

Irrelevant Operators and their Holographic Anomalies

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ABSTRACT

Irrelevant operators in a CFT modify the usual Weyl transformation of the metric. A metric beta-function turns on, which modifies the Weyl anomalies as well. In this paper, we study the relation between bulk diffeomorphisms and Weyl transformation at the boundary when a massive scalar field, which sources irrelevant operators at the boundary, is coupled to the bulk metric. Considering the effect of the backreaction generated by the scalar field, we provide a holographic description of the boundary metric beta-function and anomalies. Our results represent an additional test of the AdS/CFT correspondence.

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1 Introduction

The holographic computation of the Weyl anomaly [1] has been one of the very first non-trivial tests of the AdS/CFT correspondence. Its success relies on the equivalence between bulk diffeomorphisms and Weyl transformations at the boundary of AdS space, which, since then, has been widely studied. However, it is not so clear how the equivalence works when the asymptotics of AdS are deformed, namely when the fields present at the boundary induce a strong backreaction onto the gravitational background.

In [1] Einstein's equations with a negative cosmological constant are solved in terms of the boundary metric and it is shown that the on-shell action is divergent at the boundary. The divergences can be cancelled by the addition of local counterterms, but the regularisation spoils the conformal symmetry of the boundary theory and gives rise to a holographic Weyl anomaly. Holographic renormalisation has been further developed in [2], where scalar fields coupled to gravity in the bulk and their contributions to the Weyl anomaly are also considered. When scalar fields are added on top of a dynamical background, they induce a backreaction, and the method presented in [2] is consistent when the bulk scalars are dual to relevant or marginal operators of the boundary CFT (see also [3]). Scalars that are dual to irrelevant operators induce a stronger backreaction onto the gravitational background, and in [4] the method of holographic renormalisation has been extended to account for such backreaction, when the irrelevant operators are of non-integer conformal dimension, so that no logarithmic divergence appears in the renormalised action (i.e. there is no anomaly in the CFT). Logarithmic divergences are considered in [5], where the

conformal anomaly in the three-point function of irrelevant operators is computed in the CFT and derived from holography. However, in this case no backreaction has been considered and the gravitational background has been taken to be unperturbed by the presence of the scalar fields. Here, we want to study the effect of coupling integer dimensional scalars that source irrelevant operators on the boundary theory to a dynamical background, thus including the effect of the backreaction.

CFTs with integer dimensional irrelevant operators have peculiar properties [6]. In order to have a solution of the Wess-Zumino consistency condition for the Weyl anomaly, the presence of these operators requires a modification of the usual Weyl transformation. A metric beta-function, which depends on the sources of the irrelevant operators, has to be introduced in the Weyl transformation of the metric. As a consequence, the Weyl anomaly is deformed by the metric beta-function, i.e. the solution of the Wess-Zumino consistency condition is different in the presence of the beta-function. The geometry, however, is not subject to an RG flow, since correlation functions of irrelevant operators in [6] are computed in the undeformed CFT.¹

Here we will present a holographic description of the four dimensional CFT studied in [6]. Therefore, we will also generalise the analysis of [5] to include the case in which the bulk scalar fields are coupled to a dynamical background. However, to make contact with [6], we will be interested in describing a boundary theory that is not deformed by the irrelevant operators. To do so, the tool that we find most convenient to use is that of Penrose-Brown-Henneaux (PBH) transformations [12]. These are a particular class of bulk diffeomorphisms that reduces to Weyl transformation on the boundary. They consist of a general transformation rule for the bulk metric, and, as such, they do not require solving any equation of motion. An action evaluated on a metric that is a solution of the PBH transformation allows to study the Weyl anomaly of the boundary theory. Thus, we will extend the analysis of [12] to include massive scalar fields in the bulk and provide a holographic description of the results obtained in [6]. As we will show, the condition that the irrelevant operators at the boundary do not deform the CFT requires that we are off-shell in the bulk, thus making the PBH transformation an ideal framework for the present analysis.

Irrelevant deformations of CFTs have received attention in particular in the form of $T\bar{T}$ deformations. It is proposed that a two dimensional $T\bar{T}$ deformed CFT is dual to a three dimensional AdS space with a sharp cutoff [13]. This conjecture has then been further analysed and extended to higher dimensions [14–16]. Although we are not considering a $T\bar{T}$ deformation, the scalar field that we will add in the bulk is sourcing an integer dimensional irrelevant operator on the boundary. As we will see, this addition will still have the effect of moving the boundary into the bulk, but, since we are interested in describing an undeformed boundary theory, we will have to move the cutoff back to the AdS boundary where the undeformed CFT lives. In so doing, the solution of the PBH transformations will go off-shell, in the sense that they won't match anymore the solution of the equations of motion of a scalar field coupled to a dynamical background. Nonetheless, once this is done we will precisely recover the physics described in [6]. We will thus see that the Weyl transformation

¹Metric beta-functions in the framework of holographic RG flows are also discussed e.g. in [7–11].

of the metric is no longer the usual one, but it acquires a beta-function, and we will be able to study the modified anomaly in our holographic set-up. In the end, this analysis will sharpen the understanding of the AdS/CFT duality in the presence of a backreaction in the bulk theory.

The paper is organized as follows. In sec. 2 we review the pure gravity formulation of the PBH transformations and how the holographic Weyl anomaly is obtained. In sec. 3 we extend the PBH transformations to describe a scalar field coupled to gravity and choose the scalar field to have integer dimension $\Delta = d + 1$. We show that the transformation of the boundary metric is no longer the usual Weyl transformation and we derive the modified Weyl anomaly of the four dimensional boundary theory. These are the main results of the present paper. We then conclude and discuss possible extensions of our analysis. We report all the explicit results that are needed to derive the modified anomaly in the appendices and in the ancillary Mathematica notebook.

2 Pure gravity

We introduce now the PBH transformations for the metric, mainly reviewing [12] (see also [17–20]) to set our notation.

2.1 PBH transformations

Consider an asymptotically AdS space in $(d+1)$ dimensions with coordinates (z, x^i) such that the bulk metric has the Fefferman-Graham (FG) form²

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = \frac{dz^2}{4z^2} + \frac{1}{z} g_{ij}(z, x) dx^i dx^j, \quad (2.1)$$

where the boundary is at $z = 0$, with $g_{ij}(z = 0, x) = g_{ij}^{(0)}(x)$ being the boundary metric. In order to study the Weyl anomaly of the dual CFT, we first study the behaviour of $G_{\mu\nu}$ under diffeomorphisms. Under a general coordinate transformation

$$X^\mu = X'^\mu + \xi^\mu(X'), \quad (2.2)$$

$G_{\mu\nu}$ transforms as

$$\delta G_{\mu\nu} = G_{\mu\rho} \partial_\nu \xi^\rho + G_{\nu\rho} \partial_\mu \xi^\rho + \xi^\rho \partial_\rho G_{\mu\nu}, \quad (2.3)$$

with $\delta G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu} = G'_{\mu\nu}(z, x) - G_{\mu\nu}(z, x)$ and we require that ξ^μ is such that (2.1) is form invariant under (2.2), i.e. $\mathcal{L}_\xi G_{zz} = 0 = \mathcal{L}_\xi G_{zi}$. The solution is given by

$$\xi^z = -2z\sigma(x), \quad \xi^i = a^i(z, x) = \frac{1}{2} \partial_j \sigma(x) \int_0^z dz' g^{ij}(z', x), \quad (2.4)$$

where $\sigma(x)$ is an arbitrary function, the a^i are infinitesimal and we will work to order $\mathcal{O}(\sigma, a^i)$. The lower end of the integration is chosen so that there are no residual

²We choose an AdS space with radius $\ell = 1$. Greek letters are used for $(d+1)$ -dimensional bulk indices, while Latin letters for d -dimensional boundary indices. Our convention on the curvature is $[\nabla_\mu, \nabla_\nu] V_\rho = R_{\mu\nu\rho}{}^\sigma V_\sigma$, with $R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho$.

diffeomorphisms at the boundary $z = 0$. In other words, the boundary condition $a^i(z = 0, x) = 0$ holds. From form invariance of G_{ij} it follows that

$$\delta g_{ij} = 2\sigma(1 - z\partial_z)g_{ij} + \nabla_i a_j + \nabla_j a_i, \quad (2.5)$$

where indices are lowered with (and derivatives are covariant w.r.t.) g_{ij} . Eqs. (2.4), (2.5) define the PBH transformations.

We show now that this particular class of bulk diffeomorphisms reduces to a Weyl transformation at the boundary. Indeed, the commutator of two diffeomorphisms (2.3) is again a diffeomorphism

$$[\delta_2, \delta_1]G_{\mu\nu} = G_{\mu\rho}\partial_\nu\hat{\xi}^\rho + G_{\nu\rho}\partial_\mu\hat{\xi}^\rho + \hat{\xi}^\rho\partial_\rho G_{\mu\nu}, \quad (2.6)$$

where $\hat{\xi}^\rho$ is defined as

$$\hat{\xi}^\rho = \xi_1^\sigma\partial_\sigma\xi_2^\rho - \xi_2^\sigma\partial_\sigma\xi_1^\rho + \delta_2\xi_1^\rho - \delta_1\xi_2^\rho \quad (2.7)$$

and the last two terms are non-vanishing if we allow ξ^μ to be field dependent. If the diffeomorphism is a PBH, then it is possible to derive the PBH group property $\hat{\xi}^\mu = 0$ [19, 21], so that $[\delta_2, \delta_1]G_{\mu\nu} = 0$. Since the PBH transformations do not act on coordinates, it follows that $[\delta_2, \delta_1]g_{ij} = 0$ and the PBH transformations reduce to a Weyl transformation on the boundary metric. Indeed, from (2.5) at $z = 0$ it follows that

$$\delta g_{ij}^{(0)} = 2\sigma g_{ij}^{(0)}. \quad (2.8)$$

Now we assume the following power series expansions about the boundary³

$$a^i(z, x) = \sum_{n=1}^{\infty} a_{(n)}^i(x)z^n \quad (2.9)$$

$$g_{ij}(z, x) = \sum_{n=0}^{\infty} g_{ij}^{(n)}(x)z^n. \quad (2.10)$$

Using the PBH equations, we can determine the coefficient in the expansion of the metric in terms of covariant tensors built from the boundary metric as follows. First, we compute the $a_{(n)}$ in terms of the $g_{(n)}$, and for the first few terms we find

$$a_{(1)}^i = \frac{1}{2}g_{(0)}^{ij}\partial_j\sigma \quad (2.11)$$

$$a_{(2)}^i = -\frac{1}{4}g_{(1)}^{ij}\partial_j\sigma. \quad (2.12)$$

Indices are now lowered (raised) with the (inverse of) $g_{ij}^{(0)}$; curvatures and covariant derivatives will be w.r.t. $g_{ij}^{(0)}$. Then, combining the expansions and the $a_{(n)}$ into (2.5) we find the variation of $g_{(n)}$ as

$$\delta g_{ij}^{(0)} = 2\sigma g_{ij}^{(0)} \quad (2.13)$$

⋮

$$\delta g_{ij}^{(n)} = 2\sigma(1 - n)g_{ij}^{(n)} + \sum_{m=1}^n \left(g_{ik}^{(n-m)}\partial_j a_{(m)}^k + g_{jk}^{(n-m)}\partial_i a_{(m)}^k + a_{(m)}^k\partial_k g_{ij}^{(n-m)} \right) \quad (2.14)$$

³If d is an even integer, the expansion of the metric contains also logarithmic terms. Here we work in generic dimensions, and we do not need to include such terms.

and we see that in general $g_{(n)}$ contains $2n$ derivatives. Thus, to compute it we make the most general Ansatz for a symmetric tensor with two indices and with $2n$ derivatives, take its variation according to (2.13) and impose that it satisfies the PBH equation. In this way, for example for the first two terms we find

$$g_{ij}^{(1)} = -\frac{1}{d-2} \left(R_{ij}^{(0)} - \frac{1}{2(d-1)} R^{(0)} g_{ij}^{(0)} \right) \quad (2.15)$$

$$\begin{aligned} g_{ij}^{(2)} = & \frac{1}{d-4} \left(\frac{1}{8(d-1)} \nabla_i^{(0)} \nabla_j^{(0)} R^{(0)} - \frac{1}{4(d-2)} \square^{(0)} R_{ij}^{(0)} + \frac{1}{8(d-2)(d-1)} g_{(0)ij}^{(0)} \square^{(0)} R^{(0)} \right. \\ & - \frac{1}{2(d-2)} R^{(0)kl} R_{ikjl}^{(0)} + \frac{d-4}{2(d-2)^2} R_{ik}^{(0)} R_j^k{}^{(0)} + \frac{1}{(d-2)^2(d-1)} R_{ij}^{(0)} R^{(0)} \\ & \left. + \frac{1}{4(d-2)^2} R_{kl}^{(0)} R^{kl} g_{ij}^{(0)} - \frac{3d}{16(d-2)^2(d-1)^2} g_{ij}^{(0)} R^2 \right) \\ & + c_1 C^2 g_{ij}^{(0)} + c_2 C_{iklm}^{(0)} C_j{}^{klm} \end{aligned} \quad (2.16)$$

where C_{ijkl} is the Weyl tensor. Starting from $g_{(2)}$, the solutions will have free coefficients that are not fixed by the PBH equation. On the other hand, if one solves Einstein's equations of motion for the metric, $g_{(2)}$ is completely determined in generic dimensions. The free coefficients in the PBH solutions are thus fixed on-shell given an action.

2.2 Effective boundary action and Weyl anomalies

Consider now an action

$$S = \int_M d^{d+1} X \sqrt{G} f(R(G)), \quad (2.17)$$

where f is a local function of the curvature and its covariant derivatives and we require that $f(R)$ is such that the equations of motion are solved by asymptotically AdS_{d+1} in order to have a CFT at the boundary. Under a bulk diffeomorphism, the action S is invariant up to a boundary term

$$\delta S = \int_M d^{d+1} X \partial_\mu (\xi^\mu \mathcal{L}), \quad \mathcal{L} = \sqrt{G} f(R(G)) \quad (2.18)$$

from which we read the transformation $\delta \mathcal{L} = \partial_\mu (\xi^\mu \mathcal{L})$. If the diffeomorphism is a PBH, then one shows that $[\delta_2, \delta_1] \mathcal{L} = 0$ upon using the PBH group property [19]. In FG coordinates, the metric expansion (2.10) induces a power series expansion for \mathcal{L} as well

$$\mathcal{L} = \sqrt{g_{(0)}} z^{-d/2-1} \mathcal{L}_g, \quad \mathcal{L}_g = \sum_{n=0}^{\infty} \mathcal{L}_g^{(n)}(x) z^n. \quad (2.19)$$

Then, by virtue of $[\delta_2, \delta_1] \mathcal{L} = 0$, it is possible to show that \mathcal{L}_g satisfies a Wess-Zumino condition

$$\int_{\partial M} d^d x \sqrt{g_{(0)}} (\sigma_2(x) \delta_1 \mathcal{L}_g - \sigma_1(x) \delta_2 \mathcal{L}_g) = 0, \quad (2.20)$$

which means that \mathcal{L}_g is a candidate for the anomaly of the boundary CFT. To make the connection precise, from (2.18) we have

$$\delta S = \int_{\partial M} d^d x \xi^z \mathcal{L}|_{z=0} = -2 \int_{\partial M} d^d x z \sigma \mathcal{L}|_{z=0} \quad (2.21)$$

restricting the diffeomorphism to a PBH. Using the holographic dictionary, we interpret the variation of the bulk action as the variation of the generating functional of the CFT correlators, so that the finite piece in (2.21) gives the holographic Weyl anomaly.⁴ Thus, $\mathcal{L}_g^{(n)}(x)$ is the trace anomaly of the $d = 2n$ dimensional CFT.

Consider now as an example the following action

$$2\kappa^2 f(R(G)) = \Lambda - R(G) + \gamma (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma})(G), \quad \Lambda = -d(d-1) - 2d(d-3)\gamma, \quad (2.22)$$

where γ is a dimensionless parameter and Λ is such that AdS_{d+1} with radius $\ell = 1$ is a solution of the equations of motion (for simplicity in the following we will take $2\kappa^2 = 16\pi G_N^{(d+1)} = 1$). By writing the action in FG coordinates we find (a prime denotes a derivative w.r.t. z)

$$\begin{aligned} \mathcal{L}_g = & d(1+4\gamma) + \frac{1}{2}z(1+4\gamma) [2(1-d)g^{ij}g'_{ij} - R(g)] + \frac{1}{2}z^2 [4(1+4\gamma)g^{ij}g''_{ij} \\ & - (3-4(d-5)\gamma)g^{ik}g^{jl}g'_{ij}g'_{kl} + (1+8\gamma)(g^{ij}g'_{ij})^2 + \gamma(R_{ijkl}R^{ijkl})(g) \\ & + 8\gamma g'_{ij}R^{ij}(g)] + \gamma z^3 [4g^{im}g^{jn}g^{kl}g'_{ik}g'_{mn}g'_{jl} - 4g^{ij}g^{kl}g^{mn}g'_{ij}g'_{km}g'_{ln} \\ & - 2g'_{ij}g'_{kl}R^{ikjl}(g) - 4g^{im}g^{jn}\nabla_j g'_{ik}\nabla^k g'_{mn} + 4g^{im}g^{jn}\nabla_k g'_{ij}\nabla^k g'_{mn}] \\ & + \gamma z^4 [g^{im}g^{jn}g^{kp}g^{lq}g'_{ik}g'_{mn}g'_{jl}g'_{pq} + (g^{ik}g^{jl}g'_{ij}g'_{kl})^2 + 8g^{ik}g^{jl}g''_{ij}g''_{kl} \\ & - 8g^{im}g^{jn}g^{kl}g'_{ik}g'_{mn}g''_{jl}] \end{aligned} \quad (2.23)$$

and expanding the metric according to (2.10), we identify for instance the terms

$$\mathcal{L}_g^{(1)} = -\frac{1}{2}(1+4\gamma)R^{(0)} + \frac{1}{2}(2-d)(1+4\gamma)g^{(1)i}_i \quad (2.24)$$

$$\begin{aligned} \mathcal{L}_g^{(2)} = & \frac{1}{2}(1+12\gamma)R^{(0)ij}g^{(1)}_{ij} - \frac{1}{4}(1+4\gamma)R^{(0)}g^{(1)i}_i + \frac{3}{2}(4-d)(1+4\gamma)g^{(2)i}_i \\ & + \left(\frac{1}{8}(8-3d) + \frac{3}{2}\gamma(4-d)\right)g^{(1)i}_i g^{(1)j}_j - \frac{1}{2}(1+4\gamma)\nabla_j^{(0)}\nabla_i^{(0)}g^{(1)ij} \\ & + \left(\frac{1}{4}(3d-10) + \gamma(5d-14)\right)g^{(1)}_{ij}g^{(1)ij} + \frac{1}{2}(1+4\gamma)\square^{(0)}g^{(1)i}_i + \frac{1}{2}\gamma R^{(0)}_{ijkl}R^{(0)ijkl} \end{aligned} \quad (2.25)$$

and on the PBH solutions (2.15), (2.16) we find the trace anomaly in $d = 2, 4$ respectively:

$$\mathcal{L}_g^{(1)} = -\frac{1}{2}(1+4\gamma)R^{(0)} = -\frac{1}{2}aE_2 \quad (2.26)$$

$$\mathcal{L}_g^{(2)} = -\frac{1}{8}(1+12\gamma)R^{(0)}_{ij}R^{(0)ij} + \frac{1}{24}(1+8\gamma)R^{(0)2} + \frac{1}{2}\gamma R^{(0)}_{ijkl}R^{(0)ijkl} = -\frac{1}{16}(cC^2 - aE_4) \quad (2.27)$$

⁴Divergent terms can be cancelled with the addition of counterterms to the bulk action [22].

with $c = 1 - 4\gamma$ and $a = 1 + 4\gamma$. E_{2n} is the Euler density in $d = 2n$, and explicitly $E_4 = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2$ while $C^2 = R_{ijkl}R^{ijkl} - 2R_{ij}R^{ij} + \frac{1}{3}R^2$ with curvature w.r.t. the boundary metric. Following the classification of [23], we notice that in $d = 2$ the anomaly is entirely type A, while in $d = 4$ there is also a type B. In particular, for $\gamma = 0$ we have $a = c$, while in the presence of the quadratic term in the curvature in (2.22) then $a - c \neq 0$. Since R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms in the action would also change the values of a and c , but not their difference, (2.22) is the minimal bulk action which allows to distinguish between type A and B anomalies in the pure gravity case. Having $a \neq c$ will be useful later when we include scalar fields in the bulk. As a final comment, notice that $g_{ij}^{(n)}$ does not contribute to $\mathcal{L}_g^{(n)}$ in $d = 2n$ [18, 19, 24].

3 Adding a massive scalar field

Now we add a massive scalar field Φ in the bulk and couple it to the metric. We want to extend the PBH transformations discussed in sec. 2 to describe this system.⁵

3.1 Modified PBH transformations

From the standard holographic dictionary it is known that a bulk scalar field of mass m is dual to a scalar operator on the boundary theory with dimension Δ , related to the mass by $m^2 = \Delta(\Delta - d)$. Close to the boundary, we consider the following expansion

$$\Phi(z, x) = z^{(d-\Delta)/2} \phi(z, x), \quad \phi(z, x) = \sum_{n=0}^{\infty} \phi_{(n)}(x) z^n \quad (3.1)$$

with $\phi_{(0)}$ being the source of the boundary operator.⁶ Requiring that the bulk scalar is indeed a scalar under diffeomorphisms,

$$\Phi'(z', x') = \Phi(z, x), \quad (3.2)$$

and choosing the diffeomorphism to be a PBH (2.4), we obtain the PBH transformation for the field ϕ as

$$\delta\phi = -2\sigma \left(\frac{d-\Delta}{2} + z\partial_z \right) \phi + a^i \partial_i \phi. \quad (3.3)$$

With the expansion in (3.1), we get

$$\delta\phi_{(n)} = -\sigma(d + 2n - \Delta) + \sum_{m=0}^{n-1} (a_{(n-m)}^i \partial_i \phi_{(m)}), \quad (3.4)$$

that to lowest order yields

$$\delta\phi_{(0)} = \sigma(\Delta - d) \phi_{(0)}, \quad (3.5)$$

⁵See also [25–27] for the PBH transformation of scalar fields.

⁶As for the metric expansion, there are logarithmic terms in the expansion of $\phi(z, x)$ for even integer dimension d . We assume we do not need to include them in the present discussion.

namely the correct transformation of the source of a dimension Δ operator under Weyl transformation. Eventually, we want to make contact with the four dimensional CFT analysed in [6]. Thus, we choose $\Delta = d + 1$, and therefore the scalar field is sourcing an irrelevant operator in the CFT.

When we couple the bulk scalar to gravity, the dynamical background will backreact and the metric in FG form will be as follows

$$ds^2 = \frac{dz^2}{4z^2} + \frac{1}{z} (g_{ij}(z, x) + h_{ij}(z, x)) dx^i dx^j, \quad (3.6)$$

where h_{ij} is the backreaction, which depends explicitly on $g_{ij}(z, x)$ and $\Phi(z, x)$. We are essentially allowing for perturbations of the metric g_{ij} due to the presence of the scalar field Φ , and the metric g_{ij} is then treated as a background, unperturbed, metric. To first non-trivial order, the backreaction is quadratic in the scalar field, and from now on we will work to order $\mathcal{O}(\sigma, \phi^2)$. We impose a boundary condition for the backreaction (following [4]), namely that the backreaction does not change the boundary metric. In other words, $g_{ij}^{(0)}$ is still the boundary metric even in the presence of the backreaction. We will see the effect of this boundary condition later.

We now derive the modifications of the PBH transformations due to the presence of the backreaction by studying the behaviour of (3.6) under diffeomorphisms. The FG form of the bulk metric in (3.6) is invariant under the transformation in (2.2) for

$$\xi^z = -2z\sigma(x), \quad \xi^i = a^i(z, x) + b^i(z, x) = \frac{1}{2} \partial_j \sigma(x) \int_{\epsilon}^z dz' (g^{ij}(z', x) - h^{ij}(z', x)), \quad (3.7)$$

where b^i contains the scalar field corrections brought about by the backreaction and is therefore of order $\mathcal{O}(\sigma, \phi^2)$, while a^i is still of order $\mathcal{O}(\sigma, \phi^0)$. Notice that now we are restricting the radial integration to the region $z \geq \epsilon > 0$. This is necessary to avoid divergences in the integration. Indeed, since $\Delta = d + 1$ and h_{ij} is quadratic in the scalar field, it follows that the backreaction goes as $1/z$ about the boundary, thus making the above integration divergent at $z = 0$ and requiring that we integrate over the region $z \geq \epsilon > 0$. This effect is reminiscent of [13, 16]: the scalar field is causing the boundary to move into the bulk. Finally, from form invariance of G_{ij} we find to $\mathcal{O}(\sigma, \phi^2)$

$$\begin{aligned} \delta g_{ij} + \delta h_{ij} &= 2\sigma(1 - z\partial_z)(g_{ij} + h_{ij}) \\ &+ \nabla_i a_j + \nabla_j a_i + \nabla_i b_j + \nabla_j b_i + h_{ik} \nabla_j a^k + h_{jk} \nabla_i a^k + a^k \nabla_k h_{ij}, \end{aligned} \quad (3.8)$$

where indices are lowered with (and derivatives are covariant w.r.t.) g_{ij} . We refer to eqs. (3.7), (3.8) as the modified PBH transformations.

Given the leading asymptotic behaviour of the metric and the backreaction, we make the following Ansätze for the radial expansion of a^i and b^i

$$a^i(z, x) = \bar{a}^i(\epsilon, x) + \sum_{n=1}^{\infty} z^n a_{(n)}^i(x) \quad (3.9)$$

$$b^i(z, x) = \bar{b}^i(\epsilon, x) + \log z \tilde{b}_{(1)}^i(x) + \sum_{n=2}^{\infty} \left[z^{n-1} \left(\log z \tilde{b}_{(n)}^i(x) + b_{(n)}^i(x) \right) \right], \quad (3.10)$$

where \bar{a}^i and \bar{b}^i are constant terms in z and their appearance is due to the lower end of the integration in (3.7). For the metric and backreaction we assume

$$g_{ij}(z, x) = \sum_{n=0}^{\infty} z^n g_{ij}^{(n)}(x) \quad (3.11)$$

$$h_{ij}(z, x) = \frac{1}{z} h_{ij}^{(0)}(x) + \log z \tilde{h}_{ij}^{(1)}(x) + 0 + \sum_{n=2}^{\infty} \left[z^{n-1} \left(\log z \tilde{h}_{ij}^{(n)}(x) + h_{ij}^{(n)}(x) \right) \right], \quad (3.12)$$

where we stress that there is no term at order z^0 . This implements the boundary condition that we anticipated above, namely that the boundary metric is still given by $g_{ij}^{(0)}$ even in the presence of the backreaction [4]. The appearance of logarithmic terms is a consequence of the particular choice for the dimension of the scalar field $\Delta = d + 1$, and we are thus generalising the analysis of [4] as advocated in [5].

Using the above expansions, from the modified PBH equations we find for the first few terms (the $a_{(n)}^i$ are as before)

$$\tilde{b}_{(1)}^i = -\frac{1}{2} g_{(0)}^{im} g_{(0)}^{jn} h_{mn}^{(0)} \partial_j \sigma \quad (3.13)$$

$$\tilde{b}_{(2)}^i = -\frac{1}{2} g_{(0)}^{im} g_{(0)}^{jn} \tilde{h}_{mn}^{(1)} \partial_j \sigma \quad (3.14)$$

$$b_{(2)}^i = \frac{1}{2} \left[(g_{(0)}^{im} g_{(1)}^{jn} + g_{(1)}^{im} g_{(0)}^{jn}) h_{mn}^{(0)} + g_{(0)}^{im} g_{(0)}^{jn} \tilde{h}_{mn}^{(1)} \right] \partial_j \sigma \quad (3.15)$$

$$\tilde{b}_{(3)}^i = \frac{1}{4} \left[(g_{(0)}^{im} g_{(1)}^{jn} + g_{(1)}^{im} g_{(0)}^{jn}) \tilde{h}_{mn}^{(1)} - g_{(0)}^{im} g_{(0)}^{jn} \tilde{h}_{mn}^{(2)} \right] \partial_j \sigma \quad (3.16)$$

$$b_{(3)}^i = -\frac{1}{4} \left[h_{(2)}^{ij} - \frac{1}{2} \tilde{h}_{(2)}^{ij} + \frac{1}{2} (g_{(1)}^{ik} \tilde{h}^{(1)j}_k + g_{(1)}^{jk} \tilde{h}^{(1)i}_k) + g_{(1)}^{ik} g_{(1)}^{jl} h_{kl}^{(0)} + g_{(1)}^{ik} g^{(1)l}_k h^{(0)j}_l \right. \\ \left. + g_{(1)}^{jk} g^{(1)l}_k h^{(0)i}_l - g_{(2)}^{ik} h^{(0)j}_k - g_{(2)}^{jk} h^{(0)i}_k \right] \partial_j \sigma, \quad (3.17)$$

where indices are lowered (raised) with the (inverse of) $g_{ij}^{(0)}$; curvatures and covariant derivatives will be w.r.t. $g_{ij}^{(0)}$. For the metric and backreaction variation⁷

$$\delta h_{ij}^{(0)} = 4\sigma h_{ij}^{(0)} \quad (3.18)$$

$$\delta \tilde{h}_{ij}^{(1)} = 2\sigma \tilde{h}_{ij}^{(1)} + \nabla_i^{(0)} \tilde{b}_{(1)j} + \nabla_j^{(0)} \tilde{b}_{(1)i} \quad (3.19)$$

$$\delta g_{ij}^{(0)} = 2\sigma g_{ij}^{(0)} - 2\sigma \tilde{h}_{ij}^{(1)} + h_{ik}^{(0)} \nabla_j^{(0)} a_{(1)}^k + h_{jk}^{(0)} \nabla_i^{(0)} a_{(1)}^k + a_{(1)}^k \nabla_k^{(0)} h_{ij}^{(0)} \quad (3.20)$$

⋮

⁷The expressions that follow are written up to boundary diffeomorphisms generated by \bar{a}^i and \bar{b}^i . Since their presence does not affect the solution of the PBH equations, we disregard them for simplicity of notation.

$$\begin{aligned}
\delta\tilde{h}_{ij}^{(n+1)} &= 2\sigma(1-n)\tilde{h}_{ij}^{(n+1)} \\
&+ \sum_{m=1}^{n+1} \left(g_{ik}^{(n-m+1)} \partial_j \tilde{b}_{(m)}^k + g_{jk}^{(n-m+1)} \partial_i \tilde{b}_{(m)}^k + \tilde{b}_{(m)}^k \partial_k g_{ij}^{(n-m+1)} \right. \\
&\left. + \tilde{h}_{ik}^{(n-m+1)} \partial_j a_{(m)}^k + \tilde{h}_{jk}^{(n-m+1)} \partial_i a_{(m)}^k + a_{(m)}^k \partial_k \tilde{h}_{ij}^{(n-m+1)} \right) \quad (3.21)
\end{aligned}$$

$$\begin{aligned}
\delta g_{ij}^{(n)} + \delta h_{ij}^{(n+1)} &= 2\sigma(1-n) \left(g_{ij}^{(n)} + h_{ij}^{(n+1)} \right) - 2\sigma \tilde{h}_{ij}^{(n+1)} \\
&+ \sum_{m=1}^{n+1} \left(g_{ik}^{(n-m+1)} \partial_j b_{(m)}^k + g_{jk}^{(n-m+1)} \partial_i b_{(m)}^k + b_{(m)}^k \partial_k g_{ij}^{(n-m+1)} \right. \\
&\left. + h_{ik}^{(n-m+1)} \partial_j a_{(m)}^k + h_{jk}^{(n-m+1)} \partial_i a_{(m)}^k + a_{(m)}^k \partial_k h_{ij}^{(n-m+1)} \right). \quad (3.22)
\end{aligned}$$

A few comments are in order here. The above equation for the metric and the backreaction can be solved in the same spirit outlined in sec. 2. The term $h_{(n)}$ (and similarly $\tilde{h}_{(n)}$) is quadratic in $\phi_{(0)}$ and contains $2n$ derivatives. Once the most general expression for $g_{(n)} + h_{(n+1)}$ (or $\tilde{h}_{(n)}$) is written down, it is enough to vary it according to (3.20) and (3.5) up to $\mathcal{O}(\sigma, \phi^2)$ and impose the variation is a PBH to find the sought for expression. The solution of the backreacted Einstein's equations of motion (as in [4] but with $\Delta = d + 1$) will also satisfy the above equations.

As in the pure gravity case, the modified PBH equation fixes the expression of the backreaction only to some extent. For instance, the first term in the expansion is

$$h_{ij}^{(0)} = h_0 g_{ij}^{(0)} \phi_{(0)}^2 \quad (3.23)$$

for some coefficient h_0 , not fixed by the PBH equation. The higher order terms in the radial expansion will have more and more free coefficients, that are fixed on-shell given an action.⁸

Notice that the backreaction modifies the usual Weyl transformation of the boundary metric in (3.20). However, unlike in the pure gravity case, as it stands the modified PBH transformation does not reduce to a Weyl transformation of $g_{ij}^{(0)}$. Indeed, when the diffeomorphism is a modified PBH transformation, from (2.7) and (3.7) we find that (up to $\mathcal{O}(\sigma, \phi^2)$)

$$\hat{\xi}^z = 0, \quad \hat{\xi}^i = \epsilon (g^{ij}(\epsilon, x) - h^{ij}(\epsilon, x)) (\xi_2^z \partial_j \xi_1^z - \xi_1^z \partial_j \xi_2^z), \quad (3.24)$$

so that now we are left with a residual diffeomorphism

$$[\delta_2, \delta_1] g_{ij}^{(0)} = g_{jk}^{(0)} \nabla_i^{(0)} \hat{\xi}^k + g_{ik}^{(0)} \nabla_j^{(0)} \hat{\xi}^k. \quad (3.25)$$

Before solving the modified PBH equation for the backreaction, we thus have to address the issue of the $z = \epsilon$ cutoff.

The holographic dual of the gravitational theory discussed so far is a CFT deformed by an irrelevant operator. However, since we want to make contact with the

⁸For instance, given the action of a free massive scalar field coupled to a dynamical metric, then on-shell h_0 is proportional to the coefficient of the lowest order term in the radial expansion of the scalar field action and it is thus non-vanishing on-shell.

unperturbed CFT presented in [6], we have to move the cutoff surface back to the AdS boundary. Given that

$$\bar{b}^i(\epsilon, x) = \frac{1}{2}h_0 \log \epsilon g_{(0)}^{ij} \phi_{(0)}^2 \partial_j \sigma + \mathcal{O}(\epsilon), \quad (3.26)$$

the way to move the cutoff back to the boundary, without setting the source to zero, is to take $h_0 = 0$ and then $\epsilon = 0$. In this limit we can describe a boundary CFT in the presence of an irrelevant operator, avoiding the prescription $\phi_{(0)}^2 = 0$ advocated in [2, 3]. As a bonus, the commutator in (3.25) vanishes and the modified PBH transformation reduces to a Weyl transformation at the boundary. However, the price to pay is that the solutions of the modified PBH equations are not on-shell anymore (see footnote 8).

Now, we proceed by solving the modified PBH with $h_0 = 0$. For $\tilde{h}_{(1)}$ we make the Ansatz

$$\begin{aligned} \tilde{h}_{ij}^{(1)} = & h_1 R_{ij} \phi_{(0)}^2 + h_2 g_{ij}^{(0)} R \phi_{(0)}^2 + h_3 \nabla_i \phi_{(0)} \nabla_j \phi_{(0)} + h_4 \phi_{(0)} \nabla_i \nabla_j \phi_{(0)} \\ & + h_5 g_{ij}^{(0)} \phi_{(0)} \square \phi_{(0)} + h_6 g_{ij}^{(0)} (\nabla \phi_{(0)})^2 \end{aligned} \quad (3.27)$$

and a solution of (3.19) is given by

$$\begin{aligned} \tilde{h}_{(1)ij} = & h_1 \left(R_{ij} \phi_{(0)}^2 + (d-2) \phi_{(0)} \nabla_i \nabla_j \phi_{(0)} + g_{ij}^{(0)} \phi_{(0)} \square \phi_{(0)} - (d-1) g_{ij}^{(0)} (\nabla \phi_{(0)})^2 \right) \\ & + h_2 g_{ij}^{(0)} \left(R \phi_{(0)}^2 + 2(d-1) \phi_{(0)} \square \phi_{(0)} - d(d-1) (\nabla \phi_{(0)})^2 \right). \end{aligned} \quad (3.28)$$

Notice that the solutions parametrised by h_1 and h_2 are proportional to \hat{R}_{ij} and \hat{R} respectively, which are the curvature tensors computed from the Weyl invariant metric $g_{ij}^{(0)}/\phi_{(0)}^2$. Following [6], we will take $h_2 = 0$ for simplicity. Then, the variation of the metric in (3.20) reads

$$\delta g_{ij}^{(0)} = 2\sigma g_{ij}^{(0)} + \sigma \beta_{ij} \quad (3.29)$$

with

$$\beta_{ij} = -2h_1 \left(R_{ij} \phi_{(0)}^2 + (d-2) \phi_{(0)} \nabla_i \nabla_j \phi_{(0)} + g_{ij}^{(0)} \phi_{(0)} \square \phi_{(0)} - (d-1) g_{ij}^{(0)} (\nabla \phi_{(0)})^2 \right) \quad (3.30)$$

and we interpret it as a modification of the usual Weyl transformation of the boundary metric due to the presence of the scalar field. The holographic beta-function that we find is in agreement with [6].

Similarly, we also solve (3.21) for $\tilde{h}_{(2)}$ and (3.22) for $g_{(1)} + h_{(2)}$. The Ansatz for $h_{(2)}$ has thirty-five terms and the modified PBH equation leaves six out of the thirty-five coefficients free, while $\tilde{h}_{(2)}$ is determined in terms of h_1 . We also solve for the trace of $\tilde{h}_{(3)}$ and the trace of $h_{(3)}$ that we will need in the following. The Ansatz for the trace of $h_{(3)}$ has sixty-six terms and the modified PBH equation leaves nine coefficients free, while the trace of $\tilde{h}_{(3)}$ is determined in terms of h_1, c_1, c_2 . $g_{(1)}$ and $g_{(2)}$ are not modified by the presence of the backreaction and are still given by (2.15) and (2.16) respectively. We provide more details in app. A and in the ancillary Mathematica file.

3.2 Effective boundary action and modified Weyl anomalies

To derive the holographic dual of the modified Weyl anomaly found in [6], we extend the method outlined in sec. 2.2 to include the effect of the backreaction.

Consider an action

$$S = \int_M d^{d+1}X \sqrt{G} f(R(G), \Phi), \quad (3.31)$$

where f is a local function of the curvature and its covariant derivatives and contains also matter field Φ . We could think of S as gravitational action with a scalar field coupled to the metric and we require that $f(R(G), \Phi = 0)$ is such that AdS_{d+1} is a solution of the equations of motion. Then, we write the action in FG coordinates with backreaction (3.6) and, expanding in powers of z , we obtain the holographic anomalies by evaluating the corresponding expressions on the solutions of the modified PBH equations. However, the off-shell solution that we discussed so far with $h_0 = 0$ sets the scalar field action to zero (see footnote 8), so that eventually the scalar field contributions to the anomalies is only due to the backreaction.

Defining now $\mathcal{L} = \sqrt{G} f(R(G), \Phi)$, in the FG coordinates (3.6) we write

$$\mathcal{L} = \sqrt{g^{(0)}} z^{-d/2-1} (\mathcal{L}_g + \mathcal{L}_h) \quad (3.32)$$

$$\mathcal{L}_g = \sum_{n=0}^{\infty} \mathcal{L}_g^{(n)}(x) z^n, \quad \mathcal{L}_h = \sum_{n=0}^{\infty} \left[\log z \tilde{\mathcal{L}}_h^{(n)} + \mathcal{L}_h^{(n)} \right], \quad (3.33)$$

where \mathcal{L}_h contains the backreaction and it is thus quadratic in the scalar field; in the second line we use the expansions (3.12) with $h_0 = 0$ and \mathcal{L}_g is as in sec. 2.2. Following the reasoning of sec. 2.2, we can show that $[\delta_2, \delta_1]\mathcal{L} = 0$ upon using the PBH group property for $h_0 = 0$ and thus $\mathcal{L}_g + \mathcal{L}_h$ satisfies the following Wess-Zumino condition

$$\delta_1 \int_{\partial M} d^d x \sqrt{g^{(0)}} \sigma_2(x) (\mathcal{L}_g + \mathcal{L}_h) - \delta_2 \int_{\partial M} d^d x \sqrt{g^{(0)}} \sigma_1(x) (\mathcal{L}_g + \mathcal{L}_h) = 0, \quad (3.34)$$

where $g_{ij}^{(0)}$ now transforms with the beta-function (3.29). From (2.21) we interpret $\mathcal{L}_g^{(n)} + \mathcal{L}_h^{(n)}$ as the trace anomaly of the $d = 2n$ dimensional CFT at the boundary.⁹

Considering again the action (2.22), we now write it in the FG coordinates with backreaction (3.6) and find the expression for \mathcal{L}_h that we report in (B.1) in app. B; \mathcal{L}_g is still given by (2.23). Expanding the metric and backreaction in (3.32) according

⁹As in sec. 2.2, we neglect divergent terms that are cancelled by counterterms. In [2] the counterterms considered only cancel negative powers of the radial coordinate; with irrelevant operators there is also need of logarithmic counterterms, as considered (in flat space) in [5].

to (3.12) with $h_0 = 0$, we identify for instance

$$\begin{aligned}\mathcal{L}_h^{(1)} &= 4\gamma R^{ij} \tilde{h}_{ij}^{(1)} + \frac{1}{2}(2-d)(1+4\gamma)h^{(2)i}{}_i - (2-d)(1+8\gamma)g^{(1)ij} \tilde{h}^{(1)}{}_{ij} \\ &\quad + \frac{1}{2}(1+12\gamma-d(1+4\gamma))g^{(1)i}{}_i \tilde{h}^{(1)j}{}_j + (3-d)(1+4\gamma)\tilde{h}^{(2)i}{}_i\end{aligned}\quad (3.35)$$

$$\begin{aligned}\mathcal{L}_h^{(2)} &= -\frac{1}{4}(1+\gamma)R h^{(2)i}{}_i + 2(1+6\gamma - \frac{3}{8}d(1+4\gamma))g^{(1)j}{}_j h^{(2)i}{}_i + \frac{1}{2}(1+12\gamma)R_{ij} h^{(2)ij} \\ &\quad - (5+28\gamma - \frac{1}{2}d(3+20\gamma))g^{(1)}{}_{ij} h^{(2)ij} - \frac{3}{2}(d-4)(1+4\gamma)h^{(3)i}{}_i \\ &\quad - (5+68\gamma-d(1+12\gamma))g^{(2)ij} \tilde{h}^{(1)}{}_{ij} - 8\gamma R^{ij} g^{(1)}{}_i{}^k \tilde{h}^{(1)}{}_{jk} + 8\gamma \nabla_k \tilde{h}^{(1)}{}_{ij} \nabla^k g^{(1)ij} \\ &\quad + (5+56\gamma-d(1+12\gamma))g^{(1)}{}_i{}^k g^{(1)ij} \tilde{h}^{(1)}{}_{jk} - 8\gamma \nabla_j \tilde{h}^{(1)}{}_{ik} \nabla^k g^{(1)ij} \\ &\quad - \frac{1}{2}(4+48\gamma-d(1+8\gamma))g^{(1)i}{}_i g^{(1)jk} \tilde{h}^{(1)}{}_{jk} + \frac{1}{2}(3+28\gamma-d(1+4\gamma))g^{(2)i}{}_i \tilde{h}^{(1)j}{}_j \\ &\quad - \frac{1}{4}(3+44\gamma-d(1+4\gamma))g^{(1)}{}_{ij} g^{(1)ij} \tilde{h}^{(1)k}{}_k + \frac{1}{2}(1+4\gamma)\square h^{(2)i}{}_i \\ &\quad + \frac{1}{8}(3+28\gamma-d(1+4\gamma))g^{(1)i}{}_i g^{(1)j}{}_j \tilde{h}^{(1)k}{}_k - 4\gamma R_{ikjl} g^{(1)ij} \tilde{h}^{(1)kl} + 4\gamma R^{ij} \tilde{h}^{(2)ij} \\ &\quad - (6+32\gamma+d(1+8\gamma))g^{(1)ij} \tilde{h}^{(2)ij} + \frac{1}{2}(5+28\gamma-d(1+4\gamma))g^{(1)i}{}_i \tilde{h}^{(2)j}{}_j \\ &\quad - 2\gamma \tilde{h}^{(1)ij} \nabla_j \nabla_i g^{(1)k}{}_k + 4\gamma \tilde{h}^{(1)ij} \nabla_k \nabla_j g^{(1)}{}_i{}^k - \frac{1}{2}(1+4\gamma) \nabla_j \nabla_i h^{(2)ij} \\ &\quad - (d-7)(1+4\gamma)\tilde{h}^{(3)i}{}_i + 2\gamma R^{ij} g^{(1)k}{}_k \tilde{h}^{(1)}{}_{ij} - 2\gamma \tilde{h}^{(1)ij} \square g^{(1)}{}_{ij}\end{aligned}\quad (3.36)$$

which should contribute to the anomaly in $d = 2, 4$ respectively. Focusing on the $d = 4$ case, the candidate anomaly is $\mathcal{L}_g^{(2)} + \mathcal{L}_h^{(2)}$, which on the PBH solutions yields indeed a solution of the Wess-Zumino condition. Notice that $g^{(n)}$ appears in $\mathcal{L}_h^{(n)}$ and together with $\tilde{h}^{(3)i}{}_i$ it causes factors of $(d-4)^{-1}$ to appear in $\mathcal{L}_h^{(2)}$. Thus, at first sight $\mathcal{L}_g^{(2)} + \mathcal{L}_h^{(2)}$ is singular in $d = 4$. Nonetheless, it is possible to renormalise the free coefficients of the PBH solution for the backreaction (see app. C) so that eventually $\mathcal{L}_g^{(2)} + \mathcal{L}_h^{(2)}$ is regular in $d = 4$ and can be identified with the holographic anomaly of the boundary CFT. We thus define $\mathcal{A}^{4d} = (\mathcal{L}_g^{(2)} + \mathcal{L}_h^{(2)})|_{\text{reg}}$ as the regularised, four dimensional holographic anomaly. The term quadratic in the curvature in (2.22) allows us to separate again the pure gravity type A and B anomalies, which now receive contributions also from the scalar field. We find the following expression for the pure gravity type B anomaly and the scalar field contributions ($g_{ij} \equiv g_{ij}^{(0)}$ and $\phi \equiv \phi_{(0)}$):

$$\begin{aligned}\mathcal{A}^{4d}|_B &= -\frac{1}{16}cC^2 + ch_1 \left(\frac{3}{4} \nabla^j \nabla^i \phi \square \nabla_j \nabla_i \phi - \frac{3}{16} \square \phi \phi^2 + \frac{1}{2} \nabla_k \nabla_j \nabla_i \phi \nabla^k \nabla^j \nabla^i \phi \right. \\ &\quad - \frac{89}{144} R_i{}^k R^{ij} R_{jk} \phi^2 + \frac{35}{48} R_{ij} R^{ij} R \phi^2 - \frac{31}{324} R^3 \phi^2 - \frac{11}{16} R^{ij} R^{kl} R_{ikjl} \phi^2 \\ &\quad \left. - \frac{47}{1152} R R_{ijkl} R^{ijkl} \phi^2 - \frac{1}{18} R_i{}^m{}^n R^{ijkl} R_{jmln} \phi^2 + \frac{53}{576} R_{ij}{}^{mn} R^{ijkl} R_{klmn} \phi^2 \right)\end{aligned}$$

$$\begin{aligned}
& -\frac{7}{192}R\phi^2\Box R - \frac{5}{576}R^2\phi\Box\phi - \frac{19}{288}\phi^2\nabla_i R\nabla^i R - \frac{65}{288}R\phi\nabla_i\phi\nabla^i R \\
& + R^{jk}\phi\nabla_i R_{jk}\nabla^i\phi + \frac{1}{16}R^{jklm}\phi\nabla_i R_{jklm}\nabla^i\phi + \frac{1}{8}R_{jk}R^{jk}\nabla_i\phi\nabla^i\phi \\
& - \frac{1}{36}R^{ij}\phi^2\nabla_j\nabla_i R - \frac{11}{72}R^{ij}R\phi\nabla_j\nabla_i\phi - \frac{35}{288}\phi\Box R\Box\phi - \frac{13}{48}\nabla_i\phi\nabla^i R\Box\phi \\
& - \frac{1}{12}\phi\nabla^i R\Box\nabla_i\phi - \frac{1}{6}R\nabla^i\phi\Box\nabla_i\phi - \frac{3}{32}R\phi\Box^2\phi + \frac{1}{3}R_{ij}\phi\nabla^i R\nabla^j\phi \\
& + \frac{1}{12}R_{ij}R\nabla^i\phi\nabla^j\phi - \frac{1}{2}R^{kl}R_{ikjl}\nabla^i\phi\nabla^j\phi + \frac{1}{12}\nabla^i R\nabla_j\nabla_i\phi\nabla^j\phi \\
& - \frac{1}{18}\phi\nabla_j\nabla_i\phi\nabla^j\nabla^i R - \frac{3}{16}R\nabla_j\nabla_i\phi\nabla^j\nabla^i\phi - \frac{1}{2}R^{jk}\phi\nabla^i\phi\nabla_k R_{ij} \\
& - \frac{1}{8}R_i^k R^{ij}\phi\nabla_k\nabla_j\phi + \frac{1}{2}R^{jk}\nabla^i\phi\nabla_k\nabla_j\nabla_i\phi + \frac{1}{8}R^{ij}\phi^2\Box R_{ij} + \frac{13}{24}\phi\nabla^j\nabla^i\phi\Box R_{ij} \\
& + \frac{1}{6}R_{ij}R^{ij}\phi\Box\phi + \frac{1}{2}R_i^j\nabla^i\phi\Box\nabla_j\phi + \frac{3}{8}R^{ij}\phi\Box\nabla_j\nabla_i\phi - \frac{9}{16}\phi^2\nabla_j R_{ik}\nabla^k R^{ij} \\
& + \frac{61}{96}\phi^2\nabla_k R_{ij}\nabla^k R^{ij} + \frac{1}{2}\phi\nabla_k\nabla_j\nabla_i\phi\nabla^k R^{ij} + \frac{3}{8}R^{ij}\nabla_k\nabla_j\phi\nabla^k\nabla_i\phi \\
& + \frac{5}{4}\nabla_i R_{jk}\nabla^i\phi\nabla^k\nabla^j\phi - \frac{1}{2}\nabla^i\phi\nabla_k R_{ij}\nabla^k\nabla^j\phi + \frac{1}{2}R_{ijkl}\phi\nabla^i\phi\nabla^l R^{jk} \\
& + \frac{7}{24}R_{ikjl}\phi^2\nabla^l\nabla^k R^{ij} + \frac{5}{24}R^{ij}R_{ikjl}\phi\nabla^l\nabla^k\phi + \frac{3}{8}R_{ikjl}\nabla^j\nabla^i\phi\nabla^l\nabla^k\phi \\
& + \frac{5}{192}\phi^2\nabla_m R_{ijkl}\nabla^m R^{ijkl} - \frac{1}{36}R^2\nabla_i\phi\nabla^i\phi \Big), \tag{3.37}
\end{aligned}$$

while the pure gravity type A anomaly and the scalar field contributions read (again $g_{ij} \equiv g_{ij}^{(0)}$ and $\phi \equiv \phi_{(0)}$):

$$\begin{aligned}
\mathcal{A}^{4d}|_A = & \frac{1}{16}a E_4 + a h_1 \left(\frac{13}{32}\nabla^j\Box\phi\Box\nabla_j\phi - \frac{1}{8}\Box\phi\Box^2\phi - \frac{1}{4}\nabla^i\phi\Box^2\nabla_i\phi + \frac{7}{32}\phi\Box^3\phi \right. \\
& - \frac{1}{16}\nabla_k\nabla_j\nabla_i\phi\nabla^k\nabla^j\nabla^i\phi - \frac{1}{2}R_i^m{}_k{}^n R^{ijkl}R_{jmln}\phi^2 + \frac{5}{32}R_{ij}{}^{mn}R^{ijkl}R_{klmn}\phi^2 \\
& - \frac{1}{16}R^{jk}\phi\nabla_i R_{jk}\nabla^i\phi - \frac{83}{128}R_{jk}R^{jk}\nabla_i\phi\nabla^i\phi + \frac{263}{768}R^2\nabla_i\phi\nabla^i\phi - \frac{57}{128}\phi\Box R\Box\phi \\
& + \frac{65}{256}R_{jklm}R^{jklm}\nabla_i\phi\nabla^i\phi - \frac{89}{384}R^{ij}\phi^2\nabla_j\nabla_i R - \frac{65}{96}R^{ij}R\phi\nabla_j\nabla_i\phi \\
& + \frac{1}{3}\nabla_i\phi\nabla^i R\Box\phi + \frac{1}{4}\phi\nabla^i\phi\Box\nabla_i R + \frac{7}{32}\phi\nabla^i R\Box\nabla_i\phi - \frac{41}{32}R\nabla^i\phi\Box\nabla_i\phi \\
& + \frac{7}{192}\phi^2\Box^2 R + \frac{35}{192}R\phi\Box^2\phi - \frac{79}{192}R_{ij}\phi\nabla^i R\nabla^j\phi + \frac{61}{32}R_i^k R_{jk}\nabla^i\phi\nabla^j\phi \\
& - \frac{67}{48}R_{ij}R\nabla^i\phi\nabla^j\phi - \frac{3}{16}R^{kl}R_{ikjl}\nabla^i\phi\nabla^j\phi - \frac{1}{6}\nabla^i\phi\nabla_j\nabla_i R\nabla^j\phi \\
& - \frac{13}{24}\nabla^i R\nabla_j\nabla_i\phi\nabla^j\phi - \frac{37}{192}\phi\nabla_j\nabla_i\phi\nabla^j\nabla^i R - \frac{53}{64}R\nabla_j\nabla_i\phi\nabla^j\nabla^i\phi \\
& + \frac{97}{32}R^{jk}\phi\nabla^i\phi\nabla_k R_{ij} - \frac{15}{32}R_i^k R^{ij}\phi\nabla_k\nabla_j\phi + \frac{75}{16}R^{jk}\nabla^i\phi\nabla_k\nabla_j\nabla_i\phi \\
& + \frac{7}{64}R^{ij}\phi^2\Box R_{ij} + \frac{177}{64}\phi\nabla^j\nabla^i\phi\Box R_{ij} + \frac{73}{128}R_{ij}R^{ij}\phi\Box\phi + \frac{169}{32}R^{ij}\nabla_j\nabla_i\phi\Box\phi
\end{aligned}$$

$$\begin{aligned}
& -\frac{51}{32}R_i^j\nabla^i\phi\Box\nabla_j\phi + \frac{5}{8}R^{ij}\phi\Box\nabla_j\nabla_i\phi + \frac{27}{16}\phi\nabla_k\nabla_j\nabla_i\phi\nabla^kR^{ij} \\
& -\frac{99}{32}R^{ij}\nabla_k\nabla_j\phi\nabla^k\nabla_i\phi - \frac{11}{4}\nabla_iR_{jk}\nabla^i\phi\nabla^k\nabla^j\phi + \frac{9}{4}\nabla^i\phi\nabla_kR_{ij}\nabla^k\nabla^j\phi \\
& + \frac{11}{16}R_{ijkl}\phi\nabla^i\phi\nabla^lR^{jk} + \frac{115}{64}R_{ikjl}\phi^2\nabla^l\nabla^kR^{ij} + \frac{33}{16}R^{ij}R_{ikjl}\phi\nabla^l\nabla^k\phi \\
& - \frac{167}{32}R_{ikjl}\nabla^j\nabla^i\phi\nabla^l\nabla^k\phi + \frac{27}{128}\phi^2\nabla_mR_{ijkl}\nabla^mR^{ijkl} - \frac{61}{96}R\Box\phi\Box\phi \Big), \quad (3.38)
\end{aligned}$$

where we chose the free coefficients of the backreaction appearing in the anomaly, but h_1 , in such a way as to simplify the results; the following comments are unaffected by this choice.

The expressions (3.37) and (3.38) satisfy the Wess-Zumino condition (3.34). At first sight, they look different from the one obtained in [6]. However, we checked that the expressions are the same,¹⁰ up to variation of local counterterms in the field theory and addition of ‘ordinary’ anomalies, i.e. tensorial structures quadratic in the scalar field which lead to Weyl invariant integrands to order $\mathcal{O}(\sigma, \phi^2)$ and are therefore solutions of (3.34). These terms are present in (3.37) and (3.38) and the bulk action (2.22) does not have enough parameters to distinguish them from the modifications of the pure gravity anomalies which are solutions of (3.34) only when the beta-function is taken into account. To do so, it might be necessary to include additional higher derivative terms in the bulk action, which would complicate the computation of the anomalies and for simplicity we didn’t consider.

Notice the interplay between the a , c coefficients and the metric beta-function coefficient h_1 , as already observed in [6]. It is this feature that makes the anomalies (3.37) and (3.38) solutions of the consistency condition (3.34) in the presence of the metric beta-function. The other free coefficients of the backreaction, which appear in general in the anomaly through (3.36), do not show any interplay with the a and c coefficients; the terms that these coefficients parametrise correspond to cohomologically trivial solutions of the Wess-Zumino consistency condition in the field theory.

As a final comment, notice that in flat boundary space and for constant σ , the anomaly reduces to (after integration by parts)

$$\int d^4x \sqrt{g_{(0)}} \sigma(x) \mathcal{A}^{4d} \rightarrow \frac{1}{16} c h_1 \sigma \int d^4x \phi_{(0)} \Box^3 \phi_{(0)}, \quad (3.39)$$

i.e. the type A anomaly vanishes in flat space and the result is proportional to h_1 . We checked that including the solution parametrised by h_2 in (3.28) in the metric beta-function does not change this result. The flat space limit (3.39) is in agreement with [6].

4 Conclusions

In this paper we computed the holographic Weyl anomaly in the presence of a source for irrelevant operators of the boundary theory, when the boundary theory is a four

¹⁰Up to a factor of 2, which is missing in the normalisation of the metric beta-function in [6].

dimensional CFT. In the bulk, the massive scalar field induces a backreaction onto the metric, and, since the scalar field sources irrelevant operator on the boundary, it changes the leading asymptotic behaviour of the metric which becomes more singular as the boundary is approached. This causes logarithmic terms to appear in the radial expansion of the backreaction which affect the Weyl transformation of the boundary metric. In particular, a metric beta-function appears, which in turn modifies the holographic Weyl anomaly. The metric beta-function and the holographic Weyl anomaly are in agreement with the known CFT results, so that the present analysis provides an additional test of the AdS/CFT correspondence.

To obtain the holographic results, we studied diffeomorphisms in the bulk and used the equivalence between bulk diffeomorphisms and Weyl transformation at the boundary. We saw that the presence of a scalar field, which sources an irrelevant operator at the boundary, introduces a radial cutoff, which reflects the fact that the boundary field theory is being deformed by the irrelevant operator. Eliminating the cutoff, we were able to describe the underformed CFT and the modified PBH transformations reduce to a boundary Weyl transformation. However, this requires that the backreaction is put off-shell. It would be interesting to extend the present analysis by considering an on-shell backreaction, so that the dual theory is a CFT deformed by irrelevant operators. The metric beta-function would then indicate that the geometry is subject to an RG flow, and it would be interesting to provide its holographic description.

The holographic anomaly is obtained as the finite piece of a bulk action, evaluated on the PBH solutions. The scalar field bulk action does not contribute to the holographic anomaly, which is obtained by the gravitational action alone evaluated on the PBH solution for the metric including the backreaction due to the scalar field. In the resulting expression, in addition to the usual pure gravity Weyl anomaly there are contributions which depend on the scalar field and we calculated them to lowest non-trivial order, which is quadratic in $\phi_{(0)}$. Some of them correspond to anomalous terms seeded by the metric beta-function, while others correspond to solutions of the Wess-Zumino consistency condition to quadratic order in the scalar field but not seeded by the backreaction. The holographic anomaly is non-vanishing in flat space, where it reduces to the expected result, which in d boundary dimensions for a scalar operator of dimension Δ with source $\phi_{(0)}$ is proportional to $\phi_{(0)}\square^n\phi_{(0)}$ with integer $n = \Delta - d/2$ [5].

The present analysis could be generalised to higher dimensional theories, where the calculations will be more complicated but similar to the one we presented here. However, it does not provide the correct dual description of a two dimensional boundary CFT. Indeed, for $d = 2$ and $\Delta = 3$ the candidate holographic anomaly should be given by $\mathcal{L}_g^{(1)} + \mathcal{L}_h^{(1)}$ with $\mathcal{L}_h^{(1)}$ given by (3.35). On the PBH solutions it yields an expression which satisfies the Wess-Zumino condition (3.34), but it vanishes in flat space and therefore does not reproduce the expected result which should be proportional to $\phi_{(0)}\square^2\phi_{(0)}$. This happens because for $d = 2$ the solutions parametrised by h_1 and h_2 in (3.28) are no longer independent. The breakdown of the present description in $d = 2$ is expected, as the CFT analysis in two dimensions [6] requires the addition of a $\Delta = 4$ operator, which is eventually identified with the $T\bar{T}$ operator, together with an operator of dimension $\Delta = 3$. We checked that

including an additional scalar field in the bulk which sources a $\Delta = 4$ operator on the boundary and then demanding that the boundary CFT is not deformed by the irrelevant operators still does not reproduce the $d = 2$ metric beta-function found in [6]. This may be because the standard holographic dictionary tells that the $T\bar{T}$ operator is dual to a sharp cut-off in the bulk. However, we saw that requiring that the dual CFT is undeformed translated into eliminating the cut-off induced by the irrelevant operator. It would thus be interesting to extend the present analysis to the two dimensional case and see how the CFT results are reproduced by holographic computations.

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A Ansätze

In this appendix we collect the Ansätze that we make to solve eqs. (3.21), (3.22) for $n = 1, 2$. We report the explicit results in the ancillary Mathematica notebook.¹¹ In the following, we assume that $g_{ij} \equiv g_{ij}^{(0)}$ and curvatures and covariant derivatives are w.r.t. $g_{ij}^{(0)}$, and $\phi \equiv \phi_{(0)}$. We also choose $h_0 = 0$, as explained in the text.

The solution of the PBH equation for $g_{ij}^{(1)}$ is given in (2.15) and it is not modified by the presence of the backreaction. The most generic Ansatz for $h_{ij}^{(2)}$ has thirty-five terms and six coefficients will not be fixed by the modified PBH equation. We write the Ansatz as

$$\begin{aligned}
h_{ij}^{(2)} = & + h_{21}g_{ij}R_{kl}R^{kl}\phi^2 + h_{22}R_{ij}R\phi^2 + h_{23}R^{kl}R_{ikjl}\phi^2 + h_{24}\phi^2\nabla_j\nabla_iR + h_{25}R_{ij}\phi\Box\phi \\
& + h_{26}g_{ij}R\phi\Box\phi + \beta_1R_i^kR_{jk}\phi^2 + \beta_2g_{ij}R^2\phi^2 + \beta_3R_i^{klm}R_{jklm}\phi^2 \\
& + \beta_4g_{ij}R_{klmn}R^{klmn}\phi^2 + \beta_6R\nabla_i\phi\nabla_j\phi + \beta_5(\phi\nabla_i\phi\nabla_jR + \phi\nabla_iR\nabla_j\phi) \\
& + \beta_7R\phi\nabla_j\nabla_i\phi + \beta_8(R_j^k\phi\nabla_k\nabla_i\phi + R_i^k\phi\nabla_k\nabla_j\phi) + \beta_9\phi^2\Box R_{ij} + \beta_{10}g_{ij}\phi^2\Box R \\
& + \beta_{11}\nabla_j\nabla_i\phi\Box\phi + \beta_{12}(\nabla_j\phi\Box\nabla_i\phi + \nabla_i\phi\Box\nabla_j\phi) + \beta_{13}\phi\Box\nabla_j\nabla_i\phi \\
& + \beta_{14}g_{ij}\phi\nabla_k\phi\nabla^kR + \beta_{15}(\phi\nabla_iR_{jk}\nabla^k\phi + \phi\nabla_jR_{ik}\nabla^k\phi) \\
& + \beta_{16}(R_{jk}\nabla_i\phi\nabla^k\phi + R_{ik}\nabla_j\phi\nabla^k\phi) + \beta_{17}\phi\nabla_kR_{ij}\nabla^k\phi + \beta_{18}R_{ij}\nabla_k\phi\nabla^k\phi \\
& + \beta_{19}g_{ij}R\nabla_k\phi\nabla^k\phi + \beta_{20}\nabla_k\nabla_j\nabla_i\phi\nabla^k\phi + \beta_{21}\nabla_k\nabla_j\phi\nabla^k\nabla_i\phi \\
& + \beta_{22}g_{ij}R^{kl}\phi\nabla_l\nabla_k\phi + \beta_{23}g_{ij}\Box\phi\Box\phi + \beta_{24}g_{ij}\nabla^k\phi\Box\nabla_k\phi + \beta_{25}g_{ij}\phi\Box^2\phi \\
& + \beta_{26}g_{ij}R_{kl}\nabla^k\phi\nabla^l\phi + \beta_{27}R_{ikjl}\nabla^k\phi\nabla^l\phi + \beta_{28}R_{ikjl}\phi\nabla^l\nabla^k\phi
\end{aligned}$$

¹¹The results were obtained with the help of the xAct collection of Mathematica packages [28–32].

$$+ \beta_{29} g_{ij} \nabla_l \nabla_k \phi \nabla^l \nabla^k \phi \quad (\text{A.1})$$

and leave the coefficients h_{21}, \dots, h_{26} free. The full solution is in the ancillary notebook.

A similar Ansatz is made for $\tilde{h}_{ij}^{(2)}$ and its coefficients are fixed in terms of h_1 by the modified PBH equation.

The solution for $g_{ij}^{(2)}$ is given in (2.16) and it is not modified by the presence of the backreaction. Thus, here we focus on $h_{ij}^{(3)}$, which appears in eq. (3.36) only as $h^{(3)i}_i$ and we then do not compute $h_{ij}^{(3)}$ but rather its trace that satisfies the following PBH equation

$$\delta g^{(2)i}_i + \delta h^{(3)i}_i = -2\sigma g^{(2)i}_i + g^{(0)ij} \delta g^{(2)}_{ij} - \beta^{ij} g_{ij}^{(2)} - 2\sigma h^{(3)i}_i + g^{(0)ij} \delta h_{ij}^{(3)}. \quad (\text{A.2})$$

The most generic Ansatz for $h^{(3)i}_i$ has sixty-six terms and nine coefficients will not be fixed by the modified PBH equation. We write the Ansatz as

$$\begin{aligned} h^{(3)i}_i = & h_{31} R^{jk} \phi \nabla_i R_{jk} \nabla^i \phi + h_{32} R^{ij} \phi^2 \nabla_j \nabla_i R + h_{33} R_{ij} R^{ij} R \phi^2 + h_{34} R^{ij} R \phi \nabla_j \nabla_i \phi \\ & + h_{35} R_{ij} \phi \nabla^i R \nabla^j \phi + h_{36} R^{ij} R^{kl} R_{ikjl} \phi^2 + h_{37} R^{ij} \phi^2 \square R_{ij} + h_{38} R_{ij} R^{ij} \phi \square \phi \\ & + h_{39} R^{ij} \nabla_j \nabla_i \phi \square \phi + \gamma_1 R_i^k R^{ij} R_{jk} \phi^2 + \gamma_2 R^3 \phi^2 + \gamma_3 R R_{ijkl} R^{ijkl} \phi^2 \\ & + \gamma_4 R^{ij} R_i^{klm} R_{jklm} \phi^2 + \gamma_5 R_i^m R^{ijn} R_{jmln} \phi^2 + \gamma_6 R_{ij}^{mn} R^{ijkl} R_{klmn} \phi^2 \\ & + \gamma_7 R \phi^2 \square R + \gamma_8 R^2 \phi \square \phi + \gamma_9 R_{jklm} R^{jklm} \phi \square \phi + \gamma_{10} \phi^2 \nabla_i R \nabla^i R \\ & + \gamma_{11} R \phi \nabla_i \phi \nabla^i R + \gamma_{12} R^{jklm} \phi \nabla_i R_{jklm} \nabla^i \phi + \gamma_{13} R_{jk} R^{jk} \nabla_i \phi \nabla^i \phi \\ & + \gamma_{14} R^2 \nabla_i \phi \nabla^i \phi + \gamma_{15} R_{jklm} R^{jklm} \nabla_i \phi \nabla^i \phi + \gamma_{16} \nabla_i \phi \nabla^i \phi \square R + \gamma_{17} \phi \square R \square \phi \\ & + \gamma_{18} R \square \phi \square \phi + \gamma_{19} \nabla_i \phi \nabla^i R \square \phi + \gamma_{20} \phi \nabla^i \phi \square \nabla_i R + \gamma_{21} \phi \nabla^i R \square \nabla_i \phi \\ & + \gamma_{22} R \nabla^i \phi \square \nabla_i \phi + \gamma_{23} \phi^2 \square^2 R + \gamma_{24} R \phi \square^2 \phi + \gamma_{25} R_i^k R_{jk} \nabla^i \phi \nabla^j \phi \\ & + \gamma_{26} R_{ij} R \nabla^i \phi \nabla^j \phi + \gamma_{27} R^{kl} R_{ikjl} \nabla^i \phi \nabla^j \phi + \gamma_{28} R_i^{klm} R_{jklm} \nabla^i \phi \nabla^j \phi \\ & + \gamma_{29} \nabla^i \phi \nabla_j \nabla_i R \nabla^j \phi + \gamma_{30} \nabla^i R \nabla_j \nabla_i \phi \nabla^j \phi + \gamma_{31} \phi \nabla_j \nabla_i \phi \nabla^j \nabla^i R \\ & + \gamma_{32} R_i^{klm} R_{jklm} \phi \nabla^j \nabla^i \phi + \gamma_{33} R \nabla_j \nabla_i \phi \nabla^j \nabla^i \phi + \gamma_{34} R^{jk} \phi \nabla^i \phi \nabla_k R_{ij} \\ & + \gamma_{35} R_i^k R^{ij} \phi \nabla_k \nabla_j \phi + \gamma_{36} R^{jk} \nabla^i \phi \nabla_k \nabla_j \nabla_i \phi + \gamma_{37} \nabla^i \phi \nabla^j \phi \square R_{ij} \\ & + \gamma_{38} \phi \nabla^j \nabla^i \phi \square R_{ij} + \gamma_{39} R_i^j \nabla^i \phi \square \nabla_j \phi + \gamma_{40} \nabla^j \phi \square \nabla_j \phi \\ & + \gamma_{41} R^{ij} \phi \square \nabla_j \nabla_i \phi + \gamma_{42} \nabla^j \nabla^i \phi \square \nabla_j \nabla_i \phi + \gamma_{43} \square \phi \square^2 \phi + \gamma_{44} \nabla^i \phi \square^2 \nabla_i \phi \\ & + \gamma_{45} \phi \square^3 \phi + \gamma_{46} \phi^2 \nabla_j R_{ik} \nabla^k R^{ij} + \gamma_{47} \phi^2 \nabla_k R_{ij} \nabla^k R^{ij} \\ & + \gamma_{48} \phi \nabla_k \nabla_j \nabla_i \phi \nabla^k R^{ij} + \gamma_{49} R^{ij} \nabla_k \nabla_j \phi \nabla^k \nabla_i \phi + \gamma_{50} \nabla_i R_{jk} \nabla^i \phi \nabla^k \nabla^j \phi \\ & + \gamma_{51} \nabla^i \phi \nabla_k R_{ij} \nabla^k \nabla^j \phi + \gamma_{52} \nabla_k \nabla_j \nabla_i \phi \nabla^k \nabla^j \nabla^i \phi + \gamma_{53} R_{ijkl} \phi \nabla^i \phi \nabla^l R^{jk} \\ & + \gamma_{54} R_{ikjl} \phi^2 \nabla^l \nabla^k R^{ij} + \gamma_{55} R^{ij} R_{ikjl} \phi \nabla^l \nabla^k \phi + \gamma_{56} R_{ikjl} \nabla^j \nabla^i \phi \nabla^l \nabla^k \phi \\ & + \gamma_{57} \phi^2 \nabla_m R_{ijkl} \nabla^m R^{ijkl} \end{aligned} \quad (\text{A.3})$$

and leave the coefficients h_{31}, \dots, h_{39} free. The full solution is in the ancillary notebook.

Similarly, since $\tilde{h}^{(3)}$ appears in eq. (3.36) only as $\tilde{h}^{(3)i}_i$, we do not compute $\tilde{h}_{ij}^{(3)}$ but rather its trace that satisfies the following PBH equation

$$\delta \tilde{h}^{(3)i}_i = -2\sigma \tilde{h}^{(3)i}_i + g^{(0)ij} \delta \tilde{h}_{ij}^{(3)}. \quad (\text{A.4})$$

A similar Ansatz to (A.3) is made for $\tilde{h}^{(3)i}_i$ and its coefficients are fixed in terms of h_1, c_1, c_2 by the modified PBH equation.

B Gravitational action with backreaction

From the action (2.22) in FG coordinates with backreaction (3.6) we identify:

$$\begin{aligned}
\mathcal{L}_h = & \frac{1}{2}d(1+4\gamma)h^i_i \\
& + \frac{1}{2}z(1+4\gamma)[2(d-1)g'_{ij}h^{ij} - (d-1)g^{ij}g'_{ij}h^k_k \\
& \quad - 2(d-1)g^{ij}h'_{ij} + h^{ij}R_{ij} - \frac{1}{2}h^i_i R - \nabla^j \nabla^i h_{ij} + \square h^i_i] \\
& + z^2[-2(1+4\gamma)g''_{ij}h^{ij} + (3-4(d-5)\gamma)g^{im}g^{jn}g'_{ik}g'_{mn}h^{jk} \\
& \quad - (1+8\gamma)g^{ij}g'_{ij}g'_{kl}h^{kl} + (1+4\gamma)g^{ij}g''_{ij}h^k_k - 2\gamma R_{ikjl}\nabla^l \nabla^k h^{ij} \\
& \quad - (\frac{3}{4} - (d-5)\gamma)g^{im}g^{jn}g'_{ij}g'_{mn}h^k_k + (\frac{1}{4} + 2\gamma)(g^{ij}g'_{ij})^2 h^k_k \\
& \quad - (3-4(d-5)\gamma)g^{ik}g^{jl}g'_{ij}h'_{kl} + 2(1+4\gamma)g^{ij}h''_{ij} + 2\gamma g'_{ij}R^{ij}h^k_k \\
& \quad + (1+8\gamma)g^{ij}g^{kl}g'_{ij}h'_{kl} + 4\gamma h'_{ij}R^{ij} - 8\gamma g'_{ij}h^i_k R^{jk} - \gamma h^{ij}R_i^{klp}R_{jklp} \\
& \quad + \frac{1}{4}\gamma h^i_i R_{jklp}R^{jklp} - 2\gamma g'_{ij}\nabla^j \nabla^i h^k_k + 4\gamma g'_{ij}\nabla_k \nabla^j h^{ik} - 2\gamma g'_{ij}\square h^{ij}] \\
& + \gamma z^3[-12g^{im}g^{jn}g'_{ik}g'_{mn}g'_{jl}h^{kl} + 8g^{ij}g^{mn}g'_{ij}g'_{lm}g'_{mk}h^{kl} \\
& \quad + 4g^{im}g^{jn}g'_{ij}g'_{mn}g'_{kl}h^{kl} + 2g^{im}g^{jn}g^{kl}g'_{ik}g'_{mn}g'_{jl}h^p_p \\
& \quad - 2g^{ij}g^{km}g^{nl}g'_{ij}g'_{kl}g'_{mn}h^p_p + 12g^{im}g^{jn}g^{kl}g'_{ik}g'_{mn}h'_{jl} \\
& \quad - 8g^{ij}g^{km}g^{ln}g'_{ij}g'_{kl}h'_{mn} - 4g^{ik}g^{jl}g^{mn}g'_{ij}g'_{kl}h'_{mn} - g'_{ij}g'_{kl}h^p_p R^{ikjl} \\
& \quad - 4g'_{ij}h'_{kl}R^{ikjl} - 6g'_{ij}g'_{kl}h^i_p R^{jklp} - 4g^{km}g^{ln}h^{ij}\nabla_i g'_{kl}\nabla_j g'_{mn} \\
& \quad - 4g'_{ij}\nabla^i g'_{kl}\nabla^j h^{kl} - 8g^{il}\nabla^j h'_{ik}\nabla^k g'_{jl} + 8g^{ik}g^{jl}\nabla_k h'_{ij}\nabla^k g'_{kl} \\
& \quad + 8g'_{ij}\nabla^i g'_{kl}\nabla^l h^{jk} - 2g'_{ij}g'_{kl}\nabla^l \nabla^j h^{ik} + 2g^{ij}g^{kl}\nabla^l \nabla^k h^{ij} \\
& \quad + 8g^{kn}h^{ij}\nabla_j g'_{kl}\nabla^l g'_{in} + 4g^{in}g'_{jn}\nabla^j h^{kl}\nabla_l g'_{ik} + 4h^{ij}\nabla^k g'_{jl}\nabla^l g'_{ik} \\
& \quad - 4g^{in}g'_{ij}\nabla^k h^{jl}\nabla_l g'_{kn} - 8g^{kn}h^{ij}\nabla_l g'_{jk}\nabla^l g'_{in} - 4g^{in}g'_{ij}\nabla_l h^{jk}\nabla^l g'_{kn} \\
& \quad - 2g^{jn}h^i_i \nabla^k g'_{jl}\nabla^l g'_{kn} + 2g^{jm}g^{km}h^i_i \nabla_l g'_{jk}\nabla^l g'_{mn}] \\
& + \gamma z^4[8g^{im}g^{kn}g'_{ij}g'_{kl}g''_{mn}h^{jl} - 16g^{il}g''_{ik}g''_{jl}h^{jk} + 16g^{im}g^{jn}g'_{ik}g'_{mn}g''_{jl}h^{kl} \\
& \quad + 4g^{ik}g^{jl}g''_{ij}g''_{kl}h^k_k - 4g^{kq}g^{ir}g^{js}g'_{ik}g'_{rs}g'_{jl}g'_{pq}h^{lp} - 8g^{im}g^{jn}g^{kl}g'_{ik}g'_{mn}h''_{jl} \\
& \quad - 4g^{ik}g^{jl}g^{mr}g^{ns}g^{pq}g'_{ij}g'_{kl}g'_{mn}g'_{rs}h^{ps} - 4g^{im}g^{jn}g^{kl}g'_{ik}g'_{mn}g''_{jl}h^p_p \\
& \quad + \frac{1}{2}g^{im}g^{jn}g^{kr}g^{ls}g'_{ik}g'_{mn}g'_{jl}g'_{rs}h^p_p + \frac{1}{2}(g^{ik}g^{jl}g'_{ij}g'_{kl})^2 h^p_p - 16g^{im}g^{jn}g^{kl}g'_{mn}g'_{ik}h''_{jl} \\
& \quad + 4g^{im}g^{jn}g^{kr}g^{ls}g'_{ik}g'_{mn}g'_{jl}h'_{rs} + 4g^{im}g^{jn}g^{kr}g^{ls}g'_{ij}g'_{mn}g'_{kl}h'_{rs} + 16g^{ik}g^{jl}g''_{ij}h''_{kl}]
\end{aligned} \tag{B.1}$$

while \mathcal{L}_g is still given by (2.23). Notice that also terms of $\mathcal{O}(z^4)$ contribute to the anomaly in $d=4$, since $h''_{ij} = -z^{-2}\tilde{h}^{(1)}_{ij} + \mathcal{O}(z^{-1})$.

C Coefficients renormalisation

Evaluating (2.25) and (3.36) on the PBH solutions, we find that $\mathcal{L}_g^{(2)} + \mathcal{L}_h^{(2)}$ satisfies the Wess-Zumino condition (3.34), even though it is singular in $d = 4$. In particular, expanding around $d = 4$ we find that

$$\mathcal{L}_g^{(2)} + \mathcal{L}_h^{(2)} = \frac{1}{d-4}I^{(-1)} + I^{(0)} + (d-4)I^{(1)} + (d-4)^2I^{(2)} + \dots \quad (\text{C.1})$$

with $(g_{ij} \equiv g_{ij}^{(0)})$ and $\phi \equiv \phi_{(0)}$

$$\begin{aligned} I^{(-1)} = c h_1 & \left(\frac{7}{24} R_{ij} R^{ij} R \phi^2 - \frac{1}{24} R^3 \phi^2 - \frac{1}{2} R^{ij} R^{kl} R_{ikjl} \phi^2 + \frac{1}{24} R \phi^2 \square R - \frac{1}{12} R^2 \phi \square \phi \right. \\ & + \frac{1}{12} R^{ij} \phi^2 \nabla_j \nabla_i R + \frac{1}{3} R^{ij} R \phi \nabla_j \nabla_i \phi + \frac{1}{12} \phi \square R \square \phi + \frac{1}{6} \phi \nabla_j \nabla_i \phi \nabla^j \nabla^i R \\ & \left. - \frac{1}{4} R^{ij} \phi^2 \square R_{ij} - \frac{1}{2} \phi \nabla^j \nabla^i \phi \square R_{ij} + \frac{1}{4} R_{ij} R^{ij} \phi \square \phi - R^{ij} R_{ikjl} \phi \nabla^l \nabla^k \phi \right). \end{aligned} \quad (\text{C.2})$$

Notice that $I^{(-1)}$ is entirely type B and $\delta I^{(-1)} = -4\sigma I^{(-1)}$ under Weyl transformation up to $\mathcal{O}(\phi^2)$. However, $\mathcal{L}_g^{(2)} + \mathcal{L}_h^{(2)}$ contains also several free coefficients, which are not fixed by the PBH equations, i.e. $\{c_1, c_2, h_1, h_{21}, \dots, h_{26}, h_{31}, \dots, h_{39}\}$, and it is possible to regularise (C.1) by renormalising some of these coefficients. In particular, there is a unique shift which cancels $I^{(-1)}$ and it consists of shifting some of the free coefficients which appear in $I^{(2)}$ as follows:

$$\begin{aligned} h_{32} &\rightarrow h_{32} + \frac{c h_1}{18a(d-4)^2} & h_{33} &\rightarrow h_{33} + \frac{7c h_1}{36a(d-4)^2} & h_{34} &\rightarrow h_{34} + \frac{2c h_1}{9a(d-4)^2} \\ h_{36} &\rightarrow h_{36} - \frac{c h_1}{3a(d-4)^2} & h_{37} &\rightarrow h_{37} - \frac{c h_1}{6a(d-4)^2} & h_{38} &\rightarrow h_{38} + \frac{c h_1}{6a(d-4)^2}. \end{aligned}$$

Since the shifted coefficients only appear in $I^{(2)}$, this shift does not introduce new singularities, and of course the regularised expression satisfies the Wess-Zumino condition.

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