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Equivariant harmonic maps depend real analytically on the representation

Abstract. We consider harmonic maps into symmetric spaces of non-compact type that are equivariant for representations that induce a free and proper action on the symmetric space. We show that under suitable non-degeneracy conditions such equivariant harmonic maps depend in a real analytic fashion on the representation they are associated to. The main tool in the proof is the construction of a family of deformation maps which are used to transform equivariant harmonic maps into maps mapping into a fixed target space so that a real analytic version of the results in [4] can be applied.

1. Introduction

In this article we consider equivariant harmonic maps and study the question whether these maps depend nicely (e.g. smoothly or real analytically) on the representation they are association to.

Let us first introduce some notation. Throughout this article we let M be a closed real analytic Riemannian manifold, $\widetilde{M}$ its universal cover and $\Gamma = \pi_1(M)$ its fundamental group. Also we let G be a real semisimple Lie group without compact factors and $X = G/K$ its associated symmetric space. If $\rho \colon \Gamma \to G$ is a reductive representation of Γ in G, then by work of Corlette ([3]) and Labourie ([10]) there exists a ρ-equivariant harmonic map $f \colon \widetilde{M} \to X$. A map f is called ρ-equivariant if

$$f(\gamma m) = \rho(\gamma) f(m) \text{ for all } m \in \widetilde{M} \text{ and } \gamma \in \Gamma.$$

These maps were used by Corlette to prove a version of super rigidity. They were also used by Hitchin and Simpson in the development of the Non-Abelian Hodge correspondence which gives an identification between representation varieties and moduli spaces of Higgs bundles over Kähler manifolds.

In [4] Eells and Lemaire prove that, under suitable non-degeneracy conditions, harmonic maps between closed Riemannian manifolds depend smoothly on the metrics on both the domain and the target (see also [14]). We extend this result in two directions. First, we show it holds also in the real analytic category. Namely, we

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show in Proposition 3.3 that, under the same non-degeneracy condition, harmonic maps depend in a real analytic fashion on the metrics on the domain and target. Secondly, we show that this result can be used to prove that certain equivariant harmonic maps depend in a real analytic fashion on the representation they are associated to. For this we restrict ourselves to families of representations that satisfy a condition that we will call uniformly free and proper (see Definition 2.3). This assumption allows us to consider equivariant harmonic maps $\widetilde{M} \to X$ as maps from M to the quotients of the symmetric space X by the actions induced by the representations. In this way we can transform the setting of equivariant harmonic maps into a setting that is closer to that of [4] and Proposition 3.3.

The main result (Theorem 2.8) of this article is as follows. If $(\rho_t)_t$ is a real analytic family of representations of Γ in G that is uniformly free and proper and is such that ρ_0 is reductive and the centraliser of its image contains no hyperbolic elements, then for all t in a neighbourhood of zero there exist ρ_t-equivariant harmonic maps depending real analytically on t. Similarly, we also prove real analytic dependence on the metric on the domain M.

In general a representation $\Gamma \to G$ need not induce a free and proper action on the symmetric space X. Hence, the assumption that the family $(\rho_t)_t$ is uniformly free and proper does constitute a restriction on the applicability of the theorem. In the approach we present here, where we let equivariant harmonic maps descend to a quotient, such an assumption seems, to the author, indispensable. Hence, for a more general statement, one without the uniformly free and proper assumption, it seems likely that a different approach will be necessary.

Despite this constraint, our result is well adapted to the study of the equivariant harmonic maps that arise in higher Teichmüller theory. Namely, most representations studied in higher Teichmüller theory satisfy the Anosov condition and as observed in Remark 2.4 families of such representations always satisfy the uniformly free and proper condition.

In Sect. 2.4.1 we illustrate some applications of the main theorem to higher Teichmüller theory. More specifically, we apply our results to families of Hitchin representations. Such families satisfy all assumptions of the main theorem (see Proposition 2.12). As a result we can characterise certain sets as real analytic subsets of Teichmüller space and the set of Hitchin representations. Namely, in Corollary 2.13 we prove that the set of points at which the equivariant harmonic maps are not immersions is a real analytic subvariety of the universal Teichmüller curve crossed with the set of Hitchin representations. Similarly, in Corollary 2.14 we prove that the set of points Y in Teichmüller space and representations ρ such that Y can be realised as a minimal surface in $X/\rho(\Gamma)$ is a real analytic subvariety of Teichmüller space crossed with the set of Hitchin representations.

2. Statement of the results

We first collect some preliminary definitions and results needed to give a statement of the main theorem.

2.1. Harmonic maps

If (M, g) and (N, h) are Riemannian manifolds with M compact, then a C^1 map $f: (M, g) \to (N, h)$ is called harmonic if it is a critical point of the Dirichlet energy functional

$$E(f) = \frac{1}{2} \int_M \|df\|^2 \, \mathrm{vol}_g .$$

Here we view df as a section of the bundle $T^*M \otimes f^*TN$. A metric on this vector bundle is induced by the metrics g and h. A harmonic map satisfies the Euler–Lagrange equation $\tau(f) = 0$ where $\tau(f) = \mathrm{tr}_g \nabla df$ is the tension field of f. Here ∇ denotes the connection on $T^*M \otimes f^*TN$ induced by the Levi-Civita connections of g and h. If the domain M is not compact, then a map is called harmonic if its tension field vanishes identically.

At a critical point the Hessian of the energy functional is given by

$$\nabla^2 E(f)(X, Y) = \int_M [\langle \nabla X, \nabla Y \rangle - \mathrm{tr}_g \langle R^N(X, df \cdot)df \cdot, Y \rangle] \, \mathrm{vol}_g$$

for $X, Y \in \Gamma^1(f^*TN)$. Here ∇ denotes the connection on f^*TN induced by the Levi-Civita connection on TN and R^N denotes the Riemannian curvature tensor of (N, h). The non-degeneracy condition imposed on harmonic maps in [4] is that a harmonic map f is a non-degenerate critical point of the energy, i.e. $\nabla^2 E(f)$ is a non-degenerate bilinear form. In [15] Sunada proved that if the target is a locally symmetric space of non-positive curvature this condition is satisfied if and only if the harmonic map is unique. We collect these non-degeneracy conditions in the following lemma. Recall that we denote by G a real semisimple Lie group without compact factors and by $X = G/K$ the associated symmetric space. An element $g \in G$ is called hyperbolic if it can be written as $g = \exp_G(X)$, where $X \in \mathfrak{p}$, $X \neq 0$ and $\mathrm{Lie}(G) = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of the Lie algebra of G.

Lemma 2.1. *(Sunada) Let $\rho: \Gamma \to G$ be a representation and assume that the induced action of Γ on X is free and proper. Denote by N the locally symmetric space $N = X/\mathrm{im}\,\rho$. Assume, furthermore, that there exists a ρ-equivariant harmonic map $\tilde{f}: \tilde{M} \to X$. The map $\tilde{f}$ descends to a harmonic map $M \to N$ which we will denote by f. The following are equivalent:*

  i f is a non-degenerate critical point of E.
 ii f is the unique harmonic map in its homotopy class.
iii $\tilde{f}$ is the unique ρ-equivariant harmonic map.
 iv The centraliser of $\mathrm{im}\,\rho$ in G contains no hyperbolic elements.

Proof. We can write $\nabla^2 E(f) = - \int_M \langle J_f \cdot, \cdot \rangle \, \mathrm{vol}_g$ where

$$J_f(X) = \mathrm{tr}_g \nabla^2 X + \mathrm{tr}_g R^N(X, df \cdot)df \cdot.$$

is the Jacobi operator at f. As discussed in [4, p.35] the Hessian $\nabla^2 E(f)$ is non-degenerate precisely when $\ker J_f = 0$. It follows from [15, Proposition 3.2] that the set

$$\text{Harm}(M, N, f) = \{h \colon M \to N \mid h \text{ is harmonic and homotopic to } f\}$$

is a submanifold of $W^k(M, N)$ (the space of maps from M to N equipped with the $W^{k,2}$ Sobolev topology) with tangent space at h equal to $\ker J_h$. Because N has non-positive curvature the space $\text{Harm}(M, N, f)$ is connected ([8, Theorem 8.7.2]). We see that f is a non-degenerate critical point of the energy if and only if $\text{Harm}(M, N, f)$ contains only f. This established the equivalence of (i) and (ii).

We will demonstrate the equivalence of statements (ii), (iii) and (iv) using proofs by contraposition. We begin by proving the equivalence of (ii) and (iii). First, if a harmonic map $h \colon M \to N$ exists that is homotopic to f but also distinct from f, then h can be lifted to a ρ-equivariant harmonic map $\widetilde{h} \colon \widetilde{M} \to X$ that is different from $\widetilde{f}$. It follows that (iii) implies (ii). Now assume that (iii) does not hold and so there exists a ρ-equivariant harmonic map $\widetilde{h} \colon \widetilde{M} \to X$ that is distinct from $\widetilde{f}$. This map descends to a harmonic map $h \colon M \to N$. If this map is different from f, then we see immediately that (ii) does also not hold. If, on the other hand, $f = h$ (which happens for example if $\widetilde{h}$ is a translate of $\widetilde{f}$ by an element of im ρ), then we argue as follows. We consider the homotopy $\widetilde{F} \colon \widetilde{M} \times [0, 1] \to X$ between $\widetilde{f}$ and $\widetilde{h}$ that is given by $\widetilde{F}(x, t) = \gamma_x(t)$ where $\gamma_x \colon [0, 1] \to X$ is the unique geodesic in X that connects $\widetilde{f}(x)$ to $\widetilde{h}(x)$. For each fixed $t \in [0, 1]$ the map $\widetilde{F}(\cdot, t) \colon \widetilde{M} \to X$ is also ρ-equivariant hence the homotopy descends to a map $F \colon M \times [0, 1] \to N$. We have $F(\cdot, 0) = F(\cdot, 1) = f$ and for each $x \in M$ the path $t \mapsto F(x, t)$ is a non-constant geodesic loop. The arguments in the proof of [8, Theorem 8.7.2] show that each of the maps $F(\cdot, t) \colon M \to N$ is also harmonic. So we see there exists a family of distinct harmonic maps in the homotopy class of f and hence also in this case (ii) does not hold. We conclude that indeed (ii) and (iii) are equivalent.

Finally, we prove the equivalence of (iii) and (iv). First, suppose there exists a hyperbolic isometry $g \in G$ that centralises im ρ. Consider the map $\widetilde{h} = g \cdot \widetilde{f} \colon \widetilde{M} \to X$. Because g is an isometry the map $\widetilde{h}$ is also harmonic and because g commutes with all elements $\rho(\gamma)$, $\gamma \in \Gamma$ it follows that $\widetilde{h}$ is also ρ-equivariant. Furthermore, $\widetilde{h}$ is distinct from $\widetilde{f}$ because the isometry g has no fixed points in X. It follows that (iii) implies (iv). The other direction follows from [15, Lemma 3.4]. Namely, there it is proved that if $\widetilde{h} \colon \widetilde{M} \to X$ is a ρ-equivariant harmonic map different from $\widetilde{f}$, then a hyperbolic isometry $g \in G$ that centralises im ρ exists such that $\widetilde{h} = g \cdot \widetilde{f}$. This finishes the proof. $\square$

The main existence result in the theory of equivariant harmonic maps is the following theorem of Corlette.

Proposition 2.2. *([3]) Assume G is a real semisimple algebraic Lie group. A representation $\rho \colon \Gamma \to G$ is reductive if and only if there exists a ρ-equivariant harmonic map $f \colon \widetilde{M} \to X$.*

When G is an algebraic Lie group a representation $\rho \colon \Gamma \to G$ is called reductive if the Zariski closure of its image in G is a reductive subgroup. A generalisation

of the definition of a reductive representation and a corresponding existence result for equivariant harmonic maps if G is not an algebraic group can be found in [10].

2.2. Families of representations and metrics

We will index families of representations or metrics by open balls in $\mathbb{R}^n$. For $\epsilon > 0$ we denote by D_ϵ the open ϵ ball in $\mathbb{R}^n$ centred at 0. Let $(\rho_t)_{t \in D_\epsilon}$ be a family of representations $\rho_t \colon \Gamma \to G$. Such a family induces a natural action of Γ on $X \times D_\epsilon$ given by $\gamma \cdot (x, t) = (\rho_t(\gamma)x, t)$. We make the following properness and freeness assumption on the families of representations we will consider.

Definition 2.3. We call a family of representations uniformly free and proper if the induced action on $X \times D_\epsilon$ is free and proper.

In particular, each representation in such a family acts freely and properly on X.

Remark 2.4. Anosov representations constitute a large class of representations for which any deformation is automatically a uniformly free and proper family. These representations where introduced by Labourie in [11] and their definition was extended by Guichard and Wienhard in [5] to representations of hyperbolic groups. In our setting we can consider Anosov representations whenever M admits a metric of strictly negative curvature because then $\Gamma = \pi_1(M)$ is a hyperbolic group. The set of Anosov representations is open in $\mathrm{Hom}(\Gamma, G)$ (see [5, Theorem 5.13] or [9, Theorem 7.33]). Moreover, it follows from [9, Theorem 7.33] that continuous families of Anosov representations satisfy the uniformly free and proper assumption. So if $(\rho_t)_{t \in D_\epsilon}$ is a continuous family of representations with ρ_0 an Anosov representation, then, after possibly shrinking ϵ, the family of representations $(\rho_t)_{t \in D_\epsilon}$ is uniformly free and proper.

On families of representations we will make the following regularity assumption.

Definition 2.5. A family of representations $(\rho_t)_{t \in D_\epsilon}$ of Γ in G is called real analytic if for every $\gamma \in \Gamma$ the map $D_\epsilon \to G \colon t \mapsto \rho_t(\gamma)$ is real analytic.

Remark 2.6. A family of representations can be seen as a map from D_ϵ into $\mathrm{Hom}(\Gamma, G)$, the set of representations of Γ into G. If G is an algebraic subgroup of $\mathrm{GL}(n, \mathbb{R})$ and if S is a generating set of Γ with relations R, then $\mathrm{Hom}(\Gamma, G)$ can be realised as the closed subset of $\mathrm{GL}(n, \mathbb{R})^S$ consisting of tuples $(g_1, ..., g_n)$ satisfying the relations $r(g_1, ..., g_n) = 1$ for $r \in R$. In this way we realise $\mathrm{Hom}(\Gamma, G)$ as a real algebraic variety. We note that in this case a family of representations is real analytic if and only if the map $D_\epsilon \to \mathrm{Hom}(\Gamma, G)$ is real analytic.

Finally, for families of metrics we make the following regularity assumption.

Definition 2.7. We call a family $(g_t)_{t \in D_\epsilon}$ of Riemannian metrics on M a real analytic family of metrics if $(x, t) \mapsto g_t(x)$ induces a real analytic map $M \times D_\epsilon \to S^2 T^* M$.

2.3. Mapping spaces

If M and N are real analytic manifolds we denote by $C^{k,\alpha}(M, N)$ ($k \in \mathbb{N}, 0 < \alpha < 1$) the space of k-times differentiable maps from M to N such that the k-th derivatives are α-Hölder continuous. We equip these spaces with the topology of uniform $C^{k,\alpha}$ convergence on compact sets. To be more explicit, for a compact set $K \subset \mathbb{R}^m$ we define the $C^{k,\alpha}$ Hölder norm for functions in $C^{k,\alpha}(K, \mathbb{R}^n)$ as

$$\|f\|_{K,k,\alpha} = \max_{\substack{x \in K \\ i=1,\dots k}} \|D^i f(x)\| + \max_{x \neq y \in K} \frac{\|f(x) - f(y)\|}{\|x - y\|^\alpha}.$$

Then we define that a sequence of maps $(f_j)_{j \in \mathbb{N}}$ in $C^{k,\alpha}(M, N)$ converges to a map $f \in C^{k,\alpha}(M, N)$ if and only if for all charts (U, ϕ) of M, (V, ψ) of N and compact sets $K' \subset U$ we have $\|\psi \circ f_j \circ \phi^{-1} - \psi \circ f \circ \phi^{-1}\|_{\phi(K'),k,\alpha} \to 0$ as $j \to \infty$. If the domain manifold M is compact, then the spaces $C^{k,\alpha}(M, N)$ can be equipped with a natural real analytic Banach manifold structure.

In the statement of our results we will formulate the condition that a family of maps $(f_t)_{t \in D_\epsilon}$ is a real analytic family as the requirement that $t \mapsto f_t$ induces a real analytic map into a mapping space that is a Banach manifold. Here we follow the approach that Eells and Lemaire take in [4] for the smooth case. However, the mapping space $C^{k,\alpha}(\widetilde{M}, X)$ which naturally contains the equivariant maps $\widetilde{M} \to X$ can not be given a suitable Banach manifold structure because $\widetilde{M}$ is not compact. In order to avoid this issue we make use of the observation that equivariant maps are determined by their values on a fundamental domain. This will allow us to interpret these maps as an element of a mapping space that is a Banach manifold.

When M is a closed manifold we let $\Omega \subset \widetilde{M}$ be a bounded domain containing a fundamental domain for the action of Γ on $\widetilde{M}$. We note that a map $\widetilde{M} \to X$ that is equivariant with respect to any representation is completely determined by its restriction to Ω. Furthermore, ρ_n-equivariant maps f_n converge to a ρ-equivariant map f uniformly on compacts if and only if the restrictions $f_n|_\Omega$ converge uniformly to $f|_\Omega$.

We will consider the space of bounded functions from Ω to X. For this we equip M with a background metric and for simplicity we identify X with $\mathbb{R}^n$ via the exponential map $\exp_o \colon T_o X \to X$ based at some basepoint $o \in X$. The metric on M induces a $C^{k,\alpha}$ norm on the space of functions $\Omega \to \mathbb{R}^n \cong X$. We denote by $C_b^{k,\alpha}(\Omega, X)$ the space of functions for which this norm is bounded. This space can be equipped with the structure of a real analytic Banach manifold. For this we observe that equipped with the $C^{k,\alpha}$ norm the space $C_b^{k,\alpha}(\Omega, X)$ becomes a Banach space (note that the linear structure comes from the identification $X \cong \mathbb{R}^n$ and carries no direct geometric meaning). The Banach manifold structure is obtained by declaring this to be a global chart. One can check that the Banach manifold structure is independent of the choice of background metric on M and basepoint in X (see Lemma 3.4). We would like to note that, although the use of the identification $X \cong \mathbb{R}^n$ is somewhat ad hoc, if we replace the domain Ω by a closed manifold M, then the above construction yields the usual Banach manifold structure on the space $C^{k,\alpha}(M, X)$.

2.4. Main result

The main result of this article can be stated as follows.

Theorem 2.8. *Let* $(g_t)_{t \in D_\epsilon}$ *be a real analytic family of metrics on M and let* $(\rho_t)_{t \in D_\epsilon}$ *be a real analytic family of representations of Γ in G. We assume that the family $(\rho_t)_{t \in D_\epsilon}$ is uniformly free and proper. Suppose ρ_0 is reductive and the centraliser $Z_G(\mathrm{im}\, \rho_0)$ contains no hyperbolic elements. Then for every $k \in \mathbb{N}, 0 < \alpha < 1$ there exists a $\delta > 0$ smaller then ϵ and a unique continuous map $F \colon D_\delta \to C^{k,\alpha}(\widetilde{M}, X)$ such that each $F(t)$ is a ρ_t-equivariant harmonic map $(\widetilde{M}, g_t) \to X$ and the restricted map $F(\cdot)|_\Omega \colon D_\delta \to C_b^{k,\alpha}(\Omega, X)$ is real analytic.*

Remark 2.9. Let us briefly clarify the statement of the theorem by spelling out what it means for the map $F(\cdot)|_\Omega \colon D_\delta \to C_b^{k,\alpha}(\Omega, X)$ to be real analytic. The situation can be simplified by again using the fact that $C_b^{k,\alpha}(\Omega, X)$ admits a global chart. So we pick an identification $X \cong \mathbb{R}^n$ and use this to identify $C_b^{k,\alpha}(\Omega, X) = C_b^{k,\alpha}(\Omega, \mathbb{R}^n)$. The mapping $F(\cdot)|_\Omega$ is real analytic if for every $s \in D_\delta$ there exist maps $f_{k_1,\dots,k_n} \in C_b^{k,\alpha}(\Omega, \mathbb{R}^n)$ such that for all t in a neighbourhood of s the power series

$$P(t) = \sum_{k_1,\dots,k_n=0}^\infty f_{k_1,\dots,k_n}(t_1 - s_1)^{k_1} \cdots (t_n - s_n)^{k_n}$$

converges uniformly and $F(t)|_\Omega = P(t)$. This implies in particular that for every fixed $x \in \Omega$ the value of $F(t)(x)$ depends real analytically on $t \in D_\delta$.

Remark 2.10. The above result is also true in the smooth category. More precisely we can define, analogous to Definitions 2.5 and 2.7, the notion of smooth families of metrics and representations. Then Theorem 2.8 also holds when we replace 'real analytic' by 'smooth'. For brevity we will not prove the smooth case here but the reader can easily check that the proof goes through also in this case.

If each ρ_t is reductive and has trivial centraliser, then by applying Theorem 2.8 at each $t \in D_\epsilon$ we obtain immediately the following corollary.

Corollary 2.11. *Let* $(g_t)_{t \in D_\epsilon}$, $(\rho_t)_{t \in D_\epsilon}$ *be as in Theorem 2.8. Suppose that for every $t \in D_\epsilon$ the representation ρ_t is reductive and $Z_G(\mathrm{im}\, \rho_t) = 0$. Then for all $k \in \mathbb{N}, 0 < \alpha < 1$ there exists a unique continuous map $F \colon D_\epsilon \to C^{k,\alpha}(\widetilde{M}, X)$ such that each $F(t) \colon (\widetilde{M}, g_t) \to X$ is a ρ_t-equivariant harmonic map and the restricted map $F(\cdot)|_\Omega \colon D_\epsilon \to C_b^{k,\alpha}(\Omega, X)$ is real analytic.*

2.4.1. Hitchin representations We briefly mention how the above results can be applied when we consider Hitchin representations. In this section we let $M = S$ be a closed surface of genus $g \geq 2$ and as before $\Gamma = \pi_1(S)$. In this case the harmonicity of a map $f \colon S \to N$ depends only on the conformal class of the metric on S. Also, in this section we let G be a split real Lie group. A split real Lie group is a Lie group whose Lie algebra is a split real form of a complex Lie algebra. For example,

the non-exceptional split real Lie groups are $SL(n, \mathbb{R})$, $Sp(n, \mathbb{R})$, $SO(n, n)$ and $SO(n, n + 1)$.

Associated to Γ and G is the representation variety

$$\mathrm{Rep}(\Gamma, G) = \mathrm{Hom}_{\mathrm{red}}(\Gamma, G)/G$$

which consists of reductive representations $\Gamma \to G$ considered up to conjugation by elements in G. In [6] Hitchin proved that $\mathrm{Rep}(\Gamma, G)$ contains a connected component, now called the Hitchin component, which contains $\mathcal{T}(S)$, the Teichmüller space of S, in a natural way. A representation $\Gamma \to G$ whose conjugacy class lies in the Hitchin component is called a Hitchin representation. We denote by $\mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ the component of $\mathrm{Hom}(\Gamma, G)$ consisting of Hitchin representations.

Hitchin representations enjoy many special properties. They are reductive because, by definition, their conjugacy class lies in the Hitchin component which is a subset of $\mathrm{Rep}(\Gamma, G)$. The centraliser of the image of a Hitchin representation is trivial (see [2, Lemma 2.9]). Furthermore, in [11] Labourie showed that Hitchin representations are Anosov representations. It follows, as discussed in Remark 2.4, that continuous families of Hitchin representations satisfy the uniformly free and proper assumption as in Definition 2.3.

It follows that there exists a map $F : \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to C^{k,\alpha}(\widetilde{S}, X)$ assigning to each (J, ρ) the unique ρ-equivariant harmonic map $(\widetilde{S}, J) \to X$. A chart of $\mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ modelled on D_ϵ can be seen as real analytic family of representations that is uniformly free and proper. Furthermore, it follows from [18] that it is possible to choose metrics g_J on S representing points in Teichmüller space $J \in \mathcal{T}(S)$ depending on J in a real analytic fashion. By applying Corollary 2.11 to charts around points $(J, \rho) \in \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ we obtain the following proposition.

Proposition 2.12. *For all $k \in \mathbb{N}, 0 < \alpha < 1$ the map*

$$F : \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to C^{k,\alpha}(\widetilde{S}, X)$$

assigning to each (J, ρ) the unique ρ-equivariant harmonic map $(\widetilde{S}, J) \to X$ is continuous and the restricted map $F(\cdot, \cdot)|_\Omega : \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to C_b^{k,\alpha}(\Omega, X)$ is real analytic.

We discuss three corollaries to this result.

First we observe that the family of harmonic maps given by F can also be interpreted as a single map with the universal Teichmüller curve as domain. Namely, let $\Xi(S)$ be the universal Teichmüller curve of S. It is a trivial fibre bundle over $\mathcal{T}(S)$ with fibres homeomorphic to S. It is equipped with a complex structure such that the fibre $\Xi(S)_J$ over $J \in \mathcal{T}(S)$ together with the marking provided by the trivialization $\Xi(S)_J \cong \mathcal{T}(S) \times S$ determines the point J in Teichmüller space (see [7, Sect. 6.8]). The universal cover $\widetilde{\Xi}(S)$ is a trivial fibre bundle over $\mathcal{T}(S)$ with fibres homeomorphic to $\widetilde{S}$. Let $F' : \widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to X$ be the map which on each fibre $\widetilde{\Xi}(S)_J \times \{\rho\} \cong \widetilde{S}$ is given by the ρ-equivariant harmonic map $(\widetilde{\Xi}(S)_J, J) \to X$. It follows from Proposition 2.12 that this map is real analytic.

Corollary 2.13. *The set*

$$I = \{((J, x), \rho) \in \widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \mid$$
$$F'(J, \cdot, \rho) \colon \widetilde{\Xi}(S)_J \to X \text{ is not an immersion at } x\}$$

is a real analytic subvariety of $\widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$.

It is conjectured (see for example [12, Conjecture 9.3]) that equivariant harmonic maps associated to Hitchin representations are immersions which would correspond to the set I being empty.

Proof. We equip $\Xi(S)$ with a choice of real analytic metric. Given a point $((J, x), \rho) \in \widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ we consider the derivative of F' in the fibre direction

$$dF(J, \cdot, \rho) \colon T_x(\widetilde{\Xi}(S)_J) \to \mathrm{im}\, dF(J, \cdot, \rho).$$

Because the spaces $T_x(\widetilde{\Xi}(S)_J)$ and $\mathrm{im}\, dF(J, \cdot, \rho) \subset T_{F'(J,x,\rho)}X$ are equipped with inner products we can consider the determinant of this map. We let $d \colon \widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to \mathbb{R}$ be the map which at a point $((J, x), \rho)$ is given by the determinant of the above map. Because F' is real analytic the map d is real analytic as well. We observe that $I = d^{-1}(0)$ from which the result follows. $\square$

In a similar vein we also have the following corollary.

Corollary 2.14. *The set*

$$T = \{(J, \rho) \in \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \mid$$
$$(S, J) \text{ can be realised in } X/\rho(\Gamma) \text{ as a branched minimal surface}\}$$

is a real analytic subvariety of $\mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$.

Proof. For $J \in \mathcal{T}(S)$ and $\rho \in \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ we consider the Hopf differential of the harmonic map $F(J, \rho) \colon (\widetilde{S}, J) \to X$ which is given by $\phi_{J,\rho} = (F(J, \rho)^* m)^{2,0}$. Here m is the Riemannian metric of the symmetric space X. The Hopf differential is a holomorphic quadratic differential on $(\widetilde{S}, J)$ which vanishes if and only if the harmonic map $F(J, \rho)$ is a (branched) minimal surface. The Hopf differential $\phi_{J,\rho}$ is Γ-invariant and descends to S since $F(J, \rho)$ is ρ-equivariant. Consider the function $V \colon \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to \mathbb{R}$ given by the L^2-norm of $\phi_{J,\rho}$, namely

$$V(J, \rho) = \int_S \frac{|\phi_{J,\rho}|^2}{\sqrt{\det g_J}} |dz|^2.$$

Here g_J is a metric in the conformal class of J depending real analytically on J. It follows from Proposition 2.12 that this function is real analytic (it is for this reason that we choose the L^2-norm rather than the L^1-norm). The statement of the corollary follows from $T = V^{-1}(0)$. $\square$

The space of Fuchsian representations $\Gamma \to \mathrm{SL}(2, \mathbb{R})$ can be included in $\mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ in the following way. There exists an irreducible representation $\iota: \mathrm{SL}(2, \mathbb{R}) \to G$ that is unique up to conjugation. Then to a Fuchsian representation $\rho_0: \Gamma \to \mathrm{SL}(2, \mathbb{R})$ we associate a so-called principal Fuchsian representation $\rho = \iota \circ \rho_0$ which lies in $\mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$. This inclusion descends to the natural inclusion of Teichmüller space into the Hitchin component. A principal Fuchsian representation stabilises the totally geodesically embedded copy of $\mathbb{H}^2$ in G/K given by the inclusion $\iota': \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) \to G/K = X$ that is induced by ι. By uniqueness of equivariant harmonic maps we have that the harmonic map $(\tilde{S}, J) \to X$ equivariant for $\rho = \iota \circ \rho_0$ is given by the composition $\iota' \circ f_0$ where $f_0: (\tilde{S}, J) \to \mathbb{H}^2$ is the unique ρ_0-equivariant map. It follows from [14, Theorem 11] that f_0 is a diffeomorphism. Hence, equivariant harmonic maps associated to principal Fuchsian representations are immersions. Because being an immersion is an open condition we immediately obtain the following corollary to Proposition 2.12

Corollary 2.15. *There exists an open neighbourhood of*

$$\mathcal{T}(S) \times \{principal\ Fuchsian\ representations\} \subset \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$$

such that for any pair (J, ρ) in this neighbourhood the ρ-equivariant harmonic map $(\tilde{S}, J) \to X$ is an immersion.

3. Proof of the main result

As in the proof of Eells and Lemaire in [4], our main analytical tool will be the implicit function theorem for maps between Banach manifolds. The main difficulty to overcome is that a priori the equivariant harmonic maps are not elements of the same space of mappings. Namely, if $(\rho_t)_t$ is a family of representations, then a ρ_t-equivariant map is an element of the space $C^{k,\alpha}(M, X/\rho_t(\Gamma))$. Since the target manifold is different for each t these spaces are not equal (although they are likely to be diffeomorphic). Our aim is to modify these maps so that they can be seen as elements of a single mapping space. This will be achieved by means of a family of deformation maps which intertwine the representations ρ_0 and ρ_t. By composing with these deformation maps we can view each ρ_t-equivariant map as element of (a subset of) $C^{k,\alpha}(M, X/\rho_0(\gamma))$.

We first fix some notation. We let $(\rho_t)_{t \in D_\epsilon}$ be a real analytic family of representations that is uniformly free and proper. We denote $X_\epsilon = X \times D_\epsilon$ and by $\alpha: \Gamma \times X_\epsilon \to X_\epsilon, \alpha(\gamma)(x, t) = (\rho_t(\gamma)x, t)$ the natural action induced by $(\rho_t)_t$. We fix a base point $o \in X$ of the symmetric space and denote by $U_R = \cup_{\gamma \in \Gamma} B(\rho_0(\gamma)o, R)$ the R-neighbourhood of the $\rho_0(\Gamma)$-orbit of o. Our deformation maps will be provided by the following proposition.

Proposition 3.1. *For every $R > 0$ there exists a $\delta = \delta(R) > 0$ smaller than ϵ and family of maps $(\Phi_t: U_R \to X)_{t \in D_\delta}$ satisfying the following properties:*

i The induced map $U_R \times D_\delta \to X: (x, t) \mapsto \Phi_t(x)$ is real analytic.

ii For each $t \in D_\delta$ the set $\Phi_t(U_R)$ is open and $\Phi_t \colon U_R \to \Phi_t(U_R)$ is a real analytic diffeomorphism.

iii $\Phi_0 = \mathrm{id} \colon U_R \to U_R$.

iv For each $t \in D_\delta$ the map Φ_t intertwines the actions of ρ_0 and ρ_t, i.e. $\rho_t(\gamma) \circ \Phi_t = \Phi_t \circ \rho_0(\gamma)$ for $\gamma \in \Gamma$.

The content of this proposition is closely related to Ehresmann's fibration theorem. In fact, when the actions of the representations ρ_t on X are cocompact Proposition 3.1 can be obtained from it. Consequently the proof of Proposition 3.1 is along the same lines as the proof of the fibration theorem.

We denote by $\mathrm{pr}_X \colon X_\epsilon \to X$ and $\pi \colon X_\epsilon \to D_\epsilon$ the projections onto the first and second factor of $X_\epsilon = X \times D_\epsilon$ respectively. By $(t_1, \ldots, t_n)$ we denote the coordinates on D_ϵ and also the coordinates on the D_ϵ factor in X_ϵ. So in this notation we have $d\pi(\frac{\partial}{\partial t_i}(x, t)) = \frac{\partial}{\partial t_i}(t)$.

We first prove the following lemma.

Lemma 3.2. *Let $R > 0$. On an $\alpha(\Gamma)$-invariant neighbourhood of $U_R \times \{0\}$ in X_ϵ there exist $\alpha(\Gamma)$-invariant real analytic vector fields ξ_i (for $i = 1, \ldots, n$) that satisfy $d\pi(\xi_i(x, t)) = \frac{\partial}{\partial t_i}(t)$.*

Proof. It is possible to give a more or less explicit construction for such vector fields. However, proving they are indeed real analytic is rather cumbersome. Instead we opt to explicitly construct smooth vector fields which we then approximate by real analytic ones.

We let $\varphi \colon [0, \infty) \to [0, 1]$ be a smooth function satisfying $\varphi|_{[0, R]} \equiv 1$ and $\varphi|_{[R+1, \infty)} \equiv 0$. For $i = 1, \ldots, n$ we define smooth vector fields η_i on X_ϵ by

$$\eta_i(x, t) = \varphi(d(o, x)) \cdot \frac{\partial}{\partial t_i}.$$

Now let

$$\xi_i' = \sum_{\gamma \in \Gamma} (\alpha(\gamma))_* \eta_i.$$

The sum on the right hand side is locally finite by the uniform properness assumption on $(\rho_t)_t$. Hence, each ξ_i' is $\alpha(\Gamma)$-invariant smooth vector field on X_ϵ. We observe that $d\pi(\eta_i(x, t)) = s(x, t)\frac{\partial}{\partial t_i}$ with

$$s(x, t) = \sum_{\gamma \in \Gamma} \varphi(d(o, \rho_t(\gamma)^{-1}x)).$$

On $B(o, R) \times \{0\}$ we have that $s(x, t) \geq \varphi(d(o, x)) = 1$ and by $\alpha(\Gamma)$-invariance we have that $s \geq 1$ on $U_R \times \{0\}$.

We now approximate the smooth vector fields ξ_i' by real analytic ones. By the uniformly free and proper assumption on $(\rho_t)_t$ we have that $X_\epsilon/\alpha(\Gamma)$ is a real analytic manifold. The vector fields ξ_i' descend to smooth vector fields. On compact subsets these vector fields can be approximated arbitrarily closely in C^0 norm by real analytic vector fields (see [16] and [13]). The set $U_R \times \{0\}$ maps to a precompact

subset of $X_\epsilon/\alpha(\Gamma)$. Hence, by pulling back approximating vector fields we see that on a neighbourhood of $U_R \times \{0\}$ we can approximate ξ_i' arbitrarily closely by $\alpha(\Gamma)$-invariant real analytic vector fields. Let ξ_i'' be such approximating vector fields. For some real analytic functions s_i' we have $d\pi(\xi_i''(x, t)) = s_i'(x, t)\frac{\partial}{\partial t_i}$. By choosing the approximating vector fields ξ_i'' close enough to ξ_i' we can arrange that each s_i' is close to s and hence satisfies $s_i' > 0$ on a neighbourhood of $U_R \times \{0\}$. For $i = 1, ..., n$ we can now define $\xi_i = \xi_i''/s'$. $\square$

Proof of Proposition 3.1. Let ξ_i for $i = 1, ..., n$ be the vector fields constructed in Lemma 3.2. We denote by ψ_i^s their flows which are defined on a neighbourhood of $U_R \times \{0\}$. We consider the maximal flow domain for a combination of these flows starting at points in $X \times \{0\}$, i.e. the set

$$\Omega = \{(x, s) \in X \times \mathbb{R}^n \mid \psi_1^{s_1} \circ \cdots \circ \psi_n^{s_n}((x, 0)) \text{ is defined}\}.$$

This is an open set containing $U_R \times \{0\}$. On Ω we set

$$\Psi(x, (s_1, ..., s_n)) = \psi_1^{s_1} \circ \cdots \circ \psi_n^{s_n}((x, 0)).$$

Because $d\pi(\xi_i) = \frac{\partial}{\partial t_i}$ (when defined) we see that

$$t \mapsto \pi \circ \Psi(x, (s_1, ..., s_{i-1}, t, s_{i+1}, ..., s_n))$$

is an integral line for the vector field $\frac{\partial}{\partial t_i}$. Since these integral lines are unique and $\pi \circ \Psi(x, 0) = 0$ we find $\pi \circ \Psi(x, s) = s$ when $(x, s) \in \Omega$, i.e. $\pi \circ \Psi = \pi$. Because the vector fields ξ_i are $\alpha(\Gamma)$-invariant we observe for $\gamma \in \Gamma$ that

$$\begin{aligned}\Psi(\rho_0(\gamma)x, s) &= \psi_1^{s_1} \circ \cdots \circ \psi_n^{s_n}(\alpha(\gamma)(x, 0)) \\ &= \alpha(\gamma)[\psi_1^{s_1} \circ \cdots \circ \psi_n^{s_n}(x, 0)] = \alpha(\gamma)\Psi(x, s)\end{aligned}$$

whenever both sides are defined. We define $\beta\colon \Gamma \times \Omega \to \Omega$ as an action of Γ on Ω by $\beta(\gamma)(x, s) = (\rho_0(\gamma)x, s)$ which is the action of $\rho_0(\Gamma)$ on X times the trivial action. By the above the we see that the set Ω is $\beta(\Gamma)$-invariant and Ψ intertwines the β and α actions, i.e. $\alpha(\gamma) \circ \Psi = \Psi \circ \beta(\gamma)$ for all $\gamma \in \Gamma$.

On $X \times \{0\} \cap \Omega$ the map Ψ is simply the inclusion into X_ϵ. Combined with the fact that $\pi \circ \Psi = \pi$ we see for each $(x, 0) \in X \times \{0\} \cap \Omega$ the tangent map $d\Psi|_{(x,0)}\colon T_x X \times T_0\mathbb{R}^n \to T_x X \times T_0 D_\epsilon$ is the identity map. Hence, we can shrink Ω to a smaller open neighbourhood of $U_R \times \{0\}$ such that Ψ is a local diffeomorphism on Ω. By shrinking Ω further we can also assume Ψ to be injective. For if not, then there exist two distinct sequences $(x_n, s_n), (x_n', s_n') \in \Omega$ satisfying $\Psi(x_n, s_n) = \Psi(x_n', s_n')$ with $s_n, s_n' \to 0$ and x_n, x_n' converging to points x and x' in U_R. By $\pi \circ \Psi = \pi$ we see $s_n = s_n'$. By continuity $\Psi(x, 0) = \Psi(x', 0)$ and because when restricted to $X \times \{0\} \cap \Omega$ the map Ψ is an injection we must have $x = x'$. However, this contradicts the fact that Ψ is a local diffeomorphism. So we can arrange that Ψ is a diffeomorphism onto its image. Since Ψ intertwines β and α this can be done in such a way that Ω is still $\beta(\Gamma)$-invariant.

Since Ω is a neighbourhood of $U_R \times \{0\}$ we can, by compactness, find a $\delta > 0$ such that $\overline{B(o, R)} \times D_\delta \subset \Omega$. By $\beta(\Gamma)$-invariance we then have $U_R \times D_\delta \subset \Omega$. We

now define the family of deformation maps $\Phi_t \colon U_R \to X$ as $\Phi_t(x) = \mathrm{pr}_X \circ \Psi(x, t)$ for $t \in D_\delta$. We check that indeed $(\Phi_t)_{t \in D_\delta}$ satisfies Properties (i)-(iv). Property (i) follows since flows of real analytic vector fields are real analytic. Property (ii) follows since $\Psi \colon \Omega \to \Psi(\Omega)$ is a diffeomorphism and satisfies $\pi \circ \Psi = \pi$ hence induces diffeomorphisms between the fibres $\pi^{-1}(t) \cap \Omega$ and $\pi^{-1}(t) \cap \Psi(\Omega)$. Property (iii) follows from the fact that Ψ restricted to $X \times \{0\} \cap \Omega$ is the inclusion map and Property (iv) follows from the fact that Ψ intertwines the actions of β and α. $\qquad\square$

Using the deformation maps the problem of dependence on representations can be reduced to the problem of dependence on metrics on a fixed target manifold. In this case the results of [4] can be used. In their paper Eells and Lemaire only prove smooth dependence so for completeness we prove a version of their result in the real analytic category.

Proposition 3.3. *Let M, N be real analytic manifolds with M compact. Let $(g_t)_{t \in D_\epsilon}$ and $(h_t)_{t \in D_\epsilon}$ be real analytic families of metrics on M and N, respectively. If $f_0 \colon (M, g_0) \to (N, h_0)$ is a harmonic map such that $\nabla^2 E(f_0)$ is non-degenerate, then for every $k \in \mathbb{N}, 0 < \alpha < 1$ there exists a $\delta > 0$ and a unique real analytic map $F \colon D_\delta \to C^{k,\alpha}(M, N)$ such that $F(0) = f_0$ and each $F(t)$ is a harmonic map $(M, g_t) \to (N, h_t)$.*

Proof. For each $t \in D_\epsilon$ a C^2 map $\phi \colon (M, g_t) \to (N, h_t)$ is harmonic if and only if $\tau_t(\phi) = \mathrm{tr}_{g_t} \nabla d\phi = 0$ where ∇ is the connection on $T^*M \otimes \phi^*TN$ induced by g_t and h_t. In local coordinates $(x^i)_i$ on M and $(u^\alpha)_\alpha$ on N, $\tau_t(\phi)$ is given by

$$\tau_t(\phi)^\gamma = (g_t)_{ij} \left\{ \frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} - (\Gamma_{g_t})_{ij}^k \frac{\partial \phi^\gamma}{\partial x^k} + (\Gamma_{h_t})_{\alpha\beta}^\gamma(\phi) \frac{\partial \phi^\alpha}{\partial x_i} \frac{\partial \phi^\beta}{\partial x_j} \right\}$$

here Γ_g denote the Christoffel symbols of a metric g. We combine the tension fields for different $t \in D_\epsilon$ in a map

$$\tau \colon C^{k+2+\alpha}(M, N) \times D_\epsilon \to TC^{k+\alpha}(M, N).$$

We claim this map is real analytic. To see this we write τ as a composition of two real analytic map. First we consider the second prolongation map

$$J \colon C^{k+2+\alpha}(M, N) \to C^{k+\alpha}(M, J^2(M, N))$$

mapping a map $\phi \colon M \to N$ to its 2-jet $j^2\phi$. A diffeomorphism between a neighbourhood of the zero section in ϕ^*TN and a neighbourhood of the image of $\mathrm{graph}(\phi)$ in $M \times N$ induces charts of the two mapping spaces modelled on $\Gamma^{k+2+\alpha}(\phi^*TN)$ and $\Gamma^{k+\alpha}(J^2(M, \phi^*TN))$ respectively. In these charts the second prolongation map is a bounded linear map so in particular it is real analytic. Secondly, we consider the map $T \colon J^2(M, N) \times D_\epsilon \to TN$ given in local coordinates (induced by $(x^i)_i$ on M and $(u^\alpha)_\alpha$ on N) by

$$(x^i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, t) \mapsto (g_t)_{ij} \left\{ u_{ij}^\gamma - (\Gamma_{g_t})_{ij}^k u_k^\gamma + (\Gamma_{h_t})_{\alpha\beta}^\gamma(u) u_i^\alpha u_j^\beta \right\}.$$

By assumption the coefficients $(g_t)_{ij}$, $(\Gamma_{g_t})_{ij}^k$ and $(\Gamma_{h_t})_{\alpha\beta}^\gamma$ are real analytic functions so the map T is real analytic. It now follows from the Ω-lemma (see Lemma 3.4 below) that T induces a real analytic map

$$T_*\colon C^{k+\alpha}(M, J^2(M, N)) \times D_\epsilon \to C^{k+\alpha}(M, TN)$$
$$(\psi, t) \mapsto T(\psi(\cdot), t)$$

The map τ is a composition of T_* and J and is therefore real analytic.

As discussed in [4, p. 35] the partial derivative of τ with respect to the first factor $(D_1\tau)_{(f_0,0)}\colon TC^{k+2+\alpha}(M, N) \to T_0TC^{k+\alpha}(T, N)$ is an isomorphism of Banach spaces precisely when $\nabla^2 E(f_0)$ is non-degenerate. Hence, we can apply a real analytic version of the implicit function theorem for Banach spaces (e.g. [17]) to obtain, for $\delta > 0$ small enough, a unique real analytic map $F\colon D_\delta \to C^{k+2+\alpha}(M, N)$ such that $F(0) = f_0$ and $\tau_t(F(t)) = 0$ for all $t \in D_\delta$.   □

Lemma 3.4. *(The Ω-Lemma) Let M, N and P be real analytic manifolds with M compact. Suppose $F\colon N \to P$ is real analytic. Then F induces a real analytic map*

$$\Omega_F\colon C^{k,\alpha}(M, N) \to C^{k,\alpha}(M, P)\colon \phi \mapsto F \circ \phi$$

for all $k \in \mathbb{N}$, $0 < \alpha < 1$.

Compare with [1, Theorem 11.3]. Unfortunately, a proof of the real analytic case as stated here does not seem to be available in the literature. We give a sketch of the proof.

Sketch of proof of Lemma 3.4. By following the same steps as in [1] the statement can be reduced to a local version (cf. [1, Theorem 3.7]), i.e. it is enough to prove that if $K \subset \mathbb{R}^n$ is compact, $V \subset \mathbb{R}^p$ open and $F\colon K \times V \to \mathbb{R}^q$ real analytic, then $\Omega_F\colon C^{k,\alpha}(K, V) \to C^{k,\alpha}(K, \mathbb{R}^q)$ given by $[\Omega_F(\phi)](x) = F(x, \phi(x))$ is a real analytic map between Banach spaces. To this end we observe that since F is real analytic it can be extended to a complex analytic map $\widetilde{F}\colon U \to \mathbb{C}^q$ on an open set $U \subset \mathbb{C}^n \times \mathbb{C}^p$ containing $K \times V$. Let $\widetilde{V} \subset \mathbb{C}^p$ be an open such that $K \times V \subset K \times \widetilde{V} \subset U$. Then $\widetilde{F}$ induces a map $\Omega_{\widetilde{F}}\colon C^{k,\alpha}(K, \widetilde{V}) \to C^{k,\alpha}(K, \mathbb{C}^q)$ between complex Banach spaces which extends Ω_F. Applying the smooth version of the Ω-Lemma yields that $\Omega_{\widetilde{F}}$ is a C^1 map with derivative given by $D\Omega_{\widetilde{F}} = \Omega_{D_2\widetilde{F}}$. Since $\widetilde{F}$ is holomorphic we see that this derivative is complex linear. It now follows from [7, Theorem A5.3] that $\Omega_{\widetilde{F}}$ is a complex analytic map. We conclude that Ω_F, which is a restriction of $\Omega_{\widetilde{F}}$ to $C^{k,\alpha}(K, V)$, is real analytic.   □

We can now prove the statement of our main theorem.

Proof of Theorem 2.8. We set $N = X/\rho_0(\Gamma)$ (recall that by assumption the action of ρ_0 on X is free and proper so N is a manifold). Since ρ_0 is reductive and $Z_G(\operatorname{im}\rho_0)$ contains no hyperbolic elements there exists a unique ρ_0-equivariant harmonic map $f\colon \widetilde{M} \to X$. This map descends to a harmonic map $f\colon M \to N$. We denote by o' the point in N covered by the base point o in X. The set U_R

descends to the set $V = B(o', R)$ in N. We choose $R > 0$ large enough such that the image of f is contained in $B(o', R)$.

Let $(\Phi_t)_{t \in D_\delta}$ be the family of deformation maps as in Proposition 3.1. We denote by m the Riemannian metric on the symmetric space X. Define a family of metrics $(h_t)_{t \in D_\delta}$ on U_R by $h_t = \Phi_t^* m$. By Property (3.1.i) this is a real analytic family of metrics. We observe for $\gamma \in \Gamma$ that

$$\rho_0(\gamma)^* h_t = \rho_0^* \Phi_t^* m = \Phi_t^* \rho_t(\gamma)^* m = \Phi_t^* m = h_t.$$

Here we used Property (3.1.iv) and the fact that each ρ_t acts on X by isometries. We conclude that each h_t is $\rho_0(\Gamma)$-invariant hence the family of metrics descends to a family of metrics, also denoted $(h_t)_{t \in D_\delta}$, on V. By Lemma 2.1 the Hessian $\nabla^2 E(f)$ is non-degenerate so Proposition 3.3 yields, after shrinking δ, a unique real analytic map $G \colon D_\delta \to C^{k,\alpha}(M, V)$ such that $G(t)$ is a harmonic map from (M, g_t) to (V, h_t) for each $t \in D_\delta$. By choosing for each t a ρ_0-equivariant lift we can view G as a continuous map $G \colon D_\delta \to C^{k,\alpha}(\widetilde{M}, U_R)$. We define F by composing with the deformation maps, $F(t) = \Phi_t \circ G(t)$. By Property (3.1.iv) every map $F(t)$ is ρ_t-equivariant. By construction, each Φ_t is an open isometric embedding of (V, h_t) into (X, m) hence each $F(t)$ is also harmonic. Finally, by Property (3.1.i) we see that the map $F \colon D_\delta \to C^{k,\alpha}(\widetilde{M}, X)$ is continuous and real analytic as a map $F(\cdot)|_\Omega \colon D_\delta \to C^{k,\alpha}(\Omega, X)$. $\qquad \square$

Declarations

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