Bifurcation to Traveling Spots in Reaction-Diffusion Systems

K. Krischer[1] and A. Mikhailov[1,2]

[1]*Fritz-Haber-Institut der Max-Planck-Gesellschaft, Faradayweg 4-6, 14195 Berlin (Dahlem), Germany*

[2]*N.N. Semenov Institute for Chemical Physics, Russian Academy of Sciences, ul. Kosygina 4, 117333 Moscow, Russia*

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A bifurcation leading to the onset of translational motion of localized particlelike structures (spots) in two-dimensional excitable media with long-range inhibition and global coupling is analytically and numerically investigated. Properties of slowly traveling spots and effects of collisions between these objects are studied.

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Recent experiments and numerical simulations [1–4] brought attention to the phenomena of pattern formation in two-dimensional reaction-diffusion systems which involve localized particlelike structures (spots). A spot represents a region filled with activator and surrounded by a larger inhibitor cloud. Localized one-dimensional patterns (standing and moving pulses or filaments) have been also observed in semiconductor materials [5,6], in gas discharge phenomena [7], and in chemical systems [8]. Explicit analytical solutions for stationary and moving pulses [9–11] and for stationary two-dimensional spots [12] have been constructed, and instabilities of one-dimensional localized patterns [10,13] and the onset of breathing and static deformations of two-dimensional spots [12] have been analyzed.

The aim of the present Letter is to investigate the spontaneous onset of translational motion of stationary two-dimensional spots. This instability may be viewed as an analog of the meandering transition for spiral waves (see, e.g., [14]) or mobility onset for defects in coupled chaotic map lattices [15], which play an important role in the transition to developed reaction-diffusion turbulence. We find that the bifurcation is supercritical only in the presence of sufficiently strong global coupling. Near such a bifurcation an excitable two-dimensional medium with long-range inhibition exhibits a special kind of pattern—localized steadily moving spots filled with activator and surrounded by a large cloud of the inhibitor. As the bifurcation point is approached, their velocity gradually decreases and they transform into stable static spots.

We consider a reaction-diffusion system described by the equations:

$$\dot{u} = -u + H(u - a) - v + \nabla^2 u,$$
$$\tau\dot{v} = \mu u - v + L^2\nabla^2 v, \qquad (1)$$

where $H(z)$ is the step function, $H(z) = 0$ for $z < 0$ and $H(z) = 1$ for $z > 0$. This system represents a variant of the Rinzel-Keller model [16] with additional diffusion of the inhibitor v. We study the case when the diffusion length L of the inhibitor is much larger than the diffusion length of the activator u, which is equal to unity in our notation. Therefore, we choose $L = \varepsilon^{-1}$ and $\tau = (\varepsilon\sigma)^{-1}$, where $\varepsilon \ll 1$ and $\sigma = L/\tau$ is an independent parameter. The system has a stable uniform stationary state ($u = v = 0$) with the excitation threshold a. Global coupling is introduced similar to Refs. [8] and [17] by assuming that the threshold a depends on the total activator and inhibitor concentrations in the medium, i.e., that $a = a_0 + \alpha(s - s_0)$ where

$$S = \int (u + v)\, d\mathbf{x}, \qquad (2)$$

and α is a positive coefficient.

We look first at the one-dimensional case where stationary and propagating pulse solutions could be explicitly constructed using the singular perturbation approximation [10,11]. We generalized this construction in order to include the effects of global coupling. Figure 1 shows the obtained bifurcation diagram for localized stationary and traveling solutions of Eq. (1) in the limit $\varepsilon \to 0$. As long as parameter $\sigma = L/\tau$ is large, a stationary pulse (with velocity $c = 0$) is the only nontrivial stable solution. When σ is decreased, the stationary solution loses its stability and at $\sigma = 0.179$ it gives rise to two traveling pulses with opposite velocities. In the absence of global coupling the bifurcation is subcritical so that moving and standing pulses can coexist in a certain parameter interval (cf. [11]). Sufficiently strong global coupling makes it supercritical (it also suppresses the breathing instability of pulses not shown in Fig. 1). When the bifurcation is supercritical, the velocity of pulses is vanishingly small at the critical point and their profile is close to that of a stationary pulse.

Standing one-dimensional pulses correspond to stationary spots in two-dimensional media. However, not every moving pulse gives rise to a traveling spot in two dimensions. To suppress spots from blowing up into spatially extended patterns (i.e., into propagating excitation waves), sufficiently strong global inhibitory coupling is required.

A mathematical method for the stability analysis of two-dimensional localized patterns in excitable media with long-range inhibition has been proposed by Ohta *et al.* [12]. Since the boundary of the excited domain, forming the core of the spot pattern, is narrow when the parameter ε in Eq. (1) is small, its width can be neglected and it can be represented approximately by a

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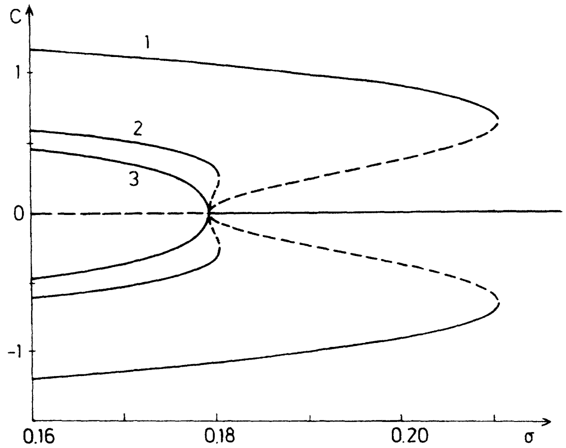

FIG. 1. Velocity of traveling pulse solutions of Eq. (1) as a function of σ in the limit of $\varepsilon \rightarrow 0$ for the one-dimensional case. Curves 1, 2, and 3 correspond to different coupling strengths ($\alpha = 0$, 0.1, and 0.2, respectively). Dashed lines indicate unstable solutions. A stationary pulse ($c = 0$) is stable for $\sigma > 0.179$.

single closed curve Γ. Since the boundary of the core is essentially an activator front, its motion is controlled by the local inhibitor concentration. The activator is produced inside the region bounded by curve Γ but can diffuse out of this region. Because the second equation in model (1) is linear, a closed dynamical equation for the motion and time-dependent deformations of the curve Γ could be constructed. Because of inhibitor diffusion, it contains retarded interactions between distant elements of the boundary curve. The stability analysis is thus reduced to the investigation of small perturbations of the steady solution of this equation describing a stationary spot. This method was used to investigate the stability of stationary spots with respect to their radial breathing and asymmetric static deformations [12]. We have applied it to analyze the instability leading to translational motion of spots in the presence of global coupling.

Omitting the derivation, which is performed following the steps outlined above in close analogy to Ref. [12], we find that the increment z of growth of perturbations representing a small shift of the activator core is given by the root of the equation $F(z) = 0$ where

$$F(z) = \sigma z - \xi\rho[I_1(\rho)K_1(\rho) - I_1(\rho\sqrt{1+z})K_1(\rho\sqrt{1+z})]. \tag{3}$$

Here $I_n(x)$ and $K_n(x)$ are the modified Bessel functions, $\rho = (1+\mu)^{1/2}R$ is the rescaled radius R of the stationary spot, and the coefficient ξ is approximately given by $\xi = 4\mu(1+\mu)^{-3/2}$. The radius is determined by the same set of matching equations as in Ref. [12] with the only change that in the presence of global coupling the threshold a in these equations becomes a function of the radius, i.e., $a = a_0 + \alpha(s - s_0)$, where s is the total excited area [when only one spot is found in the medium, $s = \pi\rho^2(1+\mu)^{-1}$]. Note that the radius of the spots does not depend on the parameter σ.

Equation $F(z) = 0$ always has a root $z = 0$ due to the invariance with respect to translations of the whole localized solution. A nontrivial perturbation mode consists of a shift of the activator core against a fixed background of the inhibitor distribution. At the onset of translational motion, the real root z associated with this mode changes its sign and becomes positive. Thus, the function $F(z)$ must have a doubly degenerate root $z = 0$ at the critical point. Requiring that $dF/dz = 0$, we find that stationary spots become absolutely unstable with respect to the onset of translational motion for $\sigma < \sigma_c$ where

$$\sigma_c = \frac{1}{4\xi\rho}\{I_1(\rho)[K_0(\rho) + K_2(\rho)] - K_1(\rho)[I_0(\rho) + I_2(\rho)]\}. \tag{4}$$

For the system without global coupling, Fig. 2 shows the boundary for the onset of translational motion given by Eq. (4) and the boundaries of instabilities with respect to breating and static deformations of the spot derived in Ref. [12]. We see that upon decreasing the control parameter σ the bifurcation leading to the onset of translational motion is preceded by the breathing instability of stationary spots. When the excitation threshold a is decreased at fixed σ, the radius of stationary spots grows, and they undergo static deformations (which may lead to the development of labyrinthine patterns, cf. [18]).

Note that the excitation threshold $a = a_0 + \alpha(s - s_0)$ is not changed by any transformation that conserves the total excited area s given by (2). If s_0 is chosen equal to the value of s for a stationary spot without global coupling, stationary spots under strong global coupling have the same shape and radius as in its absence (i.e., for $\alpha = 0$). At the bifurcation leading to the onset of translational motion, the spots are moving slowly (with $c \rightarrow 0$) and have practically the same shape and total excited area as the stationary ones. Therefore, the boundary of this bifurcation (given by curve t in Fig. 2) is not sensitive to global coupling when this choice is made. However, it stabilizes slowly moving (and stationary) spots by preventing their blowup.

We performed numerical simulations of spots in the vicinity of the boundary t of the bifurcation to translational motion for different values of the global coupling coefficient α. The numerical simulations of spots in model (1) meet certain difficulties. They are caused by a large difference in the diffusion lengths of activator and inhibitor species (in our simulations the ratio of two diffusion lengths was $\varepsilon = 0.125$). When a standard forward Euler scheme is chosen, this would lead to enormous computation times. The computational speed can be much enhanced if the grid size of the inhibitor is adjusted, so that the resolutions of activator and inhibitor are equal with respect to their characteristic lengths. Moreover, we speed up the simulations by using different time

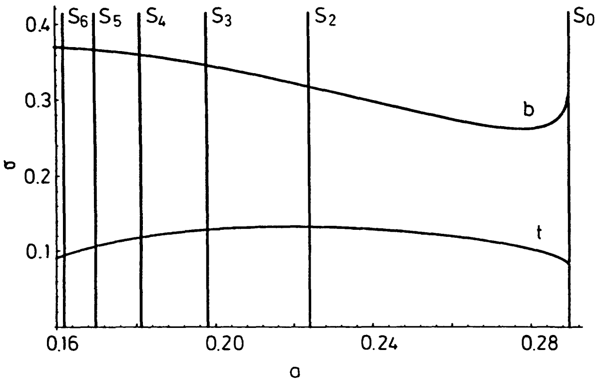

FIG. 2. Bifurcation diagram in the plane (a, σ) for the two-dimensional case without global coupling, $\varepsilon = 0.125$. Stationary spots exist to the left of the vertical boundary s_0. A stationary spot is unstable with respect to the breathing instability below boundary b; it is unstable with respect to static deformations of angular symmetry $n = 2, 3, 4, 5, \ldots$ to the left of the vertical boundaries s_n. Curve t gives the boundary of the instability resulting in translational motion.

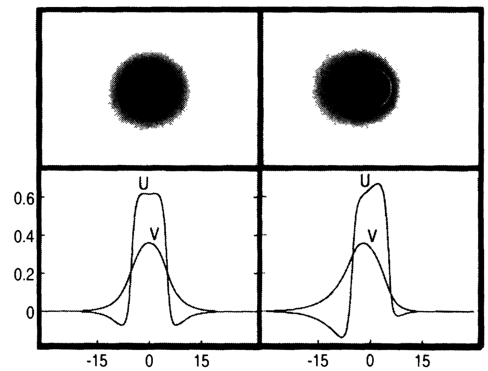

FIG. 3. Gray-level representation and profiles of a stationary spot (left, $\sigma = 1.0$) and a traveling spot close to the onset of translational motion (right, $\sigma = 0.0975$); other parameters are $\mu = 3.0$, $L = 8.0$, $a_0 = 0.275$, $\alpha = 5.0$, and $s_0 = 0.03$. The gray-scale representations show the distribution of the inhibitor; the white contours show the activator interface.

steps for the integration of u and v, each time step given by the stability boundary of the numerical scheme. In this way not only do we reduce the number of grid points for the inhibitor field by nearly 2 orders of magnitude, it also allows us to use $10^2 - 10^3$ times larger time steps. To obtain the values of v at grid points and time steps at which they are not calculated explicitly, linear interpolation is applied. A further increase in the speed of calculations is achieved by altering the original variable u to the new variable $\overline{u} = u + v$ in Eq. (1). It could be shown that this combination (exponentially) vanishes outside the spot. This means that we can use in this region an "active" procedure for the calculation of the Laplacian proposed by Barkley [19]; i.e., we do not compute it whenever the value of $\overline{u}$ falls below a preset value (which was 10^{-4} in our calculations).

Our simulations demonstrate that breathing instability is suppressed by strong global coupling and that the system exhibits a supercritical bifurcation leading to the onset of translational motion of spots. Figure 3 shows (i) a stationary spot and (ii) a slowly moving spot slightly below the critical point $\sigma_c = 0.108$ for $\alpha = 5.0$. We see that the transformation to moving spots is accompanied by a shift of the activator domain (white contour in Fig. 3) with respect to the inhibitor cloud, which makes the pattern asymmetric. The presence of strong global coupling is essential for the existence of traveling spots: If we gradually decrease the coupling coefficient α while keeping other parameters fixed, the moving spot expands, elongates in the transverse direction, and finally transforms into an extended propagating wave pattern (Fig. 4).

Steadily moving spots obtained as a result of the bifurcation represent a special type of spatiotemporal pattern in two-dimensional excitable media. We have numerically investigated collisions between such spots. Close to the bifurcation point, elastic scattering of slowly moving spots has been found [Fig. 5(a)]. The spots approach one another, slow down, change the direction of their motion, and move away. Characteristically, the regions filled with the activator do not come into contact in such a collision. Bouncing results from an increase of the inhibitor concentration in the region between two colliding activator cores. Under the same conditions, slowly moving spots were found in our simulations to scatter from no-flux boundaries of the medium.

Frontal collisions between faster moving spots found farther away from the bifurcation point [Fig. 5(b)] lead to the formation of a single bound state where the cores of two spots are fused together. After a while this bound state breaks apart and gives rise to spots which move away in a direction perpendicular to the original direction of motion of the particles. In contrast to this, an oblique collision [Fig. 5(c)] ends with the formation of one larger stable moving spot; i.e., the collision between the particles is in this case inelastic. The effect of inelastic collisions demonstrates also the multiplicity of traveling spot solutions: instead of a few smaller spots the medium

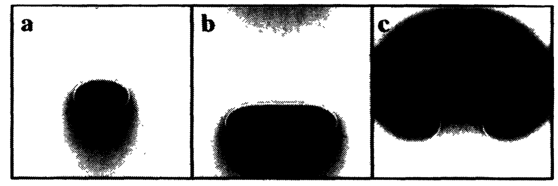

FIG. 4. Transition from a localized moving spot to an extended propagating wave pattern as the coupling strength is decreased: (a) $\alpha = 5.0$, (b) $\alpha = 1.0$, (c) $\alpha = 0$; $\sigma = 0.05$; other parameters are the same as in Fig. 3. For the activator (inhibitor) field, 400×400 (50×50) grid points were used.

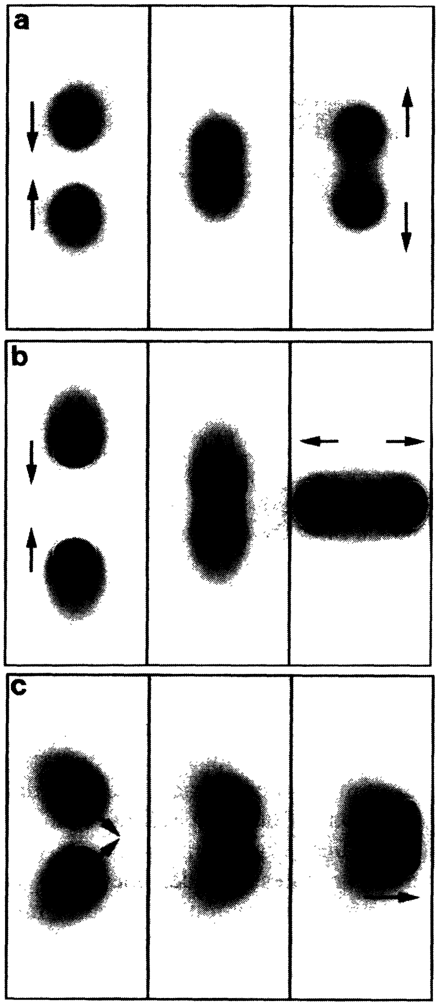

FIG. 5. Collisions between moving spots. (a) Close to the onset of translational motion ($\sigma = 0.0975$), colliding spots are elastically scattered. (b) Frontal collision of fast-moving spots ($\sigma = 0.05$) leads to a 90° change of the direction of motion. (c) Oblique collision of fast-moving spots ($\sigma = 0.05$) results in a single-moving spot of a larger size. The parameters were chosen as in Fig. 3, but $s_0 = 0.06$ (activator field: 800×400, inhibitor field: 100×50).

may form a larger spot, providing that the total excited area s remains approximately constant. When inelastic scattering is possible, a moving spot can stick to a no-flux boundary and continue motion along it.

Though our study has been confined to a particular model, we believe that its results illustrate generic properties of excitable media with long-range slow inhibition and global coupling. Traveling spots may represent fundamental elementary patterns in such distributed dynamical systems, as typical as spiral waves in more usual excitable media with short-range slow inhibition. Remarkably, a population of traveling spots would behave as a reactive gas of self-moving particles. A challenge is to construct the statistical mechanics and "chemistry" of these objects.

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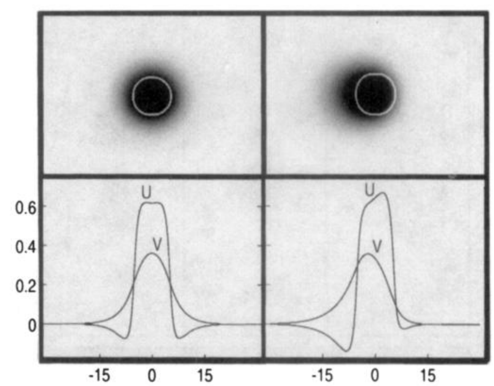

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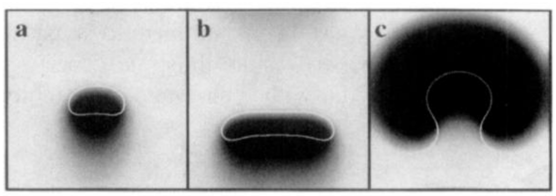

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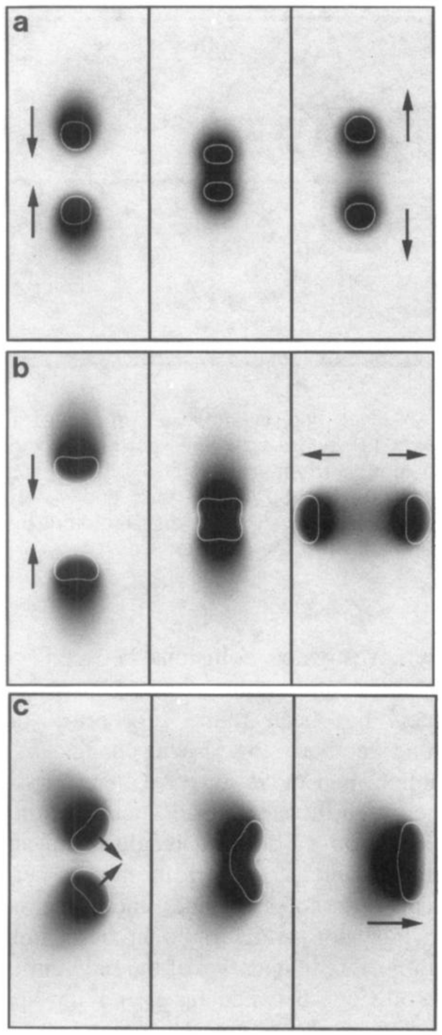

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