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Bryce Kerr

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BOUNDS OF MULTIPLICATIVE CHARACTER SUMS OVER SHIFTED PRIMES

BRYCE KERR

ABSTRACT. For integer q, let χ be a primitive multiplicative character mod q. For integer a coprime to q, we obtain a bound of the form

$$\left|\sum_{n\le N}\Lambda(n)\chi(n+a)\right|\le \frac{N}{q^{\delta}}, \quad N\ge q^{3/4+\varepsilon},$$

where $\Lambda(n)$ is the von Mangoldt function. This improves on a series of previous results.

1. Introduction

One of Vinogradov's fundamental contributions to mathematics is the method of bilinear forms. This can be viewed as some general framework to convert the problem of estimating an exponential sum over a set with arithmetic structure to bounding the norm of a linear operator. This method, combined with other techniques, allowed Vinogradov to make progress on famous problems including the zero free region of the Riemann zeta function, Waring's problem and the representation of a number as the sum of three primes. Since Vinogradov's work, the method of bilinear forms has played a central role in number theory.

Another one of the early applications of Vinogradov's method is the estimation of character sums over shifted primes. Let q be prime and χ a primitive character mod q. Consider the sum

$$S_a(q;N)=\sum_{n\le N}\Lambda(n)\chi(n+a), \tag{1}$$

where a is an integer relatively prime to q and as usual,

$$\Lambda(n)=\begin{cases}\log p, & \text{if } n \text{ is a power of a prime } p,\\ 0, & \text{otherwise,}\end{cases}$$

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is the von Mangoldt function. Vinogradov [15] showed an estimate of the form

$$|S_a(q;N)| \le \frac{N}{q^{\delta}}, \quad N \ge q^{3/4+\varepsilon}.$$

This was considered a surprising result since a direct application of the Riemann hypothesis for Dirichlet L-functions implies a nontrivial estimate only in the longer range $N \ge q^{1+\varepsilon}$. Vinogradov's estimate has been improved by Karatsuba [8] by using some ideas of Burgess and obtained an estimate of the form

$$|S_a(q;N)| \le \frac{N}{q^{\delta}}, \quad N \ge q^{1/2+\varepsilon}. \tag{2}$$

In this paper we consider the problem of estimating $S_a(q;N)$ for an arbitrary integer q. This first appears to be considered by Rakhmonov [9, 10] who has shown that nontrivial cancellations in the sums $S_a(q;N)$ occur in the range $N > q^{1+\varepsilon}$. This has been extended by Friedlander, Gong and Shparlinski [4] to $N > q^{8/9+\varepsilon}$, where the bound

$$|S_a(q;N)| \le (N^{7/8}q^{1/9} + N^{33/32}q^{-1/18})q^{o(1)}, \tag{3}$$

is given for $N \le q^{16/9}$. The current record on the longest range of parameters for which the sums $S_a(q;N)$ are estimated for general modulus is due to Rakhmonov [12] (see also [11] for sharper results in the case of cubefree modulus) who has shown

$$|S_a(q;N)| \ll N \exp\left(-0.6\sqrt{q}\right), \quad N \ge q^{5/6+\varepsilon}. \tag{4}$$

Note the much shorter range of summation in Karatsubas estimate (2) for prime modulus compared with (4). Estimating character sums for general composite modulus is typically much harder than prime modulus and this has something to do with the current state of knowledge regarding complete exponential sums over residue rings. For prime modulus one may appeal to the Weil bound to obtain square root cancellation for such sums. For arbitrary modulus it is possible to apply elementary methods to estimate complete sums, see for example [7, Chapter 12], although applying such techniques requires counting zeros of polynomials in residue rings which is often difficult to obtain good bounds.

In this paper, we overcome some of these difficulties and provide an estimate for the sums $S_a(q;N)$ in the range $N \ge q^{3/4+\varepsilon}$.

Theorem 1. *For $N \le q$, we have*

$$|S_a(q;N)| \le q^{1/9+o(1)}N^{23/27}.$$

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2. The Heath-Brown identity

The following identity is due to Heath-Brown [6] and will be used to transform sums over primes into bilinear sums.

Lemma 2. *For integers J, X, Z and n satisfying*

$$J \geq 1, \quad n < 2X \quad \text{and} \quad Z = X^{1/J},$$

we have

$$\Lambda(n) = -\sum_{j=1}^{J}(-1)^j \binom{J}{j} \sum_{\substack{m_i \leq Z \\ 1 \leq i \leq j}} \mu(m_1)\dots\mu(m_j) \sum_{\substack{n_1,\dots,n_j \\ m_1\dots m_j n_1 \dots n_j = n}} \log n_1.$$

Lemma 2 will be used to transform summation over primes to bilinear forms. Our next result allows some control over the lengths of summation in the resulting bilinear forms and is standard in applications of Heath-Brown's identity.

Lemma 3. *Let $j \geq 2$ be an integer and $M_1, \dots, M_j, N_1, \dots, N_j$ satisfy*

$$M_j \leq M_{j-1} \leq \dots \leq M_1, \quad N_j \leq N_{j-1} \leq \dots \leq N_1, \quad N \ll \prod_{i=1}^{j} M_i N_i \ll N,$$

and

$$M_1 \leq Z. \tag{5}$$

We have either

$$N_1 \geq \left(\frac{N}{Z}\right)^{1/2}, \tag{6}$$

or there exists subsets $\mathcal{I}, \mathcal{J} \subseteq \{1, \dots, j\}$ such that

$$Z \leq \prod_{i \in \mathcal{I}} M_i \prod_{i \in \mathcal{J}} N_i \leq \max\{Z^2, N^{1/3}\}. \tag{7}$$

Proof. Let $i_0 \geq 1$ be the smallest integer such that

$$\prod_{i \geq i_0} M_i \leq Z.$$

If $i_0 \neq 1$ then by (5)

$$Z \leq \prod_{i \geq i_0 - 1} M_i \leq Z^2,$$

which gives a partition of the form (7). Hence we may suppose $i_0 = 1$. Let $i_1 \geq 1$ be the smallest integer such that

$$\prod_{i \leq j} M_i \prod_{i \geq i_1} N_i \leq Z.$$

We may suppose if $i < i_1$ then

$$N_i \geq Z,$$

since otherwise

$$Z \leq \prod_{i \leq j} M_i \prod_{i \geq i_1 - 1} N_i \leq Z^2,$$

which gives a partition of the form (7). We next proceed on a case by case basis depending on the value of i_1. Suppose first that $i_1 \leq 3$. Then using

$$N \ll N_1 \dots N_j M_1 \dots M_j \ll N,$$

we have

$$\prod_{i < i_1} N_i \gg \frac{N}{Z},$$

which implies

$$N_1 \gg \frac{N^{1/2}}{Z^{1/2}}.$$

Suppose next that $i_1 \geq 4$. Then from

$$N_3^3 \leq N_1 N_2 N_3 \leq N,$$

we obtain that

$$Z \leq N_3 \leq N^{1/3},$$

which completes the proof. □

3. Pólya-Vinogradov Bound

We will use a variant of the Pólya-Vinogradov inequality with an arithmetic condition on summation which is [4, Lemma 5].

Lemma 4. *For any integers M, N, a with $(a,q)=1$ and any primitive character $\chi \pmod q$ we have*

$$\left|\sum_{\substack{M<n\le M+N \\ (n,q)=1}} \chi(n+a)\right| \le q^{1/2+o(1)} + Nq^{-1/2+o(1)}.$$

4. Burgess Bounds

In [4], the Burgess bound for the sums

$$\sum_{v_1,\ldots,v_{2r}=1}^{V} \left|\sum_{x=1}^{q} \chi\left(\prod_{i=1}^{r}(x+v_i)\right)\overline{\chi}\left(\prod_{i=r+1}^{2r}(x+v_i)\right)\right| \quad r=2,3,$$

was used to improve on Lemma 4 for small values of N. We a give further improvement by using the methods of Burgess [1, 2] to bound the sums

$$\sum_{v_1,\ldots,v_{2r}=1}^{V} \left|\sum_{x=1}^{q} \chi\left(\prod_{i=1}^{r}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=r+1}^{2r}(x+dv_i)\right)\right|, \quad r=2,3, \tag{8}$$

which will then be combined with techniques from [4] to obtain new bounds for sums of the form

$$\sum_{\substack{n\le N \\ (n,q)=1}} \chi(n+a).$$

4.1. **The case r=2.** The aim of this section is to improve on [4, Lemma 10], see Lemma 7 below. This result will not be used in the proof of Theorem 1 and may have independent interest. It is possible one may incoporate the results of this section to obtain quantitative improvements on Theorem 1.

We first recall a special case of [1, Lemma 7].

Lemma 5. *For integer q let χ be a primitive character $\pmod q$ and let*

$$f_1(x) = (x-dv_1)(x-dv_2), \quad f_2(x) = (x-dv_3)(x-dv_4).$$

Suppose at least 3 of v_1, v_2, v_3, v_4 are distinct and define

$$A_i = \prod_{j\ne i}(dv_i - dv_j).$$

Then we have

$$\left|\sum_{x=1}^{q}\chi(f_1(x))\overline{\chi}(f_2(x))\right|\leq 8^{\omega(q)}q^{1/2}(q,A_i),$$

for some $A_i\neq 0$ *with* $1\leq i\leq 4$*, where* $\omega(q)$ *counts the number of distinct prime factors of* q*.*

We use Lemma 5 and ideas from the proof of [1, Lemma 8] to show,

Lemma 6. *For any primitive character* χ *modulo* q *and any positive integer* V *we have,*

$$\sum_{v_1,\ldots,v_4=1}^{V}\left|\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{2}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=3}^{4}(x+dv_i)\right)\right|\leq (V^2q+(d,q)^3q^{1/2}V^4)q^{o(1)}.$$

Proof. We divide the outer summation of

$$\sum_{v_1,v_2,v_3,v_4=1}^{V}\left|\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{2}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=3}^{4}(x+dv_i)\right)\right|,$$

into two sets. In the first set we put all v_1,v_2,v_3,v_4 which contain at most 2 distinct numbers and we put the remaining v_1,v_2,v_3,v_4 into the second set. The number of elements in the first set is $\ll V^2$ and for these sets we estimate the inner sum trivially. This gives

$$\sum_{v_1,\ldots,v_4=1}^{V}\left|\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{2}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=3}^{4}(x+dv_i)\right)\right|\ll$$
$$qV^2+\sum_{v_1,\ldots,v_4=1}^{V}{}'\left|\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{2}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=3}^{4}(x+dv_i)\right)\right|,$$

where the last sum is restricted to v_1,v_2,v_3,v_4 which contain at least 3 distinct numbers. With notation as in Lemma 5, we have

$$\sum_{v_1,\ldots,v_4=1}^{V}{}'\left|\sum_{x=1}^{q}\chi\left(f_1(x)\right)\overline{\chi}\left(f_2(x)\right)\right|\leq q^{1/2+o(1)}\sum_{v_1,\ldots,v_4=1}^{V}{}'\sum_{\substack{i=1\\A_i\neq 0}}^{4}(A_i,q).$$

Since $A_i=\prod_{i\neq j}(dv_i-dv_j)=d^3\prod_{i\neq j}(v_i-v_j)=d^3A_i'$, we have

$$\sum_{v_1,\ldots,v_4=1}^{V}{}'\sum_{\substack{i=1\\A_i\neq 0}}^{4}(A_i,q)\leq (d^3,q)\sum_{v_1,\ldots,v_4=1}^{V}{}'\sum_{\substack{i=1\\A_i\neq 0}}^{4}(A_i',q),$$

and in [1, Lemma 8] it is shown

$$\sideset{}{'}\sum_{v_1,\ldots,v_4=1}^{V} \sum_{\substack{i=1\\A_i\neq 0}}^{4} (A_i', q) \leq V^4 q^{o(1)},$$

from which the result follows. □

We next use Lemma 6 to improve on [4, Lemma 10] with respect to the parameter d. For applications to Theorem 1, this is an important factor for summation over mid length intervals.

Lemma 7. *For any primitive character $\chi \pmod q$ and integers M, N, a and d satisfying*

$$N \leq q^{5/8} d^{-5/4}, \quad d \leq q^{1/6}, \quad (a, q) = 1,$$

we have

$$\left| \sum_{M<n\leq M+N} \chi(dn+a) \right| \leq q^{3/16+o(1)} d^{3/8} N^{1/2}.$$

Proof. We proceed by induction on N. Since the result is trivial for $N \leq q^{3/8}$, this forms the basis of the induction. We define

$$U = [0.25 N d^{3/2} q^{-1/4}], \quad V = [0.25 d^{-3/2} q^{1/4}],$$

and let

$$\mathcal{U} = \{\ 1 \leq u \leq U\ :\ (u, dq) = 1\ \}, \quad \mathcal{V} = \{\ 1 \leq v \leq V\ :\ (v, q) = 1\ \}.$$

By the inductive assumption, for any $\varepsilon > 0$ and integer $h \leq UV < N$ we have

$$\left| \sum_{M<n\leq M+N} \chi(dn+a) \right| \leq \left| \sum_{M<n\leq M+N} \chi(d(n+h)+a) \right| + 2q^{3/16+\varepsilon} d^{3/8} h^{1/2},$$

for sufficiently large q. Hence

$$\left| \sum_{M<n\leq M+N} \chi(dn+a) \right| \leq \frac{1}{\#\mathcal{U}\#\mathcal{V}} |W| + 2q^{3/16+\varepsilon} d^{3/8} (UV)^{1/2},$$

where

$$\begin{aligned} W &= \sum_{u\in\mathcal{U}} \sum_{v\in\mathcal{V}} \sum_{M<n\leq M+N} \chi(d(n+uv)+a) \\ &= \sum_{u\in\mathcal{U}} \chi(u) \sum_{M<n\leq M+N} \sum_{v\in\mathcal{V}} \chi((dn+a)u^{-1}+dv). \end{aligned}$$

We have

$$|W| \le \sum_{x=1}^{q} \nu(x) \left| \sum_{v \in \mathcal{V}} \chi(x+dv) \right|,$$

where $\nu(x)$ is the number of representations $x \equiv (dn+a)u^{-1} \pmod q$ with $M < n \le M+N$ and $u \in \mathcal{U}$. Two applications of Hölder's inequality gives

$$|W|^4 \le \left(\sum_{x=1}^{q} \nu^2(x) \right) \left(\sum_{x=1}^{q} \nu(x) \right)^2 \sum_{x=1}^{q} \left| \sum_{v \in \mathcal{V}} \chi(x+dv) \right|^4.$$

From the proof of [4, Lemma 7] we have

$$\sum_{x=1}^{q} \nu(x) = N\#\mathcal{U}, \quad \sum_{x=1}^{q} \nu^2(x) \le \left(\frac{dNU}{q} + 1 \right) NUq^{o(1)},$$

and by Lemma 6

$$\begin{aligned} \sum_{x=1}^{q} \left| \sum_{v \in \mathcal{V}} \chi(x+dv) \right|^4 &= \sum_{v_1,\dots v_4 \in \mathcal{V}} \sum_{x=1}^{q} \chi\left(\prod_{i=1}^{2} (x+dv_i) \right) \overline{\chi}\left(\prod_{i=3}^{4} (x+dv_i) \right) \\ &\le \sum_{v_1,\dots v_4 = 1}^{V} \left| \sum_{x=1}^{q} \chi\left(\prod_{i=1}^{2} (x+dv_i) \right) \overline{\chi}\left(\prod_{i=3}^{4} (x+dv_i) \right) \right| \\ &\ll V^2 q^{1+o(1)}, \end{aligned}$$

since $V \le d^{-3/2} q^{1/4}$. Combining the above bounds gives

$$|W|^4 \le \left(\frac{dNU}{q} + 1 \right) NU(N\#\mathcal{U})^2 V^2 q^{1+o(1)},$$

and since

$$\#\mathcal{U} = Uq^{o(1)}, \quad \#\mathcal{V} = Vq^{o(1)},$$

we have

$$\left| \sum_{M<n\le M+N} \chi(dn+a) \right| \le \left(\frac{d^{1/4}N}{V^{1/2}} + \frac{q^{1/4}N^{3/4}}{U^{1/4}V^{1/2}} \right) q^{o(1)} + 2q^{3/16+\varepsilon} d^{3/8} (UV)^{1/2}.$$

Recalling the choice of U and V we get

$$\left| \sum_{M<n\le M+N} \chi(dn+a) \right| \le \left(\frac{dN}{q^{1/8}} + q^{3/16} d^{3/8} N^{1/2} \right) q^{o(1)} + \frac{1}{2} q^{3/16+\varepsilon} d^{3/8} N^{1/2},$$

and since by assumption,

$$N \le q^{5/8} d^{-5/4},$$

we get for sufficiently large q

$$\left|\sum_{M<n\le M+N}\chi(dn+a)\right| \le q^{3/16}d^{3/8}N^{1/2}q^{o(1)}+\frac{1}{2}q^{3/16+\varepsilon}d^{3/8}N^{1/2}$$
$$\le q^{3/16+\varepsilon}d^{3/8}N^{1/2}.$$

□

Lemma 8. *Let* χ *be a primitive character* (mod q) *and suppose* $(a,q)=1$*. Then for* $N\le q^{43/72}$ *we have*

$$\left|\sum_{\substack{M<n\le M+N\\(n,q)=1}}\chi(n+a)\right|\le q^{3/16+o(1)}N^{1/2}.$$

Proof. We have

$$\left|\sum_{\substack{M<n\le M+N\\(n,q)=1}}\chi(n+a)\right| = \left|\sum_{d|q}\mu(d)\sum_{M/d<n\le(M+N)/d}\chi(dn+a)\right|$$
$$\le \sum_{d|q}\left|\sum_{M/d<n\le(M+N)/d}\chi(dn+a)\right|.$$

Let

$$Z=\left\lfloor\frac{N^{1/2}}{q^{3/16}}\right\rfloor,$$

then by Lemma 7 we have

$$\sum_{d|q}\left|\sum_{M/d<n\le(M+N)/d}\chi(dn+a)\right| =$$
$$\sum_{\substack{d|q\\d\le Z}}\left|\sum_{M/d<n\le(M+N)/d}\chi(dn+a)\right|+\sum_{\substack{d|q\\d>Z}}\left|\sum_{M/d<n\le(M+N)/d}\chi(dn+a)\right|$$
$$\le\sum_{\substack{d|q\\d\le Z}}q^{3/16+o(1)}d^{-1/8}N^{1/2}+\sum_{\substack{d|q\\d>Z}}\frac{N}{d}.$$

By choice of Z we get

$$\sum_{\substack{d|q \\ d\le Z}} q^{3/16+o(1)} d^{-1/8} N^{1/2} + \sum_{\substack{d|q \\ d>Z}} \frac{N}{d} \le \left(q^{3/16}N^{1/2} + \frac{N}{Z}\right) q^{o(1)} \le q^{3/16+o(1)} N^{1/2},$$

which gives the desired bound. It remains to check that the conditions of Lemma 7 are satisfied. For each $d|q$ with $d \le Z$, we need

$$\frac{N}{d} \le q^{5/8} d^{-5/4}, \quad d \le q^{1/6},$$

which on recalling the choice of Z is satisfied for $N \le q^{43/72}$. □

By partial summation we deduce from Lemma 8.

Lemma 9. *Let χ be a primitive character* (mod q) *and suppose* $(a,q) = 1$. *Then for $N \le q^{43/72}$ we have*

$$\left| \sum_{\substack{M<n\le M+N \\ (n,q)=1}} (\log n)\chi(n+a) \right| \le q^{3/16+o(1)} N^{1/2}.$$

4.2. **The case r=3.** Throughout this section we let

(9)
$$f_1(x) = (x+dv_1)(x+dv_2)(x+dv_3), \quad f_2(x) = (x+dv_4)(x+dv_5)(x+dv_6),$$

and

$$F(x) = f_1'(x)f_2(x) - f_1(x)f_2'(x), \tag{10}$$

and write $\mathbf{v} = (v_1, \dots v_6)$. We generalize the argument of Burgess [2] to give an upper bound for the cardinality of the set

$$\mathcal{A}(s,s') = \{\mathbf{v} \ : \ 0 < v_i \le V, \text{ there exists an } x \text{ such that } (s, f_1(x)f_2(x)) = 1, \ s|F(x), \ s|F'(x), \ s'|F''(x)\},$$

which will then be combined with the proof of [2, Theorem 2] to bound the sums (8). The proof of the following Lemma is the same as [2, Lemma 3].

Lemma 10. *Let $s'|s$ and consider the equations*

$$(\lambda, s) = 1, \quad (f_1(-t), s/s') = 1, \tag{11}$$

$$6(f_1(X) + \lambda f_2(X)) \equiv 6(1+\lambda)(X+t)^3 \pmod{s}, \tag{12}$$

$$6(1+\lambda) \equiv 0 \pmod{s'}. \tag{13}$$

Let

$$\mathcal{A}_1(s,s') = \{\mathbf{v},\ \lambda,\ t\ :\ 0 < v_i \le V,\ v_i \ne v_1,\ i \ge 2,\ 0 < \lambda \le s,\ 0 < t \le s/s',\ (11),\ (12),\ (13)\},$$

then we have

$$\#\mathcal{A}(s,s') \ll V^3 + \#\mathcal{A}_1(s,s').$$

We next make the substitutions

$$\begin{aligned} Y &= X + dv_1, \\ V_i &= v_i - v_1, \quad i \ge 2, \\ T &= t - dv_1 \pmod{s/s'}, \end{aligned} \tag{14}$$

so that

$$\begin{aligned} f_1(X) &= Y(Y+dV_2)(Y+dV_3) = Y^3 + d(V_2+V_3)Y^2 + d^2V_2V_3Y \\ &= g_1(Y), \end{aligned} \tag{15}$$

$$\begin{aligned} f_2(X) &= (Y+dV_4)(Y+dV_5)(Y+dV_6) = Y^3 + d\sigma_1Y^2 + d^2\sigma_2Y + d^3\sigma_3 \\ &= g_2(Y), \end{aligned} \tag{16}$$

where

$$\begin{aligned} \sigma_1 &= V_4 + V_5 + V_6, \\ \sigma_2 &= V_4V_5 + V_4V_6 + V_5V_6, \\ \sigma_3 &= V_4V_5V_6, \end{aligned} \tag{17}$$

and we see that (12) becomes

$$6(g_1(Y) + \lambda g_2(Y)) \equiv 6(1+\lambda)(Y+T)^3 \pmod{s}. \tag{18}$$

The proof of the following Lemma follows that of [2, Lemma 4].

Lemma 11. *With notation as in* (14) *and* (17)*, consider the equations*

$$(s/s', T) = 1, \quad (s/s', T - dV_3) = 1, \tag{19}$$

$$6d^2T^3(V_3^2 - \sigma_1V_3 + \sigma_2) - 18d^3\sigma_3T^2 + 18d^4V_3\sigma_3T - 6d^5V_3^2\sigma_3 \equiv 0 \pmod{s}, \tag{20}$$

$$6d^3\sigma_3 \equiv 0 \pmod{s'}, \tag{21}$$

and let

$$\mathcal{A}_2(s,s') = \{(V_3,\ V_4,\ V_5,\ V_6,\ T)\ :\ 0 < |V_i| \le V,\ 0 < T \le s/s',\ (19),\ (20),\ (21)\}.$$

Then we have

$$\#\mathcal{A}_1(s,s') \ll (d,s)V(1+V/s)\#\mathcal{A}_2(s,s').$$

Proof. We first note that (11) and (14) imply (19). Let

$$\mathcal{B}_1 = \{(V_2,V_3,V_4,V_5,V_6,T) \ : \ 0<|V_i|\le V, \quad 0<\lambda\le s, \ (\lambda,s)=1, \ 0<T\le s/s', (13), (18), (19)\},$$

so that

$$\#\mathcal{A}_1(s,s') \le V\#\mathcal{B}_1.$$

Using (15) and (16) and considering common powers of Y in (18) we get

$$6d(V_2+V_3+\lambda\sigma_1) \equiv 18(1+\lambda)T \pmod{s}, \tag{22}$$

$$6d^2(V_2V_3+\lambda\sigma_2) \equiv 18(1+\lambda)T^2 \pmod{s}, \tag{23}$$

$$6d^3\lambda\sigma_3 \equiv 6(1+\lambda)T^3 \pmod{s}. \tag{24}$$

By (22) we see that

$$6dV_2 \equiv 18(1+\lambda)T - 6dV_3 - 6d\lambda\sigma_1 \pmod{s},$$

which has $O\left((d,s)(1+V/s)\right)$ solutions in V_2. The equations (13) and (24) imply that

$$6d^3\sigma_3 \equiv 0 \pmod{s'},$$

and

$$6\lambda(d^3\sigma_3 - T^3) \equiv 6T^3 \pmod{s}. \tag{25}$$

Since $(T, s/s') = 1$ by the above equations, there are $O(1)$ possible values of λ. Finally combining (22), (23) and (25) gives (20). □

The following is [2, Lemma 2].

Lemma 12. *For any integer s and polynomial $G(X)$ with integer coefficients, we have*

$$\#\{0\le x<s, \ G(x)\equiv 0 \pmod{s}, \ (s,G'(x))|6\} \le s^{o(1)},$$

where the term $o(1)$ depends only on the degree of G.

The proof of the following Lemma follows that of [2, Lemma 5].

Lemma 13. *For $s''|(s/s')$ consider the equations*

$$(s, 6d^3\sigma_3) = s's'', \tag{26}$$

$$6d^2(V_3^2 - \sigma_1V_3 + \sigma_2) \equiv 0 \pmod{s''}, \tag{27}$$

and let

$$\mathcal{A}_3(s,s',s'') = \{(V_3,V_4,V_5,V_6) \ : \ 0<|V_i|\le V, \ (26), \ (27)\}.$$

Then we have

$$\#\mathcal{A}_2(s,s') \le s^{o(1)} \sum_{s''|s/s'} s''\#\mathcal{A}_3(s,s',s'').$$

Proof. For $s''|(s/s')$, let

$$\mathcal{A}_3'(s,s',s'') = \{\ (V_3,V_4,V_5,V_6,T) \in \mathcal{A}_2(s,s')\ :\ (s,6d^3\sigma_3) = s's''\},$$

so that

$$\#\mathcal{A}_2(s,s') = \sum_{s''|(s/s')} \#\mathcal{A}_3'(s,s',s''). \tag{28}$$

Let $S=(s',s/s')$, so that $(s'/S,s/s')=1$. For elements of $\mathcal{A}_3(s,s',s'')$, since

$$6d^3\sigma_3 \equiv 0 \pmod{Ss''}, \tag{29}$$

we have by (19), (20) and (26)

$$6d^2(V_3^2-\sigma_1V_3+\sigma_2) \equiv 0 \pmod{Ss''}, \tag{30}$$

hence (20) implies that

$$\begin{aligned}\frac{6d^2(V_3^2-\sigma_1V_3+\sigma_2)}{Ss''}T^3 &- \frac{18d^3\sigma_3}{Ss''}T^2\\ +\frac{18d^4\sigma_3V_3}{Ss''}T &- \frac{6d^5\sigma_3V_3^2}{Ss''} \equiv 0 \pmod{s/(s's'')}.\end{aligned} \tag{31}$$

Let

$$G(T) = \frac{6d^2(V_3^2-\sigma_1V_3+\sigma_2)}{Ss''}T^3 - \frac{18d^3\sigma_3}{Ss''}T^2 + \frac{18d^4\sigma_3V_3}{Ss''}T - \frac{6d^5\sigma_3V_3^2}{Ss''},$$

so that

$$3G(T)-TG'(T) = -\frac{18d^3\sigma_3}{Ss''}(T-dV_3)^2.$$

Writing $6d^3\sigma_3 = s's''\sigma'$ with $(\sigma',s)=1$, we see from (19) that for some integer y with $(y,s/s')=1$ that

$$3G(T)-TG'(T) = -\frac{3s'}{S}y.$$

If T_0 is a root of $G(T) \pmod{s/(s's'')}$ then since $(s'/S,s/s')=1$ we have

$$(G'(T_0),s/(s's''))|3,$$

hence from Lemma 12, the number of possible values for T is $\ll s''s^{o(1)}$. Finally (30) implies

$$6d^2(V_3^2-\sigma_1V_3+\sigma_2) \equiv 0 \pmod{s''},$$

and the result follows from (28). □

Lemma 14. *With notation as in Lemma 13, for integers s, s', s'' satisfying $s'|s$ and $s''|s/s'$ we have*

$$\#\mathcal{A}_3(s,s',s'') \le (d^3,s)V^4 s^{o(1)}/(s's'').$$

Proof. Bounding the number of solutions to the equation (27) trivially and recalling the definition of σ_3 from (17), we see that

(32)

$$\#\mathcal{A}_3(s,s',s'') \le V\#\{(V_4,V_5,V_6)\ :\ 0<|M_i|\le V,\ (s,6d^3V_4V_5V_6)=s's''\}.$$

Writing $s=(d^3,s)s_1$, $d^3=(d^3,s)d_1$, we see that

$$(s,6d^3V_4V_5V_6)=s's'',$$

implies

$$(s_1,6V_4V_5V_6)=s's''/(d^3,s).$$

For integers s_1,s_2,s_3, let

$$\mathcal{A}_4(s_1,s_2,s_3)=\{V_4,V_5,V_6\ :\ 0<|V_i|\le V,\ s_1|6V_4,\ s_2|6V_5,\ s_3|6V_6\},$$

so that from (32)

$$\#\mathcal{A}_3(s,s',s'') \le V \sum_{s_1s_2s_3=s's''/(d^3,s)} \mathcal{A}_4(s_1,s_2,s_3).$$

Since

$$\mathcal{A}_4(s_1,s_2,s_3) \ll \frac{V^3}{s_1s_2s_3}=\frac{(d^3,s)V^3}{s's''},$$

we see that

$$\#\mathcal{A}_3(s,s',s'') \le \frac{(d^3,s)V^4s^{o(1)}}{s's''}.$$

□

Combining the above results we get

Lemma 15. *Let $s'|s$ and*

$$\mathcal{A}(s,s')=\{\ \mathbf{v}\ :\ 0<v_i\le V, \text{there exists an } x \text{ such that } (s,f_1(x)f_2(x))=1,\ s|F(x),\ s|F'(x),\ s'|F''(x)\}.$$

Then

$$\#\mathcal{A}(s,s') \le (d,s)^4\left(\frac{V^6}{ss'}+\frac{V^5}{s'}\right)s^{o(1)}+V^3.$$

Proof. From Lemma 10, Lemma 11, Lemma 13 we see that

$$\#\mathcal{A}(s,s') \le V^3+(d,s)\left(1+\frac{V}{s}\right)V\sum_{s''|s/s'} s''\#\mathcal{A}_3(s,s',s''),$$

and from Lemma 14 we have

$$\sum_{s''|s/s'} s''\#\mathcal{A}_3(s,s',s'') \le \frac{s^{o(1)}(d^3,s)V^4}{s'},$$

which gives the desired result. □

For integer q, we define the numbers $h_1(q), h_2(q), h_3(q)$ as in [2],

$$\begin{aligned} h_1(q)^2 &= \text{smallest square divisible by } q, \\ h_2(q)^3 &= \text{smallest cube divisible by } q, \\ h_3(q) &= \text{ product of distinct prime factors of } q. \end{aligned} \tag{33}$$

The following is [2, Theorem 2].

Lemma 16. *Let χ be a primitive character mod q and let*

$$q = q_0q_1q_2q_3, \tag{34}$$

where the q_i are pairwise coprime. Let the integers l_0, l_1, l_2 satisfy

$$l_0|h_1(q_0)/h_3(q_0), \quad l_1|h_2(q_1)/h_3(q_1), \quad l_2|h_2(q_2)/h_3(q_2), \tag{35}$$

and consider the equations

$$l_0h_1(q_1q_2q_3)|F(x), \quad (F(x), h_1(q_0)) = l_0, \tag{36}$$

$$l_1h_2(q_2q_3)|F'(x), \quad (F'(x), h_2(q_1)) = l_1, \tag{37}$$

$$l_2h_2(q_3)|F''(x), \quad (F''(x), h_2(q_2)) = l_2, \tag{38}$$

and let

$$\mathcal{C} = \mathcal{C}(l_0, l_1, l_2, q_0, q_1, q_2, q_3) = \{1 \le x \le q \ : (36), \ (37), \ (38)\}. \tag{39}$$

Then we have

$$\left|\sum_{x\in\mathcal{C}} \chi(f_1(x))\overline{\chi}(f_2(x))\right| \le q^{1/2+o(1)}\frac{(q_2q_3l_1)^{1/2}l_2}{h_2(q_2)}.$$

We next present some results which are used by Burgess in the proof of [2, Theorem B] and we will use in combination with Lemma 15 and Lemma 16 in the proof of Lemma 19 below. Since these result are nontrivial and Burgess does not provide a proof, we give details for the sake of completeness.

Lemma 17. *With notation as in (33), let q, u and i be positive integers satisfying $i \in \{1, 2\}$ and $1 \le u \le q$. There exists unique integers q_0, ℓ_0 satisfying*

$$q_0|q, \quad (q_0, q/q_0) = 1, \quad \ell_0h_i(q/q_0)|u, \quad (u, h_i(q_0)) = \ell_0, \quad \ell_0|h_i(q_0)/h_3(q_0).$$

Proof. We first show q_0, l_0 exist. For integer n with prime factorization

$$n = \prod_{j=1}^{k} p_j^{\alpha_j},$$

and $i \in \{1,2\}$ we have

$$h_i(n) = \prod_{j=1}^{k} p_j^{\lfloor (\alpha_j+i)/(i+1) \rfloor}, \quad h_3(n) = \prod_{j=1}^{k} p_j. \tag{40}$$

Suppose q has prime factorization

$$q = \prod_{j=1}^{k} p_j^{\alpha_j},$$

and define β_j by

$$(u,q) = \prod_{j=1}^{k} p_j^{\beta_j}. \tag{41}$$

Let $\mathcal{K}_0$ denote the set

$$\mathcal{K}_0 = \{1 \le j \le k \ : \ \lfloor (\alpha_j+i)/(i+1) \rfloor > \beta_j \}, \tag{42}$$

and define q_0, ℓ_0 by

$$q_0 = \prod_{j \in \mathcal{K}_0} p_j^{\alpha_j}, \quad (u, h_i(q_0)) = l_0 = \prod_{j \in \mathcal{K}_0}^{k} p_j^{\beta_j}.$$

First note that

$$(q_0, q/q_0) = 1. \tag{43}$$

From (40)

$$l_0 h_i\left(\frac{q}{q_0}\right) = \prod_{j \in \mathcal{K}_0} p_j^{\beta_j} \prod_{\substack{1 \le j \le k \\ j \notin \mathcal{K}_0}} p_j^{\lfloor (\alpha_j+i)/(i+1) \rfloor},$$

and if $j \notin \mathcal{K}_0$ then

$$\lfloor (\alpha_j+i)/(i+1) \rfloor \le \beta_j,$$

which by (41) implies

$$l_0 h_i\left(q/q_0\right) | (u,q). \tag{44}$$

We have

$$\frac{h_i(q_0)}{h_3(q_0)} = \prod_{j \in \mathcal{K}_0}^{k} p_j^{\lfloor (\alpha_j+i)/(i+1) \rfloor - 1},$$

and if $j \in \mathcal{K}_0$ then

$$\lfloor (\alpha_j + i)/(i+1) \rfloor - 1 \geq \beta_j,$$

which implies

$$l_0 | h_i(q_0)/h_3(q_0). \tag{45}$$

It follows from (43), (44) and (45) that q_0, l_0 satisfy the desired properties. We next show q_0, l_0 are unique. Suppose q_1, l_1 satisfy

$$q_1 | q, \quad (q_1, q/q_1) = 1, \tag{46}$$

$$l_1 h_i(q/q_1) | u, \quad (u, h_i(q_1)) = l_1, \tag{47}$$

and

$$l_1 | h_i(q_1)/h_3(q_1). \tag{48}$$

We will show $q_1 = q_0$ and $l_1 = l_0$. From (46), there exists a subset $\mathcal{K}_1 \subseteq \{1, \ldots, k\}$ such that

$$q_1 = \prod_{j \in \mathcal{K}_1} p_j^{\alpha_j}, \quad q/q_1 = \prod_{\substack{1 \leq j \leq k \\ j \notin \mathcal{K}_1}} p_j^{\alpha_j}. \tag{49}$$

and since $(u, h_i(q_1)) = l_1$, we have

$$l_1 = \prod_{j \in \mathcal{K}_1} p_j^{\min\{\lfloor (\alpha_j + i)/(i+1), \beta_j\}}. \tag{50}$$

By (40), (47) and (49), if $j \notin \mathcal{K}_1$ then

$$\lfloor (\alpha_j + i)/(i+1) \rfloor \leq \beta_j.$$

Recalling (42), this implies

$$\{1, \ldots, k\}/\mathcal{K}_1 \subseteq \{1, \ldots, k\}/\mathcal{K}_0,$$

and hence

$$\mathcal{K}_0 \subseteq \mathcal{K}_1. \tag{51}$$

From (40), (48) and (50), if $j \in \mathcal{K}_1$ then

$$\min\{\lfloor (\alpha_j + i)/(i+1), \beta_j\} \leq \lfloor (\alpha_j + i)/(i+1) \rfloor - 1,$$

which implies

$$\beta_j < \beta_j + 1 \leq \lfloor (\alpha_j + i)/(i+1) \rfloor.$$

By (42)

$$\mathcal{K}_1 \subseteq \mathcal{K}_0,$$

which combined with (51) implies

$$\mathcal{K}_1 = \mathcal{K}_0,$$

and hence

$$q_1 = q_0, \quad l_1 = l_0,$$

which completes the proof.

□

Lemma 18. *With notation as in Lemma 16, for each $1 \le x \le q$ there exists unique integers $q_0, q_1, q_2, q_3, l_0, l_1, l_2$ satisfying*

$$q = q_0 q_1 q_2 q_3 \quad \textit{and} \quad (q_i, q_j) = 1 \quad \textit{if } i \neq j,$$

$$l_0 | h_1(q_0)/h_3(q_0), \quad l_1 | h_2(q_1)/h_3(q_1), \quad l_2 | h_2(q_2)/h_3(q_2),$$

$$l_0 h_1(q_1 q_2 q_3) | F(x), \quad (F(x), h_1(q_0)) = l_0,$$
$$l_1 h_2(q_2 q_3) | F'(x), \quad (F'(x), h_2(q_1)) = l_1,$$
$$l_2 h_2(q_3) | F''(x), \quad (F''(x), h_2(q_2)) = l_2.$$

Proof. By Lemma 17, there exists unique integers q_0, l_0 satisfying $(q_0, q/q_0) = 1$ and

$$l_0 | h_1(q_0)/h_3(q_0), \quad l_0 h_1(q/q_0) | F(x), \quad (F(x), h_1(q_0)) = l_0.$$

A second application of Lemma 17 gives unique q_1, l_1 satisfying $(q_1, q/(q_0 q_1)) = 1$ and

$$l_1 | h_2(q_1)/h_3(q_1), \quad l_1 h_2(q/(q_0 q_1)) | F'(x), \quad (F'(x), h_2(q_1)) = l_1.$$

A third application of Lemma 17 gives unique q_2, l_2 satisfying $(q_2, q/(q_0 q_1 q_2)) = 1$ and

$$l_2 h_2(q/(q_0 q_1 q_2)) | F''(x), \quad (F''(x), h_2(q_2)) = l_2, \quad l_2 | h_2(q_2)/h_3(q_2),$$

and the result follows after defining $q_3 = q/(q_0 q_1 q_2)$. □

Lemma 19. *For any primitive character χ modulo q and any integer $V < q^{1/6} d^{-2}$, we have*

$$\sum_{v_1, \ldots, v_6 = 1}^{V} \left| \sum_{x=1}^{q} \chi \left(\prod_{i=1}^{3} (x + d v_i) \right) \overline{\chi} \left(\prod_{i=4}^{6} (x + d v_i) \right) \right| \le V^3 q^{1+o(1)}.$$

Proof. Let $\mathcal{C}(\ell_0, \ldots, q_3)$ be as in (39) and f_1, f_2, F be as in (9) and (10). By Lemma 18, as $\ell_0, \ell_1, \ell_2, q_0, q_1, q_2, q_3$ range over values satisfying the conditions of Lemma 16, the sets $\mathcal{C}(\ell_0, \ldots, q_3)$ partition the set

$$\{1 \le x \le q, \quad (f_1(x) f_2(x), q) = 1\},$$

into disjoint subsets. Therefore

$$\left|\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{3}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=4}^{6}(x+dv_i)\right)\right| \leq \sum_{\substack{\ell_0,\ldots,\ell_2\\ q_0,\ldots,q_3}}\left|\sum_{x\in\mathcal{C}(\ell_0,\ldots,q_3)}\chi(f_1(x))\overline{\chi}(f_2(x))\right|,$$

where the last sum is extended over all $\ell_0,\ldots,q_3$ satisfying the conditions of Lemma 16. Summing over $1\leq v_1,\ldots,v_6\leq V$, interchanging summation and using bounds for the divisor function, we see that there exists some $\ell_0,\ldots,q_3$ such that

$$\sum_{v_1,\ldots,v_6=1}^{V}\left|\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{3}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=4}^{6}(x+dv_i)\right)\right| \leq q^{o(1)}\sum_{1\leq v_1,\ldots,v_6\leq V}\left|\sum_{x\in\mathcal{C}(\ell_0,\ldots,q_3)}\chi(f_1(x))\overline{\chi}(f_2(x))\right|. \tag{52}$$

For fixed $v_1,\ldots,v_6$, if summation over x in (52) is nonempty then there exists some $1\leq x\leq q$ such that

$$(q,f_1(x)f_2(x))=1,\ \ell_1h_2(q_2q_3)|F(x),\ \ell_1h_2(q_2q_3)|F'(x),\ \ell_2h_2(q_3)|F''(x).$$

Hence by Lemma 15 and Lemma 16,

$$\sum_{v_1,\ldots,v_6=1}^{V}\left|\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{3}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=4}^{6}(x+dv_i)\right)\right| \leq \left((q,d)^4\left(\frac{V^6}{l_1h_2(q_2q_3)l_2h_2(q_3)}+\frac{V^5}{l_2h_2(q_3)}\right)+V^3\right)\frac{(qq_2q_3l_1)^{1/2}l_2}{h_2(q_2)}q^{o(1)} \leq \left((q,d)^4V^6q^{1/2}+(q,d)^4V^5q^{2/3}+V^3q\right)q^{o(1)},$$

from the definition of l_i,h_i,q_i. The result follows since the term V^3q dominates for $V\leq q^{1/6}d^{-2}$. □

We next use Lemma 19 to improve on [4, Lemma 7].

Lemma 20. *For any primitive character χ modulo q and integers M, N, d and a satisfying*

$$N\leq q^{7/12}d^{-3/2},\quad d\leq q^{1/12},\quad (a,q)=1,$$

we have

$$\left|\sum_{M<n\leq M+N}\chi(dn+a)\right|\leq q^{1/9+o(1)}d^{2/3}N^{2/3}.$$

Proof. Using the same argument from Lemma 7, we proceed by induction on N. Since the result is trivial for $N \le q^{1/3}$, this forms the basis of our induction. Define

$$U = [0.5Nd^2q^{-1/6}], \quad V = [0.5d^{-2}q^{1/6}],$$

and let

$$\mathcal{U} = \{ \ 1 \le u \le U \ : \ (u, dq) = 1 \ \}, \quad \mathcal{V} = \{ \ 1 \le v \le V \ : \ (v, q) = 1 \ \}.$$

Fix $\varepsilon > 0$, by the inductive hypothesis, for any integer $h \le UV < N$ we have

$$\left| \sum_{M<n\le M+N} \chi(dn+a) \right| \le \left| \sum_{M<n\le M+N} \chi(d(n+h)+a) \right| + 2q^{1/9+\varepsilon}d^{2/3}h^{2/3},$$

for sufficiently large q. Hence

$$\left| \sum_{M<n\le M+N} \chi(dn+a) \right| \le \frac{1}{\#\mathcal{U}\#\mathcal{V}}|W| + 2q^{1/9+\varepsilon}d^{2/3}(UV)^{2/3},$$

where

$$W = \sum_{u\in\mathcal{U}} \sum_{v\in\mathcal{V}} \sum_{M<n\le M+N} \chi(d(n+uv)+a) = \sum_{u\in\mathcal{U}} \chi(u) \sum_{M<n\le M+N} \sum_{v\in\mathcal{V}} \chi((dn+a)u^{-1}+dv).$$

We have

$$|W| \le \sum_{x=1}^{q} \nu(x) \left| \sum_{v\in\mathcal{V}} \chi(x+dv) \right|,$$

where $\nu(x)$ is the number of representations $x \equiv (dn+a)u^{-1} \pmod q$ with $M < n \le M+N$ and $u \in \mathcal{U}$. Two applications of Hölder's inequality gives,

$$|W|^6 \le \left(\sum_{x=1}^{q} \nu^2(x) \right) \left(\sum_{x=1}^{q} \nu(x) \right)^4 \sum_{x=1}^{q} \left| \sum_{v\in\mathcal{V}} \chi(x+dv) \right|^6.$$

As in Lemma 7

$$\sum_{x=1}^{q} \nu(x) = N\#\mathcal{U}, \quad \sum_{x=1}^{q} \nu^2(x) \le \left(\frac{dNU}{q} + 1 \right) NUq^{o(1)},$$

and by Lemma 19

$$\sum_{x=1}^{q}\left|\sum_{v\in\mathcal{V}}\chi(x+dv)\right|^6 = \sum_{v_1,\dots v_4\in\mathcal{V}}\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{3}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=4}^{6}(x+dv_i)\right)$$
$$\le \sum_{v_1,\dots v_4=1}^{V}\left|\sum_{x=1}^{q}\chi\left(\prod_{i=1}^{3}(x+dv_i)\right)\overline{\chi}\left(\prod_{i=4}^{6}(x+dv_i)\right)\right|$$
$$\le V^3 q^{1+o(1)}.$$

The above bounds give

$$|W|^6 \le \left(\frac{dNU}{q}+1\right) NUq^{o(1)}(N\#\mathcal{U})^4\left(V^3q\right)q^{o(1)},$$

so that

$$\left|\sum_{M<n\le M+N}\chi(dn+a)\right| \le \left(\frac{d^{1/6}N}{V^{1/2}}+\frac{q^{1/6}N^{5/6}}{U^{1/6}V^{1/2}}\right)q^{o(1)}+2q^{1/9+\varepsilon}d^{2/3}(UV)^{2/3}.$$

Recalling the choice of U and V we get

$$\left|\sum_{M<n\le M+N}\chi(dn+a)\right| \le \frac{d^{7/6}N}{q^{1/12+o(1)}}+q^{1/9+o(1)}d^{2/3}N^{2/3}+\frac{2}{5}q^{1/9+\varepsilon}d^{2/3}N^{2/3},$$

and since

$$N \le q^{7/12}d^{-3/2},$$

we have by assumption on N and d

$$\left|\sum_{M<n\le M+N}\chi(dn+a)\right| \le q^{1/9+o(1)}d^{2/3}N^{2/3}+\frac{2}{5}q^{1/9+\varepsilon}d^{2/3}N^{2/3}$$
$$\le q^{1/9+\varepsilon}d^{2/3}N^{2/3},$$

for sufficiently large q. □

Using Lemma 20 as in the proof of Lemma 8 we get,

Lemma 21. *Let χ be a primitive character* $\pmod q$ *and suppose* $(a,q)=1$*, then for $N\le q^{23/42}$ we have*

$$\left|\sum_{\substack{M<n\le M+N\\(n,q)=1}}\chi(n+a)\right| \le q^{1/9+o(1)}N^{2/3}.$$

Proof. We have

$$\left|\sum_{\substack{M<n\le M+N\\(n,q)=1}}\chi(n+a)\right| = \left|\sum_{d|q}\mu(d)\sum_{M/d<n\le (M+N)/d}\chi(dn+a)\right| \le \sum_{d|q}\left|\sum_{M/d<n\le (M+N)/d}\chi(dn+a)\right|.$$

Let

$$Z=\left\lfloor \frac{N^{1/3}}{q^{1/9}}\right\rfloor,$$

then by Lemma 20 we have

$$\sum_{d|q}\left|\sum_{M/d<n\le (M+N)/d}\chi(dn+a)\right| = \sum_{\substack{d|q\\ d\le Z}}\left|\sum_{M/d<n\le (M+N)/d}\chi(dn+a)\right| + \sum_{\substack{d|q\\ d>Z}}\left|\sum_{M/d<n\le (M+N)/d}\chi(dn+a)\right| \le \sum_{\substack{d|q\\ d\le Z}} q^{1/9+o(1)}N^{2/3}+\sum_{\substack{d|q\\ d>Z}}\frac{N}{d}.$$

Since by choice of Z

$$\sum_{\substack{d|q\\ d\le Z}} q^{1/9+o(1)}N^{2/3}+\sum_{\substack{d|q\\ d>Z}}\frac{N}{d} \le \left(q^{1/9}N^{2/3}+\frac{N}{Z}\right)q^{o(1)} \le q^{1/9+o(1)}N^{2/3},$$

we get the desired bound. It remains to check that the conditions of Lemma 7 are satisfied. For each $d|q$ with $d\le Z$ we need

$$\frac{N}{d}\le q^{7/12}d^{-3/2}, \quad d\le q^{1/12},$$

and from the choice of Z, this is satisfied for $N\le q^{23/42}$. □

From Lemma 21 and partial summation we deduce.

Lemma 22. *Let χ be a primitive character* $\pmod q$ *and suppose* $(a,q)=1$*, then for* $N\le q^{23/42}$ *we have*

$$\left|\sum_{\substack{M<n\le M+N\\(n,q)=1}}(\log n)\chi(n+a)\right|\le q^{1/9+o(1)}N^{2/3}.$$

5. Bilinear Character Sums

Lemma 23. *Let χ be a primitive character* $\pmod q$*. Then for integers* u_1,u_2,λ *we have*

$$\left|\sum_{n=1}^{q}\chi(n+u_1)\overline{\chi}(n+u_2)e^{2\pi i\lambda n/q}\right|=\left|\sum_{n=1}^{q}\chi(n+\lambda)\overline{\chi}(n)e^{2\pi i(u_1-u_2)n/q}\right|.$$

Proof. Let

$$\tau(\chi)=\sum_{n=1}^{q}\chi(n)e^{2\pi in/q},$$

be the Gauss sum, so that

$$|\tau(\chi)|=q^{1/2}\quad\text{and}\quad\sum_{n=1}^{q}\chi(n)e^{2\pi ian/q}=\overline{\chi}(a)\tau(\chi).$$

Writing

$$\chi(n+u_1)=\frac{1}{\tau(\overline{\chi})}\sum_{\lambda_1=1}^{q}\overline{\chi}(\lambda_1)e^{2\pi i(n+u_1)\lambda_1/q},$$

and

$$\overline{\chi}(n+u_2)=\frac{1}{\tau(\chi)}\sum_{\lambda_2=1}^{q}\chi(\lambda_2)e^{2\pi i(n+u_2)\lambda_2/q},$$

we have

$$\sum_{n=1}^{q}\chi(n+u_1)\overline{\chi}(n+u_2)e^{2\pi i\lambda n/q}=$$

$$\frac{1}{\tau(\chi)\tau(\overline{\chi})}\sum_{\lambda_1=1}^{q}\sum_{\lambda_2=1}^{q}\overline{\chi}(\lambda_1)e^{2\pi i\lambda_1u_1/q}\chi(\lambda_2)e^{2\pi i\lambda_2u_2/q}\sum_{n=1}^{q}e^{2\pi in(\lambda+\lambda_1+\lambda_2)/q}.$$

By orthogonality of the exponential function

$$\sum_{\lambda_1=1}^{q}\sum_{\lambda_2=1}^{q}\overline{\chi}(\lambda_1)e^{2\pi i\lambda_1 u_1/q}\chi(\lambda_2)e^{2\pi i\lambda_2 u_2/q}\sum_{n=1}^{q}e^{2\pi i n(\lambda+\lambda_1+\lambda_2)/q}=$$

$$\chi(-1)e^{-2\pi i u_2\lambda/q}q\sum_{\lambda_1=1}^{q}\chi(\lambda_1+\lambda)\overline{\chi}(\lambda_1)e^{2\pi i\lambda_1(u_1-u_2)/q},$$

and hence

$$\begin{aligned}\left|\sum_{n=1}^{q}\chi(n+u_1)\overline{\chi}(n+u_2)e^{2\pi i\lambda n/q}\right| &= \frac{q}{|\tau(\chi)|^2}\left|\sum_{n=1}^{q}\chi(n+\lambda)\overline{\chi}(n)e^{2\pi i n(u_1-u_2)/q}\right| \\ &= \left|\sum_{n=1}^{q}\chi(n+\lambda)\overline{\chi}(n)e^{2\pi i n(u_1-u_2)/q}\right|.\end{aligned}$$

□

Lemma 24. *Let χ be a primitive character* (mod q)*. Let $h_4(q)$ denote the smallest square dividing q. For integers b,λ with $b\not\equiv 0 \pmod{q}$, there exists an integer c satisfying $(c,q)=1$ such that*

$$\left|\sum_{\substack{n=1\\(n,q)=1}}^{q}\chi\left(1+\frac{b}{n}\right)e^{2\pi i\lambda n/q}\right|\le (b,q)(4c+\lambda b,h_4(q))^{1/2}q^{1/2+o(1)}.$$

Proof. Consider first when $\lambda\equiv 0 \pmod{q}$. Then from Lemma 23 we have

$$\left|\sum_{\substack{n=1\\(n,q)=1}}^{q}\chi\left(1+\frac{b}{n}\right)e^{2\pi i\lambda n/q}\right|=\left|\sum_{n=1}^{q}|\chi(n)|e^{2\pi i bn/q}\right|=\left|\sum_{\substack{n=1\\(n,q)=1}}^{q}e^{2\pi i bn/q}\right|,$$

and from [7, Equation 3.5] we have

$$\left|\sum_{\substack{n=1\\(n,q)=1}}^{q}e^{2\pi i bn/q}\right|\le (b,q),$$

so that

$$\left|\sum_{\substack{n=1\\(n,q)=1}}^{q}\chi\left(1+\frac{b}{n}\right)e^{2\pi i\lambda n/q}\right|\le (b,q)\le (b,q)q^{1/2+o(1)}.$$

Next consider when $\lambda \not\equiv 0 \pmod{q}$. We first note that if χ is a non-trivial character (mod p), with p prime, then we have from the Weil bound, see [13, Theorem 2G]

$$\left| \sum_{\substack{n=1 \\ (n,p)=1}}^{p} \chi\left(1+\frac{b}{n}\right) e^{2\pi i \lambda n/p} \right| \ll p^{1/2}.$$

For p prime and integers λ, b, c, α, let $N(\lambda, b, c, p^\alpha)$ denote the number of solutions to the congruence

$$\lambda(n^2+bn) \equiv cb \pmod{p^\alpha}, \quad 1 \le n \le p^\alpha, \quad (n,p)=1. \tag{53}$$

We will show if $(c,p)=1$ then

$$N(\lambda, b, c, p^\alpha) \le 6(\lambda, p^\alpha)(4c+\lambda b, p^\alpha)^{1/2}. \tag{54}$$

If there exists a solution n to (53) then since $(c,p)=1$ we must have $(\lambda, p^\alpha)|(b,p^\alpha)$. Define $\ell_0, b_0, \gamma, \beta$ by

$$\lambda = p^\gamma \ell_0, \quad (\ell_0,p)=1 \quad \text{and} \quad b=p^\beta b_0, \quad (b_0,p)=1.$$

The above implies $\gamma \le \beta$ and

$$N(\lambda,b,c,p^\alpha) \le p^\gamma N_1,$$

where N_1 counts the number of solutions to the congruence

$$n^2+p^\beta b_0 n \equiv c\lambda_0^{-1} b_0 p^{\beta-\gamma} \pmod{p^{\alpha-\gamma}}, \quad 1 \le n \le p^{\alpha-\gamma}, \quad (n,p)=1. \tag{55}$$

By Hensel's Lemma

$$N(\lambda,b,c,p^\alpha) \le (2+N^*)\, p^\gamma, \tag{56}$$

where N^* counts the number of solutions to

$$n^2+p^\beta b_0 n \equiv c\lambda_0^{-1} b_0 p^{\beta-\gamma} \pmod{p^{\alpha-\gamma}}, \quad 1 \le n \le p^{\alpha-\gamma}, \tag{57}$$

and

$$2n+p^\beta b_0 \equiv 0 \mod p, \quad (n,p)=1. \tag{58}$$

Assume $N^* \neq 0$. If p is odd we may write (55) as

$$\left(n+\frac{p^\beta b_0}{2}\right)^2 \equiv c\lambda_0^{-1} b_0 p^{\beta-\gamma} + \frac{p^{2\beta} b_0^2}{4} \pmod{p^{\alpha-\gamma}}.$$

If there exists an integer n satisfying the conditions (58) then $\beta=\gamma=0$. By (58) and the change of variable $n \to n - \frac{b}{2}$, we see that N^* is bounded by the number of solutions to

$$p^2 n^2 \equiv b\left(c\lambda^{-1}+\frac{b}{4}\right) \pmod{p^\alpha}, \quad 1 \le n \le p^{\alpha-1},$$

and hence

$$N^* \leq 2(4c + b\lambda, p^\alpha)^{1/2}.$$

Combining the above with (56) establishes (54). Suppose next $p = 2$. Recall (56) and the notation used in (57), (58). If $N^* \neq 0$ then there exists an integer n satisfying

$$2n + 2^\beta b_0 \equiv 0. \mod 2,$$

which implies $\beta \geq 1$. We see that any solution to (57) and (58) satisfies

$$\left(n + 2^{\beta-1} b_0\right)^2 \equiv c\lambda_0^{-1} b_0 p^{\beta-\gamma} + p^{2(\beta-1)} b_0^2. \pmod{p^{\alpha-\gamma}}. \tag{59}$$

Suppose first $\beta \geq 2$. Then from $(n, 2) = 1$, we must have

$$c\lambda_0^{-1} b_0 p^{\beta-\gamma} + p^{2(\beta-1)} b_0^2 \equiv 1 \pmod 2,$$

which implies

$$N^* \leq 4,$$

hence from (56)

$$N(\lambda, b, c, p^\alpha) \leq 6(\lambda, p^\alpha).$$

If $\beta = 1$ then reducing both sides of (59) mod 2 and using $(n, 2) = 1$ we must have $\gamma = 1$. Arguing as in the case $p \neq 2$

$$N^* \leq 4(c + b\lambda/4, 2^\alpha)^{1/2} \leq 4(4c + b\lambda, 2^\alpha)^{1/2},$$

and hence from (56)

$$N(\lambda, b, c, p^\alpha) \leq 6(4c + b\lambda, 2^\alpha)^{1/2}(\lambda, 2^\alpha),$$

which completes the proof of the bound (54).

Recall that $h_4(q)$ denotes the smallest square dividing q. Suppose $q = p^{2\alpha}$ is an even prime power and χ a primitive character mod p. Let c be defined by

$$\chi(1 + p^\alpha) = e^{2\pi i c/p^\alpha}.$$

Since χ is primitive, we have $(c, p) = 1$. From the argument of [1, Lemma 2] (see also [7, Lemma 12.2]) we have by (54)

$$\left| \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi\left(1 + \frac{b}{n}\right) e^{2\pi i \lambda n/q} \right| \ll p^\alpha N(\lambda, b, c, p^\alpha) \ll (\lambda, q)(4c + \lambda b, h_4(q))^{1/2} q^{1/2}.$$

Suppose next $q = p^{2\alpha+1}$ is an odd prime power, with $p > 2$ and χ a primitive character mod q. Let c be defined by

$$\chi(1 + p^{\alpha+1}) = e^{2\pi i c/p^\alpha},$$

so that $(c,p)=1$. From the argument of [1, Lemma 4] (see also [7, Lemma 12.3])

$$\left|\sum_{\substack{n=1\\(n,q)=1}}^{q}\chi\left(1+\frac{b}{n}\right)e^{2\pi i\lambda n/q}\right| \ll p^{(2\alpha+1)/2}N(\lambda,b,c,p^{\alpha})+p^{\alpha}N(\lambda,b,c,p^{\alpha+1})$$

$$\ll p^{(2\alpha+1)/2}(\lambda,p^{\alpha})(4c+\lambda b,p^{\alpha})^{1/2}+p^{\alpha}(\lambda,p^{\alpha+1})(4c+\lambda b,p^{\alpha+1})^{1/2}$$

$$\ll (\lambda,q)(4c+\lambda b,h_4(q))^{1/2}q^{1/2}.$$

Finally if $q=2^{2\alpha+1}$, then from the argument of [1, Lemma 3]

$$\left|\sum_{\substack{n=1\\(n,q)=1}}^{q}\chi\left(1+\frac{b}{n}\right)e^{2\pi i\lambda n/q}\right| \ll 2^{1/2}2^{\alpha}N(\lambda,b,c,2^{\alpha})$$

$$\ll (\lambda,q)(4c+\lambda b,h_4(q))^{1/2}q^{1/2}.$$

Combining the above bounds gives the desired result when q is a prime power. For the general case, suppose χ is a primitive character $\pmod q$ and let $q=p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$ be the prime factorization of q. By the Chinese Remainder Theorem we have

$$\chi=\chi_1\chi_2...\chi_k,$$

where each χ_i is a primitive character $\pmod{p_i^{\alpha_i}}$. Let $q_i=q/p_i^{\alpha_i}$, then by the above bounds and another application of the Chinese remainder theorem (see [7, Equation 12.21]), for some absolute constant C

$$\left|\sum_{\substack{n=1\\(n,q)=1}}^{q}\chi\left(1+\frac{b}{n}\right)e^{2\pi i\lambda n/q}\right| =$$

$$\left|\sum_{\substack{n_1=1\\(n_1,p_1)=1}}^{p_1^{\alpha_1}}\cdots\sum_{\substack{n_k=1\\(n_k,p_k)=1}}^{p_k^{\alpha_k}}\chi_1\left(1+\frac{b}{\sum_{i=1}^{k}n_iq_i}\right)e^{2\pi i\lambda n_1/p_1^{\alpha_1}}...\chi_k\left(1+\frac{b}{\sum_{i=1}^{k}n_iq_i}\right)e^{2\pi i\lambda n_k/p_i^{\alpha_k}}\right|$$

$$=\left|\prod_{i=1}^{k}\left(\sum_{\substack{n_i=1\\(n_i,p_i)=1}}^{p_i^{\alpha_i}}\chi_i\left(1+\frac{b}{n_iq_i}\right)e^{2\pi i\lambda n_i/p_i^{\alpha_i}}\right)\right|$$

$$\le\prod_{i=1}^{k}C(\lambda,p_i^{\alpha_i})(\lambda,p_i^{\alpha_i})(4c_{p_i}+\lambda b,h_4(p_i^{\alpha_i}))^{1/2}p_i^{\alpha_i/2},$$

for some integers c_{p_i} satisfying $(c_{p_i}, p_i) = 1$. By the Chinese remainder theorem, there exists an integer c satisfying

$$c \equiv c_{p_i} \pmod{p_i^{\alpha_i}}, \quad 1 \le i \le k, \quad (c,q)=1.$$

Therefore, the above implies

$$\left| \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi\left(1+\frac{b}{n}\right) e^{2\pi i \lambda n/q} \right| \ll (\lambda,q)(4c+\lambda b, h_4(q))^{1/2} q^{1/2+o(1)}.$$

and the result follows from Lemma 23. □

Lemma 25. *Let K, L be natural numbers satisfying $K, L \le q$ and for any two sequences $(\alpha_k)_{k=1}^{K}$ and $(\beta_\ell)_{\ell=1}^{L}$ of complex numbers supported on integers coprime to q and any integer a coprime to q, let*

$$W = \sum_{k \le K} \sum_{\ell \le L} \alpha_k \, \beta_\ell \, \chi(k\ell + a),$$

where χ is a primitive character mod q. Then

$$W \le AB\left(KL^{1/2} + q^{1/4}K^{1/2}L + \frac{KL}{q^{1/8}}\right) q^{o(1)},$$

where

$$A = \max_{k \le K} |\alpha_k| \qquad \textit{and} \qquad B = \max_{\ell \le L} |\beta_\ell|.$$

Proof. By the Cauchy-Schwarz inequality

$$\begin{aligned} |W|^2 &\le A^2 K \sum_{k \le K} \left| \sum_{\ell \le L} \beta_\ell \chi(k\ell + a) \right|^2 \\ &\le A^2B^2K^2L + A^2K \left| \sum_{k \le K} \sum_{\substack{\ell_1, \ell_2 \le L \\ \ell_1 \ne \ell_2}} \beta_{\ell_1} \overline{\beta}_{\ell_2} \chi(k\ell_1 + a) \overline{\chi}(k\ell_2 + a) \right|. \end{aligned}$$

Let

$$W_1 = \sum_{k \le K} \sum_{\substack{\ell_1, \ell_2 \le L \\ \ell_1 \ne \ell_2}} \beta_{\ell_1} \overline{\beta}_{\ell_2} \chi(k\ell_1 + a) \overline{\chi}(k\ell_2 + a),$$

then we have

$$|W_1| \leq \frac{2B^2}{q} \sum_{\substack{\ell_1<\ell_2\leq L \\ (\ell_1,q)=1 \\ (\ell_2,q)=1}} \left| \sum_{s=1}^{q} \sum_{k\leq K} e^{-2\pi i s k/q} \sum_{\lambda=1}^{q} \chi(\lambda + a\ell_1^{-1})\overline{\chi}(\lambda + a\ell_2^{-1}) e^{2\pi i s \lambda/q} \right|$$

$$\leq \frac{2B^2}{q} \sum_{\substack{\ell_1<\ell_2\leq L \\ (\ell_1,q)=1 \\ (\ell_2,q)=1}} \sum_{s=1}^{q} \left| \sum_{k\leq K} e^{-2\pi i s k/q} \right| \left| \sum_{\lambda=1}^{q} \chi(\lambda + a\ell_1^{-1})\overline{\chi}(\lambda + a\ell_2^{-1}) e^{2\pi i s \lambda/q} \right|.$$

By Lemma 23 and Lemma 24, there exists an integer c satisfying $(c,q)=1$ such that

$$\sum_{\substack{\ell_1<\ell_2\leq L \\ (\ell_1,q)=1 \\ (\ell_2,q)=1}} \sum_{s=1}^{q} \left| \sum_{k\leq K} e^{-2\pi i s k/q} \right| \left| \sum_{\lambda=1}^{q} \chi(\lambda + a\ell_1^{-1})\overline{\chi}(\lambda + a\ell_2^{-1}) e^{2\pi i s \lambda/q} \right| \ll$$
$$q^{1/2+o(1)} S,$$

where

$$S = \sum_{\substack{1\leq \ell_1<\ell_2\leq\leq L \\ (\ell_1,q)=1 \\ (\ell_2,q)=1}} \sum_{s=1}^{q} \min\left(K, \frac{1}{||s/q||}\right)(\ell,q)(4c\ell_1\ell_2 + a(\ell_1-\ell_2)s, h_4(q))^{1/2},$$

where $h_4(q)$ denotes the smallest square dividing q. We have

$$S \ll \sum_{\substack{d|q \\ 2^i\leq 2K}} \frac{Kd}{2^i} S_{d,i},$$

where

$$S_{d,i} = \sum_{\substack{1\leq \ell_1<\ell_2\leq L \\ (\ell_1,q)=1 \\ (\ell_2,q)=1 \\ \ell_2-\ell_1\equiv 0 \mod d}} \sum_{s\leq (2^{i+1}-1)q/K} (4c\ell_1\ell_2 + a(\ell_1-\ell_2)s, h_4(q))^{1/2}.$$

Fix some d,i and consider $S_{i,d}$. If $d\neq 1$ and $d|(\ell_2-\ell_1)$ then since $(ac\ell_1\ell_2,q)=1$ we must have

$$(4c\ell_1\ell_2 + a(\ell_1-\ell_2)s, q)^{1/2} \leq 4.$$

This implies

$$S_{d,i} \ll \frac{2^i qL}{K} \sum_{\substack{1\le \ell \le L \\ \ell \equiv 0 \mod d}} 1 \ll \frac{2^i qL^2}{Kd}.$$

If $d=1$ then

$$S_{1,i} = \sum_{e|h_4(q)} e^{1/2} S_{e,1,i},$$

where $S_{e,1,i}$ counts the number of solutions to the congruence

$$4ca^{-1}\ell_1\ell_2 + (\ell_1-\ell_2)s \equiv 0 \mod e, \quad 1 \le \ell_1 < \ell_2 \le L, \quad s \le \frac{(2^{i+1}-1)q}{K}.$$

Fixing ℓ_1, ℓ_2 with $O(L^2)$ choices gives at most

$$\frac{2^{i+1}q}{Ke} + 1,$$

possibilities in remaining variable s. This implies

$$S_{1,i} \ll L^2 \sum_{e|h_4(q)} \left(\frac{2^i q}{Ke^{1/2}} + e^{1/2}\right) q^{o(1)} \ll L^2 \left(\frac{2^i q}{K} + q^{1/4}\right) q^{o(1)},$$

since

$$h_4(q) \le q^{1/2}.$$

Combining the above estimates gives

$$S \ll \left(L^2 + \frac{KL^2}{q^{3/4}}\right) q^{o(1)},$$

and hence

$$|W_1| \le B^2 \left(q^{1/2}L^2 + \frac{L^2K^2}{q^{1/4}}\right) q^{o(1)}.$$

This implies

$$|W|^2 \le A^2B^2 \left(K^2L + q^{1/4}K^{1/2}L + \frac{KL}{q^{1/8}}\right) q^{o(1)},$$

which completes the proof. □

Next, we use an idea of Garaev [5] to derive a variant of Lemma 25 in which the summation limits over ℓ depend on the parameter k.

Lemma 26. *Let K, L be natural numbers and let the sequences $(L_k)_{k=1}^K$ and $(M_k)_{k=1}^K$ of nonnegative integers be such that $M_k < L_k \le L$ for each k. For any two sequences $(\alpha_k)_{k=1}^K$ and $(\beta_\ell)_{\ell=1}^L$ of complex numbers* *supported on integers coprime to q and for any integer a coprime to q, let*

$$\widetilde{W} = \sum_{k \le K} \sum_{M_k < \ell \le L_k} \alpha_k \, \beta_\ell \, \chi(k\ell + a).$$

Then

$$\widetilde{W} \ll AB \left(KL^{1/2} + (1 + K^{1/2} q^{-1/2}) q^{1/4} K^{1/2} L \right) (Lq)^{o(1)},$$

where

$$A = \max_{k \le K} |\alpha_k| \qquad \text{and} \qquad B = \max_{\ell \le L} |\beta_\ell|.$$

Proof. For real z we denote

$$e_L(z) = \exp(2\pi i z/L).$$

For each inner sum, using the orthogonality of exponential functions, we have

$$\begin{aligned} \sum_{M_k < \ell \le L_k} \beta_\ell \, \chi(k\ell + a) &= \sum_{\ell \le L} \sum_{M_K < s \le L_k} \beta_\ell \, \chi(k\ell + a) \cdot \frac{1}{L} \sum_{-\frac{1}{2}L < r \le \frac{1}{2}L} e_L(r(\ell - s)) \\ &= \frac{1}{L} \sum_{-\frac{1}{2}L < r \le \frac{1}{2}L} \sum_{M_k < s \le L_k} e_L(-rs) \sum_{\ell \le L} \beta_\ell \, e_L(r\ell) \, \chi(k\ell + a). \end{aligned}$$

In view of [7, Bound (8.6)], for each $k \le K$ and every integer r such that $|r| \le \frac{1}{2}L$ we can write

$$\sum_{M_k < s \le L_k} e_L(-rs) = \sum_{s \le L_k} e_L(-rs) - \sum_{s \le M_k} e_L(-rs) = \eta_{k,r} \frac{L}{|r| + 1},$$

for some complex number $\eta_{k,r} \ll 1$. Thus, if we put $\widetilde{\alpha}_{k,r} = \alpha_k \, \eta_{k,r}$ and $\widetilde{\beta}_{\ell,r} = \beta_\ell \, e_L(r\ell)$, it follows that

$$\sum_{K_0 < k \le K} \sum_{M_k < \ell \le L_k} \alpha_k \, \beta_\ell \, \chi(k\ell + a) = \sum_{-\frac{1}{2}L < r \le \frac{1}{2}L} \frac{1}{|r| + 1} \sum_{k \le K} \sum_{\ell \le L} \widetilde{\alpha}_{k,r} \widetilde{\beta}_{\ell,r} \, \chi(k\ell + a).$$

Applying Lemma 25 with the sequences $(\widetilde{\alpha}_{k,r})_{k=1}^{K}$ and $(\widetilde{\beta}_{\ell,r})_{\ell=1}^{L}$, and noting that

$$\sum_{-\frac{1}{2}L < r \le \frac{1}{2}L} \frac{1}{|r| + 1} \ll \log L,$$

we derive the stated bound. □

6. Proof of Theorem 1

We first partition the sum $S_a(q;N)$ into dyadic intervals to get

$$S_a(q;N) = \sum_{2^i \le N} \tilde{S}_a(q;2^i),$$

where

$$\tilde{S}_a(q;2^i) = \sum_{\substack{n \le N \\ 2^i \le n < 2^{i+1}}} \Lambda(n)\chi(n+a).$$

This implies there exists some integer $K \le N$ such that

$$S_a(q;N) \ll N^{o(1)} \tilde{S}_a(q;K).$$

Hence to establish Theorem 1 it is sufficient to show

$$\tilde{S}_a(q;K) \ll .$$

We apply Lemma 2 with

$$J = 9, \quad Z = K^{1/9}, \tag{60}$$

to get

$$\left|\tilde{S}_a(q;K)\right|$$
$$= \left| \sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{\substack{K \le m_1 \dots m_j n_1 \dots n_j < 2K \\ m_1 \dots m_j n_1 \dots n_j \le N \\ m_1, \dots, m_j \le Z \\ (m_1 \dots m_j n_1 \dots n_j, q) = 1}} \mu(m_1) \dots \mu(m_j) \log n_1 \chi(m_1 \dots m_j n_1 \dots m_j + a) \right|.$$

Letting S_j be defined by

(61)
$$S_j = \sum_{\substack{K \le m_1 \dots m_j n_1 \dots n_j < 2K \\ m_1 \dots m_j n_1 \dots n_j \le N \\ m_1, \dots, m_j \le Z \\ (m_1 \dots m_j n_1 \dots n_j, q) = 1}} \mu(m_1) \dots \mu(m_j) \log n_1 \chi(m_1 \dots m_j n_1 \dots m_j + a),$$

we see that

$$|\tilde{S}_a(q;K)| \le \sum_{j=1}^{J} \binom{J}{j} |S_j|. \tag{62}$$

We partition summation occuring in (61) into dyadic intervals to get

$$S_j = \sum_{k_1,\dots,k_j} \sum_{\ell_1,\dots,\ell_j} \sum_{\substack{m_1\dots m_j n_1\dots n_j \le N \\ m_1,\dots,m_j \le Z \\ 2^{k_i-1}\le m_i < 2^{k_i} \\ 2^{\ell_i-1}\le n_i \le 2^{\ell_i} \\ (m_1\dots m_j n_1\dots n_j, q)=1}} \mu(m_1)\dots\mu(m_j)\log n_1 \chi(m_1\dots m_j n_1 \dots m_j + a).$$

Taking a maximum over j in (62) then a maximum over dyadic partitions, there exists positive integers $M_1,\dots,M_j,N_1,\dots,N_j$ satisfying

$$2M_1,\dots,2M_j \le Z \quad \text{and} \quad K \ll M_1\dots M_j N_1 \dots N_j \ll K,$$

such that

$$|\tilde{S}_a(q;K)| \le N^{o(1)} S,$$

where

$$S = \sum_{\substack{m_1\dots m_j n_1\dots n_j \le N \\ M_i\le m_i < 2M_i \\ N_i \le n_i < 2N_i \\ (m_1\dots m_j n_1\dots n_j, q)=1}} \mu(m_1)\dots\mu(m_j)\log n_1 \chi(m_1\dots m_j n_1 \dots n_j + a).$$

Recalling (60) and applying Lemma 3, either there exists some $1 \le i \le j$ such that

$$N_i \gg K^{4/9}, \tag{63}$$

or there exists subsets $\mathcal{I},\mathcal{J} \subseteq \{1,\dots,j\}$ such that

$$K^{1/9} \ll \prod_{i\in\mathcal{I}} M_i \prod_{i\in\mathcal{J}} N_i \ll K^{1/3}. \tag{64}$$

If (63) then we apply Lemma 21 to summation over n_i. This gives

$$S \ll \frac{q^{1/9}K^{1+o(1)}}{N_i^{1/3}} \ll q^{1/9}K^{23/27+o(1)} \ll q^{1/9}N^{23/27+o(1)},$$

provided

$$N_i \le q^{23/42}. \tag{65}$$

If (65) does not hold then we use Lemma 4. Noting that

$$N_i \le N \le q,$$

we get

$$S \ll \frac{N^{1+o(1)}q^{1/2}}{N_i} \ll \frac{N^{1+o(1)}}{q^{1/21}}.$$

In either case we have

$$S \ll \frac{N^{1+o(1)}}{q^{1/12}} + q^{1/9}N^{23/27+o(1)}. \tag{66}$$

Suppose next (64). In this case we apply Lemma 26 with parameter

$$L = \prod_{i\in\mathcal{I}} M_i \prod_{i\in\mathcal{J}} N_i, \qquad \frac{K}{L}.$$

Using (64) and the bound $K \ll N$, we get

$$\begin{aligned} S &\ll \left(\frac{K}{L^{1/2}} + q^{1/4}K^{1/2}L^{1/2} + \frac{K}{q^{1/8}}\right) q^{o(1)} \\ &\ll \left(N^{17/18} + q^{1/4}N^{2/3} + \frac{N}{q^{1/8}}\right) q^{o(1)}. \end{aligned}$$

Hence if either (63) or (64)

$$S \ll \left(\frac{N}{q^{1/21}} + q^{1/9}N^{23/27} + N^{17/18} + q^{1/4}N^{2/3}\right) N^{o(1)}.$$

We may assume $N \geq q^{3/4}$ since otherwise the above bound is trivial. In the range $q^{3/4} \leq N \leq q$ we may simplify the above to

$$S \ll q^{1/9+o(1)}N^{23/27},$$

and the result follows from (62).

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Max Planck Institute of Mathematics, Bonn, Germany
E-mail address: bryce.kerr89@gmail.com