

Higher Genus Correlation Functions in CFTs with $T\bar{T}$ Deformation

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Abstract

We construct the correlation functions of conformal field theories (CFTs) on genus two Riemann surface with the $T\bar{T}$ deformation in terms of perturbative CFT approach. Thanks to sewing construction to form higher genus Riemann surface from lower genus ones and conformal symmetry, we systematically obtain the first order $T\bar{T}$ correction to the higher genus correlation functions in the $T\bar{T}$ deformed CFTs, e.g. partition function and one/higher-point correlation functions. In the weak coupling limit, the higher genus correlation functions can be decomposed by lower genus correlation functions. Our results offer $T\bar{T}$ deformed field theories data to allow us to extract quantum chaos signals and multiple-interval Rényi entropies in deformed field theories.

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1 Introduction

The $T\bar{T}$ deformation of 2d quantum field theories (QFTs) was first introduced by Zamolodchikov [1], has attracted much research interest recently. This deformation

has many remarkable properties such as integrability [2–7]. It means that for an undeformed theory with infinite commuting conserved charges, the charges keep conservation and still commute with each other along with the $T\bar{T}$ deformation. In a sense, the integrability of the $T\bar{T}$ deformation makes it solvable.

Quantum chaotic theories are quite different from the integrable theories. It is a nature question to ask how $T\bar{T}$ deformation changes the characteristic behaviors of chaotic/integrable theories. To extract the quantum chaos signals [8], there were several objects capturing quantum chaos, for examples, spectrum form factor (SFF) [9], out of time ordered correlation function (OTOC) [10–12], operator growth [13], eigenstate thermalization hypothesis (ETH) [14–16], pole-skipping phenomenon [17], Loschmidt echo [18], and quantum entanglement entropies [19]. In deformed theories, there were some preliminary studies [20, 21] to extract quantum chaos signals in terms of quantum entanglement and OTOC. To read off quantum chaos signals from these quantities in the $T\bar{T}$ deformed theories, one has to construct the correlation functions essentially.

There were extensive attempts to construct the correlation functions in deformed theories. The $T\bar{T}$ deformed partition function on torus, namely zero-point correlation function, could be computed and the associated modular properties have been investigated in [22–24]. Furthermore the partition function with chemical potentials for KdV charges turning on was also obtained by [25, 26]. The correlation functions for stress tensor operators have been obtained in terms of random geometry [27, 28] and holography [29–32]. The deformed correlation functions for generic operators have been obtained in perturbative CFT approach [20, 21, 33, 34]. In [35], they obtained the deformed correlation functions of the deep UV theories in a non-perturbative way. In the context of a massive scalar [4] and Dirac fermion [36], integrability was used to fix renormalization ambiguities presented in correlation functions. More recently, authors of [37–39] try to construct surface charges of $T\bar{T}$ deformation to constrain the correlation functions.

In current work, one specific motivation to investigate the theories on higher genus manifold is to explore field theory data for holography, e.g. the field theory defined on torus will be important to understand the boundary theory which is the

holographic dual to the BTZ black hole [40]. The other motivation to study the correlation functions in deformed theory on the torus is associated with reading the information about multiple-interval Rényi entropy [41–44]. Motivated by these, we focus on the correlation functions in $T\bar{T}$ -deformed CFTs live on a higher genus Riemann surface. There are main two ways called by sewing prescriptions to construct high genus Riemann surface, the one is Schottky Uniformization [44–46] and the other is offered by [47–49]. In the current work, we focus on the second sewing prescription. We assume that on genus two Riemann surface, the $T\bar{T}$ deformed action can still be written as

$$S^\lambda = S_{CFT} - \frac{i}{2}\lambda \int_{\Sigma^{(2)}} d^2z T\bar{T}^\lambda(z, \bar{z}), \quad (1)$$

where $\Sigma^{(2)}$ is the genus two Riemann surface formed by sewing two tori, and z are the local coordinates on them. We focus on the deformation nearby the CFTs, which means the $T\bar{T}$ coupling λ is sufficiently small, and the conformal Ward identity still holds when we calculate the first order deformation.

The structure of this paper is as follows. In Section 2, we review general approaches to form a genus two Riemann surface by sewing two tori, and the associated Ward identity used. In Section 3, we apply the conformal Ward identity on genus two Riemann surface to calculate the first order $T\bar{T}$ deformation of the partition function. In Section 4, we calculate the first order deformation of correlation functions. Conclusions and discussions are given in the final section. In the appendices, we would like to list some relevant techniques and notations which are useful in our analysis.

2 Mathematical preparation

In this section, we review some mathematical facts about the Riemann surface which are relevant in the main content. Firstly, we present a typical approach of sewing together two tori to form a genus two Riemann surface, and review a method of constructing high genus partition function. Then we review the genus two Ward identity, which plays a crucial role in calculating the first-order $T\bar{T}$ deformation.

2.1 Sewing construction

We review a general approach to form a genus two Riemann surface by sewing together two tori [47–49]. Two complete tori T_a^2 for $a = 1, 2$ are introduced with modular parameters τ_a and local coordinates z_a , respectively. We construct the torus T_a^2 with periods $2\pi i$ and $2\pi i\tau_a$. A closed disk $\{z_a, |z_a| \leq r_a\}$ is introduced on T_a^2 , which has radius r_a and is centered at $z_a = 0$. For the disk, in order not to overlap with itself, its radius r_a must be less than half the minimum period D_a of the torus. A complex parameter ϵ is introduced to sew the two tori together, and an annulus $A_a = \{z_a, |\epsilon|/r_{\bar{a}} \leq |z_a| \leq r_a\}$ is introduced on each torus T_a^2 . The modulus of ϵ is upper bounded:

$$|\epsilon| \leq r_a r_{\bar{a}} < \frac{1}{4} D_a D_{\bar{a}}, \quad (2)$$

where we use the convention $\bar{1} = 2, \bar{2} = 1$. The genus two Riemann surface, which is denoted as $\Sigma^{(2)}$ in the following context, can be formed by identifying two annuli A_1 and A_2 as a single region A via the sewing relation $z_1 z_2 = \epsilon$. After removing the small disk $\{z_a, |z_a| \leq |\epsilon|/r_{\bar{a}}\}$ and annulus A , the remainder of the torus T_a^2 is denoted as S_a , thus $\Sigma^{(2)}$ can be divided into three parts, as shown in Fig.1:

$$\Sigma^{(2)} = S_1 \cup A \cup S_2. \quad (3)$$

By this construction, a sewn genus two Riemann surface can be completely described by five parameters $(\tau_1, \tau_2, \epsilon, r_1, r_2)$. The two radius parameters r_1 and r_2 are independent of most properties of the Riemann surface. For instance, the period matrix Ω of $\Sigma^{(2)}$, which is used to parameterize the moduli space of Riemann surfaces, can be completely determined by the parameters τ_1, τ_2 and ϵ , by a holomorphic map $F^\Omega : (\tau_1, \tau_2, \epsilon) \mapsto \Omega(\tau_1, \tau_2, \epsilon)$ in [48]. Here the parameters $\tau_1, \tau_2 \in \mathbb{H}_1$ and $\epsilon \in \{\epsilon, |\epsilon| < \frac{1}{4} D_1 D_2\} \subset \mathbb{C}$, and the period matrix $\Omega \in \mathbb{H}_2$, where \mathbb{H}_n is the Siegel upper half-space. The explicit form of $\Omega(\tau_1, \tau_2, \epsilon)$ is given by

$$2\pi i \Omega_{11} = 2\pi i \tau_1 + \epsilon [A_2 (\mathbb{1} - A_1 A_2)^{-1}] (1, 1), \quad (4)$$

$$2\pi i \Omega_{22} = 2\pi i \tau_2 + \epsilon [A_1 (\mathbb{1} - A_2 A_1)^{-1}] (1, 1), \quad (5)$$

$$2\pi i \Omega_{12} = 2\pi i \Omega_{21} = -\epsilon [(\mathbb{1} - A_1 A_2)^{-1}] (1, 1), \quad (6)$$

where $(1, 1)$ refers to the component of a matrix. A_a is the infinite matrix with components

$$A_a(m, n) = \epsilon^{\frac{m+n}{2}} (-1)^{m+1} \frac{1}{\sqrt{mn}} \frac{(m+n-1)!}{(m-1)!(n-1)!} E_{m+n}(\tau_a). \quad (7)$$

It has been proved in [48] that the holomorphic map F^Ω is injective but not surjective.

The general method of calculating the high genus partition function is to decompose it into correlation functions on lower genus Riemann surface. A Riemann surface $\Sigma^{(g)}$ can be divided into two lower genus ones by cutting it along a particular closed curve. For instance, the sewn Riemann surface with genus-2 in Fig.1 can be divided into two tori by cutting it along a non-contractible closed curve in the annulus A . By inserting the complete set of boundary states, the partition function on $\Sigma^{(2)}$ can be decomposed into set of one-point functions on two tori. We will follow the convention in [49] throughout the rest of the discussion. We will use the square bracket Vertex Operator Algebra, which was first introduced by Zhu in [50]. A new vertex operator $V[v, z]$ is defined as

$$V[v, z] = V(e^{zL(0)}v, e^z - 1). \quad (8)$$

In Vertex Operator Algebra, Virasoro generators are the Laurent modes of a particular vertex operator:

$$V[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}, \quad (9)$$

where $\tilde{\omega}$ is called conformal vector with conformal weight $L[0]\tilde{\omega} = wt[\tilde{\omega}]\tilde{\omega} = 2\tilde{\omega}$. For v is a primary state, $wt[v] = wt(v)$. \mathcal{H} is used to denote the complex vector space in which quantum states live, which can be written as a sum of subspaces spanned by states with the same conformal weight:

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_{[n]}, \quad (10)$$

where $\mathcal{H}_{[n]} = \{v \in \mathcal{H} | L[0]v = wt[v]v = nv\}$. $\{u_i^{[n]} \in \mathcal{H}_{[n]}, i = 1, \dots, \dim \mathcal{H}_{[n]}\}$ is a basis for $\mathcal{H}_{[n]}$, and its dual basis $\{\bar{u}_i^{[n]}\}$ is defined by $\langle u_i^{[n]}, \bar{u}_j^{[n]} \rangle_{\text{sq}} = \delta_{i,j}$, where $\langle, \rangle_{\text{sq}}$ is the square bracket Li-Zamolodchikov metric in [51]. The partition function $Z^{(2)}(\tau_1, \tau_2, \epsilon)$

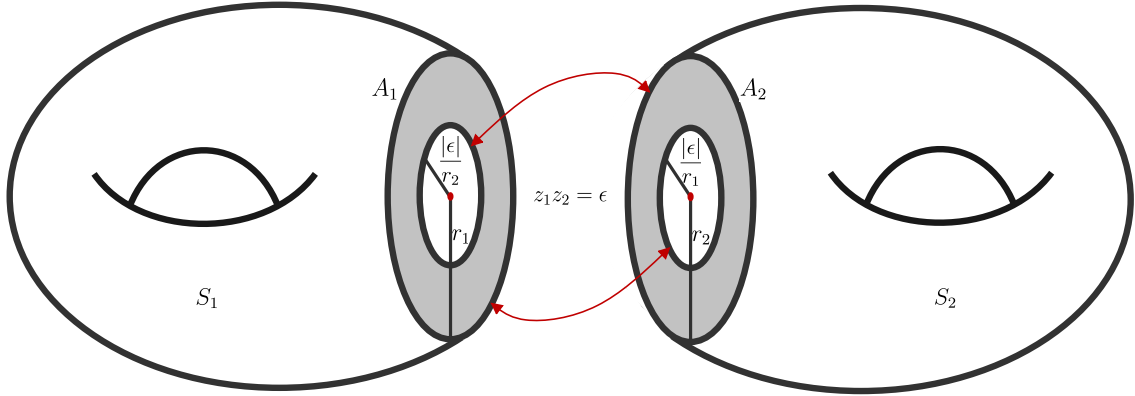


Figure 1: The method for constructing a genus two Riemann surface. The annuli A_1 and A_2 (two gray areas) are introduced on the two tori respectively, and identify them as a single region by sewing relation $z_1 z_2 = \epsilon$. The two red lines refer to the identification of the boundaries of the two annuli.

on the sewn Riemann surface $\Sigma^{(2)}$ can be decomposed into a combination of product of one-point functions on two tori. $Z^{(1)}(v, x; \tau_a)$ is used to denote a unnormalized one-point function on torus with a general state v inserted at $x \in T_a^2$. The torus one-point function is independent of coordinate x because of translation invariance. The genus two partition function of the sewing construction is given by [51]

$$Z^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{i=1}^{\dim \mathcal{H}_{[n]}} Z^{(1)}(u_i^{[n]}, \tau_1) Z^{(1)}(\bar{u}_i^{[n]}, \tau_2). \quad (11)$$

In particular, the subspace $\mathcal{H}_{[0]}$ is spanned by the vacuum state $\mathbf{1}$ in CFT, and for vacuum state the one-point function $Z^{(1)}(\mathbf{1}; \tau_a)$ is the partition function on torus T_a^2 .

There are other sewing constructions to form a genus two Riemann surface. In [44, 46], four disks are removed from the Riemann sphere, and the boundaries of each pair of disks are identified to obtain two handles. Besides, the annuli A_1 and A_2 mentioned above can be introduced on the same torus and are centered at two separate points. One can identify the two annuli by sewing relation to form a self-sewn torus [48]. One can also obtain the Riemann surface of arbitrary genus by sewing some spheres with three punctures on each ³. In this paper, our calculation is based on the sewing construction in Fig.1, and we follow the conventions in [49].

³One can refer to [52] and section 9.3 of [53] for construction details.

2.2 Genus two Ward identity

We review the genus two ward identity derived by Gilroy and Tuite in [49]. They generalize the Zhu recursion [50] to genus two correlation function, which provide an approach to represent $(n + 1)$ -point function in terms of n -point functions. The genus-2 Ward identity is a special case of their result. On the genus two Riemann surface $\Sigma^{(2)}$, some general states u_1, \dots, u_L and v_1, \dots, v_R are inserted at $x_1, \dots, x_L \in S_1 \cup A$ and $y_1, \dots, y_R \in S_2 \cup A$, respectively, which provides an unnormalized $(L + R)$ -point correlation function denoted as $Z(u_1, x_1; \dots; u_L, x_L; v_1, y_1; \dots; v_R, y_R; \tau_1, \tau_2, \epsilon)$. The conformal vector $\tilde{\omega}$ are inserted at $z \in \Sigma^{(2)}$ to obtain an $(L + R + 1)$ -point function, which satisfies the genus two Ward identity [49]

$$\begin{aligned}
& Z(\tilde{\omega}, z; \mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}; \tau_1, \tau_2, \epsilon) \\
&= D_z Z(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}; \tau_1, \tau_2, \epsilon) \\
&+ \sum_{l=1}^L \sum_{j \geq 0} {}^2\mathcal{P}_{1+j}(z, x_l; \tau_1, \tau_2, \epsilon) Z(\dots; L[j-1]u_l, x_l; \dots) \\
&+ \sum_{r=1}^R \sum_{j \geq 0} {}^2\mathcal{P}_{1+j}(z, y_r; \tau_2, \tau_1, \epsilon) Z(\dots; L[j-1]v_r, y_r; \dots), \tag{12}
\end{aligned}$$

where $(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y})$ is used to mark $(u_1, x_1; \dots; u_L, x_L; v_1, y_1; \dots; v_R, y_R)$. In what follows, we will explain the notations in eq.(12) and introduce some of our conventions to facilitate the subsequent calculation. The operator D_z contains the derivative operator of the sewing parameters $(\tau_1, \tau_2, \epsilon)$ which is defined as

$$D_z = {}^2\mathcal{F}_1(z; \tau_1, \tau_2, \epsilon) \frac{1}{2\pi i} \partial_{\tau_1} + {}^2\mathcal{F}_2(z; \tau_1, \tau_2, \epsilon) \frac{1}{2\pi i} \partial_{\tau_2} + {}^2\mathcal{F}^\Pi(z; \tau_1, \tau_2, \epsilon) \epsilon^{\frac{1}{2}} \partial_\epsilon. \tag{13}$$

The index 2 at the top left of the \mathcal{F} function represents the conformal weight of $\tilde{\omega}$ in [49]. The definitions of ${}^2\mathcal{F}_a(z; \tau_1, \tau_2, \epsilon)$ for $a = 1, 2$ are

$${}^2\mathcal{F}_a(z; \tau_1, \tau_2, \epsilon) = \begin{cases} 1 + \sum_{k \geq 1} P_{k+3}(z, \tau_a) \alpha_a(k), & z \in S_a \cup A, \\ \sum_{k \geq 1} P_{k+3}(z, \tau_{\bar{a}}) \beta_{\bar{a}}(k), & z \in S_{\bar{a}} \cup A, \end{cases} \tag{14}$$

where $P_k(z, \tau_a)$ is the Elliptic function defined in Appendix A. When the conformal vector $\tilde{\omega}$ is inserted on the different torus, the ${}^2\mathcal{F}_a(z; \tau_1, \tau_2, \epsilon)$ takes various forms.

The column vectors with components $\alpha_a(k)$ and $\beta_a(k)$ are defined as follows. Firstly, the infinite matrix $\tilde{\Lambda}_a$ is introduced with components

$$\tilde{\Lambda}_a(m, n) = \epsilon^{\frac{m+n+2}{2}} (-1)^{n+1} \begin{pmatrix} m+n+1 \\ n+2 \end{pmatrix} E_{m+n+2}(\tau_a), \quad (15)$$

where $E_n(\tau_a)$ is the Eisenstein series (see Appendix A), which is equal to zero for n odd. Then we define the Θ_a matrix by $\Theta_a = (\mathbb{1} - \tilde{\Lambda}_a \tilde{\Lambda}_a)^{-1}$, where $\mathbb{1}$ is the identity matrix. We expand the components of Θ_a by the powers of ϵ and have

$$\begin{aligned} & \Theta_a(m, n) \\ &= \delta_{m,n} + \sum_{k \geq 1} \epsilon^{\frac{m+n+2k+4}{2}} A_{k,\bar{a}}^{m,n+2} E_{k+n+2}(\tau_a) + \left(\sum_{k \geq 1} \epsilon^{\frac{m+n+2k+4}{2}} A_{k,\bar{a}}^{m,n+2} E_{k+n+2}(\tau_a) \right)^2 + \dots, \end{aligned} \quad (16)$$

where the coefficient $A_{k,a}^{m,n}$ is

$$A_{k,a}^{m,n} = (-1)^{k+n} \begin{pmatrix} m+k+1 \\ k+2 \end{pmatrix} \begin{pmatrix} k+n-1 \\ n \end{pmatrix} E_{m+k+2}(\tau_a). \quad (17)$$

By the properties of the Eisenstein series, only if $(m+n)$ is even, $A_{k,a}^{m,n}$ does not all equal to zero. The components $\alpha_a(k)$ and $\beta_a(k)$ are defined as

$$\alpha_a(k) = \sum_{l \geq 1} \epsilon^{\frac{k+l+6}{2}} \Theta_a(k, l) A_{1,\bar{a}}^{l,1}, \quad (18)$$

$$\beta_a(k) = \epsilon^{\frac{k+3}{2}} \Theta_a(k, 1). \quad (19)$$

The definition of ${}^2\mathcal{F}^\Pi(z; \tau_1, \tau_2, \epsilon)$ that appears in eq.(13) is

$${}^2\mathcal{F}^\Pi(z; \tau_1, \tau_2, \epsilon) = \epsilon^{-\frac{1}{2}} \sum_{k \geq 1} P_{k+1}(z, \tau_a) \theta_a(k) \quad (20)$$

for $z \in S_a \cup A$. θ_a is a column vector with components

$$\theta_a(k) = \epsilon^{\frac{k+1}{2}} \times \begin{cases} 1, & k = 1, \\ 0, & k = 2, \\ \sum_{l \geq 1} \epsilon^{\frac{l+1}{2}} \Theta_a(k-2, l) \left(l E_{l+1}(\tau_a) + \sum_{m \geq 1} \epsilon^{m+1} A_{m,\bar{a}}^{l,1} E_{m+1}(\tau_a) \right), & k \geq 3. \end{cases} \quad (21)$$

The function ${}^2\mathcal{P}_{1+j}(z, x; \tau_a, \tau_{\bar{a}}, \epsilon)$ that appears in Ward identity eq.(12) is referred to as the Genus Two Generalised Weierstrass Function in [49], and the index 2 at the top left of it has the same meaning as ${}^2\mathcal{F}(z; \tau_1, \tau_2, \epsilon)$. Firstly, the definition of ${}^2\mathcal{P}_1(z, x; \tau_a, \tau_{\bar{a}}, \epsilon)$ is

$${}^2\mathcal{P}_1(z, x; \tau_a, \tau_{\bar{a}}, \epsilon) = \begin{cases} P_1(z - x, \tau_a) - P_1(z, \tau_a) + \sum_{k \geq 1} P_{k+3}(z, \tau_a) [\xi_a^{(0)}(x)](k), & z, x \in S_a \cup A, \\ -\epsilon P_3(z, \tau_{\bar{a}}) + \sum_{k \geq 1} P_{k+3}(z, \tau_{\bar{a}}) [\zeta_{\bar{a}}^{(0)}(x)](k), & z \in S_{\bar{a}} \cup A, x \in S_a \cup A. \end{cases} \quad (22)$$

We define the two column vectors $\xi_a^{(0)}(x)$ and $\zeta_{\bar{a}}^{(0)}(x)$ that depend on the coordinate x with components

$$[\xi_a^{(0)}(x)](k) = \sum_{l \geq 1} \epsilon^{\frac{k+l+4}{2}} \Theta_a(k, l) \left[\frac{l(l+1)}{2} E_{l+2}(\tau_{\bar{a}}) + \sum_{m \geq 1} \epsilon^m A_{m, \bar{a}}^{l, 0} P'_m(x, \tau_a) \right], \quad (23)$$

$$[\zeta_{\bar{a}}^{(0)}(x)](k) = - \sum_{l \geq 1} \epsilon^{\frac{k+l+2}{2}} \Theta_a(k, l) \left[P'_l(x, \tau_{\bar{a}}) + \sum_{m \geq 1} \epsilon^{m+2} A_{m, \bar{a}}^{l, 2} E_{m+2}(\tau_a) \right], \quad (24)$$

where we define $P'_k(x, \tau) = P_k(x, \tau) - E_k(x, \tau)$, which is the elliptic function P_k minus the constant term of its Laurent expansion. The functions ${}^2\mathcal{P}_{1+j}(z, x; \tau_1, \tau_2, \epsilon)$ for $j \geq 1$ are defined by taking the derivative of ${}^2\mathcal{P}_1(z, x; \tau_1, \tau_2, \epsilon)$ multiple times ${}^2\mathcal{P}_{1+j}(z, x) = (j!)^{-1} \partial_x^j [{}^2\mathcal{P}_1(z, x)]$ in [49]. We introduce two column vectors $\xi_a^{(j)}(x)$ and $\zeta_{\bar{a}}^{(j)}(x)$ to write ${}^2\mathcal{P}_{1+j}(z, x)$ as

$${}^2\mathcal{P}_{1+j}(z, x; \tau_a, \tau_{\bar{a}}, \epsilon) = \begin{cases} P_{1+j}(z - x, \tau_a) + \sum_{k \geq 1} P_{k+3}(z, \tau_a) [\xi_a^{(j)}(x)](k), & z, x \in S_a \cup A, \\ \sum_{k \geq 1} P_{k+3}(z, \tau_{\bar{a}}) [\zeta_{\bar{a}}^{(j)}(x)](k), & z \in S_{\bar{a}} \cup A, x \in S_a \cup A, \end{cases} \quad (25)$$

where

$$[\xi_a^{(j)}(x)](k) = \sum_{l, m \geq 1} \epsilon^{\frac{k+l+2m+4}{2}} \Theta_a(k, l) A_{m, \bar{a}}^{l, j} P_{m+j}(x, \tau_a), \quad (26)$$

$$[\zeta_{\bar{a}}^{(j)}(x)](k) = (-1)^{1+j} \sum_{m \geq 1} \epsilon^{\frac{k+m+2}{2}} \Theta_a(k, m) \binom{m+j-1}{j} P_{m+j}(x, \tau_{\bar{a}}). \quad (27)$$

In eq.(12), we are only considering the Ward identity by inserting the holomorphic stress tensor. The Ward identity for inserting an anti-holomorphic stress tensor is similar to eq.(12), we conjugate all the functions, coordinates and parameters, and replace $L[j - 1]$ by $\bar{L}[j - 1]$.

3 Partition function

In this section, we apply the genus two Ward identity eq.(12) to calculate the first-order deformation of partition function under $T\bar{T}$ on $\Sigma^{(2)}$. The deformed Lagrangian \mathcal{L}^λ with parameter λ satisfies the flow equation

$$\partial_\lambda \mathcal{L}^\lambda(z, \bar{z}) = -T\bar{T}^\lambda(z, \bar{z}), \quad (28)$$

where $T\bar{T}^\lambda(z, \bar{z})$ is the deformed $T\bar{T}$ operator. We can expand the \mathcal{L}^λ and $T\bar{T}^\lambda$ to the power of λ :

$$\mathcal{L}^\lambda = \sum_{n=0}^{\infty} \lambda^n \mathcal{L}^{(n)}, \quad T\bar{T}^\lambda = \sum_{n=0}^{\infty} \lambda^n T\bar{T}^{(n)}. \quad (29)$$

The index (n) here and below refers to the expansion coefficient of order λ^n . From the flow equation eq.(28), we can derive a recursion relation

$$\mathcal{L}^{(n+1)} = -\frac{1}{n+1} T\bar{T}^{(n)}. \quad (30)$$

We write down the deformed action⁴ S^λ and expand it to the first-order of λ

$$S^\lambda = \int_{\Sigma^{(2)}} d^2x \mathcal{L}^\lambda = \frac{i}{2} \int_{\Sigma^{(2)}} d^2z \left(\mathcal{L}^{(0)} - \lambda T\bar{T}^{(0)} \right) + O(\lambda^2). \quad (32)$$

The deformed partition function Z^λ can be derived using the path integral formula

$$\begin{aligned} Z^\lambda &= \int D\phi e^{-S^\lambda[\phi]} \\ &= \int D\phi \left(1 + \lambda \frac{i}{2} \int_{\Sigma^{(2)}} d^2z T\bar{T}^{(0)} \right) e^{-S^{(0)}[\phi]} + O(\lambda^2) \\ &= Z^{(0)} + \lambda Z^{(0)} \frac{i}{2} \int_{\Sigma^{(2)}} d^2z \langle T\bar{T}^{(0)} \rangle^{(0)} + O(\lambda^2), \end{aligned} \quad (33)$$

⁴Since $\Sigma^{(2)}$ is formed by sewing, the integral in the action should be divided into three parts

$$\int_{\Sigma^{(2)}} d^2z = \int_{S_1} d^2z_1 + \int_A d^2z_1 + \int_{S_2} d^2z_2, \quad (31)$$

where z_1 and z_2 are the coordinates of the two tori.

Here $Z^{(0)}$ in the second term of eq.(33) comes from the normalization. $Z^{(0)}$ and $T^{(0)}$ are the undeformed partition function and stress tensor, respectively, and we will omit the index (0) below. The radius parameters r_1, r_2 affect the first-order deformation of the partition function (the domain of integration depends on r_1 and r_2 , see Fig.1). To ensure that the $T\bar{T}$ first-order deformation on $\Sigma^{(2)}$ can degenerate to that on two tori when we take the limit $\epsilon \rightarrow 0$, we set up a relationship between radius parameter and ϵ :

$$r_1 = r_2 = \sqrt{|\epsilon|}. \quad (34)$$

In current work, for simplicity, we choose the above condition which satisfies the prerequisite conditions given by eq.(2). Intuitively, the condition causes the width of the annulus A to be 0. Even though the condition eq.(34) is introduced to constrain the shape of the Riemann surface $\Sigma^{(2)}$ largely, we can still read two characteristics from the sewing parameter ϵ . The modulus $|\epsilon|$ determines the coupling degree of the two tori, and the argument $e^{i\varphi} = \epsilon/|\epsilon|$ determines the relative rotation between them. The undeformed theory is assumed to be a CFT, which means that the expectation value $\langle T\bar{T} \rangle$ (at zeroth-order of λ) can be calculated by CFT Ward identity. From the genus two Ward identity eq.(12), $\langle T\bar{T}(z, \bar{z}) \rangle$ is regular on the annulus A , and its integral over A vanishes under condition eq.(34). Finally, the first-order deformation of the partition function can be divided into two parts

$$\begin{aligned} \delta_\lambda Z &= \frac{i}{2} \int_{S_1} d^2 z_1 Z \langle T\bar{T}(z_1, \bar{z}_1) \rangle + \frac{i}{2} \int_{S_2} d^2 z_2 Z \langle T\bar{T}(z_2, \bar{z}_2) \rangle \\ &= \frac{i}{2} \int_{S_1} d^2 z_1 D_{z_1} \bar{D}_{\bar{z}_1} Z + \frac{i}{2} \int_{S_2} d^2 z_2 D_{z_2} \bar{D}_{\bar{z}_2} Z. \end{aligned} \quad (35)$$

The integral of $D_z \bar{D}_{\bar{z}}$ is calculated in detail in Appendix C.2. Here we consider the weak coupling between S_1 and S_2 , which means that $|\epsilon|$ is sufficiently small. We approximate $\delta_\lambda Z$ to $|\epsilon|^1$ and obtain:

$$\delta_\lambda Z = \sum_{a=1}^2 \left[\frac{1}{2} (\text{Im}[\tau_a] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_a} \partial_{\bar{\tau}_a} Z - \frac{i}{2} \bar{\epsilon} \partial_{\tau_a} \partial_{\bar{\epsilon}} Z \right] + \pi |\epsilon| \partial_\epsilon \partial_{\bar{\epsilon}} Z + c.c. + O(|\epsilon|^2), \quad (36)$$

where *c.c.* refers to complex conjugate term.

In particular, we consider the case of $\epsilon = 0$, which means that the two tori are completely decoupled. According to eq.(11), the genus two partition function Z

under $\epsilon = 0$ can be written as the product of two torus partition functions Z_1 and Z_2 . The decoupled first-order deformation can be obtained from eq.(36):

$$\begin{aligned}\delta_\lambda Z \Big|_{\epsilon=0} &= Z_2 \text{Im}[\tau_1] \partial_{\tau_1} \partial_{\bar{\tau}_1} Z_1 + Z_1 \text{Im}[\tau_2] \partial_{\tau_2} \partial_{\bar{\tau}_2} Z_2 \\ &= Z_2 \delta_\lambda Z_1 + Z_1 \delta_\lambda Z_2.\end{aligned}\tag{37}$$

One can read off the deformation of the torus partition function $\delta_\lambda Z_a$, which coincides with the first order deformation of the torus partition function obtained by [54] up to a normalization factor.

To close this section, we would like to add some potential applications of the deformed genus two partition function obtained here. One can apply the first order deformation of the partition function to investigate the multiple-interval Rényi entropy [41–44]. There has been a resurgence of interest in higher-genus partition functions of two-dimensional deformed CFTs, which is partly motivated by the perturbative study of entanglement entropies. The computation of entanglement entropies via the replica trick involves evaluating Rényi entropy, which can be regarded as certain higher-genus partition functions with the deformation perturbatively. Particularly interesting research applies calculations of Rényi entropy to check whether the holographic Ryu-Takayanagi formula [55, 56] exists or not in the $T\bar{T}$ deformed theories. This is a check of our basic understanding of AdS_3/CFT_2 duality with $T\bar{T}$ deformation.

4 Correlation functions

In this section, we further consider first-order deformation of correlation functions on genus two Riemann surface $\Sigma^{(2)}$. X is used to denote the product of a series of fields, which also flow under $T\bar{T}$. We expand X^λ to the power of λ :

$$X^\lambda = \sum_{n=0}^{\infty} \lambda^n X^{(n)} = X + \lambda \delta_\lambda X + O(\lambda^2).\tag{38}$$

The deformed correlation function $\langle X^\lambda \rangle^\lambda$ can be derived by the path integral formula

$$\langle X^\lambda \rangle^\lambda = \frac{1}{Z^\lambda} \int D\phi X^\lambda[\phi] e^{-S^\lambda[\phi]}$$

$$\begin{aligned}
&= \left(1 - \frac{\lambda \delta_\lambda Z}{Z}\right) \frac{1}{Z} \int D\phi \left(X + \lambda \delta_\lambda X\right) \left(1 + \lambda \frac{i}{2} \int_{\Sigma^{(2)}} d^2 z T \bar{T}^{(0)}\right) e^{-S[\phi]} + O(\lambda^2) \\
&= \langle X \rangle + \lambda \left[-\frac{\delta_\lambda Z}{Z} \langle X \rangle + \langle \delta_\lambda X \rangle + \frac{i}{2} \int_{\Sigma^{(2)}} d^2 z \langle T \bar{T}(z, \bar{z}) X \rangle \right] + O(\lambda^2). \quad (39)
\end{aligned}$$

In eq.(39), the first-order deformation $\delta_\lambda \langle X \rangle$ can be divided into three parts. The first part $-\frac{\delta_\lambda Z}{Z} \langle X \rangle$ is contributed by the deformation of the normalization factor (i.e. the partition function), which has been calculated in eq.(36). The second part $\langle \delta_\lambda X \rangle$ depends on the flow of the field X under $T\bar{T}$. Since λ is small enough, the conformal symmetry still hold and we do not take the $T\bar{T}$ flow effect of $\delta_\lambda X$ here as shown in [20, 33, 54]. In this paper we focus on the contribution of the third term $\frac{i}{2} \int_{\Sigma^{(2)}} d^2 z \langle T \bar{T}(z, \bar{z}) X \rangle$. We normalize the Ward identity eq.(12) by dividing both sides by Z :

$$\begin{aligned}
&\langle T(z) X(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}) \rangle \\
&= D_z \langle X(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}) \rangle + \langle X(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}) \rangle D_z \log Z \\
&\quad + \sum_{l=1}^L \sum_{j \geq 0} {}^2\mathcal{P}_{1+j}(z, x_l; \tau_1, \tau_2, \epsilon) \langle X(\dots; L[j-1]u_l, x_l; \dots) \rangle \\
&\quad + \sum_{r=1}^R \sum_{j \geq 0} {}^2\mathcal{P}_{1+j}(z, y_r; \tau_2, \tau_1, \epsilon) \langle X(\dots; L[j-1]v_r, y_r; \dots) \rangle. \quad (40)
\end{aligned}$$

In what follows, we will consider the $T\bar{T}$ deformed correlation function of primary fields and of stress tensors respectively. We will calculate the first-order deformation of one-point and two-point functions, and generalize to higher-point functions.

4.1 One-point function of primary field

The vertex operator of a primary state u satisfies

$$V[L[-1]u, x] = \partial_x V[u, x], \quad V[L[0]u, x] = wt[u]V[u, x], \quad V[L[j > 0]u, x] = 0. \quad (41)$$

Let X denote the product of primary fields, and Ward identity eq.(40) can be simplified to

$$\begin{aligned}
\langle T(z) X(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}) \rangle &= D_z \langle X(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}) \rangle + \langle X(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}) \rangle D_z \log Z \\
&\quad + \left(\sum_{l=1}^L \mathcal{P}_{z, x_l} + \sum_{r=1}^R \mathcal{P}_{z, y_r} \right) \langle X(\mathbf{u}, \mathbf{x}; \mathbf{v}, \mathbf{y}) \rangle, \quad (42)
\end{aligned}$$

where the operator $\mathcal{P}_{z,x}$ is defined as

$$\mathcal{P}_{z,x} = {}^2\mathcal{P}_1(z, x; \tau_a, \tau_{\bar{a}}, \epsilon) \partial_x + {}^2\mathcal{P}_2(z, x; \tau_a, \tau_{\bar{a}}, \epsilon) wt[u], \quad (43)$$

for $x \in S_a$, and $wt[u]$ is the conformal weight of u . Following the prescription in [57], we remove a small open disk $D_\delta = \{z, |z - x| < \delta\}$ centered on the singularity $z = x$, which introduces an additional boundary ∂D_δ to the domain of integration demonstrated by Fig.4. In the rest of $\Sigma^{(2)}$, the commutativity⁵ of $\mathcal{P}_{z,x}$ and $\bar{\mathcal{P}}_{\bar{z},\bar{x}}$ is preserved. After integration, we regularize it by taking the limit $\delta \rightarrow 0$ and simply discarding the divergence.

Let us begin with the first-order deformed one-point function of primary field. We insert a pair of primary states (u, \bar{u}) at $(x, \bar{x}) \in S_a$ and apply Ward identity eq.(42) to obtain

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) V \rangle \\ &= \frac{i}{2} \frac{1}{Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left[D_z \bar{D}_{\bar{z}} + D_z \bar{\mathcal{P}}_{\bar{z},\bar{x}} + \bar{D}_{\bar{z}} \mathcal{P}_{z,x} + \mathcal{P}_{z,x} \bar{\mathcal{P}}_{\bar{z},\bar{x}} \right] (Z \langle V \rangle). \end{aligned} \quad (45)$$

The integrals of $D_z \bar{\mathcal{P}}_{\bar{z},\bar{x}}$, $\bar{D}_{\bar{z}} \mathcal{P}_{z,x}$ and $\mathcal{P}_{z,x} \bar{\mathcal{P}}_{\bar{z},\bar{x}}$ is calculated in detail in Appendix C.3. Using eqs.(174)(176)(178) we obtain eq.(45) up to $|\epsilon|^1$:

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) V \rangle \\ &= \frac{1}{Z} \left\{ \sum_{a=1}^2 \left[\frac{1}{2} (\text{Im}[\tau_a] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_a} \partial_{\bar{\tau}_a} - \frac{i}{2} \bar{\epsilon} \partial_{\tau_a} \partial_{\bar{\epsilon}} \right] + \pi |\epsilon| \partial_\epsilon \partial_{\bar{\epsilon}} \right. \\ & \quad + i (\text{Re}[x] - \frac{|\epsilon|}{3} \bar{P}_1(\bar{x}, \bar{\tau}_a)) \partial_{\tau_a} \partial_{\bar{x}} + i \left(\frac{1}{2} + \frac{|\epsilon|}{3} \bar{P}_2(\bar{x}, \bar{\tau}_a) \right) wt[\bar{u}] \partial_{\tau_a} + \pi \epsilon P_1(x, \tau_a) \partial_\epsilon \partial_{\bar{x}} \\ & \quad + \pi (\log |Q(x, \tau_a)|^2 + \frac{1}{4} - \frac{|\epsilon|}{3} |P_1(x, \tau_a)|) \partial_x \partial_{\bar{x}} - \frac{\pi |\epsilon|}{3} |P_2(x, \tau_a)|^2 wt[u] wt[\bar{u}] \\ & \quad \left. + \pi (\bar{P}_1(\bar{x}, \bar{\tau}_a) + \frac{2|\epsilon|}{3} P_1(x, \tau_a) \bar{P}_2(\bar{x}, \bar{\tau}_a)) \partial_x wt[\bar{u}] \right\} (Z \langle V \rangle) + c.c. + O(|\epsilon|^2), \end{aligned} \quad (46)$$

where $\log Q(x, \tau_a)$ is introduced by eq.(80) as a primitive function of $P_1(x, \tau_a)$.

⁵The commutativity of $\mathcal{P}_{z,x}$ and $\bar{\mathcal{P}}_{\bar{z},\bar{x}}$ is broken at singularities, due to

$$\begin{aligned} \partial_x \bar{P}_1(\bar{z}, \bar{x}) &= \partial_x \frac{1}{\bar{z} - \bar{x}} = -\pi \delta^{(2)}(z - x), \\ \partial_x \bar{P}_2(\bar{z}, \bar{x}) &= \partial_x \left(\frac{1}{\bar{z} - \bar{x}} \right)^2 = -\pi \partial_{\bar{x}} \delta^{(2)}(z - x). \end{aligned} \quad (44)$$

4.2 Higher-point function of primary field

We consider the $T\bar{T}$ deformed correlation function with two pairs of primary states (u_1, \bar{u}_1) and (u_2, \bar{u}_2) inserted at (x_1, \bar{x}_1) and (x_2, \bar{x}_2) , respectively. According to eq.(42) the first-order correction depends on

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) V_1 V_2 \rangle \\ &= \frac{i}{2} \frac{1}{Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left[D_z \bar{D}_{\bar{z}} + D_z \bar{\mathcal{P}}_{\bar{z}, \bar{x}_1} + D_z \bar{\mathcal{P}}_{\bar{z}, \bar{x}_2} + \bar{D}_{\bar{z}} \mathcal{P}_{z, x_1} + \bar{D}_{\bar{z}} \mathcal{P}_{z, x_2} \right. \\ & \quad \left. + \mathcal{P}_{z, x_1} \bar{\mathcal{P}}_{\bar{z}, \bar{x}_1} + \mathcal{P}_{z, x_2} \bar{\mathcal{P}}_{\bar{z}, \bar{x}_2} + \mathcal{P}_{z, x_1} \bar{\mathcal{P}}_{\bar{z}, \bar{x}_2} + \mathcal{P}_{z, x_2} \bar{\mathcal{P}}_{\bar{z}, \bar{x}_1} \right] (Z \langle V_1 V_2 \rangle). \end{aligned} \quad (47)$$

Let us consider two different profiles. The one profile is that two points are inserted in the same torus $x_1, x_2 \in S_a$, one can obtain eq.(47) using eqs.(174)(176)(178)(182), up to $|\epsilon|^1$:

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) V_1 V_2 \rangle \\ &= \frac{1}{Z} \left\{ \sum_{a=1}^2 \left[\frac{1}{2} (\text{Im}[\tau_a] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_a} \partial_{\bar{\tau}_a} - \frac{i}{2} \bar{\epsilon} \partial_{\tau_a} \partial_{\bar{\epsilon}} \right] + \pi |\epsilon| \partial_{\epsilon} \partial_{\bar{\epsilon}} \right. \\ & \quad + \sum_{i=1}^2 \left[i (\text{Re}[x_i] - \frac{|\epsilon|}{3} \bar{P}_1(\bar{x}_i, \bar{\tau}_a)) \partial_{\tau_a} \partial_{\bar{x}_i} + i \left(\frac{1}{2} + \frac{|\epsilon|}{3} \bar{P}_2(\bar{x}_i, \bar{\tau}_a) \right) \text{wt}[\bar{u}_i] \partial_{\tau_a} \right. \\ & \quad + \pi \epsilon P_1(x_i, \tau_a) \partial_{\epsilon} \partial_{\bar{x}_i} + \pi (\log |Q(x_i, \tau_a)|^2 + \frac{1}{4} - \frac{|\epsilon|}{3} |P_1(x_i, \tau_a)|) \partial_{x_i} \partial_{\bar{x}_i} \\ & \quad \left. - \frac{\pi |\epsilon|}{3} |P_2(x_i, \tau_a)|^2 \text{wt}[u_i] \text{wt}[\bar{u}_i] + \pi (\bar{P}_1(\bar{x}_i, \bar{\tau}_a) + \frac{2|\epsilon|}{3} P_1(x_i, \tau_a) \bar{P}_2(\bar{x}_i, \bar{\tau}_a)) \partial_{x_i} \text{wt}[\bar{u}_i] \right] \\ & \quad + \pi (\log \frac{|Q(x_1, \tau_a) Q(x_2, \tau_a)|^2}{|Q(x_1 - x_2, \tau_a)|^2} + \frac{1}{2} - \frac{2|\epsilon|}{3} P_1(x_1, \tau_a) \bar{P}_1(\bar{x}_2, \bar{\tau}_a)) \partial_{x_1} \partial_{\bar{x}_2} \\ & \quad + \pi (\bar{P}_1(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) + \bar{P}_1(\bar{x}_2, \bar{\tau}_a) + \frac{2|\epsilon|}{3} P_1(x_1, \tau_a) \bar{P}_2(\bar{x}_2, \bar{\tau}_a)) \partial_{x_1} \text{wt}[\bar{u}_2] \\ & \quad + \pi (P_1(x_2 - x_1, \tau_a) + P_1(x_1, \tau_a) + \frac{2|\epsilon|}{3} P_2(x_1, \tau_a) \bar{P}_1(\bar{x}_2, \bar{\tau}_a)) \text{wt}[u_1] \partial_{\bar{x}_2} \\ & \quad \left. - \frac{2\pi |\epsilon|}{3} P_2(x_1, \tau_a) \bar{P}_2(\bar{x}_2, \bar{\tau}_a) \text{wt}[u_1] \text{wt}[\bar{u}_2] \right\} (Z \langle V_1 V_2 \rangle) + c.c. + O(|\epsilon|^2). \end{aligned} \quad (48)$$

The other profile is that two points are inserted in the different tori. We set $x_1 = x \in S_{a_1}$ and $x_2 = y \in S_{a_2}$, and insert primary states $u_1 = u$ and $u_2 = v$ at x and y respectively. The first-order correction is given by eqs.(174)(176)(178)(183):

$$\frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) V_x V_y \rangle$$

$$\begin{aligned}
&= \frac{1}{Z} \left\{ \sum_{a=1}^2 \left[\frac{1}{2} (\text{Im}[\tau_a] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_a} \partial_{\bar{\tau}_a} - \frac{i}{2} \bar{\epsilon} \partial_{\tau_a} \partial_{\bar{\epsilon}} \right] + \pi |\epsilon| \partial_{\epsilon} \partial_{\bar{\epsilon}} \right. \\
&+ \sum_{i=1}^2 \left[i (\text{Re}[x_i] - \frac{|\epsilon|}{3} \bar{P}_1(\bar{x}_i, \bar{\tau}_{a_i})) \partial_{\tau_{a_i}} \partial_{\bar{x}_i} + i \left(\frac{1}{2} + \frac{|\epsilon|}{3} \bar{P}_2(\bar{x}_i, \bar{\tau}_{a_i}) \right) \text{wt}[\bar{u}_i] \partial_{\tau_{a_i}} \right. \\
&+ \pi \epsilon P_1(x_i, \tau_{a_i}) \partial_{\epsilon} \partial_{\bar{x}_i} + \pi (\log |Q(x_i, \tau_{a_i})|^2 + \frac{1}{4} - \frac{|\epsilon|}{3} |P_1(x_i, \tau_{a_i})|) \partial_{x_i} \partial_{\bar{x}_i} \\
&- \frac{\pi |\epsilon|}{3} |P_2(x_i, \tau_{a_i})|^2 \text{wt}[u_i] \text{wt}[\bar{u}_i] + \pi (\bar{P}_1(\bar{x}_i, \bar{\tau}_{a_i}) + \frac{2|\epsilon|}{3} P_1(x_i, \tau_{a_i}) \bar{P}_2(\bar{x}_i, \bar{\tau}_{a_i})) \partial_{x_i} \text{wt}[\bar{u}_i] \left. \right] \\
&- \left. \frac{\pi}{2} (\bar{\epsilon} \bar{P}'_2(\bar{x}, \bar{\tau}_{a_1}) + \epsilon P'_2(y, \tau_{a_2})) \partial_x \partial_{\bar{y}} \right\} (Z \langle V_x V_y \rangle) + c.c. + O(|\epsilon|^2). \tag{49}
\end{aligned}$$

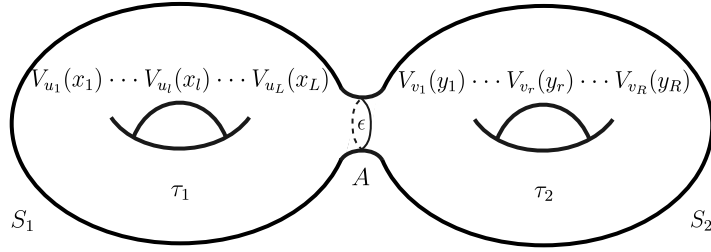


Figure 2: A primary $(L + R)$ -point function on sewn Riemann surface with genus-2.

We generalize our result to a $(L+R)$ -point function of primary fields. As shown in Fig.2, we insert L pairs of primary states $(u_1, \bar{u}_1), \dots, (u_L, \bar{u}_L)$ and R pairs of primary states $(v_1, \bar{v}_1), \dots, (v_R, \bar{v}_R)$ at $(x_1, \bar{x}_1), \dots, (x_L, \bar{x}_L) \in S_1$ and $(y_1, \bar{y}_1), \dots, (y_R, \bar{y}_R) \in S_2$, respectively. For a primary $(L + R)$ -point function⁶ $\langle X_L X_R \rangle = \langle \prod_{l=1}^L V_{u_l} \prod_{r=1}^R V_{v_r} \rangle$, the first-order correction is derived by Ward identity eq.(42) as follows:

$$\begin{aligned}
&\frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T \bar{T}(z, \bar{z}) X_L X_R \rangle \\
&= \frac{i}{2Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left\{ D_z \bar{D}_{\bar{z}} + \sum_{l=1}^L \left[D_z \bar{P}_{\bar{z}, \bar{x}_l} + \bar{D}_{\bar{z}} \mathcal{P}_{z, x_l} + \mathcal{P}_{z, x_l} \bar{P}_{\bar{z}, \bar{x}_l} \right] \right. \\
&+ \sum_{r=1}^R \left[D_z \bar{P}_{\bar{z}, \bar{y}_r} + \bar{D}_{\bar{z}} \mathcal{P}_{z, y_r} + \mathcal{P}_{z, y_r} \bar{P}_{\bar{z}, \bar{y}_r} \right] + \left[\sum_{l' \neq l''} \mathcal{P}_{z, x_{l'}} \bar{P}_{\bar{z}, \bar{x}_{l''}} + \sum_{r' \neq r''} \mathcal{P}_{z, y_{r'}} \bar{P}_{\bar{z}, \bar{y}_{r''}} \right] \\
&\left. + \sum_{l, r} \left[\bar{P}_{\bar{z}, \bar{x}_l} \mathcal{P}_{z, y_r} + \mathcal{P}_{z, x_l} \bar{P}_{\bar{z}, \bar{y}_r} \right] \right\} [Z \langle X_L X_R \rangle]. \tag{50}
\end{aligned}$$

The first term in eq.(50) is analogous to eq.(174) (we call it the zero-point contribution). The second and third terms in eq.(50) are the sum of all one-point

⁶For simplicity, the vertex operator corresponding to each pair of primary states (u_l, \bar{u}_l) at (x_l, \bar{x}_l) is denoted as V_{u_l} , and (v_r, \bar{v}_r) at (y_r, \bar{y}_r) is denoted as V_{v_r} .

contribution eqs.(175)(177). The fourth and fifth terms in eq.(50) contain all the two-point contribution eqs.(182)(183). No matter how many points are inserted, the first-order correction of correlation function contains at most two-point contribution. The contribution of three or more points can be found in higher-order correction.

4.3 One-point function of stress tensor

In the next two subsections, we consider the $T\bar{T}$ deformed correlation functions of stress tensor based on first-order perturbation theory. The holomorphic stress tensor $T(z)$ is the vertex operator of conformal vector $\tilde{\omega}$, and it is a quasi-primary field with conformal weight $wt[\tilde{\omega}] = 2$, which satisfies

$$V[L[-1]\tilde{\omega}, z] = \partial_z V[\tilde{\omega}, z], \quad V[L[0]\tilde{\omega}, z] = 2V[\tilde{\omega}, z], \quad V[L[j \geq 1]\tilde{\omega}, z] = \frac{c}{2}\mathbb{1}\delta_{j,2} \quad (51)$$

where $\mathbb{1}$ is the identity operator and c is the central charge. The one-point function of stress tensor can be obtained in undeformed CFT by Ward identity eq.(40) and operator D_x in eq.(13):

$$\langle T(x) \rangle = \frac{1}{Z} D_x Z = \frac{1}{2\pi i Z} {}^2\mathcal{F}_a(x) \partial_{\tau_a} Z + \frac{1}{2\pi i Z} {}^2\mathcal{F}_{\bar{a}}(x) \partial_{\tau_{\bar{a}}} Z + \frac{1}{Z} {}^2\mathcal{F}^\Pi(x) \epsilon^{\frac{1}{2}} \partial_\epsilon Z, \quad (52)$$

where $x \in S_a$. According to the definition of \mathcal{F} functions eqs.(14)(20), the expectation value $\langle T(x) \rangle$ is biperiodic on each torus. The first-order $T\bar{T}$ deformation $\delta_\lambda \langle T(x) \rangle$ is obtained by eq.(39):

$$\delta_\lambda \langle T(x) \rangle = -\frac{1}{Z^2} \delta_\lambda Z D_x Z + \langle \delta_\lambda T(x) \rangle + \frac{i}{2} \int_{\Sigma^{(2)}} d^2 z \langle T\bar{T}(z, \bar{z}) T(x) \rangle. \quad (53)$$

The first term is the flow effect of normalization factor, which has been calculated in eq.(36). The second term depends on specific Lagrangian. The third term $\frac{i}{2} \int_{\Sigma^{(2)}} d^2 z \langle T\bar{T}(z, \bar{z}) T(x) \rangle$ can be calculated by eq.(40):

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) T(x) \rangle \\ &= \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left\{ \frac{1}{Z} D_z \bar{D}_{\bar{z}} (D_x Z) + \frac{1}{Z} \bar{D}_{\bar{z}} \mathcal{P}_{z,x} (D_x Z) + \frac{c}{2Z} {}^2\mathcal{P}_4(z, x) \bar{D}_{\bar{z}} Z \right\}, \quad (54) \end{aligned}$$

which is discussed in detail in Appendix C.2. We approximate the first-order correction of $\langle T(x) \rangle$ to $|\epsilon|^1$ using eqs.(174)(176)(181):

$$\begin{aligned}
& \delta_\lambda \langle T(x) \rangle - \langle \delta_\lambda T(x) \rangle \\
&= -\frac{1}{2\pi} \left(1 + \frac{2|\epsilon|}{3} P_2(x, \tau_a)\right) \frac{\partial_{\tau_a} \partial_{\bar{\tau}_a} Z}{Z} - \frac{i\epsilon}{6} P_4(x, \tau_a) \frac{\partial_{\bar{\tau}_a} Z}{Z} \\
&\quad - \frac{i}{2\pi} \sum_{b=a, \bar{a}} \left[\left((\text{Im}[\tau_b] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_a} + \epsilon \text{Im}[\tau_b] P_2(x, \tau_a) \partial_\epsilon \right) \left(\frac{\partial_{\tau_b} \partial_{\bar{\tau}_b} Z}{Z} \right) \right] \\
&\quad - \frac{1}{4\pi} \sum_{b=a, \bar{a}} \partial_{\tau_a} \left[\bar{\epsilon} \frac{\partial_{\tau_b} \partial_{\bar{\epsilon}} Z}{Z} - \epsilon \frac{\partial_{\bar{\tau}_b} \partial_\epsilon Z}{Z} \right] + i\epsilon \left[(\text{Re}[x] \partial_x - i \text{Im}[\tau_a] \partial_{\tau_a} + 2) P_2(x, \tau_a) \right] \frac{\partial_{\bar{\tau}_a} \partial_\epsilon Z}{Z} \\
&\quad - i|\epsilon| (\partial_{\tau_a} + 2\pi i P_2(x, \tau_a)) \frac{\partial_\epsilon \partial_{\bar{\epsilon}} Z}{Z} + O(|\epsilon|^2). \tag{55}
\end{aligned}$$

In decoupling limit, the genus two partition function Z satisfies $Z|_{\epsilon=0} = Z_a Z_{\bar{a}}$, where Z_a denotes torus partition function on T_a^2 . According to eq.(52), the decoupled one-point function of stress tensor is $\langle T(x) \rangle|_{\epsilon=0} = \frac{1}{2\pi i Z_a} \partial_{\tau_a} Z_a = \langle T(x) \rangle_{T_a^2}$, where $\langle T(x) \rangle_{T_a^2}$ is completely defined on torus T_a^2 . One can read off the first-order deformation of $\langle T(x) \rangle_{T_a^2}$ from eq.(55) with $\epsilon = 0$:

$$\begin{aligned}
& \left(\delta_\lambda \langle T(x) \rangle - \langle \delta_\lambda T(x) \rangle \right) \Big|_{\epsilon=0} \\
&= -\frac{1}{2\pi} \frac{\partial_{\tau_a} \partial_{\bar{\tau}_a} Z_a Z_{\bar{a}}}{Z_a Z_{\bar{a}}} - \frac{i}{2\pi} \text{Im}[\tau_a] \partial_{\tau_a} \left[\frac{\partial_{\tau_a} \partial_{\bar{\tau}_a} Z_a Z_{\bar{a}}}{Z_a Z_{\bar{a}}} \right] - \frac{i}{2\pi} \text{Im}[\tau_{\bar{a}}] \partial_{\tau_a} \left[\frac{\partial_{\tau_a} \partial_{\bar{\tau}_a} Z_a Z_{\bar{a}}}{Z_a Z_{\bar{a}}} \right] \\
&= -\frac{1}{2\pi} \frac{\partial_{\tau_a} \partial_{\bar{\tau}_a} Z_a}{Z_a} - \frac{i}{2\pi} \text{Im}[\tau_a] \frac{\partial_{\tau_a}^2 \partial_{\bar{\tau}_a} Z_a}{Z_a} + \frac{i}{2\pi} \text{Im}[\tau_a] \frac{\partial_{\tau_a} \partial_{\bar{\tau}_a} Z_a}{Z_a} \frac{\partial_{\tau_a} Z_a}{Z_a} \\
&= \delta_\lambda \langle T(x) \rangle_{T_a^2} - \langle \delta_\lambda T(x) \rangle_{T_a^2}, \tag{56}
\end{aligned}$$

This result coincides with the first order deformation of the expectation value of stress tensor on torus obtained by [54] up to a normalization factor.

4.4 Higher-point function of stress tensor

We consider two types of the two-point function of stress tensor, $\langle T(x_1) T(x_2) \rangle$ and $\langle T(x_1) \bar{T}(\bar{x}_2) \rangle$ respectively. In undeformed CFT, these two expectation values can be obtained by eq.(40)

$$\begin{aligned}
\langle T(x_1) T(x_2) \rangle &= \frac{1}{Z} \left[D_{x_1} + {}^2\mathcal{P}_1(x_1, x_2) \partial_{x_2} + {}^2\mathcal{P}_2(x_1, x_2) \partial_x^2 \right] (D_{x_2} Z) + {}^2\mathcal{P}_4(x_1, x_2) \frac{c}{2}, \\
\langle T(x_1) \bar{T}(\bar{x}_2) \rangle &= \frac{1}{Z} D_{x_1} \bar{D}_{\bar{x}_2} Z. \tag{57}
\end{aligned}$$

As in the previous sections, we concentrate on using Ward identity eq.(40) to compute integrals $\frac{i}{2} \int_{\Sigma^{(2)}} d^2 z \langle T\bar{T}(z, \bar{z}) T_1 T_2 \rangle$ and $\frac{i}{2} \int_{\Sigma^{(2)}} d^2 z \langle T\bar{T}(z, \bar{z}) T_1 \bar{T}_2 \rangle$, which contribute to first-order deformation. For the first type $\langle T_1 T_2 \rangle$ we have

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) T_1 T_2 \rangle \\ &= \frac{i}{2} \frac{1}{Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left\{ \left[D_z \bar{D}_{\bar{z}} + \bar{D}_{\bar{z}} \mathcal{P}_{z, x_1} + \bar{D}_{\bar{z}} \mathcal{P}_{z, x_2} \right] (Z \langle T_1 T_2 \rangle) \right. \\ & \quad \left. + \frac{c}{2} \mathcal{P}_4(z, x_1) \bar{D}_{\bar{z}} (Z \langle T_2 \rangle) + \frac{c}{2} \mathcal{P}_4(z, x_2) \bar{D}_{\bar{z}} (Z \langle T_1 \rangle) \right\}. \end{aligned} \quad (58)$$

The one profile is that two insertion points live on the same torus $x_1, x_2 \in S_a$. the first-order correction is obtained by eqs.(57)(174)(176)(181), up to $|\epsilon|^1$:

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) T_1 T_2 \rangle \\ &= \frac{1}{Z} \left\{ \left[\sum_{b=a, \bar{a}} \left[(\text{Im}[\tau_b] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_b} \partial_{\bar{\tau}_b} + \frac{i}{2} \bar{\epsilon} \partial_{\tau_b} \partial_{\bar{\epsilon}} - \frac{i}{2} \epsilon \partial_{\bar{\tau}_b} \partial_{\bar{\epsilon}} \right] + 2\pi |\epsilon| [\partial_{\epsilon} \partial_{\bar{\epsilon}} - 2i \partial_{\bar{\tau}_a}] \right. \right. \\ & \quad - \frac{2i|\epsilon|}{3} \sum_{i=1,2} P_2(x_i, \tau_a) \partial_{\bar{\tau}_a} \left. \left(-\frac{1}{4\pi^2} \partial_{\tau_a}^2 Z + \frac{1}{\pi i} P_2(x_1 - x_2, \tau_a) \partial_{\tau_a} Z + \frac{c}{2} P_4(x_1 - x_2, \tau_a) Z \right) \right. \\ & \quad + \left. \left[(2\text{Re}[x_1 - x_2] - \frac{2|\epsilon|}{3} [P_1(x_1, \tau_a) - P_1(x_2, \tau_a)]) \partial_{\bar{\tau}_a} - 2\pi i \bar{\epsilon} [\bar{P}_1(\bar{x}_1, \bar{\tau}_a) - \bar{P}_1(\bar{x}_2, \bar{\tau}_a)] \partial_{\bar{\epsilon}} \right] \right. \\ & \quad \times \left. \left(\frac{1}{\pi} P_3(x_1 - x_2, \tau_a) \partial_{\tau_a} Z + ic P_5(x_1 - x_2, \tau_a) Z \right) \right. \\ & \quad + \epsilon \partial_{\epsilon} \left(\sum_{b=a, \bar{a}} \text{Im}[\tau_b] \partial_{\tau_b} \partial_{\bar{\tau}_b} - i \sum_{i=1,2} \text{Re}[x_i] \partial_{x_i} \partial_{\bar{\tau}_a} - 2i \partial_{\bar{\tau}_a} \right) \left[\frac{1}{2\pi i} \sum_{i=1,2} P_2(x_i, \tau_a) \partial_{\tau_a} Z \right. \\ & \quad + \left. \left(\frac{1}{2\pi i} \partial_{\tau_a} + [P_1(x_1 - x_2, \tau_a) - P_1(x_1, \tau_a)] \partial_{x_2} + 2P_2(x_1 - x_2, \tau_a) \right) P_2(x_2, \tau_a) Z \right] \\ & \quad \left. - \frac{c|\epsilon|}{12\pi} \sum_{i=1,2} P_4(x_i, \tau_a) \partial_{\tau_a} \partial_{\bar{\tau}_a} Z \right\} + O(|\epsilon|^2). \end{aligned} \quad (59)$$

The other profile is that two points are inserted in different tori $x \in S_a$ and $y \in S_{\bar{a}}$. We combine eqs.(57)(174)(176)(181) and have

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T\bar{T}(z, \bar{z}) T_x T_y \rangle \\ &= \frac{1}{Z} \left\{ \left[\sum_{b=a, \bar{a}} \left[(\text{Im}[\tau_b] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_b} \partial_{\bar{\tau}_b} + \frac{i}{2} \bar{\epsilon} \partial_{\tau_b} \partial_{\bar{\epsilon}} - \frac{i}{2} \epsilon \partial_{\bar{\tau}_b} \partial_{\bar{\epsilon}} \right] + 2\pi |\epsilon| [\partial_{\epsilon} \partial_{\bar{\epsilon}} - i \partial_{\bar{\tau}_a} - i \partial_{\bar{\tau}_{\bar{a}}}] \right. \right. \\ & \quad - \frac{i|\epsilon|}{3} P_2(x, \tau_a) \partial_{\bar{\tau}_a} - \frac{i|\epsilon|}{3} P_2(y, \tau_{\bar{a}}) \partial_{\bar{\tau}_{\bar{a}}} \left. \left(-\frac{1}{4\pi^2} \partial_{\tau_a} \partial_{\tau_{\bar{a}}} Z \right) \right. \\ & \quad + \epsilon \partial_{\epsilon} \left(\sum_{b=a, \bar{a}} \text{Im}[\tau_b] \partial_{\tau_b} \partial_{\bar{\tau}_b} - i \partial_{\bar{\tau}_a} - i \partial_{\bar{\tau}_{\bar{a}}} \right) \left[\frac{1}{2\pi i} (P_2(x, \tau_a) \partial_{\tau_a} + P_2(y, \tau_{\bar{a}}) \partial_{\tau_{\bar{a}}}) Z \right] \end{aligned}$$

$$\begin{aligned}
& + \epsilon \partial_\epsilon \frac{1}{\pi} \left(\text{Re}[x] P_3(x, \tau_a) \partial_{\tau_a} \partial_{\bar{\tau}_a} Z + \text{Re}[y] P_3(y, \tau_{\bar{a}}) \partial_{\tau_a} \partial_{\bar{\tau}_a} Z \right) \\
& - \frac{c|\epsilon|}{12\pi} \left(P_4(x, \tau_a) \partial_{\tau_a} \partial_{\bar{\tau}_a} Z + P_4(y, \tau_{\bar{a}}) \partial_{\tau_a} \partial_{\bar{\tau}_a} Z \right) \Big\} + O(|\epsilon|^2). \tag{60}
\end{aligned}$$

For the second type $\langle T_1 \bar{T}_2 \rangle$ we have

$$\begin{aligned}
& \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T \bar{T}(z, \bar{z}) T_1 \bar{T}_2 \rangle \\
& = \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \frac{1}{Z} \left\{ \left[D_z \bar{D}_{\bar{z}} + D_z \bar{\mathcal{P}}_{\bar{z}, \bar{x}_2} + \bar{D}_{\bar{z}} \mathcal{P}_{z, x_1} + \mathcal{P}_{z, x_1} \bar{\mathcal{P}}_{\bar{z}, \bar{x}_2} \right] (Z \langle T_1 \bar{T}_2 \rangle) \right. \\
& \quad + \frac{c}{2} \bar{\mathcal{P}}_4(\bar{z}, \bar{x}_2) \left[D_z + \mathcal{P}_{z, x_1} \right] (Z \langle T_1 \rangle) + \frac{c}{2} \mathcal{P}_4(z, x_1) \left[\bar{D}_{\bar{z}} + \bar{\mathcal{P}}_{\bar{z}, \bar{x}_2} \right] (Z \langle \bar{T}_2 \rangle) \\
& \quad \left. + \frac{c^2}{4} \mathcal{P}_4(z, x_1)^2 \bar{\mathcal{P}}_4(\bar{z}, \bar{x}_2) Z \right\}, \tag{61}
\end{aligned}$$

In the case of two points inserted in the same torus, one can obtain first-order correction using eqs.(57)(174)(176)(181)(182)(184)(185):

$$\begin{aligned}
& \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T \bar{T}(z, \bar{z}) T_1 \bar{T}_2 \rangle \\
& = \frac{1}{Z} \left\{ \left[\sum_{b=a, \bar{a}} \left[(\text{Im}[\tau_b] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_b} \partial_{\bar{\tau}_b} + \frac{i}{2} \bar{\epsilon} \partial_{\tau_b} \partial_{\bar{\epsilon}} - \frac{i}{2} \epsilon \partial_{\bar{\tau}_b} \partial_{\bar{\epsilon}} \right] + 2\pi |\epsilon| \partial_\epsilon \partial_{\bar{\epsilon}} + i \partial_{\tau_a} - i \partial_{\bar{\tau}_a} \right. \right. \\
& \quad - \frac{2i|\epsilon|}{3} P_2(x_1, \tau_a) \partial_{\tau_a} + \frac{2i|\epsilon|}{3} \bar{P}_2(\bar{x}_2, \bar{\tau}_a) \partial_{\tau_a} \left. \right] \left(\frac{1}{4\pi^2} \partial_{\tau_a} \partial_{\bar{\tau}_a} Z \right) \\
& \quad + \left(\sum_{b=a, \bar{a}} \text{Im}[\tau_b] \partial_{\tau_b} \partial_{\bar{\tau}_b} + i \partial_{\tau_a} - i \partial_{\bar{\tau}_a} \right) \left[\frac{1}{2\pi i} (\bar{\epsilon} \bar{P}_2(\bar{x}_2, \bar{\tau}_a) \partial_{\bar{\epsilon}} \partial_{\tau_a} - \epsilon P_2(x_1, \tau_a) \partial_\epsilon \partial_{\bar{\tau}_a}) Z \right] \\
& \quad - \frac{c|\epsilon|}{12\pi} \left[P_4(x_1, \tau_a) (\partial_{\bar{\tau}_a} - 4\pi i \bar{P}_2(\bar{x}_2, \bar{\tau}_a)) \partial_{\bar{\tau}_a} + \bar{P}_4(\bar{x}_2, \bar{\tau}_a) (\partial_{\tau_a} + 4\pi i P_2(x_1, \tau_a)) \partial_{\tau_a} \right] Z \\
& \quad - \epsilon \partial_\epsilon P_3(x_1, \tau_a) \left[\frac{1}{\pi} \text{Re}[x_1] \partial_{\tau_a}^2 + 2i (\bar{P}_1(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) + \bar{P}_1(\bar{x}_2, \bar{\tau}_a)) \partial_{\bar{\tau}_a} \right. \\
& \quad \left. - \frac{\pi c}{3} (\bar{P}_3(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) + \bar{P}_3(\bar{x}_2, \bar{\tau}_a)) \right] Z \\
& \quad - \bar{\epsilon} \partial_{\bar{\epsilon}} \bar{P}_3(\bar{x}_2, \bar{\tau}_a) \left[\frac{1}{\pi} \text{Re}[x_2] \partial_{\tau_a}^2 - 2i (P_1(x_2 - x_1, \tau_a) + P_1(x_1, \tau_a)) \partial_{\tau_a} \right. \\
& \quad \left. - \frac{\pi c}{3} (P_3(x_2 - x_1, \tau_a) + P_3(x_1, \tau_a)) \right] Z \Big\} + O(|\epsilon|^2). \tag{62}
\end{aligned}$$

In the case of two points inserted in different tori, one can obtain first-order correction from eqs.(57)(174)(176)(181)(183)(186)(187):

$$\begin{aligned}
& \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T \bar{T}(z, \bar{z}) T_x \bar{T}_y \rangle \\
& = \frac{1}{Z} \left\{ \left[\sum_{b=a, \bar{a}} \left[(\text{Im}[\tau_b] - \frac{|\epsilon|}{6\pi}) \partial_{\tau_b} \partial_{\bar{\tau}_b} + \frac{i}{2} \bar{\epsilon} \partial_{\tau_b} \partial_{\bar{\epsilon}} - \frac{i}{2} \epsilon \partial_{\bar{\tau}_b} \partial_{\bar{\epsilon}} \right] + 2\pi |\epsilon| \partial_\epsilon \partial_{\bar{\epsilon}} + i \partial_{\tau_a} - i \partial_{\bar{\tau}_a} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{2i|\epsilon|}{3}P_2(x, \tau_a)\partial_{\bar{\tau}_a} + \frac{2i|\epsilon|}{3}\bar{P}_2(\bar{y}, \bar{\tau}_a)\partial_{\tau_a} \left(\frac{1}{4\pi^2}\partial_{\tau_a}\partial_{\bar{\tau}_a}Z \right) \\
& + \left(\sum_{b=a, \bar{a}} \text{Im}[\tau_b]\partial_{\tau_b}\partial_{\bar{\tau}_b} + i\partial_{\tau_a} - i\partial_{\bar{\tau}_a} \right) \left[\frac{1}{2\pi i}(\bar{\epsilon}\bar{P}_2(\bar{y}, \bar{\tau}_a)\partial_{\bar{\epsilon}}\partial_{\tau_a} - \epsilon P_2(x, \tau_a)\partial_{\epsilon}\partial_{\bar{\tau}_a})Z \right] \\
& - \frac{c|\epsilon|}{12\pi} \left[P_4(x, \tau_a)\partial_{\bar{\tau}_a}\partial_{\tau_a} + \bar{P}_4(\bar{y}, \bar{\tau}_a)\partial_{\tau_a}\partial_{\bar{\tau}_a} \right] Z \\
& - \left. \frac{\epsilon}{\pi}\text{Re}[x]P_3(x, \tau_a)\partial_{\epsilon}\partial_{\bar{\tau}_a}\partial_{\tau_a}Z - \frac{\bar{\epsilon}}{\pi}\text{Re}[y]\bar{P}_3(\bar{x}, \bar{\tau}_a)\partial_{\bar{\epsilon}}\partial_{\tau_a}\partial_{\bar{\tau}_a}Z \right\} + O(|\epsilon|^2). \tag{63}
\end{aligned}$$

As an example, we consider the decoupling limit of the first-order correction of $\langle T_1\bar{T}_2 \rangle$ in eq.(62). According to eq.(57), $\langle T_1\bar{T}_2 \rangle|_{\epsilon=0} = \frac{1}{4\pi^2 Z_a} \partial_{\tau_a} \partial_{\bar{\tau}_a} Z_a = \langle T_1\bar{T}_2 \rangle_{T_a^2}$, which is completely defined on torus T_a^2 . Thus we can read off first-order correction of $\langle T_1\bar{T}_2 \rangle_{T_a^2}$ from eq.(62) with $\epsilon = 0$:

$$\begin{aligned}
& \left(\delta_\lambda \langle T_1\bar{T}_2 \rangle - \langle \delta_\lambda T_1\bar{T}_2 \rangle \right) \Big|_{\epsilon=0} \\
& = \frac{1}{4\pi^2 Z_a Z_{\bar{a}}} \left[\text{Im}[\tau_a]\partial_{\tau_a}\partial_{\bar{\tau}_a} + \text{Im}[\bar{\tau}_a]\partial_{\bar{\tau}_a}\partial_{\tau_a} + i\partial_{\tau_a} - i\partial_{\bar{\tau}_a} \right] \partial_{\tau_a}\partial_{\bar{\tau}_a} (Z_a Z_{\bar{a}}) - \frac{\delta_\lambda Z}{Z} \langle T_1\bar{T}_2 \rangle \Big|_{\epsilon=0} \\
& = \frac{1}{4\pi^2 Z_a} \left[\text{Im}[\tau_a]\partial_{\tau_a}^2 \partial_{\bar{\tau}_a}^2 Z_a + i(\partial_{\tau_a}^2 \partial_{\bar{\tau}_a} - \partial_{\bar{\tau}_a} \partial_{\tau_a}^2) Z_a \right] - \frac{\delta_\lambda Z_a}{Z_a} \langle T_1\bar{T}_2 \rangle_{T_a^2} \\
& = \delta_\lambda \langle T_1\bar{T}_2 \rangle_{T_a^2} - \langle \delta_\lambda T_1\bar{T}_2 \rangle_{T_a^2}. \tag{64}
\end{aligned}$$

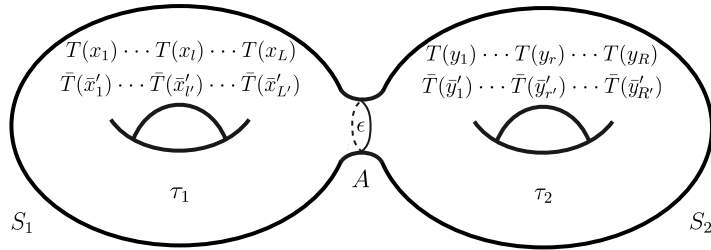


Figure 3: An $(L + R + L' + R')$ -point function of stress tensors on sewn Riemann surface with genus-2.

Now we consider the first-order deformation of a general $(L + R + L' + R')$ -point function of stress tensors, as shown in Fig.3, $(L + R)$ holomorphic stress tensor are inserted at $x_1, \dots, x_L \in S_1$, $y_1, \dots, y_R \in S_2$ and $(L' + R')$ anti-holomorphic stress tensor are inserted at $\bar{x}'_1, \dots, \bar{x}'_{L'} \in S_1$, $\bar{y}'_1, \dots, \bar{y}'_{R'} \in S_2$, respectively. For simplicity, we define⁷

$$X_L \equiv T(x_1) \cdots T(x_L), \quad X_R \equiv T(y_1) \cdots T(y_R), \tag{65}$$

⁷For simplicity, $\setminus_l X_L$ stands for deleting the l -th factor in X_L .

$$\bar{X}_{L'} \equiv \bar{T}(\bar{x}'_1) \cdots \bar{T}(\bar{x}'_{L'}), \quad \bar{X}_{R'} \equiv \bar{T}(\bar{y}'_1) \cdots \bar{T}(\bar{y}'_{R'}), \quad (66)$$

$$\backslash_l X_L \equiv T(x_1) \cdots T(x_{l-1}) T(x_{l+1}) \cdots T(x_L), \quad (67)$$

$$\backslash_{l'} \bar{X}_{L'} \equiv \bar{T}(\bar{x}'_1) \cdots \bar{T}(\bar{x}'_{l'-1}) \bar{T}(\bar{x}'_{l'+1}) \cdots \bar{T}(\bar{x}'_{L'}). \quad (68)$$

Then we use eq.(40) to calculate the first-order deformation of $\langle X_L \bar{X}_{L'} X_R \bar{X}_{R'} \rangle$:

$$\begin{aligned} & \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \langle T \bar{T}(z, \bar{z}) X_L \bar{X}_{L'} X_R \bar{X}_{R'} \rangle \\ &= \frac{i}{2} \frac{1}{Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left\{ D_z \bar{D}_{\bar{z}} + \sum_{l=1}^L \left[\bar{D}_{\bar{z}} \mathcal{P}_{z, x_l} + \frac{c}{2} \bar{D}_{\bar{z}}^2 \mathcal{P}_4^{(z, x_l)} \backslash_l \right] \right. \\ & \quad + \sum_{r=1}^R \left[\bar{D}_{\bar{z}} \mathcal{P}_{z, y_r} + \frac{c}{2} \bar{D}_{\bar{z}}^2 \mathcal{P}_4^{(z, y_r)} \backslash_r \right] + \sum_{l'=1}^{L'} \left[D_z \bar{\mathcal{P}}_{\bar{z}, \bar{x}'_{l'}} + \frac{c}{2} D_z^2 \bar{\mathcal{P}}_4^{(\bar{z}, \bar{x}'_{l'})} \backslash_{l'} \right] \\ & \quad + \sum_{r'=1}^{R'} \left[D_z \bar{\mathcal{P}}_{\bar{z}, \bar{y}'_{r'}} + \frac{c}{2} D_z^2 \bar{\mathcal{P}}_4^{(\bar{z}, \bar{y}'_{r'})} \backslash_{r'} \right] + \sum_{l, l'} \left[\mathcal{P}_{z, x_l} \bar{\mathcal{P}}_{\bar{z}, \bar{x}'_{l'}} + \frac{c^2}{4} \mathcal{P}_4^{(z, x_l)} \mathcal{P}_4^{(\bar{z}, \bar{x}'_{l'})} \backslash_{l, l'} \right] \\ & \quad + \sum_{r, r'} \left[\mathcal{P}_{z, y_r} \bar{\mathcal{P}}_{\bar{z}, \bar{y}'_{r'}} + \frac{c^2}{4} \mathcal{P}_4^{(z, y_r)} \mathcal{P}_4^{(\bar{z}, \bar{y}'_{r'})} \backslash_{r, r'} \right] + \sum_{l, r'} \left[\mathcal{P}_{z, x_l} \bar{\mathcal{P}}_{\bar{z}, \bar{y}'_{r'}} + \frac{c^2}{4} \mathcal{P}_4^{(z, x_l)} \mathcal{P}_4^{(\bar{z}, \bar{y}'_{r'})} \backslash_{l, r'} \right] \\ & \quad \left. + \sum_{r, r'} \left[\mathcal{P}_{z, y_r} \bar{\mathcal{P}}_{\bar{z}, \bar{y}'_{r'}} + \frac{c^2}{4} \mathcal{P}_4^{(z, y_r)} \mathcal{P}_4^{(\bar{z}, \bar{y}'_{r'})} \backslash_{r, r'} \right] \right\} \left[Z \langle X_L \bar{X}_{L'} X_R \bar{X}_{R'} \rangle \right]. \quad (69) \end{aligned}$$

The first term in eq.(69) is calculated in eq.(174). The terms from second to fifth contains all the one-point contribution of stress tensors eqs.(176)(181). The remaining terms contains all the two-point contribution eqs.(184)(186)(185)(187).

To close this section, we would like to add a potential application of these deformed correlation functions. To check the AdS_3/CFT_2 with $T\bar{T}$ deformation [29, 58, 59], one has to match the correlation functions in both field theory side and gravity side. As reviewed in the introduction, it is a nontrivial attempt to construct the nonperturbative deformed correlation functions. Here we apply the perturbative field theory approach to construct the generic deformed correlation functions in the boundary field theories with nontrivial topology. Since the holographic CFTs show maximal quantum chaotic behavior [10–12], to extract the chaos signals of the deformed holographic CFTs is an important step to check the holographic dictionary. In particular, one can directly apply the higher point correlation functions in deformed theory on higher genus Riemann surface to calculate OTOC and multiple-interval Rény entropies as following previously works [20, 21]. Further, one can also

do Fourier transformation of deformed two-point functions of stress tensor to look at the poles structure to read off the chaos signals, namely, pole-skipping phenomenon proposed by [17].

5 Conclusions and perspectives

To understand the quantum chaos of $T\bar{T}$ deformed conformal field theories, one has to calculate OTOC, spectrum form factor, and pole skipping phenomenon, which are associated with the correlation function in deformed field theory. It is highly nontrivial to construct the correlation function in a non-perturbative approach. Alternatively, one can follow a perturbative approach [20, 21] to earn some lessons about quantum chaos of deformed CFTs. In this work, we have applied the perturbative conformal field theory approach to construct higher genus correlation functions of $T\bar{T}$ -deformed theories to offer field theories data to achieve our final goal of understanding the quantum chaos of deformed CFTs. The most important ingredients are sewing construction and the conformal ward identity of CFTs on higher genus two-dimensional Riemann surface. Thanks to sewing construction, one can construct the higher genus correlation functions in terms of the correlation functions on the low genus Riemann surface. In the current work, we apply a particular sewing construction and perturbative conformal field theory approach to obtain the first order $T\bar{T}$ deformation of the function and correlation functions on a genus two Riemann surface. As a consistency check, we take the decoupling limit to find genus two correlation functions can be expressed by genus one partition function and correlation functions presented in the literature. To obtain the final results, we apply a systematic renormalization [57] by following the calculation given in the $T\bar{T}$ deformation in genus one CFTs [26, 54].

It is a preliminary tempt to calculate the correlation functions in the higher genus $T\bar{T}$ deformed CFTs in perturbative approach. To go beyond the first-order calculation will be a highly nontrivial project along this direction, even in the free field theory [26]. In higher-order deformation, one has to take the flow effects of $T\bar{T}$ operator into account. Further, one can follow up the resulting correlation

functions to investigate quantum chaotic signals or quantum integrability structure of $T\bar{T}$ deformed theories, as we mentioned at the beginning. We would like to report further progress in future works.

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A Elliptic functions

In this appendix, we list the elliptic functions that appear in the context and their properties. We follow the conventions in [48, 51]. The torus T^2 is defined by the identification on complex plane $z \sim z + 2\pi i$ and $z \sim z + 2\pi i\tau$. $P_k(z, \tau)$ is used to denote an elliptic function with subscript k , and its Laurent expansion in the neighborhood of $z = 0$ is

$$P_{k \geq 1}(z, \tau) = \frac{1}{z^k} + (-1)^k \sum_{n \geq k} E_n(\tau) \binom{n-1}{k-1} z^{n-k}, \quad (70)$$

where τ is modular parameter of the torus. $E_n(\tau)$ is the *Eisenstein series* for $n \geq 2$ which equals to zero for n odd, and for n even $E_n(\tau)$ can be defined as

$$E_n(\tau) = -\frac{B_n}{n!} + \frac{2}{(n-1)!} \sum_{m \geq 1} \sigma_{n-1}(m) q^m, \quad (71)$$

where B_n is the n^{th} Bernoulli number, $\sigma_{n-1}(m) = \sum_{d|m} d^{n-1}$ and $q = e^{2\pi i\tau}$. In this paper, we also use the convention $P_0 = 1$. There are simple relations among $P_1(z, \tau)$, $P_2(z, \tau)$ and classical Weierstrass functions:

$$P_2(z, \tau) = \wp(z, \tau) + E_2(\tau), \quad (72)$$

$$P_1(z, \tau) = \zeta(z, \tau) - E_2(\tau)z, \quad (73)$$

where $\wp(z, \tau)$ is *Weierstrass P-function* and $\zeta(z, \tau)$ is *Weierstrass ζ -function*. These two functions are defined as

$$\wp(z, \tau) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[\frac{1}{(z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} \right], \quad (74)$$

$$\zeta(z, \tau) = \frac{1}{z} + \sum_{(m,n) \neq (0,0)} \left[\frac{1}{z - w_{m,n}} + \frac{1}{w_{m,n}} + \frac{z}{w_{m,n}^2} \right], \quad (75)$$

where $w_{m,n} = 2\pi im + 2\pi i\tau n$ is the coordinate of the lattice. The k -th derivative of $P_1(z, \tau)$ with respect to z gives $P_{k+1}(z, \tau)$:

$$P_{k+1}(z, \tau) = \frac{(-1)^k}{k!} \partial_z^k P_1(z, \tau) = -\frac{1}{k} \partial_z P_k(z, \tau). \quad (76)$$

$P_1(z, \tau)$ is quasi-periodic, and $P_k(z, \tau)$ is periodic for $k \geq 2$, and they satisfy

$$P_1(z + 2\pi i, \tau) = P_1(z, \tau) + 2P_1(\pi i, \tau) = P_1(z, \tau), \quad (77)$$

$$P_1(z + 2\pi i\tau, \tau) = P_1(z, \tau) + 2P_1(\pi i\tau, \tau) = P_1(z, \tau) - 1, \quad (78)$$

$$P_k(z + 2\pi i, \tau) = P_k(z + 2\pi i\tau, \tau) = P_k(z, \tau). \quad (79)$$

We define $Q(z, \tau)$ function by

$$\begin{aligned} \int_0^z dz' \left(P_1(z', \tau) - \frac{1}{z'} \right) &= \log \frac{Q(z, \tau)}{z}, \\ P_1(z, \tau) &= \frac{\partial_z Q(z, \tau)}{Q(z, \tau)} = \partial_z \log Q(z, \tau), \\ \log Q(z, \tau) &= \log z - \sum_{n \geq 1} \frac{1}{n} E_n(\tau) z^n. \end{aligned} \quad (80)$$

The relationship between $Q(z, \tau)$ and *Weierstrass σ -function* is

$$Q(z, \tau) = e^{-\frac{1}{2} E_2(\tau) z^2} \sigma(z, \tau), \quad (81)$$

and *Weierstrass σ -function* is defined as

$$\sigma(z, \tau) = z \prod_{(m,n) \neq (0,0)} \left[\left(1 - \frac{z}{w_{m,n}} \right) \exp \left(\frac{z}{w_{m,n}} + \frac{z^2}{2w_{m,n}^2} \right) \right]. \quad (82)$$

$Q(z, \tau)$ is an odd function just like $\sigma(z, \tau)$, and it is quasi-periodic with

$$Q(z + 2\pi i, \tau) = -Q(z, \tau), \quad (83)$$

$$Q(z + 2\pi i\tau, \tau) = -e^{-(z+i\pi\tau)} Q(z, \tau). \quad (84)$$

B Some useful integrals

In this appendix, we discuss the details of integrals I_a , $J_a(x, \bar{x})$ and $K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)$ that appear in the context. The integrands over a torus may contain singularities. Following the prescription in [57], we regularize the domain of integration by removing some small disks ($D_{\delta_1}, \dots, D_{\delta_N}$) of radiuses ($\delta_1, \dots, \delta_N$) centered on the singularities (x_1, \dots, x_N) (see Fig.4). $f(z, \bar{z})$ is used to denote a general function on torus T^2 with these singularities, and we assume that it can be written as the divergence of some vector field F^μ :

$$f(z, \bar{z}) = \partial_\mu F^\mu(z, \bar{z}) = \partial_z F^z(z, \bar{z}) + \partial_{\bar{z}} F^{\bar{z}}(z, \bar{z}). \quad (85)$$

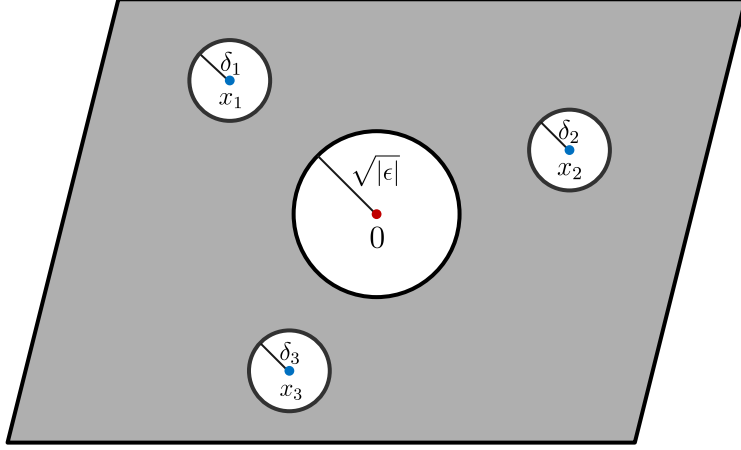


Figure 4: An example for regularized domain of integration. The red point is the origin of the torus coordinates, and the circle of radius $\sqrt{|\epsilon|}$ surrounding it is the sewing region. The blue points are the singularities of the integrand, and the small disks surrounding them have been removed. The gray area is the domain of integration.

The integral of $f(z, \bar{z})$ over the regularized domain can be calculated using the Stoke's theorem

$$\int_{T^2 \setminus D_\delta} d^2z \partial_\mu F^\mu(z, \bar{z}) = \left(\oint_{\partial T^2} - \sum_{i=1}^N \oint_{\partial D_{\delta_i}} \right) (F^z d\bar{z} - F^{\bar{z}} dz). \quad (86)$$

After integrating eq.(86), we take the limit $(\delta_1, \dots, \delta_N) \rightarrow (0, \dots, 0)$ and discard the divergent term (if it exists) to regularize the integral. In this paper, we introduce constraints on the integrand $f(z, \bar{z})$. The integrand may diverge at the boundary of the torus ∂T^2 , for instance, the elliptic function $P_k(z, \tau)$ diverges at points on the

lattice $z = 2\pi im + 2\pi i\tau n$ with $m, n \in \mathbb{Z}$. To avoid contact between the boundary of disk and the boundary of torus, we require that the integrand $f(z, \bar{z})$ be biperiodic on the torus with respect to both the variables z and \bar{z} . Under this condition we can translate the parallelogram of the torus on the complex plane to ensure that the singularities only appear inside. Next we require that the integrand can be factorized into two parts:

$$f(z, \bar{z}) = g(z)h(\bar{z}), \quad (87)$$

and we assume that $G(z) = \partial_z g(z)$ and $H(\bar{z}) = \partial_{\bar{z}} h(\bar{z})$. Since we have removed the disks around the singularities, $g(z)$ and $h(\bar{z})$ are holomorphic and antiholomorphic respectively over the domain of integration. Thus we can construct the vector fields F^μ and F'^μ as

$$F^z(z, \bar{z}) = G(z)h(\bar{z}), \quad F^{\bar{z}}(z, \bar{z}) = 0, \quad (88)$$

$$F'^z(z, \bar{z}) = 0, \quad F'^{\bar{z}}(z, \bar{z}) = g(z)H(\bar{z}). \quad (89)$$

Take the first construction for example, eq.(86) can be further written as

$$\begin{aligned} \int_{T^2 \setminus D_\delta} d^2 z f(z, \bar{z}) &= \int_{\bar{z}_0}^{\bar{z}_0 - 2\pi i} d\bar{z} [G(z) - G(z + 2\pi i\tau)] h(\bar{z}) \\ &\quad - \int_{\bar{z}_0}^{\bar{z}_0 - 2\pi i\bar{\tau}} d\bar{z} [G(z) - G(z + 2\pi i)] h(\bar{z}) \\ &\quad - \sum_{i=1}^N \int_0^{2\pi} (-i\delta_i e^{-i\theta}) d\theta G(\delta_i e^{i\theta} + x_i) h(\delta_i e^{-i\theta} + \bar{x}_i), \end{aligned} \quad (90)$$

where the biperiodic property of $h(\bar{z})$ is used in the first two terms. In the rest of this Appendix, we will apply the above method to calculate three types of integrals, which play an important role in calculating first-order corrections.

B.1 Integral $I_a(k, l)$

The first type of integral that we encounter is

$$I_a(k, l) = \int_{S_a} d^2 z_a P_k(z_a, \tau_a) \bar{P}_l(\bar{z}_a, \bar{\tau}_a), \quad (91)$$

for $k, l \geq 0$. z_a is the local coordinate of the torus T_a^2 , and we will omit its subscript a later. S_a is the remainder of the torus T_a^2 after removing the a disk of radius $\sqrt{|\epsilon|}$

centered at $z_a = 0$, and the integrand $P_k(z_a, \tau_a) \bar{P}_l(\bar{z}_a, \bar{\tau}_a)$ is regular in the domain of integration. $I_a(k, l)$ has a simple property:

$$I_a(l, k) = -\overline{I_a(k, l)}. \quad (92)$$

For $k \geq 3$ and $l \geq 2$, we can construct $F^z(z, \bar{z}) = \frac{-1}{k-1} P_{k-1}(z, \tau_a) \bar{P}_l(\bar{z}, \bar{\tau}_a)$ (and $F^{\bar{z}} = 0$) using the recursion eq.(76) of $P_k(z, \tau_a)$. In this case $P_{k-1}(z, \tau_a)$ is still bi-periodic on the torus, and thus the first two terms of eq.(90) vanish. The last term contains integral over the annulus A (of radius $\sqrt{|\epsilon|}$ centered at $z = 0$), which can be calculated using the Laurent expansion eq.(70):

$$\begin{aligned} I_a(k, l) &= \frac{-i}{k-1} \int_0^{2\pi} d\theta \left[\sqrt{|\epsilon|}^{2-k-l} e^{-i(k-l)\theta} \right. \\ &\quad + (-1)^l \sum_{n \geq l} \bar{E}_n(\bar{\tau}_a) \binom{n-1}{l-1} \sqrt{|\epsilon|}^{n+2-k-l} e^{-i(k-l+n)\theta} \\ &\quad + (-1)^{k-1} \sum_{m \geq k-1} E_m(\tau_a) \binom{m-1}{k-2} \sqrt{|\epsilon|}^{m+2-k-l} e^{-i(k-l-m)\theta} \\ &\quad + (-1)^{k+l-1} \sum_{m \geq k-1} \sum_{n \geq l} E_m(\tau_a) \bar{E}_n(\bar{\tau}_a) \binom{m-1}{k-2} \binom{n-1}{l-1} \\ &\quad \left. \times \sqrt{|\epsilon|}^{m+n+2-k-l} e^{-i(n-m+k-l)\theta} \right] \\ &= \frac{-2\pi i}{k-1} \left[\frac{1}{|\epsilon|^{k-1}} \delta_{k,l} + \sum_{n \geq l} C_{n-3,a}^{k-3,l-3} |\epsilon|^{n-l+1} \right], \end{aligned} \quad (93)$$

where the coefficient $C_{k,a}^{n,l}$ is defined as

$$C_{n,a}^{k,l} = (-1)^{k+l} \binom{n+k-l+2}{k+1} \binom{n+2}{l+2} E_{n+k-l+3}(\tau_a) \bar{E}_{n+3}(\bar{\tau}_a), \quad (94)$$

and it is easy to prove that

$$C_{n,a}^{k,l} = \frac{k+2}{l+2} \bar{C}_{n+k-l,a}^{l,k}. \quad (95)$$

From eq.(92) and eq.(95) we immediately obtain the case for $k \geq 2$ and $l \geq 3$:

$$\begin{aligned} I_a(k, l) &= \frac{-2\pi i}{l-1} \left[\frac{1}{|\epsilon|^{l-1}} \delta_{l,k} + \sum_{n \geq k} \bar{C}_{n-3,a}^{l-3,k-3} |\epsilon|^{n-k+1} \right] \\ &= \frac{-2\pi i}{k-1} \frac{1}{|\epsilon|^{k-1}} \delta_{k,l} + \frac{-2\pi i}{l-1} \sum_{n' \geq l} \bar{C}_{n'-3+k-l,a}^{l-3,k-3} |\epsilon|^{n'-l+1} \end{aligned}$$

$$= \frac{-2\pi i}{k-1} \left[\frac{1}{|\epsilon|^{k-1}} \delta_{k,l} + \sum_{n \geq l} C_{n-3,a}^{k-3,l-3} |\epsilon|^{n-l+1} \right], \quad (96)$$

which is consistent with eq.(93). When we consider the case $k = 2$ and $l = 2$, the integrand $F^z(z, \bar{z}) = -P_1(z, \tau_a) \bar{P}_2(\bar{z}, \bar{\tau}_a)$ is quasi-periodic and its integral on the torus boundary can be written as

$$\begin{aligned} \oint_{\partial T^2} F^z d\bar{z} &= \int_{\bar{z}_0}^{\bar{z}_0 - 2\pi i} d\bar{z} [P_1(z + 2\pi i \tau_a, \tau_a) - P_1(z, \tau_a)] \bar{P}_2(\bar{z}, \bar{\tau}_a) \\ &\quad - \int_{\bar{z}_0}^{\bar{z}_0 - 2\pi i \bar{\tau}} d\bar{z} [P_1(z + 2\pi i, \tau_a) - P_1(z, \tau_a)] \bar{P}_2(\bar{z}, \bar{\tau}_a) \\ &= 2\eta'_a \int_{z_0}^{z_0 + 2\pi i} d\bar{z} \bar{P}_2(\bar{z}, \bar{\tau}_a) - 2\eta_a \int_{z_0}^{z_0 + 2\pi i \tau_a} d\bar{z} \bar{P}_2(\bar{z}, \bar{\tau}_a) \\ &= 8\text{Im}[\eta_a \bar{\eta}'_a], \end{aligned} \quad (97)$$

where $\eta_a = P_1(\pi i, \tau_a) = 0$ and $\eta'_a = P_1(\pi i \tau_a, \tau_a) = -\frac{1}{2}$, thus this term has no contribution. Then we compute the integral over A and finally we obtain

$$I_a(2, 2) = -2\pi i \frac{1}{|\epsilon|} + 2\pi i \sum_{n \geq 1} n |E_{n+1}(\tau)|^2 |\epsilon|^n, \quad (98)$$

which is also consistent with eq.(93).

For $k \geq 3$ and $l = 0$, we construct $F^z = \frac{-1}{k-1} P_{k-1}(z, \tau_a)$ and obtain

$$\begin{aligned} I_a(k, 0) &= \frac{-i}{k-1} \int_0^{2\pi} d\theta \left[\sqrt{|\epsilon|}^{2-k} e^{-ik\theta} \right. \\ &\quad \left. + (-1)^{k-1} \sum_{m \geq k-1} E_m(\tau_a) \binom{m-1}{k-2} \sqrt{|\epsilon|}^{m+2-k} e^{i(m-k)\theta} \right] \\ &= 2\pi i |\epsilon| E_k(\tau_a), \end{aligned} \quad (99)$$

and for the case $k = 0$ and $l \geq 3$ we have

$$I_a(0, l) = 2\pi i |\epsilon| \bar{E}_l(\bar{\tau}_a). \quad (100)$$

For $k = 2$ and $l = 0$ we have

$$\begin{aligned} I_a(2, 0) &= \int_{\bar{z}_0}^{\bar{z}_0 - 2\pi i} d\bar{z} [P_1(z + 2\pi i \tau_a, \tau_a) - P_1(z, \tau_a)] \\ &\quad - \int_{\bar{z}_0}^{\bar{z}_0 - 2\pi i \bar{\tau}} d\bar{z} [P_1(z + 2\pi i, \tau_a) - P_1(z, \tau_a)] \\ &\quad - i \int_0^{2\pi} d\theta \left[e^{-i2\theta} + \sum_{m \geq 1} E_m(\tau_a) \sqrt{|\epsilon|}^m e^{i(m-2)\theta} \right] \\ &= 2\pi i + 2\pi i |\epsilon| E_2(\tau_a). \end{aligned} \quad (101)$$

For $k = 0$ and $l = 2$ we have

$$I_a(0, 2) = 2\pi i + 2\pi i |\epsilon| \bar{E}_2(\bar{\tau}_a). \quad (102)$$

B.2 Integral $[J_a(x, \bar{x})](k, l)$

The second type of integral is

$$[J_a(x, \bar{x})](k, l) = \int_{S_a} d^2 z_a P_k(z_a, \tau_a) \bar{\tilde{P}}_l(\bar{z}_a, \bar{x}, \bar{\tau}_a), \quad (103)$$

where \tilde{P}_l is defined as

$$\tilde{P}_l(z, x, \tau) = \begin{cases} P_1(z - x, \tau) - P_1(z, \tau) & (l = 1), \\ P_l(z - x, \tau) & (l \geq 2). \end{cases} \quad (104)$$

This integral depends on the coordinate of the singularity (x, \bar{x}) , and from the recursion eq.(76) and Laurent expansion eq.(70) we have

$$\begin{aligned} \partial_x [J_a(x, \bar{x})](k, l) &= \int_{S_a} d^2 z P_k(z, \tau_a) \partial_x \left(\frac{1}{(l-1)!} \partial_{\bar{x}}^{l-1} \frac{1}{\bar{z} - \bar{x}} \right) \\ &= \frac{2\pi i}{(l-1)!} \partial_{\bar{x}}^{l-1} P_k(x, \tau_a) \\ &= 2\pi i P_k(x, \tau_a) \delta_{l,1}, \end{aligned} \quad (105)$$

$$\partial_{\bar{x}} [J_a(x, \bar{x})](k, l) = \int_{S_a} d^2 z P_k(z, \tau_a) \partial_{\bar{x}} \bar{P}_l(\bar{z} - \bar{x}, \bar{\tau}_a) = l [J_a(x, \bar{x})](k, l+1), \quad (106)$$

where in the last step we restrict the insertion point $x \neq 0$.

For $k \geq 3$ and $l = 1$, we construct $F^z = \frac{-1}{k-1} P_{k-1}(z, \tau_a) [\bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) - \bar{P}_1(\bar{z}, \bar{\tau}_a)]$. Since $x \neq 0$, $\bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a)$ is regular around the origin and has Taylor expansion

$$\bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) = - \sum_{l \geq 1} \bar{P}_l(\bar{x}, \bar{\tau}_a) \bar{z}^{l-1}. \quad (107)$$

Using eq.(90) we have

$$\begin{aligned} [J_a(x, \bar{x})](k, 1) &= - \frac{1}{k-1} \oint_A d\bar{z} P_{k-1}(z, \tau_a) \left[\sum_{l \geq 1} \bar{P}_l(\bar{x}, \bar{\tau}_a) \bar{z}^{l-1} + \bar{P}_1(\bar{z}, \bar{\tau}_a) \right] \\ &\quad + \frac{1}{k-1} P_{k-1}(x, \tau_a) \oint_{\partial D_\delta} d\bar{z} \bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) \Big|_{\delta \rightarrow 0} \\ &= - \frac{2\pi i}{k-1} \left[P'_{k-1}(x, \tau_a) - \sum_{l \geq 1} \frac{k-1}{l} |\epsilon|^l A_{k-3,a}^{l,0} \bar{P}'_l(\bar{x}, \bar{\tau}_a) \right], \end{aligned} \quad (108)$$

where the coefficient $A_{k,a}^{l,n}$ is defined in eq.(17), and $P'_k(x, \tau) = P_k(x, \tau) - E_k(\tau)$. For $k = 2$ and $l = 1$ we have

$$\begin{aligned}
[J_a(x, \bar{x})](2, 1) &= - \oint_{\partial T^2} d\bar{z} P_1(z, \tau_a) \left[\bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) - \bar{P}_1(\bar{z}, \bar{\tau}_a) \right] \\
&\quad - \oint_A d\bar{z} P_1(z, \tau_a) \left[\sum_{l \geq 1} \bar{P}_l(\bar{x}, \bar{\tau}_a) \bar{z}^{l-1} + \bar{P}_1(\bar{z}, \bar{\tau}_a) \right] \\
&\quad + P_1(x, \tau_a) \oint_{\partial D_\delta} d\bar{z} \bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) \Big|_{\delta \rightarrow 0} \\
&= - 2\pi i \left[P_1(x, \tau_a) + \sum_{l \geq 1} |\epsilon|^l E_{l+1}(\tau_a) \bar{P}'_l(\bar{x}, \bar{\tau}_a) \right], \tag{109}
\end{aligned}$$

where the integral over ∂T^2 is calculated using the quasi-periodic property of $Q(z, \tau_a)$ in eq.(84). For $k = 0$ and $l = 1$, we construct $F^z = z [\bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) - \bar{P}_1(\bar{z}, \bar{\tau}_a)]$ and have

$$\begin{aligned}
[J_a(x, \bar{x})](0, 1) &= \oint_{\partial T^2} d\bar{z} z \left[\bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) - \bar{P}_1(\bar{z}, \bar{\tau}_a) \right] \\
&\quad + \oint_A d\bar{z} z \left[\sum_{l \geq 1} \bar{P}_l(\bar{x}, \bar{\tau}_a) \bar{z}^{l-1} + \bar{P}_1(\bar{z}, \bar{\tau}_a) \right] \\
&\quad - x \oint_{\partial D_\delta} d\bar{z} \bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) \Big|_{\delta \rightarrow 0} \\
&= 4\pi i \operatorname{Re}[x] - 2\pi i |\epsilon| \bar{P}_1(\bar{x}, \bar{\tau}_a). \tag{110}
\end{aligned}$$

For $k \geq 3$ and $l = 2$, we construct $F^z = \frac{-1}{k-1} P_{k-1}(z, \tau_a) \bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a)$ and have

$$\begin{aligned}
[J_a(x, \bar{x})](k, 2) &= \frac{1}{k-1} \oint_A d\bar{z} \sum_{l \geq 1} l \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) P_{k-1}(z, \tau_a) \bar{z}^{l-1} \\
&\quad + \frac{1}{k-1} P_{k-1}(x, \tau_a) \oint_{\partial D_\delta} d\bar{z} \bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a) \Big|_{\delta \rightarrow 0} \\
&= - 2\pi i \sum_{l \geq 1} |\epsilon|^l A_{k-3,a}^{l,0} \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a). \tag{111}
\end{aligned}$$

For $k = 2$ and $l = 2$ we have

$$\begin{aligned}
[J_a(x, \bar{x})](2, 2) &= - \oint_{\partial T^2} d\bar{z} P_1(z, \tau_a) \bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a) \\
&\quad + \frac{1}{k-1} \oint_A d\bar{z} \sum_{l \geq 1} l \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) P_1(z, \tau_a) \bar{z}^{l-1} \\
&\quad + \frac{1}{k-1} P_1(x, \tau_a) \oint_{\partial D_\delta} d\bar{z} \bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a) \Big|_{\delta \rightarrow 0} \\
&= 2\pi i \sum_{l \geq 1} |\epsilon|^l E_{l+1}(\tau_a) l \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a). \tag{112}
\end{aligned}$$

For $k = 0$ and $l = 2$, we construct $F^z = z\bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a)$ and have

$$\begin{aligned}
[J_a(x, \bar{x})](0, 2) &= \oint_{\partial T^2} d\bar{z} z \bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a) \\
&\quad - \oint_A d\bar{z} \sum_{l \geq 1} l \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) z \bar{z}^{l-1} \\
&\quad - x \oint_{\partial D_\delta} d\bar{z} \bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a) \Big|_{\delta \rightarrow 0} \\
&= 2\pi i + 2\pi i |\epsilon| \bar{P}_2(\bar{x}, \bar{\tau}_a).
\end{aligned} \tag{113}$$

These results can also be obtained by using eq.(106).

For $k \geq 3$ and $l = 4$, we construct $F^z = \frac{-1}{k-1} P_{k-1}(z, \tau_a) \bar{P}_4(\bar{z} - \bar{x}, \bar{\tau}_a)$ and obtain

$$[J_a(x, \bar{x})](k, 4) = -2\pi i \sum_{l \geq 1} |\epsilon|^l A_{k-3,a}^{l,0} \frac{l^2 + 3l + 2}{6} \bar{P}_{l+3}(\bar{x}, \bar{\tau}_a). \tag{114}$$

For $k = 2$ and $l = 4$ we have

$$[J_a(x, \bar{x})](2, 4) = 2\pi i \sum_{l \geq 1} |\epsilon|^l E_{l+1}(\tau_a) \frac{l^3 + 3l^2 + 2l}{6} \bar{P}_{l+3}(\bar{x}, \bar{\tau}_a). \tag{115}$$

For $k = 0$ and $l = 4$, we construct $F^z = z\bar{P}_4(\bar{z} - \bar{x}, \bar{\tau}_a)$ and obtain

$$[J_a(x, \bar{x})](0, 4) = 2\pi i |\epsilon| \bar{P}_4(\bar{x}, \bar{\tau}_a). \tag{116}$$

B.3 Integral $[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k, l)$

The last type of integral is

$$[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k, l) = \int_{S_a} d^2 z_a \tilde{P}_k(z_a, x_1, \tau_a) \bar{\tilde{P}}_l(\bar{z}_a, \bar{x}_2, \bar{\tau}_a), \tag{117}$$

for $k, l \geq 1$. This integral appears in the first order deformation of the two-point function (for the case where x_1 and x_2 are inserted in the same torus.) In particular, if we fix x_1 and x_2 at the same point x , the corresponding integral is

$$[K_a(x, \bar{x})](k, l) = \int_{S_a} d^2 z_a \tilde{P}_k(z_a, x, \tau_a) \bar{\tilde{P}}_l(\bar{z}_a, \bar{x}, \bar{\tau}_a), \tag{118}$$

which appears in the first-order deformation of one-point function. Using the definition of $\tilde{P}_l(z, x, \tau)$, the recursion eq.(76) and the Laurent expansion eq.(70), we

obtain some properties of $K_a(x, \bar{x})$:

$$\begin{aligned}
\partial_x[K_a(x, \bar{x})](k, l) &= \int_{S_a} d^2z \left[\partial_x P_k(z - x, \tau_a) \bar{P}_l(\bar{z}, \bar{x}, \bar{\tau}_a) + \tilde{P}_k(z_a, x, \tau_a) \partial_x \bar{P}_l(\bar{z} - \bar{x}, \bar{\tau}_a) \right] \\
&= k[K_a(x, \bar{x})](k + 1, l) + \frac{-\pi}{(l - 1)!} \int_{S_a} d^2z \partial_{\bar{z}}^{l-1} \tilde{P}_k(z, x, \tau_a) \delta^{(2)}(z - x) \\
&= k[K_a(x, \bar{x})](k + 1, l) - 2\pi i P_1(x, \tau_a) \delta_{k,1} \delta_{l,1} \\
&\quad + \frac{(-1)^{k-1} \pi^2}{(k - 1)! (l - 1)!} \int_{S_a} d^2z \delta^{(2)}(z - x) \partial_z^{k-1} \partial_{\bar{z}}^{l-2} \delta^{(2)}(z - x), \quad (119)
\end{aligned}$$

$$\begin{aligned}
\partial_{\bar{x}}[K_a(x, \bar{x})](k, l) &= l[K_a(x, \bar{x})](k, l + 1) - 2\pi i \bar{P}_1(\bar{x}, \bar{\tau}_a) \delta_{k,1} \delta_{l,1} \\
&\quad + \frac{(-1)^{l-1} \pi^2}{(k - 1)! (l - 1)!} \int_{S_a} d^2z \delta^{(2)}(z - x) \partial_z^{k-2} \partial_{\bar{z}}^{l-1} \delta^{(2)}(z - x). \quad (120)
\end{aligned}$$

The last term in each of these equations is purely divergent, and we discard it to regularize the integral. We have similar properties for $K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)$:

$$\partial_{x_1}[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k, l) = k[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k + 1, l), \quad (121)$$

$$\partial_{\bar{x}_1}[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k, l) = 2\pi i \bar{P}_1(\bar{x}_1, \bar{x}_2, \bar{\tau}_a) \delta_{k,1}, \quad (122)$$

$$\partial_{x_2}[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k, l) = 2\pi i \tilde{P}_k(x_2, x_1, \tau_a) \delta_{l,1}, \quad (123)$$

$$\partial_{\bar{x}_2}[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k, l) = l[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k, l + 1), \quad (124)$$

where we restrict the insertion points $x_1, x_2 \neq 0$ and $x_1 \neq x_2$.

First we calculate the integral $[K_a(x, \bar{x})](k, l)$, which appears in the deformed one-point function of primary field. For $k = 2$ and $l = 2$ we have

$$\begin{aligned}
[K_a(x, \bar{x})](2, 2) &= - \oint_{\partial \hat{T}^2} d\bar{z} P_1(z - x, \tau_a) \bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a) \\
&\quad - \oint_A d\bar{z} \sum_{k, l \geq 1} l P_k(x, \tau_a) \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) z^{k-1} \bar{z}^{l-1} \\
&\quad + \oint_{\partial D_\delta} d\bar{z} P_1(z - x, \tau_a) \bar{P}_2(\bar{z} - \bar{x}, \bar{\tau}_a) \Big|_{\delta \rightarrow 0} \\
&= 2\pi i \sum_{k \geq 1} |\epsilon|^k k |P_{k+1}(x, \tau_a)|^2 - 2\pi i \frac{1}{\delta} \Big|_{\delta \rightarrow 0}, \quad (125)
\end{aligned}$$

and we can simply discard the last term. For $k = 2$ and $l = 1$ we have

$$\begin{aligned}
[K_a(x, \bar{x})](2, 1) &= - \oint_{\partial \hat{T}^2} d\bar{z} P_1(z - x, \tau_a) \left[\bar{P}_1(\bar{z} - \bar{x}, \bar{\tau}_a) - \bar{P}_1(\bar{z}, \bar{\tau}_a) \right] \\
&\quad + \oint_A d\bar{z} \sum_{k \geq 1} P_k(x, \tau_a) z^{k-1} \left[\sum_{l \geq 1} \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) \bar{z}^{l-1} + \bar{P}_1(\bar{z}, \bar{\tau}_a) \right]
\end{aligned}$$

$$\begin{aligned}
& + \oint_{\partial D_\delta} d\bar{z} P_1(z-x, \tau_a) \left[\bar{P}_1(\bar{z}-\bar{x}, \bar{\tau}_a) - \bar{P}_1(\bar{x}, \bar{\tau}_a) \right] \Big|_{\delta \rightarrow 0} \\
& = -2\pi i P_1(x, \tau_a) - 2\pi i \sum_{k \geq 1} |\epsilon|^k P_{k+1}(x, \tau_a) \bar{P}'_k(\bar{x}, \bar{\tau}_a), \tag{126}
\end{aligned}$$

and we immediately obtain the case for $k = 1$ and $l = 2$:

$$\begin{aligned}
[K_a(x, \bar{x})](1, 2) & = -\overline{[K_a(x, \bar{x})](2, 1)} \\
& = -2\pi i \bar{P}_1(\bar{x}, \bar{\tau}_a) - 2\pi i \sum_{k \geq 1} |\epsilon|^k P'_k(x, \tau_a) \bar{P}_{k+1}(\bar{x}, \bar{\tau}_a). \tag{127}
\end{aligned}$$

We know from eq.(120) that $[K_a(x, \bar{x})](1, 1)$ satisfies

$$\begin{aligned}
\partial_x [K_a(x, \bar{x})](1, 1) & = [K_a(x, \bar{x})](2, 1) - 2\pi i P_1(x, \tau_a) - \pi \int_{S_a} d^2 z \delta^{(2)}(z-x) \frac{1}{z-x} \\
& = -4\pi i P_1(x, \tau_a) - 2\pi i \sum_{k \geq 1} |\epsilon|^k P_{k+1}(x, \tau_a) \bar{P}'_k(\bar{x}, \bar{\tau}_a) + A, \tag{128}
\end{aligned}$$

$$\begin{aligned}
\partial_{\bar{x}} [K_a(x, \bar{x})](1, 1) & = [K_a(x, \bar{x})](1, 2) - 2\pi i \bar{P}_1(\bar{x}, \bar{\tau}_a) - \pi \int_{S_a} d^2 z \delta^{(2)}(z-x) \frac{1}{\bar{z}-\bar{x}} \\
& = -4\pi i \bar{P}_1(\bar{x}, \bar{\tau}_a) - 2\pi i \sum_{k \geq 1} |\epsilon|^k P'_k(x, \tau_a) \bar{P}_{k+1}(\bar{x}, \bar{\tau}_a) + A, \tag{129}
\end{aligned}$$

where A is purely divergent term. We discard A and integrate them to obtain the regularized $[K_a(x, \bar{x})](1, 1)$:

$$[K_a(x, \bar{x})](1, 1) = -4\pi i \log |Q(x, \tau_a)|^2 + 2\pi i \sum_{k \geq 1} |\epsilon|^k \frac{1}{k} |P'_k(x, \tau_a)|^2. \tag{130}$$

Then we calculate the integral $[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](k, l)$, which appears in the deformed two-point function. For $k = 2$ and $l = 2$ we have

$$\begin{aligned}
[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](2, 2) & = - \oint_{\partial \hat{T}^2} d\bar{z} P_1(z-x_1, \tau_a) \bar{P}_2(\bar{z}-\bar{x}_2, \bar{\tau}_a) \\
& \quad - \oint_A d\bar{z} \sum_{k, l \geq 1} l P_k(x_1, \tau_a) \bar{P}_{l+1}(\bar{x}_2, \bar{\tau}_a) z^{k-1} \bar{z}^{l-1} \\
& \quad + \bar{P}_2(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) \oint_{\partial D_{\delta_1}} d\bar{z} P_1(z-x_1, \tau_a) \Big|_{\delta_1 \rightarrow 0} \\
& \quad + P_1(x_2 - x_1, \tau_a) \oint_{\partial D_{\delta_2}} d\bar{z} \bar{P}_2(\bar{z}-\bar{x}_2, \bar{\tau}_a) \Big|_{\delta_2 \rightarrow 0} \\
& = 2\pi i \sum_{k \geq 1} |\epsilon|^k k P_{k+1}(x_1, \tau_a) \bar{P}_{k+1}(\bar{x}_2, \bar{\tau}_a). \tag{131}
\end{aligned}$$

For $k = 2$ and $l = 1$ we have

$$\begin{aligned} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](2, 1) &= -2\pi i P_1(x_1, \tau_a) - 2\pi i P_1(x_2 - x_1, \tau_a) \\ &\quad - 2\pi i \sum_{k \geq 1} |\epsilon|^k P_{k+1}(x_1, \tau_a) \bar{P}'_k(\bar{x}_2, \bar{\tau}_a). \end{aligned} \quad (132)$$

For $k = 1$ and $l = 2$ we have

$$\begin{aligned} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 2) &= -2\pi i \bar{P}_1(\bar{x}_2, \bar{\tau}_a) - 2\pi i \bar{P}_1(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) \\ &\quad - 2\pi i \sum_{k \geq 1} |\epsilon|^k P'_k(x_1, \tau_a) \bar{P}_{k+1}(\bar{x}_2, \bar{\tau}_a). \end{aligned} \quad (133)$$

For $[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 1)$ it satisfies

$$\begin{aligned} \partial_{x_1} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 1) &= -2\pi i P_1(x_1, \tau_a) - 2\pi i P_1(x_2 - x_1, \tau_a) \\ &\quad - 2\pi i \sum_{k \geq 1} |\epsilon|^k P_{k+1}(x_1, \tau_a) \bar{P}'_k(\bar{x}_2, \bar{\tau}_a), \end{aligned} \quad (134)$$

$$\partial_{\bar{x}_1} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 1) = 2\pi i \left[\bar{P}_1(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) - \bar{P}_1(\bar{x}_1, \bar{\tau}_a) \right], \quad (135)$$

$$\partial_{x_2} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 1) = 2\pi i \left[P_1(x_2 - x_1, \tau_a) - P_1(x_2, \tau_a) \right], \quad (136)$$

$$\begin{aligned} \partial_{\bar{x}_2} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 1) &= -2\pi i \bar{P}_1(\bar{x}_2, \bar{\tau}_a) - 2\pi i \bar{P}_1(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) \\ &\quad - 2\pi i \sum_{k \geq 1} |\epsilon|^k P'_k(x_1, \tau_a) \bar{P}_{k+1}(\bar{x}_2, \bar{\tau}_a). \end{aligned} \quad (137)$$

We integrate these equations to obtain $[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 1)$:

$$\begin{aligned} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 1) &= -2\pi i \log \frac{|Q(x_1, \tau_a)Q(x_2, \tau_a)|^2}{|Q(x_1 - x_2, \tau_a)|^2} \\ &\quad + 2\pi i \sum_{k \geq 1} |\epsilon|^k \frac{1}{k} P'_k(x_1, \tau_a) \bar{P}'_k(\bar{x}_2, \bar{\tau}_a). \end{aligned} \quad (138)$$

For $k = 4$ and $l = 1$ we have

$$\begin{aligned} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](4, 1) &= -\frac{2\pi i}{3} P_3(x_1, \tau_a) - \frac{2\pi i}{3} P_3(x_2 - x_1, \tau_a) \\ &\quad - \frac{2\pi i}{3} \sum_{k \geq 1} |\epsilon|^k \frac{k^2 + 3k + 2}{2} P_{k+3}(x_1, \tau_a) \bar{P}'_k(\bar{x}_2, \bar{\tau}_a). \end{aligned} \quad (139)$$

For $k = 1$ and $l = 4$ we have

$$\begin{aligned} [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 4) &= -\frac{2\pi i}{3} \bar{P}_3(\bar{x}_2, \bar{\tau}_a) - \frac{2\pi i}{3} \bar{P}_3(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) \\ &\quad - \frac{2\pi i}{3} \sum_{k \geq 1} |\epsilon|^k \frac{k^2 + 3k + 2}{2} P'_k(x_1, \tau_a) \bar{P}_{k+3}(\bar{x}_2, \bar{\tau}_a). \end{aligned} \quad (140)$$

For $k = 4$ and $l = 2$ we have

$$[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](4, 2) = \frac{2\pi i}{3} \sum_{k \geq 1} |\epsilon|^k \frac{k^3 + 3k^2 + 2k}{2} P_{k+3}(x_1, \tau_a) \bar{P}_{k+1}(\bar{x}_2, \bar{\tau}_a). \quad (141)$$

For $k = 2$ and $l = 4$ we have

$$[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](2, 4) = \frac{2\pi i}{3} \sum_{k \geq 1} |\epsilon|^k \frac{k^3 + 3k^2 + 2k}{2} P_{k+1}(x_1, \tau_a) \bar{P}_{k+3}(\bar{x}_2, \bar{\tau}_a). \quad (142)$$

For $k = 4$ and $l = 4$ we have

$$[K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](4, 4) = \frac{2\pi i}{3} \sum_{k \geq 1} |\epsilon|^k \frac{k(k+1)^2(k+2)^2}{12} P_{k+3}(x_1, \tau_a) \bar{P}_{k+3}(\bar{x}_2, \bar{\tau}_a). \quad (143)$$

C Complete integral results

In this appendix, we demonstrate the complete integral results that appear in the first-order deformation of the partition function and the correlation function. These integrals can be classified into three types, e.g. $\mathcal{F}\bar{\mathcal{F}}$ -type, $\mathcal{F}\bar{\mathcal{P}}$ -type (or $\mathcal{P}\bar{\mathcal{F}}$ -type), and $\mathcal{P}\bar{\mathcal{P}}$ -type. The $\mathcal{F}\bar{\mathcal{F}}$ -type appears in zero-point contribution $\int d^2z D_z \bar{D}_{\bar{z}} \langle X \rangle$. The $\mathcal{F}\bar{\mathcal{P}}$ -type (or $\mathcal{P}\bar{\mathcal{F}}$ -type) appears in one-point contribution $\int d^2z D_z \bar{D}_{\bar{z}, \bar{x}_i} \langle X \rangle$. The $\mathcal{P}\bar{\mathcal{P}}$ -type appears in two-point contribution $\int d^2z \mathcal{P}_{z, x_i} \bar{\mathcal{P}}_{\bar{z}, \bar{x}_j} \langle X \rangle$. All of these integrals can be represented by I_a , $J_a(x, \bar{x})$ and $K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)$ in Appendix B, as well as column vectors α_a , β_a , θ_a , $\xi_a(x)$ and $\zeta_a(x)$ defined in Subsection 2.2. As an example, consider the following $\mathcal{P}\bar{\mathcal{P}}$ -type integral:

$$\begin{aligned} & \int_{\Sigma(2)} d^2z \left[{}^2\mathcal{P}_1(z, x_1; \tau_a, \tau_{\bar{a}}, \epsilon) {}^2\bar{\mathcal{P}}_2(\bar{z}, \bar{x}_2; \bar{\tau}_a, \bar{\tau}_{\bar{a}}, \bar{\epsilon}) \right] \\ &= \int_{S_a} d^2z \left\{ \left[\tilde{P}_1(z, x_1, \tau_a) + \sum_{k \geq 1} P_{k+3}(z, \tau_a) [\xi_a^{(0)}(x_1)](k) \right] \right. \\ & \quad \times \left. \left[\bar{\tilde{P}}_2(\bar{z}, \bar{x}_2, \bar{\tau}_a) + \sum_{l \geq 1} \bar{P}_{l+3}(\bar{z}, \bar{\tau}_a) [\bar{\xi}_a^{(1)}(\bar{x}_2)](l) \right] \right\} \\ & \quad + \int_{S_{\bar{a}}} d^2z \left\{ \left[-\epsilon P_3(z, \tau_{\bar{a}}) + \sum_{k \geq 1} P_{k+3}(z, \tau_{\bar{a}}) [\zeta_{\bar{a}}^{(0)}(x_1)](k) \right] \left[\sum_{l \geq 1} \bar{P}_{l+3}(\bar{z}, \bar{\tau}_{\bar{a}}) [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x}_2)](l) \right] \right\} \\ &= [K_a(x_1, \bar{x}_1; x_2, \bar{x}_2)](1, 2) + \sum_{k, l \geq 1} I_a(k+3, l+3) [\xi_a^{(0)}(x_1)](k) [\bar{\xi}_a^{(1)}(\bar{x}_2)](l) \\ & \quad + \sum_{k, l \geq 1} I_{\bar{a}}(k+3, l+3) [\zeta_{\bar{a}}^{(0)}(x_1)](k) [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x}_2)](l) + \sum_{k \geq 1} [J_a(x_2, \bar{x}_2)](k+3, 2) [\xi_a^{(0)}(x_1)](k) \end{aligned}$$

$$- \sum_{k \geq 1} \overline{[J_a(x_1, \bar{x}_1)](k+3, 1)} [\bar{\xi}_a^{(1)}(\bar{x}_2)](k) - \epsilon \sum_{l \geq 1} I_{\bar{a}}(3, l+3) [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x}_2)](l). \quad (144)$$

C.1 The full form of three types of integrals

The following are the $\mathcal{F}\bar{\mathcal{F}}$ -type integrals that appear in the deformed partition function:

$$\begin{aligned} & \int_{\Sigma^{(2)}} d^2 z ({}^2\mathcal{F}_a \quad {}^2\bar{\mathcal{F}}_a) \\ &= -8\pi^2 i \text{Im}[\tau_a] + 2\pi i |\epsilon| + 4\pi i |\epsilon| \sum_{k \geq 1} \text{Re}[E_{k+3}(\tau_a) \alpha_a(k)] \\ &+ \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{n,a}^{k,l} |\epsilon|^{n-l+1} \right] \alpha_a(k) \bar{\alpha}_a(l) \\ &+ \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{n,\bar{a}}^{k,l} |\epsilon|^{n-l+1} \right] \beta_{\bar{a}}(k) \bar{\beta}_{\bar{a}}(l), \end{aligned} \quad (145)$$

$$\begin{aligned} & \int_{\Sigma^{(2)}} d^2 z ({}^2\mathcal{F}_a \quad {}^2\bar{\mathcal{F}}_{\bar{a}}) \\ &= 2\pi i |\epsilon| \sum_{k \geq 1} \left[\bar{E}_{k+3}(\bar{\tau}_a) \bar{\beta}_{\bar{a}}(k) + E_{k+3}(\tau_{\bar{a}}) \beta_{\bar{a}}(k) \right] \\ &+ \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{n,a}^{k,l} |\epsilon|^{n-l+1} \right] \alpha_a(k) \bar{\beta}_{\bar{a}}(l) \\ &+ \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{n,\bar{a}}^{k,l} |\epsilon|^{n-l+1} \right] \beta_{\bar{a}}(k) \bar{\alpha}_{\bar{a}}(l), \end{aligned} \quad (146)$$

$$\begin{aligned} & \int_{\Sigma^{(2)}} d^2 z \bar{\epsilon}^{\frac{1}{2}} ({}^2\mathcal{F}_a \quad {}^2\bar{\mathcal{F}}^{\Pi}) \\ &= -2\pi i \bar{\epsilon} + 2\pi i |\epsilon| \sum_{l \geq 1} \bar{E}_{l+1}(\bar{\tau}_a) \bar{\theta}_a(l) \\ &+ \sum_{n, k \geq 1} \frac{2\pi i}{k+2} |\epsilon|^{n-1} \bar{\epsilon} \left[C_{n-2,a}^{k,-1} \alpha_a(k) + C_{n-2,\bar{a}}^{k,-1} \beta_{\bar{a}}(k) \right] \\ &+ \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{n,a}^{k,l} |\epsilon|^{n-l+1} \right] \alpha_a(k) \bar{\theta}_a(l+2) \\ &+ \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{n,\bar{a}}^{k,l} |\epsilon|^{n-l+1} \right] \beta_{\bar{a}}(k) \bar{\theta}_{\bar{a}}(l+2), \end{aligned} \quad (147)$$

$$\begin{aligned} & \int_{\Sigma^{(2)}} d^2 z \epsilon^{\frac{1}{2}} \bar{\epsilon}^{\frac{1}{2}} ({}^2\mathcal{F}^{\Pi 2} \bar{\mathcal{F}}^{\Pi}) \\ &= -4\pi i |\epsilon| + 2\pi i |\epsilon|^3 \sum_{a=1,2} \sum_{n \geq 1} |\epsilon|^{n-1} n |E_{n+1}(\tau_a)|^2 \\ &+ \sum_{a=1,2} \sum_{n, k \geq 1} \frac{4\pi i}{k+2} |\epsilon|^n \text{Re}[C_{n-2,a}^{k,-1} \bar{\epsilon} \theta_a(k+2)] \end{aligned}$$

$$+ \sum_{a=1,2} \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{n,a}^{k,l} |\epsilon|^{n-l+1} \right] \theta_a(k+2) \bar{\theta}_a(l+2). \quad (148)$$

The following are the $\mathcal{F}\bar{\mathcal{P}}$ -type integrals that appear in the deformed one-point functions of primary field and of stress tensor:

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{F}_a {}^2\bar{\mathcal{P}}_1) \\ &= 4\pi i \operatorname{Re}[x] - 2\pi i |\epsilon| \bar{P}_1(\bar{x}, \bar{\tau}_a) \\ &+ \bar{\epsilon} \sum_{k,n \geq 1} \frac{-2\pi i}{k+2} C_{k,\bar{a}}^{n-1,0} |\epsilon|^n \beta_{\bar{a}}(k) + 2\pi i |\epsilon| \sum_{l \geq 1} \bar{E}_{l+3}(\bar{\tau}_a) [\bar{\xi}_a^{(0)}(\bar{x})](l) \\ &+ \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[P'_{k+2}(x, \tau_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l A_{k,a}^{l,0} \bar{P}'_l(\bar{x}, \bar{\tau}_a) \right] \alpha_a(k) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] \alpha_a(k) [\bar{\xi}_a^{(0)}(\bar{x})](l) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] \beta_{\bar{a}}(k) [\bar{\zeta}_{\bar{a}}^{(0)}(\bar{x})](l), \end{aligned} \quad (149)$$

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{F}_{\bar{a}} {}^2\bar{\mathcal{P}}_1) \\ &= \bar{\epsilon} \sum_{k,n \geq 1} \frac{-2\pi i}{k+2} C_{k,\bar{a}}^{n-1,0} |\epsilon|^n \alpha_{\bar{a}}(k) + 2\pi i |\epsilon| \sum_{l \geq 1} \bar{E}_{l+3}(\bar{\tau}_{\bar{a}}) [\bar{\zeta}_{\bar{a}}^{(0)}(\bar{x})](l) \\ &+ \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[P'_{k+2}(x, \tau_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l A_{k,a}^{l,0} \bar{P}'_l(\bar{x}, \bar{\tau}_a) \right] \beta_a(k) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] \beta_a(k) [\bar{\xi}_a^{(0)}(\bar{x})](l) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] \alpha_{\bar{a}}(k) [\bar{\zeta}_{\bar{a}}^{(0)}(\bar{x})](l), \end{aligned} \quad (150)$$

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{F}^\Pi \epsilon^{\frac{1}{2}2} \bar{\mathcal{P}}_1) \\ &= -2\pi i \epsilon P_1(x, \tau_a) - 2\pi i \epsilon \sum_{l \geq 1} \bar{P}'_l(\bar{x}, \bar{\tau}_a) E_{l+1}(\tau_a) |\epsilon|^l + \pi i \bar{\epsilon} \sum_{n \geq k \geq 1} \bar{C}_{0,a}^{n,k} |\epsilon|^{n-k+1} \theta_a(k+2) \\ &+ \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[P'_{k+2}(x, \tau_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l A_{k,a}^{l,0} \bar{P}'_l(\bar{x}, \bar{\tau}_a) \right] \theta_a(k+2) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{l+2} \left[\frac{1}{|\epsilon|^{l+2}} \delta_{k,l} + \sum_{n \geq k} \bar{C}_{l,a}^{n,k} |\epsilon|^{n-k+1} \right] \theta_a(k+2) [\bar{\xi}_a^{(0)}(\bar{x})](l) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{l+2} \left[\frac{1}{|\epsilon|^{l+2}} \delta_{k,l} + \sum_{n \geq k} \bar{C}_{l,\bar{a}}^{n,k} |\epsilon|^{n-k+1} \right] \theta_{\bar{a}}(k+2) [\bar{\zeta}_{\bar{a}}^{(0)}(\bar{x})](l) \end{aligned}$$

$$+ \sum_{n,l \geq 1} \frac{-2\pi i}{l+2} |\epsilon|^{n-1} \left[\bar{C}_{l,a}^{n-2,-1} [\bar{\xi}_a^{(0)}(\bar{x})](l) + \bar{C}_{l,\bar{a}}^{n-2,-1} [\bar{\zeta}_a^{(0)}(\bar{x})](l) \right], \quad (151)$$

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{F}_a {}^2\bar{\mathcal{P}}_2) \\ = & 2\pi i + 2\pi i |\epsilon| \sum_l \bar{E}_{l+3}(\bar{\tau}_a) [\bar{\xi}_a^{(1)}(\bar{x})](l) \\ & + 2\pi i \bar{P}_2(\bar{x}, \bar{\tau}_a) |\epsilon| - 2\pi i \sum_{k,l \geq 1} |\epsilon|^l A_{k,a}^{l,0} \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) \alpha_a(k) \\ & + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] \alpha_a(k) [\bar{\xi}_a^{(1)}(\bar{x})](l) \\ & + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] \beta_{\bar{a}}(k) [\bar{\zeta}_a^{(1)}(\bar{x})](l), \end{aligned} \quad (152)$$

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{F}_{\bar{a}} {}^2\bar{\mathcal{P}}_2) \\ = & 2\pi i |\epsilon| \sum_l \bar{E}_{l+3}(\bar{\tau}_a) [\bar{\zeta}_a^{(1)}(\bar{x})](l) - 2\pi i \sum_{k,l \geq 1} |\epsilon|^l A_{k,a}^{l,0} \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) \beta_a(k) \\ & + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] \beta_a(k) [\bar{\xi}_a^{(1)}(\bar{x})](l) \\ & + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] \alpha_{\bar{a}}(k) [\bar{\zeta}_a^{(1)}(\bar{x})](l), \end{aligned} \quad (153)$$

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{F}^\Pi \epsilon^{\frac{1}{2}2} \bar{\mathcal{P}}_2) \\ = & 2\pi i \epsilon \sum_{l \geq 1} l \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) E_{l+1}(\tau_a) |\epsilon|^l - 2\pi i \sum_{k,l \geq 1} |\epsilon|^l A_{k,a}^{l,0} \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) \theta_a(k+2) \\ & + \sum_{k,l \geq 1} \frac{-2\pi i}{l+2} \left[\frac{1}{|\epsilon|^{l+2}} \delta_{k,l} + \sum_{n \geq k} \bar{C}_{l,a}^{n,k} |\epsilon|^{n-k+1} \right] \theta_a(k+2) [\bar{\xi}_a^{(1)}(\bar{x})](l) \\ & + \sum_{k,l \geq 1} \frac{-2\pi i}{l+2} \left[\frac{1}{|\epsilon|^{l+2}} \delta_{k,l} + \sum_{n \geq k} \bar{C}_{l,\bar{a}}^{n,k} |\epsilon|^{n-k+1} \right] \theta_{\bar{a}}(k+2) [\bar{\zeta}_a^{(1)}(\bar{x})](l) \\ & + \sum_{n,l \geq 1} \frac{-2\pi i}{l+2} |\epsilon|^{n-1} \left[\bar{C}_{l,a}^{n-2,-1} [\bar{\xi}_a^{(1)}(\bar{x})](l) + \bar{C}_{l,\bar{a}}^{n-2,-1} [\bar{\zeta}_a^{(1)}(\bar{x})](l) \right], \end{aligned} \quad (154)$$

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_4 {}^2\bar{\mathcal{F}}_a) \\ = & 2\pi i P_4(x, \tau_a) |\epsilon| + 2\pi i \sum_{k \geq 1} |\epsilon| E_{k+3}(\tau_a) [\xi_a^{(3)}(x)](k) \\ & - 2\pi i \sum_{k,l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l \bar{A}_{k,a}^{l,0} P_{l+3}(x, \tau_a) \bar{\alpha}_a(k) \\ & + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(3)}(x)](k) \bar{\alpha}_a(l) \end{aligned}$$

$$+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(3)}(x)](k) \bar{\beta}_{\bar{a}}(l), \quad (155)$$

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_{\delta}} d^2 z ({}^2\mathcal{P}_4 {}^2\bar{\mathcal{F}}_{\bar{a}}) \\ &= 2\pi i \sum_{k \geq 1} |\epsilon| E_{k+3}(\tau_{\bar{a}}) [\zeta_{\bar{a}}^{(3)}(x)](k) - 2\pi i \sum_{k,l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l \bar{A}_{k,\bar{a}}^{l,0} P_{l+3}(x, \tau_{\bar{a}}) \bar{\beta}_{\bar{a}}(k) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_{\bar{a}}^{(3)}(x)](k) \bar{\beta}_{\bar{a}}(l) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(3)}(x)](k) \bar{\alpha}_{\bar{a}}(l), \quad (156) \end{aligned}$$

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_{\delta}} d^2 z ({}^2\mathcal{P}_4 {}^2\bar{\mathcal{F}}^{\Pi} \bar{\epsilon}^{\frac{1}{2}}) \\ &= 2\pi i \sum_{l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l P_{l+3}(x, \tau_{\bar{a}}) \left[\bar{\epsilon} l \bar{E}_{l+1}(\bar{\tau}_{\bar{a}}) - \bar{A}_{k,\bar{a}}^{l,0} \bar{\theta}_{\bar{a}}(k+2) \right] \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} + \sum_{n \geq k} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_{\bar{a}}^{(3)}(x)](k) \bar{\theta}_{\bar{a}}(l+2) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} + \sum_{n \geq k} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(3)}(x)](k) \bar{\theta}_{\bar{a}}(l+2). \quad (157) \end{aligned}$$

The following are the $\mathcal{P}\bar{\mathcal{P}}$ -type integrals that appear in the deformed one-point functions of primary field:

$$\begin{aligned} & \int_{\Sigma^{(2)} \setminus D_{\delta}} d^2 z ({}^2\mathcal{P}_1 {}^2\bar{\mathcal{P}}_1) \\ &= -4\pi i \log |Q(x, \tau_{\bar{a}})|^2 + 2\pi i \sum_{k \geq 1} \frac{|\epsilon|^k}{k} |P'_k(x, \tau_{\bar{a}})|^2 \\ &+ \sum_{k,n \geq 1} \frac{-2\pi i}{k+2} |\epsilon|^n 2\text{Re} \left[C_{k,\bar{a}}^{n-1,0} \bar{\epsilon} [\zeta_{\bar{a}}^{(0)}(x)](k) \right] \\ &+ \sum_{k \geq 1} \frac{-2\pi i}{k+2} 2\text{Re} \left[\left[P'_{k+2}(x, \tau_{\bar{a}}) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l A_{k,\bar{a}}^{l,0} \bar{P}'_l(\bar{x}, \bar{\tau}_{\bar{a}}) \right] [\xi_{\bar{a}}^{(0)}(x)](k) \right] \\ &+ \sum_{n \geq 1} C_{0,\bar{a}}^{n,0} |\epsilon|^{n+3} + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_{\bar{a}}^{(0)}(x)](k) [\bar{\xi}_{\bar{a}}^{(0)}(\bar{x})](l) \\ &+ \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(0)}(x)](k) [\bar{\zeta}_{\bar{a}}^{(0)}(\bar{x})](l), \quad (158) \\ & \int_{\Sigma^{(2)} \setminus D_{\delta}} d^2 z ({}^2\mathcal{P}_1 {}^2\bar{\mathcal{P}}_2) \\ &= -2\pi i \bar{P}_1(\bar{x}, \bar{\tau}_{\bar{a}}) + \sum_{k,n \geq 1} \frac{-2\pi i}{k+2} \bar{C}_{k,\bar{a}}^{n-1,0} |\epsilon|^n [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x})](k) \end{aligned}$$

$$\begin{aligned}
& -2\pi i \sum_{l \geq 1} |\epsilon|^l \bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) \left[P'_l(x, \tau_a) + \sum_{k \geq 1} A_{k,a}^{l,0} [\xi_a^{(0)}(x)](k) \right] \\
& + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[\bar{P}'_{k+2}(\bar{x}, \bar{\tau}_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l \bar{A}_{k,a}^{l,0} P'_l(x, \tau_a) \right] [\bar{\xi}_a^{(1)}(\bar{x})](k) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(0)}(x)](k) [\bar{\xi}_a^{(1)}(\bar{x})](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(0)}(x)](k) [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x})](l), \tag{159}
\end{aligned}$$

$$\begin{aligned}
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_2 {}^2\bar{\mathcal{P}}_2) \\
& = 2\pi i \sum_k k |P_{k+1}(x, \tau_a)|^2 |\epsilon|^k - 2\pi i \sum_{k,l \geq 1} |\epsilon|^l 2\text{Re} \left[\bar{P}_{l+1}(\bar{x}, \bar{\tau}_a) A_{k,a}^{l,0} [\xi_a^{(1)}(x)](k) \right] \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(1)}(x)](k) [\bar{\xi}_a^{(1)}(\bar{x})](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(1)}(x)](k) [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x})](l). \tag{160}
\end{aligned}$$

The following are the $\mathcal{P}\bar{\mathcal{P}}$ -type integrals that appear in the deformed two-point function, and the two insertion points x_1 and x_2 are on the same torus S_a :

$$\begin{aligned}
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_1^{(x_1)} {}^2\bar{\mathcal{P}}_1^{(\bar{x}_2)}) \\
& = -2\pi i \log \frac{|Q(x_1, \tau_a) Q(x_2, \tau_a)|^2}{|Q(x_1 - x_2, \tau_a)|^2} - \pi i + 2\pi i \sum_k \frac{|\epsilon|^k}{k} P'_k(x_1, \tau_a) \bar{P}'_k(\bar{x}_2, \bar{\tau}_a) \\
& + \sum_{n \geq 1} C_{0,\bar{a}}^{n,0} |\epsilon|^{n+3} + \sum_{k,n \geq 1} \frac{-2\pi i}{k+2} |\epsilon|^n \left[C_{k,\bar{a}}^{n-1,0} \bar{\epsilon} [\zeta_{\bar{a}}^{(0)}(x_1)](k) + \bar{C}_{k,\bar{a}}^{n-1,0} \epsilon [\bar{\zeta}_{\bar{a}}^{(0)}(\bar{x}_2)](k) \right] \\
& + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[\bar{P}'_{k+2}(\bar{x}_1, \bar{\tau}_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l \bar{A}_{k,a}^{l,0} P'_l(x_1, \tau_a) \right] [\bar{\xi}_a^{(0)}(\bar{x}_2)](k) \\
& + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[P'_{k+2}(x_2, \tau_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l A_{k,a}^{l,0} \bar{P}'_l(\bar{x}_2, \bar{\tau}_a) \right] [\xi_a^{(0)}(x_1)](k) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(0)}(x_1)](k) [\bar{\xi}_a^{(0)}(\bar{x}_2)](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(0)}(x_1)](k) [\bar{\zeta}_{\bar{a}}^{(0)}(\bar{x}_2)](l), \tag{161} \\
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_1^{(x_1)} {}^2\bar{\mathcal{P}}_2^{(\bar{x}_2)}) \\
& = -2\pi i \left[\bar{P}_1(\bar{x}_2, \bar{\tau}_a) + \bar{P}_1(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) \right] + \sum_{k,n \geq 1} \frac{-2\pi i}{k+2} \bar{C}_{k,\bar{a}}^{n-1,0} \epsilon |\epsilon|^n [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x}_2)](k)
\end{aligned}$$

$$\begin{aligned}
& -2\pi i \sum_{l \geq 1} |\epsilon|^l \bar{P}_{l+1}(\bar{x}_2, \bar{\tau}_a) \left[P'_l(x_1, \tau_a) + \sum_{k \geq 1} A_{k,a}^{l,0} [\xi_a^{(0)}(x_1)](k) \right] \\
& + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[\bar{P}'_{k+2}(\bar{x}_1, \bar{\tau}_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l \bar{A}_{k,a}^{l,0} P'_l(x_1, \tau_a) \right] [\bar{\xi}_a^{(1)}(\bar{x}_2)](k) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(0)}(x_1)](k) [\bar{\xi}_a^{(1)}(\bar{x}_2)](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(0)}(x_1)](k) [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x}_2)](l), \tag{162}
\end{aligned}$$

$$\begin{aligned}
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_2^{(x_1)2} \bar{\mathcal{P}}_2^{(\bar{x}_2)}) \\
& = 2\pi i \sum_{k \geq 1} |\epsilon|^k k P_{k+1}(x_1, \tau_a) \bar{P}_{k+1}(\bar{x}_2, \bar{\tau}_a) \\
& - 2\pi i \sum_{k,l \geq 1} |\epsilon|^l \left[P_{l+1}(x_1, \tau_a) \bar{A}_{k,a}^{l,0} [\bar{\xi}_a^{(1)}(\bar{x}_2)](k) + \bar{P}_{l+1}(\bar{x}_2, \bar{\tau}_a) A_{k,a}^{l,0} [\xi_a^{(1)}(x_1)](k) \right] \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(1)}(x_1)](k) [\bar{\xi}_a^{(1)}(\bar{x}_2)](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(1)}(x_1)](k) [\bar{\zeta}_{\bar{a}}^{(1)}(\bar{x}_2)](l), \tag{163}
\end{aligned}$$

$$\begin{aligned}
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_1^{(x_1)2} \bar{\mathcal{P}}_4^{(\bar{x}_2)}) \\
& = -\frac{2\pi i}{3} \left[\bar{P}_3(\bar{x}_2, \bar{\tau}_a) + \bar{P}_3(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) \right] + \sum_{k,n \geq 1} \frac{-2\pi i}{k+2} \bar{C}_{k,\bar{a}}^{m-1,0} \epsilon |\epsilon|^n [\bar{\zeta}_{\bar{a}}^{(3)}(\bar{x}_2)](k) \\
& - 2\pi i \sum_{l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l \bar{P}_{l+3}(\bar{x}_2, \bar{\tau}_a) \left[P'_l(x_1, \tau_a) + \sum_{k \geq 1} A_{k,a}^{l,0} [\xi_a^{(0)}(x_1)](k) \right] \\
& + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[\bar{P}'_{k+2}(\bar{x}_1, \bar{\tau}_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l \bar{A}_{k,a}^{l,0} P'_l(x_1, \tau_a) \right] [\bar{\xi}_a^{(3)}(\bar{x}_2)](k) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(0)}(x_1)](k) [\bar{\xi}_a^{(3)}(\bar{x}_2)](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(0)}(x_1)](k) [\bar{\zeta}_{\bar{a}}^{(3)}(\bar{x}_2)](l), \tag{164}
\end{aligned}$$

$$\begin{aligned}
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_2^{(x_1)2} \bar{\mathcal{P}}_4^{(\bar{x}_2)}) \\
& = -2\pi i \sum_{k,l \geq 1} |\epsilon|^l \bar{A}_{k,a}^{l,0} P_{l+1}(x_1, \tau_a) [\bar{\xi}_a^{(3)}(\bar{x}_2)](k) \\
& + 2\pi i \sum_{l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l \bar{P}_{l+3}(\bar{x}_2, \bar{\tau}_a) \left[l P_{l+1}(x_1, \tau_a) - \sum_{k \geq 1} A_{k,a}^{l,0} [\xi_a^{(1)}(x_1)](k) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(1)}(x_1)](k) [\bar{\xi}_a^{(3)}(\bar{x}_2)](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_a^{(1)}(x_1)](k) [\bar{\zeta}_a^{(3)}(\bar{x}_2)](l), \tag{165} \\
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_4^{(x_1)2} \bar{\mathcal{P}}_4^{(\bar{x}_2)}) \\
& = \frac{2\pi i}{3} \sum_{l \geq 1} \frac{l(l+1)^2(l+2)^2}{12} |\epsilon|^l P_{l+3}(x_1, \tau_a) \bar{P}_{l+3}(\bar{x}_2, \bar{\tau}_a) \\
& - 2\pi i \sum_{k,l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l \bar{P}_{l+3}(\bar{x}_2, \bar{\tau}_a) A_{k,a}^{l,0} [\xi_a^{(3)}(x_1)](k) \\
& - 2\pi i \sum_{k,l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l P_{l+3}(x_1, \tau_a) \bar{A}_{k,a}^{l,0} [\bar{\xi}_a^{(3)}(\bar{x}_2)](k) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(3)}(x_1)](k) [\bar{\xi}_a^{(3)}(\bar{x}_2)](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_a^{(3)}(x_1)](k) [\bar{\zeta}_a^{(3)}(\bar{x}_2)](l). \tag{166}
\end{aligned}$$

The following are the $\mathcal{P}\bar{\mathcal{P}}$ -type integrals that appear in the deformed two-point function, and the two insertion points x and y are on different tori:

$$\begin{aligned}
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_1^{(x)2} \bar{\mathcal{P}}_1^{(\bar{y})}) \\
& = \pi i \bar{\epsilon} \left[\bar{P}'_2(\bar{x}, \bar{\tau}_a) - \sum_{l \geq 1} |\epsilon|^l (l+1) P'_l(x, \tau_a) \bar{E}_{l+2}(\bar{\tau}_a) \right] \\
& + \pi i \epsilon \left[P'_2(y, \tau_a) - \sum_{l \geq 1} |\epsilon|^l (l+1) \bar{P}'_l(\bar{y}, \bar{\tau}_a) E_{l+2}(\tau_a) \right] \\
& + \sum_{k,n \geq 1} \frac{-2\pi i}{k+2} \left[C_{k,a}^{n-1,0} \bar{\epsilon} |\epsilon|^n [\xi_a^{(0)}(x)](k) + \bar{C}_{k,\bar{a}}^{n-1,0} \epsilon |\epsilon|^n [\bar{\xi}_a^{(0)}(\bar{y})](k) \right] \\
& + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[\bar{P}'_{k+2}(\bar{x}, \bar{\tau}_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l \bar{A}_{k,a}^{l,0} P'_l(x, \tau_a) \right] [\bar{\zeta}_a^{(0)}(\bar{y})](k) \\
& + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[P'_{k+2}(y, \tau_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l A_{k,\bar{a}}^{l,0} \bar{P}'_l(\bar{y}, \bar{\tau}_a) \right] [\zeta_a^{(0)}(x)](k) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(0)}(x)](k) [\bar{\zeta}_a^{(0)}(\bar{y})](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_a^{(0)}(x)](k) [\bar{\xi}_a^{(0)}(\bar{y})](l), \tag{167} \\
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_1^{(x)2} \bar{\mathcal{P}}_2^{(\bar{y})})
\end{aligned}$$

$$\begin{aligned}
&= \pi i \epsilon \sum_l |\epsilon|^l l(l+1) \bar{P}_{l+1}(\bar{y}, \bar{\tau}_a) E_{l+2}(\tau_a) + \sum_{k, n \geq 1} \frac{-2\pi i}{k+2} \bar{C}_{k, \bar{a}}^{n-1, 0} \epsilon |\epsilon|^n [\bar{\zeta}_a^{(1)}(\bar{y})](k) \\
&\quad - 2\pi i \sum_{k, l \geq 1} |\epsilon|^l \bar{P}_{l+1}(\bar{y}, \bar{\tau}_a) A_{k, \bar{a}}^{l, 0} [\zeta_a^{(0)}(x)](k) \\
&\quad + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[\bar{P}'_{k+2}(\bar{x}, \bar{\tau}_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l \bar{A}_{k, a}^{l, 0} P'_l(x, \tau_a) \right] [\bar{\zeta}_a^{(1)}(\bar{y})](k) \\
&\quad + \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k, l} - \sum_{n \geq l} C_{k, a}^{n, l} |\epsilon|^{n-l+1} \right] [\xi_a^{(0)}(x)](k) [\bar{\zeta}_a^{(1)}(\bar{y})](l) \\
&\quad + \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k, l} - \sum_{n \geq l} C_{k, \bar{a}}^{n, l} |\epsilon|^{n-l+1} \right] [\zeta_a^{(0)}(x)](k) [\bar{\xi}_a^{(1)}(\bar{y})](l), \tag{168}
\end{aligned}$$

$$\begin{aligned}
&\int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_2^{(x)2} \bar{\mathcal{P}}_2^{(\bar{y})}) \\
&= -2\pi i \sum_{k, l \geq 1} |\epsilon|^l \left[P_{l+1}(x, \tau_a) \bar{A}_{k, a}^{l, 0} [\bar{\zeta}_a^{(1)}(\bar{y})](k) + \bar{P}_{l+1}(\bar{y}, \bar{\tau}_a) A_{k, \bar{a}}^{l, 0} [\zeta_a^{(1)}(x)](k) \right] \\
&\quad + \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k, l} - \sum_{n \geq l} C_{k, a}^{n, l} |\epsilon|^{n-l+1} \right] [\xi_a^{(1)}(x)](k) [\bar{\zeta}_a^{(1)}(\bar{y})](l) \\
&\quad + \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k, l} - \sum_{n \geq l} C_{k, \bar{a}}^{n, l} |\epsilon|^{n-l+1} \right] [\zeta_a^{(1)}(x)](k) [\bar{\xi}_a^{(1)}(\bar{y})](l), \tag{169}
\end{aligned}$$

$$\begin{aligned}
&\int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_1^{(x)2} \bar{\mathcal{P}}_4^{(\bar{y})}) \\
&= \epsilon \pi i \sum_{l \geq 1} \frac{l(l+1)^2(l+2)}{6} |\epsilon|^l \bar{P}_{l+3}(\bar{y}, \bar{\tau}_a) E_{l+2}(\tau_a) + \sum_{k, n \geq 1} \frac{-2\pi i}{k+2} \bar{C}_{k, \bar{a}}^{n-1, 0} \epsilon |\epsilon|^n [\bar{\zeta}_a^{(3)}(\bar{y})](k) \\
&\quad - 2\pi i \sum_{k, l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l \bar{P}_{l+3}(\bar{y}, \bar{\tau}_a) A_{k, \bar{a}}^{l, 0} [\zeta_a^{(0)}(x)](k) \\
&\quad + \sum_{k \geq 1} \frac{-2\pi i}{k+2} \left[\bar{P}'_{k+2}(\bar{x}, \bar{\tau}_a) - \sum_{l \geq 1} \frac{k+2}{l} |\epsilon|^l \bar{A}_{k, a}^{l, 0} P'_l(x, \tau_a) \right] [\bar{\zeta}_a^{(3)}(\bar{y})](k) \\
&\quad + \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k, l} - \sum_{n \geq l} C_{k, a}^{n, l} |\epsilon|^{n-l+1} \right] [\xi_a^{(0)}(x)](k) [\bar{\zeta}_a^{(3)}(\bar{y})](l) \\
&\quad + \sum_{k, l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k, l} - \sum_{n \geq l} C_{k, \bar{a}}^{n, l} |\epsilon|^{n-l+1} \right] [\zeta_a^{(0)}(x)](k) [\bar{\xi}_a^{(3)}(\bar{y})](l), \tag{170}
\end{aligned}$$

$$\begin{aligned}
&\int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_2^{(x)2} \bar{\mathcal{P}}_4^{(\bar{y})}) \\
&= -2\pi i \sum_{k, l \geq 1} |\epsilon|^l \bar{A}_{k, a}^{l, 0} P_{l+1}(x, \tau_a) [\bar{\zeta}_a^{(3)}(\bar{y})](k) \\
&\quad - 2\pi i \sum_{l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l \bar{P}_{l+3}(\bar{y}, \bar{\tau}_a) A_{k, \bar{a}}^{l, 0} [\zeta_a^{(1)}(x)](k)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,a}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(1)}(x)](k) [\bar{\zeta}_a^{(3)}(\bar{y})](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(1)}(x)](k) [\bar{\xi}_{\bar{a}}^{(3)}(\bar{y})](l), \tag{171}
\end{aligned}$$

$$\begin{aligned}
& \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z ({}^2\mathcal{P}_4^{(x)} {}^2\bar{\mathcal{P}}_4^{(\bar{y})}) \\
& = -2\pi i \sum_{k,l \geq 1} \frac{l^2 + 3l + 2}{6} |\epsilon|^l \left[\bar{P}_{l+3}(\bar{y}, \bar{\tau}_{\bar{a}}) A_{k,\bar{a}}^{l,0} [\zeta_{\bar{a}}^{(3)}(x)](k) + P_{l+3}(x, \tau_a) \bar{A}_{k,a}^{l,0} [\bar{\zeta}_a^{(3)}(\bar{y})](l) \right] \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\xi_a^{(3)}(x)](k) [\bar{\zeta}_a^{(3)}(\bar{y})](l) \\
& + \sum_{k,l \geq 1} \frac{-2\pi i}{k+2} \left[\frac{1}{|\epsilon|^{k+2}} \delta_{k,l} - \sum_{n \geq l} C_{k,\bar{a}}^{n,l} |\epsilon|^{n-l+1} \right] [\zeta_{\bar{a}}^{(3)}(x)](k) [\bar{\xi}_{\bar{a}}^{(3)}(\bar{y})](l). \tag{172}
\end{aligned}$$

C.2 Details of deformed partition function

In this Appendix we discuss the detailed calculation of the first-order deformed partition function. Applying eqs.(12)(13)(35) we have

$$\begin{aligned}
\delta_\lambda Z & = \frac{i}{2} \int_{\Sigma^{(2)}} d^2 z D_z \bar{D}_{\bar{z}} Z \\
& = \frac{i}{8\pi^2} \int_{\Sigma^{(2)}} d^2 z \left\{ ({}^2\mathcal{F}_1 {}^2\bar{\mathcal{F}}_1) \partial_{\tau_1} \partial_{\bar{\tau}_1} + ({}^2\mathcal{F}_1 {}^2\bar{\mathcal{F}}_2) \partial_{\tau_1} \partial_{\bar{\tau}_2} - 2\pi i \bar{\epsilon}^{\frac{1}{2}} ({}^2\mathcal{F}_1 {}^2\bar{\mathcal{F}}^\Pi) \partial_{\tau_1} \partial_{\bar{\epsilon}} \right. \\
& \quad + ({}^2\mathcal{F}_2 {}^2\bar{\mathcal{F}}_1) \partial_{\tau_2} \partial_{\bar{\tau}_1} + ({}^2\mathcal{F}_2 {}^2\bar{\mathcal{F}}_2) \partial_{\tau_2} \partial_{\bar{\tau}_2} - 2\pi i \bar{\epsilon}^{\frac{1}{2}} ({}^2\mathcal{F}_2 {}^2\bar{\mathcal{F}}^\Pi) \partial_{\tau_2} \partial_{\bar{\epsilon}} \\
& \quad \left. + 2\pi i \bar{\epsilon}^{\frac{1}{2}} ({}^2\mathcal{F}^\Pi {}^2\bar{\mathcal{F}}_1) \partial_{\bar{\epsilon}} \partial_{\bar{\tau}_1} + 2\pi i \bar{\epsilon}^{\frac{1}{2}} ({}^2\mathcal{F}^\Pi {}^2\bar{\mathcal{F}}_2) \partial_{\bar{\epsilon}} \partial_{\bar{\tau}_2} + 4\pi^2 \bar{\epsilon}^{\frac{1}{2}} \bar{\epsilon}^{\frac{1}{2}} ({}^2\mathcal{F}^\Pi {}^2\bar{\mathcal{F}}^\Pi) \partial_{\bar{\epsilon}} \partial_{\bar{\epsilon}} \right\} Z. \tag{173}
\end{aligned}$$

The $\mathcal{F}\bar{\mathcal{F}}$ -type integrals are calculated in eqs. (145)(146)(147)(148). For simplicity, we approximate the result up to $|\epsilon|^3$:

$$\begin{aligned}
\delta_\lambda Z & = \sum_{a=1,2} \left[\text{Im}[\tau_a] - \frac{|\epsilon|}{6\pi} \right] \partial_{\tau_a} \partial_{\bar{\tau}_a} Z + \left[2\pi |\epsilon| - \frac{2\pi}{3} |\epsilon|^3 \sum_{a=1,2} |E_2(\tau_a)|^2 \right] \partial_{\bar{\epsilon}} \partial_{\bar{\epsilon}} Z \\
& + \sum_{a=1,2} 2\text{Re} \left[\left[-\frac{i}{2} \bar{\epsilon} + \frac{i}{3} \bar{\epsilon} |\epsilon| \bar{E}_2(\bar{\tau}_a) + \frac{i}{2} \epsilon |\epsilon|^2 \bar{E}_2(\bar{\tau}_a) E_4(\tau_a) \right] \partial_{\tau_a} \partial_{\bar{\epsilon}} Z \right] \\
& + 2\text{Re} \left[\left[-\frac{|\epsilon|}{6\pi} \left(\epsilon^2 E_4(\tau_2) + \bar{\epsilon}^2 \bar{E}_4(\bar{\tau}_1) \right) \right] \partial_{\tau_1} \partial_{\bar{\tau}_2} Z \right] + O(|\epsilon|^4). \tag{174}
\end{aligned}$$

C.3 Details of deformed one-point function

In this Appendix we discuss the detailed calculation of the first-order deformed one-point functions. For primary one-point function, the first-order correction depends on integral eq.(45). The first term in eq.(45) has been calculated in eq.(174) already. For the second term in eq.(45), we apply eqs.(13)(43) and obtain

$$\begin{aligned}
& \frac{i}{2} \frac{1}{Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z D_z \bar{\mathcal{P}}_{\bar{z}, \bar{x}} (Z \langle V \rangle) \\
&= \frac{i}{2} \frac{1}{Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left\{ \sum_{b=a, \bar{a}} ({}^2 \mathcal{F}_b {}^2 \bar{\mathcal{P}}_1) \frac{1}{2\pi i} \partial_{\tau_b} \partial_{\bar{x}} + ({}^2 \mathcal{F}^{\text{II}} \epsilon^{\frac{1}{2} 2} \bar{\mathcal{P}}_1) \partial_\epsilon \partial_{\bar{x}} \right. \\
&\quad \left. + \sum_{b=a, \bar{a}} ({}^2 \mathcal{F}_b {}^2 \bar{\mathcal{P}}_2) \frac{1}{2\pi i} \partial_{\tau_b} wt[\bar{u}] + ({}^2 \mathcal{F}^{\text{II}} \epsilon^{\frac{1}{2} 2} \bar{\mathcal{P}}_2) \partial_\epsilon wt[\bar{u}] \right\} (Z \langle V \rangle). \quad (175)
\end{aligned}$$

The $\mathcal{F}\bar{\mathcal{P}}$ -type integrals are calculated in eqs.(149)(150)(151)(152)(153)(154), we approximate eq.(175) up to $|\epsilon|^3$:

$$\begin{aligned}
& \frac{i}{2} \frac{1}{Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z D_z \bar{\mathcal{P}}_{\bar{z}, \bar{x}} (Z \langle V \rangle) \\
&= \frac{1}{Z} \left\{ i \left[\text{Re}[x] - \frac{|\epsilon|}{3} \bar{P}_1(\bar{x}, \bar{\tau}_a) \right] \partial_{\tau_a} \partial_{\bar{x}} + i \left[\frac{1}{2} + \frac{|\epsilon|}{3} \bar{P}_2(\bar{x}, \bar{\tau}_a) \right] wt[\bar{u}] \partial_{\tau_a} \right. \\
&\quad - \frac{i}{6} \left[\epsilon^2 P_3(x, \tau_a) + 2|\epsilon| \left(\epsilon^2 E_4(\tau_a) + \bar{\epsilon}^2 \bar{E}_4(\bar{\tau}_a) \right) \bar{P}_1(\bar{x}, \bar{\tau}_a) \right] \partial_{\tau_a} \partial_{\bar{x}} \\
&\quad + \frac{i|\epsilon|}{3} \left[\epsilon^2 E_4(\tau_a) + \bar{\epsilon}^2 \bar{E}_4(\bar{\tau}_a) \right] \bar{P}_2(\bar{x}, \bar{\tau}_a) wt[\bar{u}] \partial_{\tau_a} \\
&\quad + \pi \left[\epsilon P_1(x, \tau_a) + \left(\frac{2\epsilon|\epsilon|}{3} E_2(\tau_a) - \frac{\bar{\epsilon}}{\pi} |\epsilon E_4(\tau_a)|^2 \right) \bar{P}_1(\bar{x}, \bar{\tau}_a) + \frac{\epsilon^3}{3} E_2(\tau_a) P_3(x, \tau_a) \right] \partial_\epsilon \partial_{\bar{x}} \\
&\quad \left. - \left[\frac{2\pi}{3} \epsilon |\epsilon| E_2(\tau_a) - \bar{\epsilon} |\epsilon E_4(\tau_a)|^2 \right] \bar{P}_2(\bar{x}, \bar{\tau}_a) wt[\bar{u}] \partial_\epsilon \right\} (Z \langle V \rangle) + O(|\epsilon|^4), \quad (176)
\end{aligned}$$

One can obtain the third term contribution in eq.(45) by taking complex conjugate of eq.(176). For the fourth term in eq.(45), we can divide the fourth term into four parts using eq.(43):

$$\begin{aligned}
& \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \mathcal{P}_{z, x} \bar{\mathcal{P}}_{\bar{z}, \bar{x}} \langle V \rangle \\
&= \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left\{ ({}^2 \mathcal{P}_1 {}^2 \bar{\mathcal{P}}_1) \partial_x \partial_{\bar{x}} + ({}^2 \mathcal{P}_1 {}^2 \bar{\mathcal{P}}_2) \partial_x wt[\bar{u}] \right. \\
&\quad \left. + ({}^2 \mathcal{P}_2 {}^2 \bar{\mathcal{P}}_1) wt[u] \partial_{\bar{x}} + ({}^2 \mathcal{P}_2 {}^2 \bar{\mathcal{P}}_2) wt[u] wt[\bar{u}] \right\} \langle V \rangle. \quad (177)
\end{aligned}$$

Using $\mathcal{P}\bar{\mathcal{P}}$ -type integrals eqs.(158)(160)(159) , up to $|\epsilon|^3$, then it becomes

$$\begin{aligned}
& \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \mathcal{P}_{z,x} \bar{\mathcal{P}}_{\bar{z},\bar{x}} \langle V \rangle \\
&= \left[2\pi \log |Q(x, \tau_a)|^2 + \frac{\pi}{2} \right] \partial_x \partial_{\bar{x}} \langle V \rangle + 2\text{Re} \left[\pi \bar{P}_1(\bar{x}, \bar{\tau}_a) \partial_x w t[\bar{u}] \langle V \rangle \right] \\
&\quad - \sum_{k=1}^3 \frac{2\pi}{k(k+2)} |\epsilon|^k P_x^{(k)} \bar{P}_{\bar{x}}^{(k)} \langle V \rangle + O(|\epsilon|^4). \tag{178}
\end{aligned}$$

where the operator $P_x^{(k)}$ is introduced to simplify the result:

$$P_x^{(k)} = P'_k(x, \tau_a) \partial_x - k P_{k+1}(x, \tau_a) w t[u], \tag{179}$$

for $x \in S_a$. $\log Q(x, \tau_a)$ in eq.(178) is the primitive function of $P_1(x, \tau_a)$ defined by eq.(80).

For one-point function of stress tensor, the first-order correction is shown as eq.(54). The first two terms in eq.(54) have been calculated in eq.(174) and eq.(176) respectively. The third term in eq.(54) can be further written by eq.(13) as

$$\begin{aligned}
& \frac{ic}{4Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z^2 \mathcal{P}_4 \bar{D}_{\bar{z}} Z \\
&= \frac{ic}{4Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \left\{ \sum_{b=a, \bar{a}} ({}^2\mathcal{P}_4 {}^2\bar{\mathcal{F}}_b) \frac{-1}{2\pi i} \partial_{\bar{\tau}_b} + ({}^2\mathcal{P}_4 {}^2\bar{\mathcal{F}}^{\text{II}} \bar{\epsilon}^{\frac{1}{2}}) \partial_{\bar{\epsilon}} \right\} Z, \tag{180}
\end{aligned}$$

which is obtained using eqs.(155)(156)(157), up to $|\epsilon|^3$:

$$\begin{aligned}
& \frac{ic}{4Z} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z^2 \mathcal{P}_4 \bar{D}_{\bar{z}} Z \\
&= -\frac{c}{6Z} \left\{ i|\epsilon| P_4(x, \tau_a) \partial_{\bar{\tau}_a} + i|\epsilon| \left[\epsilon^2 E_4(\tau_a) + \bar{\epsilon}^2 \bar{E}_4(\bar{\tau}_a) \right] P_4(x, \tau_a) \partial_{\bar{\tau}_a} \right. \\
&\quad \left. + 2\pi \bar{\epsilon} |\epsilon| \bar{E}_2(\bar{\tau}_a) P_4(x, \tau_a) \partial_{\bar{\epsilon}} \right\} Z + O(|\epsilon|^4). \tag{181}
\end{aligned}$$

C.4 Details of deformed two-point function

In this Appendix we discuss the detailed calculation of the first-order deformed two-point functions. For primary two-point function, the first-order correction depends on eq.(47). After equality the first lines have been calculated in eqs.(174)(176). In the second line, the integrals of $\mathcal{P}_{z,x_1} \bar{\mathcal{P}}_{\bar{z},\bar{x}_1}$ and $\mathcal{P}_{z,x_2} \bar{\mathcal{P}}_{\bar{z},\bar{x}_2}$ have been calculated in eq.(178). To calculate remaining integrals of the second line, there are two different

profiles. The one profile is that two insertion points live on the same torus $x_1, x_2 \in S_a$. Using eqs.(161)(162)(163), we approximate the result up to $|\epsilon|^3$:

$$\begin{aligned}
& \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \mathcal{P}_{z, x_1} \bar{\mathcal{P}}_{\bar{z}, \bar{x}_2} \langle V_1 V_2 \rangle \\
&= \left[\pi \log \frac{|Q(x_1, \tau_a) Q(x_2, \tau_a)|^2}{|Q(x_1 - x_2, \tau_a)|^2} + \frac{\pi}{2} \right] \partial_{x_1} \partial_{\bar{x}_2} \langle V_1 V_2 \rangle \\
&+ \left[\pi \bar{P}_1(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) + \pi \bar{P}_1(\bar{x}_2, \bar{\tau}_a) \right] \partial_{x_1} w t[\bar{u}_2] \langle V_1 V_2 \rangle \\
&+ \left[\pi P_1(x_2 - x_1, \tau_a) + \pi P_1(x_1, \tau_a) \right] w t[u_1] \partial_{\bar{x}_2} \langle V_1 V_2 \rangle \\
&- \sum_{k=1}^3 \frac{2\pi}{k(k+2)} |\epsilon|^k P_{x_1}^{(k)} \bar{P}_{\bar{x}_2}^{(k)} \langle V_1 V_2 \rangle + O(|\epsilon|^4). \tag{182}
\end{aligned}$$

The logarithmic divergence $\log |Q(x_1 - x_2, \tau_a)|^2$ in eq.(182) in the deformed two point functions has been observed in [20]. The other profile is that two insertion points live on different tori $x_1 = x \in S_a$ and $x_2 = y \in S_{\bar{a}}$, and their corresponding primary states are u and v , respectively. Using eqs.(167)(168)(169) we obtain the integral of $\mathcal{P}_{z, x} \bar{\mathcal{P}}_{\bar{z}, \bar{y}}$ up to $|\epsilon|^3$:

$$\begin{aligned}
& \frac{i}{2} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z \mathcal{P}_{z, x} \bar{\mathcal{P}}_{\bar{z}, \bar{y}} \langle V_x V_y \rangle \\
&= \sum_{k=1}^3 \frac{-\pi}{k+1} \left[\epsilon^k P'_{k+1}(y, \tau_{\bar{a}}) P_x^{(k-1)} \partial_{\bar{y}} + \bar{\epsilon}^k \bar{P}'_{k+1}(\bar{x}, \bar{\tau}_a) \partial_x \bar{P}_{\bar{y}}^{(k-1)} \right] \langle V_x V_y \rangle \\
&+ \frac{3\pi}{2} |\epsilon|^2 \left[\bar{\epsilon} \bar{E}_4(\bar{\tau}_a) P_x^{(2)} \partial_{\bar{y}} + \epsilon E_4(\tau_{\bar{a}}) \partial_x \bar{P}_{\bar{y}}^{(2)} \right] \langle V_x V_y \rangle \\
&- \frac{2\pi}{3} |\epsilon| \left[\bar{\epsilon}^2 \bar{E}_4(\bar{\tau}_a) + \epsilon^2 E_4(\tau_{\bar{a}}) \right] P_x^{(1)} \bar{P}_{\bar{y}}^{(1)} \langle V_x V_y \rangle + O(|\epsilon|^4). \tag{183}
\end{aligned}$$

For two-point function of stress tensor, there are two types to consider: $\langle T_1 T_2 \rangle$ and $\langle T_1 \bar{T}_2 \rangle$. The first-order correction of $\langle T_1 T_2 \rangle$ is shown as eq.(58), and all the integrals have been computed in eqs.(174)(176)(181) above. The first-order correction of $\langle T_1 \bar{T}_2 \rangle$ is shown as eq.(61), and all the integrals in eq.(61) have been computed in eqs.(174)(176)(182)(183)(181) except for the last three ones. In the case of two insertion points live on the same torus $x_1, x_2 \in S_a$, The integral of ${}^2\bar{\mathcal{P}}_4(\bar{z}, \bar{x}_2) \mathcal{P}_{z, x_1}$ is obtain⁸ using eqs.(164)(165), up to $|\epsilon|^3$:

$$\frac{i\epsilon}{4} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z {}^2\bar{\mathcal{P}}_4(\bar{z}, \bar{x}_2) \mathcal{P}_{z, x_1} \langle T_1 \rangle$$

⁸The integral of ${}^2\mathcal{P}_4(z, x_1) \bar{\mathcal{P}}_{\bar{z}, \bar{x}_2}$ is complex conjugate of eq.(184).

$$\begin{aligned}
&= \frac{\pi c}{6} \left[\bar{P}_3(\bar{x}_1 - \bar{x}_2, \bar{\tau}_a) + \bar{P}_3(\bar{x}_2, \bar{\tau}_a) \right] \partial_{x_1} \langle T_1 \rangle \\
&\quad + \frac{\pi c}{6} \sum_{k=1}^3 (k+1) |\epsilon|^k \bar{P}_{k+3}(\bar{x}_2, \bar{\tau}_a) P_{x_1}^{(k)} \langle T_1 \rangle + O(|\epsilon|^4). \tag{184}
\end{aligned}$$

The integral of ${}^2\mathcal{P}_4(z, x_1) {}^2\bar{\mathcal{P}}_4(\bar{z}, \bar{x}_2)$ is computed in eq.(166):

$$\begin{aligned}
&\frac{ic^2}{8} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z {}^2\mathcal{P}_4(z, x_1) {}^2\bar{\mathcal{P}}_4(\bar{z}, \bar{x}_2) \\
&= - \sum_{k=1}^3 \frac{k(k+1)^2(k+2)\pi c^2}{72} |\epsilon|^k P_{k+3}(x_1, \tau_a) \bar{P}_{k+3}(\bar{x}_2, \bar{\tau}_a) + O(|\epsilon|^4). \tag{185}
\end{aligned}$$

In the case of two insertion points live on different tori $x_1 = x \in S_a$ and $x_2 = y \in S_{\bar{a}}$, the integral of ${}^2\bar{\mathcal{P}}_4(\bar{z}, \bar{y}) \mathcal{P}_{z,x}$ is obtained using eqs.(170)(171):

$$\begin{aligned}
&\frac{ic}{4} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z {}^2\bar{\mathcal{P}}_4(\bar{z}, \bar{y}) \mathcal{P}_{z,x} \langle T_x \rangle \\
&= \left[\frac{\pi c}{6} \left(\bar{\epsilon}^2 \bar{P}_3(\bar{x}, \bar{\tau}_a) \bar{P}_4(\bar{y}, \bar{\tau}_{\bar{a}}) + \frac{\pi c}{2} \bar{\epsilon}^3 \bar{P}'_4(\bar{x}, \bar{\tau}_a) \bar{P}_5(\bar{y}, \bar{\tau}_{\bar{a}}) - 3\pi c \epsilon |\epsilon|^2 E_4(\tau_a) \bar{P}_5(\bar{y}, \bar{\tau}_{\bar{a}}) \right) \partial_x \langle T_x \rangle \right. \\
&\quad \left. + \frac{\pi c}{3} |\epsilon| \left[\epsilon^2 E_4(\tau_a) + \bar{\epsilon}^2 \bar{E}_4(\bar{\tau}_a) \right] \bar{P}_4(\bar{y}, \bar{\tau}_{\bar{a}}) P_x^{(1)} \langle T_x \rangle + O(|\epsilon|^4) \right]. \tag{186}
\end{aligned}$$

The integral of ${}^2\mathcal{P}_4(z, x) {}^2\bar{\mathcal{P}}_4(\bar{z}, \bar{y})$ is computed in eq.(172):

$$\begin{aligned}
&\frac{ic^2}{8} \int_{\Sigma^{(2)} \setminus D_\delta} d^2 z {}^2\mathcal{P}_4(z, x) {}^2\bar{\mathcal{P}}_4(\bar{z}, \bar{y}) \\
&= - \frac{\pi c^2}{6} |\epsilon| \left[\epsilon^2 E_4(\tau_a) + \bar{\epsilon}^2 \bar{E}_4(\bar{\tau}_a) \right] P_4(x, \tau_a) \bar{P}_4(\bar{y}, \bar{\tau}_{\bar{a}}) + O(|\epsilon|^4). \tag{187}
\end{aligned}$$

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