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On axioms of frobenius like structure in the theory of arrangements

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A Frobenius manifold is a manifold with a flat metric and a Frobenius algebra structure on tangent spaces at points of the manifold such that the structure constants of multiplication are given by third derivatives of a potential function on the manifold with respect to flat coordinates. In this paper we present a modification of that notion coming from the theory of arrangements of hyperplanes. Namely, given natural numbers $n > k$, we have a flat n-dimensional manifold and a vector space V with a non-degenerate symmetric bilinear form and an algebra structure on V, depending on points of the manifold, such that the structure constants of multiplication are given by $2k + 1$-st derivatives of a potential function on the manifold with respect to flat coordinates. We call such a structure a *Frobenius like structure*. Such a structure arises when one has a family of arrangements of n affine hyperplanes in $\mathbb{C}^k$ depending on parameters so that the hyperplanes move parallely to themselves when the parameters change. In that case a Frobenius like structure arises on the base $\mathbb{C}^n$ of the family.

Keywords: Frobenius structure; arrangement of hyperplanes; Gauss–Manin connection.

1. Introduction

The theory of Frobenius manifolds has multiple connections with other branches of mathematics, such as quantum cohomology, singularity theory, the theory of integrable systems, see, for example, [1–3]. The notion of a Frobenius manifold was introduced by Dubrovin in [1], see also [2–5], where numerous variants of this notion are discussed. In all alterations, a Frobenius manifold is a manifold with a flat metric and a Frobenius algebra structure on tangent spaces at points of the manifold such that the structure constants of multiplication are given by third derivatives of a potential function on the manifold with respect to flat coordinates.

In this paper we present a modification of that notion coming from the theory of arrangements of hyperplanes. Namely, given natural numbers $n > k$, we have a flat n-dimensional manifold and a vector space V with a non-degenerate symmetric bilinear form and an algebra structure on V, depending on points of the manifold, such that the structure constants of multiplication are given by $2k+1$-st derivatives of a potential function on the manifold with respect to flat coordinates. We call such a structure a *Frobenius like structure*. Such a structure arises when one has a family of arrangements of n affine hyperplanes in $\mathbb{C}^k$ depending on parameters so that the hyperplanes move parallely to themselves when the parameters change. In that case a Frobenius like structure arises on the base $\mathbb{C}^n$ of the family. On such families see, for example, [6, 7].

In Section 2, we give Dubrovin's definition of almost Frobenius structure. In Section 3, we give the definition of Frobenius like structure motivated by Dubrovin's definition and results of [8]. The definition consists of eight Axioms: 3.2.1–3.2.8. In Section 4, we consider a family of arrangements and construct a Frobenius like structure on its base.

2. Axioms of an almost Frobenius structure

An *almost Frobenius structure* of charge $d \neq 1$ on a manifold M is a structure of a Frobenius algebra on the tangent spaces $T_zM = (*_z, (\ ,\ )_z)$, where $z \in M$ [2]. It satisfies the following axioms.

Axiom 1. The metric $(\ ,\ )_z$ is flat.

Axiom 2. In the flat coordinates $z_1, \dots, z_n$ for the metric, the structure constants of the multiplication

$$\frac{\partial}{\partial z_i} *_z \frac{\partial}{\partial z_j} = N_{i,j}^k(z) \frac{\partial}{\partial z_k} \tag{2.1}$$

can be represented in the form

$$N_{i,j}^k(z) = G^{k,l} \frac{\partial^3 L}{\partial z_l \partial z_i \partial z_j}(z) \tag{2.2}$$

for some function $L(z)$. Here $(G^{i,j})$ is the matrix inverse to the matrix $(G_{i,j} := (\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}))$.

Axiom 3. The function $L(z)$ satisfies the homogeneity equation

$$\sum_{j=1}^n z_j \frac{\partial L}{\partial z_j}(z) = 2L(z) + \frac{1}{1-d} \sum_{i,j} G_{i,j} z_i z_j. \tag{2.3}$$

Axiom 4. The Euler vector field

$$E = \frac{1-d}{2} \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} \tag{2.4}$$

is the unit of the Frobenius algebra. Notice that $(E, E) = \frac{(1-d)^2}{4} \sum_{i,j} G_{i,j} z_i z_j$.

Axiom 5. There exists a vector field

$$e = \sum_{j=1}^n e_j(z) \frac{\partial}{\partial z_j}, \tag{2.5}$$

an invertible element of the Frobenius algebra T_zM for every $z \in M$, such that the operator $g(z) \mapsto (eg)(z)$ acts by the shift $\kappa \mapsto \kappa - 1$ on the space of solutions of the system of equations

$$\frac{\partial^2 g}{\partial z_i \partial z_j} = \kappa \sum_{l=1}^n N_{i,j}^l \frac{\partial g}{\partial z_l}, \qquad i,j = 1, \dots, n. \tag{2.6}$$

The solutions of that system are called *twisted periods* [2].

2.1 *Example in Section 5.2 of [2]*

Let

$$F(z_1, \ldots, z_n) = \frac{n}{4} \sum_{i<j} (z_i - z_j)^2 \log(z_i - z_j) \tag{2.7}$$

and let the metric be $(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = \delta_{i,j}$. The third derivatives of $F(z)$ give the multiplication law of tangent vectors $\partial_i := \frac{\partial}{\partial z_i}$:

$$\partial_i *_z \partial_j = -\frac{n}{2} \frac{\partial_i - \partial_j}{z_i - z_j} \quad \text{for } i \neq j, \qquad \partial_i *_z \partial_i = -\sum_{j \neq i} \partial_i *_z \partial_j. \tag{2.8}$$

The unit element of the algebra is the Euler vector field

$$E = \frac{1}{n} \sum_{j=1}^{n} z_j \partial_j. \tag{2.9}$$

The operator of multiplication in the algebra by the element $\sum_{j=1}^{n} \frac{\partial}{\partial z_j}$ equals zero. Factorizing over this direction one obtains an almost Frobenius structure on $\{z_1 + \cdots + z_n = 0\} - \cup_{i<j}\{z_i = z_j\}$.

The vector field e is

$$e = -\sum_{j=1}^{n} \frac{1}{f'(z_j)} \partial_j, \qquad f(t) := \prod_{i=1}^{n} (t - z_i). \tag{2.10}$$

Equation (2.6) takes the form

$$\partial_i \partial_j g = -\frac{\kappa}{z_i - z_j} (\partial_i g - \partial_j g), \qquad i \neq j \tag{2.11}$$

with solutions

$$g(z_1, \ldots, z_n) = \int \prod_{i=1}^{n} (t - z_i)^{\kappa} dt. \tag{2.12}$$

3. Axioms of a Frobenius like structure

3.1 *Objects*

Let an $n \times k$-matrix $(b_i^j)_{i=1,\ldots,n}^{j=1,\ldots,k}$ be given with $n > k$. For $i_1, \ldots, i_k \in [1, \ldots, n]$, denote

$$d_{i_1,\ldots,i_k} = \det(b_{i_l}^j)_{j,l=1,\ldots,k}. \tag{3.1}$$

We have $d_{\sigma i_1,\ldots,\sigma i_k} = (-1)^{\sigma} d_{i_1,\ldots,i_k}$ for $\sigma \in S_k$. We assume that $d_{i_1,\ldots,i_k} \neq 0$ for all distinct $i_1, \ldots, i_k$.

REMARK The numbers $(d_{i_1,\dots,i_k})$ satisfy the Plücker relations. For example, for arbitrary $j_1, \dots, j_{k+1}, i_2, \dots, i_k \in [1,\dots,n]$, we have

$$\sum_{m=1}^{k+1}(-1)^{m+1} d_{j_1,\dots,\widehat{j_m},\dots,j_{k+1}} d_{j_m,i_2,\dots,i_k} = 0. \tag{3.2}$$

We assume that an $\binom{n-1}{k}$-dimensional complex vector space V is given with a non-degenerate symmetric bilinear form $S(\ ,\ )$.

We assume that for every k-element unordered subset $\{i_1,\dots,i_k\} \subset [1,\dots,n]$ a non-zero vector $P_{i_1,\dots,i_k} \in V$ is given such that the vectors $\{P_{i_1,\dots,i_k}\}$ generate V as a vector space and the only linear relations among them are

$$\sum_{j=1}^{n} d_{j,i_2,\dots,i_k} P_{j,i_2,\dots,i_k} = 0, \qquad \forall i_2,\dots,i_k \in [1,\dots,n]. \tag{3.3}$$

LEMMA 3.1 For any $j \in [1,\dots,n]$, the vectors $\{P_{i_1,\dots,i_k} \mid \{i_1,\dots,i_k\} \subset [1,\dots,n] - \{j\}\}$ form a basis of V. □

We assume that a (*potential*) function $L(z_1,\dots,z_n)$ is given.

We assume that non-zero vectors $p_1(z_1,\dots,z_n),\dots,p_n(z_1,\dots,z_n) \in V$ depending on $z_1,\dots,z_n$ are given such that for any $i_2,\dots,i_k \in [1,\dots,n]$, we have

$$\sum_{j=1}^{n} d_{j,i_2,\dots,i_k} p_j(z_1,\dots,z_n) = 0. \tag{3.4}$$

3.2 *Axioms*

3.2.1 *Invariance axiom*

We assume that for any $i_2,\dots,i_k \in [1,\dots,n]$, we have

$$\sum_{j=1}^{n} d_{j,i_2,\dots,i_k} \frac{\partial L}{\partial z_j} = 0. \tag{3.5}$$

Given $z = (z_1,\dots,z_n)$, for $j = 1,\dots,n$, define a linear operator $p_j(z)*_z : V \to V$ by the formula

$$S(p_j(z) *_z P_{i_1,\dots,i_k}, P_{j_1,\dots,j_k}) = \frac{\partial}{\partial z_j} \frac{\partial^{2k} L}{\partial z_{i_1} \dots \partial z_{i_k} \partial z_{j_1} \dots \partial z_{j_k}}(z). \tag{3.6}$$

LEMMA 3.2 The operators $p_j(z)*_z$ are well-defined.

Proof. We need to check that if we substitute in (3.6) a relation of the form (3.3) instead of $P_{i_1,\dots,i_k}$ or $P_{j_1,\dots,j_k}$ or if we substitute in (3.6) a relation of the form (3.4) instead of $p_j(z)$, then the right-hand side of (3.6) is zero, but that follows from assumption (3.5). □

LEMMA 3.3 The operators $p_j(z)*_z$ are symmetric, $S(p_j(z)*_z v, w) = S(v, p_j(z)*_z w)$ for any $v, w \in V$. □

Choose a basis $\{P_\alpha \mid \alpha \in \mathcal{I}\}$ of V among the vectors $\{P_{i_1,\dots,i_k}\}$. Here each α is an unordered k-element subset $\{i_1, \dots, i_k\}$ of $[1, \dots, n]$ and $|\mathcal{I}| = \binom{n-1}{k}$. Denote $\frac{\partial^k}{\partial^k z_\alpha} := \frac{\partial^k}{\partial z_{i_1} \dots \partial z_{i_k}}$. Then (3.6) can be written as

$$S(p_j *_z P_\alpha, P_\beta) = \frac{\partial}{\partial z_j} \frac{\partial^{2k} L}{\partial^k z_\alpha \partial^k z_\beta}(z), \qquad \alpha, \beta \in \mathcal{I}. \tag{3.7}$$

Denote $S_{\alpha,\beta} = S(P_\alpha, P_\beta)$. Let $(S^{\alpha,\beta})_{\alpha,\beta\in\mathcal{I}}$ be the matrix inverse to the matrix $(S_{\alpha,\beta})_{\alpha,\beta\in\mathcal{I}}$. Introduce the matrix $(M^\gamma_{j,\alpha})_{\alpha,\gamma\in\mathcal{I}}$ of the operator $p_j(z)*_z$ by the formula $p_j(z) *_z P_\alpha = \sum_{\gamma\in\mathcal{I}} M^\gamma_{j,\alpha}(z) P_\gamma$. Then

$$M^\gamma_{j,\alpha} = \sum_{\beta\in\mathcal{I}} \frac{\partial^{2k+1} L}{\partial z_j \partial^k z_\alpha \partial^k z_\beta} S^{\beta,\gamma}. \tag{3.8}$$

3.2.2 *Unit element axiom* We assume that for some $a \in \mathbb{C}^\times$, we have

$$\frac{1}{a} \sum_{j=1}^n z_j p_j(z) *_z = \mathrm{Id} \in \mathrm{End}(V), \tag{3.9}$$

that is, $\sum_{j=1}^n z_j M^\gamma_{j,\alpha}(z) = a\, \delta^\gamma_\alpha$.

3.2.3 *Commutativity axiom* We assume that the operators $p_1(z)*_z, \dots, p_n(z)*_z$ commute. In other words,

$$\sum_\beta [M^\beta_{i,\alpha} M^\gamma_{j,\beta} - M^\beta_{j,\alpha} M^\gamma_{i,\beta}] = 0. \tag{3.10}$$

This is a quadratic relation between the $(2k+1)$-st derivatives of the potential.

3.2.4 *Relation between $p_j(z)$ and $P_{i_1,\dots,i_k}$ axiom* We assume that we have

$$p_{i_1}(z) *_z \cdots *_z p_{i_k}(z) = P_{i_1,\dots,i_k}, \tag{3.11}$$

for every z and all unordered k-element subsets $\{i_1, \dots, i_k\} \subset [1, \dots, n]$.

More precisely, let $p_j(z) = \sum_{\alpha\in\mathcal{I}} C^\alpha_j(z) P_\alpha$, $j = 1, \dots, n$, be the expansion of the vector $p_j(z)$ with respect to the basis $\{P_\alpha\}_{\alpha\in\mathcal{I}}$. Then equation (3.11) is an equation on the coefficients $C^\alpha_{i_k}(z)$ and $M^\beta_{i_m,\alpha}(z)$, where $\alpha, \beta \in \mathcal{I}$, $m = i_1, \dots, i_{k-1}$. For example, if $P_{i_1,\dots,i_k}$ is one of the basis vectors $(P_\alpha)_{\alpha\in\mathcal{I}}$, then

$$\sum_{\alpha_2,\dots,\alpha_k\in\mathcal{I}} C^{\alpha_k}_{i_k}(z) M^{\alpha_{k-1}}_{i_{k-1},\alpha_k}(z) M^{\alpha_{k-2}}_{i_{k-2},\alpha_{k-1}}(z) \dots M^{\alpha_1}_{i_1,\alpha_2}(z) = \delta^{\alpha_1}_{i_1,\dots,i_k}. \tag{3.12}$$

REMARK If $k = 1$, then this axiom says that $p_j = P_j$ for $j \in [1, \dots, n]$. In this case the commutativity axiom follows from (3.6).

Define the multiplication on V by the formula

$$P_{i_1,\dots,i_k} *_z P_{j_1,\dots,j_k} = p_{i_1}(z) *_z p_{i_2}(z) *_z \cdots *_z p_{i_k}(z) *_z P_{j_1,\dots,j_k}. \tag{3.13}$$

The multiplication is well-defined due to relations (3.3)–(3.5).

LEMMA 3.4 The multiplication on V is commutative.

Proof. Indeed, if the subsets $\{i_1,\dots,i_k\}$ and $\{j_1,\dots,j_k\}$ have a non-empty intersection, say $i_k = j_k$, then

$$\begin{aligned} P_{i_1,\dots,i_{k-1},i_k} *_z P_{j_1,\dots,j_{k-1},i_k} &= (p_{i_1} *_z \cdots *_z p_{i_{k-1}}) *_z p_{i_k} *_z (p_{j_1} *_z \cdots *_z p_{j_{k-1}}) *_z p_{i_k} \\ &= (p_{j_1} *_z \cdots *_z p_{j_{k-1}}) *_z p_{i_k} *_z (p_{i_1} *_z \cdots *_z p_{i_{k-1}}) *_z p_{i_k} = P_{j_1,\dots,j_{k-1},i_k} *_z P_{i_1,\dots,i_{k-1},i_k}. \end{aligned}$$

If the subsets $\{i_1,\dots,i_k\}$ and $\{j_1,\dots,j_k\}$ do not intersect, then

$$\begin{aligned} P_{i_1,\dots,i_k} *_z P_{j_1,\dots,j_k} &= p_{j_1} *_z \cdots *_z p_{j_{k-1}} *_z p_{i_1} *_z (p_{i_2} \cdots *_z p_{i_{k-1}} *_z p_{i_k} *_z p_{j_k}) \\ &= p_{j_1} *_z \cdots *_z p_{j_{k-1}} *_z p_{i_1} *_z P_{i_2,\dots,i_{k-1},i_k,j_k} = p_{j_1} *_z \cdots *_z p_{j_{k-1}} *_z p_{i_1} *_z P_{i_2,\dots,i_{k-1},j_k,i_k} \\ &= P_{j_1,\dots,j_k} *_z P_{i_1,\dots,i_k}. \end{aligned}$$

□

For $\alpha, \beta \in \mathcal{I}$, let $P_\alpha *_z P_\beta = \sum_{\gamma\in\mathcal{I}} N^\gamma_{\alpha,\beta}(z) P_\gamma$. If $\alpha = \{i_1,\dots,i_k\}$ then

$$N^\gamma_{\alpha,\beta}(z) = \sum_{\beta_2,\dots,\beta_k\in\mathcal{I}} M^{\beta_k}_{i_k,\beta}(z) M^{\beta_{k-1}}_{i_{k-1},\beta_k}(z) \dots M^{\beta_2}_{i_2,\beta_3}(z) M^{\gamma}_{i_1,\beta_2}(z). \tag{3.14}$$

This is a polynomial of degree k in the $(2k+1)$-st derivatives of the potential.

3.2.5 *Associativity axiom* We assume that the multiplication on V is associative, that is,

$$\sum_{\gamma\in\mathcal{I}} [N^\gamma_{\alpha_2,\alpha_3} N^\beta_{\alpha_1,\gamma} - N^\gamma_{\alpha_1,\alpha_2} N^\beta_{\gamma,\alpha_3}] = 0. \tag{3.15}$$

THEOREM 3.5 For any z, the axioms 3.2.1–3.2.5 define on the vector space V the structure $*_z$ of a commutative associative algebra with unit element $1_z = \frac{1}{a}\sum_{j=1}^n z_j p_j(z)$. The algebra is Frobenius, $S(u *_z v, w) = S(u, v *_z w)$, for all $u, v, w \in V$. The elements $p_1(z),\dots,p_n(z)$ generate $(V, *_z)$ as an algebra. □

3.2.6 *Homogeneity axiom* We assume that

$$\sum_{j=1}^n z_j \frac{\partial}{\partial z_j} L \;=\; 2kL \;+\; \frac{a^{2k+1}}{(2k)!} S(1_z, 1_z),$$

where $a \in \mathbb{C}^\times$ is the same number as in the unit element axiom.

3.2.7 *Lifting axiom* For $\kappa \in \mathbb{C}^\times$, define on the trivial bundle $V \times \mathbb{C}^n \to \mathbb{C}^n$ a connection ∇^κ by the formula:

$$\nabla_j^\kappa = \frac{\partial}{\partial z_j} - \kappa\, p_j(z)*_z, \qquad j = 1, \dots, n. \tag{3.16}$$

LEMMA 3.6 For any $\kappa \in \mathbb{C}^\times$, the connection ∇^κ is flat, that is, $[\nabla_i^\kappa, \nabla_j^\kappa] = 0$ for all i, j.

Proof. The flatness for any κ is equivalent to the commutativity of the operators $p_i(z)*_z$, $p_j(z)*_z$ and the identity $\frac{\partial}{\partial z_j}(p_i(z)*_z) = \frac{\partial}{\partial z_i}(p_j(z)*_z)$. The commutativity is our Axiom 3.2.3 and the identity follows from formula (3.8). □

A flat section $s(z_1, \dots, z_n) \in V$ of the connection ∇^κ is a solution of the system of equations

$$\frac{\partial s}{\partial z_j} - \kappa\, p_j(z) *_z s = 0, \qquad j = 1, \dots, n. \tag{3.17}$$

We say that a flat section is *liftable* if it has the form

$$s = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} \frac{\partial^k g}{\partial z_{i_1} \dots \partial z_{i_k}} d^2_{i_1,\dots,i_k} P_{i_1,\dots,i_k} \tag{3.18}$$

for a scalar function $g(z_1, \dots, z_n)$. If a section is liftable, the function $g(z)$ is not unique (an arbitrary polynomial of degree less than k can be added). Such a function g will be called a *twisted period*.

We assume that for generic κ all flat sections of ∇^κ are liftable and the set of all exceptional κ is contained in a finite union of arithmetic progressions in $\mathbb{C}$.

3.2.8 *Differential operator e axiom* We assume that there exists a differential operator

$$e = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} e_{i_1,\dots,i_k}(z) \frac{\partial^k}{\partial z_{i_1} \dots \partial z_{i_k}} \tag{3.19}$$

such that the operator $g(z) \mapsto (eg)(z)$ acts as the shift $\kappa \mapsto \kappa - 1$ on the space of solutions of equations (3.17) and (3.18). We also require that the element

$$\tilde{e} = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} e_{i_1,\dots,i_k}(z) P_{i_1,\dots,i_k} \in V \tag{3.20}$$

is invertible for every z.

Notice that the operator e annihilates all polynomials in z of degree less than k.

3.3 *Remark*

The axioms of an almost Frobenius structure in Section 2 are similar to the axioms of a Frobenius like structure in Section 3 for $k = 1$. The role of the tangent bundle TM and the vectors $\frac{\partial}{\partial z_j}$ in Section 2 is played by the trivial bundle $V \times \mathbb{C}^n \to \mathbb{C}^n$ and the vectors $p_j(z) \in V$ as well as the vectors $\frac{\partial}{\partial z_j} \in T\mathbb{C}^n$ in Section 3.

4. An example of a Frobenius like structure

4.1 *Arrangements of hyperplanes*

Consider a family of arrangements $\mathcal{C}(z) = \{H_i(z)\}_{i=1,\dots,n}$ of n affine hyperplanes in $\mathbb{C}^k$. The arrangements of the family depend on parameters $z = (z_1, \dots, z_n)$. When the parameters change, each hyperplane moves parallelly to itself. More precisely, let $t = (t_1, \dots, t_k)$ be coordinates on $\mathbb{C}^k$. For $i = 1, \dots, n$, the hyperplane $H_i(z)$ is defined by the equation

$$f_i(t, z) := \sum_{j=1}^{k} b_i^j t_j + z_i = 0, \qquad \text{where } b_i^j \in \mathbb{C} \text{ are given}.$$

When z_i changes the hyperplane $H_i(z)$ moves parallelly to itself.

We assume that every k hyperplanes intersect transversally, that is, for every distinct $i_1, \dots, i_k$, we assume that

$$d_{i_1,\dots,i_k} := \det_{l,j=1}^{k}(b_{i_l}^j) \neq 0.$$

For every $k+1$ distinct indices $i_1, \dots, i_{k+1}$, the corresponding hyperplanes have non-empty intersection if and only if

$$f_{i_1,\dots,i_{k+1}}(z) := \sum_{l=1}^{k+1} (-1)^{l-1} d_{i_1,\dots,\widehat{i_l},\dots,i_{k+1}} z_{i_l} = 0. \tag{4.1}$$

For any $1 \leqslant i_1 < \dots < i_{k+1} \leqslant n$ denote by $H_{i_1\dots,i_{k+1}}$ the hyperplane in $\mathbb{C}^n$ defined by the equation $f_{i_1,\dots,i_{k+1}}(z) = 0$. The union $\Delta = \cup_{1\leqslant i_1<\dots<i_{k+1}\leqslant n} H_{i_1,\dots,i_{k+1}} \subset \mathbb{C}^n$ is called the *discriminant*. We will consider only $z \in \mathbb{C}^n - \Delta$. In that case the arrangement $\mathcal{C}(z) = \{H_i(z)\}_{i=1,\dots,n}$ has normal crossings.

4.2 *Master function*

Fix complex numbers $a_1, \dots, a_n \in \mathbb{C}^\times$ such that $|a| := \sum_{j=1}^n a_j \neq 0$. The *master function* is the function

$$\Phi(t, z) = \sum_{j=1}^{n} a_j \log f_j(t, z).$$

For fixed z, the master function is defined on the complement to our arrangement $\mathcal{C}(z)$ as a multivalued holomorphic function, whose derivatives are rational functions. In particular, $\frac{\partial \Phi}{\partial z_i} = \frac{a_i}{f_i}$. For fixed z, define the *critical set*

$$C_z := \left\{ t \in \mathbb{C}^k - \cup_{j=1}^n H_j(z) \;\Big|\; \frac{\partial \Phi}{\partial t_j}(t, z) = 0,\ j = 1, \dots, k \right\}$$

and the algebra of functions on the critical set

$$A_z := \mathbb{C}(\mathbb{C}^k - \cup_{j=1}^n H_j(z)) \Big/ \left\langle \frac{\partial \Phi}{\partial t_j}(t, z),\ j = 1, \dots, k \right\rangle,$$

where $\mathbb{C}(\mathbb{C}^k - \cup_{j=1}^n H_j(z))$ is the algebra of regular functions on $\mathbb{C}^k - \cup_{j=1}^n H_j(z)$. It is known that $\dim A_z = \binom{n-1}{k}$. For example this follows from Theorem 2.4 and Lemmas 4.1, 4.2 in [9].

Denote by $*_z$ the operation of multiplication in A_z.

The *Grothendieck bilinear form* on A_z is the form

$$(g,h)_z := \frac{1}{(2\pi i)^k} \int_\Gamma \frac{gh\, dt_1 \wedge \cdots \wedge dt_k}{\prod_{j=1}^k \frac{\partial \Phi}{\partial t_j}} \quad \text{for } g,h \in A_z.$$

Here $\Gamma = \{t \mid |\frac{\partial \Phi}{\partial t_j}| = \epsilon,\ j = 1,\dots,k\}$, where $\epsilon > 0$ is small. The cycle Γ is oriented by the condition $d \arg \frac{\partial \Phi}{\partial t_1} \wedge \cdots \wedge d \arg \frac{\partial \Phi}{\partial t_k} > 0$, The pair $\big(A_z, (\ ,\ )_z\big)$ is a Frobenius algebra.

4.3 *Algebra A_z*

For $i = 1,\dots,n$, denote

$$p_i(z) = \left[\frac{\partial \Phi}{\partial z_i}(t,z)\right] = \left[\frac{a_i}{f_i(t,z)}\right] \in A_z, \qquad i = 1,\dots,n. \tag{4.2}$$

For any $i_1,\dots,i_k \in [1,\dots,n]$, denote

$$P_{i_1,\dots,i_k} := p_{i_1}(z) *_z \cdots *_z p_{i_k}(z) \in A_z \text{ and } w_{i_1,\dots,i_k} := d_{i_1,\dots,i_k} p_{i_1}(z) *_z \cdots *_z p_{i_k}(z) \in A_z.$$

THEOREM 4.1

(i) The elements $p_1(z),\dots,p_n(z)$ generate A_z as an algebra and for any $i_2,\dots,i_k$ we have

$$\sum_{j=1}^n d_{j,i_2,\dots,i_k} p_j(z) = 0. \tag{4.3}$$

(ii) The element

$$1_z := \frac{1}{|a|} \sum_{j=1}^n z_j p_j(z) \tag{4.4}$$

is the unit element of the algebra A_z.

(iii) The set of all elements $w_{i_1,\dots,i_k}$ generates A_z as a vector space. The only linear relations between these elements are

$$w_{i_1,\dots,i_k} = (-1)^\sigma w_{\sigma i_1,\dots,\sigma i_k}, \quad \forall \sigma \in S_k, \tag{4.5}$$
$$\sum_{j=1}^n w_{j,i_2,\dots,i_k} = 0, \qquad \forall\, i_2,\dots,i_k,$$

in particular, the linear relations between these elements do not depend on z.

(iv) The Grothendieck form has the following matrix elements:

$$(w_{i_1,\dots,i_k}, w_{j_1,\dots,j_k})_z = 0, \qquad \text{if } |\{i_1,\dots,i_k\}\cap\{j_1,\dots,j_k\}| < k-1, \tag{4.6}$$

$$(w_{i_1,\dots,i_k}, w_{i_1,\dots,i_{k-1},i_{k+1}})_z = \frac{(-1)^{k+1}}{|a|}\prod_{l=1}^{k+1} a_{i_l} \quad \text{if } i_1,\dots,i_{k+1} \text{ are distinct},$$

$$(w_{i_1,\dots,i_k}, w_{i_1,\dots,i_k})_z = \frac{(-1)^{k}}{|a|}\prod_{l=1}^{k} a_{i_l} \sum_{m\notin\{i_1,\dots,i_k\}} a_m \quad \text{if } i_1,\dots,i_k \text{ are distinct}\,.$$

Proof. The statements of the theorem are the statements of Lemmas 3.4, 6.2, 6.3, 6.6, 6.7 in [8] transformed with the help of the isomorphism α of Theorem 2.7 in [8], see also Corollary 6.16 in [8] and Theorems 2.12 and 2.16 in [10]. □

This theorem shows that the elements $w_{i_1,\dots,i_k}$ define on A_z a module structure over $\mathbb{Z}$ independent of z. With respect to this $\mathbb{Z}$-structure the Grothendieck form is constant.

REMARK The presence of this $\mathbb{Z}$-structure in A_z reflects the presence of the $\mathbb{Z}$-structure in the Orlik–Solomon algebra of the arrangement $\mathcal{C}(z)$, see [10, 11].

LEMMA 4.2 Multiplication in A_z is defined by the formulas

$$p_{i_1}(z) *_z w_{i_2,\dots,i_{k+1}} = \frac{d_{i_2,\dots,i_{k+1}}}{f_{i_1,i_2,\dots,i_{k+1}}(z)} \sum_{\ell=1}^{k+1} (-1)^{\ell+1} a_{i_\ell} w_{i_1,\dots,\widehat{i_\ell},\dots,i_{k+1}}, \tag{4.7}$$

if $i_1 \notin \{i_2,\dots,i_{k+1}\}$,

$$p_{i_1}(z) *_z w_{i_1,i_2,\dots,i_k} = -\sum_{m\notin\{i_1,\dots,i_k\}} p_{i_1}(z) *_z w_{m,i_2,\dots,i_k}, \tag{4.8}$$

where $f_{i_1,i_2,\dots,i_{k+1}}(z)$ is defined in (4.1).

Proof. This is Lemma 6.8 in [8]. □

THEOREM 4.3 For any $i_0 \in [1,\dots,n]$, the unit element $1_z \in A_z$ is given by the formula

$$1_z = \frac{1}{|a|^k} \sum_{\substack{1\leqslant i_1<\dots<i_k\leqslant n \\ i_0\notin\{i_1,\dots,i_k\}}} \frac{(f_{i_0,i_1,\dots,i_k}(z))^k}{\prod_{m=0}^{k}(-1)^m d_{i_0,i_1,\dots,\widehat{i_m},\dots,i_k}} w_{i_1,\dots,i_k}. \tag{4.9}$$

Proof. This is Theorem 6.12 in [8]. □

4.4 *Potential functions*

Denote

$$Q(z) := (1_z, 1_z)_z. \tag{4.10}$$

The function $(-1)^k Q(z)$ is called in [8] the *potential of first kind*. Theorem 3.11 in [8] says that for any $i_1, \dots, i_k, j_1, \dots, j_k \in [1, \dots, n]$, we have

$$(p_{i_1}(z) *_z \cdots *_z p_{i_k}(z), p_{j_1}(z) *_z \cdots *_z p_{j_k}(z))_z \;=\; \frac{|a|^{2k}}{(2k)!}\,\frac{\partial^{2k} Q}{\partial z_{i_1} \dots \partial z_{i_k} \partial z_{j_1} \dots \partial z_{j_k}}. \tag{4.11}$$

THEOREM 4.4 We have

$$Q(z) = \frac{1}{|a|^{2k+1}} \sum_{1 \leqslant i_1 < \cdots < i_{k+1} \leqslant n} \left(\prod_{m=1}^{k+1} \frac{a_{i_m}}{d^2_{i_1,\dots,\widehat{i_m},\dots,i_{k+1}}} \right) (f_{i_1,\dots,i_{k+1}}(z))^{2k}. \tag{4.12}$$

Compare formulas (4.22) and (5.58) in [8].

Proof. Denote by $Q^\vee$ the right-hand side of (4.12). Both $Q(z)$ and $Q^\vee(z)$ are homogeneous polynomials of degree $2k$ in $z_1, \dots, z_n$. The $2k$-th derivatives of $Q(z)$ are given by formulas (4.11) and (4.5). A direct verification shows that the $2k$-th derivatives of $Q^\vee(z)$ also are given by the same formulas (4.5). Hence $Q(z) = Q^\vee(z)$. □

Introduce the *potential function*

$$L(z_1, \dots, z_n) = \frac{1}{(2k)!} \sum_{1 \leqslant i_1 < \cdots < i_{k+1} \leqslant n} \left(\prod_{m=1}^{k+1} \frac{a_{i_m}}{d^2_{i_1,\dots,\widehat{i_m},\dots,i_{k+1}}} \right) (f_{i_1,\dots,i_{k+1}}(z))^{2k} \log(f_{i_1,\dots,i_{k+1}}(z)). \tag{4.13}$$

In [8] this function was called the *potential of second kind*.

THEOREM 4.5 ([8]) For any $i_0, i_1, \dots, i_k, j_1, \dots, j_k \in [1, \dots, n]$, we have

$$(p_{i_0}(z) *_z p_{i_1}(z) *_z \cdots *_z p_{i_k}(z), p_{j_1}(z) *_z \cdots *_z p_{j_k}(z))_z \;=\; \frac{\partial^{2k+1} L}{\partial z_{i_0} \partial z_{i_1} \dots \partial z_{i_k} \partial z_{j_1} \dots \partial z_{j_k}}. \tag{4.14}$$

This is Theorem 6.20 in [8].

LEMMA 4.6 For any $i_2, \dots, i_k \in [1, \dots, n]$, we have

$$\sum_{j=1}^{n} d_{j,i_2,\dots,i_k} \frac{\partial L}{\partial z_j} = 0. \tag{4.15}$$

Proof. The lemma follows from the Plücker relations (3.2). □

LEMMA 4.7 We have

$$\sum_{j=1}^{n} z_j \frac{\partial L}{\partial z_j} \;=\; 2k\,L \;+\; \frac{|a|^{2k+1}}{(2k)!}\,(1_z, 1_z)_z. \tag{4.16}$$

Proof. The lemma follows from the fact that each $f_{i_1,\dots,i_{k+1}}$ is a homogeneous polynomial in $z_1, \dots, z_n$ of degree one and from Theorem 4.4. □

4.5 *Bundle of algebras A_z*

The family of algebras A_z form a vector bundle over the space of parameters $z \in \mathbb{C}^n - \Delta$. The $\mathbb{Z}$-module structures of fibers canonically trivialize the vector bundle $\pi : \sqcup_{z\in\mathbb{C}^n-\Delta} A_z \to \mathbb{C}^n - \Delta$.

Let $s(z) \in A_z$ be a section of that bundle,

$$s(z) = \sum_{i_1,\dots,i_k} s_{i_1,\dots,i_k}(z) w_{i_1,\dots,i_k}, \tag{4.17}$$

where $s_{i_1,\dots,i_k}(z)$ are scalar functions. Define the *combinatorial connection* on the bundle of algebras by the rule

$$\frac{\partial s}{\partial z_j} := \sum_{i_1,\dots,i_k} \frac{\partial s_{i_1,\dots,i_k}}{\partial z_j} w_{i_1,\dots,i_k}, \qquad \forall\, j. \tag{4.18}$$

This derivative does not depend on the presentation (4.17). The combinatorial connection is *flat*. The combinatorial connection is compatible with the Grothendieck bilinear form, that is

$$\frac{\partial}{\partial z_j}\,(s_1,\ s_2)_z \;=\; \left(\frac{\partial}{\partial z_j} s_1,\ s_2\right)_z \;+\; \left(s_1,\ \frac{\partial}{\partial z_j} s_2\right)_z$$

for any j and sections s_1, s_2.

For $\kappa \in \mathbb{C}^\times$, introduce a connection ∇^κ on the bundle of algebras π by the formula

$$\nabla^\kappa_j := \frac{\partial}{\partial z_j} \;-\; \kappa\, p_j(z) *_z, \qquad j = 1, \dots, n.$$

This family of connections is a deformation of the combinatorial connection.

THEOREM 4.8 For any $\kappa \in \mathbb{C}^\times$, the connection ∇^κ is flat,

$$[\nabla^\kappa_j, \nabla^\kappa_m] = 0, \qquad \forall\, j, m.$$

Proof. This statement is a corollary of Theorems 5.1 and 5.9 in [9]. □

4.6 *Flat sections*

Given $\kappa \in \mathbb{C}^\times$, the flat sections

$$s(z) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} s_{i_1,\dots,i_k}(z) w_{i_1,\dots,i_k} \in A_z \tag{4.19}$$

of the connection ∇^κ are given by the following construction.

The function $e^{\kappa\Phi(t,z)} = (\prod_{i=1}^n f_i(t,z)^{a_i})^\kappa$ defines a rank one local system $\mathcal{L}_\kappa$ on $U(\mathcal{C}(z)) := \mathbb{C}^k - \cup_{i=1}^n H_i(z)$ the complement to the arrangement $\mathcal{C}(z)$, whose horizontal sections over open subsets of $U(z)$ are uni-valued branches of $e^{\kappa\Phi(t,z)}$ multiplied by complex numbers, see, for example, [12]. The vector bundle

$$\sqcup_{x\in\mathbb{C}^n-\Delta} H_k(U(\mathcal{C}(x)), \mathcal{L}_\kappa|_{U(\mathcal{C}(x))}) \to \mathbb{C}^n - \Delta, \tag{4.20}$$

called the *homology bundle*, has the canonical flat *Gauss–Manin connection*.

THEOREM 4.9 Assume that κ is such that $\kappa|a| \notin \mathbb{Z}_{\leqslant 0}$ and $\kappa a_j \notin \mathbb{Z}_{\leqslant 0}$ for $j = 1, \dots, n$, then for every flat section $s(z)$, there is a locally flat section of the homology bundle $\gamma(z) \in H_k(U(\mathcal{C}(x))$, such that the coefficients $s_{i_1,\dots,i_k}(z)$ in (4.19) are given by the following formula:

$$s_{i_1,\dots,i_k}(z) = \int_{\gamma(z)} e^{\kappa\Phi(t,z)} d\log f_{i_1} \wedge \cdots \wedge d\log f_{i_k}, \qquad \forall\, 1 \leqslant i_1 < \cdots < i_k \leqslant n. \tag{4.21}$$

Proof. This is Lemma 5.7 in [9]. The information on κ see in Theorem 1.1 in [13]. □

Theorem 4.9 identifies the connection ∇^κ with the Gauss–Manin connection on the homology bundle. The flat sections are labelled by the locally constant cycles $\gamma(z)$ in the complement to our arrangement. The space of flat sections is identified with the degree k homology group of the complement to the arrangement $\mathcal{C}(z)$. The monodromy of the flat sections is the monodromy of that homology group.

4.7 *Special case*

Assume that $a_j = 1$ for all $j \in [1, \dots, n]$. Then $|a| = n$. Denote

$$\begin{aligned} g(z_1, \dots, z_n, \kappa) &= \kappa^{-k} \int_{\gamma(z)} e^{\kappa\Phi(t,z)} dt_1 \wedge \cdots \wedge dt_k \\ &= \kappa^{-k} \int_{\gamma(z)} \left(\prod_{j=1}^n f_j(t,z) \right)^\kappa dt_1 \wedge \cdots \wedge dt_k. \end{aligned} \tag{4.22}$$

LEMMA 4.10 If $a_j = 1$ for all $j \in [1, \dots, n]$, then the flat section of Theorem 4.9 can be written in the form

$$s(z) = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} \frac{\partial^k g(z, \kappa)}{\partial z_{i_1} \dots \partial z_{i_k}} d^2_{i_1,\dots,i_k} P_{i_1,\dots,i_k}, \tag{4.23}$$

cf. (3.18). □

COROLLARY 4.11 If $a_j = 1$ for all $j \in [1, \dots, n]$ and $\kappa \notin \frac{1}{n}\mathbb{Z}_{\leqslant 0}$, then all flat sections of the connection ∇^κ have the form (4.22) and (4.23).

THEOREM 4.12 Assume that $a_j = 1$ for all $j \in [1, \dots, n]$, then there exists a differential operator

$$e = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} e_{i_1,\dots,i_k}(z) \frac{\partial^k}{\partial z_{i_1} \dots \partial z_{i_k}} \tag{4.24}$$

such that the operator $g(z) \mapsto (eg)(z)$ acts as the shift $\kappa \mapsto \kappa - 1$ on the space of all flat sections (4.22) and (4.23). Moreover, we have

$$\tilde{e} := \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} e_{i_1,\dots,i_k}(z) P_{i_1,\dots,i_k} = \prod_{j=1}^{n} \left[\frac{1}{f_j}\right] \in A_z, \tag{4.25}$$

and the element $\tilde{e}$ is invertible in A_z.

Proof. We have

$$g(z_1, \dots, z_n, \kappa - 1) = \int_{\gamma(z)} \left(\prod_{j=1}^{n} f_j(t, z)\right)^{\kappa} \frac{1}{\prod_{j=1}^{n} f_j(t, z)} dt_1 \wedge \dots \wedge dt_k.$$

Hence it is enough to find the functions $e_{i_1,\dots,i_k}(z)$ such that

$$\frac{1}{\prod_{j=1}^{n} f_j(t, z)} = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} e_{i_1,\dots,i_k}(z) \frac{1}{f_{i_1}(t, z) \dots f_{i_k}(t, z)}. \tag{4.26}$$

Hence, for every z, the coefficients must be of the form

$$e_{i_1,\dots,i_k} = d_{i_1,\dots,i_k} \operatorname{Res}_{f_{i_1}(t,z)=0} \dots \operatorname{Res}_{f_{i_k}(t,z)=0} \left(\frac{dt_1 \wedge \dots \wedge dt_k}{\prod_{j=1}^{n} f_j(t, z)}\right). \tag{4.27}$$

Conversely, if we choose such coefficients, then formula (4.26) holds. Clearly, the element $\tilde{e} = \prod_{j=1}^{n} \left[\frac{1}{f_j}\right]$ is invertible in A_z. □

4.8 *Summary*

Fix $a_1, \ldots, a_n \in \mathbb{C}^\times$, such that $|a| \neq 1$. Then the matrix (b^i_j), algebra A_z, bilinear form $(\ ,\ )_z$, vectors $p_j(z)$, vectors $P_{i_1,\ldots,i_k}$, element 1_z, function $L(z_1, \ldots, z_n)$ of Sections 4.1–4.4 satisfy Axioms 3.2.1–3.2.6.

If in addition we assume that $a_j = 1$ for all $j \in [1, \ldots, n]$, then these objects also satisfy Axioms 3.2.7 and 3.2.8.

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