Supplemental Material: Higgs mode stabilization by photo-induced long-range interactions in a superconductor

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Cooper's problem. The wavefunction of the Cooper pair has the form [S1]

$$|\Phi\rangle = \sum_{0<\xi_{\mathbf{k}}<\epsilon_{\mathrm{D}}} g(\mathbf{k}) c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} |\mathrm{FS}\rangle \,, \tag{S1}$$

where $|\mathrm{FS}\rangle$ represents the Fermi sea at $T = 0$K. By substituting this into the full system Hamiltonian ($H_{\mathrm{mat}} + H_{\mathrm{long}}$) and taking the $q_0 \to 0$ limit, we find that $g(\mathbf{k})$ satisfies

$$Eg(\mathbf{k}) = 2\xi_{\mathbf{k}} g(\mathbf{k}) - \sum_{\mathbf{k}'} (\tilde{V} + 2\tilde{U}\delta_{\mathbf{k},\mathbf{k}'}) g(\mathbf{k}'), \tag{S2}$$

where E is the two-electron state energy and $\tilde{V} = V/\mathcal{S}$. We rearrange the terms to get

$$\left(E - 2\xi_{\mathbf{k}} + 2\tilde{U}\right) g(\mathbf{k}) = \tilde{V} \sum_{\mathbf{k}'} g(\mathbf{k}'), \tag{S3}$$

where the RHS is independent from $\mathbf{k}$. For the s-wave two-electron state, the RHS is nonzero. We follow the usual procedure of cancelling out common factors from both sides and perform the integration in $\mathbf{k}$-space [S1] to obtain the s-wave binding energy

$$E - 2\xi_{\mathbf{k}} \approx -2\tilde{U} - 2\epsilon_{\mathrm{D}} \mathrm{e}^{\frac{-2}{N(0)V}}. \tag{S4}$$

For the two-electron states with higher angular momenta, the RHS is zero as the intrinsic attraction has no component in any higher angular momentum channels. Therefore, the binding energy in a higher momentum channel is $-2\tilde{U}$. These results are as shown in Fig. 2(c) of the main text.

Anderson's pseudospin formalism. We study the BCS reduced Hamiltonian and investigate the collective modes in BCS superconductors at $T = 0$K. This problem is more conveniently represented by mapping the electron operators to pseudospin operators [S2, S3],

$$S^z_{\mathbf{k}} = (c^{\dagger}_{\mathbf{k},\uparrow} c_{\mathbf{k},\uparrow} + c^{\dagger}_{-\mathbf{k},\downarrow} c_{-\mathbf{k},\downarrow} - 1)/2 \tag{S5}$$

$$S^+_{\mathbf{k}} = c^{\dagger}_{\mathbf{k},\uparrow} c^{\dagger}_{-\mathbf{k},\downarrow} \tag{S6}$$

$$S^-_{\mathbf{k}} = c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow}. \tag{S7}$$

The BCS MF Hamiltonian in this representation is

$$H_{\mathrm{MF}} = -\sum_{\mathbf{k}} \mathbf{B}_{\mathbf{k}} \cdot \mathbf{S}_{\mathbf{k}}, \text{ where} \tag{S8}$$

$$\mathbf{B}_{\mathbf{k}} = -2\xi_{\mathbf{k}} \mathbf{e}_z + 2\sum_{\mathbf{k}'} (\tilde{V} + 2\tilde{U}\delta_{\mathbf{k},\mathbf{k}'}) \langle \mathbf{S}^{\perp}_{\mathbf{k}'} \rangle, \tag{S9}$$

and $\mathbf{S}^{\perp}_{\mathbf{k}}$ refers to the component of $\mathbf{S}_{\mathbf{k}}$ in the xy-plane. For the ground state,

$$\mathbf{B}^0_{\mathbf{k}} = -2\xi_{\mathbf{k}} \mathbf{e}_z + 2\Delta_{\mathbf{k}} \mathbf{e}_x. \tag{S10}$$

Here, we have assumed that $\mathbf{B}^0_{\mathbf{k}}$ lies in the xz-plane without loss of generality. We then go beyond the ground state and consider perturbations $\delta\mathbf{S}_{\mathbf{k}} = \mathbf{S}_{\mathbf{k}} - \mathbf{S}^0_{\mathbf{k}}$ on top of the MF ground state. Here, $\mathbf{S}^0_{\mathbf{k}}$ is the expectation value of $\mathbf{S}_{\mathbf{k}}$ in the MF ground state. From Eq. (S8), we obtain the equation of motion for $\mathbf{S}_{\mathbf{k}}$,

$$\frac{\mathrm{d}\mathbf{S}_{\mathbf{k}}}{\mathrm{d}t} = \mathbf{S}_{\mathbf{k}} \times \mathbf{B}_{\mathbf{k}}. \tag{S11}$$

We linearise it to get the equation of motion for $\delta\mathbf{S}_{\mathbf{k}}$,

$$\frac{\mathrm{d}\delta\mathbf{S}_{\mathbf{k}}}{\mathrm{d}t} = \delta\mathbf{S}_{\mathbf{k}} \times \mathbf{B}^0_{\mathbf{k}} + \mathbf{S}_{\mathbf{k}} \times \delta\mathbf{B}_{\mathbf{k}}, \tag{S12}$$

where $\delta\mathbf{B}_{\mathbf{k}} = \mathbf{B}_{\mathbf{k}} - \mathbf{B}^0_{\mathbf{k}}$. We re-formulate this as an eigenvalue problem:

$$\begin{aligned}
&\omega^2 \phi_{\mathbf{k}} \\
&= \sum_{\mathbf{k}'} \mathcal{M}_{\mathbf{k}\mathbf{k}'} \phi_{\mathbf{k}'} \\
&= \left(B^0_{\mathbf{k}}\right)^2 \phi_{\mathbf{k}} - B^0_{\mathbf{k}} \sum_{\mathbf{k}'} (\tilde{V} + 2\tilde{U}\delta_{\mathbf{k},\mathbf{k}'}) \phi_{\mathbf{k}'} \\
&- \cos\theta_{\mathbf{k}} \sum_{\mathbf{k}'} (\tilde{V} + 2\tilde{U}\delta_{\mathbf{k},\mathbf{k}'}) B^0_{\mathbf{k}'} \cos\theta_{\mathbf{k}'} \phi_{\mathbf{k}'} \\
&+ \cos\theta_{\mathbf{k}} \sum_{\mathbf{k}',\mathbf{k}''} (\tilde{V} + 2\tilde{U}\delta_{\mathbf{k},\mathbf{k}'})(\tilde{V} + 2\tilde{U}\delta_{\mathbf{k}',\mathbf{k}''}) \cos\theta_{\mathbf{k}'} \phi_{\mathbf{k}''},
\end{aligned} \tag{S13}$$

where $S^y_{\mathbf{k}} \propto \phi_{\mathbf{k}}$, $\tan\theta_{\mathbf{k}} = -\Delta_{\mathbf{k}}/\xi_{\mathbf{k}}$ and $B^0_{\mathbf{k}} = |\mathbf{B}^0_{\mathbf{k}}|$.

For excitonic modes the perturbations $\Phi^{\mathrm{exc}}_k = (..., \phi^{\mathrm{exc}}_{\mathbf{k}}, ...)$ are local in each energy shell. Φ^{exc}_k satisfies $\phi^{\mathrm{exc}}_{\mathbf{k}} = 0 \quad \forall |\mathbf{k}| \neq k$ and $\sum_{|\mathbf{k}|=k} \phi^{\mathrm{exc}}_{\mathbf{k}} = 0$. The energy of the excitonic modes index by $\mathbf{k}$ is

$$\varepsilon^{\mathrm{exc}}_{\mathbf{k}} = \sqrt{\left(B^0_{\mathbf{k}}\right)^2 - 2B^0_{\mathbf{k}}\tilde{U}\left(1 + \cos^2\theta_{\mathbf{k}}\right) + 4\tilde{U}^2\cos^2\theta_{\mathbf{k}}}. \tag{S14}$$

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The energy is lowest on the Fermi surface, $\varepsilon^{\rm exc}_{k_F} = 2\sqrt{\Delta^2_{k_F} - \tilde{U}\Delta_{k_F}}$. We emphasise that this energy is still higher than twice the superconducting gap without the long-range interactions.

The rest of the eigen-modes are rotationally symmetric in k-space. The Nambu-Goldstone mode appears as a zero-energy mode that is symmetric about the Fermi surface and satisfies $\phi^{\rm NG}_{\mathbf{k}} \propto \Delta_{\mathbf{k}}$. This condition confirms that the Nambu-Goldstone mode indeed corresponds to the phase fluctuation of the order parameter. Through the Anderson-Higgs mechanism, this mode is absorbed into the longitudinal component of the gauge field and appears as plasmon oscillations [S2].

The next lowest energy-mode is the zero-momentum Higgs mode of amplitude fluctuations. The mode function, $\Phi^{\rm H}$, is antisymmetric about the Fermi surface. We obtain its excitation energy numerically, using the antisymmetry of the mode function. We show analytically that this Higgs excitation energy is lower than the excitonic excitation energy by plugging an approximate mode function (inspired by the Higgs mode function in superconductors with only local electron attractions), $\phi^{\rm ap}_{\mathbf{k}} \propto \Delta_{\mathbf{k}}/[(E_{\mathbf{k}} - \tilde{U})\xi_{\mathbf{k}}]$ into Eq. (S13),

$$\sum_{\mathbf{k}'} \mathcal{M}_{\mathbf{k}\mathbf{k}'}\phi^{\rm ap}_{\mathbf{k}'} = \left(4\Delta^2_{\mathbf{k}} - 4\tilde{U}\frac{\Delta^2_{\mathbf{k}}}{E_{\mathbf{k}}}\right)\phi^{\rm ap}_{\mathbf{k}} \leq (\varepsilon^{\rm exc}_{k_F})^2\phi^{\rm ap}_{\mathbf{k}}. \tag{S15}$$

This shows that the true zero-momentum Higgs mode function results in an energy lower than $\varepsilon^{\rm exc}_{k_F}$ for finite attractive long-range interactions.

In Fig. 2(b) of the main text, we showed that $\varepsilon^{\rm H}$ is very close to $\varepsilon^{\rm exc}_{k_F}$. Here, we note that this difference grows with the strength of the long-range interactions. In addition, we note that the range of the induced interactions is, though long, still finite in reality. The finite range slightly weakens the pairing in higher angular momentum channels relative to the s-wave channel [S4], thus we should expect the separation between the Higgs mode and the excitonic modes to be marginally larger.

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