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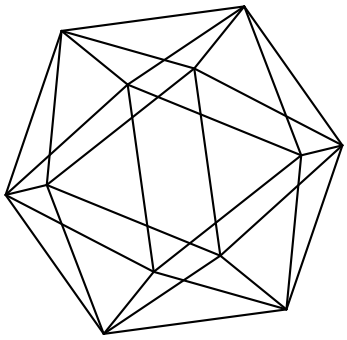

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NEW MODULI COMPONENTS OF RANK 2 BUNDLES ON PROJECTIVE SPACE

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Abstract. We present a new family of monads whose cohomology is a stable rank two vector bundle on $\mathbb{P}^3$. We also study the irreducibility and smoothness together with a geometrical description of some of these families. Such facts are used to prove that the moduli space of stable rank two vector bundles with trivial determinant and second Chern class equal to 5 has exactly three irreducible components.

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1. Introduction

In [24] Maruyama proved that the rank r stable vector bundles on a projective variety X with fixed Chern classes $c_1, ..., c_r$ can be parametrized by an algebraic quasi-projective variety, denoted by $\mathcal{B}_X(r, c_1, ..., c_r)$. Although this result has been known for almost 40 years, there are just a few concrete examples and established facts about such varieties, even for cases like $X = \mathbb{P}^3$ and $r = 2$. For instance, $\mathcal{B}_{\mathbb{P}^3}(2, 0, 1)$ was studied by Barth in [2], $\mathcal{B}_{\mathbb{P}^3}(2, 0, 2)$ was described by Harthorne in [12], $\mathcal{B}_{\mathbb{P}^3}(2, -1, 2)$ studied by Harthorne and Sols in [15] and by Manolache in [23], while $\mathcal{B}_{\mathbb{P}^3}(2, -1, 4)$ was described by Banica and Manolache in [1]. This probably happened due to the fact that the questions of irreducibility (solved in [27] and [28]), and smoothness (answered in [20]) of the so-called *instanton component* of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, c_2)$ remained opened until 2014.

In this paper, we continue the study of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, n)$, which we will simply denote by $\mathcal{B}(n)$ from now on, with the goal of providing new examples of families of vector bundles, and understanding their geometry. It is more or less clear from the table in [14, Section 5.3] that $\mathcal{B}(1)$ and $\mathcal{B}(2)$ should be irreducible, while $\mathcal{B}(3)$ and $\mathcal{B}(4)$ should have exactly two irreducible components; see [11] and [7], respectively, for the proof of the statements about $\mathcal{B}(3)$ and $\mathcal{B}(4)$. For $n \geq 5$, two families of irreducible components have been studied, namely the *instanton components*, whose generic point corresponds to an instanton bundle, and the *Ein components*, whose generic point corresponds to a bundle

given as cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0$$

where $b \geq a \geq 0$ and $c > a + b$. All of the components of $\mathcal{B}(n)$ for $n \leq 4$ are of either of these types; here we focus on a new family of bundles that appear as soon as $n \geq 5$.

More precisely, we study the family of vector bundles in $\mathcal{B}(a^2 + k)$ for each $a \geq 2$ and $k \geq 1$ which arise as cohomologies of monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to (4 + 2k) \cdot \mathcal{O}_{\mathbb{P}^3} \to k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

which will be denoted by $\mathcal{G}(a, k)$. We provide a bijection between such monads and monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

where $\tilde{E}$ is a rank 4 instanton bundle of charge k. When $k = 1$ these facts, are used to prove our first main result.

Main Theorem 1. *For each $a \geq 2$ not equal to 3, $\mathcal{G}(a, 1)$ is a nonsingular open subset of an irreducible component of $\mathcal{B}(a^2 + 1)$ of dimension*

$$4 \cdot \binom{a+3}{3} - a - 1.$$

Our second main result provides a complete description of all the irreducible components of $\mathcal{B}(5)$.

Main Theorem 2. *The moduli space $\mathcal{B}(5)$ has exactly 3 irreducible components, namely:*

(i) *the* instanton component*, of dimension 37, which consists of those bundles given as cohomology of monads of the form*

$$0 \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to 12 \cdot \mathcal{O}_{\mathbb{P}^3} \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \text{ and} \tag{1}$$

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \to 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 3 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(2) \to 0; \tag{2}$$

(ii) *the* Ein component*, of dimension 40, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \tag{3}$$

iii) *the closure of the family* $\mathcal{G}(2,1)$*, of dimension* 37*, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \text{ and} \tag{4}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 6 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{5}$$

Indeed, Hartshorne and Rao proved in [14] that every stable rank 2 bundle on $\mathbb{P}^3$ with Chern classes $c_1(E) = 0$ and $c_1(E) = 5$ is the cohomology of one of the monads listed above. Rao showed in [26] that bundles given as cohomology of monads of the form (2) lie in the closure of the family of instanton bundles of charge 5, which was shown to be irreducible firstly by Coanda, Tikhomirov and Trautmann in [8]; see also [27]. The irreducibility of the family of bundles which arise as cohomology of monads of the form (3) was established by Ein in [10].

Finally, our first main result yields the third component, and we also show that the family of bundles given by the monads of the form (5) lies in the closure of the family $\mathcal{G}(2,1)$.

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Notation and Conventions. In this work, $\mathbb{K}$ is an algebraically closed field of zero characteristic; $\mathbb{P}^3 := \text{Proj}(\mathbb{K}[x,y,z,w])$. We will not make any distinction between vector bundles and locally free sheaves, and $H^i(F)$ will denote the i-th cohomology group of the sheaf F on $\mathbb{P}^3$. Since we are working with rank 2 vector bundles on $\mathbb{P}^3$, and Gieseker stability is equivalent to $\mu-$stability, we will not make any distinction between these two concepts. If V is a vector space over $\mathbb{K}$, we will denote by $G(k,V)$ the grassmannian variety of k-dimensional subspaces of V.

2. Monads

Recall that a monad is a complex of vector bundles of the form:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{6}$$

such that α is injective, and β is surjective. We call the sheaf $E := \ker\beta/\operatorname{im}\alpha$ the cohomology of the monad (6). When α is locally left invertible, then E is a vector bundle.

The notion of monad is important in the study of vector bundles on $\mathbb{P}^3$ because Horrocks proved in [16] that every vector bundle on $\mathbb{P}^3$ is cohomology of a monad of the form (6) with A, B and C being sums of line bundles.

For completeness, we include in this section some useful results about monads that will be required in this work. The following lemma gives a relation between isomorphism classes of monads and its cohomology vector bundles; a proof can be found in [25, Lemma 4.1.3].

Lemma 1. *Let E and E' be, respectively, cohomology of the following monads:*

$$M: \qquad A \xrightarrow{a} B \xrightarrow{b} C \tag{7}$$

$$M': \qquad A' \xrightarrow{a'} B' \xrightarrow{b'} C' \tag{8}$$

If one has that $\operatorname{Hom}(B,A') = \operatorname{Hom}(C,B') = \mathrm{H}^1(X, C^\vee\otimes A') = \mathrm{H}^1(X, B^\vee\otimes A') = \mathrm{H}^1(X, C^\vee\otimes B') = \mathrm{H}^2(X, C^\vee\otimes A') = 0$ *then there exists a bijection between the set of all morphisms from E to E' and the set of all morphisms of monads from (7) to (8).*

The following important corollary will be used several times in what follows, and a proof can also be found in [25, Lemma 4.1.3, Corollary 2].

Corollary 2. *Consider the monad*

$$M: \qquad A \xrightarrow{a} B \xrightarrow{b} C$$

and its dual monad:

$$M^\vee: \qquad C^\vee \xrightarrow{b^\vee} B^\vee \xrightarrow{a^\vee} A^\vee.$$

If these monads satisfy the hypothesis of Lemma 1, and there exists an isomorphism $f : E \to E^\vee$ between its cohomology bundles such that $f^\vee = -f$, then there are isomorphisms $h : C \to A^\vee$, and $q : B \to B^\vee$, such that $q^\vee = -q$, and $h\circ b = a^\vee\circ q$.

Recall that every locally free sheaf E on $\mathbb{P}^3$ is the cohomology of a monad of the form [16]:

$$0 \to \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(a_i) \to \oplus_{j=1}^{s} \mathcal{O}_{\mathbb{P}^3}(b_j) \to \oplus_{k=1}^{t} \mathcal{O}_{\mathbb{P}^3}(c_k) \to 0 \tag{9}$$

In this work we will be interested in rank 2 locally free sheaves with vanishing first Chern class. Under these conditions, we have $E^\vee \simeq E$, thus the monad (9) is self dual, which implies that $t = r$, $s = 2r + 2$, and $\{a_i\} = \{-c_k\}$. In addition, the middle entry of the monad is also self dual, so that (9) reduces to

$$0 \to \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(a_i) \to \oplus_{j=1}^{r+1} \left(\mathcal{O}_{\mathbb{P}^3}(b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_j)\right) \to \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(-a_i) \to 0.$$

Finally, recall also that r coincides with the number of generators of $\mathrm{H}^1_*(E) = \bigoplus_{p\in\mathbb{Z}} \mathrm{H}^1(E(p))$ as a graded module over the ring of homogeneous polynomials in four variables, while a_i are the degrees of these generators, cf. [19, Theorem 2.3].

3. Symplectic instanton bundles

Instanton bundles are a particularly important class of stable rank 2 vector bundles due to their many remarkable properties and applications in mathematical physics. Besides this, instanton bundles form the only known irreducible component of the moduli space $\mathcal{B}(c)$ for every $c \in \mathbb{N}$.

We will now present the main results concerning instanton sheaves that will be used below. We start by recalling the definition of instanton sheaves on $\mathbb{P}^3$, cf. [17, Introduction] for further information on these objects.

Definition 3. *An* instanton sheaf *on $\mathbb{P}^3$ is a torsion free coherent sheaf E with $c_1(E) = 0$ satisfying the following cohomological conditions:*

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

The integer $n := c_2(E)$ is called the charge *of E. When E is locally free, we say that E is an* instanton bundle.

Recall also that a locally free sheaf E is *symplectic* if it admits a symplectic structure, that is, there exists an isomorphism $\varphi : E \to E^\vee$, such that $\varphi^\vee = -\varphi$. A *symplectic instanton bundle* is a pair (E, φ) consisting of an instanton bundle E together with a symplectic structure φ on it; two symplectic instanton bundles (E, φ) and (E', φ') are isomorphic if there exists a bundle isomorphism $g : E \xrightarrow{\sim} E'$ such that $\varphi = g^\vee \circ \varphi' \circ g$.

The cokernel N of any monomorphism of sheaves $\mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega^1_{\mathbb{P}^3}(1)$ is called a *null correlation sheaf*. Such sheaves are precisely the rank 2 instanton sheaves of charge 1, and are parametrized by the projective space $\mathbb{P}H^0(\Omega^1_{\mathbb{P}^3}(2)) \simeq \mathbb{P}^5$. If N is locally free, we say that N is a *null correlation bundle*.

For the purposes of this paper, it is important to study rank 4 instanton bundles of charge 1. Some of the following facts might be well known, but for lack of a reference we include proofs here.

Lemma 4. *(i) Every rank 4 instanton bundle E of charge 1 over $\mathbb{P}^3$ fits into an exact sequence:*

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\mu} E \xrightarrow{\nu} N \to 0 \tag{10}$$

where N is a null correlation sheaf fitting into an exact triple:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s} \Omega^1_{\mathbb{P}^3}(1) \to N \to 0. \tag{11}$$

(ii) In addition, if N is locally free, then it is a null correlation bundle, and the triple (10) splits: $E \simeq N \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}$. Respectively, if N is not locally free, then it fits into an exact triple

$$0 \to N \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_l(1) \to 0, \tag{12}$$

where l is some projective line in $\mathbb{P}^3$.
(iii) There are exact triples induced by (10) and (11):

$$0 \to 3 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{S^2\mu} S^2E \to \operatorname{coker}(S^2\mu) \to 0, \qquad 0 \to 2 \cdot N \to \operatorname{coker}(S^2\mu) \to S^2N \to 0, \tag{13}$$

$$0 \to \wedge^2\Omega^1_{\mathbb{P}^3}(2) \xrightarrow{\zeta} \Omega^1_{\mathbb{P}^3}(1) \otimes N \xrightarrow{\eta} S^2N \to 0. \tag{14}$$

Proof. From Definition 3 and Beilinson spectral sequence [25, Ch. II] it follows that E is the cohomology sheaf of a monad of the type

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} 6 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{15}$$

Since $\ker(\mathrm{H}^0(\beta) : \mathbb{K}^6 \twoheadrightarrow \mathbb{K}^4) = \mathbb{K}^2$ and $\ker(4 \cdot \mathcal{O}_{\mathbb{P}^3} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(1)) = \Omega^1_{\mathbb{P}^3}(1)$, (15) yields an exact triple of bundles

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\lambda} \ker\beta \xrightarrow{\theta} \Omega^1_{\mathbb{P}^3}(1) \to 0 \tag{16}$$

which together with the triple $0 \to 2\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{i} \ker\beta \xrightarrow{\theta} E \to 0$ also coming from the monad (15) leads to the exact triples (10) and (11) in which $\mu = \delta \circ \lambda$, $N = \operatorname{coker}\mu$ and $s = \theta \circ i$.

Next, the morphism s in (14) as a section $s \in \mathrm{H}^0(\Omega^1_{\mathbb{P}^3}(2)) \simeq \wedge^2 V^\vee$, where $V := (\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1)))^\vee$, can be considered as a skew-symmetric homomorphism $^\sharp s : V \to V^\vee$, thus leading to a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\quad s \quad} & \Omega^1_{\mathbb{P}^3}(1) \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ V \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\ ^\sharp s\ } & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}, \end{array}$$

where j_1 and j_2 are natural maps. Extending the vertical morphisms in this diagram to the corresponding Euler exact sequences, we see that the whole diagram then extends to a commutative diagram with these Euler exact sequences as columns. In case when $^\sharp s$ is an isomorphism the sheaf N is locally free; respectively, in case when $^\sharp s$ has rank 2, the leftmost column of the extended diagram is just the triple (12).

Next, remark that the exactness of the two triples (13), in case when N is locally free, follows from (10) by standard linear algebra. Respectively, in case when N is locally free, (13) again follows from (10) and (16) by linear algebra and standard diagram chasing. (Here we take into account that N is torsion free.)

As for the triple (14), the morphisms ζ and η in it are well defined on $\mathbb{P}^3$. The exactness of (14) on $\mathbb{P}^3 \setminus l$ again follows from (11) by linear algebra. Besides, standard diagram chasing shows that ζ is injective and η is surjective as morphisms of $\mathcal{O}_{\mathbb{P}^3}$-sheaves, and that $\eta \circ \zeta = 0$. Whence, the sheaf $F = \frac{\ker\eta}{\operatorname{im}\zeta}$ is either 0 or supported in the line l. In the last case we have an extension of $\mathcal{O}_{\mathbb{P}^3}$-sheaves $0 \to \wedge^2\Omega^1_{\mathbb{P}^3}(2) \xrightarrow{\zeta} \ker\eta \to F \to 0$. Now one easily checks that, since $\wedge^2\Omega^1_{\mathbb{P}^3}(2)$ is a locally free sheaf and $\operatorname{codim}\operatorname{Supp}F = 2$, the last triple splits, so that a torsion free $\mathcal{O}_{\mathbb{P}^3}$-sheaf $\Omega^1_{\mathbb{P}^3}(1) \otimes N$ has a torsion subsheaf F, a contradiction. Hence, $F = 0$ and the triple (14) is exact. □

Corollary 5. *In the conditions of Lemma 4,* $h^0(S^2E) = 3,\ h^1(S^2E) = 5,\ h^2(S^2E) = 0.$

Proof. Tensoring the triple (12) with $\Omega^1_{\mathbb{P}^3}(1)$ we obtain an exact triple $0 \to \Omega^1_{\mathbb{P}^3}(1) \otimes N \to 2 \cdot \Omega^1_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_l(1) \oplus \mathcal{O}_l \to 0$, whence $h^0(\Omega^1_{\mathbb{P}^3}(1) \otimes N) = h^2(\Omega^1_{\mathbb{P}^3}(1) \otimes N) = 0,\ h^1(\Omega^1_{\mathbb{P}^3}(1) \otimes N) = 5$. This together with the isomorphism $\wedge^2\Omega^1_{\mathbb{P}^3}(2) \simeq T_{\mathbb{P}^3}(-2)$, the equalities $h^i(T_{\mathbb{P}^3}(-2)) = 0,\ i \geq 0$, and the triple (14) yields $h^0(S^2N) = h^2(S^2N) =$

$0,\ h^1(S^2N) = 5$. On the other hand, from (11) we obtain $h^i(N) = 0,\ i \geq 0$. The last equalities together with both triples (13) lead to the assertion of Corollary. □

Fix a projective line $l \subset \mathbb{P}^3$, and let N_l denote the non locally free null correlation sheaf associted with l, as given in sequence (12). Note that

$$\dim \operatorname{Ext}^1(N_l, 2 \cdot \mathcal{O}_{\mathbb{P}^3}) = 2 \cdot h^2(N_l(-4)) = 2 \cdot h^1(\mathcal{O}_l(-3)) = 4,$$

so we must understand how many locally free extensions of N_l by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$ do exist.

Lemma 6. *For each line $l \subset \mathbb{P}^3$, the corresponding non locally free null correlation sheaf N_l admits a unique, up to isomorphism, locally free extension by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$.*

Proof. Without loss of generality, choose homogeneous coordinates $[x : y : z : w]$ in $\mathbb{P}^3$, and assume that l is the projective line given by $\{x = y = 0\}$, so that N_l is the cohomology of the following monad:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_l} 4 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{17}$$

where

$$\beta := \begin{pmatrix} x & y & z & w \end{pmatrix} \text{ and } \alpha_l := \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix}.$$

An extension of N_l by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$ will be the cohomology of a monad given as an extension of the monad (17) by the monad $0 \to 0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to 0 \to 0$; such extension is of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} 6 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{18}$$

where

$$\tilde{\beta} := \begin{pmatrix} x & y & z & w & 0 & 0 \end{pmatrix} \text{ and } \tilde{\alpha} := \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ \sigma_1 \\ \sigma_2 \end{pmatrix},$$

with $\sigma_1, \sigma_2 \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$; this pair can be written, as a column vector, in the following form

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = v_1 \cdot x + v_2 \cdot y + v_3 \cdot z + v_4 \cdot w,$$

where $v_j \in \mathbb{K}^2$, for $j = 1, \ldots, 4$.

Let $\tilde{A}$ be the 6×6 matrix

$$\tilde{A} := \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & A \end{pmatrix},$$

where $\mathbf{1}_4$ is the 4×4 identity matrix, and A is an invertible 2×2 matrix. Note that $\tilde{\beta}\tilde{A}^{-1} = \tilde{\beta}$, while

$$\tilde{A}\tilde{\alpha} = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ \sigma_1' \\ \sigma_2' \end{pmatrix},$$

where

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = Av_1 \cdot x + Av_2 \cdot y + Av_3 \cdot z + Av_4 \cdot w.$$

We claim that the cohomology of (18) is locally free if and only if the vectors v_3 and v_4 are linearly independent. Indeed, if v_3 and v_4 are linearly independent, then there exists a 2×2 matrix A such that

$$Av_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } Av_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that we have an isomorphism of monads

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{\alpha}} & 6 \cdot \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\tilde{\beta}} & \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\ & & \| & & \downarrow \tilde{A} & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{\alpha}'} & 6 \cdot \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\tilde{\beta}} & \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0, \end{array} \tag{19}$$

where

$$\tilde{\alpha}' = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ \sigma_1'' + z \\ \sigma_2'' + w \end{pmatrix},$$

with σ_1'' and σ_2'' depending only on x, y. It is then easy to see that the cohomology of the monad in the lower line of diagram (19) is locally free.

For the converse statement, there are 3 cases to be considered.

(1) If either $v_3 = \lambda \cdot v_4$ with $v_4 \neq 0$, then choose the 2×2 matrix A such that $Av_4 = (1, 0)$; the morphism $\tilde{\alpha}'$ in the lower line of diagram (19) is then given by

$$\tilde{\alpha}' = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ \sigma_1'' + \lambda z + w \\ \sigma_2'' \end{pmatrix},$$

with σ_1'' and σ_2'' depending only on x, y, hence the cohomology of this monad is reflexive, with a singularity at the point $[0 : 0 : 1 : -\lambda]$.

(2) If $v_3 \neq 0$ and $v_4 = 0$, then choose the 2×2 matrix A such that $Av_3 = (1, 0)$; the morphism $\tilde{\alpha}'$ in the lower line of diagram (19) is then given by

$$\tilde{\alpha}' = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ \sigma_1'' + z \\ \sigma_2'' \end{pmatrix},$$

with σ_1'' and σ_2'' depending only on x, y, hence the cohomology of this monad is reflexive, with a singularity at the point $[0 : 0 : 0 : 1]$.

(3) Finally, if $v_3 = v_4 = 0$, then choose the 2×2 matrix A such that $Av_3 = (1, 0)$; the morphism $\tilde{\alpha}'$ in the lower line of diagram (19) is then given by

$$\tilde{\alpha}' = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ \sigma_1'' \\ \sigma_2'' \end{pmatrix},$$

with σ_1'' and σ_2'' depending only on x, y, hence the cohomology of this monad is torsion free with a singularity on the line l (in fact, it is isomorphic to $N_l \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}$).

To prove that the uniqueness of the locally free extension, simply note that a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} 6 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

with morphisms given by

$$\beta := \begin{pmatrix} x & y & z & w & 0 & 0 \end{pmatrix} \text{ and } \alpha := \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ ax + by + z \\ cx + dy + w \end{pmatrix},$$

is isomorphic to a monad with α of the form

$$\alpha := \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ z \\ w \end{pmatrix},$$

and β given as above. □

We conclude that the moduli space of rank 4 instanton bundles of charge 1, denoted $\mathcal{I}(4, 1)$, is isomorphic to $\mathbb{P}^5$. Indeed, let $G \subset \mathbb{P}^5$ denote the Grassmanian of lines in $\mathbb{P}^3$; the points in the complement $U := \mathbb{P}^5 \setminus G$ correspond to the *split instantons*, of the form $N \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}$; the points $l \in G$ correspond to the unique extention of N_l by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$.

Lemma 7. *If E is a rank 4 instanton bundle of charge 1, then $h^0(\mathrm{End}(E)) = 5$ and $\mathrm{Aut}(E) \simeq \mathbb{K}^* \times \mathrm{GL}(2)$.*

Proof. Twist sequence (10) by $E^\vee$ and pass to cohomology to obtain

$$0 \to H^0(E^\vee)^{\oplus 2} \to H^0(\mathrm{End}(E)) \to H^0(N \otimes E^\vee) \to 0, \tag{20}$$

since $H^1(E^\vee) = 0$ as $E^\vee$ is also a rank 4 instanton bundle of charge 1.

Next, twisting (10) by N and passing to cohomology yields $h^0(N \otimes E^\vee) = h^0(N \otimes N) = 1$, because N is simple and $h^0(N) = h^1(N) = 0$. Since $h^0(E^\vee) = 2$, sequence (20) implies that

$$h^0(\mathrm{End}(E)) = h^0(N \otimes E^\vee) + 2 \cdot h^0(E^\vee) = 5.$$

Given an isomorphism $g : E \xrightarrow{\sim} E$, its composition with the monomorphism $\rho : 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to E$ must factor through $2 \cdot \mathcal{O}_{\mathbb{P}^3}$, since $h^0(N) = 0$. Thus there exists $M \in \mathrm{GL}(2)$ such that the left square of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 2 \cdot \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\rho} & E & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle g} & & \vdots & & \\ 0 & \longrightarrow & 2 \cdot \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\rho} & E & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

commutes. The isomorphism $N \xrightarrow{\sim} N$ obtained by completing the previous diagram must be a multiple of the identity, since N is simple. Therefore, we have constructed a map from $\mathrm{Aut}(E)$ to $\mathbb{K}^* \times \mathrm{GL}(2)$.

Conversely, given $(\lambda, M) \in \mathbb{K}^* \times \mathrm{GL}(2)$, we obtain an isomorphism $g \in \mathrm{Aut}(E)$ just by completing the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 2 \cdot \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\rho} & E & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M} & & \vdots & & \downarrow{\scriptstyle \lambda \cdot \mathbf{1}_N} & & \\ 0 & \longrightarrow & 2 \cdot \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\rho} & E & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

□

In particular, since $\mathrm{End}(E) = \Lambda^2 E \oplus S^2 E$, it follows from Corollary 5 and Lemma 7 that $h^0(\Lambda^2 E) = 2$.

In fact, one can show that every rank 4 instanton bundle of charge 1 is naturally isomorphic to its dual. Indeed, this is clear for the split instantons $E = N \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}$.

Otherwise, let E be given by (10) with a non locally free null correlation sheaf N_l; dualizing (10), we obtain:

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to E^\vee \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{E}xt^1(N_l, \mathcal{O}_{\mathbb{P}^3}) \to 0.$$

Since, by sequence (12), $\mathcal{E}xt^1(N_l, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{O}_l(1)$, the kernel of the last morphism coincides with N_l itself, and the first part of the previous exact sequence yields:

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to E^\vee \to N_l \to 0.$$

implying, by Lemma 6, that $E^\vee \simeq E$.

Lemma 8. *Every rank 4 instanton bundle E of charge 1 admits a unique symplectic structure, up to isomorphism.*

Proof. If $E = N \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}$, the claim is clear: a symplectic form on E is the sum of the (unique) symplectic structure on N plus a symplectic structure on $2 \cdot \mathcal{O}_{\mathbb{P}^3}$, and the latter can be transformed into the standard symplectic form, given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If E is non split, we can assume, following the proof of Lemma 6, that E is the cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} 6 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

with morphisms given by

$$\beta := \begin{pmatrix} x & y & z & w & 0 & 0 \end{pmatrix} \text{ and } \alpha := \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ z \\ w \end{pmatrix}.$$

One can check that the following skew-symmetric matrix

$$J := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

induces a symplectic structure on E which is unique up to change of bases. □

The exact sequence (10) yields an exact sequence in cohomology for every $a \geq 1$:

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^3}(a))^{\oplus 2} \to H^0(E(a)) \to H^0(N(a)) \to 0.$$

It follows that

$$H^0(E(a)) \simeq H^0(N(a)) \oplus H^0(\mathcal{O}_{\mathbb{P}^3}(a))^{\oplus 2}, \tag{21}$$

so every section $\sigma \in H^0(E(a))$ can be represented as a triple $(\sigma_N, \sigma_1, \sigma_2)$ with $\sigma_N \in H^0(N(a))$ and $\sigma_1, \sigma_2 \in H^0(\mathcal{O}_{\mathbb{P}^3}(a))$. In this representation, the action of $\mathrm{Aut}(E)$ on $H^0(E(a))$ is given by

$$(\lambda, M) \cdot (\sigma_N, \sigma_1, \sigma_2) = (\lambda \cdot \sigma_N, \sigma_1', \sigma_2'), \text{ where } \begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = M \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}. \tag{22}$$

4. Modified instanton monads

We will now study monads of the following form, with $a \geq 2$ and $k \geq 1$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{23}$$

which we call *modified instanton monads*. The family of isomorphism classes of bundles arising as cohomology of such monads will be denoted by $\mathcal{G}(a, k)$. Note that, by now, $\mathcal{G}(a, k)$ could possibly be empty.

Proposition 9. *For each $a \geq 2$ and $k \geq 1$, the family $\mathcal{G}(a, k)$ is non-empty and contains stable bundles, while every $E \in \mathcal{G}(a, k)$ is μ-semistable. In addition, every $E \in \mathcal{G}(a, 1)$ is stable.*

Proof. Let F be an rank 2 instanton bundle of charge k. Let $a \geq 2$ and take $\sigma \in \mathrm{H}^0(F(2a))$ and $(\sigma)_0 = X$ (such σ always exists if F is a 't Hooft instanton bundle, for instance). Let Y be a complete intersection of two surfaces of degree a and $X \cap Y = \emptyset$. According to [14, Lemma 4.8], there exists a bundle E and a section $\tau \in \mathrm{H}^0(E(a))$ such that $(\tau)_0 = Y \cup X$ which is given as cohomology of a monad of the form (23). In addition, since F is stable, X is not contained in any surface of degree a, hence neither is $Y \cup X$, and E is also stable.

It is straightforward to check that every $E \in \mathcal{G}(a,k)$ satisfies $h^0(E(-1)) = 0$, thus E is μ-semistable.

Now fix $k = 1$, and assume that there is $E \in \mathcal{G}(a,1)$ satisfying $h^0(E) \neq 0$. Setting $K := \ker \beta$, it follows that $h^0(K) \neq 0$, hence the quotient $K' := K/\mathcal{O}_{\mathbb{P}^3}$ fits into the following exact sequence

$$0 \to K' \to 5 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta'} \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0.$$

By [5, Theorem 2.7] K' is μ-stable. However, the monomorphism $\alpha : \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to K$ induces a monomorphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to K'$; by the μ-stability of K', we should have

$$-1 < \mu(K') = -\frac{a+1}{3} \implies a < 2,$$

providing the desired contradiction. □

Next, we provide a cohomological characterization for modified instanton bundles.

Proposition 10. *A vector bundle E on $\mathbb{P}^3$ is the cohomology of a monad of the form (23) if and only if $\mathrm{H}^1_*(E)$ has one generator in degree $-a$ and k generators in degree -1, and its Chern classes are $c_1(E) = 0$, and $c_2(E) = a^2 + k$.*

Proof. The "only if" part is straightforward. If E is a self dual vector bundle on $\mathbb{P}^3$ with one generator in degree $-a$ and k generators in degree -1, then by [19, Theorem 2.3], E is cohomology of a monad of the type:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \oplus_{i=1}^{2k+4} \mathcal{O}_{\mathbb{P}^3}(k_i) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Computing the Chern class give us $c_2(E) = a^2 + k - \sum_{i=1}^{6} k_i^2$, since $c_2(E) = a^2 + k$, we have $k_i = 0$ for all i. □

The modified instanton bundles are also related to usual instanton bundles of higher rank in a very important way. The precise relationship is outlined in the next couple of lemmas, and then summarized in Proposition 14 below.

Lemma 11. *Given a vector bundle $E \in \mathcal{G}(a,k)$, there exists a rank 4 instanton bundle $\tilde{E}$ of charge k, and sections $\sigma \in H^0(\tilde{E}(a))$, $\tau \in H^0(\tilde{E}^\vee(a))$ such that the complex:*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0 \tag{24}$$

is a monad whose cohomology is isomorphic to E.

Proof. Define $\tilde{\alpha} = \alpha \circ i$ and $\tilde{\beta} = \pi \circ \beta$ where $i : k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1)$ is the inclusion and $\pi : \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to k \cdot \mathcal{O}_{\mathbb{P}^3}(-1)$ is the projection. It is clear that $\tilde{\alpha}$ is injective and $\tilde{\beta}$ is surjective. We then get the following monad, whose cohomology is a rank 4 instanton $\tilde{E}$ of charge k:

$$0 \to k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{25}$$

Now we need to construct the morphisms σ and τ. It is straightforward to check that the chain of inclusions: $\operatorname{im}\tilde{\alpha} \subseteq \operatorname{im}\alpha \subseteq \ker\beta \subseteq \ker\tilde{\beta}$ holds. For this reason, we have:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{O}_{\mathbb{P}^3}(a) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \ker\beta & \longrightarrow & (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\tilde{\beta} & \xrightarrow{i} & (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} & \longrightarrow & k\cdot\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

where f_1 is the inclusion. It follows that $\operatorname{coker} f_1 \simeq \mathcal{O}_{\mathbb{P}^3}(a)$. In addition, we also obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{im}\tilde{\alpha} & \xrightarrow{i} & \ker\beta & \longrightarrow & \ker\beta/\operatorname{im}\tilde{\alpha} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \omega} & & \\
0 & \longrightarrow & \operatorname{im}\tilde{\alpha} & \xrightarrow{i} & \ker\tilde{\beta} & \longrightarrow & \tilde{E} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{O}_{\mathbb{P}_3}(a) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0, & & & &
\end{array}$$

where ω is the inclusion. Thus $\operatorname{coker}\omega \simeq \mathcal{O}_{\mathbb{P}^3}(a)$, and we obtain an epimorphism $\tau : \tilde{E} \to \mathcal{O}_{\mathbb{P}^3}(a)$.

Now, by the isomorphism theorem we have $\dfrac{\ker\beta}{\operatorname{im}\alpha} \simeq \dfrac{\frac{\ker\beta}{\operatorname{im}\tilde{\alpha}}}{\frac{\operatorname{im}\alpha}{\operatorname{im}\tilde{\alpha}}}$, so there exists an epimorphism $f_2 : \ker\beta/\operatorname{im}\tilde{\alpha} \to E$ fitting into the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & k\cdot\mathcal{O}_{\mathbb{P}_3}(-1) & \xrightarrow{\tilde{\alpha}} & \ker\beta & \longrightarrow & \ker\beta/\operatorname{im}\tilde{\alpha} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow f_2 & & \\
0 & \longrightarrow & k\cdot\mathcal{O}_{\mathbb{P}_3}(-1)\oplus\mathcal{O}_{\mathbb{P}_3}(-a) & \longrightarrow & \ker\beta & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{O}_{\mathbb{P}_3}(-a) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

It follows that $\ker f_2 \simeq \mathcal{O}_{\mathbb{P}_3}(-a)$, so there exists a monomorphism $\sigma' : \mathcal{O}_{\mathbb{P}^3}(-a) \to \ker\beta/\operatorname{im}\tilde{\alpha}$. Composing it with ω, we obtain a monomorphism $\sigma := \omega\circ\sigma' : \mathcal{O}_{\mathbb{P}^3}(-a) \to \tilde{E}$. An epimorphism τ is constructed in a similar way. We have therefore constructed the monad:

$$0 \to \mathcal{O}_{\mathbb{P}_3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}_3}(a) \to 0,$$

whose cohomology is precisely the bundle E. □

Lemma 12. *If a bundle E is the cohomology of a monad of the form (24), then E is also isomorphic to the cohomology of a monad of the form (23), i.e. $E \in \mathcal{G}(a,k)$.*

Proof. Let $\tilde{E}$ be an rank 4 instanton bundle of charge k over $\mathbb{P}^3$, so that $\tilde{E}$ is cohomology of a monad of the type:

$$0 \to k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} k\cdot\mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Take $\tau,\sigma \in \mathrm{H}^0(\tilde{E}(a))$ satisfying $\tau\circ\sigma = 0$. We thus have the following exacts sequences:

$$0 \to \ker\tilde{\beta} \to (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} k\cdot\mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

$$0 \to k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} \ker\tilde{\beta} \to \tilde{E} \to 0,$$

$$0 \to \ker\tau \to \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \ker\tau \to E \to 0,$$

where $E \simeq \ker\tau/\operatorname{im}\sigma$.

First, define a morphism $f_2 : \ker\tau \oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) \to \ker\tilde{\beta}$ as follows: given x and y local sections of $\ker\tau$ and $k\cdot\mathcal{O}_{\mathbb{P}^3}(-1)$, repectively, we set $f_2(x,y) := x + \tilde{\alpha}(y)$, where

x in the right hand side of the equality is regarded as a local section of $\tilde{E} \simeq \ker\tilde{\beta}/\operatorname{im}\tilde{\alpha}$. We thus obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & \ker\tau\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & \ker\tau & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle f_2} & & \downarrow & & \\
0 & \longrightarrow & k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{\alpha}} & \ker\tilde{\beta} & \longrightarrow & \tilde{E} & \longrightarrow & 0,
\end{array}$$

from which we obtain the exact sequence:

$$0 \to \ker\tau\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{f_2} \ker\tilde{\beta} \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0.$$

We can then compose f_2 with the inclusion $\ker\tilde{\beta} \subseteq (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3}$, obtaining a monomorphism $\tilde{f}_2$ fitting into the diagram below:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker\tau\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & = & \ker\tau\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & & & & \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \tilde{f}_2} & & & & \\
0 & \longrightarrow & \ker\tilde{\beta} & \xrightarrow{i} & (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\tilde{\beta}} & k\cdot\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & \operatorname{coker}\tilde{f}_2 & \longrightarrow & k\cdot\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with the third line obtained via the Snake Lemma; it follows that $\operatorname{coker}\tilde{f}_2 \simeq \mathcal{O}_{\mathbb{P}^3}(a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(1)$.

Let $\beta : (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(1)$ denote the natural quotient morphism. Making

$$\alpha := \tilde{f}_2 \circ (\sigma, \mathbf{1}_{k\cdot\mathcal{O}_{\mathbb{P}^3}(-1)}) : \mathcal{O}_{\mathbb{P}^3}(-a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) \to (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3},$$

we get the monad:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

whose cohomology is isomorphic to E. □

Next, we argue that the instanton bundle $\tilde{E}$ obtained in Proposition 11 is symplectic.

Lemma 13. *If $\tilde{E}$ is a rank 4 instanton bundle of charge k that fits in a monad of the form (24), such that the cohomology is a vector bundle, then $\tilde{E}$ admits a symplectic structure, and τ is determined by σ.*

Proof. Since E is a rank 2 vector bundle with $c_1(E) = 0$, there is a (unique up to scale) symplectic isomorphism $\varphi : E \xrightarrow{\simeq} E^\vee$. By Corollary 2, there is an isomorphism of monads:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma} & \tilde{E} & \xrightarrow{\tau} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0 \\ & & \simeq \downarrow g & & \simeq \downarrow \varphi & & \simeq \downarrow h & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\tau^\vee} & \tilde{E}^\vee & \xrightarrow{\sigma^\vee} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0 \end{array}$$

such that $\varphi^\vee = -\varphi$, so $(\tilde{E}, \varphi)$ is a symplectic instanton bundle, and $\tau = \sigma^\vee \circ \varphi$. □

Putting Lemmas 11, 12 and 13 together, we obtain the following statement.

Proposition 14. *A rank 2 bundle E is the cohomology of a monad of the form:*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

if and only if it is also the cohomology of a monad of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\sigma^\vee \circ \varphi} \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

where $(\tilde{E}, \varphi)$ is a rank 4 symplectic instanton bundle of charge k.

As a first application of Proposition 14 we provide an alternative, more manageable description of the set $\mathcal{G}(a, k)$.

In order to fix the notation, note that every automorphism $f \in \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1))$ can be represented by a $(k+1) \times (k+1)$ matrix :

$$f = \begin{pmatrix} f_{1,1} & 0 & \cdots & 0 \\ f_{2,1} & f_{2,2} & \cdots & f_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k+1,1} & f_{k+1,2} & \cdots & f_{k+1,k+1} \end{pmatrix}$$

where each $f_{j,1} \in H^0(\mathcal{O}_{\mathbb{P}^3}(a-1))$ with $j = 2, \ldots, k+1$, and $f_1, 1$ and $f_{l,m}$ are constants for $l, m = 2, 3, \ldots, (k+1)$ such that:

$$f_{1,1} \cdot \det \begin{pmatrix} f_{2,2} & \cdots & f_{2,k+1} \\ \vdots & \ddots & \vdots \\ f_{k+1,2} & \cdots & f_{k+1,k+1} \end{pmatrix} \neq 0.$$

We will denote:

$$\tilde{f} = \begin{pmatrix} f_{2,2} & \cdots & f_{2,k+1} \\ \vdots & \ddots & \vdots \\ f_{k+1,2} & \cdots & f_{k+1,k+1} \end{pmatrix};$$

clearly, $\tilde{f} \in \mathrm{Aut}(k \cdot \mathcal{O}_{\mathbb{P}^3}(-1))$.

Similarly, every $h \in \mathrm{Aut}(k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a))$ can be represented by a $(k+1)\times(k+1)$ matrix:

$$h = \begin{pmatrix} h_{1,1} & \cdots & h_{1,k} & 0 \\ h_{2,1} & \cdots & h_{2,k} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{k+1,1} & h_{k+1,2} & \cdots & h_{k+1,k+1} \end{pmatrix}$$

where each $h_{k+1,j} \in H^0(\mathcal{O}_{\mathbb{P}^3}(a-1))$ for $j = 1, \ldots, k$, and $h_{k+1,k+1}$ and $h_{l,m}$ are constants for $l, m = 1, 2, \ldots, k$, such that:

$$h_{k+1,k+1} \cdot \det \begin{pmatrix} h_{1,1} & \cdots & h_{1,k} \\ \vdots & \ddots & \vdots \\ h_{k,1} & \cdots & h_{k,k} \end{pmatrix} \neq 0.$$

We will denote:

$$\tilde{h} = \begin{pmatrix} h_{1,1} & \cdots & h_{1,k} \\ \vdots & \ddots & \vdots \\ h_{k,1} & \cdots & h_{k,k} \end{pmatrix}.$$

Clearly, $\tilde{h} \in \mathrm{Aut}(k \cdot \mathcal{O}_{\mathbb{P}^3}(1))$.

Now let $\mathcal{P}(a,k)$ be the set of pairs $((\tilde{E},\varphi),\sigma)$ consisting of a rank 4 symplectic instanton bundle $(\tilde{E},\varphi)$ of charge k, and a nowhere vanishing section $\sigma \in \mathrm{H}^0(\tilde{E}(a))$, equipped with the following equivalence relation: $((\tilde{E},\varphi),\sigma) \sim ((\tilde{E}',\varphi'),\sigma')$ if and only if there are an isomorphism of symplectic bundles $g : (\tilde{E},\varphi) \xrightarrow{\sim} (\tilde{E}',\varphi')$, and a constant $\lambda \in \mathbb{K}^*$ such that $g \circ \sigma = \lambda\sigma$. We will denote each equivalence class in $\mathcal{P}(a,k)$ by $[(\tilde{E},\varphi),\sigma]$.

Theorem 15. *There exists a bijection between $\mathcal{G}(a,k)$ and $\mathcal{P}(a,k)$.*

Proof. From each pair $((\tilde{E},\varphi),\sigma)$ we build the monad

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\sigma^\vee \circ \varphi} \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

whose cohomology, by Proposition 14, yields an element $[E] \in \mathcal{G}(a,k)$. Two equivalent pairs $((\tilde{E},\varphi),\sigma)$ and $((\tilde{E}',\varphi'),\sigma')$ yield isomorphic monads

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma} & \tilde{E} & \xrightarrow{\sigma^\vee\circ\varphi} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \lambda} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma'} & \tilde{E}' & \xrightarrow{\sigma'^\vee\circ\varphi'} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0, \end{array}$$

thus $[E] = [E']$.

Conversely, any $[E] \in \mathcal{G}(a,k)$ is the cohomology of a monad of the form (23), from which we can obtain, via Proposition 14, a pair $((\tilde{E},\varphi),\sigma)$. Any two monads whose cohomologies are isomorphic to E are also isomorphic, by Lemma 1; since E is rank 2 vector bundle with zero first Chern class, then Corollary 2 implies the existence of a skew symmetric isomorphism of monads:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\alpha} & (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^3}(a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\alpha'} & (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\beta'} & \mathcal{O}_{\mathbb{P}^3}(a)\oplus k\cdot\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0. \end{array}$$

It then follows that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{\alpha}} & (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\tilde{\beta}} & k\cdot\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \tilde{h}} & & \\ 0 & \longrightarrow & k\cdot\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{\alpha}'} & (4+2k)\cdot\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\tilde{\beta}'} & k\cdot\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \end{array}$$

provides an isomorphism of monads, since $\tilde{f}, g, \tilde{h}$ are isomorphisms, which in turn induces an isomorphism $g : \tilde{E} \xrightarrow{\sim} \tilde{E}'$.

In addition, we also have the following isomorphism of monads

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma} & \tilde{E} & \xrightarrow{\tau} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{1,1}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h_{k+1,k+1}} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma'} & \tilde{E}' & \xrightarrow{\tau'} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0, \end{array} \tag{26}$$

which implies that $g\sigma = f_{1,1}\cdot\sigma'$.

Corollary 2 tells us that $\tilde{E}$ and $\tilde{E}'$ admit symplectic structures φ and φ', respectively, and it only remains for us to show that $(\tilde{E},\varphi)$ and $(\tilde{E}',\varphi')$ are isomorphic as symplectic bundles. By Lemma 13, one can take $\tau = \sigma^\vee\circ\varphi$ and $\tau' = \sigma'^\vee\circ\varphi'$ in equation (26), so that the commutation of the right square in that diagram yields $\sigma'^\vee\circ\varphi' = h_{k+1,k+1}\cdot\sigma^\vee\circ\varphi$. Since $\sigma'^\vee = f_{1,1}^{-1}\cdot\sigma^\vee\circ g^\vee$, we conclude that $f_{1,1} = h_{k+1,k+1}$ and $g^\vee\circ\varphi'\circ g = \varphi$, as desired. $\square$

For our second application of Proposition 14, we focus our attention on the case $k = 1$ to obtain the following important formula for the case $k = 1$.

Lemma 16. *For every $E \in \mathcal{G}(a, 1)$ with $a \geq 2$, it holds*

$$h^1(\mathrm{End}(E)) = 4 \cdot \binom{a+3}{3} - a - 1 + \varepsilon(a),$$

where $\varepsilon(a) = 1$ when $a = 3$, and $\varepsilon(a) = 0$ when $a \neq 3$.

Proof. Since E is a self dual rank 2 bundle, we have $\mathrm{End}(E) \simeq S^2E \oplus \Lambda^2 E = S^2E \oplus \mathcal{O}_{\mathbb{P}^3}$, thus $h^1(\mathrm{End}(E)) = h^1(S^2E)$. We will compute the latter.

Take $E \in \mathcal{G}(a, 1)$ and consider a monad of the form (24) whose cohomology sheaf is isomorphic to E as a complex $M^\bullet$ with terms $M^{-1} = \mathcal{O}_{\mathbb{P}^3}(-a)$, $M^0 = \tilde{E}$, $M^1 = \mathcal{O}_{\mathbb{P}^3}(a)$. Proceed to the double complex $M^\bullet \otimes M^\bullet$, and to its total complex $T^\bullet$. The last complex naturally decomposes into its symmetric and antisymmetric parts; the symmetric part is the complex

$$0 \to \tilde{E}(-a) \to S^2\tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3} \to \tilde{E}(a) \to 0, \tag{27}$$

whose middle cohomology sheaf is isomorphic to S^2E. Therefore the monad (27) can be broken into two short exact sequences

$$0 \to K \to S^2\tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3} \to \tilde{E}(a) \to 0 \text{ and } 0 \to \tilde{E}(-a) \to K \to S^2E \to 0.$$

Since $h^0(\tilde{E}(-a)) = h^0(S^2E) = 0$, it follows that $h^0(K) = 0$; in addition, $h^1(\tilde{E}(a)) = h^2(S^2\tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3}) = 0$ (use Corollary 5) implies that $h^2(K) = 0$. It then follows that

$$h^1(S^2E) = h^1(K) + h^2(\tilde{E}(-a)) = h^1(K) + \varepsilon(a), \tag{28}$$

since $h^1(\tilde{E}(-a)) = 0$ for $a \geq 2$.

To complete our calculation, consider the exact sequence

$$0 \to H^0(S^2\tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3}) \to H^0(\tilde{E}(a)) \to H^1(K) \to H^1(S^2\tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3}) \to 0.$$

Since $h^0(S^2\tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3}) = 4$ and $h^1(S^2\tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3}) = 5$ by Corollary 5, we conclude that

$$h^1(K) = h^0(\tilde{E}(a)) + 1 = h^0(N(a)) + 2 \cdot h^0(\mathcal{O}_{\mathbb{P}^3}(a)) + 1,$$

which, together with the equality in equation (28), yields the desired formula. □

It is interesting to observe that the right hand side of the formula in Lemma 16 yields the expected value when $a = 2$ and $a = 3$, respectively 37 and 77; when $a \geq 4$, one can check that $4 \cdot \binom{a+3}{3} - a - 1 > 8(a^2 + 1) - 3$.

5. The structure of $\mathcal{P}(a,1)$

Motivated by Lemma 16, we now aim at showing that the set $\mathcal{P}(a,1)$ has the structure of an irreducible, nonsingular, quasi-projective variety whose dimension matches the formula in the statement of the lemma. We will not make any distinction between a vector bundle E and its class of isomorphisms $[E]$ and we will denote both of them by the letter E without the brackets.

Recall that a null correlation bundle is, by definition, the cokernel of a nonzero morphism $\eta \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1), \Omega_{\mathbb{P}^3}(1))$ up to a scalar factor, so that the moduli space of null correlation sheaves can be identified with $\mathbb{P}(\mathrm{H}^0(\Omega_{\mathbb{P}^3}(2))) \simeq \mathbb{P}^5$. Denoting by N_η the null correlation sheaf defined by $\eta \in \mathbb{P}(\mathrm{H}^0(\Omega_{\mathbb{P}^3}(2)))$, we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(a-1)) \xrightarrow{\eta} \Omega_{\mathbb{P}^3}(a+1) \longrightarrow N_\eta(a) \longrightarrow 0.$$

Therefore, by the long exact sequence of cohomology, there exists a natural isomorphism of $\mathrm{H}^0(N_\eta(a))$ with the quotient vector space $\mathrm{H}^0(\Omega_{\mathbb{P}^3}(a+1))/\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a-1))$.

Setting $V := \mathbb{P}(\mathrm{H}^0(\Omega_{\mathbb{P}^3}(2)))$, consider the morphism

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a-1)) \otimes \mathcal{O}_V(-1) \xrightarrow{\tilde{\eta}} \mathrm{H}^0(\Omega_{\mathbb{P}^3}(a+1)) \otimes \mathcal{O}_V$$

given by multiplication by the coordinates. This is clearly injective, and its cokernel is a vector bundle over V, denoted by $\mathbf{N}_a$, whose fibre over $\eta \in V$ is $\mathrm{H}^0(\mathrm{coker}\,\eta(a)) \simeq \mathrm{H}^0(N_\eta(a))$.

From Lemmas 4 and 6, we know that each rank 4 instanton bundle $\tilde{E}$ of charge 1 corresponds to a unique null correlation sheaf $N := \tilde{E}/2 \cdot \mathcal{O}_{\mathbb{P}^3}$. Since $\tilde{E}$ admits a unique symplectic structure, the splitting in cohomology given in equation (21) implies that any pair $((\tilde{E},\varphi),\sigma)$, consisting of a sympectic rank 4 instanton bundle of charge 1 and a section $\sigma \in H^0(\tilde{E}(a))$, can be regarded as a point of the product $\mathbf{N}_a \times H^0(\mathcal{O}_{\mathbb{P}^3}(a))^{\oplus 2}$, namely $((N,\sigma_N),(\sigma_1,\sigma_2))$ in the notation of equation (22).

What is more, equation (22) also implies that two equivalent pairs $((\tilde{E},\varphi),\sigma)$ and $((\tilde{E}',\varphi'),\sigma')$ will correspond to points $((N,\sigma_N),(\sigma_1,\sigma_2))$ and $((N,\lambda\sigma_N),(\sigma_1',\sigma_2'))$, respectively, where

$$\lambda \begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = M \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix};$$

here, (λ, M) is the pair representing the symplectic isomorphism $(\tilde{E},\varphi) \xrightarrow{\sim} (\tilde{E}',\varphi')$ under the isomorphism of Lemma 7. In other words, an equivalence class $[(\tilde{E},\varphi),\sigma] \in \mathcal{P}(a,1)$

defines a unique point in the Grassmanian $G(2, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a)))$ of 2-dimensional subspaces of $H^0(\mathcal{O}_{\mathbb{P}^3}(a))$.

Proposition 17. *$\mathcal{P}(a,1)$ is an irreducible, rational, nonsingular quasi-projective variety of dimension*

$$5 + h^0(N(a)) + 2 \cdot (h^0(\mathcal{O}_{\mathbb{P}^3}(a)) - 2) = 4 \cdot \binom{a+3}{3} - a - 1.$$

Proof. We start by defining the following map, using the notation of the previous paragraph:

$$\pi : \mathcal{P}(a,1) \to \mathbb{P}^5 \times G(2, h^0(\mathcal{O}_{\mathbb{P}^3}(a)))$$
$$[(\tilde{E}, \varphi), \sigma] \mapsto (\tilde{E}/2 \cdot \mathcal{O}_{\mathbb{P}^3}, \langle \sigma_1, \sigma_2 \rangle).$$

This is clearly well defined, and we check that it is surjective. Given a null correlation sheaf $N \in \mathbb{P}^5$, let $\tilde{E}$ be the unique locally free extension of N by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$, and let φ be its unique symplectic structure.

Next, take $\langle \sigma_1, \sigma_2 \rangle \in G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(a)))$, and note that the set $\{\sigma_1 = \sigma_2 = 0\}$ is a complete intersection curve C (of degree a^2) in $\mathbb{P}^3$. One can find a section $\sigma_N \in H^0(N(a))$ whose zero locus, being a curve of degree a^2+1, does not intersect C. The triple $(\sigma_N, \sigma_1, \sigma_2)$ thus obtained defines a nowhere vanishing section $\sigma \in H^0(\tilde{E}(a))$.

Clearly, the set $\pi^{-1}(N, \langle \sigma_1, \sigma_2 \rangle)$ consists of all those sections $\sigma_N \in H^0(N(a))$ which do not vanish along the curve $C := \{\sigma_1 = \sigma_2 = 0\}$, so it is an open subset of $H^0(N(a))$. It follows that $\mathcal{P}(a,1)$ can be regarded as an open subset of the product $\mathbf{N}_a \times G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(a)))$, showing that $\mathcal{P}(a,1)$ is an irreducible, nonsingular quasi-projective variety of the given dimension.

Finally, note that $\mathbf{N}_a$ is rational, since it is the total space of a vector bundle over $\mathbb{P}^5$. Hence the product $\mathbf{N}_a \times G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(a)))$ is rational, and so is $\mathcal{P}(a,1)$. □

Noting that the dimension of $\mathcal{P}(a,1)$ matches $h^1(\mathrm{End}(E))$ for $a = 2$ and $a \geq 4$, as calculated in Lemma 16, we have therefore completed the proof of the first main result of this paper.

Theorem 18. *For $a = 2$ and $a \geq 4$, the rank 2 bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

fill out an open subset of an irreducible component of $\mathcal{B}(a^2+1)$ *of dimension*

$$4\cdot\binom{a+3}{3}-a-1.$$

In particular, for the case $a=2$, we conclude that rank 2 bundles given as cohomology of monads of the form (4) yield an open subset of an irreducible component of $\mathcal{B}(5)$ with expected dimension 37.

6. Monads of the form (5)

We finally tackle the set

$$\mathcal{H}=\{[E]\in\mathcal{B}(5)\mid E \text{ is cohomology of a monad of the form (5)}\}.$$

We prove:

Proposition 19. *The set* $\mathcal{H}$ *satisfies the condition*

$$\dim(\mathcal{H}\setminus(\mathcal{G}(a,1)\cap\mathcal{H}))\leq 36. \tag{29}$$

Proof. Let E be the cohomology bundle of the following monad:

$$0\to\mathcal{O}_{\mathbb{P}^3}(-2)\oplus 2\cdot\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{\alpha}\mathcal{O}_{\mathbb{P}^3}(-1)\oplus 6\cdot\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(1)\xrightarrow{\beta}2\cdot\mathcal{O}_{\mathbb{P}^3}(1)\oplus\mathcal{O}_{\mathbb{P}^3}(2)\to 0. \tag{30}$$

Since the bundle $2\cdot\mathcal{O}_{\mathbb{P}^3}(-1)$ is a uniquely defined subbundle of the bundle $\mathcal{O}_{\mathbb{P}^3}(-2)\oplus 2\cdot\mathcal{O}_{\mathbb{P}^3}(-1)$ (respectively, $\mathcal{O}_{\mathbb{P}^3}(-1)$ is a uniquely defined quotient bundle of $\mathcal{O}_{\mathbb{P}^3}(-1)\oplus 6\cdot\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(1)$), there is a well-defined morphism

$$\tilde{\alpha}:\ 2\cdot\mathcal{O}_{\mathbb{P}^3}(-1)\hookrightarrow\mathcal{O}_{\mathbb{P}^3}(-2)\oplus 2\cdot\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{\alpha}\mathcal{O}_{\mathbb{P}^3}(-1)\oplus 6\cdot\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(1)\twoheadrightarrow\mathcal{O}_{\mathbb{P}^3}(-1). \tag{31}$$

Consider first the case

$$\tilde{\alpha}\neq 0. \tag{32}$$

It follows that $\tilde{\alpha}$ is a surjection, hence the kernel of the composition map is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(-1)$. In this case we obtain a morphism $\alpha_1=\alpha|_{\ker\tilde{\alpha}}:\mathcal{O}_{\mathbb{P}^3}(-1)\to 6\cdot\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(1)$. Thus similar to (31) there is a well-defined morphism

$$\alpha':\ \mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{\alpha_1}6\cdot\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(1)\twoheadrightarrow 6\cdot\mathcal{O}_{\mathbb{P}^3},$$

together with its dual morphism β' as in (25) with $k=1$, so that, eventually, we obtain the anti self dual monads (25) with $k=1$ and (24) with $\tilde{E}$ being a rank 4 instanton

bundle of charge 1, which implies that $E \in \mathcal{G}(2,1)$. This means that the condition (32) is equivalent to $[E] \in \mathcal{H} \cap \mathcal{G}(2,1)$, that is:

$$[E] \in \mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2,1)) \quad \Longleftrightarrow \quad \tilde{\alpha} = 0.$$

We therefore proceed to the case

$$\tilde{\alpha} = 0.$$

This condition implies that $\mathrm{im}(\alpha_0) \subset 6 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$, where $\alpha_0 := \alpha|_{2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1)}$. Moreover, since α is a subbundle morphism, it follows that $\mathrm{im}(\alpha_0) \not\subset \mathcal{O}_{\mathbb{P}^3}(1)$, so that there is a well-defined injective morphism

$$\bar{\alpha} : \ 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} 6 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \twoheadrightarrow 6 \cdot \mathcal{O}_{\mathbb{P}^3}.$$

Again similar to the anti self dual monads (25) and (24) we obtain the anti self dual monads

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 6 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\alpha_0^\vee} 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

$$E_1 := \ker \alpha_0^\vee / \mathrm{im} \alpha_0,$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\gamma} E_1 \xrightarrow{\gamma^\vee} \mathcal{O}_{\mathbb{P}^3}(2) \to 0, \quad E = \ker \gamma^\vee / \mathrm{im} \gamma, \tag{33}$$

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\bar{\alpha}} 6\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\bar{\alpha}^\vee} 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \quad E_2 := \ker \bar{\alpha}^\vee / \mathrm{im} \bar{\alpha}, \tag{34}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\delta} E_1 \xrightarrow{\delta^\vee} \mathcal{O}_{\mathbb{P}^3}(-1) \to 0, \quad E_2 \simeq \ker \delta^\vee / \mathrm{im} \delta, \tag{35}$$

where γ and δ are the induced morphisms and E_2 is a rank 2 bundle with $c_1(E_2) = 0$ and $c_2(E_2) = 2$.

The monad (33) induces an exact triple

$$0 \to E \to F \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{36}$$

where $F := \mathrm{coker}\, \gamma$ and ε is the induced morphism. Consider the composite morphisms

$$\delta' : \ \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\delta} E_1 \twoheadrightarrow F, \quad E' := \mathrm{coker}\, \delta',$$

and

$$\delta'' : \ \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\delta'} F \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}^3}(2).$$

Here $\delta'' \neq 0$, since otherwise by (36) $h^0(E(-1)) \neq 0$, contrary to the stability of E. Hence,

$$\mathrm{coker}\, \delta'' = \mathcal{O}_{\mathbb{P}^2_a}(2)$$

for some projective plane $\mathbb{P}^2_a$ in $\mathbb{P}^3$, and we have an induced exact triple:

$$0 \to E \to E' \to \mathcal{O}_{\mathbb{P}^2_a}(2) \to 0. \tag{37}$$

Besides, (33) and (35) yield exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\gamma'} E_3 \xrightarrow{\lambda} E' \to 0, \tag{38}$$

$$0 \to E_2 \xrightarrow{\mu} E_3 \xrightarrow{\nu} \mathcal{O}_{\mathbb{P}^3}(-1) \to 0.$$

where $E_3 := \operatorname{coker}\delta$ and $\gamma', \lambda, \mu, \nu$ are the induced morphisms. Note that (35) implies that $h^0(E_2(-2)) = 0$, hence by (38) the composition $\lambda \circ \mu$ is a nonzero morphism. Moreover, one easily sees that this morphism is injective. Therefore, since E' is a rank 2 sheaf, it follows that the composition $\nu \circ \gamma' : \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3}(-1)$ is a nonzero morphism and $\operatorname{coker}(\nu \circ \gamma') = \mathcal{O}_{\mathbb{P}^2_b}(-1)$ for some projective plane $\mathbb{P}^2_b$ in $\mathbb{P}^3$. We thus obtain an exact triple

$$0 \to E_2 \xrightarrow{\lambda\circ\mu} E' \xrightarrow{\theta} \mathcal{O}_{\mathbb{P}^2_b}(-1) \to 0, \tag{39}$$

where θ is the induced morphism. Now remark that the triple (37) does not split, since otherwise, as E_2 is locally free, the composition $\mathcal{O}_{\mathbb{P}^2_a}(2) \hookrightarrow E' \xrightarrow{\theta} \mathcal{O}_{\mathbb{P}^2_b}(-1)$ is nonzero which is impossible. Thus $\mathbb{P}^2_a = \mathbb{P}^2_b =: \mathbb{P}^2$ and the triple (37) as an extension is given by a nonzero element

$$s \in \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(2), E) \simeq H^0(\mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}^2}(2), E)) \simeq H^0(E|_{\mathbb{P}^2}(-1)).$$

Remind that, since E is cohomology of (30) by [14, Table 5.3, page 804] it has spectrum $(-1, 0, 0, 0, 1)$ and then follows that

$$h^1(E(-3)) = 0, \quad h^1(E(-2)) = 1. \tag{40}$$

The zero-scheme $Z = (s)_0$ of this section s is 0-dimensional. Indeed, otherwise $h^0(E|_{\mathbb{P}^2}(-2)) \neq 0$, which contradicts to the cohomology sequence of the exact triple $0 \to E(-3) \to E(-2) \to E|_{\mathbb{P}^2}(-2) \to 0$ as $h^0(E(-2)) = 0$ by the stability of E and the first equalty in (40). Besides, the cohomology sequence of the last triple twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$ in view of the stability of E and the second equalty in (40) yields:

$$h^0(E|_{\mathbb{P}^2}(-1)) = 1. \tag{41}$$

Furthermore, applying the functor $- \otimes \mathcal{O}_{\mathbb{P}^2}$ to the triple (37) we obtain an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \xrightarrow{s} E|_{\mathbb{P}^2} \to E'|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(2) \to 0.$$

By (41), the leftmost morphism s here is the above section of $E|_{\mathbb{P}^2}(-1)$, so that $\operatorname{coker}(s) \simeq \mathcal{I}_{Z,\mathbb{P}^2}(-1)$, and the last sequence yields an exact triple

$$0 \to \mathcal{I}_{Z,\mathbb{P}^2}(-1) \to E'|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(2) \to 0.$$

Apply to this sequence the functor $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^2}(2))$. Since $\dim Z = 0$, it follows that $\mathcal{H}om(\mathcal{I}_{Z,\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathcal{O}_{\mathbb{P}^2}(3)$, and we obtain an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{H}om(E'|_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2)) \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0.$$

Hence, $\dim \operatorname{Hom}(E', \mathcal{O}_{\mathbb{P}^2}(2)) = h^0(\mathcal{H}om(E'|_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2))) = 11$ and therefore

$$\mathbb{P}(\operatorname{Hom}(E', \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}.$$

This equality will be used below.

We now proceed to the study of the sheaf E_2 defined in (34). The results obtained here will complete the proof of Proposition 19.

Consider the space $\Pi = \mathbb{P}(\operatorname{Hom}(4 \cdot \mathcal{O}_{\mathbb{P}^3}, 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1)))$ and its first determinantal subvariety $\Delta = \{\mathbf{k}\varphi \in \Pi \mid \varphi : 4 \cdot \mathcal{O}_{\mathbb{P}^3} \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \text{ is not surjective}\}$. It is known that

$$\operatorname{codim}_\Pi \Delta = \operatorname{rk}(4 \cdot \mathcal{O}_{\mathbb{P}^3}) - \operatorname{rk}(2 \cdot \mathcal{O}_{\mathbb{P}^3}(1)) + 1 = 3. \tag{42}$$

Consider the monad (34), and suppose that the homomorphism

$$h^0(\bar{\alpha}^\vee) : \ H^0(6 \cdot \mathcal{O}_{\mathbb{P}^3}) \to H^0(2 \cdot \mathcal{O}_{\mathbb{P}^3}(1))$$

has rank at most 4. This means that the morphism $\bar{\alpha}^\vee$ factors through a morphism $\varphi : \ 4 \cdot \mathcal{O}_{\mathbb{P}^3} \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1)$. By the universal property of the space Π, we obtain an embedding $i : \mathbb{P}^3 \hookrightarrow \Pi$ such that, by (42), $\emptyset \neq i^{-1}(i(\mathbb{P}^3) \cap \Delta) = \{x \in \mathbb{P}^3 \mid \bar{\alpha}^\vee|_x \text{ is not surjective}\}$. This contradicts to the surjectivity of $\bar{\alpha}^\vee$. Hence, $h^0(\bar{\alpha}^\vee)$ has rank at least 5, and the monad (34) implies that

$$h^0(E_2) \leq 1.$$

We now analyze both cases, namely: (i) $h^0(E_2) = 1$; (ii) $h^0(E_2) = 0$.

(i) $h^0(E_2) = 1$. Since E_2 is a rank 2 bundle with $c_1(E_2) = 0$ and $c_2(E_2) = 2$ (see (34)), it follows that the zero scheme of the section $0 \neq s \in H^0(E_2)$ is a projective line, say, l in $\mathbb{P}^3$ with some locally complete intersection (shortly: l.c.i.) double structure $l^{(2)}$ on it satisfying the triple

$$0 \to \mathcal{O}_l(2) \to \mathcal{O}_{l^{(2)}} \to \mathcal{O}_l \to 0. \tag{43}$$

We thus obtain an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} E_2 \to \mathcal{I}_{l^{(2)}} \to 0. \tag{44}$$

Note that the set of l.c.i. double structures on a given line $l \in G(2,4)$ is the set of epimorphisms $\psi :\ N^\vee_{l,\mathbb{P}^3} \simeq 2\mathcal{O}_l(-1) \twoheadrightarrow \mathcal{O}_l(2)$ (here, $N_{l,\mathbb{P}^3}$ denotes the normal bundle of l), understood up to scalar multiple, i.e. an open dense subset U_l of the projective space $\mathbb{P}(\mathrm{Hom}(2\cdot\mathcal{O}_l(-1),\mathcal{O}_l(2))) \simeq \mathbb{P}^7$, hence $\dim U_l = 7$. Thus space D of all possible l.c.i. double structures $l^{(2)}$ on lines in $\mathbb{P}^3$ has a projection $\rho :\ D \to G(2,4),\ l^{(2)} \mapsto l$ with fibre $\rho^{-1}(l) = U_l$, so that

$$\dim D = \dim G(2,4) + \dim U_l = 11.$$

Next, for a given $l^{(2)} \in D$ the set of isomorphim classes of locally free sheaves E_2 defined as extensions (44) constitutes an open dense subset $V_{l^{(2)}}$ of the projective space $\mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_{l^{(2)}},\mathcal{O}_{\mathbb{P}^3})) \simeq \mathbb{P}^3$. To compute this space, apply to the triple

$$0 \to \mathcal{I}_{l^{(2)}} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{l^{(2)}} \to 0 \tag{45}$$

the functor $\mathcal{H}om(-,\mathcal{O}_{\mathbb{P}^3})$. We obtain $\mathcal{E}xt^1(\mathcal{I}_{l^{(2)}},\mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^2(\mathcal{O}_{l^{(2)}},\mathcal{O}_{\mathbb{P}^3})$, and therefore

$$\mathrm{Ext}^1(\mathcal{I}_{l^{(2)}},\mathcal{O}_{\mathbb{P}^3}) \simeq H^0(\mathcal{E}xt^1(\mathcal{I}_{l^{(2)}},\mathcal{O}_{\mathbb{P}^3})) \simeq H^0(\mathcal{E}xt^2(\mathcal{O}_{l^{(2)}},\mathcal{O}_{\mathbb{P}^3})). \tag{46}$$

Applying the same functor to (43) and using the isomorphisms $\mathcal{E}xt^2(\mathcal{O}_l,\mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{O}_l(2)$, and $\mathcal{E}xt^2(\mathcal{O}_l(2),\mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{O}_l$, we obtain an exact triple

$$0 \to \mathcal{O}_l(2) \to \mathcal{E}xt^2(\mathcal{O}_{l^{(2)}},\mathcal{O}_{\mathbb{P}^3}) \to \mathcal{O}_l \to 0$$

which together with (46) yields $\mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_{l^{(2)}},\mathcal{O}_{\mathbb{P}^3})) \simeq \mathbb{P}^3$, hence $\dim V_{l^{(2)}} = 3$. Now, denoting by B the space of isomorphism classes of locally free sheaves E_2 defined as extensions (44), we obtain a well defined morphism $\tau :\ B \to D,\ [E_2] \mapsto l^{(2)} = (s)_0$ for $0 \neq s \in H^0(E_2)$ with fibre $\tau^{-1}(l^{(2)}) = V_{l^{(2)}}$. Hence,

$$\dim B = \dim D + \dim V_{l^{(2)}} = 3 + 11 = 14. \tag{47}$$

Now, for any pair $([E_2],\mathbb{P}^2) \in B \times \check{\mathbb{P}}^3$, consider the space $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1),E_2)$ of extensions (39):

$$0 \to E_2 \to E' \to \mathcal{O}_{\mathbb{P}^2}(-1) \to 0. \tag{48}$$

Since E_2 is locally free, one has

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1),E_2) \simeq H^0(\mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}^2}(-1),E_2)) \simeq H^0(E_2|_{\mathbb{P}^2}(2)). \tag{49}$$

For $l = (\rho \circ \tau)([E_2])$ denote $\check{l} = \{\mathbb{P}^2 \in \check{\mathbb{P}}^3 \mid \mathbb{P}^2 \ni l\}$. Consider the two cases: (a) $\mathbb{P}^2 \in \check{l}$; and (b) $\mathbb{P}^2 \notin \check{l}$.

(a) $\mathbb{P}^2 \in \check{l}$. In this case one sees using (43) that $\mathcal{T}or_1(\mathcal{O}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathcal{O}_l(3)$ and the scheme $\bar{l} = l^{(2)} \cap \mathbb{P}^2$ is described by the triple $0 \to \mathcal{O}_Y \to \mathcal{O}_{\bar{l}} \to \mathcal{O}_l \to 0$, where Y is a 0-dimensional scheme of length 3 supported on l. Thus, after applying the functor $- \otimes \mathcal{O}_{\mathbb{P}^2}(2)$ to the exact sequence (45), we obtain an exact triple

$$0 \to \mathcal{O}_l(3) \to \mathcal{I}_{l^{(2)}} \otimes \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{I}_{Y,\mathbb{P}^2}(1) \to 0.$$

Since $Y \subset l$, it follows that $h^0(\mathcal{I}_{Y,\mathbb{P}^2}(1)) = 1$, hence the last triple yields $h^0(\mathcal{I}_{l^{(2)}} \otimes \mathcal{O}_{\mathbb{P}^2}(2)) = 5$. Therefore, the triple

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to E_2|_{\mathbb{P}^2}(2) \to \mathcal{I}_{l^{(2)}} \otimes \mathcal{O}_{\mathbb{P}^2}(2) \to 0, \tag{50}$$

obtained by applying the functor $- \otimes \mathcal{O}_{\mathbb{P}^2}(2)$ to (45), yields

$$h^0(E_2|_{\mathbb{P}^2}(2)) = 11. \tag{51}$$

(b) $\mathbb{P}^2 \notin \check{l}$. In this case $W = l^{(2)} \cap \mathbb{P}^2$ is a 0-dimensional scheme of length 2 supported at the point $l \cap \mathbb{P}^2$, and the triple (50) becomes: $0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to E_2|_{\mathbb{P}^2}(2) \to \mathcal{I}_{W,\mathbb{P}^2}(2) \to 0$. From this triple we obtain

$$h^0(E_2|_{\mathbb{P}^2}(2)) = 10. \tag{52}$$

Consider the space Σ_1 of isomorphism classes of sheaves E' obtained as extensions (48). One has a natural projection $\pi_1 : \Sigma_1 \to B \times \check{\mathbb{P}}^3$ with fibre described as $\pi_1^{-1}([E_2], \mathbb{P}^2) = \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2))$. Now by (49), (51) and (52) this fibre has dimension 10, respectively, 9 in case (a), respectively, (b) above. Hence in view of (47) we have

$$\dim \Sigma_1 = 26. \tag{53}$$

Now return to the triple (37). Consider the space W_1 parametrising the surjections $e_1 : E' \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)$ (up to a scalar multiple) for $[E'] \in \Sigma_1$ and $\mathbb{P}^2 = pr_2(\pi([E']))$, where $pr_2 : B \times \check{\mathbb{P}}^3 \to \check{\mathbb{P}}^3$ is the projection. We thus obtain a surjective morphism $p_1 : W_1 \to \Sigma_1$ with fibre $p_1^{-1}(E)$ being an open dense subset in $\mathbb{P}(\mathrm{Hom}(E', \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}$, where $\mathbb{P}^2 = pr_2(\pi_1([E']))$. Thus by (53)

$$\dim W_1 = 36. \tag{54}$$

On the other hand, the triple (37) means that there is a morphism

$$q : W_1 \to \mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2,1)),\ \mathbb{K}e_1 \mapsto \ker(e_1 : E' \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)). \tag{55}$$

(ii) $h^0(E_2) = 0$. This means that E_2 is stable, i.e. $[E_2] \in \mathcal{B}(2)$. It is well-known (see [12, §9, Lemma 9.5]) that each bundle $[E_2] \in \mathcal{B}(2)$ fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to E_2 \to \mathcal{I}_Y(1) \to 0,$$

where Y is a divisor of the type (3,0) on some smooth quadric surface in $\mathbb{P}^3$. Moreover, for given E_2, this divisor is not unique, but varies in a 1-dimensional linear series without fixed points. Therefore, for any pair $([E_2], \mathbb{P}^2) \in \mathcal{B}(2) \times \check{\mathbb{P}}^3$ one can choose a nontrivial section $s \in E_2|_{\mathbb{P}^2}(1)$ such that its zero scheme $Z = (s)_0$ is a 0-dimensional scheme of length 3, and therefore $h^0(\mathcal{I}_{Z,\mathbb{P}^2}(3)) = 7$. This together with the exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \xrightarrow{s} E_2|_{\mathbb{P}^2}(2) \to \mathcal{I}_{Z,\mathbb{P}^2}(3) \to 0$$

yields $h^0(E_2|_{\mathbb{P}^2}(2)) = 10$, hence in view of (49) we obtain

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2)) \simeq \mathbb{P}^9. \tag{56}$$

Now, as above, consider the space Σ_0 of isomorphism classes of sheaves E' obtained as extensions (48) with $[E_2] \in \mathcal{B}(2)$. One has a natural projection $\pi_0 : \Sigma_0 \to \mathcal{B}(2) \times \check{\mathbb{P}}^3$ with fibre described as $\pi_0^{-1}([E_2], \mathbb{P}^2) = \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2))$. Now by (56) this fibre has dimension 9, and we obtain

$$\dim \Sigma_0 = \dim \mathcal{B}(2) + \dim \check{\mathbb{P}}^3 + \dim \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2)) = 13 + 3 + 9 = 25. \tag{57}$$

Again return to the triple (37). Consider the space W_0 parametrising the surjections $e_0 :\ E' \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)$ (up to a scalar multiple) for $[E'] \in \Sigma_0$ and $\mathbb{P}^2 = pr_2(\pi([E']))$, where $pr_2 : \mathcal{B}(2) \times \check{\mathbb{P}}^3 \to \check{\mathbb{P}}^3$ is the projection. We thus obtain a surjective morphism $p_0 :\ W_0 \to \Sigma_0$ with fibre $p_0^{-1}(E)$ being an open dense subset in $\mathbb{P}(\mathrm{Hom}(E', \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}$, where $\mathbb{P}^2 = pr_2(\pi_0([E']))$. Thus by (57)

$$\dim W_0 = 35. \tag{58}$$

On the other hand, the triple (37) means that there is a morphism

$$q : W_0 \to (\mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H})),\ \mathbb{K}e_0 \mapsto \ker(e_0 : E' \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)). \tag{59}$$

Note that, for any E_2 in (37) we have either $h^0(E_2) = 1$ or $h^0(E_2) = 0$. This means that the morphism

$$q : W_1 \cup W_0 \to (\mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H}))$$

defined in (55) and (59) is surjective. Hence (29) follows from (54) and (58). □

7. Components of $\mathcal{B}(5)$

We finally have at hand all the ingredients needed to complete the proof of our second main result, namely the characterization of the irreducible components of $\mathcal{B}(5)$. We will proof the following result.

Theorem 20. *The moduli space $\mathcal{B}(5)$ has exactly 3 irreducible components, namely:*

(i) *the* instanton component*, of dimension 37, which consists of those bundles given as cohomology of monads of the form*

$$0 \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to 12 \cdot \mathcal{O}_{\mathbb{P}^3} \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \text{ and} \tag{60}$$

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \to 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 3 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(2) \to 0; \tag{61}$$

(ii) *the* Ein component*, of dimension 40, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \tag{62}$$

iii) *the closure of the family $\mathcal{G}(2,1)$, of dimension 37, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \text{ and} \tag{63}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 6 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{64}$$

The first ingredient of the proof is the fact, proved by Hartshorne and Rao, that every bundle in $\mathcal{B}(5)$ is cohomology of one of the above monads, cf. [14, Table 5.3, page 804].

Recall that for each stable rank 2 bundle E on $\mathbb{P}^3$ with vanishing first Chern class, the number $\alpha(E) := h^1(E(-2)) \bmod 2$ is called the Atiyah–Rees α-invariant of E, see [12, Definition in page 237]. Hartshorne showed [12, Corollary 2.4] that this number is invariant on the components of the moduli space of stable vector bundles on $\mathbb{P}^3$. One can

easily check that the cohomologies of monads of the form (60) and (61) have α-invariant equal to 0, while the cohomologies of the other three types of monads have α-invariant equal to 1.

Rao showed in [26] that the family of bundles obtained as cohomology of monads of the form (61) is irreducible, of dimension 36, and it lies in a unique component of $\mathcal{B}(5)$. Since instanton bundles of charge 5, i.e. the cohomologies of monads of the form (60), yield an irreducible family of dimension 37, it follows that the set

$$\mathcal{I} := \{[E] \in \mathcal{B}(5) \mid \alpha(E) = 0\}$$

forms a single irreducible component of $\mathcal{B}(5)$, of dimension 37, whose generic point corresponds to an instanton bundle. In addition, every $[E] \in \mathcal{I}$ satisfies $\mathrm{H}^1(\mathrm{End}(E)) = 37$; this was originaly proved by Katsylo and Ottaviani for instanton bundles [22], and by Rao for the cohomologies of monads of the form (61) [26, Section 3]. Therefore, we also conclude that $\mathcal{I}$ is nonsingular. This completes the proof of the first part of the Main Theorem.

Our next step is to analyse those bundles with Atiyah–Rees invariant equal to 1.

Hartshorne proved in [13, Theorem 9.9] that the family of stable rank 2 bundles E with $c_1(E) = 0$ and $c_2(E) = 5$ whose spectrum is $(-2, -1, 0, 1, 2)$ form an irreducible, nonsigular family of dimension 40. Such bundles are precisely those given as cohomologies of monads of the form (62), cf. [14, Table 5.3, page 804], which is a particular case of a class of monads studied by Ein in [10]. From these references, we conclude that the closure of the family of vector bundle arising as cohomology of monads of the form (62) is an oversized irreducible component of $\mathcal{B}(5)$ of dimension 40.

We proved above that the bundles arising as cohomology of monads of the form (63) form a third irreducible component of dimension 37, while those bundles arising as cohomology of monads of the form (64), denoted by $\mathcal{H}$, form an irreducible family of dimension 36. It follows that latter must lie either in $\overline{\mathcal{G}(2,1)}$ or in $\overline{\mathcal{E}}$, the closures $\mathcal{G}(2,1)$ and $\mathcal{E}$, respectively, within $\mathcal{B}(5)$.

Proposition 21. $\mathcal{H} \subset \overline{\mathcal{G}(2,1)}$.

Proof. Suppose by contradiction that there exists a vector bundle $E \in \mathcal{H} \cap \overline{\mathcal{E}}$. By the inferior semi-continuity of the dimension of the cohomology groups of coherent sheaves, one has that $h^1(E(-2)) \geq 3$. However, one can check from the display of the monad (64) that $\dim \mathrm{H}^1(E(-2)) = 1 < 3$. It follows that the family $\mathcal{H}$ must lie in $\overline{\mathcal{G}(2,1)}$. □

This last proposition finally concludes the proof of Main Theorem 2. We summarize all the information in the theorem, and the discrete invariants of stable rank 2 bundles with $c_1 = 0$ and $c_2 = 5$ in the following table.

TABLE 1. Irreducible components of $\mathcal{B}(5)$

Component	**Dimension**	**Monads**	**Spectra**	α**-invariant**
Instanton	37	(1)	(0,0,0,0,0)	0
		(2)	(-1,-1,0,1,1)	
Ein	40	(3)	(-2,-1,0,1,2)	1
Modified Instanton	37	(4)	(-1,0,0,0,1)	1
		(5)		

In order to give a complete description of the vector bundles $E \in \overline{\mathcal{G}(2,1)}$, we include here its cohomology table. Knowing the spectrum of an arbitrary $E \in \overline{\mathcal{G}(2,1)}$ (given in the table above) allows us to conclude that $h^1(E(k)) = 0$ for $k \leq -3$, and to compute $h^1(E(-2)) = 1$ and $h^1(E(-1)) = 5$. Serre duality tells us that $h^2(E(k)) = 0$ for $k \geq -1$, while stability implies that $h^0(E(k)) = 0$ for $k \leq 0$, and $h^3(E(k)) = 0$ for $k \geq -4$; it follows that $h^1(E) = -\chi(E) = 8$.

TABLE 2. $h^i(E(l))$ for $E \in \overline{\mathcal{G}(2,1)}$

i\l	**-4**	**-3**	**-2**	**-1**	**0**
3	0	0	0	0	0
2	8	5	1	0	0
1	0	0	1	5	8
0	0	0	0	0	0

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