Operator product expansion coefficients from the nonperturbative functional renormalization group

Félix Rose,[1,2] Carlo Pagani,[3,4] and Nicolas Dupuis[5]

[1] Max Planck Institute of Quantum Optics, Hans-Kopfermann-Str. 1, 85748 Garching, Germany
[2] Munich Center for Quantum Science and Technology (MCQST), Schellingstraße 4, 80799 Munich, Germany
[3] Université Grenoble Alpes, Centre National de la Recherche Scientifique, Laboratoire de Physique et Modélisation des Milieux Condensés, 38000 Grenoble, France
[4] Institute of Physics, Johannes Gutenberg University, Staudingerweg 7, 55099 Mainz, Germany
[5] Sorbonne Université, CNRS, Laboratoire de Physique Théorique de la Matière Condensée, LPTMC, F-75005 Paris, France

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Using the nonperturbative functional renormalization group (FRG) within the Blaizot-Méndez-Galain-Wschebor approximation, we compute the operator product expansion (OPE) coefficient c_{112} associated with the operators $\mathcal{O}_1 \sim \varphi$ and $\mathcal{O}_2 \sim \varphi^2$ in the three-dimensional O(N) universality class and in the Ising universality class ($N = 1$) in dimensions $2 \leq d \leq 4$. When available, exact results and estimates from the conformal bootstrap and Monte-Carlo simulations compare extremely well to our results, while FRG is able to provide values across the whole range of d and N considered.

CONTENTS

I. INTRODUCTION

The nonperturbative functional renormalization group (FRG) provides us with a versatile technique to study strongly correlated systems. It has been used in many models of quantum and statistical field theory ranging from statistical physics and condensed matter to high-energy physics and quantum gravity [1–3]. Besides the interest in models where perturbative approaches or numerical methods are difficult for various reasons, there is an ongoing effort to characterize and quantify the efficiency and the accuracy of the FRG approach by considering well-known models of statistical physics. It is now proven that the FRG yields very accurate values of the critical exponents associated with the Wilson-Fisher fixed point of O(N) models [4, 5], comparable with the best estimates from field-theoretical perturbative RG [6, 7], Monte Carlo simulations [8–13] or conformal bootstrap [14–17]. The FRG also allows the computation of universal quantities defined away from the critical point, such as universal scaling functions [18–21] or universal amplitude ratios [22], again in remarkable agreement with Monte Carlo simulations when available.

On the other hand, the operator product expansion (OPE) has received little attention in the framework of the FRG until recently [23–29]. Wilson and Kadanoff suggested independently that in a quantum field theory the product of two operators in the short distance limit is equivalent to an infinite sum of operators multiplied by possibly singular functions when inserted in any correlation function [30–33]. The validity of the OPE has been proven to all orders in pertubation theory [34] and can be established in full generality in the case of conformal field theories [35]. Indeed, the OPE has been fundamental in the study of conformal field theories in two and higher dimensions [36, 37]. In this context, the conformal bootstrap program [37–40] has lead to a large number of precise results. The OPE has been instrumental as well in studies regarding quantum chromodynamics [41] and condensed matter, where it has been used to derive the thermodynamic properties of quantum gases [42, 43].

Despite the fact that both the FRG formalism and the OPE offer non-perturbative approaches to quantum field theory, it is not yet clear to what extent these two aspects can be usefully combined to extract information regarding the non-perturbative regime of a field theory.

From the perspective of perturbation theory, the FRG provides a useful framework that allows one to prove the existence of the OPE perturbatively [23–27]. Moreover, by following the proposal of Cardy relating the OPE coefficients to the second order terms in the expansion of the beta functions around a fixed point [44], the standard perturbative renormalization group has been used to derive certain OPE coefficients within the ϵ expansion [45]; we refer to [46] for an FRG perspective on these issues based on a geometric approach to theory space.

In principle, one may reconstruct from the FRG the full operator product and express the latter as an OPE [28, 29]. However, this may be rather cumbersome in practice. For a conformally invariant fixed point theory [47], a further possibility explored in [29] consists in extracting the OPE coefficient from three-point functions. It has been shown that within this approach it is possible to calculate the OPE coefficients in the epsilon expansion.

The main quantities of interest in the FRG are the effective action, defined as the Legendre transform of the free energy, and the one-particle irreducible (1PI) vertices. Taking the Wilson-Fisher fixed point of the O(N) model as an example, we show how the OPE coefficient c_{112} associated with the operators $\mathcal{O}_1 \sim \varphi$ and $\mathcal{O}_2 \sim \varphi^2$ can be deduced from a small number of low-order 1PI vertices. One difficulty in the computation of OPE coefficients is that the latter are determined by the full momentum dependence of the vertices in the critical regime. For this reason, one has to go beyond the derivative expansion in order to accurately determine the OPE coefficients. The latter can be computed in the so-called Blaizot-Méndez-Galain-Wschebor (BMW) approximation that enables the determination of the momentum dependence of the correlation functions [48–50]. This approximation scheme has been used in the past to obtain the spectral function of the "Higgs" amplitude mode in the $(2+1)$-dimensional O(N) model [51] providing an estimate of the Higgs mass that has been confirmed by subsequent numerical simulations of lattice models [52–54].

The outline of the paper is as follows. In Section II we recall the relation between the OPE coefficients and the two- and three-point functions in momentum space, focusing on the coefficient c_{112} in the d-dimensional O(N) model. We then show how to relate c_{112} to the 1PI vertices. Finally we briefly describe the nonperturbative FRG formalism and the BMW approximation. In Section III the results obtained from a numerical solution of the flow equations are discussed for the three-dimensional O(N) model and the Ising university class ($N = 1$) in dimensions $2 \le d \le 4$, and compared with exact values in some particular cases and estimates from conformal bootstrap and Monte Carlo as well as ϵ and large-N expansions.

II. OPE COEFFICIENTS IN THE EFFECTIVE ACTION FORMALISM

A. Correlation functions in momentum space

We consider a critical, conformally invariant, theory. For fields $\mathcal{O}_a(x)$ (be them composite or not) with scaling dimensions Δ_a, the two- and three-point correlation functions are given by

$$\langle \mathcal{O}_a(x)\mathcal{O}_a(y)\rangle = \frac{1}{|x-y|^{2\Delta_a}} \tag{1}$$

and

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\rangle = \frac{c_{abc}}{x_{12}^{\Delta_a+\Delta_b-\Delta_c} x_{23}^{\Delta_b+\Delta_c-\Delta_a} x_{13}^{\Delta_a+\Delta_c-\Delta_b}} \tag{2}$$

where $x_{12} = |x_1 - x_2|$, etc. Equation (1) assumes a proper normalization of the fields and the coefficient c_{abc} in (2) can be identified with the OPE coefficient [55]. Since in practice we shall work in momentum space, it is convenient to consider the Fourier transformed correlation functions. For the two-point one,

$$\langle \mathcal{O}_a(p)\mathcal{O}_a(-p)\rangle = \int_x \frac{e^{-ipx}}{|x|^{2\Delta_a}} = \frac{A_d(\Delta_a)}{|p|^{d-2\Delta_a}}, \tag{3}$$

where

$$A_d(\Delta) = 4^{d/2-\Delta}\pi^{d/2}\frac{\Gamma(d/2-\Delta)}{\Gamma(\Delta)} \tag{4}$$

with $\Gamma(x)$ the gamma function and d the dimension. The Fourier transform $\langle \mathcal{O}_a(p_1)\mathcal{O}_a(p_2)\mathcal{O}(p_3)\rangle$ is given by a complicated expression but it is sufficient to consider the limit $|p_1| \gg |p_2|$, where

$$\langle \mathcal{O}_a(p_1)\mathcal{O}_b(p_2)\mathcal{O}_c(-p_1-p_2)\rangle \simeq \frac{c_{abc}A_d((\Delta_a-\Delta_b+\Delta_c)/2)A_d(\Delta_b)}{|p_1|^{d-\Delta_a+\Delta_b-\Delta_c}|p_2|^{d-2\Delta_b}}, \tag{5}$$

to extract the coefficient c_{abc} [29]. Equation (5) entails that the OPE coefficient c_{abc} can be deduced from the three-point function (5) once the fields have been properly normalized in order to satisfy (3).

B. O(N) model and Wilson-Fisher fixed point

In the following we consider the O(N) model in d dimensions defined by the action

$$S[\varphi] = \int_x \left\{ \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{r_0}{2}\varphi^2 + \frac{u_0}{4!N}\left(\varphi^2\right)^2 \right\} \tag{6}$$

and regularized by a UV momentum cutoff Λ. $\varphi = (\varphi_1, \cdots, \varphi_N)$ is a N-component field. The model can be tuned to its critical point by varying r_0. The correlation functions are then scale and conformal invariant in the momentum range $|p| \ll p_G$ where $p_G \sim u_0^{1/(d-4)}$ is the Ginzburg scale. In the following we shall only be interested in the critical point and the scaling limit $|p| \ll p_G$; we refer to [56] for an overview of the various regimes of the $O(N)$ model.

We focus on the operators

$$\begin{aligned}\mathcal{O}_1(x) &= \mathcal{N}_1 \varphi_i(x), \\ \mathcal{O}_2(x) &= \mathcal{N}_2 \frac{\varphi(x)^2}{2}\end{aligned} \tag{7}$$

(the index i is arbitrary) and the OPE coefficient c_{112}. Note that the correlation functions $\langle \mathcal{O}_1 \mathcal{O}_1 \rangle$ and $\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_2 \rangle$ are independent of i at criticality and in the whole disordered phase. $\mathcal{N}_1$ and $\mathcal{N}_2$ are normalization constants that ensure that $\langle \mathcal{O}_1(p) \mathcal{O}_1(-p) \rangle$ and $\langle \mathcal{O}_2(p) \mathcal{O}_2(-p) \rangle$ are given by (3) in the scaling limit $|p| \ll p_G$. The scaling dimension

$$\Delta_1 = [\varphi_i] = \frac{d-2+\eta}{2} \tag{8}$$

is related to the anomalous dimension η while

$$\Delta_2 = [\varphi^2] = d - \frac{1}{\nu} \tag{9}$$

where ν is the correlation-length exponent.

To deal with the composite field $\mathcal{O}_2$, in addition to the linear source J we introduce in the partition function a source h coupled to φ^2,

$$\mathcal{Z}[J,h] = \int \mathcal{D}[\varphi]\, e^{-S[\varphi] + \int_x (J\varphi + h\varphi^2)}. \tag{10}$$

The correlation functions of interest, besides the propagator $G(x-y) = \langle \varphi_i(x) \varphi_i(y) \rangle_c$ for $h=0$ ($\langle \cdots \rangle_c$ stands for the connected correlation function), are the scalar susceptibility

$$\chi_s(x-y) = \langle \varphi(x)^2 \varphi(y)^2 \rangle_c = \frac{\delta^2 \ln \mathcal{Z}[J,h]}{\delta h(x) \delta h(y)} \bigg|_{J=h=0} \tag{11}$$

and the three-point function

$$\begin{aligned}\chi(x,y,z) &= \langle \varphi_i(x) \varphi_i(y) \varphi(z)^2 \rangle_c \\ &= \frac{\delta^3 \ln \mathcal{Z}[J,h]}{\delta J_i(x) \delta J_i(y) \delta h(z)} \bigg|_{J=h=0}.\end{aligned} \tag{12}$$

Here and in the following, there is no implicit summation over the index i. The computation of $G(p)$ and $\chi_s(p)$ allows us to determine the normalization constants $\mathcal{N}_1$ and $\mathcal{N}_2$ since at criticality

$$\begin{aligned}G(p) &= \frac{1}{\mathcal{N}_1^2} \frac{A_d(\Delta_1)}{|p|^{d-2\Delta_1}}, \\ \chi_s(p) &= \frac{4}{\mathcal{N}_2^2} \frac{A_d(\Delta_2)}{|p|^{d-2\Delta_2}}\end{aligned} \tag{13}$$

for $p \to 0$. The knowledge of $\chi(p_1, p_2, -p_1-p_2)$ then yields the OPE coefficient c_{112} using (5).

C. Effective action

The effective action

$$\Gamma[\phi, h] = -\ln \mathcal{Z}[J,h] + \int_x \sum_i J_i \phi_i \tag{14}$$

is defined as the Legendre transform of the free energy [57]. The source J and the order parameter field ϕ are related by

$$\phi_i(x) = \frac{\delta \ln \mathcal{Z}[J,h]}{\delta J_i(x)} \quad \text{or} \quad J_i(x) = \frac{\delta \Gamma[\phi,h]}{\delta \phi_i(x)}. \tag{15}$$

All correlation functions for $h=0$ can be obtained from the one-particle irreducible (1PI) vertices

$$\begin{aligned}&\Gamma^{(n,m)}_{i_1 \cdots i_n}(x_1 \cdots x_n, y_1 \cdots y_m) = \\ &\qquad \frac{\delta^{n+m} \Gamma[\phi,h]}{\delta \phi_{i_1}(x_1) \cdots \delta \phi_{i_n}(x_n) \delta h(y_1) \cdots \delta h(y_m)} \bigg|_{\phi=h=0}\end{aligned} \tag{16}$$

where, assuming the absence of spontaneously broken symmetry, we have set $\phi = 0$. In particular, the propagator

$$G(p) = \left[\Gamma^{(2,0)}_{ii}(p) \right]^{-1} \tag{17}$$

is related to the inverse of the two-point vertex computed with a vanishing source $h=0$. The other two correlation functions of interest are given by [51]

$$\begin{aligned}\chi_s(p) &= -\Gamma^{(0,2)}(p), \\ \chi(p_1,p_2) &= -G(p_1) \Gamma^{(2,1)}_{ii}(p_1,p_2) G(p_2),\end{aligned} \tag{18}$$

where we have used the fact that $\Gamma^{(1,1)}_i$ vanishes when evaluated for $\phi = 0$. To alleviate the notations we do not write the last argument of the three-point vertices, e.g., $\Gamma^{(2,1)}_{ii}(p_1,p_2) \equiv \Gamma^{(2,1)}_{ii}(p_1,p_2,-p_1-p_2)$.

We are now in a position to relate the OPE coefficient c_{112} to the 1PI vertices at criticality. From Eqs. (13) we obtain the normalization constants

$$\mathcal{N}_1^2 = A_d(\Delta_1) \lim_{p \to 0} \frac{\Gamma^{(2,0)}_{ii}(p)}{|p|^{d-2\Delta_1}}, \tag{19}$$

$$\mathcal{N}_2^2 = -4 A_d(\Delta_2) \lim_{p \to 0} \frac{|p|^{2\Delta_2 - d}}{\Gamma^{(0,2)}(p)}. \tag{20}$$

Considering (5) in the limit $p_2 = 0$ and $p_1 \to 0$, we finally deduce

$$c_{112} = -\frac{\mathcal{N}_2}{2\mathcal{N}_1^2} \frac{A_d(\Delta_1)}{A_d(\Delta_2/2)} \lim_{p \to 0} \frac{\Gamma^{(2,1)}_{ii}(p,0)}{|p|^{\Delta_2 - 2\Delta_1}}. \tag{21}$$

Equations (19) to (21) are the basic ingredients to determine the OPE coefficient c_{112} in the effective action formalism. In Appendices A and B, we show that they yield the known results

$$c_{112} = \sqrt{\frac{2}{N}} \tag{22}$$

for the free theory ($u_0 = 0$), and

$$c_{112} = \sqrt{\frac{2}{N}} \left[\frac{\Gamma(d-2)}{(d/2-2)} \frac{\sin(\pi d/2)}{\pi} \right]^{1/2} \frac{1}{\Gamma(d/2-1)} \tag{23}$$

in the large-N limit and for $d < 4$.

D. FRG formalism and BMW approximation

The nonperturbative FRG allows one to compute the effective action beyond standard perturbation theory [1–3]. Fluctuations are regularized by the infrared regulator term

$$\Delta S_k[\varphi] = \frac{1}{2} \int_q \sum_i \varphi_i(-q) R_k(q) \varphi_i(q), \tag{24}$$

where the momentum scale k varies from the UV cutoff Λ down to zero. A possible choice for the cutoff function R_k is

$$R_k(q) = Z_k q^2 r\left(\frac{q^2}{k^2}\right), \tag{25}$$

with the function $r(y)$ taken to be for instance

$$r_{\rm W}(y) = \frac{\alpha}{e^y - 1} \quad \text{or} \quad r_{\rm E}(y) = \alpha e^{-y}/y. \tag{26}$$

$r_{\rm W}(y)$ and $r_{\rm E}(y)$ define respectively the so-called Wetterich and exponential regulators. In either case, α is a constant of order one and Z_k a field renormalization factor which varies as $k^{-\eta}$ at criticality [58]. Thus the regulator suppresses fluctuations with momenta $|q| \lesssim k$ but leaves unaffected those with $|q| \gtrsim k$. The partition function

$$\mathcal{Z}_k[J,h] = \int \mathcal{D}[\varphi]\, e^{-S[\varphi]-\Delta S_k[\varphi]+\int_x (J\varphi + h\varphi^2)} \tag{27}$$

is now k dependent. The scale-dependent effective action

$$\Gamma_k[\phi,h] = -\ln \mathcal{Z}_k[J,h] + \int_x \sum_i J_i \phi_i - \Delta S_k[\phi] \tag{28}$$

is defined as a slightly modified Legendre transform which includes the subtraction of $\Delta S_k[\phi]$. Assuming that for $k = \Lambda$ the fluctuations are completely frozen by the regulator term,

$$\Gamma_\Lambda[\phi,h] = S[\phi] - \int_x h\phi^2. \tag{29}$$

On the other hand, the effective action of the O(N) model (6) is given by $\Gamma_{k=0}$ since $R_{k=0}$ vanishes. The

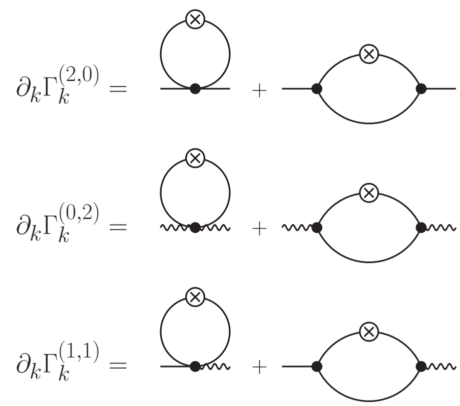

Figure 1. Diagrammatic representation of the RG equations of $\Gamma_k^{(2,0)}$, $\Gamma_k^{(0,2)}$ and $\Gamma_k^{(1,1)}$. Signs and symmetry factors are not shown. The vertex $\Gamma_k^{(n,m)}$ is represented by a black dot with n solid lines and m wavy lines and the solid lines connecting vertices stand for the propagator $G_k = (\Gamma_k^{(2,0)} + R_k)^{-1}$. The cross stands for $\partial_k R_k$.

FRG approach aims at determining $\Gamma_{k=0}$ from Γ_Λ using Wetterich's equation [59–61]

$$\partial_k \Gamma_k[\phi,h] = \frac{1}{2} \mathrm{Tr} \left\{ \partial_k R_k (\Gamma_k^{(2,0)}[\phi,h] + R_k)^{-1} \right\}. \tag{30}$$

The infinite hierarchy of flow equations satisfied by the k-dependent 1PI vertices $\Gamma_k^{(n,m)}$ can be obtained from (30) by taking functional derivatives wrt ϕ and h. The presence of the source h in addition to the field ϕ allows one to follow the flow of composite fields, an approach which proved to be useful in tackling a wide range of issues [51, 62–70].

In the BMW approximation [48–50], one considers the flow equations of the 1PI vertices in a uniform field ϕ even if one is eventually interested in the vanishing field configuration. These equations are shown diagrammatically in Fig. 1 for $\Gamma_k^{(2,0)}$, $\Gamma_k^{(0,2)}$ and $\Gamma_k^{(1,1)}$. Since the regulator $\partial_k R_k$ in Eq. (30) restricts the loop momentum to small values $|q| \lesssim k$, whereas the regulator term ΔS_k ensures that the vertices are regular functions of p_i^2/k^2, one can set $q = 0$ in the vertices $\Gamma_k^{(n,m)}$. Noting then that a vertex with a vanishing momentum can be related to a lower-order vertex, e.g.,

$$\begin{aligned} \Gamma^{(3,0)}_{k,ijl}(\mathbf{p},-\mathbf{p},0;\phi) &= \frac{\partial \Gamma^{(2,0)}_{k,ij}(\mathbf{p};\phi)}{\partial \phi_l}, \\ \Gamma^{(2,1)}_{k,ij}(\mathbf{p},0,-\mathbf{p};\phi) &= \frac{\partial \Gamma^{(1,1)}_{k,i}(\mathbf{p};\phi)}{\partial \phi_j}, \end{aligned} \tag{31}$$

we obtain a closed set of equations satisfied by $\Gamma_k^{(2,0)}(\mathbf{p},\phi)$, $\Gamma_k^{(0,2)}(\mathbf{p},\phi)$ and $\Gamma_k^{(1,1)}(\mathbf{p},\phi)$; see Ref. [51] for the explicit expressions. These equations, together with the expression (31) of $\Gamma_k^{(2,1)}(\mathbf{p},0,-\mathbf{p};\phi)$ are sufficient to obtain the vertices necessary to determine the normalization constants $\mathcal{N}_1$, $\mathcal{N}_2$ and the OPE coefficient c_{112}.

III. NUMERICAL RESULTS

The flow equations are integrated numerically, see e.g. Refs. [50, 51] for details. We work with dimensionless variables, $\tilde{p} = p/k$ and $\tilde{\rho} = Z_k k^{2-d}\rho$. The field dependence of the potential and the vertices is discretized on a finite and evenly spaced grid $\tilde{\rho} \in [0, \tilde{\rho}_{\max}]$ comprising N_ρ points, while the momentum dependence of the vertices is approximated by Chebyschev polynomials of order N_p defined on $[0, \tilde{p}_{\max}]$. The integration of the flow with respect to the RG scale k is done with an adaptive step integration. Convergence of the results with respect to the parameters has been verified; their typical range are $N_\rho = 40$–80, $\tilde{\rho}_{\max} = 4$–8, $\tilde{p}_{\max} = 4$–10 and $N_p = 20$–30 with the precise value depending on d and N.

For each universality class set by d and N and each choice of the cutoff function (25) parameterized by α, the critical point is found by tuning the initial condition of the flow. This enables the computation of $G(p)$, $\chi_s(p)$ and $\Gamma^{(2,1)}(p, 0)$ [Eqs. (13) and (31)] at criticality, from which one fits the values of the critical exponents η and ν (or equivalently $\Delta_{1,2}$) and normalization constants $\mathcal{N}_{1,2}$, yielding c_{112} through Eq. (21).

A crucial question is that of the regulator dependence. Indeed, while Eq. (30) is exact, any approximation scheme such as BMW introduces a regulator dependence to the results. In order to provide a meaningful prediction for a physical quantity $Q(\alpha)$, a choice of the regulator must be made. The usual rationale is the so-called principle of minimum sensitivity (PMS), according to which the best value of α is that for which the regulator dependence of $Q(\alpha)$ is minimal, i.e., for which $\partial_\alpha Q(\alpha) = 0$, or failing that for which $|\partial_\alpha Q(\alpha)|$ is minimal.

However, the PMS for c_{112}, shown for the $3d$ Ising universality ($N = 1$) class in Fig. 2, does not provide a satisfactory result. Indeed, for a given regulator, c_{112} is a monotonous concave function of α, with no extremum or inflection point, varying by about 3% over the range of regulators considered. As a consequence, we choose the regulator that fulfills the PMS for the anomalous dimension η. The value thus obtained for c_{112} depends only weakly on the family of regulators considered, with a variation of about 1.15% between the Wetterich and exponential regulators. The regulator dependency is slightly smaller than the 1.2% difference with the conformal bootstrap estimate.

As a side note, we point out a recent proposal for an alternative way to fix the regulator dependence for conformally invariant theories, the principle of maximal conformality (PMC) [71]. Conformal invariance implies a set of (modified) Ward identities associated with scale and special conformal transformations (SCT). While scale invariance is always fulfilled at the fixed point, invariance under SCT is broken within the derivative expansion at high order. PMC suggests to choose the regulator that minimizes the symmetry breaking. While in the present case it is not straightforward to implement the PMC for BMW, because the Ward identities are either trivially fulfilled or involve high-order vertices that cannot be computed using the BMW approximation, its implementation for the derivative expansion shows that the PMS for η and PMC yield very close results, providing a further argument in favor of our regulator choice.

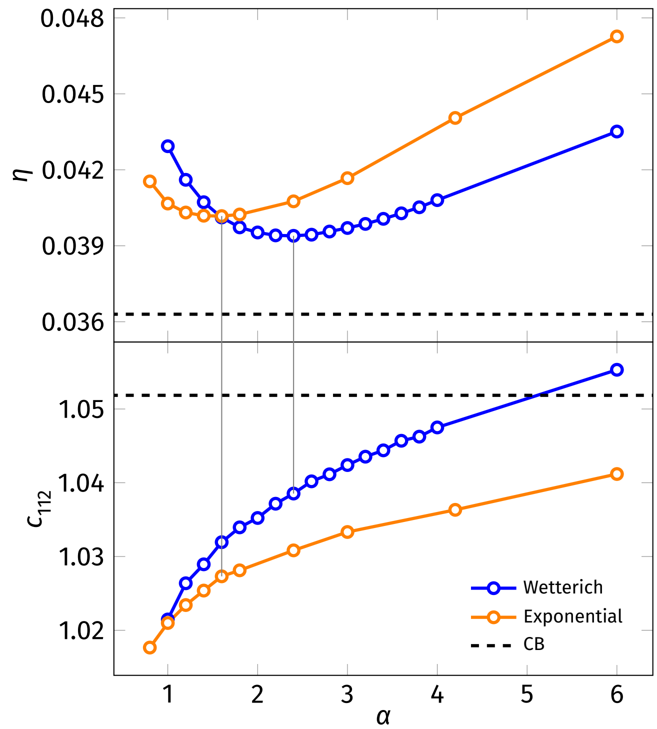

Figure 2. Dependence of the critical exponent η (top) and the OPE coefficient c_{112} (bottom) of the $3d$ Ising universality class ($N = 1$) on the coefficient α for the Wetterich (blue) and exponential (orange) regulators. For reference, the conformal bootstrap estimates $\eta_{\rm CB} = 0.036308$ and $c_{112,\rm CB} = 1.0518537$ are shown as black dashed lines [14]. Applying the PMS yields optimal parameters $\alpha_{\rm E} = 1.6$ and $\alpha_{\rm W} = 2.4$ for the exponential and Wetterich regulators (respectively subscript E and W), shown by gray vertical lines, with corresponding values $\eta_{\rm E} = 0.0402$, $c_{112,\rm E} = 1.027$, $\eta_{\rm W} = 0.0394$ and $c_{112,\rm W} = 1.039$. The value we retain for c_{112} is $c_{112,\rm W}$, given that it corresponds to the extremal value of η across both families of regulators.

A. Ising university class in dimensions $2 \le d \le 4$

We first consider the OPE coefficients of the Ising university class ($N = 1$) for dimensions d between the lower and upper critical dimensions $d = 2$ and 4, for which the results are shown in Fig. 3 and Table I. The FRG results can be compared to the exact values in $d = 2$ and 4, the conformal bootstrap [14, 72] and Monte Carlo [73, 74] estimates in $d = 3$ and the ϵ expansion up to third or-

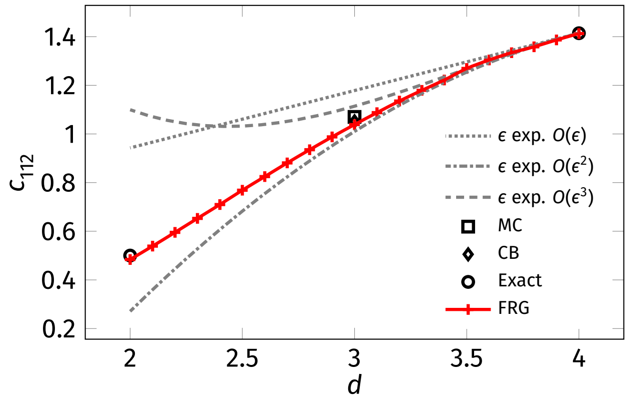

Figure 3. OPE coefficient c_{112} of the Ising university class as a function of the dimension d. The solid red crosses are obtained from FRG, with the full line a guide to the eye. The black symbols correspond to reference estimates from Monte Carlo (square) [73] and conformal bootstrap (diamond) [14] and the exact values in $d = 2$ and 4. The dashed gray line are given by the ϵ expansion [Eq. (32)] about $d = 4$ to order $\mathcal{O}(\epsilon)$ (dotted), $\mathcal{O}(\epsilon^2)$ (dashdotted) and $\mathcal{O}(\epsilon^3)$ (dashed) [77].

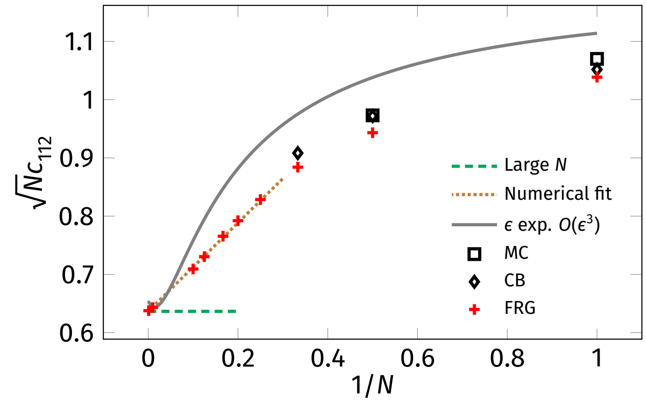

Figure 4. Rescaled OPE coefficient $\sqrt{N}c_{112}$ of the three-dimensional O(N) model as a function of the inverse number of field components $1/N$. The solid red crosses are obtained from FRG. The horizontal dashed green line shows the large-N result to leading order. The gray line is given by the ϵ expansion to order $\mathcal{O}(\epsilon^3)$ [77]. The black symbols correspond to estimates from Monte Carlo (square) [73, 74] and conformal bootstrap (diamond) [14, 72]. The brown dotted line is the numerical fit (34) to the FRG large-N data.

der [75–77],

$$c_{112} = \sqrt{2}\left(1 - \frac{1}{6}\epsilon - \frac{77}{648}\epsilon^2 + \frac{3726\zeta(3) - 1915}{34992}\epsilon^3\right) + \mathcal{O}(\epsilon^4) \qquad (32)$$

where $\zeta(3)$ is Apéry's constant.

Compared to the best results (exact in $d = 2$ and 4, conformal bootstrap in $d = 3$), the FRG has an error smaller than 2%, whereas the error from the ϵ expansion is 10% in $d = 3$ and 100% in $d = 2$. The ϵ expansion is accurate for $d \gtrsim 3.5$, since for $d = 3.5$ both order $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$ agree with FRG with a relative difference of respectively 1.2% and 0.15%. However, it has not yet converged for $d = 3$ as the relative error to all estimates (FRG, conformal bootstrap, Monte Carlo) increases from 4% at order $\mathcal{O}(\epsilon^2)$ to 6% at order $\mathcal{O}(\epsilon^3)$. By contrast, the FRG is able to interpolate smoothly between dimensions $d = 2$ and 4. As the dimension is increased, c_{112} increases monotonously, with an almost linear behavior between $d = 2$ and $d = 3$.

d	2	3	4
FRG	0.484	1.039	1.413
ϵ exp. [77]	1.100	1.114	$\sqrt{2} \simeq 1.414$
MC [73]		1.07(3)	
CB [14]		1.0518537(41)	
Exact [55]	$1/2 = 0.5$		$\sqrt{2} \simeq 1.414$

Table I. OPE coefficient c_{112} of the Ising universality class for dimensions $d = 2, 3, 4$. We compare the numerical FRG results to the ϵ expansion to order $\mathcal{O}(\epsilon^3)$, conformal bootstrap and Monte Carlo estimates and the exact values for the 2d and 4d Ising universality classes.

In $d = 4$, the FRG within the BMW approximation scheme gives the exact analytic value of c_{112}. The small difference ($\sim 0.1\%$) between the numerical result and the exact value seen in Table I arises from the fitting of the critical exponents and the normalization constants. This serves as an estimate of this numerical error: in lower dimensions, it is much smaller than the difference to the best estimates.

B. Three-dimensional O(N) model

We now focus on the three-dimensional O(N) model. Given that the leading large-N result (23) evaluates to

$$c_{112} = \frac{2}{\pi}\frac{1}{\sqrt{N}}, \qquad (33)$$

we consider rather than c_{112} the rescaled OPE coefficient $\sqrt{N}c_{112}$ that has a well-defined large-N limit. FRG results and estimates from the ϵ expansion, conformal bootstrap and Monte Carlo are shown in Fig. 4 and Table II.

For the existing values $N = 1, 2, 3$, FRG differs from conformal bootstrap by respectively 1.2%, 3% and 2%. FRG accurately reproduces the large-N behavior: for $N = 1000$, the FRG estimate $\sqrt{N}c_{112} = 0.638$ differs from the exact large-N result $\lim_{N\to\infty}\sqrt{N}c_{112} = 2/\pi \simeq 0.637$ by 0.1%, which is about the order of magnitude corresponding to a $1/N$ correction. This is expected as it is known that the relevant vertices are exact in the large-N limit [48, 51]. By contrast, the ϵ expansion to order $\mathcal{O}(\epsilon^3)$ gives $\lim_{N\to\infty}\sqrt{N}c_{112} = (5 + 2\zeta(3))/8\sqrt{2} \simeq 0.654$

N	FRG	ϵ exp.	MC	CB
1	1.039	1.114 [77]	1.07(3) [73]	1.0518537(41) [14]
2	0.943	1.038 [77]	0.9731(14) [74]	0.97193(92) [14]
3	0.884	0.975 [77]		0.908467(102) [72]
4	0.829	0.924 [77]		
5	0.792	0.882 [77]		
6	0.766	0.847 [77]		
8	0.731	0.795 [77]		
10	0.709	0.758 [77]		
100	0.643	0.645 [77]		
1000	0.638	0.653 [77]		

Table II. Rescaled OPE coefficient $\sqrt{N}c_{112}$ of the three-dimensional O(N) model for different numbers of field components N. We compare the FRG results to the ϵ expansion to order $\mathcal{O}(\epsilon^3)$, conformal bootstrap and Monte Carlo estimates. The exact large-N result is $\lim_{N\to\infty}\sqrt{N}c_{112} = 2/\pi \simeq 0.637$.

with a 2.7% error and even predicts a non-monotonous dependence on $1/N$ for large N [77].

Furthermore, the numerical FRG data suggests that the next-to-leading order correction can be fitted as

$$c_{112} = \frac{2}{\pi}\frac{1}{\sqrt{N}} + \frac{\kappa}{N^{3/2}}, \tag{34}$$

with $\kappa \simeq 0.76$ (see Fig. 4). Though no result for κ is known to the authors, this estimate might prove useful as a comparison reference for a next-to-leading order calculation.

IV. CONCLUSION

We have shown how to extract the OPE coefficients of a conformal theory within the framework of FRG, by determining three-point vertices in specific momentum configurations. We have used our approach to determine the c_{112} coefficient, corresponding to the simplest possible operator product, in the O(N) universality class for various d and N. This provides the first non-perturbative determination of the OPE coefficients based on field theory, aside from the lattice computations in [73, 74] and the conceptually very different conformal bootstrap.

While the accuracy of FRG can be sometimes difficult to gauge in the absence of a small expansion parameter, the fact that the results compare extremely well with the values, when available, obtained from Monte Carlo and conformal bootstrap increases confidence in the validity of the method. Furthermore, FRG is able to give estimates for cases not studied before by those methods, such as non-integer dimensions or $N > 3$. It is a testament to the versatility of FRG that, in this specific case, tuning such parameters as d or N demands relatively little effort, and that contrary to the ϵ-expansion, only valid close to $d = 4$, FRG gives a consistent prediction across all d between 2 and 4.

Lastly, we note that the OPE can be invoked and used in very different settings from the critical O(N) theories investigated in this work. Indeed, at no point in establishing and solving the FRG equations do we invoke conformal invariance. Thus, the OPE can be considered away from a fixed point or at a non-equilibrium fixed point, when many methods holding for equilibrium critical theories are not available. Our work suggests that the FRG may constitute the right framework to tackle these issues thanks to its aforementioned versatility.

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Appendix A: c_{112} in the free case

When u_0 vanishes, the functional integral over the field φ can be done exactly and yields the partition function

$$Z[J,h] = e^{\frac{N}{2}\mathrm{Tr}\ln G[h] + \frac{1}{2}\int_{x,y}\sum_i J_i(x)G[x,y;h]J_i(y)}, \tag{A1}$$

where $G[h]$ denotes the propagator in the presence of an arbitrary external source h:

$$G^{-1}[x,y;h] = \big(-\nabla_x^2 + 2h(x)\big)\delta(x-y). \tag{A2}$$

The expectation value of the field is given by

$$\phi_i(x) = \int_y G[x,y;h]J_i(y), \tag{A3}$$

and the effective action is simply

$$\Gamma[\phi,h] = \int_x \left\{\frac{1}{2}(\partial_\mu\phi)^2 - h\phi^2\right\} - \frac{N}{2}\mathrm{Tr}\ln G[h]. \tag{A4}$$

We thus obtain

$$\Gamma^{(2,0)}(p) = p^2, \tag{A5}$$

$$\Gamma^{(0,2)}(p) = -2N\int_q \frac{1}{q^2(p+q)^2} \simeq -\frac{2NB_d}{|p|^{4-d}}, \tag{A6}$$

$$\Gamma^{(2,1)}(p_1,p_2) = -2, \tag{A7}$$

where

$$B_d = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)\Gamma(d/2-1)^2}{\Gamma(d-2)}. \tag{A8}$$

The last expression for $\Gamma^{(0,2)}(p)$ is obtained for $p \to 0$ and $d < 4$. From (A5) and (19) we deduce $\Delta_1 = d/2 - 1$ (in agreement with the vanishing anomalous dimension in the free theory) and

$$\mathcal{N}_1^2 = A_d(d/2-1). \tag{A9}$$

On the other hand, comparing (A6) with (20) yields $\Delta_2 = d-2$ (in agreement with $\nu = 1/2$) and

$$\begin{aligned} \frac{1}{\mathcal{N}_2^2} &= \frac{N}{2}\frac{B_d}{A_d(d-2)} \\ &= \frac{N}{32\pi^d}\Gamma(d/2-1)^2. \end{aligned} \tag{A10}$$

From (21) and (A7) we finally obtain (22).

Appendix B: c_{112} in the large-N limit

Following the Appendix of Ref. [51], we introduce the field $\rho = \varphi^2$ and a Lagrange multiplier λ to write the partition function $\mathcal{Z}[h] \equiv \mathcal{Z}[J=0,h]$ of the O(N) model as

$$\begin{aligned} \mathcal{Z}[h] &= \int \mathcal{D}[\varphi,\rho,\lambda] \exp\biggl\{ -\int_x \biggl[\frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 \\ &\quad + \left(\frac{r_0}{2} - h\right)\rho + \frac{u_0}{4!N}\rho^2 + i\frac{\lambda}{2}(\varphi^2 - \rho) \biggr] \biggr\} \\ &= \int \mathcal{D}[\boldsymbol{\varphi},\lambda] \exp\biggl\{ \int_{\mathbf{x}} \biggl[\frac{3N}{2u_0}(2h + i\lambda - r_0)^2 \\ &\quad - \frac{1}{2}\left[(\nabla\varphi)^2 + i\lambda\varphi^2 \right] \biggr\}. \end{aligned} \tag{B1}$$

Then we split the field φ into a σ field and an $(N-1)$-component field π. Integrating over the π field, we obtain the action

$$\begin{aligned} S[\sigma,\lambda,h] &= \int_x \biggl[-\frac{3N}{2u_0}(2h + i\lambda - r_0)^2 \\ &\quad + \frac{1}{2}\left[(\nabla\sigma)^2 + i\lambda\sigma^2 \right] + \frac{N-1}{2}\mathrm{Tr}\ln g^{-1}[\lambda], \end{aligned} \tag{B2}$$

where

$$g^{-1}[x,x';\lambda] = [-\nabla_x^2 + i\lambda(x)]\delta(x-x') \tag{B3}$$

is the inverse propagator of the field π_i in the fluctuating λ field. In the limit $N \to \infty$, the action becomes proportional to N (if one rescales the σ field, $\sigma \to \sqrt{N}\sigma$); the saddle point approximation becomes exact for the partition function $\mathcal{Z}[h]$ and the Legendre transform of the free energy coincides with the action S [78]. This implies that the effective action is simply equal to $S[\sigma,\lambda,h]$:

$$\begin{aligned} \Gamma[\sigma,\lambda,h] &= \int_x \biggl\{ -\frac{3N}{2u_0}(2h + i\lambda - r_0)^2 \\ &\quad + \frac{1}{2}\left[(\nabla\sigma)^2 + i\lambda\sigma^2 \right] \biggr\} + \frac{N}{2}\mathrm{Tr}\ln g^{-1}[\lambda] \end{aligned} \tag{B4}$$

(we use $N-1 \simeq N$ for large N). We can eliminate the Lagrange multiplier λ using

$$\left. \frac{\delta\Gamma[\sigma,\lambda,h]}{\delta\lambda(x)} \right|_{\lambda=\lambda[\sigma,h]} = 0 \tag{B5}$$

to obtain the effective action $\Gamma[\sigma,h] \equiv \Gamma[\sigma,\lambda[\sigma,h],h]$, which is the starting point to compute the vertices $\Gamma^{(n,m)}$ in the large-N limit.

In Ref. [51] it was shown that, at criticality,

$$\begin{aligned} &\Gamma^{(2,0)}(p) = p^2, \\ &\Gamma^{(0,2)}(p) = -\frac{12N}{u_0} + \left(\frac{6N}{u_0}\right)^2 \Gamma^{(2)}_{\lambda\lambda}(p)^{-1}, \end{aligned} \tag{B6}$$

where

$$\begin{aligned} \Gamma^{(2)}_{\lambda\lambda}(p) &= \frac{3N}{u_0} + \frac{N}{2}\Pi(p), \\ \Pi(q) &= \int_q \frac{1}{q^2(p+q)^2} \simeq \frac{B_d}{|p|^{4-d}} \end{aligned} \tag{B7}$$

for $p \to 0$ and $d < 4$, where B_d is defined in (A8). Calculating the three-point function along the same lines, one finds

$$\begin{aligned} \Gamma^{(2,1)}(p_1,p_2) &= -\frac{6N}{u_0}\Gamma^{(2)}_{\lambda\lambda}(p_1+p_2)^{-1} \\ &\simeq -\frac{12|p_1+p_2|^{4-d}}{u_0 B_d} \end{aligned} \tag{B8}$$

for $p_1 + p_2 \to 0$ and $d < 4$. From (19) and (20) one then obtains $\Delta_1 = d/2 - 1$ and $\Delta_2 = 2$ (in agreement with the large-N results $\eta = 0$ and $\nu = 1/(d-2)$ to leading order) and

$$\begin{aligned} \mathcal{N}_1^2 &= A_d(d/2-1), \\ \mathcal{N}_2^2 &= -\frac{u_0^2 B_d A_d(2)}{18N}. \end{aligned} \tag{B9}$$

Equation (21) then gives

$$c_{112} = \left[-\frac{2A_d(2)}{NB_d} \right]^{1/2} \frac{1}{A_d(1)} \tag{B10}$$

and in turn (23) using standard properties of the Gamma function.

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