THE COMPLEXITIES OF NON-PERTURBATIVE COMPUTATIONS

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ABSTRACT. The article studies the behavior of equations of motions of Green's functions under different running coupling constants in strongly coupled gauge field theories in terms of the Kolmogorov complexity.

CONTENTS

1. INTRODUCTION

Perturbative methods of calculations for the study of scattering amplitudes in Quantum Field Theory are in terms of formal expansions in powers of weak enough running coupling constants together with Feynman diagrams as coefficients. The number of loops in Feynman diagrams in each term of the expansion are associated with the powers of the running couplings. These running couplings, which have been generated by regularization and renormalization techniques, are taken to be small in perturbative methods to control the behavior of those perturbative expansions. The computational complexities of perturbative methods can be evaluated in terms of the number of Feynman diagrams in expansions. In this setting, the complexity increases factorially in terms of the growing number of Feynman diagrams and nested loops which contribute to higher orders of the perturbative expansion. [9, 10, 38, 39]

If the values of the bare or running coupling constants are insufficiently small, then the perturbative setting is not a useful approach to deal with full scattering amplitudes. In this situation, one eventually reaches to a quantum phase transition at some critical couplings such that in the parameter space near this phase transition perturbative methods become unreliable. The fundamental challenge in strongly coupled gauge field theories is the appearance of these non-perturbative aspects which have been already encoded by fixed point equations of Green's functions known as Dyson–Schwinger equations. The values of β-functions, which govern the behavior of running couplings, is the original factor to classify (systems) of Dyson–Schwinger equations under linear and non-linear settings. Numerical techniques and lattice models can provide some approximations for these equations [27, 28, 30]. Therefore work on building some new advanced mathematical modelings for the study of real time dynamics of these non-perturbative situations could improve our knowledge about the foundations of non-perturbative gauge field theories beyond Standard Model.

Key words and phrases. Combinatorial Dyson–Schwinger equations; Kolmogorov complexity; Feynman graphons; Manin renormalization Hopf algebra of the Halting problem.
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The Connes–Kreimer renormalization Hopf algebra of Feynman diagrams has been applied by Kreimer's school to obtain a new reformulation of Dyson–Schwinger equations in the language of Hochschild Cohomology Theory and Combinatorics. It is shown that the unique solution of each Dyson–Schwinger equation DSE in a given gauge field theory Φ determines a connected graded free commutative Hopf subalgbera $H_{\rm DSE}$ of $H_{\rm FG}(\Phi)$ such that the β-function of the physical theory can determine the cocommutativity or non-cocommutativity of this Hopf subalgebra [5, 19, 40]. This approach has been applied to study (systems) of linear and non-linear Dyson–Schwinger equations under a combinatorial setting [4, 13, 16, 18, 34, 43]. In addition, a new geometric setting for the study of Dyson–Schwinger equations has also been formulated where recently, a new Functional Analysis model for the description of the evolution of solutions of these equations has been built in terms of some tools in Noncommutative Geometry and Infinite Combinatorics. These investigations have provided some new knowledge about the geometry of strongly coupled gauge field theories. [31, 33, 35, 36]

Applications of the Theory of Computation to Quantum Field Theory have been recently improved by Manin where he applied the Connes–Kreimer approach to the BPHZ perturbative renormalization to analyze the problem about computability. He lifted the minimal subtraction process of infinities in iterated Feynman integrals onto the level of flowcharts as decorated graphs in the Theory of Computation. Then he formulated the Halting problem in the language of the renormalization Hopf algebra [24, 25, 26]. The Manin renormalization Hopf algebra of the Halting problem can encode the level of non-computability via counterterms generated by the BPHZ renormalization treatment at the level of flowcharts [24, 25]. In this direction, the Manin's Hopf algebraic approach to the Halting problem has been applied to show a fundamental relation between Dyson–Schwinger equations and intermediate algorithms. Each Dyson–Schwinger equation DSE determines the quotient Hopf algebra $H_{\rm FG}(\Phi)/H_{\rm DSE}$ such that its dual can determine a Lie subgroup of the complex Lie group $\mathbb{G}_\Phi$ of homomorphisms on $H_{\rm FG}(\Phi)$. This class of Lie subgroups enable us to study renormalization of Dyson–Schwinger equations ([31, 34, 35]) in terms of intermediate algorithms in Galois theory of algorithms ([41, 42]). In this setting, we can observe a deep relation between the level of complexities in the computation of renormalized values of Feynman integrals or non-perturbative parameters generated by solutions of (strongly coupled) Dyson–Schwinger equations and the level of (non-)computability in the Halting problem in the Theory of Computation. [12, 22, 23, 32]

The basic idea of the Kolmogorov complexity is to describe the concept of randomness. The Kolmogorov complexity of a finite binary string is the length of the shortest description of this string. In general, we can define the Kolmogorov complexity with respect to partial recursive functions. For a given partial recursive function f, it is possible to define a program which takes an input s and generates an output $f(s)$. The output may not be defined for some s and the program may not halt on such s or generate any output. For any binary string s with respect to a given partial recursive function f, this complexity is defined by

$$K_f(s) := \min\{|p| : \ p \in \{0,1\}^*, \ f(p) = s\}, \tag{1.1}$$

[21]. The values of complexities are strongly connected with the selected descriptive language (i.e. partial recursive function). Thanks to the Invariance Theorem in the Theory of Computation ([1, 21]), there exists a universal partial recursive function U such that for any other partial recursive function f, there is a constant $c_f > 0$ which satisfies $K_U(s) \leq K_f(s) + c_f$ for all strings. We call a string v Kolmogorov random iff $K_U(v) \geq |v|$. For each n, there exist 2^n strings of length n where we have $2^n - 1$ strings of length less than n. It enables us to show the existence of Kolmogorov random strings of every length. Deciding the Kolmogorov randomness is as hard as deciding the Halting problem in programs. [1, 37]

In the Theory of Computation the algorithm's flexibility is a fundamental parameter in dealing with the computation of quantities in a problem. A feasible algorithm has a polynomial-time asymptotic scaling while an infeasible algorithm has a super-polynomial scaling. The Halting problem, as an undecidable problem, provides a way to study the feasibility of a program in the case that whether the program will finish running or continue to run forever [1, 29]. Manin has reformulated the Halting problem in terms of the BPHZ perturbative renormalization where he applied a modified version of the Connes–Kreimer renormalization Hopf algebra on rooted trees decorated by partial recursive functions (i.e. flowcharts). His setting enables us to measure the flexibility of algorithms in terms of counterterms generated by perturbative renormalization of flowcharts. [24, 25, 26]

Some new applications of Infinite Combinatorics to Quantum Field Theory have been discovered recently where solutions of Dyson–Schwinger equations are studied in the language of the theory of graphons. In this setting, a particular class of graphons originated from sequences of sparse graphs has been built to provide a new analytic generalization for Feynman diagrams and their infinite formal expansions. For this purpose we used the rooted tree representations of Feynman diagrams to associate an unlabeled graphon class (i.e. Feynman graphon) $[W_\Gamma]$ to each Feynman diagram Γ. We then lift the topology of graphons (i.e. cut-distance topology) onto the level of the Connes–Kreimer renormalization Hopf algebra to obtain an enrichment of this fundamental Hopf algebra in Quantum Field Theory. We also applied random graphs and n-adic metric to interpret the unique solution X_{DSE} of each given Dyson–Schwinger equation DSE as the convergent limit of the sequence $\{Y_m\}_{m\geq 0}$ of its partial sums with respect to the cut-distance topology. [34, 35]

Thanks to the Manin renormalization Hopf algebra of the Halting problem, combinatorial Dyson–Schwinger equations and Feynman graphon models, in this research work we plan to study non-perturbative aspects of strongly coupled gauge field theories in the context of the Theory of Computation and the Complexity Theory. We are going to build a new generalized version of the Kolmogorov complexity which works on Dyson–Schwinger equations. This new complexity is capable of evaluating the complexities of Dyson–Schwinger equations under different running coupling constants (derived by changing the scale of the bare coupling constant) in a given strongly coupled gauge field theory. In fact our Kolmogorov complexity works on a new constructive world $\mathcal{S}^{\Phi,g}$ which is the collection of all Dyson–Schwinger equations of a given gauge field theory Φ with the bare coupling constant g. We will show that the required properties for $\mathcal{S}^{\Phi,g}$ to be a suitable constructive world can be provided by applying Feynman graphon models of Dyson–Schwinger equations. For this purpose, we study combinatorial Dyson–Schwinger equations in the context of graph functions (i.e. Theorem 2.6). Then we lift the notion of graph complexity onto the level of Feynman diagrams and solutions of Dyson–Schwinger equations. Thereafter we address the structure of a multi-scale Renormalization Group which enables us to encode the behavior of all Dyson–Schwinger equations in a given gauge field theory under changing scales of the bare and running coupling constants (i.e. Theorem 2.13). This Renormalization Group setting is useful to describe a strongly coupled Dyson–Schwinger equations via a cut-distance convergent sequence of Dyson–Schwinger equations under weaker running coupling constants (i.e. Corollary 2.15). In addition, we applied this multi-scale Renormalization Group to formulate a new generalized version of the Kolmogorov complexity on $\mathcal{S}^{\Phi,g}$ (i.e. Definitions 3.3, 3.4). This complexity enables us to compare Dyson–Schwinger equations in $\mathcal{S}^{\Phi,g}$ with respect to their complexities under different rescaling of the bare and running coupling constants. At the final step, we will show how this complexity is related to the Manin renormalization of the Halting problem (i.e. Corollaries 3.10, 3.11).

2. DYSON–SCHWINGER EQUATIONS UNDER RUNNING COUPLING CONSTANTS

For a free field with mass m the interaction part of the Lagrangian must be quadratic at most such as $\frac{1}{2}m^2\phi^2$. In this case the Klein–Gordon equation

$$-\partial^2\phi + m^2\phi = 0 \tag{2.1}$$

is the equation of motion. For an interacting field with mass m the interaction part can have cubic or greater dependence on the field such as $\frac{1}{2}m^2\phi^2 + \frac{\lambda g}{4}\phi^4$ where λg is the (running) coupling constant and the equation of motion is determined by solving the corresponding Euler–Lagrange equation. It is given by

$$-\partial^2\phi + m^2\phi + \lambda g\phi^3 = 0. \tag{2.2}$$

Therefore we can consider the total Lagrangian in terms of the combination of its free part and its interaction part. [9, 10]

Suppose Φ is an interacting (strongly coupled) gauge field theory with the Lagrangian $L_\Phi \in \mathbb{R}[[g]]$ with respect to the bare coupling constant g which is invariant under the change $\phi \longmapsto -\phi$. The interaction part of this Lagrangian (with the action functional $S[\phi]$) has the general form $I_\Phi(\phi) := \sum_{k\geq 2} I_k(\phi)$ such that for all k, $I_k = O(g)$. This interaction part contributes to fixed point equations of Green's functions where we can generate equations of motion.

The N-point Green's functions, as the building blocks of the path integral formalism, encode all possible interactions between N particles. These correlation functions are defined by the functional expectational values with the general form

$$G^N(x_1, ..., x_N) = \frac{\int \prod_j \mathcal{D}\phi_j e^{iS[\phi_j]}\phi_j(x_1)...\phi_j(x_N)}{\int \prod_j \mathcal{D}\phi_j e^{iS[\phi_j]}}, \tag{2.3}$$

$$\mathcal{D}\phi_j = \prod_{i=1}^{N} d\phi_j(x_i),$$

such that the path integral measure is over all possible values of the fields ϕ_j at all space-time points. The functional integral $\prod_j \mathcal{D}\phi_j$ is the product of integrals over different fields ϕ_j at each space-time point. In addition, we need to add new terms $\pm i\epsilon$ in the denominators of the propagators to address in and out vacuum in the action functional setting. Then we will have a well-defined path integral and related propagators. The partition functions $Z[J]$, given by

$$Z[J] = \int \prod_j \mathcal{D}\phi_j e^{iS[\phi_j]} e^{i\int d^4x \phi_j(x)J(x)}, \tag{2.4}$$

as the result of adding external source terms $J(x)$ to the action functional, can generate all N-point Green's functions as functional derivatives of $Z[J]$. In other words,

$$G^N(x_1, ..., x_N) = \frac{(-i)^N}{Z[0]} \frac{\delta^N Z[J]}{\delta J(x_1)...\delta J(x_N)}|_{J=0}, \tag{2.5}$$

[9, 10, 11]. This machinery is useful for perturbative calculations where the action functional can be divided into interacting and non-interacting parts, $S[\phi] = S_0[\phi] + gS_{\text{int}}[\phi]$. We can consider this formalism for any running coupling constants λg generated by regularization and renormalization techniques. For small enough coupling constants, we can expand $e^{i\lambda g S_{\text{int}}[\phi]}$ in the powers of λg to obtain a perturbative formulation for Green's functions. In general, the coupling constant is an effective constant which depends on the squared-momentum-transfer Q^2. This dependence is very strong in gauge field theories such as low energy QCD, which study strong interactions, where gluons carry color and simultaneously they can couple to other gluons. In very small distances and high values of Q^2 the inter-quark couplings is vanishing asymptotically such that quarks can be free. However at large distances, where the inter-quark couplings increase, non-perturbative situations could happen which make impossible to separate quarks from hadrons. For strong running couplings where β-functions are non-zero, Green's functions can not be handled in terms of perturbative methods where we need to deal with non-linear Dyson–Schwinger equations and their non-perturbative behavior. In physical theories with vanishing β-function, it is possible to study Dyson–Schwinger equations under a linear setting. [10, 27, 28, 30]

Feynman diagrams are the original tools for the study of Green's functions. A Feynman diagram is a collection of oriented decorated edges and vertices such that vertices present interactions and edges present elementary particles or virtual particles. Edges which have only beginning or ending vertex (i.e. external edges) are symbols for elementary particles while edges which have beginning and ending vertices (i.e. internal edges) are symbols for virtual particles. The whole diagram obeys the law about the conservation of momenta. Feynman rules allow us to associate an iterated integral to each Feynman diagram where these integrals suffer from sub-divergences. Usually loops in Feynman diagrams are symbols for these sub-divergences. For a given physical theory, a class of Feynman diagrams, namely One Particle Irreducible (1PI) diagrams, play a fundamental role for the construction of more complicated diagrams. A Feynman diagram is called 1PI if the graph remains connected after removing any arbitrary internal edge (or cutting a single propagator line) from it. [9, 10, 11]

The gauge invariance of the classical Lagrangian determines Ward identities between Green's functions of the quantized theory. In general the gauge invariance of the partition function can be applied to provide some identities among Green's functions such as Slavnov–Taylor identity in QCD and Ward–Takahashi identity in QED. It is possible to describe Dyson–Schwinger equations as the result of these equations with translational invariance. Thanks to Feynman rules of the physical theory, the 1PI Green's functions can be presented in terms of formal expansions of Feynman diagrams with respect to the types of particles and interactions. In other words,

$$G^{e_i} = 1 - \sum_{\mathrm{res}(\Gamma)=e_i} (\lambda g)^{|\Gamma|} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}, \tag{2.6}$$

$$G^{v_j} = 1 + \sum_{\mathrm{res}(\Gamma)=v_j} (\lambda g)^{|\Gamma|} \frac{\Gamma}{\mathrm{Sym}(\Gamma)},$$

such that the inverse of these formal expansions can also be defined as geometric series. In these relations, $\mathrm{Sym}(\Gamma)$ is the symmetry factor of Γ determined by the cardinal of the automorphism group $\mathrm{Aut}(\Gamma)$ and $\mathrm{res}(\Gamma)$ is a graph as the result of shrinking all internal edges of Γ into one decorated vertex. [18, 19, 30, 43]

Combinatorial Dyson–Schwinger equations are the result of the formulation of the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams $H_{\mathrm{FG}}(\Phi)$ [5, 18]. This fundamental Hopf algebra, derived from the Bogoliubov–Zimmermann's forest formula in perturbative renormalization process ([6, 44]), has led us to study non-perturbative aspects of gauge field theories in the context of Hochschild Cohomology Theory, combinatorial methods and Noncommutative Geometry [4, 19, 34, 36].

Thanks to the Kreimer's renormalization coproduct ([6]), the grafting operator B^+_γ, as the linear operator which performs any possible insertion of Feynman diagrams Γ into the arbitrary primitive (1PI) Feynman diagram γ with respect to types of external edges in Γ and vertices in γ, is the Hochschild one cocyle corresponding to γ.

Definition 2.1. For a family of Hochschild one cocyles generated by a given family $\{\gamma_n\}_{n\geq 1}$ of primitive (1PI) Feynman diagrams, the recursive equation

$$\mathrm{DSE}(\lambda g): \quad X = \mathbb{I} + \sum_{n\geq 1} (\lambda g)^n w_n B^+_{\gamma_n}(X^{n+1}) \tag{2.7}$$

is a combinatorial reformulation for a class of analytic Dyson–Schwinger equations. In this equation λg is the running coupling constant for $0 < \lambda \leq 1$ and for each $n \geq 1$, w_n is a constant. [5, 18, 19, 40]

The unique solution of the equation (2.7) can be presented in terms of the (infinite) formal expansion $X = \sum_{n\geq 0}(\lambda g)^n X_n$ of finite Feynman diagrams such that $X_0 = \mathbb{I}$ and for each $n \geq 1$, we have

$$X_n = \sum_{j=1}^{n} w_j B^{+}_{\gamma_j}\big(\sum_{k_1+\cdots+k_{j+1}=n-j,\ k_i\geq 0} X_{k_1}...X_{k_{j+1}}\big), \tag{2.8}$$

[4, 19]. The non-linear version of these equations, derived from non-zero β-functions, generate graded commutative non-cocommutative Hopf subalgebras of the renormalization Hopf algebra while the linear version of these equations, derived from vanishing β-functions, generate graded commutative cocommutative Hopf subalgebras. Higher order perturbation method and lattice models have provided powerful tools for the study of linear Dyson–Schwinger equations [27, 28, 30]. In addition, systems of Dyson–Schwinger equations have also been studied under this Hopf algebraic setting [4, 12, 13, 18, 19, 32].

For strong running coupling constants λg, Dyson–Schwinger equation (2.7) should be considered under a non-perturbative regime where its unique solution X is actually an infinite graph. Recently, analytic graphs (i.e. graph functions or graphons) in Infinite Combinatorics have been applied to find a new interpretation for solutions of Dyson–Schwinger equations. In this new approach we have associated a new class of graphs which have a continuum set of vertices to Dyson–Schwinger equations [33, 34]. The theory of graphons aims to study graph limits of sequences of finite weighted graphs via measure theoretic, combinatorial and topological tools [20]. Topology of graphons (known as cut-distance topology) enables us to obtain a compact Hausdorff topological space of finite (simple) graphs such that graph limits are in the boundary region of this space [14]. Applications of graphons have been developed for the study of dense and sparse graphs in various fields of research [3, 7, 20].

Definition 2.2. For a given σ-finite measure space (Ω, μ_Ω), a graphon W is a measurable symmetric bounded real valued function on $\Omega \times \Omega$. W is called a bigraphon, if we remove the symmetric condition.

As an example, we can consider the probability space $\Omega = [0, 1]$ with the Borel σ-field and the Lebesgue measure as the base space and define graphons as symmetric Lebesgue measurable functions $W : [0, 1] \times [0, 1] \to [0, 1]$. Invertible measure preserving transformations ρ on $[0, 1]$ allow us to define a graphon under different relabeling or rearrangements such that they are called labeled graphons. Up to this class of transformations, we can define an unlabeled graphon as the class

$$[W] := \{W^\rho : \ W^\rho(x, y) := W(\rho(x), \rho(y)), \ \ \rho\} \tag{2.9}$$

of labeled graphons associated to W. The cut-distance between labeled graphons W_1, W_2 is defined by

$$\delta_\square(W_1, W_2) :=$$

$$\inf_{\rho_1,\rho_2} \sup_{A,B\subsetneq[0,1]} \Big| \int_{A\times B} W_1^{\rho_1}(x, y) - W_2^{\rho_2}(x, y)dxdy\Big| \tag{2.10}$$

which is a pseudometric on the space of unlabeled graphons. We can generate a metric from the cut-distance via the weakly isomorphic equivalence relation which is defined in the following.

Definition 2.3. We call graphons W_1, W_2 weakly isomorphic iff there exist measure preserving transformations η_1, η_2 such that $W_1^{\eta_1} = W_2^{\eta_2}$ almost everywhere.

It is easy to show that W_1, W_2 are weakly isomorphic iff $\delta_{\square}(W_1, W_2) = 0$.

Example 2.4. *(i) The transformations $\rho_n : x \mapsto nx, \pmod 1$ for each natural number $n \geq 1$ are measure preserving which are not invertible with respect to the Lebesgue measure. However graphons W^{ρ_n} are weakly isomorphic.*

(ii) The continued fractions $\eta_n : x \mapsto \frac{1}{n}x$, which are not measure preserving with respect to the Lebesgue measure, can be seen as measure preserving transformations with respect to the Gauss measure which is equivalent to the Lebesgue measure.

Now we can update the class (2.9) of labeled graphons to contain also weakly isomorphic graphons. Therefore up to the weakly isomorphic equivalence relation, the space of unlabeled graphons is a compact metric space with respect to the cut-distance (2.10). We have

$$\delta_{\square}([W_1], [W_2]) = \inf_{\rho_1, \rho_2} \|W_1^{\rho_1} - W_2^{\rho_2}\|_{\square}, \tag{2.11}$$

such that for each labeled graphon W,

$$\|W\|_{\square} := \sup_{A,B \subsetneq [0,1]} \left| \int_{A \times B} W(x, y) dx dy \right| \tag{2.12}$$

[7, 14, 20]. This metric allows us to define the notion of convergence for sequences of finite weighted graphs. Graph limits of sequences of dense graphs are non-zero graphons ([20]) while graph limits of sequences of sparse graphs are graphons which are weakly isomorphic to the 0-graphon with respect to the Lebesgue measure. In this case, the theory of graphons for sparse graphs has been developed where by changing the base measure space, rescaling the canonical graphons or other measure theoretic techniques, we can obtain non-zero graphons as the cut-distance convergent limit of sequences of sparse graphs. [3, 7]

The renormalization Hopf algebra of Feynman diagrams has a universal toy model in terms of non-planar rooted trees. Using a suitable collection of decorations allows us to encode types of interactions and elementary particles in vertices and edges of rooted trees. Therefore it would be possible to project Feynman diagrams of a given gauge field theory Φ to some decorated rooted trees in the Connes–Kreimer Hopf algebra $H_{\mathrm{CK}}(\Phi)$. There exists an injective Hopf algebraic homomorphism $\Psi_{\Phi} : H_{\mathrm{FG}}(\Phi) \to H_{\mathrm{CK}}(\Phi)$ such that each Feynman diagram Γ can be represented by a decorated non-planar rooted tree t_{Γ} [5, 6, 17, 43]. We applied these decorated rooted trees (as simple weighted graphs) to associate an unlabeled graphon class $[W_{\Gamma}]$ (named it as Feynman graphon) to each Feynman diagram Γ in terms of the pixel picture presentation of t_{Γ}. Then we defined the cut-distance convergence for sequences of Feynman diagrams where we need to apply some rescaling methods to obtain non-zero Feynman graphons from the graph limits of these sequences of sparse type of graphs. [33, 34]

For a fixed probability measure space (Ω, μ_{Ω}), set $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ as the vector space generated by unlabeled Feynman graphons. Linear combinations of Feynman graphons can be defined via renormalization of the canonical graphons in terms of rescaled or stretched procedures.

Theorem 2.5. *Thanks to the Kreimer's renormalization coproduct, for a given gauge field theory Φ, there exists a graded Hopf algebraic structure on $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$.*

Proof. The structure of this Hopf algebra is discussed in [34, 35] and here we address only the general information. We can lift loop numbers or number of internal edges as the graduation parameter on Feynman diagrams onto the level of Feynman graphons to graduate this class of graphons with respect to the number of sub-intervals in their corresponding partitions of the unital interval. For each $n \geq 1$, $\mathcal{S}^{\Phi,(n)}_{\mathrm{graphon}}$ contains all Feynman graphons W_{Γ} corresponding to Feynman diagrams $\Gamma \in H^{(n)}_{\mathrm{FG}}(\Phi)$. For each unlabeled Feynman graphon class $[W_\Gamma]$ define

$$\Delta_{\text{graphon}}([W_\Gamma]) = \sum [W_\gamma] \otimes [W_{\Gamma/\gamma}], \tag{2.13}$$

such that the sum is taken over Feynman graphons associated to all disjoint unions of 1PI Feynman sub-diagrams (i.e. superficially divergent subdiagrams) γ in Γ. Therefore $\mathcal{S}^{\Phi}_{\text{graphon}}$ is a graded connected free commutative non-cocommutative bialgebra which can be equipped by an antipode derived from the coproduct (2.13) and the graduation parameter. □

The enrichment of this Hopf algebra with respect to the cut-distance topology gives us a new powerful tool for the study of solutions of Dyson–Schwinger equations and the renormalization program for these expansions whenever the running couplings are strong [34, 35, 36].

Random graphs are determined by some random processes where for example we have a collection of probabilities for the existence of edges among vertices. Random graph models can be generated by graphon processes where changing the base measure space or rescaling methods will lead us to different random graph models [3, 20]. One immediate application of our Feynman graphon model (i.e. Theorem 2.5) is to study Dyson–Schwinger equations in terms of cut-distance convergent sequences of random graphs. The one interesting random graph model for our purpose is derived from Feynman graphons which contribute to solutions of Dyson–Schwinger equations.

Theorem 2.6. *The unique solution $X_{\text{DSE}} = \sum_{n\geq 0} (\lambda g)^n X_n$ of each combinatorial Dyson–Schwinger equation DSE in a given (strongly coupled) gauge field theory Φ can be described as the cut-distance convergent limit of a sequence of random graphs corresponding to some Feynman graphons in $\mathcal{S}^{\Phi}_{\text{graphon}}$.*

Proof. We work on Feynman graphon models built on the probability space $\Omega = [0, 1]$ with the Borel σ-field and the Lebesgue measure as the base measure space. The proof is valid for Feynman graphons on any other σ-finite measure spaces.

Consider the sequence $\{Y_m\}_{m\geq 1}$ of partial sums of the formal expansion $X_{\text{DSE}} := \sum_{n\geq 0} (\lambda g)^n X_n$ such that $Y_m := (\lambda g)^1 X_1 + \cdots + (\lambda g)^m X_m$. Thanks to Theorem 2.5, finite expansions Y_m and the infinite graph $X_{\text{DSE}} \in H_{\text{FG}}(\Phi)[[\lambda g]]$ have their own Feynman graphon models presented by $[W_{Y_m}]$ and $[W_{X_{\text{DSE}}}]$. We first build the non-zero Feynman graphon $W_{X_{\text{DSE}}}$ and then we explain the structures of random graphs R_{Y_m} generated by W_{Y_m}. At the final step, thanks to Proposition 4.6 in [34], we will see that the sequence $\{R_{Y_m}\}_{m\geq 0}$ is cut-distance convergent to the Feynman graphon $W_{X_{\text{DSE}}}$.

Thanks to the graduation parameter on Feynman diagrams in terms of the number of independent loops or the number of internal edges, the n-adic metric on Feynman diagrams in $H_{\text{FG}}(\Phi)$ (given in [13]) is defined by

$$d_{\text{adic}}(\Gamma_1, \Gamma_2) := 2^{-\text{val}(\Gamma_1 - \Gamma_2)}, \tag{2.14}$$

$$\text{val}(\Gamma) := \text{Max}\{n \in \mathbb{N} : \ \Gamma \in \bigoplus_{k\geq n} H^{(k)}_{\text{FG}}(\Phi)\}.$$

It defines a new function $F_{\text{adic}, X_{\text{DSE}}}$ on the set $V(X_{\text{DSE}})$ of all vertices of X_{DSE} as follows

$$F_{\text{adic}, X_{\text{DSE}}} : V(X_{\text{DSE}}) \times V(X_{\text{DSE}}) \to \mathbb{R}, \tag{2.15}$$

$$(v_i, v_j) \longmapsto d_{\text{adic}}(Y_{i_0}, Y_{j_0})$$

such that

$$i_0 := \text{Min}\{s : \ v_i \in Y_s\}, \quad j_0 := \text{Min}\{t : \ v_j \in Y_t\}. \tag{2.16}$$

The value $F_{\text{adic},X_{\text{DSE}}}(v_i, v_j)$ can be seen as the weight of the edge v_iv_j in X_{DSE}. In addition, for every vertex $v_i \in V(X_{\text{DSE}})$ define

$$w_i := d_{\text{adic}}(X_{i_0}, \mathbb{I}) \in [0, 1] \tag{2.17}$$

as the weight of v_i. Finite expansions $a_n := \sum_{1\leq k\leq n} w_k$ determine subintervals in the set of real numbers which can be projected onto the unital interval to determine a partition $\{I_n := [a_{n-1}, a_n), \ \forall n\}$ for $[0, 1)$.

Now the non-zero Feynman graphon $W_{X_{\text{DSE}}}$ can be defined by graph functions with the general form

$$W_{X_{\text{DSE}}} : [0, 1] \times [0, 1] \to [0, 1], \tag{2.18}$$

$$W_{X_{\text{DSE}}}(x, y) := d_{\text{adic}}(Y_{i_0}, Y_{j_0})$$

whenever $(x, y) \in I_{i_0} \times I_{j_0}$ and otherwise it has the zero value.

Thanks to the injective Hopf algebra homomorphism $\Psi_\Phi : H_{\text{FG}}(\Phi) \to H_{\text{CK}}(\Phi)$, for each $m \geq 1$, set

$$t_{Y_m} := \mathbb{I} + t_{X_1} + ... + t_{X_m} \tag{2.19}$$

as the decorated tree representation of the partial sum Y_m. For each $m \geq 1$, it is a disjoint union of decorated rooted trees which has $n_m = |t_{Y_m}|$ vertices. We can embed these vertices into the unital closed interval to determine points $v_1, ..., v_{n_m}$ in $[0, 1]$ by using a chosen poset embedding θ_m. Thanks to the n-adic metric (2.14), we can build a random graph R_{Y_m} such that with the probability $d_{\text{adic}}(\Gamma_{k_i}, \Gamma_{k_j})$, there exists an edge between v_i and v_j whenever $\Psi_\Phi^{-1} \circ \theta_m^{-1}(v_i) \in X_{k_i}$ and $\Psi_\Phi^{-1} \circ \theta_m^{-1}(v_j) \in X_{k_j}$ in the partial sum Y_m.

On the one hand, we can see that random graphs R_{Y_m} are built in terms of the Feynman graphon (2.18). On the second hand, it is shown in Proposition 4.6 in [34] that the sequence $\{Y_m\}_{m\geq 0}$ is cut-distant convergent to the infinite graph X_{DSE}. This means that the sequence $\{W_{Y_m}\}_{m\geq 0}$ of the Feynman graphons corresponding to the partial sums is cut-distance convergent to the Feynman graphon $W_{X_{\text{DSE}}}$ given by (2.18). On the third hand, when m tends to infinity random graphs R_{Y_m} allows us to define an infinite random graph $R_{X_{\text{DSE}}}$ for the infinite graph X_{DSE}. Now it is enough to modify discussions in Appendix D, part D.1 (page 59) in [14] for our built random graphs to observe that the sequence $\{R_{Y_m}\}_{m\geq 0}$ is cut-distance convergent to the Feynman graphon $W_{X_{\text{DSE}}}$. □

Remark 2.7. (i) We call X_{DSE} as the large Feynman diagram corresponding to the non-zero Feynman graphon $W_{X_{\text{DSE}}}$.

(ii) Different choices of the base measure space or rescaling methods enable us to study non-perturbative expansions under different random graph models.

(iii) Dyson–Schwinger equations $\text{DSE}_1, \text{DSE}_2$ are called weakly isomorphic iff their corresponding Feynman graphons $W_{X_{\text{DSE}_1}}$ and $W_{X_{\text{DSE}_2}}$ are weakly isomorphic.

Theorem 2.6 informs us that passing from the perturbative part of a given (strongly coupled) gauge field theory to its non-perturbative part can be studied via a class of analytic graphs with have continuum sets of vertices.

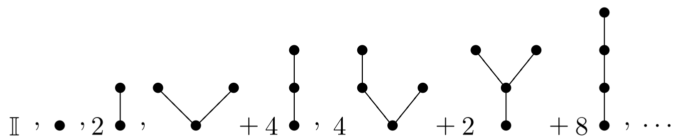
Example 2.8. *Consider the combinatorial Dyson–Schwinger equation*

$$\text{DSE}: \ \ X = \mathbb{I} + B^+(X^2) \tag{2.20}$$

in the Connes–Kreimer Hopf algebra of non-planar rooted trees. Thanks to (2.8), its unique solution $X = \sum_{n\geq 0} X_n$ *is given by the recursive relations*

$$X_{n+1} = \sum_{k=0}^{n} B^+(X_k X_{n-k}), \ \ X_0 = \mathbb{I}. \tag{2.21}$$

We have

(2.22)


It is possible to define an order relation on the set of any forest or disjoint union of non-planar rooted trees to see it as a poset. The sequence $\{Y_m\}_{m\geq 1}$ of partial sums gives an order on all vertices. Set $V(X) := \{v_{ij_i}\}_{i,j}$ as the set of all vertices in the formal expansion X such that $v_{ij_i} \in Y_i$ are vertices in the partial sum Y_i. Thanks to the method explained in the proof of Theorem 2.6, the weight $w_{v_{ij_i}}$ for any vertex v_{ij_i} which is in Y_j and not in Y_{j-1} is $d_{\text{adic}}(Y_j, \mathbb{I}) = 1/2^j$. Therefore we have

$$a_1 = 1/2,\ a_2 = 1/2 + 4(1/2^2),\ a_3 = 1/2 + 4(1/2^2) + 15(1/2^3),$$

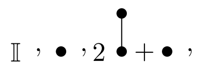
$$a_4 = 1/2 + 4(1/2^2) + 15(1/2^3) + 56(1/2^4),\ \cdots . \tag{2.23}$$

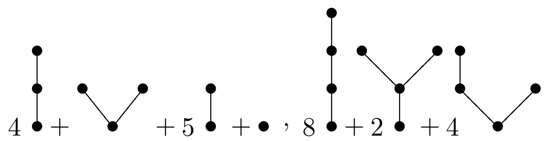
Project real subintervals $[a_{n-1}, a_n)$ into the unital interval to determine a partition $\{I_n : n\}$ for $[0, 1)$. Now apply (2.18) to define the corresponding non-zero Feynman graphon.

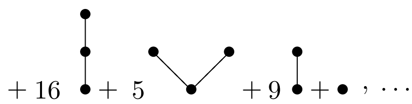
Example 2.9. *Consider the combinatorial Dyson–Schwinger equation*

$$\text{DSE}:\quad X = \mathbb{I} + \sum_{n\geq 0} B^+(X^{n+1}). \tag{2.24}$$

Thanks to (2.8), its unique solution $X = \sum_{n\geq 0} X_n$ is given by

(2.25)


(2.26)


(2.27)


Set $V(X) := \{v_{ij_i}\}_{i,j}$ as the set of all vertices in the formal expansion X such that $v_{ij_i} \in Y_i$ are vertices in the partial sum Y_i. Thanks to the method explained in the proof of Theorem 2.6, the weight $w_{v_{ij_i}}$ for any vertex v_{ij_i} which is in Y_j and not in Y_{j-1} is determined by the n-adic distance between $\mathbb{I}$ and each partial sum. The beginning terms of the above sequence shows us that $d_{\text{adic}}(Y_i, \mathbb{I}) = 1/2$ at least for $i = 1, 2, 3, 4$. Therefore we have

$$a_1 = 1/2,\ \ a_2 = 1/2 + 5(1/2),\ \ a_3 = 1/2 + 5(1/2) + 26(1/2),$$

$$a_4 = 1/2 + 5(1/2) + 26(1/2) + 138(1/2),\ \cdots . \tag{2.28}$$

Project real subintervals $[a_{n-1}, a_n)$ into the unital interval to obtain a partition $\{I_n\}_n$ for $[0, 1)$. Now apply (2.18) to define the corresponding non-zero Feynman graphon.

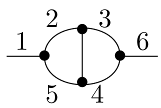
Thanks to the details given in the proof of Theorem 2.6 and the above examples, we can observe that the numbers of vertices in the partial sums of the unique solution of a given Dyson–Schwinger equation determine the required partition of $[0,1]$ for defining the corresponding Feynman graphon and also, the random graph models. Dyson–Schwinger equations with more vertices in their solutions have more graph complexities where the process for determining their corresponding random graph models requires more initial data. In this setting, since vertices in decorated rooted trees are symbols for nested loops in their corresponding Feynman diagrams, therefore random graph models R_{Y_m} built in terms of rooted tree representations t_{Y_m} are actually useful to compare complexities of disjoint unions of Feynman diagrams in terms of the scattering of their nested loops.

In Complexity Theory, the number of spanning trees in a finite graph determines the complexity of the graph. There are different methods to approximate some upper bounds for this fundamental parameter in graphs via the theory of random graphs, normalized Laplacian eigenvalues and other computational algorithms. According to one of these methods, it is possible to evaluate the complexity of an (extremely large) finite graph under a recursive algorithm in terms of cyclic edges. For a given graph G with the set S_G of all its spanning trees, each cyclic edge e defines a partition $S_G^{(1)} \sqcup S_G^{(2)}$ such that $S_G^{(1)}$ is the collection of spanning trees of G which do not contain e and $S_G^{(2)}$ is the collection of spanning trees of G which have e. In addition, the graph $G - \{e\}$ is connected while sets $S_G^{(1)}$ and $S_{G-\{e\}}$ are equivalent. This model of partitions can be formulated via an algorithm which starts with a given graph and produces two graphs at the end of the first stage. At each stage, the algorithm chooses only an edge belonging to a proper cycle (i.e. a non-trivial trail with only first and last vertices as repeated vertices). The elementary contraction on a multiple graph generates a graph which can have a (self-)loop where the algorithm can decide to continue. The algorithm chooses one proper cyclic edge (if it exists) from each graph in each subsequent stage to apply for the recurrence process. At the final step of the algorithm, a collection of graphs can be produced which have no proper cycle. The complexity of the initial input graph can be evaluated in terms of the number of the output graphs and their complexities. [1, 2, 8, 37]

On the one hand, this recursive algorithm terminates in a set of graphs with no proper loops. Therefore we can use it to determine spanning trees in an extremely large Feynman diagram together with many nested loops. We can also modify this algorithm to determine spanning trees (or forests) of each component of linear combinations of Feynman diagrams. On the other hand, the complexity of a given Feynman diagram in perturbative Quantum Field Theory can be measured in terms of the number of nested or overlapping sub-divergences which appear as loops in the graph. The Bogoliubov–Zimmermann's forest formula ([44]) encodes renormalization under an inductive machinery in terms of the step by step removal of subdivergences in Feynman diagrams. In each step of this renormalization machinery, some nested loops in the original graph can be shrunk to a point. Therefore we can observe a similarity between these two different machineries.

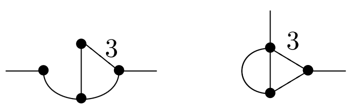
Example 2.10. *For example, consider the following Feynman diagram*

(2.29)

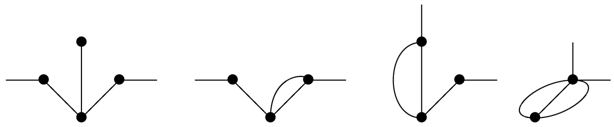


in a simplified toy model. We apply the above explained recursive algorithm. The algorithm might choose the edge 2 *as the first cyclic edge. Then the algorithm delivers the following two graphs*

(2.30)



. Then the algorithm chooses the edge 3 in both diagrams in (2.30) as the next cyclic edges. Then the following four diagrams can be produced

(2.31) 

, which do not have any more cyclic edges. At this step the algorithm stops and the graph complexity of the original Feynman diagram (2.29) would be the sum of the spanning trees in each diagram of the collection (2.31).

Thanks to Theorem 2.6, we can modify the above algorithm of counting spanning trees in a connected graph to the unique solution of a given strongly coupled Dyson–Schwinger equation and its corresponding partial sums to describe the complexity of this non-perturbative expansion under a recursive machinery.

Lemma 2.11. *The graph complexity of each Dyson–Schwinger equation can be described in terms of the sequence of graph complexities of the partial sums of its solution.*

Proof. For each m, the complexity of $Y_m := \mathbb{I} + (\lambda g)X_1 + \cdots + (\lambda g)^m X_m$ can be determined in terms of the number of different spanning forests (or trees) which live in the linear combination of Feynman diagrams X_i which contribute to Y_m. We can perform the above recursive algorithm for each Feynman diagram X_i which contribute to Y_m (for each m) such that the complexity of Y_m would be bounded by the sum of the complexities of its components. When m tends to infinity, we can achieve a sequence which presents the behavior of complexities of Feynman diagrams when the partial sums converge to X_{DSE} with respect to the cut-distance topology. □

We have plan to lift this notion of complexity of a given (strongly coupled) Dyson–Schwinger equation $\mathrm{DSE}(g)$ via the complexities of its partial sums onto the level of the collection of all Dyson–Schwinger equations $\mathrm{DSE}(\lambda g)$ under different running coupling constants λg for $0 < \lambda \leq 1$. It will lead us to describe the complexity of the non-perturbative infinite graph $X_{\mathrm{DSE}}(g)$ in terms of the complexity of a sequence $\{X_{\mathrm{DSE}}(\lambda_n g)\}_{n\geq 0}$ of large Feynman diagrams under weaker running couplings. For this purpose we require to control the dynamics of all Dyson–Schwinger equations under different running coupling constants in a given strongly coupled gauge field theory where the rescaling of running couplings are strongly connected to changing the scale of the bare coupling constant. A general setting for the construction of the Renormalization Group can be found in [9, 11] and here we address its generalization to a multi-scale version which works on Dyson–Schwinger equations.

Definition 2.12. Set $\mathcal{S}^{\Phi,g}$ as the collection of all Dyson–Schwinger equations in different running couplings λg derived from fixed point equations of Green's functions in a given gauge field theory Φ with the bare coupling constant g. A Dyson–Schwinger equation is called an effective equation under the scale Λ_1 of another Dyson–Schwinger equation under the scale Λ_2, if their corresponding Green's functions are to be the same in the original Lagrangian L^Φ.

It is shown in [36] that each Dyson–Schwinger equation at a fixed initial scale has a unique effective version at a new scale. These effective equations are the key tools for us to define our Renormalization Group which is only a semigroup because there is no identity element.

Theorem 2.13. *For a given (strongly coupled) gauge field theory Φ with the Lagrangian $L^\Phi(g)$, there exists a multi-scale Renormalization Group which governs changing the scales of the bare and running coupling constants in the solutions of Dyson–Schwinger equations in $\mathcal{S}^{\Phi,g}$.*

Proof. Set M_r as the set of scales of the running couplings. For scales $\Lambda_1, \Lambda_2, \Lambda_3 \in M_r$ such that $\Lambda_1 < \Lambda_2 < \Lambda_3$, define the scale map $R^r_{\_\_}$ on $\mathcal{S}^{\Phi,g}$ which satisfies the property

$$R^r_{\Lambda_1\Lambda_2} R^r_{\Lambda_2\Lambda_3} = R^r_{\Lambda_1\Lambda_3}. \tag{2.32}$$

For each equation DSE, $R^r_{\Lambda_1\Lambda_2}\mathrm{DSE}$ is the effective Dyson–Schwinger equation at the rescaled running coupling Λ_2 of the equation DSE at the original scale Λ_1 of running coupling constant. Now define an action of the semigroup $\mathbb{R}^+_{\leq 1}$ on $\mathcal{S}^{\Phi,g} \times M_r$ given by

$$s \circ (\mathrm{DSE}, \Lambda) = (R^r_{\Lambda, s\Lambda}\mathrm{DSE}, s\Lambda) \tag{2.33}$$

such that $R^r_{\Lambda,s\Lambda}\mathrm{DSE}$ is the Dyson–Schwinger equation obtained by changing the scale $\Lambda \mapsto s\Lambda$ of the running coupling constant. The equation

$$R^r_{s\Lambda\ \Lambda}\mathrm{DSE} := (R^r_{\Lambda,s\Lambda}\mathrm{DSE}, s\Lambda) \tag{2.34}$$

is the corresponding unique effective equation in the effective Lagrangian $L^{\Phi}_{s\Lambda}(g)$. The resulting Renormalization Group allows us to study the dynamics of Dyson–Schwinger equations under the rescaling of running coupling constants.

Set M_b as the set of scales of the bare coupling constant g. For scales $\tau_1, \tau_2, \tau_3 \in M_b$ such that $\tau_1 < \tau_2 < \tau_3$, define the scale map $R^b_{\_\_}$ on $\mathcal{S}^{\Phi,g}$ which satisfies the property

$$R^b_{\tau_1\tau_2} R^b_{\tau_2\tau_3} = R^b_{\tau_1\tau_3}. \tag{2.35}$$

For a given equation DSE, define a new Dyson–Schwinger equation $R^b_{\tau_1\tau_2}\mathrm{DSE}$ which is the effective Dyson–Schwinger equation at the rescaled bare coupling τ_2 of the equation DSE at the initial scale τ_1 of the bare coupling constant. Now define an action of the semigroup $\mathbb{R}^+_{\leq 1}$ on $\mathcal{S}^{\Phi,g} \times M_b$ given by

$$t \circ (\mathrm{DSE}, \tau) = (R^b_{\tau,t\tau}\mathrm{DSE}, t\tau) \tag{2.36}$$

such that $R^b_{\tau,t\tau}\mathrm{DSE}$ is the Dyson–Schwinger equation obtained by changing the scale $\tau \mapsto t\tau$ of the bare coupling constant. The equation

$$R^b_{t\tau\ \tau}\mathrm{DSE} := (R^b_{\tau,t\tau}\mathrm{DSE}, t\tau) \tag{2.37}$$

is the corresponding unique effective equation in the effective Lagrangian $L^{\Phi}(t\tau g)$. The resulting Renormalization Group allows us to study the dynamics of Dyson–Schwinger equations under the rescaling of the bare coupling constant g.

The formulas (2.33) and (2.36) enable us to build our multi-scale Renromalization Group on $\mathcal{S}^{\Phi,g}$ to control the rescaling of the momentum parameter with respect to the rescaling of the bare coupling constant and independent of any regularization scheme. In other words, we can define an action of the semigroup $\mathbb{R}^+_{\leq 1}$ on $\mathcal{S}^{\Phi,g} \times M_b \times M_r$ via

$$\nu \circ (\mathrm{DSE}, \tau g, \Lambda_{\tau g}) = (R^{\mathrm{multi}}_{(\tau g, \Lambda_{\tau g}),(\nu\tau g, \nu\Lambda_{\tau g})}\mathrm{DSE}, (\nu\tau g, \nu\Lambda_{\tau g})) \tag{2.38}$$

such that $R^{\mathrm{multi}}_{(\tau g,\Lambda_{\tau g}),(\nu\tau g,\nu\Lambda_{\tau g})}\mathrm{DSE}$ is the Dyson–Schwinger equation obtained by changing the scale $\tau \mapsto \nu\tau$ of the bare coupling constant g and then changing the scale $\Lambda_{\tau g} \mapsto \nu\Lambda_{\tau g}$ of the running coupling constant. The multi-scale map

$$R^{\mathrm{multi}}_{(\nu\tau g,\nu\Lambda_{\tau g})\ (\tau g,\Lambda_{\tau g})}\mathrm{DSE} := (R^{\mathrm{multi}}_{(\tau g,\Lambda_{\tau g}),(\nu\tau g,\nu\Lambda_{\tau g})}\mathrm{DSE}, (\nu\tau g, \nu\Lambda_{\tau g})) \tag{2.39}$$

on $\mathcal{S}^{\Phi,g}$ is the corresponding unique effective Dyson–Schwinger equation in the effective Lagrangian $L^{\Phi}_{\nu\Lambda_{\tau g}}(\nu\tau g)$. □

This multi-scale Renormalization Group allows us to study the behavior of Dyson–Schwinger equations under changing the scales of the bare and running couplings where the scale of the momentum parameter (i.e. running couplings) should depends naturally on the scale of the bare coupling.

Definition 2.14. The triple $(\mathrm{DSE}, \tau g, \Lambda_{\tau g})$ in $\mathcal{S}^{\Phi,g} \times M_b \times M_r$ presents the Dyson–Schwinger equation DSE as an infinite polynomial on the re-scaled bare and running coupling constants. Thanks to Feynman rules, its unique solution is an infinite formal expansion of Feynman integrals with increasing powers of the re-scaled bare coupling constant τg (as coefficients) in the effective Lagrangian $L^{\Phi}(\tau g)$. Each Feynman integral in the expansion is defined in terms of the momentum parameter at the initial scale $\Lambda_{\tau g}$.

Corollary 2.15. *For a given strongly coupled gauge field theory Φ with the bare coupling constant $g \geq 1$, the unique non-perturbative solution of a Dyson–Schwinger equation $\mathrm{DSE}(g)$ in $\mathcal{S}^{\Phi,g}$ can be described as the cut-distance convergent limit of a sequence of large Feynman diagrams under weaker running coupling constants.*

Proof. We require a strictly increasing rational sequence $\{\alpha_n\}_{n\geq 1}$ of running coupling constants which converges to g to build a new sequence of Dyson–Schwinger equations under different rescaling of the bare coupling constant. For simplicity we work with the sequence $\{\frac{n}{n+1}g\}_{n\geq 1}$ such that Theorem 2.13 leads us to make a new sequence $\{R^b_{g,\frac{n}{n+1}g}\mathrm{DSE}\}_{n\geq 1}$ of Dyson–Schwinger equations in $\mathcal{S}^{\Phi,g}$ with respect to the rescaled bare coupling constants $\frac{n}{n+1}g$ for each $n \geq 1$.

For each n, the equation $R^b_{g,\frac{n}{n+1}g}\mathrm{DSE}$ has the unique solution

$$X_{\mathrm{DSE}}(\frac{n}{n+1}g) = \sum_{m\geq 0} (\frac{n}{n+1}g)^m X_m \tag{2.40}$$

determined by the recursive relations (2.8).

Thanks to the Feynman graphon model approach to Dyson–Schwinger equations (i.e. Theorem 2.6), we want to show that the sequence $\{X_{\mathrm{DSE}}(\frac{n}{n+1}g)\}_{n\geq 1}$ is cut-distance convergent to X_{DSE}. This is equivalent to show that the sequence $\{[W_{X_{\mathrm{DSE}}(\frac{n}{n+1}g)}]\}_{n\geq 1}$ of unlabeled Feynman graphons is convergent to $[W_{X_{\mathrm{DSE}}}]$.

We suppose $g = 1$. For any fixed $n, m \geq 1$, set $\alpha_{nm} := (\frac{n}{n+1}g)^m < 1$. Feynman graphons $W_{\alpha_{nm}X_m}$ can be defined in terms of the Feynman graphon W_{X_m}. To show this fact we only need to consider the Feynman graphon $V_{nm} := \alpha_{nm}W_{X_m} + (1-\alpha_{nm})W_{X_m}$. This Feynman graphon can be defined in terms of dividing the unital interval into two parts

$$I_1^{nm} := [0, \alpha_{nm}), \;\; I_2^{nm} := [\alpha_{nm}, 1] \tag{2.41}$$

such that $V_{nm} = W_{X_m}$ on I_1^{nm} and it is the zero value on I_2^{nm}. As we can observe whenever n tends to infinity, the Lebesgue measure of the interval I_2^{nm} goes to zero which means that $V_{nm} = W_{X_m}$ almost everywhere. In other words, $W_{\alpha_{nm}X_m}$ is weakly isomorphic to W_{X_m} whenever n tends to infinity.

For $g \geq 1$, real values α_{nm} enable us to define partitions on some closed interval $[a, b]$ where we need only to renormalize our Feynman graphon models to embed them inside the unital interval.

In addition, we can change the base measure of our Feynman graphon model from Lebesgue measure to Gauss measure and apply continued fractions $\eta_{nm} : x \mapsto (\frac{n+1}{n})^m x$ as Gauss measure preserving transformations to show that for any fixed $n \geq 1$, Feynman graphons $W_{(\frac{n}{n+1})^m X_m}$ and W_{X_m} are weakly isomorphic.

Therefore for each $m \geq 1$, the Feynman graphons $W_{\alpha_{nm}Y_m}$ and W_{Y_m} corresponding to the partial sum Y_m under different running couplings α_{nm} are weakly isomorphic. In addition, thanks to Proposition 4.6 in [34] we know that the sequence $\{Y_m\}_{m\geq 0}$ is cut-distant convergent to the infinite graph X_{DSE}. Therefore the sequence $\{W_{\alpha_{nm}Y_m}\}_{n\geq 1}$ is cut-distance convergent to the Feynman graphon $W_{X_{\mathrm{DSE}}}$. We can lift this process to unlabeled classes of Feynman graphons. □

Corollary 2.16. *The complexity of each Dyson–Schwinger equation $\mathrm{DSE}(g)$ in a given gauge field theory Φ with strong bare coupling constant g can be described in terms of* *the complexity of a sequence of large Feynman diagrams under weaker running coupling constants.*

Proof. This is a direct result of Lemma 2.11 and Corollary 2.15. □

3. Kolmogorov complexity

Intermediate algorithms enable us to analyze the construction of anytime algorithms by computing an initial potentially highly suboptimal solution and then improve the computed suboptimal solution as time allows [41, 42]. For example, at this level, we can see Dyson–Schwinger equations in a given (strongly coupled) gauge field theory Φ as generators of intermediate algorithms derived from Lie subgroups of the complex infinite dimensional Lie group $\mathbb{G}_\Phi$. In this setting, the Kolmogorov complexity can be useful to determine the length of the shortest intermediate algorithm which produces an object as the output.

For a given collection of alphabets and a given partial recursive function f on the collection of all possible strings generated by those alphabets, a description of a string τ_1 is defined via a string τ_2 which satisfies $f(\tau_2) = \tau_1$. The Kolmogorov complexity K_f with respect to f is defined by

$$K_f(\tau_1) := \begin{cases} \min\{|\tau_2| : f(\tau_2) = \tau_1\} \\ \infty, \quad \text{otherwise} \end{cases}. \tag{3.1}$$

The initial idea of Turing machines (among other computing models in Theoretical Computer Science) is to find an abstract model for the computation of any calculable decimal number. Generally speaking, a Turing machine can be defined in terms of a finite set of states, a finite set of symbols which contains the blanck, an input vocabulary collection built from symbols, an initial state, a transition function and a set of final states. The standard model of Turing machines can accept an input or reject it or fail into some loops without any final state. The collection of accepted inputs makes a language for a given Turing machine. The Church–Turing Thesis discussed that each problem computed with other computing modelings can be also computed via a model of Turing machines. In this setting, a system is called Turing complete if it can compute all possible computable functions. The (un-)decidability of a problem can be considered in terms of the search for a suitable Turing machine. For example, recognizing prime numbers is decidable while the Halting problem in Turing machines, which aims to determine whether a machine halt on a given input or not, is undecidable. Functional reduction is a standard approach to reduce a problem to another which might leads us to understand its (un-)decidability. However the Halting problem is unsolvable which means that there is no Turing machine which determines whether any arbitrary Turing machine halts or not. [1, 29]

In Complexity Theory it is shown the existence of the Turing machine $\mathbb{T}$ such that for all partial computable functions f, we can determine an intermediate algorithm p (as the collection of all programs that express it) such that for all y, $\mathbb{T}(p, y) = f(y)$. The notion of complexity can be defined in terms of the choice of the universal Turing machine. The difference in complexities under two universal Turing machines is actually upper bounded in terms of a constant. This constant depends on the choice of that pair of universal machines. It is also shown that for all n, there exists some v with $|v| = n$ such that $K_{\mathbb{T}}(v) \geq n$. Such v is called the Kolmogorov random. [1, 21, 37]

Each Quantum Field Theory can be encoded by a finite set of different types of elementary particles which contribute to the physical system and a finite set of different types of interactions among those particles. For example, in QCD we have three different types of particles (i.e. quark, gluon, ghost) and five types of interactions among them. For any given (strongly coupled) gauge field theory, if we change the scale of the bare coupling $g \mapsto \lambda g$, then we can replace each equation $\mathrm{DSE}(g)$ (derived from Green's functions (2.6)) with a continuum family of Dyson–Schwinger equations $\mathrm{DSE}(\lambda g)$ as rescaled versions of the original equation $\mathrm{DSE}(g)$ such that $0 < \lambda \leq 1$. This continuum family might be useful for building a new continuum approach to non-perturbative Quantum Field Theory in the context of Feynman graphon models. However the density of rational numbers in $\mathbb{R}$ is useful for us to replace the continuous rescaling $g \mapsto \lambda g$ with the sequence $\{g \mapsto \alpha_n g\}_{n\geq 1}$ of rational re-scalings with respect to any arbitrary strictly sequence $\{\alpha_n\}_{n\geq 1}$ of rational numbers which converges to λ. As it was shown in the proof of Corollary 2.15, the Feynman graphon classes associated to the solutions $X_{\mathrm{DSE}(\alpha_n g)}$ are weakly isomorphic to the Feynman graphon $W_{X_{\mathrm{DSE}(g)}}$ whenever n tends to infinity. Therefore up to the weakly isomorphic relation on Feynman graphons and Dyson–Schwinger equations (i.e. Definition 2.3 and Remark 2.7), we can consider $\mathcal{S}^{\Phi,g}$ as a countable infinite collection of classes of Dyson–Schwinger equations which can encode all Dyson–Schwinger equations under different running couplings generated by rescaling process of the bare coupling constant g. These classes of fixed point equations of Green's functions can encode all infinite number of interactions which might exist among elementary particles of the physical system under different rescaling of the bare coupling constant.

Definition 3.1. In Complexity Theory, an infinite constructive world is a countable set equipped with a class of structural numberings (by sequences of bits) which make it computable. Computable functions provide natural maps between constructive worlds.

For example, programs (i.e. sequences of bits) make a constructive world where the conditional complexity can be defined as the minimal length of a program p which computes x in terms of the initial information y. In other words,

$$K_A(x|y) := \mathrm{Min}_{A(p,y)=x}\, l(p) \tag{3.2}$$

such that A is a way of programming. The unconditional complexity can be defined with respect to the initial information y_0, namely,

$$K_A(x) = K_A(x|y_0). \tag{3.3}$$

In this setting, the logarithmic Kolmogorov complexity of x is actually the length of the shortest program which can generate x where the way of programming A is called the optimal Kolmogorov numbering. The Kolmogorov complexity, which is not computable, can be described as the lower bound of a sequence of computable functions. The Kolmogorov order, which is also non-computable, enables us to arrange objects of the constructive world in terms of the increasing order of their Kolmogorov complexities. [1, 21, 22, 23, 37]

Thanks to the Feynman graphon models of Dyson–Schwinger equations (given by Theorem 2.6 and Corollary 2.15), the multi-scale Renormalization Group given by Theorem 2.13, and the graph complexities of solutions of Dyson–Schwinger equations (given by Lemma 2.11, Corollary 2.16) in terms of the number of spanning trees (or forests) in their corresponding infinite formal expansions of Feynman diagrams, in this part we plan to work with $\mathcal{S}^{\Phi,g}$ as a new constructive world.

Lemma 3.2. *There exist structural numberings for the collection $\mathcal{S}^{\Phi,g}$.*

Proof. Consider the finite ordered family $G(\Phi) := \{G^{z_t} : z_t \in \{e_i, v_j\}\}$ of all different types of Green's functions (i.e. relations (2.6)) in the given physical theory Φ. We use the notation DSE_{z_t} for any Dyson–Schwinger equation generated by fixed point equations of the Green's function G^{z_t}. Up to the weakly isomorphic relation between Dyson–Schwinger equations (i.e. Remark 2.7), each Green's function G^{z_t} can generate an infinite countable family of Dyson–Schwinger equations under different running coupling constants. Thanks to Theorem 2.13, for any equation DSE_{z_t} of these equations and any given strictly increasing sequence $\{\alpha_n\}_{n\geq 1}$ of rational numbers which converges to g, we can define the map

$$n \mapsto R^{\mathrm{b}}_{\alpha_n g\ g}\mathrm{DSE}_{z_t}. \tag{3.4}$$

Up to the weakly isomorphic relation, this map provides a structural numbering for the collection $\mathcal{S}^{\Phi,g}_{z_t}$ of all Dyson–Schwinger equations generated by the Green's function G^{z_t}

under different running couplings. Now it is enough to lift this structural numbering onto the collection $\mathcal{S}^{\Phi,g} := \sqcup_{z_t} \mathcal{S}^{\Phi,g}_{z_t}$. □

Definition 3.3. Consider the constructive world $\mathcal{S}^{\Phi,g}$ with respect to the bare coupling constant g. For a given strictly increasing sequence $\{\alpha_n\}_{n\geq 1}$ of rational numbers which converges to g for $g \leq 1$ (or converges to $g - \lfloor g \rfloor$ for $g > 1$), define

$$u^g_{\{\alpha_n\}_{n\geq 1}} : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}, \tag{3.5}$$

$$(n, \mathrm{DSE}) \longmapsto R^{\mathrm{multi}}_{(\alpha_n g, \alpha_n \Lambda_g)\ (g,\Lambda_g)} \mathrm{DSE}.$$

Thanks to Definition 2.12 and Theorem 2.13, the equation $R^{\mathrm{multi}}_{(\alpha_n g, \alpha_n \Lambda_g)\ (g,\Lambda_g)} \mathrm{DSE}$ is the unique effective Dyson–Schwinger equation at the new scale $\alpha_n g$ of the bare coupling constant in the effective Lagrangian $L^{\Phi}_{\alpha_n \Lambda_g}(\alpha_n g)$ corresponding to the original equation DSE. For each n, the partial recursive function $u^g_{\{\alpha_n\}_{n\geq 1}}$ changes the scale $g \mapsto \alpha_n g$ of the bare coupling constant g where the scale of running couplings automatically will be changed $\Lambda_g \mapsto \alpha_n \Lambda_g$ in terms of the new rescaled bare coupling.

Definition 3.4. The Kolmogorov complexity of an equation $\mathrm{DSE}(\lambda g) \in \mathcal{S}^{\Phi,g}$ at the scale λg (of the bare coupling constant g) with respect to the partial recursive function $u^g_{\{\alpha_n\}_{n\geq 1}}$ is given by

$$K_{u^g_{\{\alpha_n\}_{n\geq 1}}}(\mathrm{DSE}(\lambda g)) := \tag{3.6}$$

$$\mathrm{Min}\{n \in \mathbb{Z}_+ :\ \ u^g_{\{\alpha_n\}_{n\geq 1}}(n, \mathrm{DSE}'(\Lambda_g)) \subseteq \mathrm{DSE}(\lambda g)\}.$$

The inclusion $\subseteq$ in the above relation in Definition 3.4 means that the unique solution of the effective Dyson–Schwinger equation $R^{\mathrm{multi}}_{(\alpha_n g, \alpha_n \Lambda_g)\ (g,\Lambda_g)} \mathrm{DSE}'$, which is an infinite formal expansion of Feynman diagrams, can be embedded as a Feynman subgraph into the large Feynman diagram $X_{\mathrm{DSE}(\lambda g)}$. Therefore it should have less complexity than the original graph. In addition, it is important to note that the sequence $\{\alpha_n\}_{n\geq 1}$ determines a way of programming in our definition of the Kolmogorov complexity. By changing the scales of the bare coupling constant via the components α_n, the scale of running couplings will be also changed. Therefore our generalization of the Kolmogorov complexity can check the complexities of Dyson–Schwinger equations under different scales of running couplings with respect to the initial rescaling of the bare coupling constant.

Lemma 3.5. *The partial recursive functions given by Definition 3.3 determines the Kolmogorov total order on $\mathcal{S}^{\Phi,g}$.*

Proof. Thanks to Definition 3.4, the Kolmogorov total order $\prec$ is defined in a way to arrange all Dyson–Schwinger equations in the physical theory Φ under an increasing order with respect to their graph complexities. In other words,

$$\mathrm{DSE}_1(\lambda_1 g) \prec \mathrm{DSE}_2(\lambda_2 g) \Longleftrightarrow \tag{3.7}$$

$$K_{u^g_{\{\alpha_n\}_{n\geq 1}}}(\mathrm{DSE}_1(\lambda_1 g)) \leq K_{u^g_{\{\beta_n\}_{n\geq 1}}}(\mathrm{DSE}_2(\lambda_2 g))$$

such that $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are strictly increasing sequences of rational numbers which converge to $\lambda_1 g, \lambda_2 g$, respectively. The total order (3.7) can provide the bijection $\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}$ between the constructive world $\mathcal{S}^{\Phi,g}$ and $\mathbb{Z}_+$. □

Thank to Lemma 3.5 now we have a machinery to determine less complicated Dyson–Schwinger equations in terms of changing the scales of the bare and running couplings.

The Invariance Theorem in the Theory of Computation ([21, 22]) tells us that for every transducer description v and any word w, the complexity of w does not exceed the sum of $|v|$ and the complexity of w with respect to the transducer described by v. A general version of this fundamental fact has been discussed in [15].

Corollary 3.6. *For a given rescaling $g \mapsto \lambda g$ of the bare coupling constant g with the corresponding sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$, the Invariance Theorem guarantees the existence of some constant $c_0 > 0$ such that for all Dyson–Schwinger equations $\mathrm{DSE}(\lambda g)$ in $\mathcal{S}^{\Phi,g}$, we have*

$$c_0 K_{u^g_{\{\alpha_n\}_{n\geq 1}}}(\mathrm{DSE}(\lambda g)) \leq$$

$$\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(\mathrm{DSE}(\lambda g)) \leq K_{u^g_{\{\alpha_n\}_{n\geq 1}}}(\mathrm{DSE}(\lambda g)). \tag{3.8}$$

This statement is valid also for the sequence $\{\beta_n\}_{n\geq 1}$.

Lemma 3.5 allows us to study $\mathcal{S}^{\Phi,g}$ as a poset such that any given partial recursive map $\sigma : \mathcal{S}^{\Phi,g} \to \mathcal{S}^{\Phi,g}$ can generate a permutation on all Dyson–Schwinger equations as objects of our constructive world. We can define a new map

$$\sigma_{\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}} := \mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}} \circ \sigma \circ \mathbf{K}^{-1}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}} \tag{3.9}$$

with the corresponding permutation

$$\mathrm{D}(\sigma_{\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}}) := \mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(\mathrm{Dom}(\sigma)) \subseteq \mathbb{Z}_+ \tag{3.10}$$

with respect to the rescaling $g \mapsto \lambda_1 g$ via the sequence $\{\alpha_n\}_{n\geq 1}$ and the rescaling $g \mapsto \lambda_2 g$ via the sequence $\{\beta_n\}_{n\geq 1}$.

Remark 3.7. For any rescaling $g \mapsto \lambda g$ via the sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$, set

$$\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(\mathrm{DSE}(\lambda g)) := k^{\lambda}_{\mathrm{DSE}}. \tag{3.11}$$

There exists some constant c_λ such that for each $m \geq 1$,

$$\begin{aligned} \sigma^m_{\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}}(k^{\lambda}_{\mathrm{DSE}}) &= \mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(\sigma^m(\mathrm{DSE}(\lambda g))) \\ &\leq c_\lambda \mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(m). \end{aligned} \tag{3.12}$$

In Complexity Theory, it is shown that for any partial recursive function $f : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ and any $x \in \mathrm{Dom}(f)$ there exist some constants such that

$$K(f(x)) \leq c_f K(x) \leq c'_f x, \tag{3.13}$$

[1, 21].

We can lift this inequality onto the level of the constructive world $\mathcal{S}^{\Phi,g}$ and the Kolmogorov complexity $\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}$ of Dyson–Schwinger equations of the physical theory under ruunning couplings.

Corollary 3.8. *The Kolmogorov complexity*

$$\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}$$

determines some upper and lower boundaries for the permutation $\sigma_{\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}}$ given by the formula (3.9) with respect to any given partial recursive map σ on $\mathcal{S}^{\Phi,g}$.

Proof. The collection $\{\sigma^n(\mathrm{DSE}(\lambda g)) : \quad n \in \mathbb{Z}_+\}$ determines a recursively enumerable subset of $\mathcal{S}^{\Phi,g}$ which can be used as the domain for a new partial recursive function $A : \mathcal{S}^{\Phi,g} \to \mathbb{Z}_+$ given by

$$A(\mathrm{DSE}(\tau g)) = n, \quad \text{if } \sigma^n(\mathrm{DSE}(\lambda g)) = \mathrm{DSE}(\tau g). \tag{3.14}$$

Thanks to the inequality (3.13), there exists some constant $c > 0$ such that

$$\begin{aligned} \mathbf{K}^{-1}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(n) &= \mathbf{K}^{-1}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(A(\mathrm{DSE}(\tau g))) \\ &\leq c\, \mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(\mathrm{DSE}(\tau g)) \qquad (3.15) \\ &= c\, \mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(\sigma^n(\mathrm{DSE}(\lambda g))). \end{aligned}$$

Thanks to Corollary 3.6 and the inequality (3.15), for each rescaling $g \mapsto \lambda g$ there exist constants $c_1(\lambda), c_2(\lambda)$ such that

$$\begin{aligned} & c_1(\lambda)\, \mathbf{K}^{-1}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(n) \\ \leq\ & \sigma^n_{\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}}(k^\lambda_{\mathrm{DSE}}) \leq c_2(\lambda)\, \mathbf{K}^{-1}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}(n). \qquad (3.16) \end{aligned}$$

□

Manin formulated a fundamental Hopf algebra of an enriched programming method on isomorphism classes of certain descriptions in $P(\mathbb{Z}_+, \mathbb{Z}_+)$ as an object of the Constructive Universe Category. The Constructive Universe Category is a (bi)monoidal category closed with respect to the direct product and disjoint union as monoidal structures. One interesting subject is that some characters of this Hopf algebra can encode the Halting problem. Manin modified the Connes–Kreimer perturbative renormalization approach to the BPHZ formalism for the level of this Hopf algebra. The Birkhoff–Hopf factorizations of those characters of the Manin's Hopf algebra which correspond to the Halting problem at a point $k \in \mathbb{Z}_+$ enable us to measure the amount of computability of partial recursive functions in $P(\mathbb{Z}_+, \mathbb{Z}_+)$. In this setting, whether a positive integer k belongs to the domain of a given partial recursive function or not can be transformed to whether an analytic complex function, which depends on k and the chosen partial recursive function, has a singularity at $z = 1$. [24, 25]

In the rest of this part, we are going to lift the Manin approach to the Halting problem onto the level of the constructive world $\mathcal{S}^{\Phi,g}$. This study shows the application of the Kolmogorov complexity $\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\beta_n\}_{n\geq 1}}}$ in the formulation of those characters on the Hopf algebra $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ of Feynman graphons (i.e. Theorem 2.5) which encode the Halting problem at each Dyson–Schwinger equation as an object in the constructive world $\mathcal{S}^{\Phi,g}$.

Lemma 3.9. *The Halting problem for any partial recursive function $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}$ can be described via fixed points of a class of permutations on Dyson–Schwinger equations.*

Proof. Thanks to Feynman graphon models of Feynman diagrams and Dyson–Schwinger equations (i.e. Theorems 2.5, 2.6 and the proof of Corollary 2.15), we can define the linear combinations of Dyson–Schwinger equations in terms of the linear combinations of their corresponding Feynman graphons. It is only enough to recognize each Dyson–Schwinger equation DSE (as an object of $\mathcal{S}^{\Phi,g}$) via its corresponding Feynman graphon $W_{X_{\mathrm{DSE}}}$. It allows us to equip $\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}$ with a total recursive structure of the additive group without torsion with the zero element $\mathbf{0}$. For Dyson–Schwinger equations $\mathrm{DSE}_1, \mathrm{DSE}_2$, define $\mathrm{DSE}_1 \oplus \mathrm{DSE}_2 := \mathrm{DSE}_{12}$, such that DSE_{12} is an object in $\mathcal{S}^{\Phi,g}$ with the corresponding

Feynman graphon model $W_{X_{\mathrm{DSE}_1}+X_{\mathrm{DSE}_2}}$. This Feynman graphon is identified as the cut-distance convergent limit of the sequence $\{W_{Y_m^{(1)}+Y_m^{(2)}}\}_{m\geq 1}$ of Feynman graphons of the combinations of partial sums.

In other words, thanks to Definition 2.1, if the equation DSE_1 is generated by the family $\{\gamma_n^{(1)}\}_{n\geq 1}$ and the equation DSE_2 is generated by the family $\{\gamma_n^{(2)}\}_{n\geq 1}$, then DSE_{12} is the Dyson–Schwinger equation generated by the disjoint union $\{\gamma_n^{(1)}\}_{n\geq 1} \sqcup \{\gamma_n^{(2)}\}_{n\geq 1}$ of primitive (1PI) Feynman diagrams.

We extend f to a new function $g_f : \mathbb{Z}_+ \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \to (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\})$ such that

$$g_f(n, \mathrm{DSE}(\lambda g)) := \mathbf{0}, \ \text{ if } \ (n, \mathrm{DSE}(\lambda g)) \notin \mathrm{Dom}(f). \tag{3.17}$$

Now define a new permutation τ_f from $\mathbb{Z}_+ \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\})$ to $\mathbb{Z}_+ \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\})$,

$$\tau_f(n, \mathrm{DSE}_1(\lambda g), \mathrm{DSE}_2(\tau g)) := \tag{3.18}$$

$$(g_f(n, \mathbf{0}), \mathrm{DSE}_1(\lambda g) + g_f(n, \mathrm{DSE}_2(\tau g)), \mathrm{DSE}_2(\tau g)).$$

$(\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}, \oplus)$ has no torsion which informs the correspondence between finite orbits of τ_f and fixed points of this permutation. Therefore we can build a new partial recursive permutation σ_f with the domain

$$\mathrm{Dom}(\sigma_f) := (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \times \mathrm{Dom}(f). \tag{3.19}$$

On the one hand, the Halting problem for the partial recursive function f is to recognize whether a selected pair $(k, \mathrm{DSE}(\lambda g))$ belongs to the domain of f or not. This problem can be determined in terms of the fixed points of the permutation τ_f. On the other hand, thanks to [24, 25] and the definition of g_f (i.e. Formula (3.17)), we can show that the complement to $\mathrm{Dom}(\sigma_f)$ in the constructive world $(\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\})$ covers the fixed points of the permutation τ_f. □

Corollary 3.10. *- In the constructive world $\mathcal{S}^{\Phi,g}$, for any arbitrary rescaling $g \mapsto \lambda g$ with the corresponding sequence $\{\alpha_n\}_{n\geq 1}$, if the σ-orbit of the equation $\mathrm{DSE}(\lambda g) \in \mathrm{Dom}(\sigma)$ is finite, then*

$$\Psi(k^{\lambda}_{\mathrm{DSE}}, \sigma, u^g_{\{\alpha_n\}_{n\geq 1}}, z) :=$$

$$\frac{1}{(k^{\lambda}_{\mathrm{DSE}})^2} + \sum_{n\geq 1} \frac{z^{\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\alpha_n\}_{n\geq 1}}}(\mathrm{DSE}(\alpha_n g))}}{(\sigma^n_{\mathbf{K}_{u^g_{\{\alpha_n\}_{n\geq 1}}, u^g_{\{\alpha_n\}_{n\geq 1}}}}(k^{\lambda}_{\mathrm{DSE}}))^2}$$

is a rational function in the complex variable z. All poles of this formal series, which are of the first order, live at roots of unity.

- If the σ-orbit of the equation $\mathrm{DSE}(\lambda g)$ is infinite, then the function $\Psi(k^{\lambda}_{\mathrm{DSE}}, \sigma, u^g_{\{\alpha_n\}_{n\geq 1}}, z)$ is the Taylor series of an analytic function on the region $|z| < 1$ which is continuous at the boundary of this region.

Proof. It is enough to apply Proposition 4 in [26] to the constructive world $\mathcal{S}^{\Phi,g}$, the partial recursive function $u^g_{\{\alpha_n\}_{n\geq 1}}$ (given by Definition 3.3) and the value $k^{\lambda}_{\mathrm{DSE}}$ (defined in Remark 3.7). □

We can reduce a given partial recursive map $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \to \mathcal{S}^{\Phi,g}$ to the partial recursive permutation $\sigma_f : \mathrm{Dom}(\sigma_f) \subset \mathcal{S}^{\Phi,g} \to \mathrm{Dom}(\sigma_f) \subset \mathcal{S}^{\Phi,g}$. The Halting problem for the constructive world $\mathcal{S}^{\Phi,g}$ can be described as the problem of recognizing whether a positive integer number k (as the rescaling parameter for the bare coupling constant g) belongs to the domain $\mathrm{Dom}(\sigma_f)$ or not. Now thanks to Lemma 3.9 and Corollary 3.10, it is possible to deform the Halting problem to the problem of whether the corresponding analytic function $\Psi(k, \sigma_f, u^g_{\{\alpha_n\}_{n\geq 1}}, z)$ of a complex parameter z has a pole at $z = 1$ or not.

Corollary 3.11. *The BPHZ renormalization of Feynman graphons encodes the Halting problem for a given partial recursive map* $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}$.

Proof. Feynman graphon models of Dyson–Schwinger equations allow us to embed the constructive world $\mathcal{S}^{\Phi,g}$ into the Hopf algebra $\mathcal{S}^{\Phi}_{\text{graphon}}$ generated by all Feynman graphons (i.e. Theorem 2.5). The BPHZ renormalization of large Feynman diagrams has been considered in [34, 35]. Consider the character $\varphi_k : \mathcal{S}^{\Phi}_{\text{graphon}} \to A_{\text{dr}}$ which sends each Feynman graphon class $[W_{X_{\text{DSE}}}]$ to the value $\Psi(k^{\lambda}_{\text{DSE}}, \sigma_f, u^g_{\{\alpha_n\}_{n\geq 1}}, z)$. Thanks to the Birkhoff factorization on the regularization algebra, we have $A_{\text{dr}} = \mathcal{A}_+ \oplus \mathcal{A}_-$ such that $\mathcal{A}_+$ is the unital algebra of analytic functions in the region $|z| < 1$ which are continuous on the boundary $|z| = 1$ and $\mathcal{A}_- := (1-z)^{-1}\mathbb{C}[(1-z)^{-1}]$.

Now if we apply Theorem 2.6, Lemma 3.9 and Corollary 3.10, then the question about the existence of a pole at $z = 1$ for the analytic function Ψ can be described as the question of whether k^{λ}_{DSE} belongs to $\text{Dom}(\sigma_f)$ or not. □

4. Conclusion

Our main task in this work is to provide a new theoretical model for the description of the computational complexities of Dyson–Schwinger equations under different running coupling constants.

(i) We have shown the existence of a new class of partial recursive functions such as $u^g_{\{\alpha_n\}_{n\geq 1}}$ (i.e. Definition 3.3) with respect to rescaling of the strong bare coupling constant g and the multi-scale Renormalization Group given by Theorem 2.13. These partial recursive functions are applied to build a generalization of the Kolmogorov complexity (i.e. Definition 3.4). This new complexity is capable to identify an order on the collection of all Dyson–Schwinger equations under different running coupling constants. The dynamics of the well-defined map $u^g_{\{\alpha_n\}_{n\geq 1}}$ can be studied by the multi-scale Renormalization Group (i.e. Theorem 2.13) on the constructive world $\mathcal{S}^{\Phi,g}$.

(ii) We can define the Kolmogorov complexity K_w for Dyson–Schwinger equations with respect to other functions w of the set of Kolmogorov optimal functions. The concept of "optimal" means that for any partial recursive function $v : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \to \mathcal{S}^{\Phi,g}$ there exists a constant $c_{v,w} > 0$ such that for each Dyson–Schwinger equation $\text{DSE}(\lambda g)$,

$$K_w((n, \text{DSE}(\lambda g))) \leq c_{v,w} K_v((n, \text{DSE}(\lambda g))). \tag{4.1}$$

(iii) Thanks to Theorem 2.13, Corollary 3.11, which determines the amount of non-computability via the Halting problem at the level of Feynman graphons, the Kolmogorov total order given by Lemma 3.5 and Corollary 3.8, those partial recursive functions derived from R^{multi}_{--} namely,

$$v^g : (n, \text{DSE}) \longmapsto R^{\text{multi}}_{(\lambda\alpha(n)g, \lambda\Lambda_{\alpha(n)g})\ (\alpha(n)g, \Lambda_{\alpha(n)g})}\text{DSE} \tag{4.2}$$

can be considered as the truth candidates for this optimality such that α is a bijection on $\mathbb{Z}_+$ which can change the scale of the bare coupling constant.

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