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A framework for fitting quadratic-bilinear systems with applications to models of electrical circuits

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Abstract: We propose a method for fitting quadratic-bilinear models from data. Although the dynamics characterizing the original model consist of general analytic nonlinearities, we rely on lifting techniques for equivalently embedding the original model into the quadratic-bilinear structure. Here, data are given by generalized transfer function values that can be sampled from the time-domain steady-state response. This method is an extension of methods that perform bilinear, or quadratic inference, separately. It is based on first identifying the underline minimal linear model with the interpolatory method known as the Loewner framework, and then on inferring the best supplementing nonlinear (quadratic and bilinear) operators, by solving an optimization problem. The proposed data-driven method finds applications in engineering serving the scopes of robust simulation, design, and control. Model examples of electrical circuits with nonlinear components (diodes) were used to test the method's performance.

Keywords: Data-Driven, Non-Intrusive Modeling, Model Order Reduction, Nonlinear Dynamics, Quadratic-Bilinear Systems, System Identification, Lift and Learn Approach.

1. INTRODUCTION

System Identification (SI) and data-driven Model Order Reduction (MOR) are two fairly-established methodologies aiming to discover, from data, robust surrogates of dynamical systems. Such model discovery can be performed non-intrusively, without exact access to the system's structure or matrices scaling the various terms. In the case of SI, the aim is to discover known classes of dynamical systems that constitute an appropriate mathematical formalism capable of describing dynamical phenomena. We refer the reader to van Overschee and de Moor (1996) and to Ljung (1999) for more details on various methodologies. In the case of MOR, the need for approximating the underlying dynamical system is dictated mainly by the increased dimension underhand (the number of internal variables). Conventional MOR methods are intrusive in that they require an explicit formulation of the dynamical system to be reduced (in terms of matrices or various operators). We refer the reader to Benner et al. (2017) and to Antoulas et al. (2020), for more details. However, data-driven MOR methods are generally non-intrusive since they require only data (snapshots of the states, input-output measurements, etc.) and not the full/detailed description of the model. Methods that fall into this category range from DMD (dynamic mode decomposition), OpInf (operator inference), to LF (the Loewner framework). Such data-driven methods can identify surrogate models without having exact access to the original operators. Starting with Ho and Kalman (1966), many algorithms were developed to identify linear dynamical systems in the state-space realization:

$$\left\{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \quad y(t) = \mathbf{C}\mathbf{x}(t), \right. \tag{1}$$

where $\mathbf{x}$ is the state variable of dimension n, while the system matrices are given by $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B}, \mathbf{C}^T \in \mathbb{R}^{n\times 1}$. We refer to Antoulas (2005), for more details on various methodologies. In recent years, the ideas for developing methods for linear systems have steadily expanded to certain classes of nonlinear systems. We consider general analytical nonlinear systems in the following state-space representation:

$$\left\{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) + \mathbf{B}u(t), \quad y(t) = \mathbf{C}\mathbf{x}(t), \right. \tag{2}$$

where $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the nonlinearity that will be approximated using Carleman linearization in Carleman (1932), such as in Weber and Mathis (2018), or by lifting techniques developed and applied by Gu (2011) and Breiten (2013); Benner and Breiten (2015) in the MOR community. These yield an equivalent system with polynomial (quadratic) structure. In such phenomena, the scaling and superposition principles that hold for linear models, do not hold here anymore, making the analysis fairly difficult. Many phenomena of nonlinear nature can not be accurately approximated using linearization methods

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(that could be performed only locally). In addition, linear models that approximate nonlinear are usually of high complexity. Two sub-classes of polynomial models that belong to the class described in Eq. (2) are represented by quadratic and bilinear models. Their dynamics are as:

- Quadratic case: $\mathbf{f}_q(\mathbf{x}(t), u(t)) = \mathbf{Q}(\mathbf{x}(t) \otimes \mathbf{x}(t))$, and
- Bilinear case: $\mathbf{f}_b(\mathbf{x}(t), u(t)) = \mathbf{N}\mathbf{x}(t)u(t)$,

or with linear combinations of these two classes that appear in the case of quadratic-bilinear (QB) systems. Here, $\otimes$ denotes the Kronecker product. For cases when the non-linear operator of the original system is not directly written as $(\alpha f_q + \beta f_b)$, we can employ lifting techniques to embed the original nonlinear dynamics into the required format, without any approximation. This is performed by using specifically tailored lifting transformations. More specifically, auxiliary variables and equations are introduced in order to reformulate the equations in the desired form. This allows applying conventional MOR methods to more general nonlinear systems. Specific lifting transformations were discussed in Gu (2011); Breiten (2013); Kramer and Willcox (2019); Gosea (2022).

A viable alternative is to employ non-intrusive techniques and use data-driven methods like the Loewner framework (LF), for which low-order models can be constructed directly from data. It is to be noted that LF has been recently extended to fit certain classes of nonlinear systems using direct numerical simulation (DNS) data, such as bilinear systems in Antoulas et al. (2016), and quadratic-bilinear (QB) systems in Gosea and Antoulas (2018); Antoulas et al. (2019). However, for these methods, data used in the computation process cannot be easily inferred from practical experiments (challenging to obtain in practice); therefore, the motivation for developing the current extension. Another data-driven method that has emerged in recent years is operator inference (OpInf), which uses time-domain state measurements (snapshots of the state variable), and then fits a particular nonlinear model (quadratic or quadratic-bilinear) by computing the appropriate matrices. Details on OpInf can be found in Peherstorfer and Willcox (2016); Benner et al. (2020b,a).

In this contribution, we propose a data-driven method that can be used to infer reduced QB models that can perform as surrogate models of the original. This can be viewed as an extension of the methods in Karachalios et al. (2021a); Gosea et al. (2021), which dealt with bilinear, and quadratic inference, separately. One aspect that distinguishes the OpInf framework from the extended Loewner-based frameworks is that the former requires measurements of the whole state variable. Our framework requires only input-output measurements (measurements of generalized transfer functions).

The paper is structured as follows; after the introduction, Section 2 introduces the class of QB systems with its generalized transfer functions, and the classical Loewner framework is briefly mentioned. Next, the newly-proposed method is introduced in Section 3. Then, in Section 4, we first go through a simple example of a nonlinear circuit to show various reformulations of its structure. Section 5 includes a detailed numerical study for applying the method to a nonlinear ladder circuit, while Section 6 provides the conclusions and future research plans.

2. QUADRATIC-BILINEAR SYSTEMS AND THE LOEWNER FRAMEWORK

2.1 State-space format and properties of QB systems

We analyze in what follows dynamical systems as in Eq. (2), with quadratic-bilinear (QB) nonlinearities for which $\mathbf{f}$ is given as: $\mathbf{f}(\mathbf{x}(t), u(t)) = \mathbf{Q}(\mathbf{x}(t) \otimes \mathbf{x}(t)) + \mathbf{N}\mathbf{x}(t)u(t)$. More precisely, let the state-space representation of such a system be given as:

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{Q}(\mathbf{x}(t) \otimes \mathbf{x}(t)) + \mathbf{N}\mathbf{x}(t)u(t) + \mathbf{B}u(t), \\ y(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{3}$$

where $\mathbf{x}(0) = \mathbf{x}_0 = \mathbf{0}$ and the matrix $\mathbf{E} \in \mathbb{R}^{n\times n}$ is non-singular, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{Q} \in \mathbb{R}^{n\times n^2}$, $\mathbf{N} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times 1}$ and $\mathbf{C} \in \mathbb{R}^{1\times n}$. Moreover, assume that $\mathbf{Q}$ satisfies the following property $\mathbf{Q}(\mathbf{v} \otimes \mathbf{w}) = \mathbf{Q}(\mathbf{w} \otimes \mathbf{v})$.

The first two generalized symmetric transfer functions functions of a QB system as in Eq. (3) (sometimes referred to as Volterra kernels in the frequency domain) are (here, we denote the resolvent as: $\mathbf{\Phi}(s) = (s\mathbf{E} - \mathbf{A})^{-1} \in \mathbb{C}^{n\times n}$):

$$\begin{aligned} H_1(s_1) &= \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{B}, \\ H_2(s_1, s_2) &= \mathbf{C}\mathbf{\Phi}(s_1 + s_2)\mathbf{Q}(\mathbf{\Phi}(s_1)\mathbf{B} \otimes \mathbf{\Phi}(s_2)\mathbf{B}) \\ &\quad + \frac{1}{2}\mathbf{C}\mathbf{\Phi}(s_1 + s_2)\mathbf{N}(\mathbf{\Phi}(s_1)\mathbf{B} + \mathbf{\Phi}(s_2)\mathbf{B}). \end{aligned} \tag{4}$$

For more details on deriving such functions we refer the reader to Breiten (2013) and to Gosea and Antoulas (2018); Antoulas et al. (2019). An important property of these particular functions (sometimes called symmetric transfer functions) is that their samples can be inferred from the spectrum of the observed output when using a purely oscillatory control input.

2.2 The Loewner framework for linear systems

We start with a brief summary of LF for fitting linear systems as in Eq. (10). The starting point for LF is to collect measurements corresponding to the (first) transfer function, which can be inferred in practice from the first harmonic. The data are first partitioned into two disjoint subsets, as:

$$\begin{aligned} &\text{right data} : (\lambda_j; w_j),\ j = 1, \ldots, k,\ \text{and}, \\ &\text{left data} : (\mu_i; v_i),\ i = 1, \ldots, k, \end{aligned} \tag{5}$$

find the function $\mathbf{H}(s)$, such that the following interpolation conditions are (approximately) fulfilled:

$$\mathbf{H}(\mu_i) = v_i, \quad \mathbf{H}(\lambda_j) = w_j. \tag{6}$$

The Loewner matrix $\mathbb{L} \in \mathbb{C}^{k\times k}$ and the shifted Loewner matrix $\mathbb{L}_s \in \mathbb{C}^{k\times k}$ are defined as follows

$$\mathbb{L}_{(i,j)} = \frac{v_i - w_j}{\mu_i - \lambda_j},\ \mathbb{L}_{s(i,j)} = \frac{\mu_i v_i - \lambda_j w_j}{\mu_i - \lambda_j}, \tag{7}$$

while the data vectors $\mathbb{V}, \mathbb{W}^T \in \mathbb{R}^k$ are introduced as

$$\mathbb{V}_{(i)} = v_i, \quad \mathbb{W}_{(j)} = w_j,\ \text{for}\ i, j = 1, \ldots, k. \tag{8}$$

The Loewner model is hence constructed as follows:

$$\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_s, \quad \mathbf{B} = \mathbb{V}, \quad \mathbf{C} = \mathbb{W}.$$

Provided that enough data is available, the pencil $(\mathbb{L}_s, \mathbb{L})$ is often singular. In these cases, a singular value decomposition (SVD) of Loewner matrices is needed to compute projection matrices $\mathbf{X}_r, \mathbf{Y}_r \in \mathbb{C}^{k\times r}$. Here, $r < n$ represents the truncation index.

Then, the system matrices corresponding to a projected Loewner model of dimension r with $(\cdot)^*$ to denote the conjugate-transpose can be computed using the matrices $\mathbf{X}_r$ and $\mathbf{Y}_r$, as:

$$\hat{\mathbf{E}} = -\mathbf{X}_r^*\mathbb{L}\mathbf{Y}_r,\quad \hat{\mathbf{A}} = -\mathbf{X}_r^*\mathbb{L}_s\mathbf{Y}_r,\quad \hat{\mathbf{B}} = \mathbf{X}_r^*\mathbb{V},\quad \hat{\mathbf{C}} = \mathbb{W}\mathbf{Y}_r, \tag{9}$$

and therefore, directly finds a state-space realization corresponding to the reduced-order system of equations

$$\left\{\hat{\mathbf{E}}\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\mathbf{x}(t) + \hat{\mathbf{B}}u(t),\quad \hat{y}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t).\right. \tag{10}$$

More implementation details and properties on the LF procedure can be found in Antoulas et al. (2017); Karachalios et al. (2018, 2021a).

3. THE PROPOSED HYBRID METHOD BASED ON THE LOEWNER FRAMEWORK AND LS SOLVES

The idea is to recover all the operators from measurements. The Loewner framework is capable of recovering the linear part by providing a fitted realization of dimension r as in Eq. (9), denoted with $(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$. Now, based on these matrices, introduce $\hat{\mathbf{\Phi}}(s) = (s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1} \in \mathbb{C}^{r\times r}$.

The sampling domain is denoted with $\Omega = \{(\zeta_1^{(i)}, \zeta_2^{(i)}) \in \mathbb{C}^2 | 1 \leq i \leq K\}$. For a particular pair of sampling points $(\zeta_1^{(i)}, \zeta_2^{(i)}) \in \mathbb{C}^2$, we define the following quantities:

$$\begin{aligned}
\hat{\mathcal{O}}(\zeta_1^{(i)}, \zeta_2^{(i)}) &:= \hat{\mathbf{C}}\hat{\mathbf{\Phi}}(\zeta_1^{(i)} + \zeta_2^{(i)}) \in \mathbb{C}^{1\times r},\\
\hat{\mathcal{R}}_q(\zeta_1^{(i)}, \zeta_2^{(i)}) &:= (\hat{\mathbf{\Phi}}(\zeta_1^{(i)})\hat{\mathbf{B}} \otimes \hat{\mathbf{\Phi}}(\zeta_2^{(i)})\hat{\mathbf{B}}) \in \mathbb{C}^{r^2\times 1},\\
\hat{\mathcal{R}}_b(\zeta_1^{(i)}, \zeta_2^{(i)}) &:= (\hat{\mathbf{\Phi}}(\zeta_1^{(i)})\hat{\mathbf{B}} + \hat{\mathbf{\Phi}}(\zeta_2^{(i)})\mathbf{B}) \in \mathbb{C}^{r\times 1}.
\end{aligned} \tag{11}$$

It is to be noted that the vectors introduced in Eq. (11) are computed solely in terms of the matrices $(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ corresponding to the Loewner surrogate linear model.

Let $\mathbf{v} \in \mathbb{C}^K$ be the vector of data measurements, i.e., containing samples of the second symmetric transfer function $H_2(s_1, s_2)$ evaluated on the Ω grid. More precisely, let $\mathbf{v}_i = H_2(\zeta_1^{(i)}, \zeta_2^{(i)})$. Now, since we would like to fit a reduced-order QB model to interpolate the 2D data, it follows that $H_2(\zeta_1^{(i)}, \zeta_2^{(i)}) = \hat{H}_2(\zeta_1^{(i)}, \zeta_2^{(i)})$. We can write:

$$\underbrace{\hat{H}_2(\zeta_1^{(i)}, \zeta_2^{(i)})}_{\mathbf{v}_i\in\mathbb{C}} = \underbrace{\hat{\mathbf{C}}\hat{\mathbf{\Phi}}(\zeta_1^{(i)} + \zeta_2^{(i)})}_{\hat{\mathcal{O}}(\zeta_1^{(i)},\zeta_2^{(i)})\in\mathbb{C}^{1\times r}}\hat{\mathbf{Q}}\underbrace{(\hat{\mathbf{\Phi}}(\zeta_1^{(i)})\hat{\mathbf{B}} \otimes \hat{\mathbf{\Phi}}(s_2)\hat{\mathbf{B}})}_{\hat{\mathcal{R}}_q(\zeta_1^{(i)},\zeta_2^{(i)})\in\mathbb{C}^{r^2\times 1}} \tag{12}$$

$$+ (1/2)\underbrace{\mathbf{C}\mathbf{\Phi}(\zeta_1^{(i)} + \zeta_2^{(i)})}_{\hat{\mathcal{O}}(\zeta_1^{(i)},\zeta_2^{(i)})\in\mathbb{C}^{1\times r}}\hat{\mathbf{N}}\underbrace{(\hat{\mathbf{\Phi}}(\zeta_1^{(i)})\hat{\mathbf{B}} + \hat{\mathbf{\Phi}}(\zeta_2^{(i)})\mathbf{B})}_{\hat{\mathcal{R}}_b(\zeta_1^{(i)},\zeta_2^{(i)})\in\mathbb{C}^{r\times 1}}, \tag{13}$$

and hence it follows that:

$$\hat{\mathbf{H}}_2(\zeta_1^{(i)}, \zeta_2^{(i)}) = \hat{\mathcal{O}}(\zeta_1^{(i)}, \zeta_2^{(i)})\hat{\mathbf{Q}}\hat{\mathcal{R}}_q(\zeta_1^{(i)}, \zeta_2^{(i)}) \tag{14}$$

$$+ (1/2)\hat{\mathcal{O}}(\zeta_1^{(i)}, \zeta_2^{(i)})\hat{\mathbf{N}}\hat{\mathcal{R}}_b(\zeta_1^{(i)}, \zeta_2^{(i)}) \tag{15}$$

Definition 3.1. Given a matrix $\mathbf{X} \in \mathbb{C}^{m\times n}$, we denote with $\mathrm{vec}(\mathbf{X})$ the vector $(mn) \times 1$ computed as follows:

$$\mathrm{vec}(\mathbf{X}) = [\,\mathbf{X}(1,:)\ \cdots\ \mathbf{X}(m,:)\,]^T \in \mathbb{C}^{mn}, \tag{16}$$

where the MATLAB notation $\mathbf{X}(k,:) \in \mathbb{C}^{1\times n}$ was used to refer to the kth row of $\mathbf{X}$.

The vectorization procedure adapted to the data-driven problem shown here is presented in Eq. (18). Let us denote with $\mathcal{T} \in \mathbb{C}^{K\times(r^3+r^2)}$, the matrix for which the ith row is given by $\mathcal{T}(i,:) = [\mathcal{T}_q(i,:)\ \mathcal{T}_b(i,:)] \in \mathbb{C}^{1\times(r^3+r^2)}$

$$\begin{cases}\mathcal{T}_q(i,:) = \left[\hat{\mathcal{O}}(\zeta_1^{(i)}, \zeta_2^{(i)}) \otimes \hat{\mathcal{R}}_q^T(\zeta_1^{(i)}, \zeta_2^{(i)})\right] \in \mathbb{C}^{K\times r^3},\\ \mathcal{T}_b(i,:) = \left[\hat{\mathcal{O}}(\zeta_1^{(i)}, \zeta_2^{(i)}) \otimes \hat{\mathcal{R}}_b^T(\zeta_1^{(i)}, \zeta_2^{(i)})\right] \in \mathbb{C}^{K\times r^2}.\end{cases} \tag{17}$$

Now, from Eq. (18), by varying the index i such as $1 \leq i \leq K$, it follows that we can put together a linear system of equations in $r^3 + r^2$ unknowns as follows:

$$\mathcal{T}\mathbf{z} = \mathbf{v}, \tag{19}$$

where $\mathbf{z} = \left[\mathrm{vec}(\hat{\mathbf{Q}})^T\ (1/2)\mathrm{vec}(\hat{\mathbf{N}})^T\right]^T$ is the vector of variables which contains the entries of the operators corresponding to the surrogate reduced-order QB system. In order to ensure an over-determined linear system of equations, we clearly need to have enough data measurements, i.e., the condition $K \geq r^3 + r^2$ needs to hold true. Then, due to the low-order r, we can employ a direct solution of system Eq. (19), e.g., by means of the Moore-Penrose pseudo-inverse or by using Gaussian elimination. However, in most cases, the matrix $\mathcal{T}$ is not of full column rank, and hence direct solves need to be carefully dealt with (by introducing regularization techniques). In what follows, we will use a truncated singular value decomposition (tSVD) approach. This is an attractive and powerful method since it uses the optimal rank-k approximation of the SVD (in the 2 norm). Such approach has been already used for applying OpInf Benner et al. (2020a), together with the Tikhonov regularization scheme Peherstorfer and Willcox (2016) and tQR (truncated QR decomposition).

4. A SIMPLE MODEL OF A NON-LINEAR CIRCUIT

We consider a simple circuit constructed from two blocks connected in series. Each block contains a capacitor in parallel with a diode. This is modeled as a dynamical system in two variables given by the voltage drops on each block. The input is provided by the current through the circuit, while the observed output is the sum of the two variables. More precisely, we can write the differential equations characterizing the dynamics as follows:

$$\begin{cases} C_1\dfrac{dV_1(t)}{dt} = I(t) - I_{r_1}\left(e^{\frac{1}{V_{t_1}}V_1(t)} - 1\right),\\ C_2\dfrac{dV_2(t)}{dt} = I(t) - I_{r_2}\left(e^{\frac{1}{V_{t_2}}V_2(t)} - 1\right),\end{cases} \tag{20}$$

while the output is $y(t) = V_1(t) + V_2(t)$. The capacitance values are denoted with $C_i, 1 \leq i \leq 2$, while other constants are denoted with I_{r_i} and V_{t_i}. Next, we introduce $x_1(t) = V_1(t)/V_{t_1}$, $x_2(t) = V_2(t)/V_{t_2}$, and let $a = \frac{1}{C_1V_{t_1}}$, $b = \frac{1}{C_2V_{t_2}}$, $c = I_{r_1}$, $d = I_{r_2}$. Using all of these, the nonlinear system in Eq. (20) is rewritten as:

$$\begin{cases}\dot{x}_1(t) = aI(t) - ac\left(e^{x_1(t)} - 1\right),\\ \dot{x}_2(t) = bI(t) - bd\left(e^{x_2(t)} - 1\right),\\ y(t) = x_1(t)V_{t_1} + x_2(t)V_{t_2}.\end{cases} \tag{21}$$

4.1 Analysis based on Taylor series truncation

The methods discussed here are inexact, i.e., based on approximation (e.g., on truncating the Taylor series associated with the nonlinearity). First, by using a truncated

$$\begin{aligned}\mathbf{v}_i = \hat{H}_2(\zeta_1^{(i)},\zeta_2^{(i)}) &= \hat{\mathcal{O}}(\zeta_1^{(i)},\zeta_2^{(i)})\hat{\mathbf{Q}}\hat{\mathcal{R}}_q(\zeta_1^{(i)},\zeta_2^{(i)}) + (1/2)\hat{\mathcal{O}}(\zeta_1^{(i)},\zeta_2^{(i)})\hat{\mathbf{N}}\hat{\mathcal{R}}_b(\zeta_1^{(i)},\zeta_2^{(i)}) \\ &= \left(\hat{\mathcal{O}}(\zeta_1^{(i)},\zeta_2^{(i)})\otimes\hat{\mathcal{R}}_q^T(\zeta_1^{(i)},\zeta_2^{(i)})\right)vec(\hat{\mathbf{Q}}) + (1/2)\left(\hat{\mathcal{O}}(\zeta_1^{(i)},\zeta_2^{(i)})\otimes\hat{\mathcal{R}}_b^T(\zeta_1^{(i)},\zeta_2^{(i)})\right)vec(\hat{\mathbf{N}}) \\ &= \underbrace{\left[\underbrace{\hat{\mathcal{O}}(\zeta_1^{(i)},\zeta_2^{(i)})\otimes\hat{\mathcal{R}}_q^T(\zeta_1^{(i)},\zeta_2^{(i)})}_{=\mathcal{T}_q(i,:)}\Bigg|\underbrace{\hat{\mathcal{O}}(\zeta_1^{(i)},\zeta_2^{(i)})\otimes\hat{\mathcal{R}}_b^T(\zeta_1^{(i)},\zeta_2^{(i)})}_{=\mathcal{T}_b(i,:)}\right]}_{=\mathcal{T}(i,:)\,\in\,\mathbb{C}^{1\times(r^3+r^2)}}\cdot\underbrace{\left[\frac{\mathrm{vec}(\hat{\mathbf{Q}})}{(1/2)\mathrm{vec}(\hat{\mathbf{N}})}\right]}_{=\mathbf{z}\,\in\,\mathbb{C}^{(r^3+r^2)\times 1}}\end{aligned} \tag{18}$$

Taylor series given by the formula $e^{x_i(t)} \approx 1 + x_i(t)$, and substitute it in Eq. (21), it remains the following linear dynamical system:

$$\begin{cases}\dot{x}_1(t) = aI(t) - acx_1(t),\\ \dot{x}_2(t) = bI(t) - bdx_2(t),\\ \;y(t) = x_1(t)V_{t_1} + x_2(t)V_{t_2} + R\cdot I(t).\end{cases} \tag{22}$$

Then, identify the following linear realization:

$$\mathbf{A}_L = \begin{bmatrix}-ac & 0\\ 0 & -bd\end{bmatrix},\;\; \mathbf{B}_L = \begin{bmatrix}a\\ b\end{bmatrix},\;\;\; \mathbf{C}_L = [V_{t_1}\;\; V_{t_2}].$$

Next, we discuss Carleman's linearization in Carleman (1932). It is a method used to embed a nonlinear system of differential equations of finite dimension into a system of bilinear differential equations of infinite dimension. However, truncation is often performed. For instance, by introducing the new state $\mathbf{x}^C$ (linear and quadratic in $\mathbf{x}$), and by ignoring higher powers, we obtain

$$\mathbf{x}^C = \begin{bmatrix}\mathbf{x}\\ \mathbf{x}\otimes\mathbf{x}\end{bmatrix} = \begin{bmatrix}x_1 & x_2 & x_1^2 & x_1x_2 & x_2x_1 & x_2^2\end{bmatrix}^T, \tag{23}$$

and appropriately compute the derivatives of the last four entries as, for example, of:

$$\begin{aligned}\dot{x}_3^C(t) &= \frac{d}{dt}x_1^2(t) = 2x_1(t)\dot{x}_1(t)\\ &= 2a\mathbf{x}_1(t)I(t) - 2acx_1^2(t) - acx_1^3(t).\end{aligned} \tag{24}$$

Finally, the approximation steps follows, i.e., we neglect the powers in $x_i(t)$ higher than 3, and write:

$$\begin{aligned}&\dot{x}_3^C(t) \approx 2ax_1(t)I(t) - 2acx_1^2(t)\\ &\quad\Rightarrow \dot{x}_3^C(t) \approx 2ax_1^C(t)I(t) - 2acx_2^C(t),\end{aligned} \tag{25}$$

which includes a linear term, i.e., $x_2^C(t)$, and a bilinear term: $x_1^C(t)I(t)$. The same procedure is applied for the other entries of the derivative of the new state vector $\dot{\mathbf{x}}^C(t)$. Hence, an approximate bilinear systems is derived: $\{\dot{x}^C(t) = \mathbf{A}\mathbf{x}^C(t) + \mathbf{N}\mathbf{x}^C(t)u(t) + \mathbf{B}u(t),\;\; y(t) = \mathbf{C}\mathbf{x}^C(t).$

4.2 Polynomial lifting

The first step towards implementing a lifting approach is to identify the quantities that depend non-linearly on the original states. For example, let the auxiliary variables be:

$$x_3(t) := e^{x_1(t)} - 1, \quad\; x_4(t) := e^{x_2(t)} - 1. \tag{26}$$

In this way, we also enforce zero initial conditions. The augmented system is hence written as:

$$\begin{cases}\dot{x}_1(t) = aI(t) - acx_3(t),\;\;\dot{x}_2(t) = bI(t) - bdx_4(t),\\ \dot{x}_3(t) = -acx_3(t) - acx_3^2(t) + aI(t) + ax_3(t)I(t)\\ \dot{x}_4(t) = -bdx_4(t) - bdx_4^2(t) + bI(t) + bx_4(t)I(t)\\ \;y(t) = x_1(t)V_{t_1} + x_2(t)V_{t_2} + R\cdot I(t).\end{cases} \tag{27}$$

The system in Eq. (27) can be written in general QB form:

$$\begin{cases}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{Q}(\mathbf{x}(t)\otimes\mathbf{x}(t)) + \mathbf{N}\mathbf{x}(t)I(t) + \mathbf{B}I(t),\\ y(t) = \mathbf{C}\mathbf{x}(t),\end{cases}$$

where the matrices are defined next:

$$\mathbf{A} = \begin{bmatrix}0&0&-ac&0\\0&0&0&-bd\\0&0&-ac&0\\0&0&0&-bd\end{bmatrix}, \mathbf{N} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&a&0\\0&0&0&b\end{bmatrix},\; \mathbf{B} = \begin{bmatrix}a\\b\\a\\b\end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix}V_{t_1}\\V_{t_2}\\0\\0\end{bmatrix}^T, \mathbf{Q} = \begin{bmatrix}0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&-ac&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-bd\end{bmatrix}$$

The first two symmetric transfer functions can be written:

$$\begin{aligned}H_1(s) &= \mathbf{C}\boldsymbol{\Phi}(s)\mathbf{B} = \frac{V_{t_1}}{C_1V_{t_1}s + I_{r_1}} + \frac{V_{t_2}}{C_2V_{t_2}s + I_{r_2}},\\ H_2(s,s) &= \mathbf{C}\boldsymbol{\Phi}(2s)\mathbf{Q}\left[\boldsymbol{\Phi}(s)\mathbf{B}\otimes\boldsymbol{\Phi}(s)\mathbf{B}\right] + \mathbf{C}\boldsymbol{\Phi}(2s)\mathbf{N}\boldsymbol{\Phi}(s)\mathbf{B}\\ &= -\frac{I_{r_1}V_{t_1}}{2\,(I_{r_1} + C_1\,V_{t_1}\,s)^2\,(I_{r_1} + 2\,C_1\,V_{t_1}\,s)}\\ &\quad - \frac{I_{r_2}V_{t_2}}{2\,(I_{r_2} + C_2\,V_{t_2}\,s)^2\,(I_{r_2} + 2\,C_2\,V_{t_2}\,s)}\end{aligned}$$

The parameters can be recovered from the LF by considering the 2nd kernel as a univariate rational function, but the pole residue form needs a special treatment and is usually quite challenging. Similar studies have been proposed in Karachalios et al. (2019); Gosea et al. (2021) for inferring bilinear or quadratic systems respectively, where an improvement towards bilinear identification was shown in Karachalios et al. (2021b) and in the current study.

5. NUMERICAL EXPERIMENTS: A NONLINEAR RC LADDER CIRCUIT

We analyze the nonlinear RC-ladder electronic circuit first introduced in Chen (1999). Different variants of this model were also mentioned in other MOR works,i.e., Gu (2011) and Breiten (2013). This nonlinear first-order system models a resistor-capacitor network that exhibits a nonlinear behaviour caused by the nonlinear resistors consisting of a parallel connected resistor with a diode. As presented in Chen (1999), the underlying model is given by a single-input single-output (SISO) system of the form:

$$\dot{x}(t) = \begin{bmatrix}-g(x_1(t)) - g(x_1(t) - x_2(t))\\ g(x_1(t) - x_2(t)) - g(x_2(t) - x_3(t))\\ \cdots\\ g(x_{k-1}(t) - x_k(t)) - g(x_k(t) - x_{x+1}(t))\\ \cdots\\ g(x_{N-1}(t) - x_N(t))\end{bmatrix} + \begin{bmatrix}u(t)\\0\\\cdots\\0\\\cdots\\0\end{bmatrix}, \tag{28}$$

with $y(t) = x_1(t)$, where the mapping g is given by $g:\mathbb{R}\to\mathbb{R}$ defined as $g(x_i) = g_D(x_i) + x_i$, which combines the effect of a diode and a resistor. The non-linearity g_D models a diode as a nonlinear resistor, based on the classical Shockley model:

$$g_D(x_i) = i_S(\exp(u_Px_i) - 1), \tag{29}$$

with material parameters $i_S > 0$ and $u_P > 0$. For this benchmark, the parameters are selected as follows: $i_S = 1$ and $u_P = 40$ as in Chen (1999). By substituting these values into Eq. (29), we get that $g_D(x_i) = \exp(40x_i) - 1$, and hence it follows that $g(x_i) = \exp(40x_i) + x_i - 1$.

Bilinear treatment via Carleman's approach. The original nonlinear system is transformed into a bilinear system by means of the Carleman linearization, as originally shown in Chen (1999) and Breiten (2013). The matrices are given:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \frac{1}{2}\mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \otimes \mathbf{v} \end{bmatrix},$$
$$\mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B} & \mathbf{0} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}^T. \tag{30}$$

Consequently, the resulting bilinear system has dimension n^2+n with n as the number of circuit blocks of the original system. More details on the structure of the involved matrices in Eq. (30) can be found in Breiten (2013).

Lifting to Quadratic-Bilinear form. Analogues with the example in Sec. 4.2, the original RC-ladder model can be lifted to an equivalent quadratic-bilinear model. The introduced additional state variables $\mathbf{x_1} = v_1$ and $x_i = v_i - v_{i+1}$ followed by introducing the additional state variables $z_1 = e^{-40v_1} - 1$ and $z_i = e^{40x_i}$ can transform equivalently the original system Eq. (28) to a quadratic-bilinear form Eq. (3) with dimension $2n$ Breiten (2013).

In Fig. 1 the original nonlinear system along with the equivalent quadratic-bilinear and the approximated bilinear are depicted. The numerical difference between the original and the quadratic-bilinear has reached machine precision where the bilinear for this amplitude starts to differ significantly.

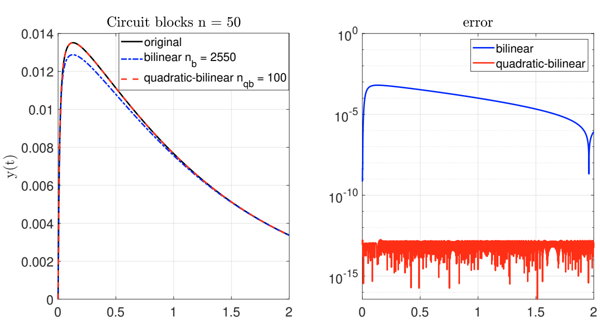

Fig. 1. The original RC-ladder model with $n = 50$ circuit blocks. The lifted QB model is equivalent with the original nonlinear. The bilinear model of dimension n^2+n offers good approximation only for relative small input amplitudes.

The aim is to fit a nonlinear model of low-order by measuring the symmetric Volterra kernels from input-output time-domain simulations. As we infer the operators from the 2nd kernel, a double tone input is considered. The scheme for kernel separation and harmonic indexing is the same as in Karachalios et al. (2021b). By simulating the nonlinear model in the time-domain under the excitation of a double-tone harmonic input, an accurate separation of kernels can be achieved (thanks to the Fourier transform). Here, as we want to illustrate the efficacy of the proposed method by inferring the operators from the 2nd Kernel, we assume a perfect measurement setup.

Step 1: The first harmonic can be measured under excitation with a single tone input. Then, measurements of the first kernel $H_1(s_1)$ (e.g., the magnitude and phase) can be derived. The Loewner framework constructs a low-order rational interpolant and identifies the minimal linear sub-system of order r. In Fig. 2 and on the left pane, the Loewner singular value decay offers the criterion for reduction. The order $r = 10$ was chosen, as the 11th singular value is close to machine precision (this is an indicator for recovering the original linear dynamics). Denote with $\Sigma_{\mathtt{lin}} : (\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ the linear reduced system of order $r = 10$.

In Fig. 2, the approximation results are depicted.

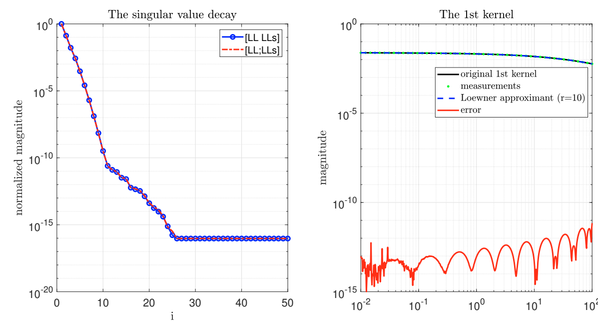

Fig. 2. Left pane: The Loewner singular value decay (used to decide on the reduced order). Right pane: The reduced-order Loewner interpolant computed from the 1st kernel that can reach machine precision approximation.

Step 2: By having access to the reduced matrices of the linear system $\hat{\Sigma}_{\mathtt{lin}}$, we can infer the remaining nonlinear quadratic and bilinear operators from Eq. (18). The second harmonic can be measured with a double tone excitation. Thus, measurements of the second kernel $H_2(s_1, s_2)$ over the whole complex domain of definition can be collected repetitively. It is important here to mention that the amount of measurements is related to the reduced dimension r, where at least $K \geq (r^3+r^2)$ measurements ensures enough data, for the solution of Eq. (19). By enforcing quadratic symmetries (the matrix $\hat{\mathbf{Q}}$ is set to satisfy the property $\hat{\mathbf{Q}}(\mathbf{w} \otimes \mathbf{v}) = \hat{\mathbf{Q}}(\mathbf{v} \otimes \mathbf{w})$), the complexity can be further reduced. Using some algebraic adjustments, another simplification can be performed by replacing the symmetric Kronecker product with the asymmetric one ($\otimes'$ as in Benner et al. (2020b)). Solving for the vector $\mathbf{z}$ yields a reduced-order QB system $\hat{\Sigma}_{QB} = (\hat{\mathbf{A}}, \hat{\mathbf{Q}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$. Finally, in Fig. 4 the time-domain solution is compared to

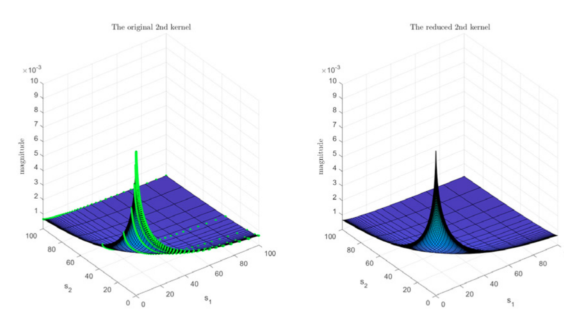

Fig. 3. Left pane: The original 2nd kernel along with the measurements (green dots). Right pane: The 2nd kernel of the reduced system. $\|H_2(s_1, s_2) - \hat{H}_2(s_1, s_2)\|_\infty \sim 10^{-8}$.

the original nonlinear response.

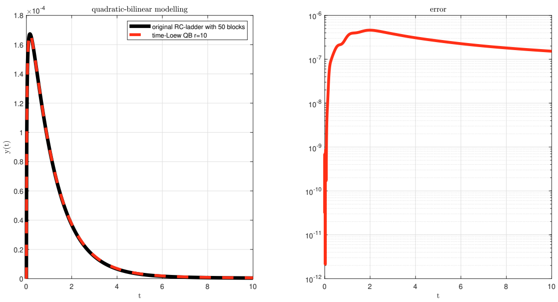

Fig. 4. Left pane: The original nonlinear RC-ladder along with the approximant that is constructed with the new LoewQB method. Right pane: The absolute error over a dense grid.

The proposed method successfully constructs a reduced quadratic-bilinear model from input-output time-domain data that accurately approximates the response of the original nonlinear system. The simulations are performed with the Runge Kutta method (e.g., using `ODE45` in Matlab) with input $u(t) = 0.01e^{-t}$, and the maximum error is $\|(y(t) - y_r(t))\|_\infty \sim 10^{-6}$.

6. CONCLUSION AND FUTURE ENDEAVOURS

A data-driven method that constructs nonlinear QB models from input-output time-domain data was presented. The new approach is based on the Loewner and Volterra frameworks and the equivalence between the nonlinear systems and their QB representation. The 2nd symmetric kernel can be inferred as the 2nd harmonic; hence, the problem of estimating the nonlinear operators can be resolved into a linear LS system. Although this LS system may be under-determined, it contains enough information for evaluating the nonlinear operators with the Moore-Penrose pseudo-inverse, thus providing good approximations. The use of higher harmonics (e.g., 3rd kernel, etc.) that results in a nonlinear optimization problem, along with the use of other regularization techniques as the one proposed by Tikhonov, will be the topics of future research in connection with the analysis of noisy data.

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