function $\mathcal{L}(z)$ and anti-holomorphic function $\bar{\mathcal{L}}(\bar{z})$. The deformed metric is parametrized by $\mathcal{L}_\mu$ and $\bar{\mathcal{L}}_\mu$. The coordinate transformation implies the deformed parameter $\mathcal{L}_\mu$ and $\bar{\mathcal{L}}_\mu$ obey

$$\partial_{\bar{w}}\mathcal{L}_\mu + \mu\bar{\mathcal{L}}_\mu\partial_w\mathcal{L}_\mu = 0, \tag{3.16}$$

$$\partial_w\bar{\mathcal{L}}_\mu + \mu\mathcal{L}_\mu\partial_w\bar{\mathcal{L}}_\mu = 0. \tag{3.17}$$

Since these gauge connections satisfy the equation of motions, the deformed metric is still the solution of AdS_3. In [21], it is also shown that the deformed parameters are following

$$\mathcal{L} = \frac{\mathcal{L}_\mu(1-\mu\bar{\mathcal{L}}_\mu)^2}{(1-\mu^2\mathcal{L}_\mu\bar{\mathcal{L}}_\mu)^2}, \quad \bar{\mathcal{L}} = \frac{\bar{\mathcal{L}}_\mu(1-\mu\mathcal{L}_\mu)^2}{(1-\mu^2\mathcal{L}_\mu\bar{\mathcal{L}}_\mu)^2}. \tag{3.18}$$

We prefer to use the coordinates $\theta = (w+\bar{w})/2, t = (w-\bar{w})/2$, where t represents the time direction while θ represents a circle at the boundary with the identification $\theta \sim \theta + 2\pi$. In this coordinate, the gauge fields can be written as

$$A_r = \frac{1}{r}L_0, \quad \bar{A}_r = -\frac{1}{r}L_0 \tag{3.19}$$

$$A_\theta = \frac{1-\mu\bar{\mathcal{L}}_\mu}{1-\mu^2\mathcal{L}_\mu\bar{\mathcal{L}}_\mu}(rL_{-1} + \frac{1}{r}\mathcal{L}_\mu L_1), \quad A_t = KA_\theta, \tag{3.20}$$

$$\bar{A}_\theta = -\frac{1-\mu\mathcal{L}_\mu}{1-\mu^2\mathcal{L}_\mu\bar{\mathcal{L}}_\mu}(\frac{1}{r}\bar{\mathcal{L}}_\mu L_{-1} + rL_1), \quad \bar{A}_t = \bar{K}\bar{A}_\theta. \tag{3.21}$$

where we define

$$K = \frac{1+\mu\bar{\mathcal{L}}_\mu}{1-\mu\bar{\mathcal{L}}_\mu}, \quad \bar{K} = -\frac{1+\mu\mathcal{L}_\mu}{1-\mu\mathcal{L}_\mu}. \tag{3.22}$$

In the Bañados geometry, the parameters $\mathcal{L}$ and $\bar{\mathcal{L}}$ relate to the charges. The new parameters $\mathcal{L}_\mu$ and $\bar{\mathcal{L}}_\mu$ do not play the role of charges in the deformed geometry. The deformed spectrum and angular momentum can be obtained from gravity side [3, 21], which take the form

$$\mathcal{E} = \frac{1}{\mu}\left(1 - \sqrt{1 - 2\mu(\mathcal{L}+\bar{\mathcal{L}}) + \mu^2(\mathcal{L}-\bar{\mathcal{L}})^2}\right), \quad \mathcal{J} = \mathcal{L} - \bar{\mathcal{L}}. \tag{3.23}$$

In the Chern-Simons AdS_3 gravity with the mixed boundary condition, the boundary surface integral gives the correct spectrum[23]. In analogy with the un-deformed case (3.5), we can define the charges

$$q = \frac{\mathcal{E}+\mathcal{J}}{2}, \quad \bar{q} = \frac{\mathcal{E}-\mathcal{J}}{2}, \tag{3.24}$$

In section 4, the q and $\bar{q}$ are turned out the the surface charges. The charges reduced to the Virasoro charges when $\mu \to 0$. As a consequence, we have three ways to parametrize the deformed gauge fields, by $(\mathcal{L}, \bar{\mathcal{L}})$, $(\mathcal{L}_\mu, \bar{\mathcal{L}}_\mu)$ and $(q, \bar{q})$. The relations between different parameters are

$$\frac{1-\mu\bar{\mathcal{L}}_\mu}{1-\mu^2\mathcal{L}_\mu\bar{\mathcal{L}}_\mu} = \frac{1}{2}\left[1+\mu(\mathcal{L}-\bar{\mathcal{L}})+\sqrt{1-2\mu(\mathcal{L}+\bar{\mathcal{L}})+\mu^2(\mathcal{L}-\bar{\mathcal{L}})^2}\right] = 1-\mu q, \tag{3.25}$$

$$\frac{1-\mu\mathcal{L}_\mu}{1-\mu^2\mathcal{L}_\mu\bar{\mathcal{L}}_\mu} = \frac{1}{2}\left[1-\mu(\mathcal{L}-\bar{\mathcal{L}})+\sqrt{1-2\mu(\mathcal{L}+\bar{\mathcal{L}})+\mu^2(\mathcal{L}-\bar{\mathcal{L}})^2}\right] = 1-\mu \bar{q}. \tag{3.26}$$

In the latter of this paper, we will use different parameters to simplify the expressions, and they can be transformed to each other with the help of the above relations.

Finally, the deformed gauge connection can be expressed in terms of $(q, \bar{q})$

$$A_\theta = r(1-\mu\bar{q})L_{-1} + \frac{1}{r}qL_1, \quad A_t = K\Big(r(1-\mu\bar{q})L_{-1} + \frac{1}{r}qL_1\Big), \tag{3.27}$$

$$\bar{A}_\theta = -\frac{1}{r}\bar{q}L_{-1} - r(1-\mu q)L_1, \quad \bar{A}_t = -\bar{K}\Big(\frac{1}{r}\bar{q}L_{-1} + r(1-\mu q)L_1\Big), \tag{3.28}$$

with

$$K = \frac{1+\mu(\bar{q}-q)}{1-\mu(q+\bar{q})}, \quad \bar{K} = -\frac{1-\mu(\bar{q}-q)}{1-\mu(q+\bar{q})}. \tag{3.29}$$

Moreover, (3.16) and (3.17) imply the parameters $(q, \bar{q})$ satisfy the equations

$$\partial_t q = \partial_\theta(Kq), \tag{3.30}$$

$$\partial_t \bar{q} = \partial_\theta(\bar{K}\bar{q}), \tag{3.31}$$

from which we can also see that the deformed charges are no longer holomorphic or antiholomorphic. When taking the limit $\mu \to 0$, the gauge connection would reduce to the undeformed case. For the deformed geometry, the radial degree of freedom can also be eliminated through the gauge transformation (3.11), the induced gauge connections are

$$a_\theta = (1-\mu\bar{q})L_{-1} + qL_1, \quad a_t = K\Big((1-\mu\bar{q})L_{-1} + qL_1\Big), \tag{3.32}$$

$$\bar{a}_\theta = -\bar{q}L_{-1} - (1-\mu q)L_1, \quad \bar{a}_t = -\bar{K}\Big(\bar{q}L_{-1} + (1-\mu q)L_1\Big). \tag{3.33}$$

These gauge fields are defined on the boundary. We then will apply the induced gauge fields to study the symmetry of $T\bar{T}$ deformation in the next section.

4 Surface charges and their algebra

In this section, we would like to calculate the surface charges induced by asymptotic symmetries of Chern-Simons theory with $T\bar{T}$ deformation. Firstly, we have to find the residual gauge symmetry generators of the $T\bar{T}$ deformed gauge fields. In terms of the symmetry generators, we calculate the surface charges defined in section 2. Finally, we obtain the algebra of the deformed surface charges.

4.1 Boundary condition and symmetries of the deformed gauge fields

For the deformed gauge fields, we assume the variation of the charges $q, \bar{q}$ induced by gauge transformation of the deformed gauge field as following forms

$$\lambda: \quad q \to q + \delta_\lambda q, \quad \bar{q} \to \bar{q} + \delta_\lambda \bar{q}, \tag{4.1}$$

$$\bar{\lambda}: \quad q \to q + \delta_{\bar{\lambda}} q, \quad \bar{q} \to \bar{q} + \delta_{\bar{\lambda}} \bar{q}. \tag{4.2}$$

Where λ and $\bar{\lambda}$ are defined as the left-moving part and the right-moving part respectively

$$\lambda = \sum_{i=-1}^{1} \lambda_i L_i, \quad \bar{\lambda} = \sum_{i=-1}^{1} \bar{\lambda}_i L_i. \tag{4.3}$$

Then the variation of the gauge fields can be expressed as

$$\delta_\lambda a_\theta = -\mu \delta_\lambda \bar{q} L_{-1} + \delta_\lambda q L_1, \tag{4.4}$$

$$\delta_\lambda a_t = \delta_\lambda \left(K(1-\mu\bar{q}) \right) L_{-1} + \delta_\lambda (Kq) L_1, \tag{4.5}$$

$$\delta_{\bar{\lambda}} \bar{a}_\theta = -\delta_{\bar{\lambda}} \bar{q} L_{-1} + \mu \delta_{\bar{\lambda}} q L_1, \tag{4.6}$$

$$\delta_{\bar{\lambda}} \bar{a}_t = -\delta_{\bar{\lambda}} \left(\bar{K} \mu \bar{q} \right) L_{-1} - \delta_{\bar{\lambda}} (\bar{K}(1-\mu q)) L_1, \tag{4.7}$$

where the variation of $K, \bar{K}$ can also expressed in terms of the variation of $q, \bar{q}$

$$\delta_\lambda K = \frac{2\mu \left(\mu \bar{q} \delta_\lambda q + (1-\mu q) \delta_\lambda \bar{q} \right)}{1 - \left(\mu \left(\bar{q} + q \right) \right)^2}, \tag{4.8}$$

$$\delta_{\bar{\lambda}} \bar{K} = -\frac{2\mu \left((1-\mu\bar{q}) \delta_{\bar{\lambda}} q + \mu q \delta_{\bar{\lambda}} \bar{q} \right)}{\left(1 - \mu \left(\bar{q} + q \right) \right)^2}. \tag{4.9}$$

Then, we have to find the relations between $(\delta_\lambda q, \delta_\lambda \bar{q})$ and $(\delta_{\bar{\lambda}} q, \delta_{\bar{\lambda}} \bar{q})$ by using the mixed boundary condition. In our setting, the gauge fields can reproduce the deformed metric through

$$g_{\mu\nu} = \frac{1}{2} \mathrm{Tr}[(A_\mu - \bar{A}_\mu)(A_\nu - \bar{A}_\nu)]. \tag{4.10}$$

The induced boundary metric at the finite cutoff surface $r = r_c$ turns out to be a flat one

$$ds^2 \Big|_{r=r_c} = \frac{1}{\mu}(d\theta^2 - dt^2), \tag{4.11}$$

where we have invoked the holographic relation $r_c^2 = 1/\mu$ in $T\bar{T}$ deformation [3]. It follows that the variation of the metric on the boundary should be vanishing

$$\delta g_{\mu\nu}\Big|_{r=r_c} = \mathrm{Tr}[(\delta_\lambda A_\mu - \delta_{\bar{\lambda}}\bar{A}_\mu)(A_\nu - \bar{A}_\nu)]\Big|_{r=r_c} = 0. \tag{4.12}$$

It leads to

$$(\delta_{\bar{\lambda}}\bar{q} - \delta_\lambda\bar{q}) + (\delta_\lambda q - \delta_{\bar{\lambda}} q) = 0, \tag{4.13}$$

$$(\delta_{\bar{\lambda}}\bar{q} - \delta_\lambda\bar{q}) - (\delta_\lambda q - \delta_{\bar{\lambda}} q) = 0, \tag{4.14}$$

$$\delta_\lambda(K(1-\mu\bar{q})) + \delta_{\bar{\lambda}}(\mu K\bar{q}) + \delta_\lambda(\mu K q) + \delta_{\bar{\lambda}}(\bar{K}(1-\mu q)) = 0, \tag{4.15}$$

$$\delta_\lambda(K(1-\mu\bar{q})) + \delta_{\bar{\lambda}}(\mu K\bar{q}) - \delta_\lambda(\mu K q) - \delta_{\bar{\lambda}}(\bar{K}(1-\mu q)) = 0. \tag{4.16}$$

The unique solution for these equations implies the constraints

$$\delta_\lambda q - \delta_{\bar{\lambda}} q = 0, \tag{4.17}$$

$$\delta_{\bar{\lambda}}\bar{q} - \delta_\lambda\bar{q} = 0. \tag{4.18}$$

These relations mean that the gauge transformations on the left-moving part and right-moving part are entangled.

Under the infinitesimal gauge transformation, variation of the deformed gauge fields are

$$\delta_\lambda a = d\lambda + [a,\lambda], \quad \delta_{\bar{\lambda}}\bar{a} = d\bar{\lambda} + [\bar{a},\bar{\lambda}]. \tag{4.19}$$

The gauge transformation that preserve the asymptotic behavior of $a_\theta, \bar{a}_\theta$ gives

$$-\mu\delta_\lambda\bar{q} = \lambda'_{-1} + \lambda_0(1-\mu\bar{q}), \tag{4.20}$$

$$0 = \lambda'_0 + 2\Big(\lambda_{-1}q - \lambda_1(1-\mu\bar{q})\Big), \tag{4.21}$$

$$\delta_\lambda q = \lambda'_1 - \lambda_0 q, \tag{4.22}$$

$$-\delta_{\bar{\lambda}}\bar{q} = \bar{\lambda}'_{-1} - \bar{\lambda}_0\bar{q}, \tag{4.23}$$

$$0 = \lambda'_0 + 2\Big(\bar{\lambda}_1\bar{q} - \bar{\lambda}_{-1}(1-\mu q)\Big), \tag{4.24}$$

$$\mu\delta_{\bar{\lambda}} q = \bar{\lambda}'_1 + \bar{\lambda}_0(1-\mu q). \tag{4.25}$$

The gauge transformation that preserves the asymptotic behavior of $a_t, \bar{a}_t$ gives

$$\delta_\lambda\left(K(1-\mu\bar{q})\right) = \partial_t\lambda_{-1} + K\lambda_0(1-\mu\bar{q}), \tag{4.26}$$

$$0 = \partial_t\lambda_0 + 2K\Big(\lambda_{-1}q - \lambda_1(1-\mu\bar{q})\Big), \tag{4.27}$$

$$\delta_\lambda\left(Kq\right) = \partial_t\lambda_1 - K\lambda_0 q, \tag{4.28}$$

$$-\delta_{\bar{\lambda}}(\bar{K}\bar{q}) = \partial_t\bar{\lambda}_{-1} - \bar{K}\bar{\lambda}_0\bar{q}, \tag{4.29}$$

$$0 = \partial_t\bar{\lambda}_0 + 2\bar{K}\Big(\bar{\lambda}_1\bar{q} - \bar{\lambda}_{-1}(1-\mu q)\Big), \tag{4.30}$$

$$-\delta_{\bar{\lambda}}\left(\bar{K}(1-\mu q)\right) = \partial_t\bar{\lambda}_1 + \bar{K}\bar{\lambda}_0(1-\mu q). \tag{4.31}$$

For later convenience, one can choose following parameters

$$\epsilon = \lambda_{-1} - \mu\bar{\lambda}_{-1}, \quad \bar{\epsilon} = \bar{\lambda}_1 - \mu\lambda_1. \tag{4.32}$$

The gauge transformation generators are parametrized by $(\epsilon, \bar{\epsilon})$. According to above equations, we can express the variation of the charges $\delta_\lambda q$ and $\delta_{\bar{\lambda}}\bar{q}$ in terms of the parameters $(\epsilon, \bar{\epsilon})$

$$\begin{aligned}
\delta_\lambda q = \delta_{\bar{\lambda}} q =&\epsilon(\theta)C' + \bar{\epsilon}(\theta)A' + \frac{1}{2}\bar{\epsilon}'(\theta)B'' - \frac{1}{2}\epsilon'(\theta)(B'' - 4C) \\
&- \frac{1}{2}\bar{\epsilon}''(\theta)(D' - B') - \frac{3}{2}\epsilon''(\theta)B' - \frac{1}{2}\bar{\epsilon}'''(\theta)D - \frac{1}{2}\epsilon'''(\theta)(2B + 1), \qquad (4.33)\\
\delta_\lambda \bar{q} = \delta_{\bar{\lambda}} \bar{q} =& - \epsilon(\theta)\bar{A}' - \bar{\epsilon}(\theta)\bar{C}' - \frac{1}{2}\epsilon'(\theta)\bar{B}'' + \frac{1}{2}\bar{\epsilon}'(\theta)(\bar{B}'' - 4\bar{C}) \\
&+ \frac{1}{2}\epsilon''(\theta)(\bar{D}' - \bar{B}') + \frac{3}{2}\bar{\epsilon}''(\theta)\bar{B}' + \frac{1}{2}\epsilon'''(\theta)\bar{D} + \frac{1}{2}\bar{\epsilon}'''(\theta)(2\bar{B} + 1), \qquad (4.34)
\end{aligned}$$

where we have introduced the following auxiliary variables to simplify the expressions

$$A =\bar{A} = \frac{\mu q\bar{q}}{1 - \mu\left(\bar{q} + q\right)}, \tag{4.35}$$

$$B =\frac{\mu\left(2\bar{q} - \mu(q + \bar{q})^2\right)}{2(1 - \mu(q + \bar{q}))^2}, \quad \bar{B} = \frac{\mu\left(2q - \mu(q + \bar{q})^2\right)}{2(1 - \mu(q + \bar{q}))^2}, \tag{4.36}$$

$$C =\frac{(1 - \mu q)q}{1 - \mu(q + \bar{q})}, \quad \bar{C} = \frac{(1 - \mu\bar{q})\bar{q}}{1 - \mu(q + \bar{q})}, \tag{4.37}$$

$$D = - \bar{D} = \frac{\mu\left(q - \bar{q}\right)}{\left(1 - \mu\left(\bar{q} + q\right)\right)^2}. \tag{4.38}$$

One can turn to the Appendix A for details about solving the variation of the charges under the gauge transformation.

From (4.26)-(4.31), the variables ϵ and $\bar{\epsilon}$ satisfy

$$-\partial_t\epsilon =\epsilon' - \frac{2\mu\bar{q}\left(1 - \mu\bar{q}\right)}{\left(1 - \mu\left(\bar{q} + q\right)\right)^2}\left(\epsilon' + \bar{\epsilon}'\right), \tag{4.39}$$

$$\partial_t\bar{\epsilon} =\bar{\epsilon}' - \frac{2\mu q\left(1 - \mu q\right)}{\left(1 - \mu\left(\bar{q} + q\right)\right)^2}\left(\epsilon' + \bar{\epsilon}'\right). \tag{4.40}$$

The details for deriving these equation are given in Appendix B. In [53], they got the same result using the Killing vector that leaves all components of the metric invariant at the boundary. For the undeformed case, namely, $\mu \to 0$, the equations reduce to that the ϵ and $\bar{\epsilon}$ are holomorphic and antiholomorphic functions, respectively, which indeed corresponds to the infinitesimal conformal transformation. The equations (4.39) and (4.40) can be identified as the $T\bar{T}$ deformed conformal Killing equations.

4.2 Surface charges and their algebra

Given the gauge transformation preserving the asymptotic behavior of gauge fields, we can obtain the associated surface charge

$$\begin{aligned}\mathcal{Q}_{\epsilon,\bar{\epsilon}} =& \int_{\partial\Sigma} \Big(\mathrm{Tr}(\lambda a_\theta) + \mathrm{Tr}(\bar{\lambda}\bar{a}_\theta)\Big) d\theta \\ =& 2\int_0^{2\pi} \Big(q(\theta)\epsilon(\theta) - \bar{q}(\theta)\bar{\epsilon}(\theta)\Big) d\theta,\end{aligned} \tag{4.41}$$

where the $(\lambda, \bar{\lambda})$ are the gauge generators we have solved in Appendix A. In the second step, we have dropped a total derivative term, which does not contribute to the charge. For convenience, we can define the charges

$$Q = \frac{1}{2}\mathcal{Q}_\epsilon = \int_0^{2\pi} q(\theta)\epsilon(\theta)d\theta, \tag{4.42}$$

$$\bar{Q} = \frac{1}{2}\mathcal{Q}_{\bar{\epsilon}} = \int_0^{2\pi} \bar{q}(\theta)\bar{\epsilon}(\theta)d\theta. \tag{4.43}$$

The variation of the charges under the symmetry transformation can be expressed as Poisson bracket algebra

$$\delta_\lambda q = \delta_{\bar{\lambda}} q = \frac{1}{2}\delta_{\lambda,\bar{\lambda}} q = \frac{1}{2}\{\mathcal{Q}_{\epsilon,\bar{\epsilon}}, q\} = \int_0^{2\pi} \Big(\{q(\theta'), q(\theta)\}\epsilon(\theta') - \{\bar{q}(\theta'), q(\theta)\}\bar{\epsilon}(\theta')\Big) d\theta', \tag{4.44}$$

$$\delta_\lambda \bar{q} = \delta_{\bar{\lambda}} \bar{q} = \frac{1}{2}\delta_{\lambda,\bar{\lambda}} \bar{q} = \frac{1}{2}\{\mathcal{Q}_{\epsilon,\bar{\epsilon}}, \bar{q}\} = \int_0^{2\pi} \Big(\{q(\theta'), \bar{q}(\theta)\}\epsilon(\theta') - \{\bar{q}(\theta'), \bar{q}(\theta)\}\bar{\epsilon}(\theta')\Big) d\theta'. \tag{4.45}$$

They generate the surface charges, which allows us to identify the Poisson structure on the phase space of asymptotically AdS_3 solutions with $T\bar{T}$ deformation. According to (4.33) and (4.34), after performing integration by parts and dropping some total derivative terms, we obtain the Poisson brackets

$$i\{q(\theta'), q(\theta)\} = C'\delta(\theta-\theta') - \frac{1}{2}(B'' - 4C)\delta'(\theta-\theta') - \frac{3}{2}B'\delta''(\theta-\theta') - \frac{1}{2}(2B+1)\delta'''(\theta-\theta'), \tag{4.46}$$

$$i\{\bar{q}(\theta'), q(\theta)\} = -A'\delta(\theta-\theta') - \frac{1}{2}B''\delta'(\theta-\theta') + \frac{1}{2}(D' - B')\delta''(\theta-\theta') + \frac{1}{2}D\delta'''(\theta-\theta'), \tag{4.47}$$

$$i\{q(\theta'), \bar{q}(\theta)\} = -\bar{A}'\delta(\theta-\theta') - \frac{1}{2}\bar{B}''\delta'(\theta-\theta') + \frac{1}{2}(\bar{D}' - \bar{B}')\delta''(\theta-\theta') + \frac{1}{2}\bar{D}\delta'''(\theta-\theta'), \tag{4.48}$$

$$i\{\bar{q}(\theta'), \bar{q}(\theta)\} = \bar{C}'\delta(\theta-\theta') - \frac{1}{2}(\bar{B}'' - 4\bar{C})\delta'(\theta-\theta') - \frac{3}{2}\bar{B}'\delta''(\theta-\theta') - \frac{1}{2}(2\bar{B}+1)\delta'''(\theta-\theta'). \tag{4.49}$$

The modes expansion of the charges is following

$$Q_n = \int_0^{2\pi} q(\theta) e^{-in\theta} d\theta, \quad \bar{Q}_m = \int_0^{2\pi} \bar{q}(\theta) e^{-im\theta} d\theta. \tag{4.50}$$

Then the Poisson brackets algebra is following

$$i\{Q_n, Q_m\} = (n-m)C_{n+m} - \frac{1}{2}mn(n-m)B_{n+m} + \frac{1}{2}n^3\delta_{n+m,0}, \tag{4.51}$$

$$\begin{aligned} i\{Q_n, \bar{Q}_m\} =& (n+m)A_{n+m} - \frac{1}{2}mn(n+m)B_{n+m} + \frac{1}{2}m^2 n D_{n+m} \\ =& (n+m)\bar{A}_{n+m} - \frac{1}{2}mn(n+m)\bar{B}_{n+m} + \frac{1}{2}mn^2\bar{D}_{n+m}, \end{aligned} \tag{4.52}$$

$$i\{\bar{Q}_n, \bar{Q}_m\} = (n-m)\bar{C}_{n+m} - \frac{1}{2}mn(n-m)\bar{B}_{n+m} + \frac{1}{2}n^3\delta_{n+m,0}, \tag{4.53}$$

where

$$A_{n+m} = \bar{A}_{n+m} = \int_0^{2\pi} A(\theta) e^{-i(m+n)\theta} d\theta = \int_0^{2\pi} \frac{\mu q\bar{q}}{1-\mu(q+\bar{q})} e^{-i(m+n)\theta} d\theta, \tag{4.54}$$

$$B_{n+m} = \int_0^{2\pi} B(\theta) e^{-i(m+n)\theta} d\theta = \frac{\mu}{2}\int_0^{2\pi} \frac{2\bar{q} - \mu(q+\bar{q})^2}{(1-\mu(q+\bar{q}))^2} e^{-i(m+n)\theta} d\theta, \tag{4.55}$$

$$C_{n+m} = \int_0^{2\pi} C(\theta) e^{-i(m+n)\theta} d\theta = \int_0^{2\pi} \frac{(1-\mu q)q}{1-\mu(q+\bar{q})} e^{-i(m+n)\theta} d\theta, \tag{4.56}$$

$$\bar{B}_{n+m} = \int_0^{2\pi} \bar{B}(\theta) e^{-i(m+n)\theta} d\theta = \frac{\mu}{2}\int_0^{2\pi} \frac{2q - \mu(q+\bar{q})^2}{(1-\mu(q+\bar{q}))^2} e^{-i(m+n)\theta} d\theta, \tag{4.57}$$

$$\bar{C}_{n+m} = \int_0^{2\pi} \bar{C}(\theta) e^{-i(m+n)\theta} d\theta = \int_0^{2\pi} \frac{(1-\mu\bar{q})\bar{q}}{1-\mu(q+\bar{q})} e^{-i(m+n)\theta} d\theta, \tag{4.58}$$

$$D_{n+m} = -\bar{D}_{n+m} = \int_0^{2\pi} D(\theta) e^{-i(m+n)\theta} d\theta = \int_0^{2\pi} \frac{\mu(q-\bar{q})}{(1-\mu(q+\bar{q}))^2} e^{-i(m+n)\theta} d\theta. \tag{4.59}$$

This algebra coincides with the result in [53]. In [53], the authors consider the 3D gravity in a box with Dirichlet boundary conditions on a finite boundary. The boundary charges associated with the boundary preserving vector give the deformed Virasoro algebra. We obtain the same result from Chern-Simons gravity with mixed boundary conditions. The quantization of this algebra is also studied in [53, 73]. Taking the limit of $\mu \to 0$, this algebra reduces to the Virasoro algebra. For the non-zero μ, the deformed algebra turns on a deformation of the Virasoro algebra. The central charge can be restored by multiplying the third order of (m, n) by the Chern-Simons level $k = c/6$.

An analogy to the Virasoro algebra, we find the zero modes of the charges give the Hamiltonian and momentum

$$H = Q_0 + \bar{Q}_0, \quad P = Q_0 - \bar{Q}_0. \tag{4.60}$$

From (3.30) and (3.31), we can obtain

$$
\begin{aligned}
i\{H, Q_n\} =&\partial_t Q_n, \quad i\{H, \bar{Q}_m\} = \partial_t \bar{Q}_m, && (4.61)\\
i\{P, Q_n\} =& - nQ_n, \quad i\{P, \bar{Q}_m\} = -m\bar{Q}_m. && (4.62)
\end{aligned}
$$

In order to see the effect of the $T\bar{T}$ deformation, we consider the perturbative expansion of this algebra for small μ. After restoring the central charge, we can obtain the first order expansion

$$
\begin{aligned}
i\{Q_n, Q_m\} =&(n-m)Q_{n+m} + \frac{c}{12}n^3\delta_{n+m,0} && (4.63)\\
&+ \mu\Big((n-m)(Q\bar{Q})_{n+m} - \frac{c}{12}mn(n-m)Q_{m+n}\Big) + O(\mu^2), && (4.64)\\
i\{Q_n, \bar{Q}_m\} =&\mu\Big((n+m)(Q\bar{Q})_{n+m} - \frac{c}{12}(mn^2 Q_{n+m} + nm^2\bar{Q}_{n+m})\Big) + O(\mu^2), && (4.65)\\
i\{\bar{Q}_n, \bar{Q}_m\} =&(n-m)\bar{Q}_{n+m} + \frac{c}{12}n^3\delta_{n+m,0} && (4.66)\\
&+ \mu\Big((n-m)(Q\bar{Q})_{n+m} - \frac{c}{12}mn(n-m)\bar{Q}_{m+n}\Big) + O(\mu^2), && (4.67)
\end{aligned}
$$

where

$$
(Q\bar{Q})_{n+m} = \sum_{k\in\mathbb{Z}} Q_k \bar{Q}_{n+m-k}. \tag{4.68}
$$

The leading order reproduces the Virasoro algebra. The first-order correction provides a coupling between Q_n and $\bar{Q}_m$.

To close this section, we would like to point out that these charges are not conserved except for the energy and momentum[1]. Even for Virasoro only L_0 is conserved. For Virasoro the general L_n have simple time dependence, and these charges therefore impose symmetry relations on correlators. In principle, the algebra in the cutoff case also imposes symmetry or algebra relations, but these are harder to work with due to the nonlinear properties of the algebra. It remains to be seen whether the deformed algebra is "useful" or not.

5 Comments on the charges

The charges or the symmetry generators Q_n and $\bar{Q}_m$ do not remain constant with time evolutes. In other words, the non-zero modes of the charges are not conserved. As explained in [74], the charges we obtained are not invariant but covariant under this algebra. One can refine charges by adding an explicitly time-dependent factor, such that the refined charges become the conserved ones.

[1]We would like to thank Per Kraus for his comments on this point.

With following the strategy [74], we assume the refined charges take the following general form

$$\tilde{Q}_n = \int_0^{2\pi} q e^{-in(\theta+X)} d\theta, \quad \tilde{\bar{Q}}_m = \int_0^{2\pi} \bar{q} e^{-im(\theta+\bar{X})} d\theta \tag{5.1}$$

where X and $\bar{X}$ depend on (θ, t). Then the time derivative of the charges become

$$\partial_t \tilde{Q}_n = \int_0^{2\pi} \left(\partial_\theta (Kq) e^{in(\theta+X)} + inq e^{in(\theta+X)} \partial_t X \right) d\theta, \tag{5.2}$$

$$\partial_t \tilde{\bar{Q}}_m = \int_0^{2\pi} \left(\partial_\theta (\bar{K}\bar{q}) e^{im(\theta+\bar{X})} + im\bar{q} e^{im(\theta+\bar{X})} \partial_t \bar{X} \right) d\theta, \tag{5.3}$$

which would be vanishing if we set the integrand to be a total derivative on the right hand side. One of the simple settings is

$$\partial_\theta (Kq) e^{in(\theta+X)} + inq e^{in(\theta+X)} \partial_t X = \partial_\theta (Kq e^{in(\theta+X)}), \tag{5.4}$$

$$\partial_\theta (\bar{K}\bar{q}) e^{im(\theta+\bar{X})} + imq e^{im(\theta+\bar{X})} \partial_t \bar{X} = \partial_\theta (\bar{K}\bar{q} e^{im(\theta+\bar{X})}). \tag{5.5}$$

We then obtain the equations for $X, \bar{X}$

$$\partial_t X - K\partial_\theta X = K, \quad \partial_t \bar{X} - \bar{K}\partial_\theta \bar{X} = \bar{K}. \tag{5.6}$$

According to the coordinate transformation (3.13), we can write the differential operator on the left hand side as

$$\partial_t - K\partial_\theta = -\frac{2}{1-\mu\bar{\mathcal{L}}(\bar{z})}\partial_{\bar{z}}, \quad \partial_t - \bar{K}\partial_\theta = \frac{2}{1-\mu\mathcal{L}(z)}\partial_z. \tag{5.7}$$

where the $(z, \bar{z})$ are the original coordinates in the Bañados geometry. In these coordinates, we can write down the general solutions

$$X = -\frac{\bar{z}}{2} - \frac{\mu}{2}\int \bar{\mathcal{L}}(\bar{z}) d\bar{z} - f(z), \tag{5.8}$$

$$\bar{X} = -\frac{z}{2} - \frac{\mu}{2}\int \mathcal{L}(z) dz - \bar{f}(\bar{z}). \tag{5.9}$$

where the $f(z)$ and $\bar{f}(\bar{z})$ are arbitrary functions of z and $\bar{z}$ respectively. Substituting back into (5.1), we obtain the refined charges

$$\tilde{Q}_n = \int_0^{2\pi} q e^{-in\left(\theta - \frac{\bar{z}}{2} - \frac{\mu}{2}\int \bar{\mathcal{L}}(\bar{z}) d\bar{z} - f(z)\right)} d\theta, \tag{5.10}$$

$$\tilde{\bar{Q}}_m = \int_0^{2\pi} \bar{q} e^{-im\left(\theta - \frac{z}{2} - \frac{\mu}{2}\int \mathcal{L}(z) dz - \bar{f}(\bar{z})\right)} d\theta. \tag{5.11}$$

It is convenient to express these refined charges in terms of the original coordinates $(z, \bar{z})$. From the coordinate transformation (3.13), we get

$$\theta = \frac{(w+\bar{w})}{2} = \frac{z+\bar{z}}{2} + \frac{\mu}{2}\left(\int \mathcal{L}(z)dz + \int \bar{\mathcal{L}}(\bar{z})d\bar{z}\right). \tag{5.12}$$

On a constant time slice, we find the relations

$$d\theta = \frac{1-\mu^2\mathcal{L}(z)\bar{\mathcal{L}}(\bar{z})}{1-\mu\mathcal{L}(z)}dz, \quad d\theta = \frac{1-\mu^2\mathcal{L}(z)\bar{\mathcal{L}}(\bar{z})}{1-\mu\bar{\mathcal{L}}(\bar{z})}d\bar{z}. \tag{5.13}$$

The periodicity of θ would lead to the period of z and $\bar{z}$

$$z \sim z + \frac{2\pi}{\kappa}, \quad \bar{z} \sim \bar{z} + \frac{2\pi}{\bar{\kappa}}, \tag{5.14}$$

where $\kappa, \bar{\kappa}$ are constants and they depend on the explicit form of $\mathcal{L}$ and $\bar{\mathcal{L}}$. In general, we can not give the specific formula of $\kappa, \bar{\kappa}$. If $\mathcal{L}$ and $\bar{\mathcal{L}}$ are constants, namely the BTZ black holes, we have

$$\kappa = \frac{4-\mu^2(M^2-J^2)}{4-2\mu(M-J)}, \quad \bar{\kappa} = \frac{4-\mu^2(M^2-J^2)}{4-2\mu(M+J)}. \tag{5.15}$$

Once the μ vanishes, the κ and $\bar{\kappa}$ go back to the undeformed period.

Finally, the refined charges end up with

$$\tilde{Q}_n = \int_0^{\frac{2\pi}{\kappa}} \mathcal{L}(z)e^{-in\left(\frac{z}{2}+\frac{\mu}{2}\int \mathcal{L}(z)dz + f(z)\right)}dz, \tag{5.16}$$

$$\tilde{\bar{Q}}_m = \int_0^{\frac{2\pi}{\kappa}} \bar{\mathcal{L}}(\bar{z})e^{-im\left(\frac{\bar{z}}{2}+\frac{\mu}{2}\int \bar{\mathcal{L}}(\bar{z})d\bar{z} + \bar{f}(\bar{z})\right)}d\bar{z}. \tag{5.17}$$

In particular,

$$f(z) = (\kappa - \frac{1}{2})z - \frac{\mu}{2}\int \mathcal{L}(z)dz, \tag{5.18}$$

$$\bar{f}(\bar{z}) = (\bar{\kappa} - \frac{1}{2})\bar{z} - \frac{\mu}{2}\int \bar{\mathcal{L}}(\bar{z})d\bar{z}, \tag{5.19}$$

the charges can be expressed as

$$\tilde{Q}_n = \int_0^{\frac{2\pi}{\kappa}} \mathcal{L}(z)e^{-in\kappa z}dz, \tag{5.20}$$

$$\tilde{\bar{Q}}_m = \int_0^{\frac{2\pi}{\kappa}} \bar{\mathcal{L}}(\bar{z})e^{-im\bar{\kappa}\bar{z}}d\bar{z}. \tag{5.21}$$

The refined charges are the same as the Virasoro ones, and their algebra becomes a field-dependent Virasoro algebra. This result coincides with the conclusion in [21].

In addition, one can choose different $f(z)$ and $\bar{f}(\bar{z})$ which lead to different refined charges and corresponding algebras. The similar situation happens in the field theory calculation shown in [52]. It will be interesting future problem to connect the this ambiguity to the one shown in the field theory side [52].

6 Conclusion and discussion

It is proposed that the $T\bar{T}$ deformed 2D CFTs dual to the cutoff AdS_3 with Dirichlet boundary condition or equivalently a mixed boundary condition. The mixed boundary condition can be realized by a field-dependent coordinate transformation from the Brown-Henneaux boundary condition [21]. The Chern-Simons formalism of AdS_3 is a powerful tool to explore the holographic aspects of the AdS_3 with various boundary conditions. In this paper, we apply the Chern-Simons formalism to study the charges of $T\bar{T}$ deformed CFT. We start from the Bañados geometry, which is the most general AdS_3 solution with Brown-Henneaux boundary condition. The deformed Chern-Simons gauge connections were obtained through the field-dependent coordinate transformation. An analogy to the Bañados geometry, we parametrize the deformed gauge connections by two independent functions, which corresponds to the $T\bar{T}$ deformed charges.

After gauge fixing of the deformed gauge fields, the residual gauge symmetries can be found. The left-moving gauge fields and the right-moving gauge fields are entangled. The residual gauge generators can be parametrized by $\epsilon = \lambda_{-1} - \mu\bar{\lambda}_{-1}$ and $\bar{\epsilon} = \bar{\lambda}_1 - \mu\lambda_1$. Then the variation of the charges gives the algebra of the charges under the gauge transformation concerning $\epsilon, \bar{\epsilon}$. The resulting charge algebra is a non-linear deformation of the Virasoro algebra. We expand the Poisson bracket algebra of the charges perturbatively around $\mu = 0$ and the leading order reproduces the Virasoro algebra. The first-order correction induces coupling between the deformed charges Q and $\bar{Q}$.

In [21], the asymptotic symmetry of AdS_3 with the mixed boundary condition is described by two commuting copies of field-dependent Virasoro algebra. The different algebra structure was obtained in [53] when they consider the asymptotic symmetry of the deformed metric on the finite surface $r = r_c$ with Dirichlet boundary condition. In [52], it turns out that there are some uncertain winding terms in the deformed charge algebra. We show the difference between the two algebras offered by [21] and [53] comes from the ambiguous definition of the deformed charges.

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A Solving the variation of the charges

In this appendix, we treat in more detail how to solve the generator of the residual gauge transformation. We will also extract the variation of the charges under the gauge transformation. For convenience, we rewrite the equations as following

$$-\mu\delta_\lambda\bar{q} = \lambda'_{-1} + \lambda_0(1-\mu\bar{q}), \tag{A.1}$$
$$0 = \lambda'_0 + 2(\lambda_{-1}q - \lambda_1(1-\mu\bar{q})), \tag{A.2}$$
$$\delta_\lambda q = \lambda'_1 - \lambda_0 q, \tag{A.3}$$
$$-\delta_{\bar{\lambda}}\bar{q} = \bar{\lambda}'_{-1} - \bar{\lambda}_0\bar{q}, \tag{A.4}$$
$$0 = \lambda'_0 + 2\Big(\bar{\lambda}_1\bar{q} - \bar{\lambda}_{-1}(1-\mu q)\Big), \tag{A.5}$$
$$\mu\delta_{\bar{\lambda}}q = \bar{\lambda}'_1 + \bar{\lambda}_0(1-\mu q), \tag{A.6}$$
$$\delta_\lambda q = \delta_{\bar{\lambda}}q, \quad \delta_\lambda\bar{q} = \delta_{\bar{\lambda}}\bar{q}. \tag{A.7}$$

First of all, from (A.1), (A.3), (A.4) and (A.6), by eliminating $\delta_\lambda q$ and $\delta_{\bar{\lambda}}\bar{q}$ we can get the equations for $\lambda_0, \bar{\lambda}_0$

$$\mu\left(\bar{\lambda}_0(\theta) - \lambda_0(\theta)\right)\bar{q}(\theta) + \epsilon'(\theta) + \lambda_0(\theta) = 0, \tag{A.8}$$
$$\bar{\epsilon}'(\theta) + \bar{\lambda}_0(\theta)(1-\mu q(\theta)) + \mu\lambda_0(\theta)q(\theta) = 0, \tag{A.9}$$

Solving these equations, we obtain

$$\lambda_0(\theta) = (\zeta(\theta)-1)\bar{\epsilon}'(\theta) - \zeta(\theta)\epsilon'(\theta), \tag{A.10}$$
$$\bar{\lambda}_0(\theta) = \left(\bar{\zeta}(\theta)-1\right)\epsilon'(\theta) - \bar{\zeta}(\theta)\bar{\epsilon}'(\theta), \tag{A.11}$$

where

$$\zeta(\theta) = \frac{1-\mu q}{1-\mu\left(\bar{q}+q\right)}, \quad \bar{\zeta} = \frac{1-\mu\bar{q}}{1-\mu\left(\bar{q}+q\right)}. \tag{A.12}$$

Substituting these solutions into (A.2) and (A.5), the equations are following

$$\frac{2\lambda_{-1}(\theta)\left(\bar{\zeta}(\theta)-1\right) - 2\mu\lambda_1(\theta)\bar{\zeta}(\theta)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} + \zeta'(\theta)\left(\bar{\epsilon}'(\theta)-\epsilon'(\theta)\right) + (\zeta(\theta)-1)\bar{\epsilon}''(\theta) - \zeta(\theta)\epsilon''(\theta) = 0, \tag{A.13}$$

$$\frac{2\bar{\lambda}_1(\theta)(\zeta(\theta)-1) - 2\mu\bar{\lambda}_{-1}(\theta)\zeta(\theta)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} + \bar{\zeta}'(\theta)\left(\epsilon'(\theta)-\bar{\epsilon}'(\theta)\right) + \left(\bar{\zeta}(\theta)-1\right)\epsilon''(\theta) - \bar{\zeta}(\theta)\bar{\epsilon}''(\theta) = 0. \tag{A.14}$$

Combining with the definition of ϵ and $\bar{\epsilon}$,

$$\epsilon(\theta) - (\lambda_{-1}(\theta) - \mu\bar{\lambda}_{-1}(\theta)) = 0, \quad \bar{\epsilon}(\theta) - (\bar{\lambda}_1(\theta) - \mu\lambda_1(\theta)) = 0, \tag{A.15}$$

The solution implies one can express $\lambda_{-1}, \lambda_1, \bar{\lambda}_{-1}, \bar{\lambda}_1$ in terms of the new parameters ϵ and $\bar{\epsilon}$, which read

$$\begin{aligned}\lambda_{-1}(\theta) =& \frac{\zeta(\theta)\bar{\zeta}(\theta)\epsilon(\theta)}{\bar{\zeta}(\theta)+\zeta(\theta)-1} + \frac{(\zeta(\theta)-1)\bar{\zeta}(\theta)\bar{\epsilon}(\theta)}{\bar{\zeta}(\theta)+\zeta(\theta)-1} \\ &+ \frac{1}{2}\mu\left(\bar{\zeta}(\theta)\bar{\zeta}'(\theta)-(\zeta(\theta)-1)\zeta'(\theta)\right)\epsilon'(\theta) + \frac{1}{2}\mu\left((\zeta(\theta)-1)\zeta'(\theta)-\bar{\zeta}(\theta)\bar{\zeta}'(\theta)\right)\bar{\epsilon}'(\theta) \\ &+ \frac{1}{2}\mu\left(\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}(\theta)-\zeta(\theta)^2+\zeta(\theta)\right)\epsilon''(\theta) + \frac{1}{2}\mu\left((\zeta(\theta)-1)^2-\bar{\zeta}(\theta)^2\right)\bar{\epsilon}''(\theta),\end{aligned} \tag{A.16}$$

$$\begin{aligned}\lambda_1(\theta) =& \frac{\epsilon(\theta)\zeta(\theta)\left(\bar{\zeta}(\theta)-1\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} + \frac{(\zeta(\theta)-1)\bar{\epsilon}(\theta)\left(\bar{\zeta}(\theta)-1\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} \\ &+ \frac{1}{2}\epsilon'(\theta)\left(\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}'(\theta)-\zeta(\theta)\zeta'(\theta)\right) + \frac{1}{2}\bar{\epsilon}'(\theta)\left(\zeta(\theta)\zeta'(\theta)-\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}'(\theta)\right) \\ &- \frac{1}{2}\left(\zeta(\theta)-\bar{\zeta}(\theta)\right)\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)\bar{\epsilon}''(\theta) \\ &- \frac{1}{2}\left(-\bar{\zeta}(\theta)+\zeta(\theta)+1\right)\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)\epsilon''(\theta),\end{aligned} \tag{A.17}$$

$$\begin{aligned}\bar{\lambda}_{-1}(\theta) =& \frac{\epsilon(\theta)(\zeta(\theta)-1)\left(\bar{\zeta}(\theta)-1\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} + \frac{(\zeta(\theta)-1)\bar{\epsilon}(\theta)\bar{\zeta}(\theta)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} \\ &+ \frac{1}{2}\epsilon'(\theta)\left(\bar{\zeta}(\theta)\bar{\zeta}'(\theta)-(\zeta(\theta)-1)\zeta'(\theta)\right) + \frac{1}{2}\bar{\epsilon}'(\theta)\left((\zeta(\theta)-1)\zeta'(\theta)-\bar{\zeta}(\theta)\bar{\zeta}'(\theta)\right) \\ &+ \frac{1}{2}\left(-\bar{\zeta}(\theta)+\zeta(\theta)-1\right)\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)\bar{\epsilon}''(\theta) \\ &- \frac{1}{2}\left(\zeta(\theta)-\bar{\zeta}(\theta)\right)\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)\epsilon''(\theta),\end{aligned} \tag{A.18}$$

$$\begin{aligned}\bar{\lambda}_1(\theta) =& \frac{\epsilon(\theta)\zeta(\theta)\left(\bar{\zeta}(\theta)-1\right)}{\bar{\zeta}(\theta)+\zeta(\theta)-1} + \frac{\zeta(\theta)\bar{\epsilon}(\theta)\bar{\zeta}(\theta)}{\bar{\zeta}(\theta)+\zeta(\theta)-1} \\ &+ \frac{1}{2}\epsilon'(\theta)\left(\mu\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}'(\theta)-\mu\zeta(\theta)\zeta'(\theta)\right) + \frac{1}{2}\mu\bar{\epsilon}'(\theta)\left(\zeta(\theta)\zeta'(\theta)-\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}'(\theta)\right) \\ &+ \frac{1}{2}\mu\left(\zeta(\theta)-\bar{\zeta}(\theta)\right)\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)\bar{\epsilon}''(\theta) + \frac{1}{2}\mu\left((\bar{\zeta}(\theta)-1)^2-\zeta(\theta)^2\right)\epsilon''(\theta).\end{aligned} \tag{A.19}$$

Finally, substituting back into (A.3) and (A.4), one can obtain

$$\begin{aligned}
\delta q =&\bar{\delta} q = \frac{\bar{\epsilon}(\theta)\left((\bar{\zeta}(\theta)-1)\bar{\zeta}(\theta)\zeta'(\theta)+(\zeta(\theta)-1)\zeta(\theta)\bar{\zeta}'(\theta)\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)^2} \\
&+\frac{\epsilon(\theta)\left(\left(\bar{\zeta}(\theta)-1\right)^2\zeta'(\theta)+\zeta(\theta)^2\bar{\zeta}'(\theta)\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)^2} \\
&+\frac{1}{2}\epsilon'(\theta)\left(\frac{4\zeta(\theta)\left(\bar{\zeta}(\theta)-1\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)}+\bar{\zeta}'(\theta)^2+\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}''(\theta)-\zeta(\theta)\zeta''(\theta)-\zeta'(\theta)^2\right) \\
&+\frac{1}{2}\bar{\epsilon}'(\theta)\left(-\bar{\zeta}'(\theta)^2-\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}''(\theta)+\zeta(\theta)\zeta''(\theta)+\zeta'(\theta)^2\right) \\
&-\frac{3}{2}\epsilon''(\theta)\left(\zeta(\theta)\zeta'(\theta)-\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}'(\theta)\right)+\frac{1}{2}\left(\left(2-3\bar{\zeta}(\theta)\right)\bar{\zeta}'(\theta)+(3\zeta(\theta)-1)\zeta'(\theta)\right)\bar{\epsilon}''(\theta) \\
&-\frac{1}{2}\left(\bar{\zeta}(\theta)^2-\zeta(\theta)^2+(\zeta(\theta))-\bar{\zeta}(\theta)\right)\bar{\epsilon}'''(\theta)-\frac{1}{2}\epsilon'''(\theta)\left(\zeta(\theta)^2-(\bar{\zeta}(\theta)-2)\bar{\zeta}(\theta)-1\right),
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
\delta\bar{q} =&\bar{\delta}\bar{q} = \frac{\epsilon(\theta)\left(-(\bar{\zeta}(\theta)-1)\bar{\zeta}(\theta)\zeta'(\theta)-(\zeta(\theta)-1)\zeta(\theta)\bar{\zeta}'(\theta)\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)^2} \\
&-\frac{\bar{\epsilon}(\theta)\left(\bar{\zeta}(\theta)^2\zeta'(\theta)+(\zeta(\theta)-1)^2\bar{\zeta}'(\theta)\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)^2} \\
&+\frac{1}{2}\epsilon'(\theta)\left(-\bar{\zeta}'(\theta)^2-\bar{\zeta}(\theta)\bar{\zeta}''(\theta)+\zeta(\theta)\zeta''(\theta)-\zeta''(\theta)+\zeta'(\theta)^2\right) \\
&+\frac{1}{2}\bar{\epsilon}'(\theta)\left(-\frac{4(\zeta(\theta)-1)\bar{\zeta}(\theta)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)}+\bar{\zeta}'(\theta)^2+\bar{\zeta}(\theta)\bar{\zeta}''(\theta)-\zeta(\theta)\zeta''(\theta)+\zeta''(\theta)-\zeta'(\theta)^2\right) \\
&+\frac{1}{2}\epsilon''(\theta)\left(\left(1-3\bar{\zeta}(\theta)\right)\bar{\zeta}'(\theta)+(3\zeta(\theta)-2)\zeta'(\theta)\right)+\frac{3}{2}\left(\bar{\zeta}(\theta)\bar{\zeta}'(\theta)-(\zeta(\theta)-1)\zeta'(\theta)\right)\bar{\epsilon}''(\theta) \\
&+\frac{1}{2}\epsilon'''(\theta)\left(\zeta(\theta)^2-\bar{\zeta}(\theta)^2+\bar{\zeta}(\theta)-\zeta(\theta)\right)+\frac{1}{2}\left(\bar{\zeta}(\theta)^2-(\zeta(\theta)-2)\zeta(\theta)-1\right)\bar{\epsilon}'''(\theta)
\end{aligned} \tag{A.21}$$

Fortunately, it is convenient to introduce the auxiliary variables

$$A = \bar{A} = \frac{(\zeta(\theta)-1)\left(\bar{\zeta}(\theta)-1\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} = \frac{\mu q\bar{q}}{1-\mu\left(\bar{q}+q\right)}, \tag{A.22}$$

$$B = \frac{1}{2}\Big(\zeta(\theta)^2-(\bar{\zeta}(\theta)-2)\bar{\zeta}(\theta)-2\Big) = \frac{\mu\left(2\bar{q}-\mu(q+\bar{q})^2\right)}{2(1-\mu(q+\bar{q}))^2}, \tag{A.23}$$

$$\bar{B} = \frac{1}{2}\Big(\bar{\zeta}(\theta)^2-(\zeta(\theta)-2)\zeta(\theta)-2\Big) = \frac{\mu\left(2q-\mu(q+\bar{q})^2\right)}{2(1-\mu(q+\bar{q}))^2}, \tag{A.24}$$

$$C = \frac{\zeta(\theta)\left(\bar{\zeta}(\theta)-1\right)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} = \frac{(1-\mu q)q}{1-\mu(q+\bar{q})}, \tag{A.25}$$

$$\bar{C} = \frac{(\zeta(\theta)-1)\bar{\zeta}(\theta)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)} = \frac{(1-\mu\bar{q})\bar{q}}{1-\mu(q+\bar{q})}, \tag{A.26}$$

$$D = -\bar{D} = \bar{\zeta}(\theta)^2-\bar{\zeta}(\theta)-\zeta(\theta)^2+\zeta(\theta) = \frac{\mu\left(q-\bar{q}\right)}{\left(1-\mu\left(\bar{q}+q\right)\right)^2}, \tag{A.27}$$

Then

$$A' = \bar{A}' = \frac{\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}(\theta)\zeta'(\theta)+(\zeta(\theta)-1)\zeta(\theta)\bar{\zeta}'(\theta)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)^2}, \tag{A.28}$$

$$B' = \zeta(\theta)\zeta'(\theta)-\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}'(\theta), \tag{A.29}$$

$$B'' = \zeta'(\theta)^2+\zeta(\theta)\zeta''(\theta)-\left(\bar{\zeta}(\theta)-1\right)\bar{\zeta}''(\theta)-\bar{\zeta}'(\theta)^2, \tag{A.30}$$

$$\bar{B}' = \bar{\zeta}(\theta)\bar{\zeta}'(\theta)-(\zeta(\theta)-1)\zeta'(\theta), \tag{A.31}$$

$$\bar{B}'' = \bar{\zeta}'(\theta)^2+\bar{\zeta}(\theta)\bar{\zeta}''(\theta)-(\zeta(\theta)-1)\zeta''(\theta)-\zeta'(\theta)^2, \tag{A.32}$$

$$C' = \frac{\left(\bar{\zeta}(\theta)-1\right)^2\zeta'(\theta)+\zeta(\theta)^2\bar{\zeta}'(\theta)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)^2}, \tag{A.33}$$

$$\bar{C}' = \frac{\bar{\zeta}(\theta)^2\zeta'(\theta)+(\zeta(\theta)-1)^2\bar{\zeta}'(\theta)}{\mu\left(\bar{\zeta}(\theta)+\zeta(\theta)-1\right)^2}, \tag{A.34}$$

$$D' = -\bar{D}' = \left(2\bar{\zeta}(\theta)-1\right)\bar{\zeta}'(\theta)+(1-2\zeta(\theta))\zeta'(\theta). \tag{A.35}$$

Finally, $\delta_\lambda q$ and $\delta_{\bar{\lambda}}\bar{q}$ can be formulated as

$$\begin{aligned}\delta_\lambda q = \delta_{\bar{\lambda}} q =& \epsilon(\theta)C'+\bar{\epsilon}(\theta)A'+\frac{1}{2}\bar{\epsilon}'(\theta)B''-\frac{1}{2}\epsilon'(\theta)(B''-4C)\\ &-\frac{1}{2}\bar{\epsilon}''(\theta)(D'-B')-\frac{3}{2}\epsilon''(\theta)B'-\frac{1}{2}\bar{\epsilon}'''(\theta)D-\frac{1}{2}\epsilon'''(\theta)(2B+1),\end{aligned} \tag{A.36}$$

$$\begin{aligned}\delta_\lambda \bar{q} = \delta_{\bar{\lambda}}\bar{q} =& -\epsilon(\theta)\bar{A}'-\bar{\epsilon}(\theta)\bar{C}'-\frac{1}{2}\epsilon'(\theta)\bar{B}''+\frac{1}{2}\bar{\epsilon}'(\theta)(\bar{B}''-4\bar{C})\\ &+\frac{1}{2}\epsilon''(\theta)(\bar{D}'-\bar{B}')+\frac{3}{2}\bar{\epsilon}''(\theta)\bar{B}'+\frac{1}{2}\epsilon'''(\theta)\bar{D}+\frac{1}{2}\bar{\epsilon}'''(\theta)(2\bar{B}+1).\end{aligned} \tag{A.37}$$

B Constraints on the gauge transformation

In this appendix, we would like to derive the evolution equation of the parameters ϵ, ϵ' from the following equations

$$\begin{aligned}
\delta_\lambda \left(K(1-\mu\bar{q})\right) =& \partial_t \lambda_{-1} + K\lambda_0(1-\mu\bar{q}), && \text{(B.1)}\\
0 =& \partial_t\lambda_0 + 2K\left(\lambda_{-1}q - \lambda_1(1-\mu\bar{q})\right), && \text{(B.2)}\\
\delta_\lambda (Kq) =& \partial_t\lambda_1 - K\lambda_0 q, && \text{(B.3)}\\
-\delta_{\bar\lambda}(\bar{K}\bar{q}) =& \partial_t\bar\lambda_{-1} - \bar{K}\bar\lambda_0\bar{q}, && \text{(B.4)}\\
0 =& \partial_t\bar\lambda_0 + 2\bar{K}\left(\bar\lambda_1\bar{q} - \bar\lambda_{-1}(1-\mu q)\right), && \text{(B.5)}\\
-\delta_{\bar\lambda}\left(\bar{K}(1-\mu q)\right) =& \partial_t\bar\lambda_1 + \bar{K}\bar\lambda_0(1-\mu q). && \text{(B.6)}
\end{aligned}$$

Firstly, by using the definition of $K, \bar{K}$, one can find the relations

$$K(1-\mu\bar{q}) + \mu\bar{K}\bar{q} = 1, \quad \mu K q + \bar{K}(1-\mu q) = 1 \tag{B.7}$$

then

$$\begin{aligned}
\delta_\lambda\left(K(1-\mu\bar{q})\right) + \mu\delta_{\bar\lambda}(\bar{K}\bar{q}) = 0, && \text{(B.8)}\\
\delta_{\bar\lambda}\left(\bar{K}(1-\mu q)\right) + \mu\delta_\lambda(Kq) = 0. && \text{(B.9)}
\end{aligned}$$

Combining (B.1), (B.4), (B.8), (4.17), and (4.18), one can obtain

$$\partial_t\epsilon = K(1-\mu\bar{q})\lambda_0 + \mu\bar{K}\bar{q}\bar\lambda_0. \tag{B.10}$$

From (B.3) and (B.6), one can get

$$\partial_t\bar\epsilon = \bar{K}(1-\mu q)\bar\lambda_0 + \mu K q\lambda_0. \tag{B.11}$$

Finally, plugging (A.10) and (A.11) into (B.10) and (B.11), one can arrive at

$$\begin{aligned}
-\partial_t\epsilon =& \epsilon' - \frac{2\mu\bar{q}\left(1-\mu\bar{q}\right)}{\left(1-\mu\left(\bar{q}+q\right)\right)^2}\left(\epsilon' + \bar\epsilon'\right), && \text{(B.12)}\\
\partial_t\bar\epsilon =& \bar\epsilon' - \frac{2\mu q\left(1-\mu q\right)}{\left(1-\mu\left(\bar{q}+q\right)\right)^2}\left(\epsilon' + \bar\epsilon'\right). && \text{(B.13)}
\end{aligned}$$

In addition, from (A.2), (A.5),(B.2) and (B.5), the λ_0 and $\bar\lambda_0$ obey

$$\begin{aligned}
\partial_t\lambda_0 =& \frac{1+\mu(\bar{q}-q)}{1-\mu(q+\bar{q})}\lambda_0', && \text{(B.14)}\\
\partial_t\lambda_0 =& -\frac{1-\mu(\bar{q}-q)}{1-\mu(q+\bar{q})}\bar\lambda_0'. && \text{(B.15)}
\end{aligned}$$

References

[1] F. A. Smirnov and A. B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys. B* **915** (2017) 363 [1608.05499].

[2] A. Cavaglià, S. Negro, I. M. Szécsényi and R. Tateo, *$T\bar{T}$-deformed 2D Quantum Field Theories*, *JHEP* **10** (2016) 112 [1608.05534].

[3] L. McGough, M. Mezei and H. Verlinde, *Moving the CFT into the bulk with $T\overline{T}$*, *JHEP* **04** (2018) 010 [1611.03470].

[4] G. Bonelli, N. Doroud and M. Zhu, *$T\bar{T}$-deformations in closed form*, *JHEP* **06** (2018) 149 [1804.10967].

[5] S. Datta and Y. Jiang, *$T\bar{T}$ deformed partition functions*, *JHEP* **08** (2018) 106 [1806.07426].

[6] O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, *Modular invariance and uniqueness of $T\bar{T}$ deformed CFT*, *JHEP* **01** (2019) 086 [1808.02492].

[7] S. Dubovsky, V. Gorbenko and M. Mirbabayi, *Asymptotic fragility, near AdS_2 holography and $T\overline{T}$*, *JHEP* **09** (2017) 136 [1706.06604].

[8] S. Dubovsky, V. Gorbenko and G. Hernández-Chifflet, *$T\overline{T}$ partition function from topological gravity*, *JHEP* **09** (2018) 158 [1805.07386].

[9] T. Ishii, S. Okumura, J.-I. Sakamoto and K. Yoshida, *Gravitational perturbations as $T\bar{T}$-deformations in 2D dilaton gravity systems*, *Nucl. Phys. B* **951** (2020) 114901 [1906.03865].

[10] J. Cardy, *The $T\overline{T}$ deformation of quantum field theory as random geometry*, *JHEP* **10** (2018) 186 [1801.06895].

[11] N. Callebaut, J. Kruthoff and H. Verlinde, *$T\overline{T}$ deformed CFT as a non-critical string*, *JHEP* **04** (2020) 084 [1910.13578].

[12] A. J. Tolley, *$T\overline{T}$ deformations, massive gravity and non-critical strings*, *JHEP* **06** (2020) 050 [1911.06142].

[13] R. Conti, S. Negro and R. Tateo, *The $\mathrm{T}\overline{\mathrm{T}}$ perturbation and its geometric interpretation*, *JHEP* **02** (2019) 085 [1809.09593].

[14] R. Conti, S. Negro and R. Tateo, *Conserved currents and $T\bar{T}_s$ irrelevant deformations of 2D integrable field theories*, *JHEP* **11** (2019) 120 [1904.09141].

[15] P. Kraus, J. Liu and D. Marolf, *Cutoff AdS_3 versus the $T\overline{T}$ deformation*, *JHEP* **07** (2018) 027 [1801.02714].

[16] V. Shyam, *Background independent holographic dual to $T\bar{T}$ deformed CFT with large central charge in 2 dimensions*, *JHEP* **10** (2017) 108 [1707.08118].

[17] W. Donnelly, E. LePage, Y.-Y. Li, A. Pereira and V. Shyam, *Quantum corrections to finite radius holography and holographic entanglement entropy*, *JHEP* **05** (2020) 006 [1909.11402].

[18] I. R. Klebanov and E. Witten, *AdS / CFT correspondence and symmetry breaking*, *Nucl. Phys. B* **556** (1999) 89 [hep-th/9905104].

[19] E. Witten, *Multitrace operators, boundary conditions, and AdS / CFT correspondence*, hep-th/0112258.

[20] I. Papadimitriou, *Multi-Trace Deformations in AdS/CFT: Exploring the Vacuum Structure of the Deformed CFT*, *JHEP* **05** (2007) 075 [hep-th/0703152].

[21] M. Guica and R. Monten, *$T\bar{T}$ and the mirage of a bulk cutoff*, *SciPost Phys.* **10** (2021) 024 [1906.11251].

[22] H. Ouyang and H. Shu, *$T\bar{T}$ deformation of chiral bosons and Chern–Simons AdS_3 gravity*, *Eur. Phys. J. C* **80** (2020) 1155 [2006.10514].

[23] M. He and Y.-h. Gao, *$T\bar{T}/J\bar{T}$-deformed WZW models from Chern-Simons AdS_3 gravity with mixed boundary conditions*, *Phys. Rev. D* **103** (2021) 126019 [2012.05726].

[24] G. Giribet, *$T\bar{T}$-deformations, AdS/CFT and correlation functions*, *JHEP* **02** (2018) 114 [1711.02716].

[25] W. Donnelly and V. Shyam, *Entanglement entropy and $T\overline{T}$ deformation*, *Phys. Rev. Lett.* **121** (2018) 131602 [1806.07444].

[26] B. Chen, L. Chen and P.-X. Hao, *Entanglement entropy in $T\overline{T}$-deformed CFT*, *Phys. Rev. D* **98** (2018) 086025 [1807.08293].

[27] H.-S. Jeong, K.-Y. Kim and M. Nishida, *Entanglement and Rényi entropy of multiple intervals in $T\overline{T}$-deformed CFT and holography*, *Phys. Rev. D* **100** (2019) 106015 [1906.03894].

[28] S. Grieninger, *Entanglement entropy and $T\overline{T}$ deformations beyond antipodal points from holography*, *JHEP* **11** (2019) 171 [1908.10372].

[29] G. Jafari, A. Naseh and H. Zolfi, *Path Integral Optimization for $T\bar{T}$ Deformation*, *Phys. Rev. D* **101** (2020) 026007 [1909.02357].

[30] B. Chen, L. Chen and C.-Y. Zhang, *Surface/state correspondence and $T\overline{T}$ deformation*, *Phys. Rev. D* **101** (2020) 106011 [1907.12110].

[31] E. A. Mazenc, V. Shyam and R. M. Soni, *A $T\bar{T}$ Deformation for Curved Spacetimes from 3d Gravity*, 1912.09179.

[32] Y. Li and Y. Zhou, *Cutoff AdS_3 versus $T\overline{T}$ CFT_2 in the large central charge sector: correlators of energy-momentum tensor*, *JHEP* **12** (2020) 168 [2005.01693].

[33] Y. Li, *Comments on large central charge $T\bar{T}$ deformed conformal field theory and cutoff AdS holography*, 2012.14414.

[34] P. Caputa, S. Datta, Y. Jiang and P. Kraus, *Geometrizing $T\overline{T}$*, *JHEP* **03** (2021) 140 [2011.04664].

[35] S. Hirano, T. Nakajima and M. Shigemori, *$T\overline{T}$ Deformation of stress-tensor correlators from random geometry*, *JHEP* **04** (2021) 270 [2012.03972].

[36] Y. Jiang, *A pedagogical review on solvable irrelevant deformations of 2D quantum field theory*, *Commun. Theor. Phys.* **73** (2021) 057201 [1904.13376].

[37] B. Le Floch and M. Mezei, *KdV charges in $T\bar{T}$ theories and new models with super-Hagedorn behavior*, *SciPost Phys.* **7** (2019) 043 [1907.02516].

[38] J. Cardy, *$T\bar{T}$ deformation of correlation functions*, *JHEP* **12** (2019) 160 [1907.03394].

[39] S. He and H. Shu, *Correlation functions, entanglement and chaos in the $T\overline{T}/J\overline{T}$-deformed CFTs*, *JHEP* **02** (2020) 088 [1907.12603].

[40] S. He, J.-R. Sun and Y. Sun, *The correlation function of (1,1) and (2,2) supersymmetric theories with $T\bar{T}$ deformation*, *JHEP* **04** (2020) 100 [1912.11461].

[41] J. Kruthoff and O. Parrikar, *On the flow of states under $T\overline{T}$*, 2006.03054.

[42] S. He and Y. Sun, *Correlation functions of CFTs on a torus with a $T\overline{T}$ deformation*, *Phys. Rev. D* **102** (2020) 026023 [2004.07486].

[43] S. He, Y. Sun and Y.-X. Zhang, *$T\bar{T}$-flow effects on torus partition functions*, 2011.02902.

[44] S. He, *Note on higher-point correlation functions of the $T\bar{T}$ or $J\bar{T}$ deformed CFTs*, Sci. China Phys. Mech. Astron. **64** (2021) 291011 [2012.06202].

[45] J. Cardy, *$T\overline{T}$ deformations of non-Lorentz invariant field theories*, 1809.07849.

[46] E. Marchetto, A. Sfondrini and Z. Yang, *$T\bar{T}$ Deformations and Integrable Spin Chains*, Phys. Rev. Lett. **124** (2020) 100601 [1911.12315].

[47] J. Cardy and B. Doyon, *$T\overline{T}$ deformations and the width of fundamental particles*, 2010.15733.

[48] M. Medenjak, G. Policastro and T. Yoshimura, *$T\bar{T}$-Deformed Conformal Field Theories out of Equilibrium*, Phys. Rev. Lett. **126** (2021) 121601 [2011.05827].

[49] Y. Jiang, $\mathrm{T}\overline{\mathrm{T}}$*-deformed 1d Bose gas*, 2011.00637.

[50] B. Chen, J. Hou and J. Tian, *Note on the nonrelativistic $T\bar{T}$ deformation*, Phys. Rev. D **104** (2021) 025004 [2012.14091].

[51] G. Jorjadze and S. Theisen, *Canonical maps and integrability in $T\bar{T}$ deformed 2d CFTs*, 2001.03563.

[52] M. Guica and R. Monten, *Infinite pseudo-conformal symmetries of classical $T\bar{T}$, $J\bar{T}$ and JT_a - deformed CFTs*, 2011.05445.

[53] P. Kraus, R. Monten and R. M. Myers, *3D Gravity in a Box*, 2103.13398.

[54] J. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, Commun. Math. Phys. **104** (1986) 207.

[55] M. Banados, *Three-dimensional quantum geometry and black holes*, AIP Conf. Proc. **484** (1999) 147 [hep-th/9901148].

[56] M. Banados, T. Brotz and M. E. Ortiz, *Boundary dynamics and the statistical mechanics of the (2+1)-dimensional black hole*, Nucl. Phys. B **545** (1999) 340 [hep-th/9802076].

[57] J. Cotler and K. Jensen, *A theory of reparameterizations for AdS_3 gravity*, JHEP **02** (2019) 079 [1808.03263].

[58] M. Henneaux and S.-J. Rey, *Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, JHEP **12** (2010) 007 [1008.4579].

[59] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [1008.4744].

[60] M. Gutperle and P. Kraus, *Higher Spin Black Holes*, *JHEP* **05** (2011) 022 [1103.4304].

[61] A. P. Balachandran, G. Bimonte, K. S. Gupta and A. Stern, *Conformal edge currents in Chern-Simons theories*, *Int. J. Mod. Phys. A* **7** (1992) 4655 [hep-th/9110072].

[62] M. Banados, *Global charges in Chern-Simons field theory and the (2+1) black hole*, *Phys. Rev. D* **52** (1996) 5816 [hep-th/9405171].

[63] T. Regge and C. Teitelboim, *Role of Surface Integrals in the Hamiltonian Formulation of General Relativity*, *Annals Phys.* **88** (1974) 286.

[64] G. Compère, W. Song and A. Strominger, *New Boundary Conditions for AdS3*, *JHEP* **05** (2013) 152 [1303.2662].

[65] G. Compère, P. Mao, A. Seraj and M. M. Sheikh-Jabbari, *Symplectic and Killing symmetries of AdS_3 gravity: holographic vs boundary gravitons*, *JHEP* **01** (2016) 080 [1511.06079].

[66] D. Grumiller and M. Riegler, *Most general AdS_3 boundary conditions*, *JHEP* **10** (2016) 023 [1608.01308].

[67] H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo et al., *Soft Heisenberg hair on black holes in three dimensions*, *Phys. Rev. D* **93** (2016) 101503 [1603.04824].

[68] H. Afshar, D. Grumiller, W. Merbis, A. Perez, D. Tempo and R. Troncoso, *Soft hairy horizons in three spacetime dimensions*, *Phys. Rev. D* **95** (2017) 106005 [1611.09783].

[69] A. Pérez, D. Tempo and R. Troncoso, *Boundary conditions for General Relativity on AdS_3 and the KdV hierarchy*, *JHEP* **06** (2016) 103 [1605.04490].

[70] E. Ojeda and A. Pérez, *Boundary conditions for General Relativity in three-dimensional spacetimes, integrable systems and the KdV/mKdV hierarchies*, *JHEP* **08** (2019) 079 [1906.11226].

[71] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, *Nucl. Phys. B* **311** (1988) 46.

[72] O. Coussaert, M. Henneaux and P. van Driel, *The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant*, Class. Quant. Grav. **12** (1995) 2961 [gr-qc/9506019].

[73] S. Datta and Y. Jiang, *Characters of irrelevant deformations*, JHEP **07** (2021) 162 [2104.00281].

[74] C. Bunster, M. Henneaux, A. Perez, D. Tempo and R. Troncoso, *Generalized Black Holes in Three-dimensional Spacetime*, JHEP **05** (2014) 031 [1404.3305].