The universality of islands outside the horizon

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Abstract

We systematically calculate the quantum extremal surface (QES) associated with Hawking radiation for general D-dimensional ($D \geq 2$) asymptotically flat (or AdS) eternal black holes using the island formula. We collect the Hawking radiation particles by a non-gravitational bath and find that a QES exists in the near-horizon region outside the black hole when $c \cdot G_{(D)}$ is smaller enough where c is the central charge of the conformal matter and $G_{(D)}$ the D-dimensional Newton constant. The locations of the QES in these backgrounds are obtained and the late-time radiation entropy saturates the two times of black hole entropy. Finally, we numerically check that the no island configuration exists once $c \cdot G_{(D)}$ exceeds a certain upper bound in two-dimensional generalized dilaton theories (GDT).

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1 Introduction

The black hole information paradox [1,2] is one of the most fundamental problems in contemporary physics. Resolving it has been regarded as the crux of understanding quantum gravity. According to Hawking's original calculations, the radiation of a black hole behaving like thermal radiation implies that the entanglement entropy outside the black hole is monotonically increasing. This result contradicts the expectation of the unitarity of the black hole evaporation process, which is commonly reckoned to be compatible only with the evolution of radiation entropy satisfying the so-called "Page curve" [3,4]. Whereas, the original calculations of the entanglement entropy in [3,4] depends on a postulate that the Hilbert space is factorizable. Recent research indicates that the bulk locality is absent in the gravitational system and the boundary system encodes all the bulk information [5–9]. Without the bulk locality, the whole system can not be divided into the black hole and Hawking radiation intrinsically. In this scenario, one can only collect the bulk information at the asymptotic boundary and then get a constant fine-grained entropy [6,8,9]. However, the bulk locality can be restored by gluing a non-gravitational system, which is called "bath" conventionally, to the black hole [6] with transparent boundary conditions, and one can thus calculate the fine-grained entropy of the Hawking radiation absorbed by the bath. The price of doing so is that the conservation of stress tensor is broken and the graviton obtains mass [10–13]. Thanks to the break of the stress tensor conservation, in the AdS/CFT literature, a new approach, known as the island formula, has been applied to compute the radiation entropy of evaporating black holes and yield the Page curve [14–18].

The island formula somehow stems from the investigations of the quantum corrections [19–21] of the Ryu-Takayanagi (RT) formula [22, 23]. It is well known that the RT formula, as a significant crystallization of the AdS/CFT correspondence [24–26], provides a powerful holographic way to evaluate the entanglement entropy of boundary conformal field theory (CFT). Nevertheless, the RT formula is a classical formula, as was proposed in [21], when one wishes to count the bulk quantum effects, it should give way to the QES prescription. The QES extremizes the generalized entropy which is the sum of area and bulk entanglement entropy. In terms of the prescription of minimal quantum extremal surface, the island formula for computing the fine-grained entanglement entropy of the Hawking radiation is proposed as [16]

$$S_{\mathrm{Rad}}(A) = \min\left\{\operatorname*{ext}_{I}\left[\frac{\mathrm{Area}(\partial I)}{4G_N} + S_{\mathrm{matter}}(A\cup I)\right]\right\}. \tag{1}$$

Here $S_{\mathrm{Rad}}(A)$ is the generalized entropy for the radiation in the region A, I called island is a bulk region whose boundary ∂I is the minimal quantum extremal surface. The entanglement entropy from matter part contains the UV divergence which is proportional to the island area, subject to a UV cut-off scale [27, 28], and the Newton constant G_N must be renormalized [29]. The S_{matter} corresponds to the finite contribution of the matter entanglement entropy. The validity of this formula is provided by the bulk locality, thus the coupling of the non-gravitational bath is necessary [7]. Note that (1) can be also derived from the replica trick for gravitational theories [14, 30].

Although the island formula was originally used to reproduce the Page curve of the evaporating black hole in Jackiw-Teitelboim (JT) gravity [16, 18], the correlational research has been extended to many aspects so far. As an incomplete summary, except for the well-known doubly holographic model as well as the replica wormhole [13, 31–34], for instance, more on evaporation models and details are explored in [35–40]. Meanwhile, higher dimensional black hole cases are considered in [41–48] as well as higher derivative gravity [49, 50]. Interestingly, the page curve can be realized in the moving mirror scenario [51–53], and other quantum information or thermodynamic quantities except entanglement entropy are investigated within island formula [54–62].

As pointed in [63], within the framework of the so-called "black hole couples thermal baths" model, the island appears outside the horizon for an external black hole in 2D JT gravity. The radiation entropy approaches $2S_{\mathrm{BH}}$ in the late time limit. There were several case-by-case studies, to confirm the above behavior of QES with the approximation that the central charge of thermal bath is smaller than the inverse of Newton constant associated with a black hole.

In this paper, we would like to systematically study QES for various two-dimensional external black holes including asymptotically flat and AdS cases, and higher-dimensional cases. In asymptotically AdS cases, we couple a flat bath at the boundary of the spacetime, while in asymptotically flat cases, we couple the flat bath at some finite location and then cut off the spacetime region outside it. The Page curves in these kind of models display the information transformation from the gravitational system to the flat bath carried by the Hawking radiation. As mentioned above, coupling a flat bath makes the graviton massive. Unfortunately, the validity of the QES formula and the entanglement wedge reconstruction in both asymptotically flat spacetime and massive gravity theory is still an open question. The island formula and Page curve have been investigated in asymptotically flat spacetime without non-gravitational bath [35,36,42–48,64,65]. In this work, we assume the QES formula is applicable in the asymptotically flat spacetime with massive graviton. In these generic gravitational backgrounds, we try to extract universal features for the existence of QES and islands. We find that once the combination $c \cdot G_{(D)}$ of central charge and Newton constant stays within a certain region, the QES and island configuration in such generic gravitational background always exists outside nearby the black hole event horizon, not inside the horizon. We further do the analytical and numerical self-consistency checks in several GDT.

The organization of this paper is as follows. In Section 2, we set up the generic "black hole couples thermal baths" model and obtain certain constraints in terms of the existence of QES. To close this section, we do the numerically self-consistency checks and go beyond the $c \cdot G_{(D)} \ll 1$ limit in 2D eternal black holes. The summary and prospect are given in section 3. Our conventions, useful formulae, and some calculation details are presented in the appendices.

2 Island formula in eternal black holes

2.1 Setup and assumptions

Let us consider a D-dimensional ($D \geq 2$) gravitational system, which consists of a non-extremal asymptotically flat (or AdS) black hole and a thermal bath with which it reaches thermal equilibrium. The whole system is assumed to be filled with conformal matter with central charge c, and the black hole's metric is assumed under the Schwarzschild gauge as follows

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{D-2}^2. \tag{2}$$

Here $\mathrm{d}\Omega_{D-2}^2$ is the unit metric on $\mathbb{S}^{\mathrm{D}-2}$. $f(r)$ is allowed to have multiple roots and r_h ($f(r_h) = 0$) represents the largest one (i.e., the location of the outermost horizon). The black hole's Hawking temperature and entropy are

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi}, \quad S = \frac{A(r_h)}{4G_{(D)}}, \tag{3}$$

respectively. Thereinto, κ is surface gravity of the outermost horizon, $G_{(D)}$ is D-dimensional Newton constant, and $A(r)$ is a model-dependent function which stands for the area of the $(D-2)$-sphere at radius r in $D \geq 3$ dimensional Einstein gravity and represents the value of the dilaton field at r in two-dimensional dilaton gravity [66], etc.

The Penrose diagram of the full system might be depicted as Fig.1, and the coordinate transformations between Kruskal coordinates and Schwarzschild coordinates in the four wedges of the Penrose diagram of the black hole are set to following

$$\text{I}: \quad \hat{u} = \kappa^{-1}\mathrm{e}^{\kappa(t_R + r^*(r_R))}, \qquad \hat{v} = -\,\kappa^{-1}\mathrm{e}^{-\kappa(t_R - r^*(r_R))} \qquad (r_R > r_h), \tag{4}$$

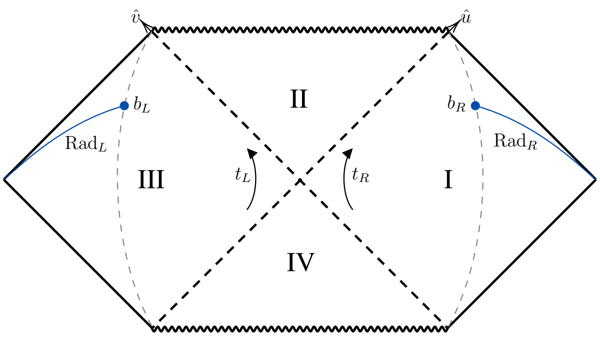
$$\text{II}: \quad \hat{u} = \kappa^{-1}\mathrm{e}^{\kappa(t_R + r^*(r_R))}, \qquad \hat{v} = \kappa^{-1}\mathrm{e}^{-\kappa(t_R - r^*(r_R))} \qquad (r_R < r_h), \tag{5}$$

$$\text{III}: \quad \hat{u} = -\kappa^{-1}\mathrm{e}^{-\kappa(t_L - r^*(r_L))}, \qquad \hat{v} = \kappa^{-1}\mathrm{e}^{\kappa(t_L + r^*(r_L))} \qquad (r_L > r_h), \tag{6}$$

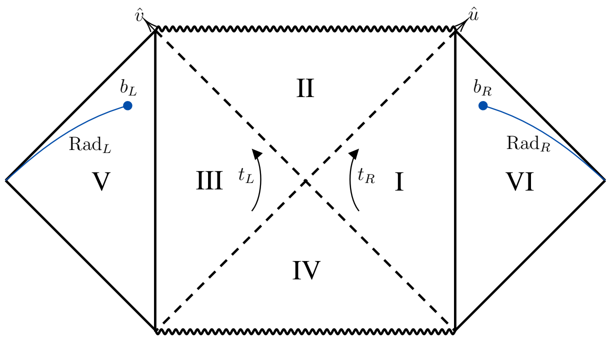
$$\text{IV}: \quad \hat{u} = -\kappa^{-1}\mathrm{e}^{\kappa(t_L + r^*(r_L))}, \qquad \hat{v} = -\,\kappa^{-1}\mathrm{e}^{-\kappa(t_L - r^*(r_L))} \qquad (r_L < r_h), \tag{7}$$

where $r^* \equiv \int^r f(\tilde{r})^{-1}\mathrm{d}\tilde{r}$ is tortoise coordinate. The transformations above give the length element in Kruskal coordinates

$$\mathrm{d}s^2 = -\mathrm{e}^{2\rho}\mathrm{d}\hat{u}\mathrm{d}\hat{v} + r_{R(L)}^2\mathrm{d}\Omega_{D-2}^2 \qquad \left(\mathrm{e}^{2\rho} \equiv f(r_{R(L)})\mathrm{e}^{-2\kappa r^*(r_{R(L)})}\right). \tag{8}$$



(a) Asymptotically flat black hole



(b) Asymptotically AdS black hole+Flat thermal baths

Figure 1: Penrose diagrams of the whole gravitational system (*Left*: Asymptotically flat black hole with single horizon and singularity. *right*: Asymptotically AdS black hole with single horizon and singularity). Each point on diagrams represents a $(D-2)$-dimensional sphere. The dotted gray lines in (a) are boundaries of non-gravitational baths which used to collect the Hawking radiation. The blue lines stand for collecting region with boundaries $b_{L(R)}$ in a schwarzschild time slice. We consider the symmetric case that $t_{b_L} = t_{b_R} = t_b$ and $r^*_{b_L} = r^*_{b_R} = r^*_b$.

As shown in Fig.1(b), we have adopted the customary approach to deal with the black hole and the thermal bath in the case of asymptotically AdS: D-dimensional flat spacetimes $\mathbb{R}^{1,\mathrm{D}-1}$ will be used as auxiliary thermal baths to be glued to both sides of the two-sided black hole [63][5]. When the spacetime is asymptotically flat, the prevalent method is to select the region far away

[5] We follow the prescription in [63] but generalize it to higher-dimensions. Firstly the tortoise coordinates are normalized by requiring $\lim\limits_{r_{R(L)}\to\infty} r^*_{R(L)} = 0$, such that the right (left) bath corresponds to $r^*_{R(L)} > 0$. The

from the black hole as the thermal bath [35, 64]. As discussed in the introduction, the bulk locality is absent without a flat bath, and as a result the island formula is inapplicable. To solve this problem, we couple a flat bath to collect Hawking radiation at a certain Schwarzschild coordinate $r \to r_{b+}$, as shown in Fig.1(a). Note that the dependence on the character of the bath in equation (9) is only the location r_b and the Weyl factor $W(r_b)$ of b. With the benefit of the continuity of the metric at r_b, these two quantities of our models are the same as those models without flat bath [42–47, 64, 65, 67].

It is also important to emphasize that, when $D \geq 3$, the s-wave approximation [42, 68] has been taken into account in the calculations of S_{matter} below. The entanglement entropy of matter between two shells S_1 and S_2 becomes

$$\begin{aligned} S_{\text{matter}}(S_1, S_2) =& \frac{c}{6} \log d^2(S_1, S_2) \\ =& \frac{c}{6} \log \left| \left(\hat{u}(S_1) - \hat{u}(S_2)\right)\left(\hat{v}(S_1) - \hat{v}(S_2)\right) \sqrt{\mathrm{W}(S_1)\mathrm{W}(S_2)} \right| , \end{aligned} \tag{9}$$

when the quantum state of total system is vacuum in $(\hat{u}, \hat{v})$ coordinates. In the above, $W(S_1)$ and $W(S_2)$ are warped factors of the metric at S_1 and S_2 under the $(\hat{u}, \hat{v})$ coordinates, respectively.

2.2 Without island, the radiation entropy diverges linearly

In this section, we evaluate the entanglement entropy of the Hawking radiation at late times in the missing island construction. It shows that "information loss" is a common phenomenon for black holes we are considering.

Without island, the only contribution of (1) is coming from the collecting regions of the Hawking radiation (see Rad_L and Rad_R in Fig.1). The collecting region on the right (left) is the region outside the shell $r^*_{R(L)} = r^*_{b_{R(L)}}$ in time slices of $(t_{R(L)}, r^*_{R(L)})$ coordinates and we shall choose the symmetric configuration $r^*_{b_L} = r^*_{b_R} = r^*_b$ and $t_{b_L} = t_{b_R} = t_b$ in the following calculations. Assuming that the state of total system is vacuum in $(\hat{u}, \hat{v})$ coordinates, The formula can be further reduced to the entanglement entropy of the interval $[b_L, b_R]$ by (9), that is

Kruskal coordinates thus can be extended to the baths (V and VI):

$$\begin{aligned} \mathrm{V}: \quad & \hat{u} = -\kappa^{-1} \mathrm{e}^{-\kappa(t_L - r^*_L)}, && \hat{v} = \kappa^{-1} \mathrm{e}^{\kappa(t_L + r^*_L)} && (r^*_L > 0), \\ \mathrm{VI}: \quad & \hat{u} = \kappa^{-1} \mathrm{e}^{\kappa(t_R + r^*_R)}, && \hat{v} = -\kappa^{-1} \mathrm{e}^{-\kappa(t_R - r^*_R)} && (r^*_R > 0). \end{aligned}$$

Meanwhile, we assume that the two-sided black hole is truncated at $r_R = \Lambda$ and $r_L = \Lambda$ respectively, and the metric of the right (left) bath is set to following

$$\mathrm{d}s^2 = f(\Lambda) \left(-\mathrm{d}t^2_{R(L)} + (\mathrm{d}r^*_{R(L)})^2 \right) + \left(\sqrt{f(\Lambda)} r^*_{R(L)} + \Lambda \right)^2 \mathrm{d}\Omega^2_{D-2}$$

to ensure that two metrics (black hole and bath) are continuously connected at the cut-off. Note that this metric is flat.

$$S_{\text{Rad}} = \frac{c}{6}\log\left|\left(\hat{u}(b_L)-\hat{u}(b_R)\right)\left(\hat{v}(b_L)-\hat{v}(b_R)\right)\sqrt{W(b_L)W(b_R)}\right|, \tag{10}$$

where

$$W(b_{R(L)}) = \begin{cases} -f(r_b)\mathrm{e}^{-2\kappa r_b^*}, & \text{for asymptotically flat black holes,} \\ -f(\Lambda)\mathrm{e}^{-2\kappa r_b^*}, & \text{for asymptotically AdS black holes.} \end{cases} \tag{11}$$

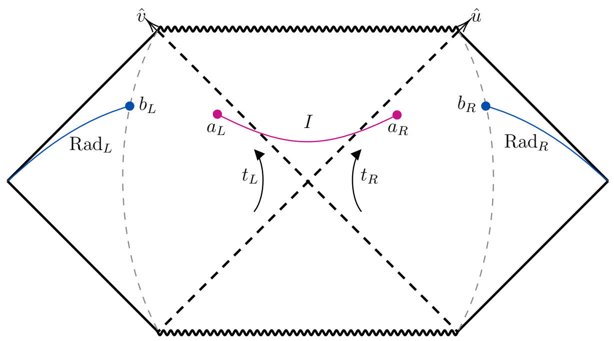
Simple calculation shows that

$$\begin{aligned} S_{\text{Rad}} &= \left\{ \begin{array}{ll} \frac{c}{6}\log\left(4\kappa^{-2}f(b)\cosh^2\kappa t_b\right), & \text{for asymptotically flat black holes} \\ \frac{c}{6}\log\left(4\kappa^{-2}f(\Lambda)\cosh^2\kappa t_b\right), & \text{for asymptotically AdS black holes} \end{array} \right\} \\ &\simeq \frac{c}{3}\kappa t_b + \text{time independent terms.} \end{aligned} \tag{12}$$

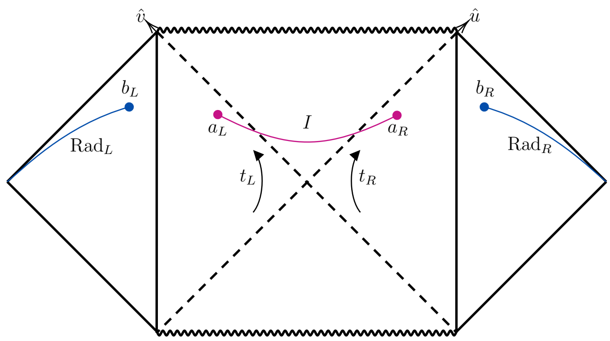
Notice that (12) holds for all black holes we are considering. The linear growth of radiation entropy when the island contribution is missing obviously contradicts the Page curve and thus leads to the information paradox for the black hole.

2.3 Island emerges outside the horizon and saves the entropy bound

In this section, we shall reconsider the entropy of the Hawking radiation by counting the contribution of the island. It is easy to verify that the equation determining the location of QES has no solution inside the horizon. Therefore the basic configuration is set as shown in Fig.2. As shown in Fig.2, we are continuing with the symmetric structure used in the previous section. The two boundaries of island are marked a_L and a_R respectively, and $t_{a_L} = t_{a_R} = t_a$, $r_{a_L} = r_{a_R} = r_a$. After taking s-wave approximation for $D \geq 3$, it shows that the entanglement



(a) Asymptotically flat black hole with an island



(b) Asymptotically AdS black hole with an island+Flat thermal baths

Figure 2: Penrose diagrams with islands (*Left*: Asymptotically flat black hole with single horizon and singularity. *right*: Asymptotically AdS black hole with single horizon and singularity). The pink lines are islands whose boundaries are outside the horizon. The dotted gray lines in (a) are boundaries of collecting region for the Hawking radiation. The blue lines stand for collecting region with boundaries $b_{L(R)}$ in a Schwarzschild time slice. We consider the symmetric case that $t_{b_L} = t_{b_R} = t_b$, $t_{a_L} = t_{a_R} = t_a$, $r_{a_L} = r_{a_R} = r_a$, $r^*_{b_L} = r^*_{b_R} = r^*_b$.

entropy of conformal matter in $\{\text{Rad}\cup I\}$ can be well approximated by twice of the entanglement entropy in the single interval $[a_R, b_R]$ when t_b and $t_a \to \infty$ [69]

$$S_{\text{Rad}} = \frac{A(a_R)}{2G_{(D)}} + \frac{c}{3}\log\left|\big(\hat{u}\left(a_R\right) - \hat{u}\left(b_R\right)\big)\big(\hat{v}\left(a_R\right) - \hat{v}\left(b_R\right)\big)\sqrt{W(a_R)W(b_R)}\right|, \tag{13}$$

where $W(a_R) = -f(r_a)\mathrm{e}^{-2\kappa r_a^*}$ and $W(b_R)$ is (11). Eq.(13) can be expressed in (t_R, r_R) coordinates

$$S_{\text{Rad}} = \frac{c}{3}\log\left|\kappa^{-2}\big(f(r_a)f(r_b)\mathrm{e}^{-2\kappa(r_a^*+r_b^*)}\big)^{\frac{1}{2}}\left(2\mathrm{e}^{\kappa(r_a^*+r_b^*)}\cosh[\kappa(t_b - t_a)] - (\mathrm{e}^{2\kappa r_a^*} + \mathrm{e}^{2\kappa r_b^*})\right)\right| + \frac{A(r_a)}{2G_{(D)}}, \tag{14}$$

for asymptotically flat black holes. $\quad f(r_b) \to f(\Lambda)$ for asymptotically AdS black holes.

It's easy to find t_a should be equal to t_b when we extremise S_{Rad} with respect to t_a, then we arrive at a simpler expression compared to (14),

$$S_{\text{Rad}} = \frac{A(r_a)}{2G_{(D)}} + \frac{2c}{3}\log\left[\frac{\mathrm{e}^{\kappa r_b^*} - \mathrm{e}^{\kappa r_a^*}}{\kappa}\right] + \frac{c}{6}\log\left[f(r_a)f(r_b)\mathrm{e}^{-2\kappa\left(r_a^*+r_b^*\right)}\right], \tag{15}$$

for asymptotically flat black holes. $\quad f(r_b) \to f(\Lambda)$ for asymptotically AdS black holes.

Taking partial derivative of S_{Rad} with respect to r_a, we meet the algebra equation of determining the location of QES (r_a here)

$$\partial_{r_a} S_{\text{Rad}} = \frac{A'(r_a)}{2G_{(D)}} - \frac{2c}{3}\frac{\kappa}{f(r_a)\left(\mathrm{e}^{\kappa(r_b^*-r_a^*)} - 1\right)} + \frac{c}{6}\frac{f'(r_a) - 2\kappa}{f(r_a)} = 0, \tag{16}$$

which is the same for both asymptotically flat black holes and asymptotically AdS black holes. There are some model-independent properties of the solution that can be extracted from (16), notwithstanding this algebra equation of r_a may be precisely solved only after $f(r)$ and $A(r)$ are given. The key point essentially comes from the fact that the near-horizon geometry is common to all non-extreme black holes. To show them clearly, let's rewrite (16) as follows

$$Y(r) \equiv \frac{3A'(r)}{2}\cdot\left(\frac{2\mathrm{e}^{\kappa r^*(r)}}{f(r)}\left(\frac{\kappa}{\mathrm{e}^{\kappa r_b^*} - \mathrm{e}^{\kappa r^*(r)}}\right) + \frac{2\kappa - f'(r)}{2f(r)}\right)^{-1} = c\cdot G_{(D)}, \tag{17}$$

where the subscript a has been omitted for brevity. The zero points of $\partial_{r_a} S_{\text{Rad}}$ now become the points of intersection between the horizontal line $y = c\cdot G_{(D)}$ and the curve $y = Y(r)$ $\big(r_h < r < r_b$ for asymptotically flat and $r_h < r < \Lambda$ for asymptotically AdS$\big)$ on the $r - y$ plane, as shown in Fig.4. Let's focus on the behavior of $Y(r)$ near r_h. A rough estimation can be made since $f(r) \approx 2\kappa(r - r_h)$ and $r^*(r) \approx \frac{1}{2\kappa}\log\left[\frac{r}{r_h} - 1\right]$ for $r \gtrsim r_h$. $Y(r)$ can thus be approximated to

$$Y(r) \approx \frac{3}{2}A'(r_h)\Big(X\cdot r_h^{-1}\mathrm{e}^{-\kappa r_b^*}(\frac{r}{r_h} - 1)^{-\frac{1}{2}} - \frac{f''(r_h)}{4\kappa}\Big)^{-1} \sim \sqrt{\frac{r}{r_h} - 1}, \tag{18}$$

where X is an undetermined constant. The approximate behavior of function Y near r_h is sufficient for us to draw two following conclusions:

Table 1: Approximations of location of quantum extremal surface for several black holes

Black hole	$A(r)$	$f(r)$	$r_a - r_h \approx$
Witten(CGHS)	$e^{2\lambda r}$	$1-e^{-2\lambda(r-r_h)}$	$\frac{2c^2G_{(2)}^2}{9\lambda}\left(e^{2\lambda(r_h+r_b)}-e^{4\lambda r_h}\right)^{-1}$
JT	$\frac{r}{L}$	$\frac{r^2-r_h^2}{L^2}$	$\frac{2c^2G_{(2)}^2L^2}{9r_h}e^{-2\frac{r_h}{L^2}r_b^*}$
BTZ	$2\pi r$	$\frac{r^2-r_h^2}{L^2}$	$\frac{c^2G_{(3)}^2}{18\pi^2r_h}e^{-2\frac{r_h}{L^2}r_b^*}$
4d-Schwarzschild	$4\pi r^2$	$1-\frac{r_h}{r}$	$\frac{c^2G_{(4)}^2}{144\pi^2r_h^2(r_b-r_h)}e^{1-\frac{r_b}{r_h}}$
4d-non-extremal RN	$4\pi r^2$	$(1-\frac{r_+}{r})(1-\frac{r_-}{r})$	$\frac{c^2G_{(4)}^2}{144\pi^2r_+^2(r_b-r_+)}\left(\frac{r_b-r_-}{r_+-r_-}\right)^{\frac{r_-^2}{r_+^2}}e^{-\frac{(r_b-r_+)(r_+-r_-)}{r_+^2}}$

Conclusion 1. *There must be a quantum extremal surface located in the near-horizon region outside the black hole,*

$$r_a = r_h + \frac{8\kappa(c\cdot G_{(D)})^2}{9A'(r_h)^2}\exp\Big\{-2\kappa r_b^* - 2\rho(r_h)\Big\} + \mathcal{O}\Big(\big(c\cdot G_{(D)}\big)^3\Big), \tag{19}$$

when $c\cdot G_{(D)} \ll 1$.

Conclusion 2. *There has to be an upper bound on* $c\cdot G_{(D)}$ *to have an island configuration.*

The second conclusion can be a direct corollary to the *boundedness theorem*, since $Y(r)$ is a continuous function on the closed interval $[r_h, r_b]$ ($[r_h, \Lambda]$ for asymptotically AdS). While for the conclusion 1, firstly, the approximate behavior of Y guarantees that when $c\cdot G_{(D)} \ll 1$ there must be a point of intersection near r_h, which is graphically obvious.[6] Secondly, the approximate formula (19) is obtained by Taylor expansion of the local inverse function of Y near r_h.[7] As listed in Table 1, we calculate the approximate locations of QESs for several common black holes by (19) and compare them with existing results [42–44, 64, 65, 67]. Note that those results are calculated in the models with gravitational bath which is different from our models. However, the coupling of a non-gravitational bath doesn't affect the result mathematically as discussed in the last subsection. Explicitly, our results coincide with qualitative results in literature [64], and exactly match the quantitative results in [42].[8]

Substituted the approximate solution (19) into (15), the late-time radiation entropy after

[6] One may worry that we may miss some other points of intersection. Indeed, for asymptotically flat black holes, it's not hard to find that there is another intersection near r_b, which we call $r_{a'}$. However, when considering the constraint that $c\cdot G_{(D)} \ll 1$, the leading order contribution of the island formula comes from the area term, and since $r_{a'} > r_a$, we have $S_{\mathrm{Rad}}(r_{a'}) > S_{\mathrm{Rad}}(r_a)$. The root near r_b is thus discarded.

[7] For details, please refer to Appendix A.

[8] In addition, Eq.(19) can also reproduce the results in [44,67] and differ by a scale factor from those of [43,65].

including the island contribution can be obtained as[9]

$$\begin{aligned} S_{\rm Rad}[\text{with island}] =& \frac{A(r_h)}{2G_{(D)}} + \frac{c}{3}\log d^2(r_h, r_b) - \frac{4\kappa c^2 G_{(D)}}{9A'(r_h)} \exp\Big\{ - 2\kappa r_b^* - 2\rho(r_h)\Big\} + \mathcal{O}(c^3 G_{(D)}^2) \\ =& 2S_{\rm BH} + \mathcal{O}(c) \quad (c\cdot G_{(D)} \ll 1). \end{aligned} \tag{20}$$

Note that the above approximation formula for the late-time radiation entropy also coincide with results in [42, 67]. Based on above results, we can reproduce the Page curve for generic non-extremal spherically symmetric black holes as Fig.3. Under the constraint $c\cdot G_{(D)} \ll 1$, the estimation of the Page time also has a concise and uniform form, $t_{\rm Page} \sim \frac{6S_{\rm BH}}{c\kappa} = \frac{3S_{\rm BH}}{\pi c T_{\rm H}}$ for all black holes that meet the requirements. Note that, as shown in the next section, once the condition $c\cdot G_{(D)} \ll 1$ is broken, the late-time radiation entropy after considering the island contribution does not saturate near $2S_{\rm BH}$, but has a significant deviation. This suggests that the estimation for the Page time will also change.

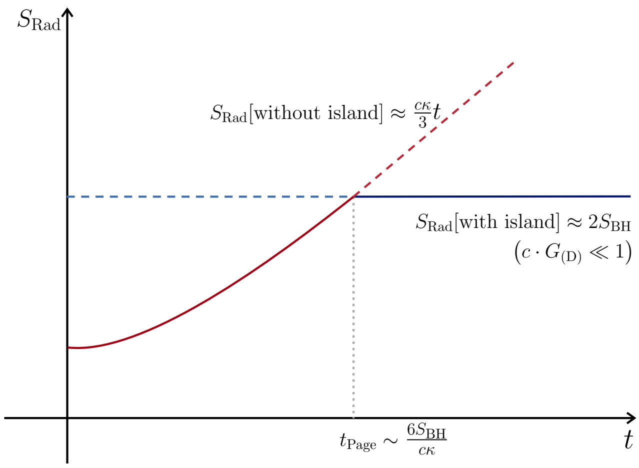


Figure 3: The Page curve for D-dimensional non-extremal spherically symmetric black holes. When the contribution of the island is not considered, the late-time radiation entropy increases linearly (red dashed line); After considering the island's contribution and the condition $c\cdot G_{(D)} \ll 1$, the late-time radiation entropy is approximatively saturated at $2S_{\rm BH}$ (blue solid line).

2.4 Go beyond $c\cdot G_{(D)} \ll 1$: Examples in two-dimensional dilaton gravity

In the previous section, we show that when $c\cdot G_{(D)} \ll 1$, there must be a QES located in the near-horizon region outside the black hole, and the late-time radiation entropy given by it is saturated near $2S_{\rm BH}$ (with sub-leading corrections of order c). It is natural to ask how does the island change when $c\cdot G_{(D)} \ll 1$ is no longer satisfied. One can expect that the location of the QES might be model-dependent and the late-time radiation entropy may deviate from $2S_{\rm BH}$ significantly. In this section, we shall numerically solve the equation (16) in eternal black hole solutions of two-dimensional GDT to look at the change of the island as $c\cdot G_{(2)}$ varies.

[9]Similar to (19), the derivation is a little tricky, please refer to Appendix B for details.

2.4.1 Eternal black holes in GDT

The action of the GDT in 2 dimensions is given by [66]

$$I_{\rm GDT}=\frac{1}{16\pi G_{(2)}}\int_{\mathcal{M}}\sqrt{-g}\left(\phi R+U(\phi)\left(\nabla\phi\right)^2+V(\phi)\right)\mathrm{d}^2x+\frac{1}{8\pi G_{(2)}}\int_{\partial\mathcal{M}}\sqrt{-h}\left(\phi K-\mathcal{L}_{\rm c.t.}\right)\mathrm{d}x. \tag{21}$$

Note that in the above equation, $U(\phi)$ and $V(\phi)$ are arbitrary functions of dilaton field ϕ. The boundary term in the action involving the extrinsic curvature K and a counterterm $\mathcal{L}_{\rm c.t.}$[10] plays two main roles [71]: 1) It makes the variational properties of the action compatible with the semi-classical approximation of the path integral. 2) It renders the Euclidean on-shell action finite and gives the correct black hole thermodynamics.

Given proper functions U and V, one can in principle obtain a series of physically reasonable solutions. Notably, a family of eternal black hole solutions have been given in [66, 71]: Under the Schwarzschild gauge of the metric and the time-independent presupposition of the dilaton field

$$\mathrm{d}s^2=-f(r)\mathrm{d}t^2+f(r)^{-1}\mathrm{d}r^2,\quad \phi=\phi(r), \tag{22}$$

the equations of motion corresponding to (21)

$$\partial_\phi U(\phi)\left(\nabla\phi\right)^2+2U(\phi)\nabla^2\phi-\partial_\phi V(\phi)=R, \tag{23}$$

$$U\nabla_a\phi\nabla_b\phi-\nabla_a\nabla_b\phi+g_{ab}\left[\nabla^2\phi-\frac{1}{2}U(\phi)\left(\nabla\phi\right)^2-\frac{1}{2}V(\phi)\right]=0 \tag{24}$$

can be solved as

$$r=\int^{\phi}\mathrm{e}^{Q(\phi')}\mathrm{d}\phi'+C, \tag{25}$$

$$f(r)\equiv F\big(\phi(r)\big)=\big(W(\phi)-16\pi G_{(2)}M\big)\,\mathrm{e}^{Q(\phi)}, \tag{26}$$

where

$$Q(\phi)=Q_0-\int^{\phi}U(\phi')\mathrm{d}\phi', \tag{27}$$

$$W(\phi)=W_0+\int^{\phi}V(\phi')\mathrm{e}^{Q(\phi')}\mathrm{d}\phi'. \tag{28}$$

Here C, Q_0, W_0 are integration constants and M is the mass parameter[11] of the black hole as shown in appendix C to preserve the thermodynamic relation with the black hole temperature T and entropy S,

$$T=\beta^{-1}=\frac{f'(r_h)}{4\pi}=\frac{\partial_\phi W}{4\pi}\bigg|_{\phi_h}\qquad \big(\phi_h\equiv\phi(r_h)\big), \tag{29}$$

$$S=\frac{\phi_h}{4G_{(2)}}, \tag{30}$$

[10]The exact form of $\mathcal{L}_{\rm c.t.}$ depends on the selection of $U(\phi)$ and $V(\phi)$, please refer to Appendix C for details. One can also refer to [70] for details.

[11]M is also the conserved charge associated with the Killing vector ∂_t and coincides with the ADM mass if $\lim_{\phi\to\infty}W(\phi)\mathrm{e}^{Q(\phi)}=1$.

where r_h means the location of the outermost horizon.

2.4.2 Numerical results

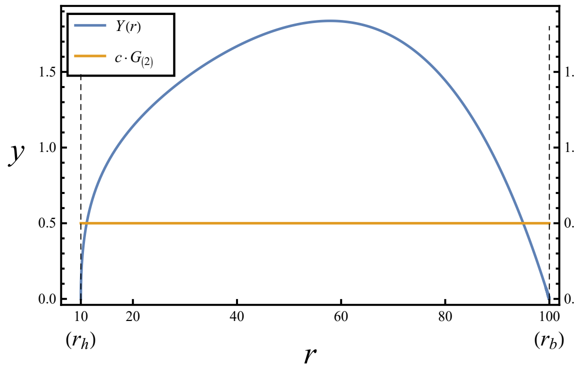
To show the behavior of the QES and its corresponding late-time radiation entropy with respect to $c \cdot G_{(2)}$, we mainly focus on the following concrete cases: (Weyl-related) Witten (or CGHS) black hole [72–74], (Weyl-related) Schwarzschild black hole [75], (Weyl-related) black hole attractor [76], JT black hole [77, 78], and AdS-Schwarzschild black hole [79, 80]. The first six black holes are (asymptotically) flat[12] and the prefix "Weyl-related" means that the metric of the theory is related to the original theory by a Weyl transformation (see Appendix D). The metrics, dilaton profiles and corresponding U, V functions are summarized in Table 2.

Table 2: Serval eternal black hole solutions in two-dimensional GDT.

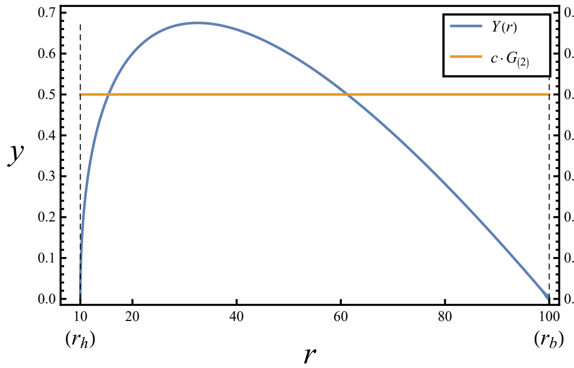
Black hole	$U(\phi)$	$V(\phi)$	$\phi(r)$	$f(r)$
Witten (CGHS)	ϕ^{-1}	$4\lambda^2\phi$	$e^{2\lambda r}$	$1 - e^{-2\lambda(r-r_h)}$
Weyl-related Witten (CGHS)	0	$4\lambda^2$	$2\lambda r$	$2\lambda(r - r_h)$
Schwarzschild	$(2\phi)^{-1}$	$2\lambda^2$	$\lambda^2 r^2$	$1 - \frac{r_h}{r}$
Weyl-related Schwarzschild	0	$2\lambda^2\phi^{-\frac{1}{2}}$	$2\lambda r$	$\sqrt{2\lambda r} - \sqrt{2\lambda r_h}$
Black hole attractor	0	$4\lambda^2\phi^{-1}$	$2\lambda r$	$\log \frac{r}{r_h}$
Weyl-related black hole attractor	ϕ^{-1}	$4\lambda^2$	$e^{2\lambda r}$	$2\lambda e^{-2\lambda r}(r - r_h)$
JT	0	$\frac{2}{L^2}\phi$	$\frac{r}{L}$	$\frac{r^2-r_h^2}{L^2}$
AdS-Schwarzschild	$(2\phi)^{-1}$	$2\lambda^2 + \frac{6}{L^2}\phi$	$\lambda^2 r^2$	$\frac{(r-r_h)(r^2+r_h r+r_h^2+L^2)}{L^2 r}$

We first draw Y-functions (17) for black holes mentioned above. As demonstrated in Fig.4, for (asymptotically) flat black holes, $Y(r)$ is a concave function that is continuous and consistently greater than or equals to 0 on the closed interval $[r_h, r_b]$ (0 is evaluated at two endpoints). This indicates that when $0 < c \cdot G_{(2)} < \max_{r_h<r<r_b}[Y(r)]$, there must be two roots, one is closer to the horizon (denoted as a) and the other is closer to the boundary of the collecting region (denoted as a'). By comparing the late-time radiation entropy given by the two roots, as shown in Fig.5, a' is discarded due to the larger entropy given. When $c \cdot G_{(2)} = \max_{r_h<r<r_b}[Y(r)]$, a coincides with a'. When $c \cdot G_{(2)} > \max_{r_h<r<r_b}[Y(r)]$, Eq.(16) has no solution and the island structure is thus destroyed, as stated in conclusion 2.

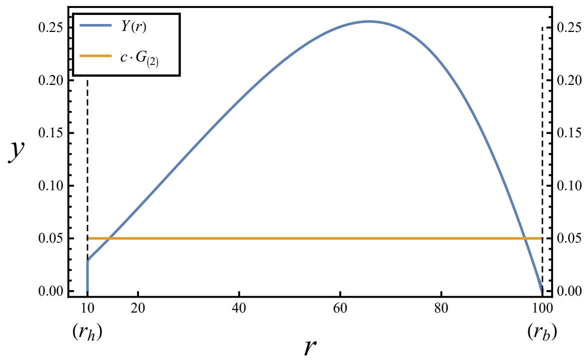
[12]The curvature for the Weyl-related Witten (CGHS) black hole is zero.



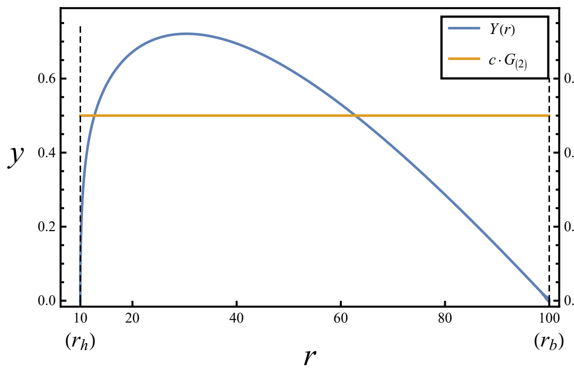
(a) $Y(r)$ for Witten (CGHS) black hole



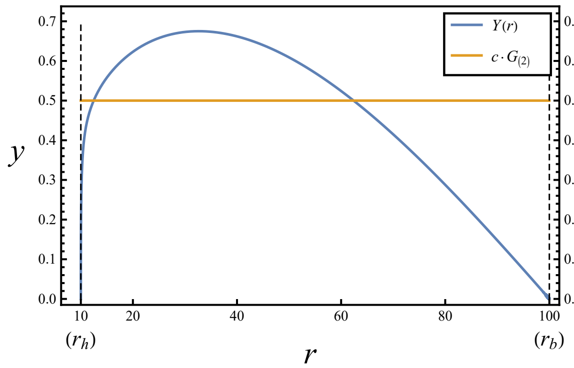
(b) $Y(r)$ for Weyl-related Witten (CGHS) black hole



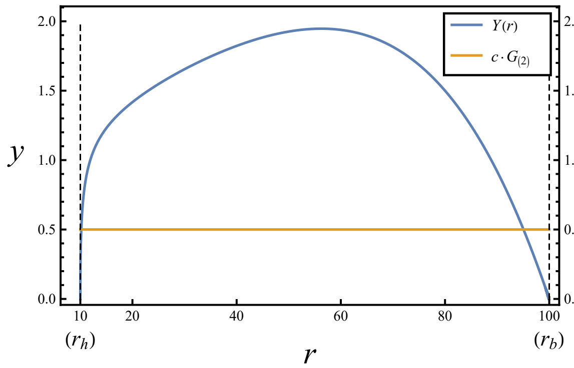
(c) $Y(r)$ for Schwarzschild black hole



(d) $Y(r)$ for Weyl-related Schwarzschild black hole



(e) $Y(r)$ for black hole attractor



(f) $Y(r)$ for Weyl-related black hole attractor

Figure 4: Y-functions (17) for (asymptotically) flat black holes. We draw these diagrams by setting $r_h = 10$, $\lambda = 10^{-2}$, $r_b = 10r_h$,

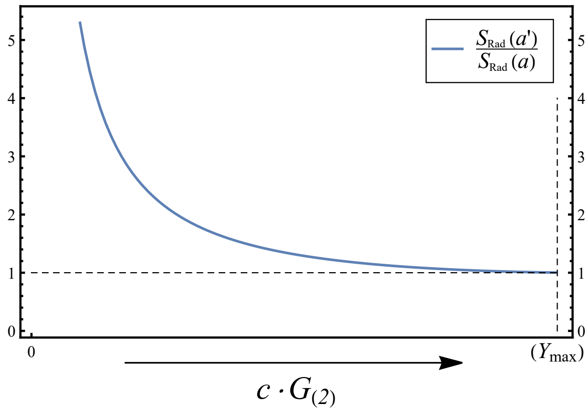
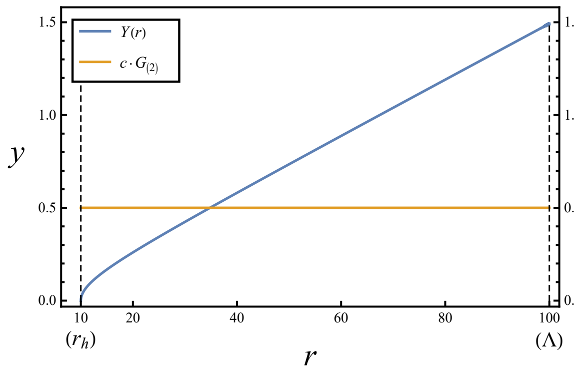
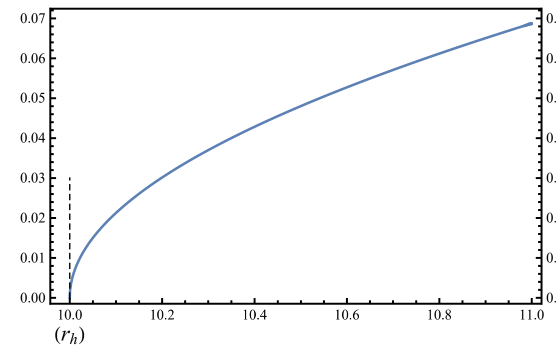


Figure 5: The ratio of the late-time radiation entropy given by the two roots of (16) for (asymptotically) flat black holes. a represents the root near r_h and a' represents the root near r_b. The numerical result shows that the root near the horizon will always be the boundary of island, if there is one.

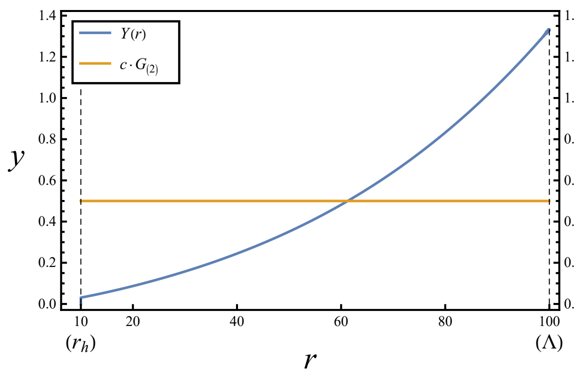
The situation will be changed for (asymptotically) AdS black holes. As shown in Fig.6, $Y(r)$ is monotonically increasing from zero on the interval $[r_h, \Lambda]$, which indicates that eq.(16) has one and only one root if and only if $0 < c \cdot G_{(2)} \leq Y(\Lambda)$. Therefore, as stated in conclusion 2, for (asymptotically) AdS black holes, $c \cdot G_{(2)}$ has an upper bound after the truncation is given. For JT and AdS-Schwarzschild black holes this is given by the value of $Y(r)$ at cut-off. It can be seen that no matter it is asymptotic flatness (Fig.4) or asymptotic AdS (Fig.6), the



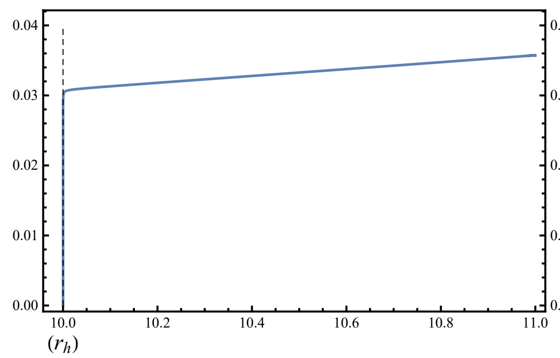
(a) $Y(r)$ for JT black hole



(b) Enlarged view of Fig.6(a) at r_h



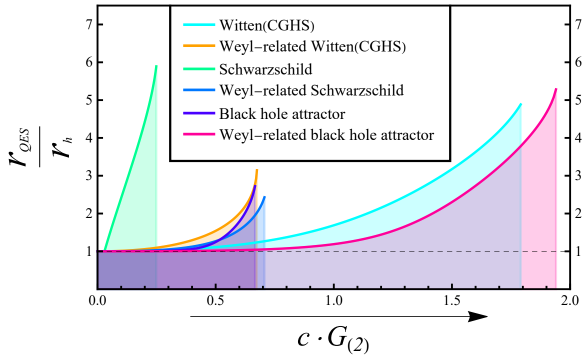
(c) $Y(r)$ for AdS-Schwarzschild black hole



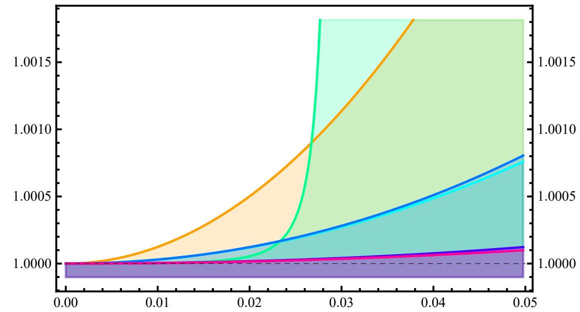
(d) Enlarged view of Fig.6(c) at r_h

Figure 6: Y-functions (17) for (asymptotically) AdS black holes. We draw these diagrams by setting $r_h = 10$, $L = 100$, $r_b^* = 0$, $\Lambda = 10r_h$ ($\lambda = 10^{-2}$ for AdS-Schwarzschild).

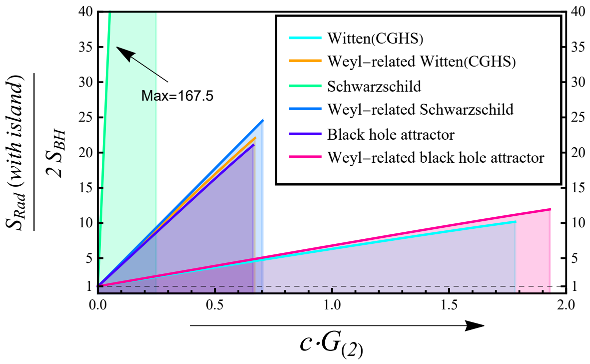
behavior of Y-function near r_h is similar to that of the square root function $\sqrt{\frac{r}{r_h} - 1}$, which is



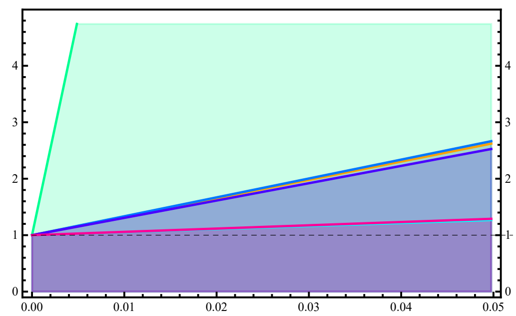
(a) r_{QES}/r_h for asymptotically flat black holes



(b) Enlarged view of Fig.7(a) at $c \cdot G_{(2)} \sim 0$



(c) $S_{\mathrm{Rad}}/2S_{\mathrm{BH}}$ for asymptotically flat black holes



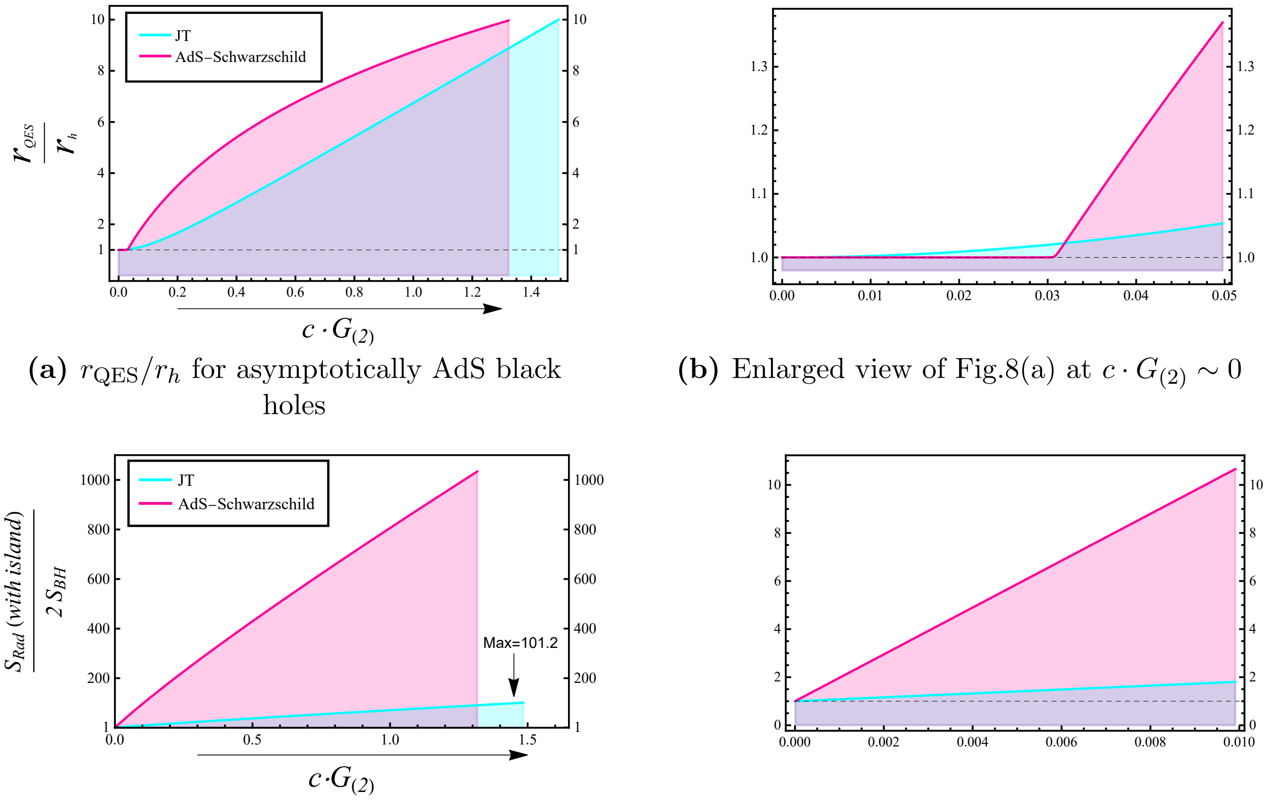
(d) Enlarged view of Fig.7(c) at $c \cdot G_{(2)} \sim 0$

Figure 7: Curves of the QESs and their corresponding late-time radiation entropy with respect to $c \cdot G_{(2)}$ (for asymptotically flat black holes). Diagrams are plotted with setting $r_h = 10$, $\lambda = 10^{-2}$, $r_b = 10r_h$, $G_{(2)} = 1/8\pi$.

consistent with the previous analysis.

By solving the intersection of $y = Y(r)$ and $y = c \cdot G_{(2)}$ numerically, we obtain a series of curves of r_{QES} with respect to $c \cdot G_{(2)}$, see Fig.7(a),(b) for (asymptotically) flat black holes and Fig.8(a),(b) for (asymptotically) AdS black holes. The corresponding late-time radiation entropy for (asymptotically) flat and AdS black holes are plotted as Fig.7(c),(d) and Fig.8(c),(d) respectively.

According to the results in Fig.7 and 8, we may summarize the behavior of the quantum extremum surface and its corresponding late-time radiation entropy with respect to $c \cdot G_{(2)}$: When $c \cdot G_{(2)} \ll 1$ (or $c \cdot G_{(2)} \sim 0$), the location of the QES and the late-time radiation entropy are described by (19) and (20) respectively. It results in the QES located in the near-horizon region of the black hole and is a square function of $c \cdot G_{(2)}$, and the radiation entropy is approximately equal to two times the black hole entropy and is a linear function of c. When $c \cdot G_{(2)}$ gradually increases, the QES will gradually move away from the horizon, and the radiation entropy will obviously deviate from the black hole entropy. When $c \cdot G_{(2)}$ grows beyond a certain limit, assuming that r_b (or Λ for asymptotically AdS black holes) has been fixed, the equation governing the location of the QES (16) will have no solution and the island



(a) $r_{\rm QES}/r_h$ for asymptotically AdS black holes

(b) Enlarged view of Fig.8(a) at $c \cdot G_{(2)} \sim 0$

(c) $S_{\rm Rad}/2S_{\rm BH}$ for asymptotically AdS black holes

(d) Enlarged view of Fig.8(c) at $c \cdot G_{(2)} \sim 0$

Figure 8: Curves of the QESs and their corresponding late-time radiation entropy with respect to $c \cdot G_{(2)}$ (for asymptotically AdS black holes). Diagrams are plotted with setting $r_h = 10$, $L = 100$, $r_b^* = 0$ $\Lambda = 10r_h$, $G_{(2)} = 1/8\pi$ ($\lambda = 10^{-2}$ for the AdS-Schwarzschild black hole).

configuration will be destroyed.

3 Conclusions and prospect

In this paper, we systematically study the QES associated with the Hawking radiation collected by a non-gravitational bath for general D-dimensional ($D \geq 2$) asymptotically flat (or AdS) eternal black holes using the island formula. We focus on the non-extremal black hole with spherical symmetry. In this case, the near-horizon geometry is common to all non-extreme black holes and we can use the s-wave approximation in higher dimensional ($D \geq 3$) calculating of the matter field entropy. We have obtained the following conclusions:

- When $c \cdot G_{(D)} \ll 1$, thanks to the common near horizon structure, there must be a quantum extremal surface (QES) located in the near-horizon region outside the black hole,

$$r_{\text{QES}} = r_h + \frac{8\kappa(c \cdot G_{(D)})^2}{9A'(r_h)^2} \exp\Big\{ -2\kappa r_b^* - 2\rho(r_h)\Big\} + \mathcal{O}\big((c \cdot G_{(D)})^3\big), \tag{31}$$

  and the late time radiation entropy saturates $2S_{\text{BH}}$. The formula (31) is compatible with various known results in [42–44, 64, 65, 67] and the late time behaviour of the radiation entropy is in good agreement with the previous studies [14–18].

- We go beyond the $c \cdot G_{(D)} \ll 1$ limit and scan the parameter space numerically to analyze the location of the QES and its corresponding radiation entropy. It can be shown generally by the boundedness theorem that there must be an upper bound on $c \cdot G_{(D)}$ to have an island configuration. Besides, the numerical results manifest that the location of the QES is just out of the event horizon when $c \cdot G_{(D)} \ll 1$. As the value of $c \cdot G_{(D)}$ increases, the QES gradually goes away from the black hole event horizon and the radiation entropy bound will obviously deviate from $2S_{\text{BH}}$.

It will be interesting to extend our analysis to the near extremal black hole and black hole without spherical symmetry, such as the planar or axisymmetric black hole. Another thing to reconsider is the gravitational effects of the bath in the asymptotically flat black hole because the thermal bath in this case is a gravitational system intrinsically. In asymptotically AdS couple to a gravitating bath, one finds that there is a new saddle point of the bulk geometry in the replica calculation, namely a wormhole connecting the black hole and the gravitational bath [81]. After the Page time, this configuration is the dominant contribution and this phenomenon can be regarded as a realization of ER=EPR [82]. Besides, it is important to proof the QES formula still works in asymptotically flat spacetime with or without some non-gravitational reference system. The most interesting future problem is to see how an island is generated dynamically after the page time during the black hole evaporation process. To our knowledge, one can qualitatively reproduce the page curve behavior in several asymptotically flat (or AdS) eternal black holes. However, they can not tell how the black hole information is restored in a concrete way. We are ignorant of the details of the black hole evaporation process even in the semi-classical level. To dynamically generate the island will be an important aspect to reveal the mystery of the black hole information paradox.

Acknowledgements

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A Derivation of Eq.(19)

In this appendix, we present the details of the derivation of (19). The key is to find the second derivative of the local inverse of Y around r_h (the first derivative is zero), which can be expressed in terms of the derivative of the primitive function

$$\left(Y^{-1}\right)''\Big|_{Y(r_h)} = -\frac{Y''}{\left(Y'\right)^3}\Big|_{r_h}. \tag{32}$$

Firstly, it's useful to set $Y = \frac{3}{2}A'Z^{-1}$, and thereinto,

$$Z \equiv \alpha\beta + \gamma, \quad \alpha \equiv \frac{2\mathrm{e}^{\kappa r^*(r)}}{f}, \quad \beta \equiv \frac{\kappa}{\mathrm{e}^{\kappa r_b^*} - \mathrm{e}^{\kappa r^*(r)}}, \quad \gamma \equiv \frac{2\kappa - f'}{2f}. \tag{33}$$

Z is blow up when $r \to r_h$, as is evident from the following limits

$$\lim_{r\to r_h} \alpha \sim \lim_{r\to r_h} \frac{1}{\sqrt{\frac{r}{r_h} - 1}} = \infty, \quad \lim_{r\to r_h} \beta = \kappa \mathrm{e}^{-\kappa r_b^*} \equiv \beta_h, \quad \lim_{r\to r_h} \gamma = \frac{-f''(r_h)}{f'(r_h)} \equiv \gamma_h. \tag{34}$$

Let's write down the derivative of Y

$$Y' = \frac{3}{2}A''Z^{-1} - \frac{3}{2}A'\frac{Z'}{Z^2}, \quad Y'' = \frac{3}{2}A'''Z^{-1} - 3A''\frac{Z'}{Z^2} - \frac{3}{2}A'\left(\frac{Z''}{Z^2} - 2\frac{(Z')^2}{Z^3}\right), \tag{35}$$

where

$$\begin{aligned} Z' =& \alpha'\beta + \alpha\beta' + \gamma' \\ =& \alpha f^{-1}\beta(\kappa - f') + \alpha^2\frac{\beta^2}{2} + f^{-1}\left(-\frac{f''}{2} - \gamma f'\right), \end{aligned} \tag{36}$$

and

$$\begin{aligned} Z'' =& \alpha''\beta + 2\alpha'\beta' + \alpha\beta'' + \gamma'' \\ =& \alpha f^{-2}\beta(\kappa - f')(\kappa - 2f') - \alpha f^{-1}\beta f'' + \alpha^2 f^{-1}\beta^2(\kappa - f') \\ &+ \alpha^2 f^{-1}\beta^2\frac{(\kappa - f')}{2} + \alpha^3\frac{\beta^3}{2} + f^{-2}\left(f'f'' + 2\gamma(f')^2\right) - f^{-1}\left(\frac{1}{2}f''' + \gamma f''\right). \end{aligned} \tag{37}$$

In the above we have used the derivatives of α, β, and γ up to the second-order

$$\alpha' = \alpha f^{-1}(\kappa - f'), \qquad \alpha'' = \alpha f^{-2}(\kappa - f')(\kappa - 2f') - \alpha f^{-1} f'', \tag{38}$$

$$\beta' = \alpha \frac{\beta^2}{2}, \qquad \beta'' = \alpha f^{-1} \beta^2 \frac{(\kappa - f')}{2} + \alpha^2 \frac{\beta^3}{2}, \tag{39}$$

$$\gamma' = f^{-1}\Big(-\frac{f''}{2} - \gamma f' \Big), \qquad \gamma'' = f^{-2}\Big(f' f'' + 2\gamma (f')^2 \Big) - f^{-1}\Big(\frac{1}{2} f''' + \gamma f'' \Big). \tag{40}$$

According to (36-37) and the limits of α, β, and γ, we have

$$\frac{Z'}{Z^2} = \frac{\mathrm{e}^{-\kappa r^*(r)}}{2} \cdot \frac{\beta(\kappa - f') + \mathcal{O}(\alpha^{-1})}{\beta^2 + \mathcal{O}(\alpha^{-1})}, \tag{41}$$

$$\frac{Z''}{Z^2} = \frac{\mathrm{e}^{-\kappa r^*(r)}}{2f} \cdot \frac{\beta(\kappa - f')(\kappa - 2f') + \mathcal{O}(\alpha^{-1})}{\beta^2 + \mathcal{O}(\alpha^{-1})}, \tag{42}$$

$$\frac{(Z')^2}{Z^3} = \frac{\mathrm{e}^{-\kappa r^*(r)}}{2f} \cdot \frac{\beta^2(\kappa - f')^2 + \mathcal{O}(\alpha^{-1})}{\beta^3 + \mathcal{O}(\alpha^{-1})}, \tag{43}$$

which leads to the following two limits

$$\begin{aligned} &\lim_{r \to r_h} f \mathrm{e}^{\kappa r^*(r)} Y''(r) \\ =& -\frac{3}{2} A'(r_h) \lim_{r \to r_h} \Big(\frac{1}{2} \cdot \frac{\beta(\kappa - f')(\kappa - 2f') + \mathcal{O}(\alpha^{-1})}{\beta^2 + \mathcal{O}(\alpha^{-1})} - \frac{\beta^2(\kappa - f')^2 + \mathcal{O}(\alpha^{-1})}{\beta^3 + \mathcal{O}(\alpha^{-1})} \Big) \\ =& -\frac{3\kappa^2 A'(r_h)}{4\beta_h}, \end{aligned} \tag{44}$$

$$\begin{aligned} &\lim_{r \to r_h} f \mathrm{e}^{\kappa r^*(r)} \big(Y'\big)^3 \\ =& \frac{-3^3}{2^6} \big(A'(r_h)\big)^3 \lim_{r \to r_h} f \mathrm{e}^{-2\kappa r^*(r)} \cdot \frac{\beta^3(\kappa - f')^3 + \mathcal{O}(\alpha^{-1})}{\beta^6 + \mathcal{O}(\alpha^{-1})} \\ =& \frac{3^3 \kappa^3}{4^3} \big(A'(r_h)\big)^3 \mathrm{e}^{2\rho(r_h)} \beta_h^{-3}. \end{aligned} \tag{45}$$

The second derivative of the local inverse of Y at r_h can thus be obtained by using above limits

$$\big(Y^{-1}\big)''\Big|_{Y(r_h)} = \lim_{r \to r_h} \frac{-Y''}{\big(Y'\big)^3} = \kappa \left(\frac{3}{4} A'(r_h) \exp\left\{ \rho(r_h) + \kappa r_b^* \right\} \right)^{-2}. \tag{46}$$

In the light of the Taylor expansion of $Y(r)^{-1}$ at $Y(r_h) = 0$, we finally arrive at the approximation of r_a upto the second-order of $c^2 G_{(D)}^2$

$$\begin{aligned} r_a =& r_h + \frac{1}{2} \big(Y^{-1}\big)'' \Big|_{Y(r_h)} \cdot \big(c \cdot G_{(D)}\big)^2 + \mathcal{O}\Big(\big(c \cdot G_{(D)}\big)^3 \Big) \\ =& r_h + \frac{8\kappa (c \cdot G_{(D)})^2}{9 A'(r_h)^2} \exp\Big\{ -2\kappa r_b^* - 2\rho(r_h) \Big\} + \mathcal{O}\Big(\big(c \cdot G_{(D)}\big)^3 \Big). \end{aligned} \tag{47}$$

B Derivation of Eq.(20)

In this appendix, we demonstrate the derivation of (20). Firstly (15) is essentially

$$S_{\text{Rad}}(r) = \frac{A(r)}{2G_{(D)}} + \frac{c}{3} \log d^2(r, r_b) \quad (t = t_b), \tag{48}$$

where the first term is the area term, which is easy to evaluate according to (19)

$$\begin{aligned}\frac{A(r_a)}{2G_{(D)}} =&\frac{A(r_h)}{2G_{(D)}} + \frac{A'(r_h)}{2G_{(D)}}(r_a - r_h) + \mathcal{O}\big((r_a - r_h)^2\big)\\ =&\frac{A(r_h)}{2G_{(D)}} + \frac{4\kappa c^2 \cdot G_{(D)}}{9A'(r_h)}\exp\Big\{-2\kappa r_b^* - 2\rho(r_h)\Big\} + \mathcal{O}(c^3 G_{(D)}^2).\end{aligned} \tag{49}$$

We then focus on the matter term. Since the approximate behavior of Y near r_h is (18), meanwhile $Y(r) = -\frac{3}{2}A'(r)\left(\partial_r \log d^2(r, r_b)\right)^{-1}$, which gives that the approximate behavior of $\log d^2(r, r_b)$ near r_h is $C_h - \sqrt{\frac{r}{r_h} - 1}$, where $C_h \equiv \log d^2(r_h, r_b)$. We can thus obtain its approximation by Taylor expansion of its local inverse function at r_h

$$r = r_h + \frac{1}{2}\Big(\big(\log d^2(r, r_b)\big)^{-1}\Big)''\Big|_{r_h} \cdot \Big(\log d^2(r, r_b) - C_h\Big)^2. \tag{50}$$

The key step again becomes finding the second derivative of the inverse function at r_h

$$\Big(\big(\log d^2(r, r_b)\big)^{-1}\Big)'' = -\frac{\big(\log d^2(r, r_b)\big)''}{\Big(\big(\log d^2(r, r_b)\big)'\Big)^3} = -\frac{Z'}{Z^3}, \tag{51}$$

where Z is defined as (33). In terms of the limits obtained in Appendix A, it's not difficult to find that

$$\begin{aligned}\lim_{r\to r_h}\Big(\big(\log d^2(r, r_b)\big)^{-1}\Big)'' =& \lim_{r\to r_h}\left(\frac{f\mathrm{e}^{-2\kappa r^*}}{4}\cdot\frac{\beta(f'-\kappa) + \mathcal{O}(\alpha^{-1})}{\beta^3 + \mathcal{O}(\alpha^{-1})}\right)\\ =&\frac{1}{4\kappa}\exp\big\{2\kappa r_b^* + 2\rho(r_h)\big\}.\end{aligned} \tag{52}$$

We then inversely solve for the approximation of $\log d^2(r_a, r_b)$ based on (50) and (52)

$$\begin{aligned}\log d^2(r_a, r_b) \approx& C_h - \sqrt{\frac{2(r_a - r_h)}{\Big(\big(\log d^2(r, r_b)\big)^{-1}\Big)''\Big|_{r_h}}}\\ =&C_h - \frac{8\kappa c \cdot G_{(D)}}{3A'(r_h)}\exp\big\{-2\kappa r_b^* - 2\rho(r_h)\big\}.\end{aligned} \tag{53}$$

Combine (49) with (53), the final answer arrives

$$\begin{aligned}S_{\mathrm{Rad}}(\text{with island}) =&\frac{A(r_a)}{2G_{(D)}} + \frac{c}{3}\log d^2(r_a, r_b)\\ \approx&\frac{A(r_h)}{2G_{(D)}} + \frac{c}{3}\log d^2(r_h, r_b) - \frac{4\kappa c^2 \cdot G_{(D)}}{9A'(r_h)}\exp\Big\{-2\kappa r_b^* - 2\rho(r_h)\Big\} + \mathcal{O}(c^3 G_{(D)}^2).\end{aligned} \tag{54}$$

C Black hole thermodynamics in GDT

In this appendix, we derive the thermodynamic quantities for the 2d dilaton gravity models with action (21).[13] We start with the corresponding Euclidean version

$$\begin{aligned} I_{\mathrm{E}} = & -\frac{1}{16\pi G_{(2)}}\int_{\mathcal{M}}\sqrt{g}\left(\phi R+U(\phi)\left(\nabla\phi\right)^2+V(\phi)\right)\mathrm{d}^2x \\ & -\frac{1}{8\pi G_{(2)}}\int_{\partial\mathcal{M}}\sqrt{h}\phi K\mathrm{d}x+\frac{1}{8\pi G_{(2)}}\int_{\partial\mathcal{M}}\sqrt{h}\mathcal{L}_{\mathrm{c.t.}}\mathrm{d}x, \end{aligned} \tag{55}$$

where $\mathcal{M}$ is spacetime region outside the black hole and the corresponding boundary $\partial\mathcal{M}$ is $\{r=r_h\}\bigcup\{r=r_{\mathrm{reg.}}\}$. Note that $r_{\mathrm{reg.}}$ is a regulator and should be removed by taking the limit $r_{\mathrm{reg.}}\to\infty$.

The boundary counterterm $\mathcal{L}_{\mathrm{c.t.}}$, as we will see below, should be of form

$$\mathcal{L}_{\mathrm{c.t.}}=\sqrt{W(\phi)\mathrm{e}^{-Q(\phi)}}, \tag{56}$$

where the definitions of $W(\phi)$ and $Q(\phi)$ are (28) and (27),respectively. We show this point by reproducing the correct thermodynamics of the black hole.

We start with evaluating the Euclidean action for the black hole solution (25–26). The bulk contribution reads

$$\begin{aligned} I_{\mathrm{E}}^{\mathrm{bulk}} = & \frac{-1}{16\pi G_{(2)}}\int_{\mathcal{M}}\sqrt{g}\Big\{\phi R+U(\phi)\big(\nabla\phi\big)^2+V(\phi)\Big\}\mathrm{d}^2x \\ = & \frac{-1}{16\pi G_{(2)}}\int_0^{\beta}\mathrm{d}\tau\int_{r_h}^{r_{\mathrm{reg.}}}\mathrm{d}r\Big\{-\phi f''(r)+U(\phi)f(r)\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^2+V(\phi)\Big\} \\ = & \frac{-\beta}{16\pi G_{(2)}}\int_{\phi_h}^{\phi_{\mathrm{reg.}}}\mathrm{d}\phi\Big\{-\phi\partial_\phi^2W+\phi U\partial_\phi W+\phi W\partial_\phi U-16\pi G_{(2)}M\phi\partial_\phi U \\ & +U\big(W-16\pi G_{(2)}M\big)+\partial_\phi W\Big\} \\ = & \frac{-\beta}{16\pi G_{(2)}}\Big\{-\phi\partial_\phi W\Big|_{\phi_h}^{\phi_{\mathrm{reg.}}}+2W\Big|_{\phi_h}^{\phi_{\mathrm{reg.}}}+\phi_{\mathrm{reg.}}U(\phi_{\mathrm{reg.}})\big(W(\phi_{\mathrm{reg.}})-16\pi G_{(2)}M\big)\Big\}. \end{aligned} \tag{57}$$

Let us next consider the on-shell Gibbons-Hawking-York(GHY) term

$$\begin{aligned} I_{\mathrm{E}}^{\mathrm{GHY}} = & -\frac{1}{8\pi G_{(2)}}\int_{\partial\mathcal{M}}\sqrt{h}\phi K\mathrm{d}x \\ = & -\frac{\beta}{8\pi G_{(2)}}\sqrt{f(r_{\mathrm{reg.}})}\cdot\phi_{\mathrm{reg.}}K(r_{\mathrm{reg.}}) \\ = & \frac{-\beta}{16\pi G_{(2)}}\Big\{\phi_{\mathrm{reg.}}\partial_\phi W(\phi_{\mathrm{reg.}})-\phi_{\mathrm{reg.}}U(\phi_{\mathrm{reg.}})\big(W(\phi_{\mathrm{reg.}})-16\pi G_{(2)}M\big)\Big\}. \end{aligned} \tag{58}$$

It's clear to see that the on-shell bulk term plus the on-shell GHY term equals

$$\frac{\beta}{16\pi G_{(2)}}\Big\{2W(\phi_h)-2W(\phi_{\mathrm{reg.}})-\phi_h\partial_\phi W(\phi_h)\Big\}. \tag{59}$$

[13] Certain assumptions have been made, 1) $\lim_{r\to+\infty}\phi=+\infty$. 2) $\lim_{\phi\to+\infty}W(\phi)=+\infty$. 3) $\mathrm{e}^Q\neq 0$ for finite ϕ, in this derivation.

The above equation is divergent when $r_{\text{reg.}} \to \infty$ since we have assumed that $\lim_{\phi\to\infty} W(\phi) = \infty$.

The final contribution in (55) is the boundary counterterm

$$\begin{aligned} I_{\text{E}}^{\text{c.t.}} =& \frac{1}{8\pi G_{(2)}} \int_{\partial\mathcal{M}} \sqrt{h}\mathcal{L}_{\text{c.t.}} \mathrm{d}x \\ =& \frac{\beta}{8\pi G_{(2)}} \sqrt{\big(W(\phi_{\text{reg.}}) - 16\pi G_{(2)} M\big)\, \mathrm{e}^{Q(\phi_{\text{reg.}})}} \cdot \sqrt{W(\phi_{\text{reg.}}) \mathrm{e}^{-Q(\phi_{\text{reg.}})}} \\ =& \frac{\beta}{8\pi G_{(2)}} \Big\{ W(\phi_{\text{reg.}}) - 8\pi G_{(2)} M + \mathcal{O}\big(W(\phi_{\text{reg.}})^{-1}\big) \Big\}. \end{aligned} \tag{60}$$

Summing over above contribution and letting $\phi_{\text{reg.}} \to \infty$, the total on-shell action reads

$$I_{\text{E}}^{\text{total}} = \beta M - S, \tag{61}$$

where we have used the definitions of black hole temperature (29) and Wald entropy (30). The free energy of black hole in the canonical ensemble reads

$$\mathcal{F} = -\frac{1}{\beta} \log \mathcal{Z} \sim -\frac{1}{\beta} \log \mathrm{e}^{-I_{\text{E}}^{\text{total}}} = M - TS. \tag{62}$$

D Models related by Weyl transformation

An intriguing feature of the 2d-dilaton gravity model is that one can eliminate (or recover) the kinetic term of the dilaton in the original theory by applying a Weyl transformation [83–85]. Let's consider a new metric $\hat{g}_{\mu\nu}$ related to $g_{\mu\nu}$ by

$$g_{\mu\nu} = \mathrm{e}^{-2\Omega} \hat{g}_{\mu\nu}, \quad \Omega = \frac{1}{2} \int^{\phi} U(\phi') \mathrm{d}\phi'. \tag{63}$$

The bulk term of (21) can be re-expressed as follows in terms of $\hat{g}$

$$\begin{aligned} \int_{\mathcal{M}} \sqrt{-g} \big(\phi R + U(\phi) \big(\nabla\phi\big)^2 + V(\phi)\big) \mathrm{d}^2 x =& \int_{\mathcal{M}} \sqrt{-\hat{g}} \big(\phi \hat{R} + \mathrm{e}^{-2\Omega} V(\phi)\big) \mathrm{d}^2 x \\ &+ \int_{\partial\mathcal{M}} \sqrt{-\hat{h}}\ \phi U(\phi) \hat{n}^{\mu} \hat{\nabla}_{\mu} \phi \mathrm{d}x, \end{aligned} \tag{64}$$

where $\hat{R}$ and $\hat{\nabla}$ are Ricci scalar and covariant derivative corresponding to $\hat{g}_{\mu\nu}$, $\hat{n}^{\mu}$ is the unit vector normal to $\partial\mathcal{M}$. Ignoring the boundary term of no interest, we arrive a simpler theory with vanished kinetic term of dilaton. Three things are noteworthy about the new theory (64): 1) It gives a linear dilaton solution, i.e., $\phi_{\text{new}}(r) = \mathrm{e}^{-Q_0} r$. 2) The new metric solution is closely related to the original one. We have $f_{\text{new}}(r) \equiv F_{\text{new}}(\phi_{\text{new}}) = \mathrm{e}^{-2\Omega} F|_{\phi=\phi_{\text{new}}}$, where $F(\phi)$ is solution of the original theory. 3) The black hole thermodynamic quantities are invariant under the Weyl transformation.[14]

[14] The reason comes from the following observation: $W(\phi)$ is invariant under the transformation (63).

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