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Model-order reduction for hyperbolic relaxation systems

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Abstract: We propose a novel framework for model-order reduction of hyperbolic differential equations. The approach combines a relaxation formulation of the hyperbolic equations with a discretization using shifted base functions. Model-order reduction techniques are then applied to the resulting system of coupled ordinary differential equations. On computational examples including in particular the case of shock waves we show the validity of the approach and the performance of the reduced system.

Keywords: hyperbolic relaxation systems; model order reduction; numerical methods.

1 Introduction

Model-order reduction has been successfully applied to large-scale systems of ordinary differential equations as well as problems governed by elliptic or parabolic differential equations, see, e.g., [1–7]. There is a large variety of methods out there for these problem classes, and all of them base on the idea that the solution space as a subset of either a large finite dimensional space, or possibly an infinite dimensional function space is well approximated by a finite dimensional linear subspace of relatively low dimension. There are several different methods to determine a suitable subspace and several methods to use it for a reduced order model. Some model order reduction methods only take the description of the system, to create the projection onto that subspace, and some use data created from solving the full system. A crucial point in the interest and usefulness of a reduced model is, that one is not interested in one single solution for one single equation, but for a collection of solutions or equations. Sometimes this collection is obtained by a parameter in the differential equation, sometimes by a varying input function, or by considering different starting values.

A way to quantify how reducible an equation is can be done by understanding, how well the solution space is approximated by the best n-dimensional linear subspace. This concept is referred to as the Kolmogorov n-width in the literature. This is also studied for specific hyperbolic problems and the best approximation space in this setting is not satisfying. Therefore, we need to rethink the general strategy for **nonlinear** hyperbolic problem. So far a general method is not available. Several approaches have been proposed to provide a suitable finite dimensional approximation space. In particular, in the case of linear hyperbolic system the solution can be expressed as a linear semigroup on suitable spaces. Then, an approximation by finite dimensional subspaces is feasible [8–12]. For linear hyperbolic problems the transport speed is constant and known *a priori*. This allows to exploit the idea of shifted base functions. Several different approaches exist and they have partially been extended to the nonlinear case [4, 8, 11, 13]. In the nonlinear case, a major obstacle has been the loss of regularity of the solutions in the presence of shocks. Those also move at a speed determined through a possibly nonlinear relation out of the solution itself. This time-dependence in the approximate finite

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dimensional space has been dealt with by time dependent space transformation as part of the reduced model. There is a large body of literature addressing different solutions to this well established problem [14–20]. They all use very different methods to create a possibly nonlinear subspace approximating the solution space.

We propose a method to treat loss of regularity due to shocks as well as the nonlinear transport speed. To that end, we first lift the solution space and then find a linear subspace exploiting known techniques. The lifting is done in two steps, first a hyperbolic relaxation [21–23] and then a discretization using suitable spacetime Ansatz functions. The hyperbolic relaxation methods use a suitable reformulation of the nonlinear flux at the expense of an enlarged system. This in turn allows keeping possible discontinuous solutions, but reduces the transport part to a linear transport. The linear part ensures further that the new system formally has fixed transport speeds. The latter system is therefore amendable for treatment within model order reduction as shown in this work. We propose to capture the movement of discontinuities by suitable *moving* approximations. On those approximations we perform a suitable model order reduction. Based on the continuous formulation, we discuss possible numerical discretizations and show computational results in the case of shocks.

2 Reducibility of scalar nonlinear hyperbolic equations

We consider a scalar nonlinear hyperbolic differential equation for the unknown $U = U(t,x)$ on the torus $\mathbb{T} = [-1,1] \subset \mathbb{R}$ as solution to

$$\partial_t U(t,x) + \partial_x f(U(t,x)) = 0 \tag{1}$$

subject to the initial conditions $u_0 \colon \mathbb{T} \to \mathbb{R}$

$$u(0,x) = u_0(x). \tag{2}$$

The flux function $f \in C^2(\mathbb{R};\mathbb{R})$ is assumed to be nonlinear. Even for smooth initial data (2) u_0 the solution u may exhibit discontinuities in finite time [24]. Therefore, weak entropy solutions to (1) have been introduced and we refer to [24] for more details on well-posedness of weak solutions. This presentation is concerned with finding a reduced model to this system in the sense of approximating the solution on a lower dimensional manifold. For (linear) elliptic differential equations the lower dimensional manifold can be shown to be a linear subspace. Then, the model-order reduction can be successfully applied. However, for nonlinear hyperbolic systems this approach is not straight forward. For general nonlinear problems the typical way to create a reduced order model is to first solve the system at certain instances (in time) using a high-dimensional solution technique. This information is used to define a linear subspace of the solution space. This space becomes the search space in which the equation is solved resulting in a so called reduced system. This in turn, is used to approximate the solution for different parameters or input functions. In our setting, we assume the flux function f is given; however, the initial condition u_0 could vary. Therefore, a suitable reduced modeling technique should allow generating a reduced system which is able to approximate the solution to the original equation for different initial conditions.

In order to derive the discretization with the space-time ansatz function on the relaxation, we consider as an example the linear case first where we already have a linear transport operator. Let

$$f(U) = \lambda U \tag{3}$$

with coefficient $\lambda \neq 0$. The explicit solution to (1), (2) on $\mathbb{T}$ is given by

$$U(t,x) = u_0(x - \lambda t). \tag{4}$$

Classical model order reduction of partial differential equations is based on the idea that the numerical solution is computed as an approximation of the type

$$U(t,x) \approx \sum u_j(t)\phi_j(x), \tag{5}$$

for a set of basis functions ϕ_j, like for example finite elements. In general reduced solutions to the PDE are also described in a similar fashion but with different basis functions. Those are chosen in such a way that we do not need so many basis functions. In other words model reduction tries to extract a lower dimensional space which represents the solution of the given problem well. Assuming that we choose u_0 to be a compactly supported local finite element basis function, the solution $U(t,x)$ which is just the transported u_0 has a support that moves through the entire space over time. The collection of these functions evaluated at discrete time instances would fast span a large dimensional space within the finite element space. This leaves not too much hope to find a low dimensional subspace. This has been recognized as a problem for hyperbolic systems for a while [19] and a few techniques have been used to overcome that. The most promising approach being to use an ansatz is where the basis function contains a time dependent spatial shift. For linear problems as the speed is fixed, that can be done in a straightforward manner. However, for nonlinear equations the spatial transformation is part of finding the right reduced system and it is still a work in progress [25]. In this paper, we use the idea of the spatial shift not to create the right reduced order model, but to discretize the full model in order to get a large scale ordinary differential equation that no longer suffers from the transport phenomena. The large dimensional ansatz space is given by a set of basis function ϕ_j but evaluated at a fixed spatial shift:

$$U(t,x) = \sum_{j=1}^{N} u_j(t)\phi_j(x - \lambda t). \tag{6}$$

The initial condition u_0 is expanded in a truncated series of N coefficients

$$u_0(x) = \sum_{j=1}^{N} u_{0,j}\phi_j(x), \tag{7}$$

for some functions $\{\phi_j\}_{j=1}^{N}$. The explicit solution (6) then yields the *exact* solution for $u_j(t) = u_{0,j}$ on the linear transport problem.

If we choose a function $u_0(x)$ as the correct linear combination of a linear subspace as in Eq. (7) and if we consider the solution $u(t,x)$ within the one-dimensional manifold spanned by u_0

$$\{u(x,t) = u_0(x - \lambda t)\} \tag{8}$$

we obtain the exact solution. This approach can be extended to linear transport equations with nonlinear right-hand side, as e.g.,

$$\partial_t u(t,x) + \lambda \partial_x u(t,x) = g(u(t,x)). \tag{9}$$

This approach however does **not** extend to nonlinear equations. It is important to note, that the previous approach only works if λ is constant. However, in the case of nonlinear flux $U \to f(U)$ the characteristic $\frac{\mathrm{d}x}{\mathrm{d}t}(t)$ depends on the value of the initial datum u_0 at x_0:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'\left(U(t,x(t)),\ x(0) = x_0 \text{ and } U(t,x(t)) = u_0(x_0)\right). \tag{10}$$

In the nonlinear case it is challenging to determine the correct shift. There is a large effort in the literature and for certain problems this strategy has been applied successfully [15–18, 20]. In the following, we propose to develop a general method allowing to have a fixed shift in the base functions.

3 Semi-discretization compatible to model order reduction

Our approach is robust with respect to the type of nonlinear flux function and the initial condition used, since we do not track the speed. To this end, a stiff relaxation approximation (11) is considered, e.g., in [21–23, 26–32]. For the scalar problem (1) a relaxation approximation reads

$$\partial_t u(t,x) + \partial_x v(t,x) = 0 \tag{11a}$$

$$\partial_t v(t,x) + \lambda^2 \partial_x u(t,x) = -\frac{1}{\epsilon}(v(t,x) - f(u(t,x))). \tag{11b}$$

Here, $\lambda > 0$ is a positive fixed parameter that fulfills the subcharacteristic condition

$$\lambda \geq \max_{x\in\mathbb{T}} |f'(u_0(x))| \tag{12}$$

and $\epsilon > 0$ is the (small) relaxation parameter. At the expense of an additional variable $v = v(t,x)$ the relaxation system (11) introduces a *linear*, *hyperbolic* approximation to Eq. (1). Using a Chapman–Enskog expansion in ϵ a formal computation shows that

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = \epsilon \partial_x \left((\lambda^2 - f'(u(t,x))^2) \partial_x u(t,x) \right) + O(\epsilon^2). \tag{13}$$

Hence, u given by (11) is a viscous approximation to the solution U of Eq. (1). However, it needs to be pointed out, that (11) is linear hyperbolic and therefore a similar decomposition as shown above might be possible.

The eigenvalues of the linear part in Eq. (11) are λ and $-\lambda$, respectively. For small values of ϵ we expect $v \approx f(u)$ and therefore we set the following initial conditions for (u_0, v_0)

$$u(0,x) = u_0(x) \text{ and } v(0,x) = f(u_0(x)). \tag{14}$$

Diagonalizing system (11) using the variables

$$w^{\pm}(t,x) = v(t,x) \pm \lambda\, u(t,x) \tag{15}$$

and

$$v(t,x) = \frac{1}{2}(w^+(t,x) + w^-(t,x)),\ u(t,x) = \frac{1}{2\lambda}(w^+(t,x) - w^-(t,x)), \tag{16}$$

respectively, yields the following system

$$\partial_t w^+ + \lambda \partial_x w^+ = -\frac{1}{\epsilon}\left(\frac{w^+ + w^-}{2} - f\left(\frac{w^+ - w^-}{2\lambda}\right)\right), \tag{17}$$

$$\partial_t w^- - \lambda \partial_x w^- = -\frac{1}{\epsilon}\left(\frac{w^+ + w^-}{2} - f\left(\frac{w^+ - w^-}{2\lambda}\right)\right), \tag{18}$$

Their corresponding initial conditions are

$$w^+(0,x) = f(u_0(x)) + \lambda u_0(x) \quad w^-(0,x) = f(u_0(x)) - \lambda u_0(x). \tag{19}$$

Following the procedure of the linear case we introduce $\{\phi_j(\cdot)\}_{j=1}^N$ a set of N differentiable functions $\phi_j\colon \mathbb{T} \to \mathbb{R}$ for $j = 1, \dots, N$. The initial data $w_0^{\pm}$ is then expanded using the truncated series

$$w_0^{\pm}(x) = \sum_{j=1}^{N} \alpha_j^{\pm}(0)\phi_j(x), \tag{20}$$

and the solution is expanded using the translated base functions

$$w^+(t,x) \approx \sum_{j=1}^{N} \alpha_j^+(t)\phi_j(x - \lambda t) \text{ and } w^-(t,x) \approx \sum_{j=1}^{N} \alpha_j^-(t)\phi_j(x + \lambda t), \tag{21}$$

respectively. Note that in the case $f(u) = \lambda u$ we in fact have that (21) is exact. However, due to the nonlinearity of the right-hand side of (17) and contrary to the linear case the previous ansatz (21) is in general *not* the exact solution to (17) and (19).

A series expansion of the original variables (u, v) is obtained applying the linear transformation (16). Hence, using ansatz (21) in Eq. (11) we obtain

$$\partial_t u + \partial_x v = \frac{1}{2\lambda}\left(\sum_{j=1}^{N} \dot\alpha_j^+(t)\phi_j(x - \lambda t) - \sum_{j=1}^{N} \dot\alpha_j^-(t)\phi_j(x + \lambda t)\right) = 0, \tag{22}$$

$$\partial_t v + \lambda^2 \partial_x u = \frac{1}{2}\left(\sum_{j=1}^{N} \dot\alpha_j^+(t)\phi_j(x-\lambda t) + \sum_{j=1}^{N} \dot\alpha_j^-(t)\phi_j(x+\lambda t)\right) = -\frac{1}{\epsilon}(v - f(u)). \tag{23}$$

Here, we did not expand v and u in terms of ϕ_j in the right-hand side of Eq. (23) for the sake of readability. Define the family of matrices $t \to M(t) \in \mathbb{R}^{N,N}$ by

$$M_{jk}(t) = \int_{\mathbb{T}} \phi_j(x)\phi_k(x + 2\lambda t)\mathrm{d}x,\ t \geq 0, \tag{24}$$

and the projected initial data $b_k^\pm$ for $k = 1, \dots, N$ as

$$b_k^\pm = \int_{\mathbb{T}} \phi_k(x)\left(f\left(u_0(x)\right) \pm \lambda u_0(x)\right)\mathrm{d}x. \tag{25}$$

Then, the following system for the evolution of the coefficients $\alpha^\pm = \left(\alpha_j^\pm\right)_{j=1}^N$ is obtained

$$M(0)\dot\alpha^+(t) - M(t)\dot\alpha^-(t) = 0 \tag{26}$$

$$M(0)\dot\alpha^+(t) + M(t)\dot\alpha^-(t) = -\frac{2}{\epsilon}\left(\frac{1}{2}\left(M(0)\alpha^+(t) + M(t)\alpha^-(t)\right) - \tilde F(t, \alpha^\pm(t))\right), \tag{27}$$

where $\tilde F = (\tilde F_1, \dots, \tilde F_N)$ and where

$$\tilde F_j(t, \alpha^\pm(t)) := \int_{\mathbb{T}} \phi_j(x) f(\tilde u(t, x + \lambda t))\mathrm{d}x, \tag{28}$$

$$\tilde u(t,x) := \frac{1}{2\lambda}\left(\sum_{j=0}^{N} \alpha_j^+(t)\phi_j(x-\lambda t) - \alpha_j^-(t)\phi_j(x+\lambda t)\right). \tag{29}$$

This is a result of multiplying (23) and (22) by $\phi_j(x - \lambda t)$ for all j and integrating it over x on $\mathbb{T}$. The initial data is given by

$$M(0)\alpha^+(0) = b^+ \text{ and } M(0)\alpha^-(0) = b^-, \tag{30}$$

following from multiplying by basis function and integration. Summarizing, for fixed N and $\epsilon > 0$, the stiff system (26), (27) and (30) determine the coefficients $\alpha^\pm(t)$ and u given by Eqs. (21) and (16), i.e.,

$$u^N(t,x) = \frac{1}{2\lambda}\left(\sum_{j=0}^{N} \alpha_j^+(t)\phi_j(x-\lambda t) - \alpha_j^-(t)\phi_j(x+\lambda t)\right). \tag{31}$$

Note that it is not clear *a priori*, if $M(t)$ for $t \geq 0$ is invertible, and therefore the governing equations are not necessarily an ordinary differential equation, but possibly a differential algebraic equation. This point will be discussed in more detail in the forthcoming section. For the further considerations, assume

$$(Assumption)\ \forall t \geq 0:\ M(t) \text{ is invertible.} \tag{32}$$

Summarizing, under assumption (32) system (26) and (27) with initial conditions (30) yield the approximation (31) to the solution $U = U(t,x)$ of the nonlinear conservation law (1) on $\mathbb{T}$. The proposed approximation (31) contains different approximation errors that have to be addressed in a numerical scheme. First, the solution is projected on the space spanned by the N functions ϕ_j. Since we expect discontinuities, the choice of suitable functions ϕ_j is critical to the approximation error. Second, the derivation shows that u^N given by (31) in fact approximates the relaxation solution u to system (11) for some fixed ϵ. However, analytically, the sequence of weak solution u^ϵ to Eq. (11) converges weakly to the weak solution U to Eq. (1) as $\epsilon \to 0$ [21]. The interplay of the obtained numerical errors with the choice of the parameters ϵ and N will be investigated in the numerical results below.

3.1 Properties of system (26)–(30)

Using the notation $\alpha = (\alpha^+, \alpha^-)$, we obtain

$$\begin{pmatrix} M(0) & -M(t) \\ M(0) & M(t) \end{pmatrix} \frac{d}{\mathrm{d}t}\alpha(t) = -\frac{1}{\epsilon}\begin{pmatrix} 0 \\ [M(0), M(t)]\alpha(t) - 2\tilde{F}(t, \alpha(t)) \end{pmatrix}. \tag{33}$$

The left hand side of Eq. (33) consists of a 2×2 block matrix. This matrix is invertible provided that for all $t \geq 0 M(t)$ is invertible. In this case the inverse is explicitly given by

$$\frac{1}{2}\begin{pmatrix} M^{-1}(0) & M^{-1}(0) \\ M^{-1}(t) & M^{-1}(t) \end{pmatrix} \tag{34}$$

By suitable choice of $\{\phi_j(\cdot)\}_j$ we can guarantee that $M(0)$ is invertible. In fact, if for all $j, k = 1, \dots, N$

$$\int_{\mathbb{T}} \phi_j(x)\phi_k(x)\mathrm{d}x = \delta_{j,k} \tag{35}$$

holds true, then $M(0)$ is the identity matrix. Provided that ϕ_j is continuously differentiable we obtain under assumption (35) that $M(t)$ is invertible for $t > 0$ sufficiently small. Then, we obtain local existence and uniqueness of solutions α. However, the following simple example shows that $M(t)$ is not necessarily invertible for all $t > 0$. Consider $\mathbb{T} = [-1, 1]$, $N = 2$, $2\lambda = 1$, $\phi_1(x) = \sin(x\pi)$ and $\phi_2(x) = \sin(2x\pi)$. Then, $M(0) = Id$ and $M(\frac{1}{2}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

3.1.1 Case of compactly supported translated base functions

Consider a compactly supported function $\phi_0 \colon \mathbb{T} \to \mathbb{R}$. For fixed $\Delta x = \frac{2}{N}$ sufficiently small, define the family of base functions

$$\phi_j(x) := \phi_0\left(x - (j-1)\Delta x\right), \; j = 1, \dots, N. \tag{36}$$

By definition of ϕ_j the base functions fulfill $\phi_j(x) = \phi_k\left(x - (j-k)\Delta x\right)$. For $j = 1, \dots, N$, $k = 2, \dots, N$ we have

$$M_{j,k}(t) = \int_{\mathbb{T}} \phi_j(x)\phi_k(x + 2\lambda t)\mathrm{d}x = \int_{\mathbb{T}} \phi_j(x)\phi_{k-1}(x + 2\lambda t - \Delta x)\mathrm{d}x = M_{j,k-1}\left(t - \frac{\Delta x}{2\lambda}\right) \tag{37}$$

which implies that

$$M(t) = P^k\, M\left(t - k\frac{\Delta x}{2\lambda}\right), \; k = 1, \dots, \text{ and } t \in \left[k\frac{\Delta x}{2\lambda}, (k+1)\frac{\Delta x}{2\lambda}\right]. \tag{38}$$

The permutation matrix P is given by

$$P_{i,\mathrm{mod}(j+1,N)} = \delta_{i,j}, \; i, j = 1, \dots N. \tag{39}$$

Hence, the family of matrices $M(t)$ is for all $t \geq 0$ uniquely defined by $t \to M(t)$ for $t \in [0, \frac{\Delta x}{2\lambda})$. Since ϕ_0 is defined on $\mathbb{T}$, we obtain that $M(t)$ is a circulant matrix, i.e., for $i, j = 1, \dots, N$,

$$M_{i,j}(t) = M_{\mathrm{mod}(i+1,N),\mathrm{mod}(j+1,N)}(t). \tag{40}$$

The family of matrices $M(t)$ is therefore uniquely defined by a family of vectors $\vec{c} = \vec{c}(t) \in \mathbb{R}^N$ with $c_j(t) = M_{1,j}(t)$ for $j = 1, \dots, N$ and $t \geq 0$. For circulant matrices the eigenvalues Λ_M and $m = 0, \dots, N-1$ are

$$\Lambda_m(t) = \sum_{k=0}^{N-1} c_{k+1}(t)\; \exp\left(-2\pi i\, \frac{mk}{N}\right). \tag{41}$$

The explicit eigenvalues (41) determine possible t such that $M(t)$ is not invertible. We illustrate this on two examples. Let $\phi_0(x) = \chi_{[-\frac{\Delta x}{2},\frac{\Delta x}{2}]}(x)$ and let ϕ_j be defined by Eq. (36). Then, there exists $\Delta x > 0$ and N such that the support $\Omega_j := supp_x \phi_j(x)$ fulfills

$$\Omega_i \cap \Omega_j = \emptyset,\ i \neq j, \text{ and } \cup_{j=1}^N \Omega_j = \mathbb{T}. \tag{42}$$

For this choice of $\{\phi_j\}_{j=1}^N$ the vector $c(0) = (\Delta x, 0, \dots, 0)^{\mathrm{T}}$ and $c\left(\frac{\Delta x}{4\lambda}\right) = \frac{\Delta x}{2}(1, 1, 0, \dots, 0)^T$. Hence, if N is even, then $\Lambda_m\left(\frac{\Delta x}{4\lambda}\right) = 0$ for $m = \frac{N}{2}$ and hence $M\left(\frac{\Delta x}{4\lambda}\right)$ is not invertible.

Similarly, if the support of ϕ_0 is of size Δx, i.e., $\phi_0(x) = \chi_{[-\Delta x, \Delta x]}(x)$, then $c(0) = (c_0, c_1, c_2, 0, \dots, 0)^T$ with $c_0 > c_j > 0$, $j = 2, 3$ and $M(0)$ is invertible. However, at time $t = \left(\frac{\Delta x}{4\lambda}\right)$ and $N \geq 4$ even, we obtain $\Lambda_m = 0$ for $m = \frac{N}{4}$. In the following, we discuss properties of the matrix M for the basis functions used in the numerical results later on.

Hence, from now on we assume that ϕ_0 is given by

$$\phi_0(x) = \begin{cases} 2x & x \in [0, \Delta x] \\ 4\Delta x - 2x & x \in [\Delta x, 2\Delta x] \\ 0 & x \notin [0, 2\Delta x] \end{cases} \tag{43}$$

and ϕ_j for $j \geq 1$ are given by (36). There is equivalence if the circulant matrix is singular.

Theorem 3.1. *A circulant matrix obtained from the vector* $c = [c_0, c_1, \dots, c_n]$ *is singular if and only if* $f(x) = \sum_{i=0}^{n-1} c_i x^i$ *and* $1 - x^n$ *have a common zero* [33].

The matrix $M(t)$ resulting from the given basis function is nonsingular almost everywhere. It is only nonsingular at discrete time instances and then there is only one zero eigenvalue:

Theorem 3.2. *The matrix* $M(t)$ *given by* (37) *for* ϕ *given by* (43) *is nonsingular on the interval* $[0, \frac{1}{\lambda N}]$ *as long as* $t \neq t^* = \frac{1}{2\lambda N}$ *and the nullspace at* t^* *is only one-dimensional.*

Proof. In order to proof that the matrix is nonsingular we apply Theorem 3.1. The matrix is a circulant matrix composed of the vector $c = [c_1, \dots, c_N]$, where $c_j = \int_{\mathbb{T}} \phi_1(x)\phi_j(x + 2\lambda t)\mathrm{d}x$. In the given interval we have $c_j = 0$ except for c_1, c_2, c_3, c_N. It is well known that the circulant matrix composed by $c_1, \dots c_N$ has up to sign the same determinant as $c_N, c_1, \dots c_{N-1}$. Therefore we can consider this matrix instead. Hence, the polynomial is given by

$$c_N + c_1 x + c_2 x^2 + c_3 x^3.$$

Next, we show that no root of unity is a zero of that polynomial except at time $t = t^*$. □

Lemma 3.3. *For* $c_j(t) = \int_{\mathbb{T}} \phi_1(x)\phi_j(x + 2\lambda t)\mathrm{d}x$ *the polynomial* $p(x, t) = c_N(t) + c_1(t)x + c_2(t)x^2 + c_3(t)x^3$ *has only a root of unity if* $t = t^*$.

Proof. Assume that ω is a root of unity and also a root of $p(x, t)$. Then ω is either complex, equal to 1 or −1. However, $\omega = 1$ cannot be a root of p as all c_j are positive. If $\omega = -1$ is a root we have that $c_N(t) - c_1(t) + c_2(t) - c_3(t) = 0$. It is straightforward by the definition of c_j to show that $c_N(0) - c_1(0) + c_2(0) - c_3(0) < 0$ and $c_N(\frac{1}{\lambda N}) - c_1(\frac{1}{\lambda N}) + c_2(\frac{1}{\lambda N}) - c_3(\frac{1}{\lambda N}) > 0$. Further, the derivative is positive in the given interval and therefore it has exactly one zero in this interval. This is at $t = t^*$. If ω is complex, then also $\bar{\omega}$ has to be a root of $p(x, t)$ and then we obtain

$$p(x, t) = (x - \omega)(x - \bar{\omega})(\alpha + \beta x)$$

for some β and α. Comparing the coefficients we get that

$$c_1 = \beta - 2\alpha\Re(\omega) \tag{44}$$

$$c_2 = \alpha - 2\beta\Re(\omega) \tag{45}$$

$$c_3 = \beta \tag{46}$$

$$c_N = \alpha \tag{47}$$

and we obtain

$$\Re(\omega) = \frac{c_3 - c_1}{2c_N} = \frac{c_N - c_2}{2c_3}.$$

This fraction is always less or equal to -1 and therefore ω can only be -1 which has been treated before. □

3.1.2 Differential algebraic nature of system (33)

As discussed in the previous section $M(t)$ could be singular for base functions fulfilling (42). For the choices discussed above $M(t)$ is singular only at a single point in time t^* within the interval $\left[0, \frac{\Delta x}{2\lambda}\right]$, i.e., for the last example $t^* = \frac{\Delta x}{4\lambda}$. Furthermore, there exists a vector e such that $M(t^*)e = 0$ and for all vectors v orthogonal to that we have $M(t^*)v \neq 0$ unless $v = 0$

Let V, W be the $N \times (N-1)$ dimensional orthogonal matrices and f the vector orthogonal to W such that $W^T M(t)V$ is invertible and $f^T M(t)V = 0$. Then, decompose α^- into

$$\alpha^-(t) = \alpha_0^-(t)e + V\bar{\alpha}^-(t). \tag{48}$$

For $\vec{\beta} = \left(\alpha^+, \bar{\alpha}^-, \alpha_0^-\right)$ problem (33) reads

$$\begin{bmatrix} M(0) & -M(t)V & -M(t)e \\ W^T M(0) & W^T M(t)V & W^T M(t)e \\ f^T M(0) & 0 & f^T M(t)e \end{bmatrix} \frac{d}{\mathrm{d}t}\vec{\beta}(t) = \frac{1}{\epsilon}\begin{pmatrix} 0 \\ [M(0), M(t)]\vec{\beta}(t) - 2\tilde{F}(t, \vec{\beta}(t)) \end{pmatrix}. \tag{49}$$

This system is not an ordinary differential equation at $t = t^*$, since $M(t^*)e = 0$. The resulting system is a semi–explicit differential algebraic equation. We introduce a small parameter $\rho > 0$ and regularize Eq. (49) by

$$\begin{bmatrix} M(0) & -M(t)V & -M(t)e \\ W^T M(0) & W^T M(t)V & W^T M(t)e \\ f^T M(0) & 0 & f^T M(t)e + \rho \end{bmatrix} \frac{d}{\mathrm{d}t}\vec{\beta} = \frac{1}{\epsilon}\begin{pmatrix} 0 \\ [M(0), M(t)]\vec{\beta}(t) - 2\tilde{F}(t, \vec{\beta}(t)) \end{pmatrix}. \tag{50}$$

or in terms of α, we have

$$\begin{bmatrix} M(0) & -M(t) \\ M(0) & M(t) + \rho f e^T \end{bmatrix} \frac{d}{\mathrm{d}t}\alpha = \frac{1}{\epsilon}\begin{pmatrix} 0 \\ [M(0), M(t)]\alpha(t) - 2\tilde{F}(t, \alpha(t)) \end{pmatrix}. \tag{51}$$

For $\rho > 0$ the matrix is invertible and its inverse is given by

$$\begin{bmatrix} M(0)^{-1}(I + M(t)(2M(t) + \rho f e^T)^{-1}) & M(0)^{-1}M(t)(2M(t) + \rho f e^T)^{-1} \\ -(2M(t) + \rho f e^T)^{-1} & (2M(t) + \rho f e^T)^{-1} \end{bmatrix}. \tag{52}$$

3.2 Temporal discretization and model order reduction

Fix a positive parameter $\rho > 0$ and consider system (51) subject to initial conditions (30). Consider a temporal grid $t^n = \Delta t\, n$ for $n = 0, \ldots,$ where for simplicity we consider an equi-distant grid in time. Denote by $\alpha_n^\pm = \alpha^\pm(t^n)$. Furthermore, denote by

$$N(t) := M(t) + \rho f e^T.$$

We rewrite (51) as

$$\frac{d}{dt}(M(0)\alpha^+ - M(t)\alpha^-) = -\dot{M}(t)\alpha^-, \tag{53}$$

$$\frac{d}{dt}(M(0)\alpha^+ + N(t)\alpha^-) = \dot{M}(t)\alpha^- - \frac{2}{\epsilon}\left(\frac{1}{2}(M(0)\alpha^+ + M(t)\alpha^-) - \tilde{F}(t, \alpha^\pm(t))\right), \tag{54}$$

where $\tilde{F}$ is given by Eq. (28) and $\dot{M}_{i,j}(t) = \int_{\mathbb{T}} \phi_j(x)\phi_k'(x + 2\lambda t)2\lambda \mathrm{d}x$. An implicit discretization of (54) is preferable to resolve small scales of ϵ. Since the term $\tilde{F}$ is an integral term in both α^+ and α^- a fully implicit discretiaztion is computationally too costly. We therefore proceed using a semi-implicit discretization, i.e.,

$$\left(M_0\alpha^+_{n+1} - M_{n+1}\alpha^-_{n+1}\right) - \left(M_0\alpha^+_n - M_n\alpha^-_n\right) = -\Delta t\dot{M}_n\alpha^-_n \tag{55}$$

$$\left(M_0\alpha^+_{n+1} + N_{n+1}\alpha^-_{n+1}\right) - \left(M_0\alpha^+_n + N_n\alpha^-_n\right) = \Delta t\dot{M}_n\alpha^-_n - \frac{\Delta t}{\epsilon}\left(M_0\alpha^+_{n+1} + M_{n+1}\alpha^-_{n+1} - \tilde{F}_n\right), \tag{56}$$

$$\tilde{F}_n = 2\tilde{F}\left(t^n, \alpha^\pm_n\right), \tag{57}$$

leading to the following system

$$\begin{bmatrix} M_0 & -M_{n+1} \\ M_0 & N_{n+1} \end{bmatrix} \begin{bmatrix} \alpha^+_{n+1} \\ \alpha^-_{n+1} \end{bmatrix} = \begin{bmatrix} M_0\alpha^+_n - M_n\alpha^-_n - \Delta t\dot{M}_n\alpha^-_n \\ \frac{\epsilon}{\epsilon + \Delta t}\left(M_0\alpha^+_n + N_n\alpha^-_n + \Delta t\dot{M}_n\alpha^-_n\right) + \frac{\Delta t}{\epsilon + \Delta t}\tilde{F}_n \end{bmatrix} \tag{58}$$

As in Eq. (33) the left-hand side of Eq. (58) consists of a 2×2 block matrix

$$R_{n+1} := \begin{bmatrix} M_0 & -M_{n+1} \\ M_0 & N_{n+1} \end{bmatrix}$$

which is invertible provided that $M(0)$ is invertible and ρ is non-negative. In this case its inverse is given by Eq. (52) evaluated at $t = t^n$. Furthermore, $\tilde{F}_n$ and $\dot{M}_n = \dot{M}(t^n)$ needs to be discretized using a numerical quadrature formula of sufficient high-order. Note that the previous formulation can be formally evaluated for all values of ϵ (even $\epsilon = 0$). However, since $\tilde{F}_n = \tilde{F}\left(t^n, \alpha^+_n, \alpha^-_n\right)$ the previous scheme requires a time step restriction of the type

$$\Delta t \le C\epsilon \tag{59}$$

for some constant C to be stable. Clearly, this leads to small steps for sufficiently small ϵ. The only way to circumvent this restriction is to discretize $\tilde{F}$ implicit. Since our focus is on the model order reduction for system (58), we leave the efficient computation of the fully implicit scheme for future investigation. For the sake of completeness we also state the alternative fully explicit discretization as

$$\left(M_0\alpha^+_{n+1} - M_{n+1}\alpha^-_{n+1}\right) - \left(M_0\alpha^+_n - M_n\alpha^-_n\right) = -\Delta t\dot{M}_n\alpha^-_n \tag{60}$$

$$\left(M_0\alpha^+_{n+1} + N_{n+1}\alpha^-_{n+1}\right) - \left(M_0\alpha^+_n + N_n\alpha^-_n\right) = \Delta t\dot{M}_n\alpha^-_n - \frac{2\Delta t}{\epsilon}\left(\frac{1}{2}\left(M_0\alpha^+_n + M_n\alpha^-_n\right) - \tilde{F}_n\right) \tag{61}$$

$$\tilde{F}_n = \tilde{F}\left(t^n, \alpha^\pm_n\right). \tag{62}$$

The same restriction (59) applies for this discretization.

Note that the original scheme [22] does **not** require a time step restriction of the order of ϵ. This has also been exploited numerically in so-called implicit–explicit schemes (IMEX) where the stiff term has been discretized implicitly, see, e.g., [31, 34]. However, as in system (11) it is crucial that the nonlinearity is linear in the underlying variable, here v. This allows also in the numerical scheme to have an explicit scheme without time-step restriction even in the stiff case. On the contrary, after expansion in a series system (51) is nonlinear in the coefficients α. Hence, an implicit or IMEX discretization cannot be solved analytically. So far, the basis functions ϕ, however couple the coefficients and this coupling prevents an analytically evaluation.

3.3 Projection based model order reduction for system (26) and (27)

The previous formulation (58) is amendable for model order reduction. Hence, we approximate $\alpha^{\pm}(t) \in \mathbb{R}^N$ within a lower dimensional linear subspace of $\mathbb{R}^N$, meaning there exists V_+ and V_- such that $\alpha_{\pm}(t) \approx V_{\pm}V_{\pm}^T\alpha^{\pm}(t)$ and therefore an $\hat{\alpha}^{\pm}(t)$ exists such that $\alpha^{\pm}(t) \approx V_{\pm}\hat{\alpha}^{\pm}(t)$.

Using this approximation we get the following

$$M(0)V_+\dot{\hat{\alpha}}^+(t) - M(t)V_-\dot{\hat{\alpha}}^-(t) = 0 \tag{63}$$

$$M(0)V_+\dot{\hat{\alpha}}^+(t) + M(t)V_-\dot{\hat{\alpha}}^-(t) = -\frac{2}{\epsilon}\left(\frac{1}{2}\left(M(0)V_+\hat{\alpha}^+(t) + M(t)V_-\hat{\alpha}^-(t)\right) - \tilde{F}(t, V_{\pm}\hat{\alpha}^{\pm}(t))\right), \tag{64}$$

which is then projected to get a system of ordinary differential equation in a lower dimension. We use a Galerkin projection for simplicity. However, we solve for $V_{\pm}\hat{\alpha}^{\pm}$ and then multiply the equation by the transpose of the projections matrices $V_{\pm}$:

$$\dot{\hat{\alpha}}^+(t) = -\frac{1}{2}V_+^T M^{-1}(0)\frac{2}{\epsilon}\left(\frac{1}{2}\left(M(0)V_+\hat{\alpha}^+(t) + M(t)V_-\hat{\alpha}^-(t)\right) - \tilde{F}(t, V_{\pm}\hat{\alpha}^{\pm}(t))\right) \tag{65}$$

$$\dot{\hat{\alpha}}^-(t) = -V_-^T M^{-1}(t)\frac{1}{\epsilon}\left(\frac{1}{2}\left(M(0)V_+\hat{\alpha}^+(t) + M(t)V_-\hat{\alpha}^-(t)\right) - \tilde{F}(t, V_{\pm}\hat{\alpha}^{\pm}(t))\right), \tag{66}$$

As above, if $M(t)$ is not invertible we replace it by $N(t)$.

In order to gain computation speed solving Eqs. (65) and (66) over the full system (51), we require the right hand side to be evaluated fast and do not need the computation of vectors of the full size. This can be done for arbitrary nonlinear flux function and arbitrary basis functions ϕ, but it is not a trivial problem. However, this paper is concerned with the proof of concept of the general method, namely the fact that the solution of α in $\mathbb{R}^{2N}$ lives in a low-dimensional space and this fact can be exploited to create a reduced model with standard methods for α.

4 Computational results

The theoretical findings are exemplified on a series of linear and nonlinear numerical examples. All computational results are obtained on torus $\mathbb{T} = [-1, 1]$. The matrix $M(t)$ defined by Eq. (24) and the jth component of the right-hand side $\tilde{F}$ are given by

$$M_{j,k}(t) = \int_{-1+\lambda t}^{1+\lambda t} \phi_j(x - \lambda t)\phi_k(x + \lambda t)\mathrm{d}x,\ \tilde{F}_j(t, \alpha) = \int_{-1+\lambda t}^{1+\lambda t} \phi_j(x - \lambda t) f(\tilde{u}(t, x))\mathrm{d}x, \tag{67}$$

where $\tilde{u}$ is given by Eq. (29). As base function we choose compactly supported, piecewise linear functions fulfilling property (42). We divide the torus in cells $[j-1, j]\Delta x$ where for fixed N we set $\Delta x = \frac{2}{N}$ and $j = 1, \ldots N$. Then, the set of base functions $\{\phi_j : j = 1, \ldots, N\}$ are defined by

$$\phi_j(x) = \begin{cases} x-(j-1)\Delta x & x \in [j-1, j]\Delta x \\ (j+1)\Delta x - x & x \in [j, j+1]\Delta x \\ 0 & \text{else} \end{cases} \tag{68}$$

As shown above, the matrix $M(0)$ is invertible for the previous choice of ϕ_j. Further, $M(t)$ is invertible for $t > 0$, except at the discrete time $t_\ell = \frac{2\ell+1}{2\lambda N}$. The number of time steps is denoted by $N_{\Delta t}$.

The further parameters are set as follows:

$$\rho = \epsilon, ; \Delta t = \frac{1}{2}\epsilon \text{ and } T = N_{\Delta t}\Delta t. \tag{69}$$

Since the basis function is only nonzero on a small interval, we exploit this in the numerical implementation within the MATLAB® built in function integral. At any time $t = t_n$ the matrix M is computed, we use the fact that our basis functions ϕ is simply shifted as indicated above. This implies that we have to compute only a single row of the matrix as the matrix is a circulant matrix. To be more precise, as only four of the values are potentially nonzero, we only have to compute those. Besides M and N we have to also compute $\tilde{F}$ at time t_n. As $\alpha_n^\pm$ are given we can define the function $\tilde{u}(t_n, x)$ and $\tilde{F}$ compute via a quadrature rule, which we do by using the build in MATLAB® function integral. Once we have the initial values for $\alpha^\pm$ and the possibility to evaluate M and $\tilde{F}$ we use (58) to compute further timesteps of $\alpha^\pm$. Since we are only interested in the qualitative behavior, we do not discuss the possibilities for improving this numerical computation In the numerical results, we will first show that the approach using translated base functions yields qualitative and quantitative correct solutions in the case of linear transport with and without nonlinear source terms, showing that this discretization produces feasible solution. We then show, that a linear subspace in the solution space of $\alpha^\pm$ produces correct results and with that the reducibility of the ordinary differential equation in α. Secondly, we show that also for nonlinear transport the proposed method yields a good qualitative and quantitative agreement with standard results by finite-volume methods. The latter however are not amendable for model order reduction. In the case of strong shocks the reduction in dimension of the reduced model order system is however not as significant as in the linear case. However, the computed reduced order system is able to correctly reproduce solutions to different initial data. This example shows, that the chosen formulation is amendable for model order reduction even in the nonlinear case and in the case of discontinuous solutions.

4.1 Linear transport with nonlinear source

In order to validate the Ansatz (21) we present numerical results for linear transport equation with nonlinear source term:

$$\partial_t w(t,x) + \lambda \partial_x w(t,x) = \gamma w^2(t,x) + \delta w \tag{70}$$

$$w(0,x) = w_0(x) \neq 0 \tag{71}$$

Eq. (70) contains three parameters $\lambda \neq 0$, and $\gamma, \delta \in \mathbb{R}$. The case $\gamma = \delta = 0$ corresponds to a linear transport equation. On the full space $x \in \mathbb{R}$ the explicit solution to Eqs. (70) and (71) is given by

$$w(t,x) = \frac{1}{\frac{e^{-\delta t}}{w_0(x-\lambda t)} - \frac{\gamma}{\delta}(1-e^{-\delta t})}. \tag{72}$$

for t sufficiently small such that (72) is well-defined. Due to the finite speed of propagation a numerical comparison of approximation errors with the exact solution is possible provided that $supp w_0(x) \subset\subset \mathbb{T}$. In this case the exact solution $w(t,x)$ is given by Eq. (72) for $x \in \mathbf{S}(t)$ where $\mathbf{S}(t) := \{x : x - \lambda t \in supp w_0\}$ and

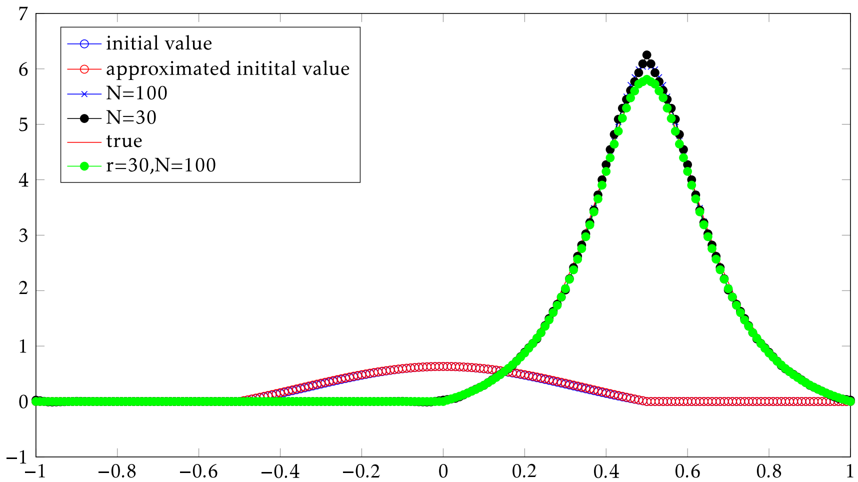

Figure 1: Simulation result for a linear transport equation with a nonlinear right hand side with a solution ansatz as in Eq. (73) for two different values of *N* and also a simulated reduced system arising from the larger system and a reduced order equivalent to the size of the smaller one.

$w = 0$ zero else. The exact solution $w(t, x)$ is defined for any t such that $\mathbf{S}(t) \subset \mathbb{T}$. In the case of the linear equation our Ansatz reduces to

$$w(t, x) = \sum_{i=1}^{N} \alpha_i(t)\phi_i(x - \lambda t) \tag{73}$$

In the numerical test shown in Figure 1 we choose $\gamma = 2, \delta = 1$ and simulate until $T = 0.5$. We compute the solution for $N = 100$ and a reduced model projected on a $r = 30$ dimensional subspace. We compare the analytical solution, the solution on the subspace with $N = 100$ and the solution with $N = 30$ modes. The initial value is given by $w_0(x) = e^{-4x^2} - e^{-1}$ on $(-1/2, 1/2)$ and zero otherwise. Its approximation on the subspace is also shown. As seen in Figure 1, we observe very good agreement between the reduced basis approximation and the analytical solution. Clearly, higher-dimensional subspaces provide better agreement than lower dimensional ones. This example indicates that the use of translate base functions leads to qualitative and quantitative correct results in the linear case.

4.2 Relaxation approach for a linear problems

Consider the same linear flux as in the previous example. Here, we apply the relaxation formulation with $\lambda = a$ and $\epsilon = 10^{-3}$ to the linear problem. Clearly, this is not necessary in order to solve the linear problem but the numerical result following illustrates that no additional numerical approximation error appears.

The initial condition is

$$u_0 = \sin(\pi x)$$

and the analytical solution is given by $u(t, x) = u_0(x - t)$. In Figure 2, we show initial condition and analytical solution at final time $T = 1$. Figure 2 shows that using $N = 40$ basis functions the numerical solution is indistinguishable from the analytical solution. Further, we observe that in this particular example a reduced system of dimension two can already capture the complete behavior because that there is only linear transport. For sake of completeness we also show the result with only a single base function that is equal to zero. The test case only contains smooth data and solution and as expected the a low dimensional reduced base formulation recovers the behavior well.

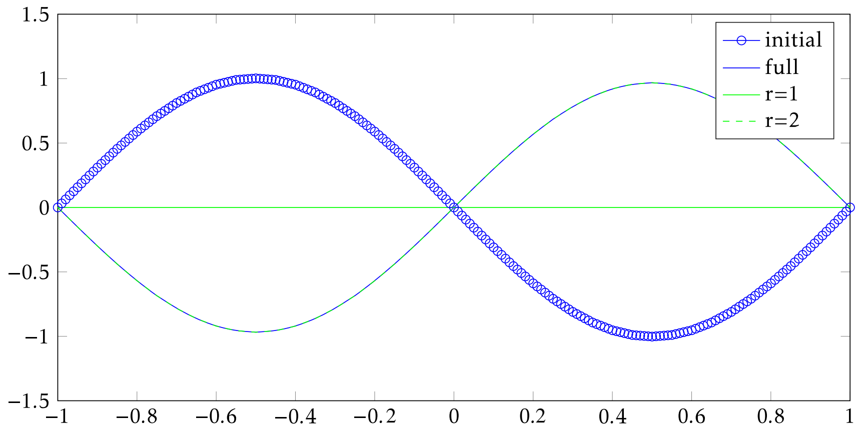

Figure 2: Relaxation formulation applied to a linear flux $f(u) = u$ and smooth initial data. Shown are the numerical solution at time $T = 1$ with $N = 40$ piecewise defined basis function and $r = 2$ and $r = 1$ reduced basis functions, respectively.

4.3 Burgers equation and approximation of Shock solution

We consider the relaxation formulation for Burgers equation, i.e., the flux is given by $f(u) = \frac{1}{2}u^2$. Smooth periodic initial data

$$u_0(x) = \frac{1}{2} + \sin(\pi x)$$

on $\mathbb{T}$ is considered. It is known that at time $T = 1$ a shock is formed due to the nonlinear transport. In Figure 3, we show the quality of the proposed approximation for different numbers of base functions N. We choose λ larger than the norm of the initial data, i.e.,

$$\lambda = 2$$

and

$$\epsilon = 10^{-3}$$

for this test. In the subfigures of Figure 3, initial data and the solution at terminal time $T = 1$ is shown. We observe that for N sufficiently large the expected shock is recovered in detail. For small N, we observe a Gibb's phenomenon due to the strong discontinuity of the underlying solution. To compare the solution we also included a figure showing the result of a first-order finite volume scheme applied to the same relaxation formulation. In particular, we observe that the size of the jump discontinuity is the same for the proposed approximation and the finite-volume scheme. The latter is taken from reference [22] and the spatial discretization is given by $\Delta x = 1/320$. Since $\epsilon > 0$ we observe in all simulations a slight decay of the maxima and minima over time. For smaller values of ϵ the decay of the extreme values is expected to be smaller. However, the time step of the proposed method scales with ϵ and this leads to inefficiencies in the numerical scheme. Compared to the method [22], we cannot resolve in the regime $\Delta t > \epsilon$.

In the previous convergence experiment, we left ϵ constant and increased only the number of basis functions N. We also did an experiment with varying parameter ϵ shown in Figure 4. Here, we can see that below 10^{-3} the results do not yield any qualitative difference. Therefore, we will use $\epsilon = 10^{-3}$ in the following experiments.

In Figure 6 (the red curve), we observe decay of singular values in the solution such that we can derive efficient model order reduction formulations in the classical sense. As expected the decay is not as significant as in elliptic or parabolic problems.

4.4 Model order reduction for strong shocks

As a second example with nonsmooth data, we consider Burgers' equations and initial data of the type

$$u_0(x) = a(\chi_{0,1/2}(x) - 1),$$

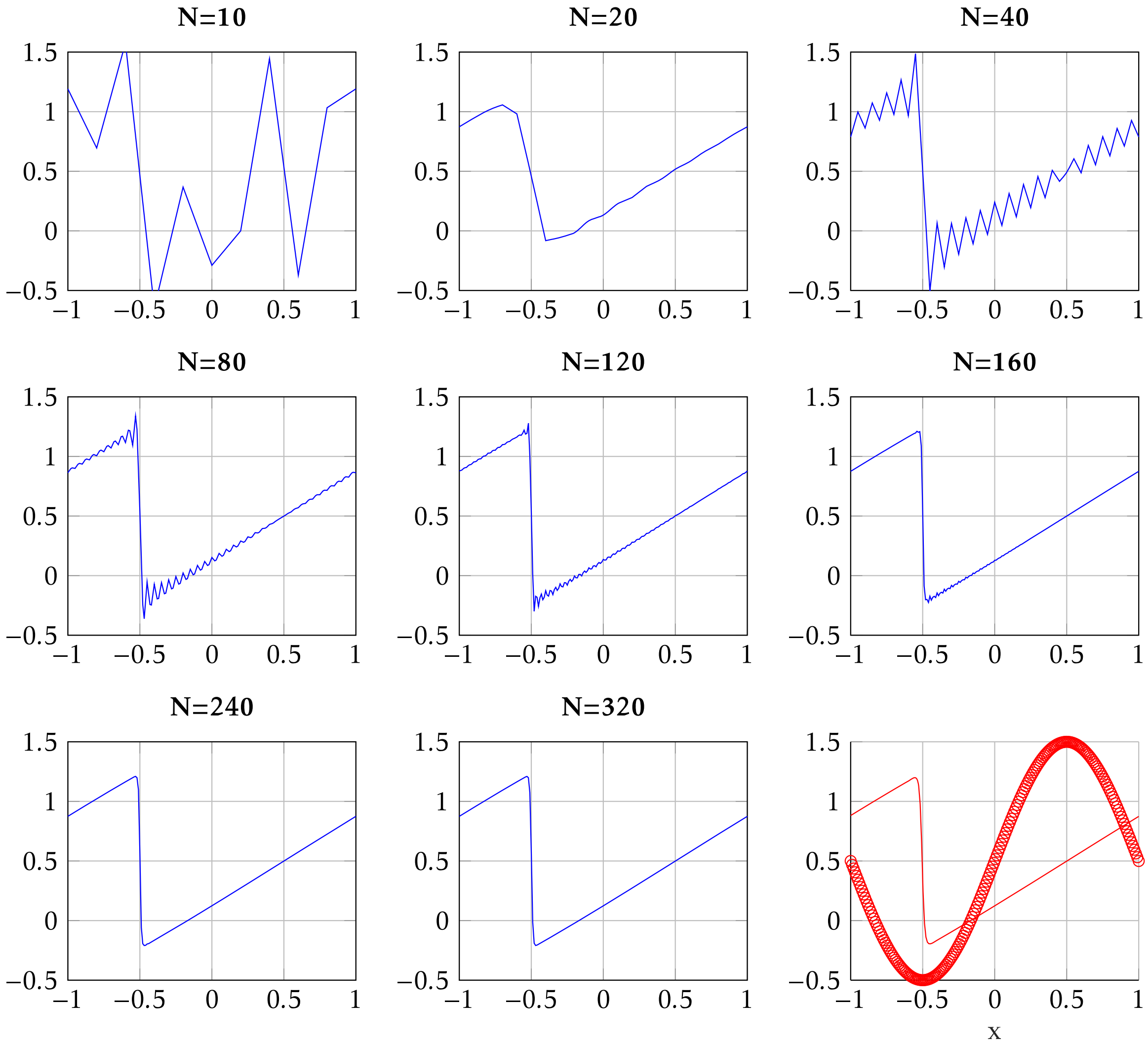

Figure 3: Relaxation approximation to a solution to Burgers equation with smooth initial data (shown in red with circles). At time $T = 1$ a shock develops that is captured by the proposed approximation for N sufficiently large (shown as continuous lines). In red a comparison with a first-order finite volume scheme with $N = 320$ discretization points in space.

for a parameter $a > 0$. The value of a controls the size of the jump discontinuity. The solution u to Burgers equation and the given initial data consists of a shock wave followed by a rarefaction. The latter wave is a linear function. The parameter a also controls the speed of propagation of the shock wave due to the Rankine–Hugoniot condition: the speed is $s = -a\frac{1}{4}$. For the numerical test we set $\epsilon = 10^{-3}$ and $\Delta t = \frac{\epsilon}{10}$. In Figure 5 the initial condition and its approximation with a discretization of $N = 160$ are shown in the left part of the figure. Small oscillations due to the strong discontinuity are visible. On the right we show the solution for two reduced model order approximations as well as the full model and a reference solution. The latter is computed as in the previous section using a second-order finite volume scheme with $\Delta x = \frac{1}{160}$.

In Figure 6, we investigate the previously observed oscillatory behavior. To this end we show the absolute values of the normalized singular values of the reduced solutions in several cases, namely the case of smooth initial data, the discontinuous case as well as the linear combination of both. The normalized singular values give an estimate on the size of the reduced problem necessary to capture the full dynamics. Here, a strong decay

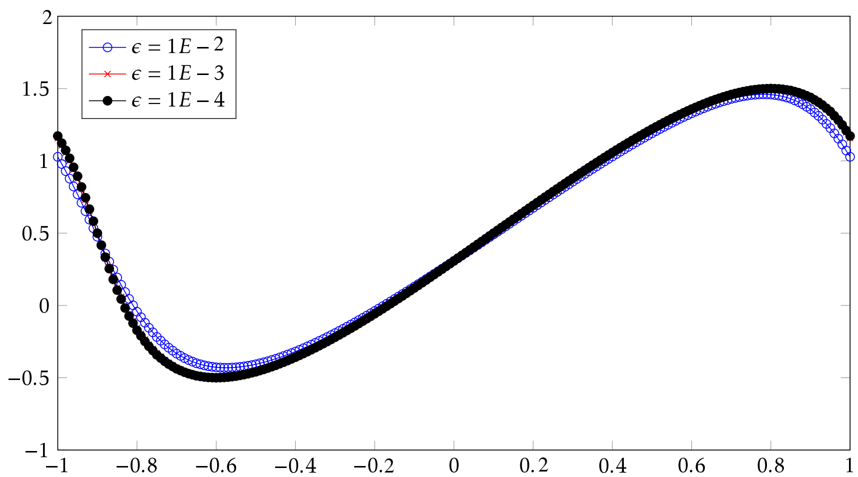

Figure 4: Solution of Burgers equation at time $T = 0.2$ for a discretization with $N = 80$ basis functions and different values of ϵ.

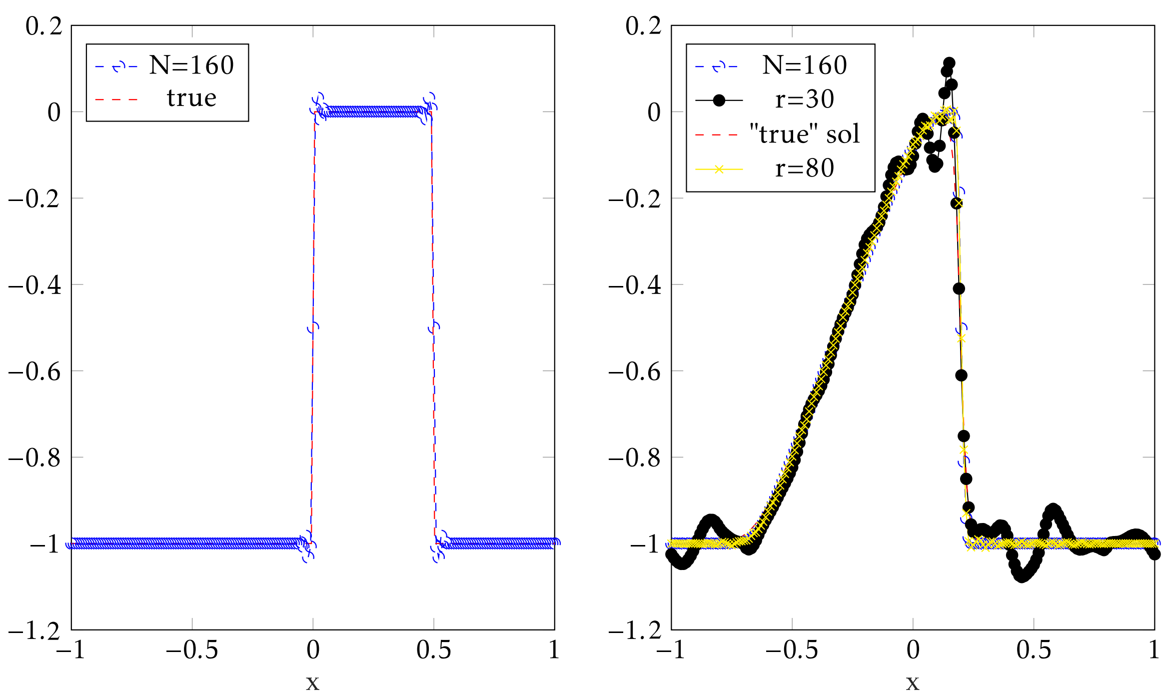

Figure 5: Initial data and approximation with $N = 160$ base functions. The Gibbs phenomenon is observed at the discontinuity (left). For a small set of base functions this phenomenon is also visible at terminal time $T = 0.6$ (right). For 80 base functions we observe agreement with a classical finite-volume solution.

in the singular values would allow for a small reduced problem. In the case of a low number of base functions, i.e., a small number of singular values, almost no decay in their absolute values are observed. However, for smooth solutions a decrease in the absolute value of the singular values is observed. The decrease is more pronounced compared with nonsmooth initial data and compared to the mixed case. Even so the decay in the absolute values of the singular values is mild compared with examples of parabolic and elliptic equations, the results of Figure 5 show that model order reduction is still possible. In the forthcoming section, the dominant modes are the basis of the reduced model.

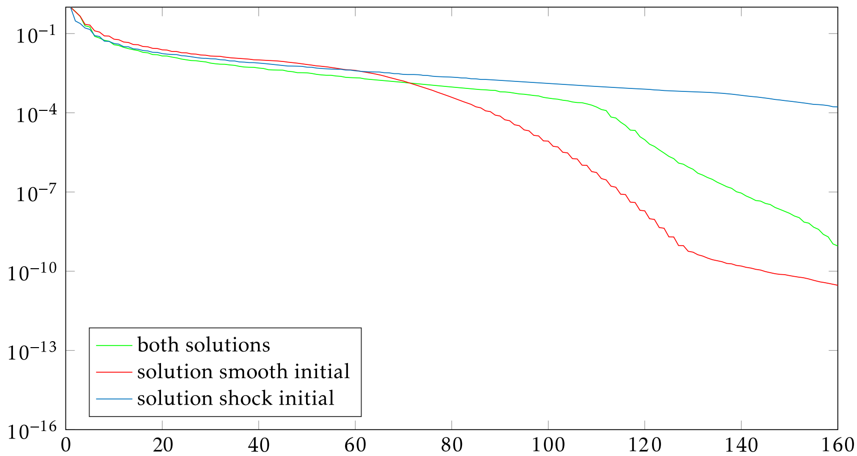

Figure 6: Singular values of a matrix of solutions u at different time instances computed with $N = 160$ base functions and for different initial conditions. Decay. We show the singular values for three examples: the solution to a smooth initial profile developing a shock (red), a solution to discontinuous initial data and for a linear combination of those initial conditions (green). In all cases and compared with the distribution of singular values for a solution to parabolic or elliptic problems we observe no exponential decay.

4.5 Model order reduction for different initial conditions

We use a reduced model obtained from a combination of the above initial conditions to predict model output for *different* initial conditions. We consider the solutions to the two different initial conditions given in the previous section. A reduced model from the dominant basis functions of the first two problems is obtained. The solution to this reduced system for initial data given by Eq. (74) is compared with classical finite-volume integration. The initial condition is chosen as a linear combination of two previous initial conditions.

$$u_0 = \sin(\pi x) + a\left(\chi_{0,1/2}(x) - 1\right) \tag{74}$$

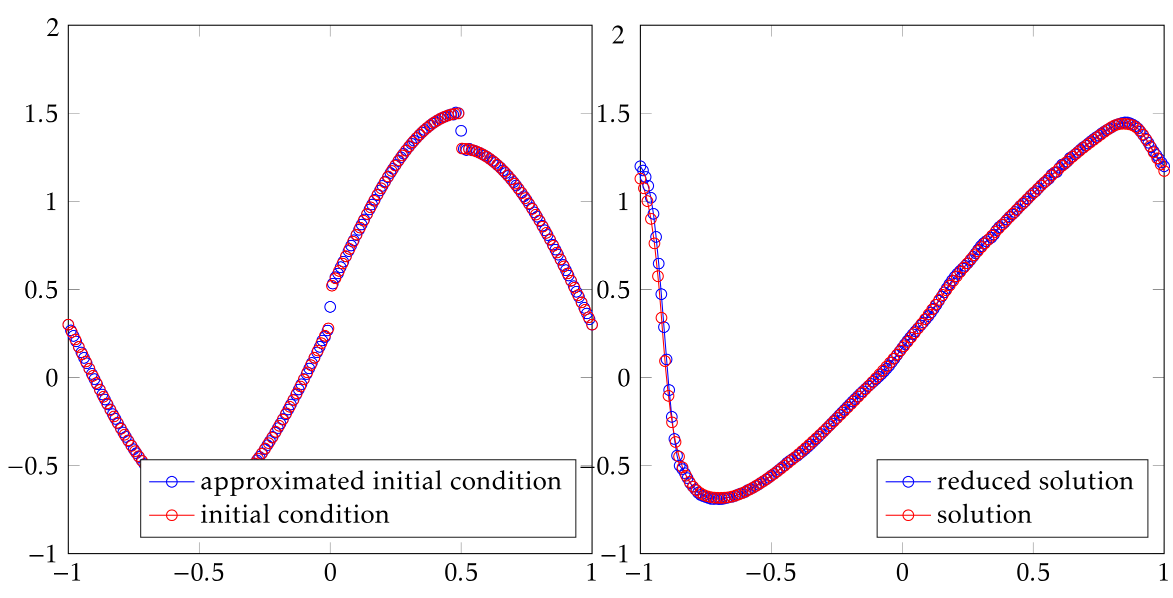

Figure 7: Left: Initial condition on the full space and reduced space with $r = 80$ out of $N = 160$ base functions. Right: Solution at time $T = 0.3$ on full and reduced space.

with $a = 0.2$. Note that the solution $u(T, x)$ is **not** a linear combination of the two previous solutions due to the nonlinear nature of the problem. Hence, we generate computational efficiency by reducing the size of the Ansatz space needed to solve for a given initial datum. Results are shown in Figure 7. The initial condition is sinoidal with additional discontinuities. The reduced system solution at $T = 0.3$ as well as the finite volume comparison show good qualitative and quantitative agreement. In this example we set the number of base functions as $N = 160$, the dimension of the reduced space $r = 80$. The further parameters are $\epsilon = 10^{-3}$ as above and $\Delta t = \frac{\epsilon}{10}$. The results confirm that the chosen approach allows to efficiently apply a model order reduction to hyperbolic problems.

5 Summary

We proposed a relaxation formulation of hyperbolic conservation laws that allows using shifted base functions for a formulation that is amendable for model-order reduction. The resulting discretized scheme is reduced using snapshots in time and shows qualitative good approximation properties even in the case of shock waves. The approach has been tested on linear hyperbolic problems with nonlinear source terms, but known exact solution as well as nonlinear hyperbolic problems with strong shocks. A numerical investigation of the approximation quality, the singular value decay as well as comparisons with classical finite-volume schemes have been conducted.

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References

[1] A. C. Antoulas, C. A. Beattie, and S. Gugercin, *Interpolatory Methods for Model Reduction*, Philadelphia, PA, Computational Science & Engineering, Society for Industrial and Applied Mathematics, 2020.
[2] A. C. Antoulas, D. C. Sorensen, and S. Gugercin, "A survey of model reduction methods for large-scale systems," *Contemp. Math.*, vol. 280, pp. 193–219, 2001.
[3] M. Grepl, "Reduced-basis approximation a posteriori error estimation for parabolic partial differential equations," PhD thesis, Massachussetts Institute of Technology (MIT), Cambridge, USA, 2005.
[4] B. Haasdonk and M. Ohlberger, "Reduced basis method for explicit finite volume approximations of nonlinear conservation laws," in *Proc. 12th International Conference on Hyperbolic Problems: Theory, Numerics, Application*, College Park, MD, United States, Citeseer, 2008.
[5] B. Haasdonk and M. Ohlberger, "Reduced basis method for finite volume approximations of parametrized linear evolution equations," *ESAIM: Math. Model. Numer. Anal.*, vol. 42, pp. 277–302, 2008.
[6] A. T. Maday, Y. Patera and G. Turinici, "A priori convergence theory for reduced-basis approximations of single-parameter elliptic partial differential equations," *J. Sci. Comput.*, vol. 17, pp. 437–446, 2002.
[7] A. Quarteroni, G. Rozza, and A. Manzoni, "Certified reduced basis approximation for parametrized partial differential equations and applications," *J. Math. Ind.*, vol. 1, pp. 1–49, 2011.
[8] X. Dai and Y. Maday, "Stable parareal in time method for first-and second-order hyperbolic systems," *SIAM J. Sci. Comput.*, vol. 35, pp. A52–A78, 2013.
[9] C. Himpe and M. Ohlberger, "Model reduction for complex hyperbolic networks," in *2014 European Control Conference (ECC)*, Strasbourg, France, IEEE, 2014, pp. 2739–2743.
[10] F. Laakmann and P. Petersen, "Efficient approximation of solutions of parametric linear transport equations by ReLU DNNs," *Adv. Comput. Math.*, vol. 47, pp. 1–32, 2021.

[11] K.-S. Moon, A. Szepessy, R. Tempone, and G. Zouraris, *Hyperbolic Differential Equations and Adaptive Numerics*, Berlin, Heidelberg, Springer Berlin Heidelberg, 2001, pp. 231–280.
[12] N. Sarna and S. Grundel, *Model Reduction of Time-dependent Hyperbolic Equations Using Collocated Residual Minimisation and Shifted Snapshots*, e-prints 2003.06362, arXiv, 2020. cs.NA.
[13] J. Reiss, P. Schulze, J. Sesterhenn, and V. Mehrmann, "The shifted proper orthogonal decomposition: a mode decomposition for multiple transport phenomena," *SIAM J. Sci. Comput.*, vol. 40, pp. A1322–A1344, 2018.
[14] P. Benner, M. Ohlberger, A. Cohen, and K. Willcox, *Model Reduction and Approximation*, Philadelphia, PA, Society for Industrial and Applied Mathematics, 2017.
[15] N. Cagniart, Y. Maday, and B. Stamm, "Model order reduction for problems with large convection effects," in *Contributions to Partial Differential Equations and Applications*, Cham, Springer International Publishing, 2019, pp. 131–150.
[16] V. Ehrlacher, D. Lombardi, O. Mula, and F.-X. Vialard, "Nonlinear model reduction on metric spaces. Application to one-dimensional conservative PDEs in Wasserstein spaces," *ESAIM Math. Model. Numer. Anal.*, vol. 54, 2019.
[17] N. J. Nair and M. Balajewicz, "Transported snapshot model order reduction approach for parametric, steady-state fluid flows containing parameter-dependent shocks," *Int. J. Numer. Methods Eng.*, vol. 117, pp. 1234–1262, 2019.
[18] B. Peherstorfer, "Model reduction for transport-dominated problems via online adaptive bases and adaptive sampling," *SIAM J. Sci. Comput.*, vol. 42, pp. A2803–A2836, 2020.
[19] T. Taddei, S. Perotto, and A. Quarteroni, "Reduced basis techniques for nonlinear conservation laws," *ESAIM: M2AN*, vol. 49, pp. 787–814, 2015.
[20] G. Welper, "Interpolation of functions with parameter dependent jumps by transformed snapshots," *SIAM J. Sci. Comput.*, vol. 39, pp. A1225–A1250, 2017.
[21] S. Bianchini, "Hyperbolic limit of the Jin-Xin relaxation model," *Commun. Pure Appl. Math.*, vol. 59, pp. 688–753, 2006.
[22] S. Jin and Z. Xin, "The relaxation schemes for systems of conservation laws in arbitrary space dimensions," *Commun. Pure Appl. Math.*, vol. 48, pp. 235–276, 1995.
[23] R. Natalini, "Recent mathematical results on hyperbolic relaxation problems," *Quaderno IAC*, vol. 7, pp. 128–198, 1999.
[24] C. M. Dafermos, "Hyperbolic conservation laws in continuum physics," in *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 325, 2nd ed., Berlin, Springer-Verlag, 2005.
[25] U. Pallaske, "Ein verfahren zur ordnungsreduktion mathematischer prozessmodelle," *Chem. Ing. Tech.*, vol. 59, pp. 604–605, 1987.
[26] D. Aregba-Driollet and R. Natalini, "Convergence of relaxation schemes for conservation laws," *Int. J. Phytoremediation*, vol. 61, pp. 163–193, 1996.
[27] A. Chalabi, "Convergence of relaxation schemes for hyperbolic conservation laws with stiff source terms," *Math. Comput.*, vol. 68, pp. 955–970, 1999.
[28] A. Klar, "Relaxation scheme for a lattice–Boltzmann-type discrete velocity model and numerical Navier–Stokes limit," *J. Comput. Phys.*, vol. 148, pp. 416–432, 1999.
[29] H. Liu, J. Wang, and G. Warnecke, "The lip + -stability and error estimates for a relaxation scheme," *SIAM J. Numer. Anal.*, vol. 38, pp. 1154–1170, 2001.
[30] L. Pareschi and G. Russo, "High order asymptotically strong-stability-preserving methods for hyperbolic systems with stiff relaxation," in *Hyperbolic Problems: Theory, Numerics, Applications*, Berlin, Heidelberg, Springer, 2003, pp. 241–251.
[31] L. Pareschi and G. Russo, "Implicit-explicit Runge-Kutta schemes and applications to hyperbolic systems with relaxation," *J. Sci. Comput.*, vol. 25, pp. 129–155, 2005.
[32] W.-A. Yong, "An interesting class of partial differential equations," *J. Math. Phys.*, vol. 49, p. 03350321, 2008.
[33] Z. Chen, "On nonsingularity of circulant matrices," *Lin. Algebra Appl.*, vol. 612, pp. 162–176, 2021.
[34] S. Boscarino, L. Pareschi, and G. Russo, "Implicit-explicit Runge-Kutta schemes for hyperbolic systems and kinetic equations in the diffusion limit," *SIAM J. Sci. Comput.*, vol. 35, pp. A22–A51, 2013.