

Double extension for commutative n -ary superalgebras with a skew-symmetric invariant form

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Abstract

The method of double extension, introduced by A. Medina and Ph. Revoy, is a procedure which decomposes a Lie algebra with an invariant symmetric form into elementary pieces. Such decompositions were developed for other algebras, for instance for Lie superalgebras and associative algebras, Filippov n -algebras and Jordan algebras.

The aim of this note is to find a unified approach to such decompositions using the derived bracket formalism. More precisely, we show that any commutative n -ary superalgebra with a skew-symmetric invariant form can be obtained inductively by taking orthogonal sums and generalized double extensions.

1 Preliminaries

Let $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ be a finite dimensional \mathbb{Z}_2 -graded vector space over \mathbb{K} , where we assume for simplicity that $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . We denote by $\bar{a} \in \mathbb{Z}_2$ the parity of a homogeneous element $a \in \mathfrak{a}_{\bar{i}}$. A bilinear form $(,)$ on \mathfrak{a} is called *even* if the corresponding linear map $\mathfrak{a} \otimes \mathfrak{a} \rightarrow \mathbb{K}$ is even. A bilinear form is called *skew-symmetric* if

$$(a, b) = -(-1)^{\bar{a}\bar{b}}(b, a)$$

for any homogeneous elements $a, b \in \mathfrak{a}$. From now on we assume that $(,)$ is an even non-degenerate skew-symmetric form on \mathfrak{a} . This is

- $(,)|_{\mathfrak{a}_{\bar{0}} \times \mathfrak{a}_{\bar{0}}}$ is a non-degenerate skew-symmetric form,
- $(,)|_{\mathfrak{a}_{\bar{1}} \times \mathfrak{a}_{\bar{1}}}$ is a non-degenerate symmetric form,
- $(,)|_{\mathfrak{a}_{\bar{0}} \times \mathfrak{a}_{\bar{1}}} = 0$.

Definition 1. • An n -ary superalgebra is a vector space \mathfrak{a} together with an n -linear map $\mathfrak{a} \times \cdots \times \mathfrak{a} \longrightarrow \mathfrak{a}$. We denote this map by $(a_1, \dots, a_n) \mapsto \{a_1, \dots, a_n\}$.

- An n -ary superalgebra is called *commutative* if

$$\{a_1, \dots, a_i, a_{i+1}, \dots, a_n\} = (-1)^{\bar{a}_i \bar{a}_{i+1}} \{a_1, \dots, a_{i+1}, a_i, \dots, a_n\} \quad (1)$$

for all homogeneous $a_i, a_{i+1} \in \mathfrak{a}$.

- A commutative n -ary superalgebra is called *invariant with respect to the form* $(,)$ if the following holds:

$$(a_0, \{a_1, \dots, a_n\}) = (-1)^{\bar{a}_0 \bar{a}_1} (a_1, \{a_0, a_2, \dots, a_n\}) \quad (2)$$

for all homogeneous $a_i \in \mathfrak{a}$.

We will write a *commutative invariant n -ary superalgebra* as a shorthand for a *commutative n -ary superalgebra that is invariant with respect to the form $(,)$* . Let \mathfrak{a} be a commutative n -ary superalgebra.

Definition 2. • An n -ary subalgebra in \mathfrak{a} is a vector subspace $\mathfrak{b} \subset \mathfrak{a}$ such that $\{\mathfrak{b}, \dots, \mathfrak{b}\} \subset \mathfrak{b}$. An *ideal* in \mathfrak{a} is a vector subspace $\mathfrak{i} \subset \mathfrak{a}$ such that $\{\mathfrak{a}, \dots, \mathfrak{a}, \mathfrak{i}\} \subset \mathfrak{i}$.

- Let \mathfrak{a} and \mathfrak{b} be two n -ary algebras. An even linear map $\phi : \mathfrak{a} \rightarrow \mathfrak{b}$ is called a *homomorphism* of n -ary algebras if

$$\phi(\{a_1, \dots, a_n\}_{\mathfrak{a}}) = \{\phi(a_1), \dots, \phi(a_n)\}_{\mathfrak{b}}.$$

The vector space $\ker \phi \subset \mathfrak{a}$ is an ideal in \mathfrak{a} .

- A commutative n -ary superalgebra is called *simple* if it is not trivial one dimensional and it does not have any proper ideals.

- An invariant commutative n -ary superalgebra is called *irreducible* if it is not a direct sum of two non-degenerate ideals. (An ideal \mathfrak{i} is called *non-degenerate* if $(,)|_{\mathfrak{i}}$ is non-degenerate.)

The main tool that we use in this paper is the derived bracket construction. Let \mathfrak{a} be a \mathbb{Z}_2 -graded vector space and $(,)$ be as above. We denote by $S^n \mathfrak{a}$ the n -th symmetric power of \mathfrak{a} and we put $S^* \mathfrak{a} = \bigoplus_n S^n \mathfrak{a}$. The superspace $S^* \mathfrak{a}$ possesses a natural structure $[,]$ of a Poisson superalgebra that is defined in the following way:

$$\begin{aligned} [x, y] &:= (x, y), \quad x, y \in \mathfrak{a}; \\ [v, w_1 \cdot w_2] &:= [v, w_1] \cdot w_2 + (-1)^{vw_1} w_1 \cdot [v, w_2], \\ [v, w] &= -(-1)^{vw} [w, v], \end{aligned}$$

where v, w, w_i are homogeneous elements in $S^* \mathfrak{a}$. The super-Jacobi identity has the following form:

$$[v, [w_1, w_2]] = [[v, w_1], w_2] + (-1)^{\bar{v}\bar{w}_1} [w_1, [v, w_2]].$$

Let us take any element $\mu \in S^{n+1} \mathfrak{a}$. Then we can define an n -ary superalgebra on \mathfrak{a} in the following way:

$$\{a_1, \dots, a_n\} := [a_1, [\dots, [a_n, \mu] \dots]], \quad a_i \in \mathfrak{a}. \quad (3)$$

Clearly, $\{a_1, \dots, a_n\} \in \mathfrak{a}$. We will denote the corresponding superalgebra by (\mathfrak{a}, μ) and we will call the element μ the *derived potential* of (\mathfrak{a}, μ) . The n -ary superalgebras (\mathfrak{a}, μ) are commutative and invariant with respect to the form $(,)$, see [Vor1] and [V]. We have the following observation (see [V] for details).

Proposition 1. *Any commutative invariant n -ary superalgebras on \mathfrak{a} can be obtained by construction (3).*

2 Double extension for invariant n -ary superalgebras

Let \mathfrak{g} be an invariant commutative n -ary superalgebra and $\mu \in S^{n+1} \mathfrak{g}$ be its derived potential. Let \mathfrak{h} be any commutative n -ary superalgebra with the multiplication $\nu \in S^n \mathfrak{h}^* \otimes \mathfrak{h}$. We can identify the vector spaces $S^n \mathfrak{h}^* \otimes \mathfrak{h}$ with the subspace $S^n \mathfrak{h}^* \cdot \mathfrak{h} \subset S^*(\mathfrak{h} \oplus \mathfrak{h}^*)$ and consider ν as a derived potential for the invariant superalgebra $\mathfrak{h} \oplus \mathfrak{h}^*$. The even non-degenerate skew-symmetric invariant form $(,)$ on $\mathfrak{h} \oplus \mathfrak{h}^*$ is given by:

$$(\alpha, x) := \alpha(x), \quad (x, \alpha) := -(-1)^{\bar{\alpha}\bar{x}}(\alpha, x),$$

$\alpha \in \mathfrak{h}^*$, $x \in \mathfrak{h}$ are homogeneous elements.

Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*$ be a vector space with the non-degenerate skew-symmetric bilinear form $(,)$ that is the sum of the non-degenerate skew-symmetric bilinear forms on \mathfrak{g} and $\mathfrak{h} \oplus \mathfrak{h}^*$.

Definition 3. The commutative invariant n -ary superalgebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*$ with the derived potential

$$L = \mu + \nu + \sum_{i=1}^{n+1} \psi_i, \quad \text{where } \mu \in S^{n+1} \mathfrak{g}, \quad (4)$$

$$\nu \in S^n \mathfrak{h}^* \cdot \mathfrak{h}, \quad \psi_i \in S^i \mathfrak{h}^* \cdot S^{n-i+1} \mathfrak{g}$$

is called a **generalized double extension** of \mathfrak{g} by \mathfrak{h} via ψ_i , $i = 1, \dots, n+1$.

The main observation of this section is:

Theorem 1. *Assume that \mathfrak{a} is an irreducible but not simple commutative invariant n -ary superalgebra. Then \mathfrak{a} is isomorphic to a certain generalized double extension.*

Proof. Let us take a maximal non-trivial ideal \mathfrak{i} of \mathfrak{a} . Clearly, \mathfrak{i}^\perp is a minimal ideal in \mathfrak{a} . Furthermore, since $\mathfrak{i}^\perp \cap \mathfrak{i}$ is also an ideal and \mathfrak{a} is irreducible, we see that $\mathfrak{i}^\perp \subset \mathfrak{i}$. Therefore, \mathfrak{i}^\perp is a minimal isotropic ideal in \mathfrak{a} . Let us take a subspace \mathfrak{h} in \mathfrak{a} such that \mathfrak{h} is isotropic, $\mathfrak{h} \cap \mathfrak{i}^\perp = \{0\}$ and $(,)|_{\mathfrak{h} \oplus \mathfrak{i}^\perp}$ is non-degenerate. Since, $(,)|_{\mathfrak{h} \oplus \mathfrak{i}^\perp}$ is non-degenerate, we have $\mathfrak{a} = \mathfrak{i} \oplus \mathfrak{h}$. Consider the vector space $\mathfrak{w} = \mathfrak{i}^\perp \oplus \mathfrak{h}$. We have the decompositions $\mathfrak{a} = \mathfrak{w} \oplus \mathfrak{w}^\perp$ and $\mathfrak{i} = \mathfrak{i}^\perp \oplus \mathfrak{w}^\perp$.

Denote by $\lambda \in S^{n+1}\mathfrak{a}$ the derived potential of \mathfrak{a} . Since, $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{i}^\perp \oplus \mathfrak{w}^\perp$, we have:

$$S^{n+1}\mathfrak{a} = \bigoplus_{i+j+k=n+1} S^i(\mathfrak{h}) \cdot S^j(\mathfrak{i}^\perp) \cdot S^k(\mathfrak{w}^\perp).$$

Therefore, we have the following decomposition of the derived potential:

$$\lambda = \sum_{i+j+k=n+1} \lambda_{ijk}, \text{ where } \lambda_{ijk} \in S^i(\mathfrak{h}) \cdot S^j(\mathfrak{i}^\perp) \cdot S^k(\mathfrak{w}^\perp).$$

Furthermore, for any $b \in \mathfrak{i}^\perp$ we have

$$[b, \lambda] \in \bigoplus_{i+j+k=n+1} S^{i-1}(\mathfrak{h}) \cdot S^j(\mathfrak{i}^\perp) \cdot S^k(\mathfrak{w}^\perp).$$

Since \mathfrak{i}^\perp is an ideal we have

$$[a_1, \dots, [a_{n-1}, [b, \lambda]]] \in \mathfrak{i}^\perp \text{ for } a_p \in \mathfrak{a}.$$

Therefore, λ_{ijk} can be non-trivial only in the following two cases:

- $i = 0$;
- $i = 1$ and $k = 0$.

In other words, we get

$$\lambda \in \left(\mathfrak{h} \cdot S^n(\mathfrak{i}^\perp) \right) \oplus \left(\bigoplus_{j+k=n+1} S^j(\mathfrak{i}^\perp) \cdot S^k(\mathfrak{w}^\perp) \right). \quad (5)$$

We put

$$\begin{aligned} \mu &:= \lambda_{0,0,n+1} \in S^{n+1}(\mathfrak{w}^\perp); & \nu &:= \lambda_{1,n,0} \in \mathfrak{h} \cdot S^n(\mathfrak{i}^\perp); \\ \psi_i &:= \lambda_{0,i,n-i+1} \in S^i(\mathfrak{i}^\perp) \cdot S^{n+1-i}(\mathfrak{w}^\perp), & i &= 1, \dots, n+1; \\ \mathfrak{g} &:= \mathfrak{w}^\perp. \end{aligned}$$

We also can identify \mathfrak{i}^\perp with \mathfrak{h}^* . We see that (\mathfrak{a}, λ) is a double extension of (\mathfrak{g}, μ) by (\mathfrak{h}, ν) via ψ_i . The result follows. \square

Proposition 2. The n -ary superalgebra (\mathfrak{g}, μ) is isomorphic to $\mathfrak{i}/\mathfrak{i}^\perp$. Indeed, assume that $a_i \in \mathfrak{i}$. Then

$$[a_1, \dots, [a_n, \nu]] = 0, \quad [a_1, \dots, [a_n, \psi_i]] \in \mathfrak{i}^\perp.$$

Hence,

$$\{\tilde{a}_1, \dots, \tilde{a}_n\}_{\mathfrak{i}/\mathfrak{i}^\perp} = \{a_1, \dots, a_n\}_{(\mathfrak{g}, \mu)},$$

where \tilde{a}_i is the image of a_i in $\mathfrak{i}/\mathfrak{i}^\perp$.

Corollary 1. Assume that \mathfrak{a} is irreducible and not simple, \mathfrak{i} is a maximal non-trivial ideal and \mathfrak{h} is an isotropic subalgebra in \mathfrak{a} such that $\mathfrak{a} = \mathfrak{i} \oplus \mathfrak{h}$. Then \mathfrak{a} is isomorphic to a certain generalized double extension with $\psi_n = \psi_{n+1} = 0$. In this case the generalized double extension is called **double extension**.

Proof. Consider (5) holds. Let us take $x_i \in \mathfrak{h}$. The result follows from the following observations:

$$[x_1, \dots, [x_n, \mu + \sum_{i=1}^{n-1} \psi_i]] = 0$$

$$[x_1, \dots, [x_n, \nu]] \in \mathfrak{h}, \quad [x_1, \dots, [x_n, \psi_n]] \in \mathfrak{g}, \quad [x_1, \dots, [x_n, \psi_{n+1}]] \in \mathfrak{h}^*.$$

Since \mathfrak{h} is a subalgebra, we have $\psi_n = \psi_{n+1} = 0$. The proof is complete. \square

Corollary 2. Assume that \mathfrak{a} is an irreducible but not simple skew-symmetric invariant n -ary algebra and \mathfrak{i} is a maximal non-trivial ideal of codimension 1. Then \mathfrak{a} is isomorphic to a certain double extension with $\nu = \psi_i = 0$ for all $i \neq 1$.

Proof. In this case the statement 5 has the following form:

$$\lambda \in \mathfrak{i}^\perp \cdot S^n(\mathfrak{w}^\perp) \oplus S^{n+1}(\mathfrak{w}^\perp). \square$$

2.1 Lie algebras

In this section we show that our definition coincides with the definitions given in [MR] in case of Lie algebras.

Definition 4. A *derivation* of an n -ary algebra (V, μ) is a linear map $D : V \rightarrow V$ such that

$$D(\{v_1, \dots, v_n\}) = \sum_j \{v_1, \dots, D(v_j), \dots, v_n\}.$$

We denote by $\text{IDer}(\mu)$ the vector space of all derivations of the algebra (V, μ) preserving the form $(,)$. The following proposition was proven in [V]:

Proposition 3. *Let us take any $w \in S^2(V)$ and $\mu \in S^{n+1}(V)$. We have:*

$$\text{IDer}(\mu) = \{w \in S^2(V) \mid \text{ad } w(\mu) = 0\}.$$

In [MR] the following theorem was proven:

Theorem 2. [Medina, Revoy] *Let \mathfrak{g} be a Lie algebra with an invariant symmetric form B . Let us take any Lie algebra \mathfrak{h} with a homomorphism θ of \mathfrak{h} to $\mathcal{D}\text{er}(\mathfrak{g})$ such that $\theta(\mathfrak{h})$ preserves the form B . Then $\mathfrak{d} = \mathfrak{h}^* \oplus \mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra with respect to the following multiplication:*

$$\begin{aligned} & [(f_1, w_1, s_1), (f_2, w_2, s_2)] = \\ & (\text{ad}^*(s_2)f_1 - \text{ad}^*(s_1)f_2 + \omega(w_1, w_2), [w_1, w_2] + \theta(s_1)(w_2) - \theta(s_2)(w_1), [s_1, s_2]). \end{aligned}$$

Here $\omega(w_1, w_2)(s) := B(\theta(s)w_1, w_2)$. The Lie algebra \mathfrak{d} possesses an invariant form given by the sum of B and the natural symmetric form on $\mathfrak{h}^* \oplus \mathfrak{h}$. (In [MR] the algebra \mathfrak{d} is called a **double extension** of \mathfrak{g} by \mathfrak{h} .)

Conversely, any Lie algebra with an invariant symmetric form can be obtained inductively by direct sums and double extensions.

We can simplify the definition of the double extension from [MR] using the derived bracket construction. Let \mathfrak{g} be a Lie algebra with an invariant symmetric form and μ be its derived potential. Consider a Lie algebra \mathfrak{h} with the multiplication $\nu \in \bigwedge^2 \mathfrak{h}^* \otimes \mathfrak{h}$. We have the non-degenerate symmetric form on $\mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g}$: it is the sum of the form on \mathfrak{g} and the natural non-degenerate symmetric form on $\mathfrak{h}^* \oplus \mathfrak{h}$. As above we denote by $[\cdot, \cdot]$ the corresponding Poisson bracket on $S^*(\mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g})$. (Again we can consider $\mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g}$ as a pure odd vector space with a non-degenerate skew-symmetric form.)

There is a one-to-one correspondence between elements $\psi \in \mathfrak{h}^* \otimes \bigwedge^2 \mathfrak{g}$ and linear maps $\theta : \mathfrak{h} \rightarrow \bigwedge^2 \mathfrak{g} \simeq \mathfrak{so}(\mathfrak{g})$. This correspondence is given by $\psi \mapsto \theta_\psi$, where $\theta_\psi(x) = [x, \psi]$.

Theorem 3. Let $\mathfrak{d} = \mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g}$, and \mathfrak{g} , \mathfrak{h} and $\theta = \theta_\psi$ be a double extension of \mathfrak{g} by \mathfrak{h} via θ_ψ in the sense of Theorem 2. Then in terms of Proposition 1 the Lie algebra \mathfrak{d} has the derived potential

$$\mu + \nu + \psi,$$

and we have

$$[\mu + \nu + \psi, \mu + \nu + \psi] = 0. \tag{6}$$

Conversely, if the condition (6) holds then the Lie algebra $\mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g}$ is a double extension of \mathfrak{g} by \mathfrak{h} via θ_ψ in the sense of Theorem 2.

Proof. Assume that \mathfrak{d} is a double extension of \mathfrak{g} by \mathfrak{h} via θ_ψ . An easy computation shows that the derived potential of \mathfrak{d} is equal to $\mu + \nu + \psi$. Since \mathfrak{d} is a Lie algebra, we have (6).

Conversely, let us take Lie algebras \mathfrak{g} and \mathfrak{h} and an element ψ as above such that (6) holds. We need to show that $\theta = \theta_\psi$ is a homomorphism of \mathfrak{h} to $\mathcal{D}er(\mathfrak{g})$ that preserves the form on \mathfrak{g} . Since \mathfrak{g} and $\mathfrak{h} \oplus \mathfrak{h}^*$ are Lie algebras we have $[\mu, \mu] = 0$ and $[\nu, \nu] = 0$. Therefore, we also have

$$[\mu + \nu + \psi, \mu + \nu + \psi] = 2[\mu, \psi] + 2[\psi, \nu] + [\psi, \psi] = 0. \quad (7)$$

Note that $[\mu, \nu] = 0$. Since $[\mu, \psi] \in \mathfrak{h}^* \otimes \wedge^3 \mathfrak{g}$ and $[\psi, \nu], [\psi, \psi] \in \wedge^2 \mathfrak{h}^* \otimes \wedge^2 \mathfrak{g}$ we see that (7) is equivalent to

$$[\mu, \psi] = 0, \quad 2[\psi, \nu] + [\psi, \psi] = 0 \quad (8)$$

Let us show that

- from $[\mu, \psi] = 0$ it follows that $\theta(x)$ is a derivation of \mathfrak{g} preserving (\cdot, \cdot) ;
- from $2[\psi, \nu] + [\psi, \psi] = 0$ it follows that the map θ is a homomorphism from \mathfrak{h} to $\mathcal{D}er(\mathfrak{g})$

We have for any $x \in \mathfrak{h}$:

$$0 = [x, [\mu, \psi]] = [[x, \mu], \psi] - [\mu, [x, \psi]] = -[\mu, [x, \psi]] = -[\mu, \theta(x)].$$

Now we apply Proposition 3.

Let us study the second equation in (8). We need to show that the following holds:

$$\theta([x, [y, \nu]]) = [\theta(x), \theta(y)] \text{ for all } x, y \in \mathfrak{h}.$$

Indeed,

$$\theta([x, [y, \nu]]) = [\psi, [x, [y, \nu]]] = [[\psi, x], [y, \nu]] - [x, [[\psi, y], \nu]] + [x, [y, [\psi, \nu]]], \quad x, y \in \mathfrak{h}.$$

Notice that $[\psi, x] \in \wedge^2 \mathfrak{g}$. Therefore, $[[\psi, x], [y, \nu]] = 0$. Similarly, $[x, [[\psi, y], \nu]] = 0$. Therefore,

$$\theta([x, [y, \nu]]) = [x, [y, [\psi, \nu]]], \quad x, y \in \mathfrak{h}.$$

Our statement follows from the following observation:

$$-\frac{1}{2}[x, [y, [\psi, \psi]]] = [x, [\psi, [y, \psi]]] = [[x, \psi], [y, \psi]] + 0 = [\theta(x), \theta(y)]. \quad \square$$

Similar idea may be used to define a double extension for other types of quadratic algebras.

2.2 Solvable skew-symmetric invariant n -ary algebras

Define by induction the following subalgebras in \mathfrak{a} .

$$\mathfrak{a}^{(1)} := \{\mathfrak{a}, \dots, \mathfrak{a}\}, \quad \mathfrak{a}^{(2)} := \{\mathfrak{a}^{(1)}, \dots, \mathfrak{a}^{(1)}\}, \dots$$

By induction, we see that $\mathfrak{a}^{(k+1)} \subset \mathfrak{a}^{(k)}$ and more precisely that $\mathfrak{a}^{(k+1)}$ is an ideal in $\mathfrak{a}^{(k)}$. We call n -ary superalgebra \mathfrak{a} *solvable* if there exists an integer K such that $\mathfrak{a}^{(K)} = \{0\}$.

Proposition 4. *Subalgebras and homomorphic images of a solvable commutative n -ary superalgebra are solvable.*

Proof. Let \mathfrak{b} be a subalgebra in a solvable n -ary superalgebra \mathfrak{a} . Then we see

$$\mathfrak{b}^{(1)} = \{\mathfrak{b}, \dots, \mathfrak{b}\} \subset \{\mathfrak{a}, \dots, \mathfrak{a}\} = \mathfrak{a}^{(1)}.$$

By induction we have $\mathfrak{b}^{(k)} \subset \mathfrak{a}^{(k)}$ for all k . Hence, \mathfrak{b} is solvable.

Let $\phi : \mathfrak{a} \rightarrow \mathfrak{h}$ be a homomorphism. Denote by \mathfrak{b} the image of ϕ . Then,

$$\mathfrak{b}^{(1)} = \{\phi(\mathfrak{a}), \dots, \phi(\mathfrak{a})\} = \phi(\{\mathfrak{a}, \dots, \mathfrak{a}\}) = \phi(\mathfrak{a}^{(1)}).$$

Again by induction we have $\mathfrak{b}^{(k)} = \phi(\mathfrak{a}^{(k)})$. Hence, \mathfrak{b} is solvable. The proof is complete. \square

In particular, all solvable skew-symmetric n -ary algebras can be obtained inductively by such double extensions.

Clearly, $\mathfrak{a}^{(1)} = \{\mathfrak{a}, \dots, \mathfrak{a}\}$ is an ideal in \mathfrak{a} and any proper subspace \mathfrak{i} in \mathfrak{a} such that $\mathfrak{a}^{(1)} \subset \mathfrak{i}$ is an ideal in \mathfrak{a} . Hence, there exists a maximal ideal of codimension 1. The result follows from Theorem 1.

The second observation follows from $\mathfrak{g} \simeq \mathfrak{i}/\mathfrak{i}^\perp$, hence \mathfrak{g} is solvable.

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References

- [ABB2] *Albuquerque, H., Barreiro, E., Benayadi, S.* Quadratic Lie superalgebras with a reductive even part. *J. Pure Appl. Algebra* 213 (2009), no. 5, 724-731.
- [BBB] *Bajo, I., Benayadi, S., Bordemann, M.* Generalized double extension and descriptions of quadratic Lie superalgebras arXiv: math-ph/0712.0228 (2007)

- [BB] *Benamor, H., Benayadi, S.* Double extension of quadratic Lie superalgebras. Comm. Algebra 27 (1999), no. 1, 67-88.
- [Bord] *Bordemann, M.* Nondegenerate invariant bilinear forms on nonassociative algebras. Acta Math. Univ. Comenian. (N.S.) 66 (1997), no. 2, 151-201.
- [FF] *Figueroa-O'Farrill, J.* Metric Lie n -algebras and double extensions. arXiv:0806.3534
- [HW] *Hanlon P., Wachs M. L.* On Lie k -algebras, Adv. in Math. 113 (1995) 206-236.
- [KoSch1] *Kosmann-Schwarzbach, Y.* Derived brackets. Letters in Mathematical Physics 69: 61-87, 2004.
- [MR] *Medina, A., Revoy, Ph.* Algèbres de Lie et produit scalaire invariant. (French) [Lie algebras and invariant scalar products] Ann. Sci. cole Norm. Sup. (4) 18 (1985), no. 3, 553-561.
- [V] *Vishnyakova E.* Commutative n -ary superalgebras with an invariant skew-symmetric form. arXiv:1409.4342.
- [Vor1] *Voronov, Th.* Higher derived brackets and homotopy algebras. J. Pure Appl. Algebra 202 (2005), no. 1-3, 133-153.

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