

Is a typical bi-Perron algebraic unit a pseudo-Anosov dilatation?

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ABSTRACT. In this note, we deduce a partial answer to the question in the title. In particular, we show that asymptotically almost all bi-Perron algebraic unit whose characteristic polynomial has degree at most $2n$ do not correspond to dilatations of pseudo-Anosov maps on a closed orientable surface of genus n for $n \geq 10$. As an application of the argument, we also obtain a statement on the number of closed geodesics of the same length in the moduli space of area one abelian differentials for low genus cases.

Thurston's celebrated theorem [19] classified self-homeomorphisms of compact orientable surfaces up to isotopy. As long as a homeomorphism does not admit a finite power which is isotopic to the identity and does not admit a finite union of disjoint simple closed curves which are preserved up to isotopy, it is isotopic to a so-called *pseudo-Anosov map*. Associated to a pseudo-Anosov map is a positive number called its *dilatation*. The logarithm of the dilatation of a pseudo-Anosov map is the topological entropy of the map. See [21] for details.

We call a number λ to be a *pseudo-Anosov dilatation* if it is the dilatation constant for a pseudo-Anosov map on some compact orientable surface. We call an algebraic integer λ *bi-Perron* if all the Galois conjugates of λ are contained in an annulus of outer radius λ itself and inner radius $1/\lambda$. Fried [7] showed that every pseudo-Anosov dilatation is a bi-Perron algebraic unit, and he conjectured that this is also a sufficient condition: every bi-Perron algebraic unit is realized as the dilatation constant of some pseudo-Anosov map.

This turned out to be a difficult question and there are many related works. Thurston [20] gave an answer to an $\text{Out}(F_n)$ -version of this question. In [20], Thurston also gave a few examples of constructions of pseudo-Anosov maps for given bi-Perron algebraic units (here the numbers are given as the slopes of some post-critically finite piecewise-linear self-homeomorphisms of the unit interval). These examples motivated the authors' previous work to construct pseudo-Anosov maps in [2] for a bi-Perron algebraic unit given as the leading eigenvalue of a Perron-Frobenius matrix satisfying some additional properties. Both constructions in [20] and [2] are similar to the one given in [3]. See also, for instance, [1], [15], [17], [14], [11], [12], [13], [18] for various constructions of pseudo-Anosov maps.

In this short note, we formulate an easier version of Fried's question, and give a statistical answer to it. To state our result precisely, we introduce some notation.

Let $n \geq 2$ be fixed, and R be any positive real number. Define $\mathcal{B}_n(R)$ to be the set of bi-Perron algebraic units no larger than R whose characteristic polynomial has degree at most $2n$. Here, by the characteristic polynomial of a bi-Perron algebraic unit λ , we mean the monic palindromic integral polynomial whose leading root is λ and has the lowest degree among all such polynomials. And let $\mathcal{D}_n(R)$ be the set of dilatations no larger than R of pseudo-Anosov maps with orientable invariant foliations on a closed orientable surface S_n of genus n . Similarly, let $\mathcal{D}'_n(R)$ be the set of dilatations no larger than R of pseudo-Anosov maps, not necessarily with orientable invariant foliation, on a closed orientable surface with genus n .

We remark that $\mathcal{D}_n(R)$ is contained in $\mathcal{B}_n(R)$, and $\mathcal{D}'_n(R)$ is contained in $\mathcal{B}_{3n-3}(R)$. A pseudo-Anosov dilatation λ on a surface of genus n is a root of an integral palindromic polynomial of degree at most $2n$ if its invariant foliations are orientable. This is because λ is the leading eigenvalue of the induced symplectic action on the homology group of the surface, which is \mathbb{Z}^{2n} . If we do not require the invariant foliation to be orientable, the upper bound on degree is $6n - 6$: we can reduce this case to the case of orientable foliation by taking a double cover of the surface, and this bound follows from the fact that a quadratic differential on a surface of genus n has at most $2n$ zeros which is due to Gauss-Bonnet together with the Riemann-Hurwitz formula.

Note that Fried's conjecture is equivalent to $\mathcal{B}_n(R)$ being contained in $\mathcal{D}_m(R)$ for some large enough m . But a priori, m could be arbitrarily large and we do not know how to prove or disprove the claim. Instead we show the following.

Theorem 1. *Let $\mathcal{B}_n(R)$, $\mathcal{D}'_n(R)$ and $\mathcal{D}_n(R)$ be as above. Then*

(1)

$$\lim_{R \rightarrow \infty} \frac{|\mathcal{D}_m(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} = 0,$$

where $4m - 3 \leq n(n + 1)/2$. In particular, $\lim_{R \rightarrow \infty} \frac{|\mathcal{D}_n(R)|}{|\mathcal{B}_n(R)|} = 0, \forall n \geq 6$.

(2)

$$\lim_{R \rightarrow \infty} \frac{|\mathcal{D}'_m(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} = 0,$$

where $6m - 6 \leq n(n + 1)/2$. In particular, $\lim_{R \rightarrow \infty} \frac{|\mathcal{D}'_n(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} = 0, \forall n \geq 10$.

Here $|A|$ means the cardinality of A for a finite set A .

Theorem 1 says that asymptotically almost all bi-Perron algebraic units whose characteristic polynomial has degree at most $2n$ do not correspond to dilatations of pseudo-Anosov maps on a surface of genus n for all $n \geq 10$,

and for $n \geq 6$ if the invariant foliation is further assumed to be orientable. It would be interesting to see if the statement still holds for lower genera.

Let Γ_n be the set of all periodic orbits for the Teichmüller flow on the moduli space of area one abelian differentials on S_n . Then there exists a surjective map $\Gamma_n \rightarrow \cup_{R>0} \mathcal{D}_n(R)$ defined by $\gamma \mapsto e^{\ell(\gamma)}$ where $\ell(\gamma)$ is the length of the orbit $\gamma \in \Gamma_n$. Let $\Gamma_n(R)$ be the preimage of $\mathcal{D}_n(R)$ of this map.

By $f \sim g$ we mean $\exists C$ such that $\frac{1}{C}f(x) \leq g(x) \leq Cf(x)$ when $x \gg 0$. By $f \lesssim g$ we mean $f = O(g)$ when $x \rightarrow \infty$. At our best knowledge, the following theorem is independently due to Eskin-Mirzakhani-Rafi [5] and Hamenstädt [9] (which is rephrased for our purpose).

Theorem 2 ([5], [9]). $|\Gamma_n(R)| \sim \frac{R^{4n-3}}{(4n-3)\log R}$.

We give a brief explanation of the above statement. In [5] and [9], it was stated that when restricted to each connected component of the strata of the moduli space of area one abelian differentials on S_n ,

$$\begin{aligned} |\Gamma_n(R)| &= |\{\gamma \in \Gamma_n : e^{\ell(\gamma)} \leq R\}| \\ &= |\{\gamma \in \Gamma_n : \ell(\gamma) \leq \log R\}| \\ &\sim \frac{e^{(2n+\ell-1)\log R}}{(2n+\ell-1)\log R} = \frac{R^{2n+\ell-1}}{(2n+\ell-1)\log R}. \end{aligned}$$

Here ℓ is the maximum number of zeros of an area one abelian differential on S_n , so it is $2n-2$. See [16], [22], [4] and [5] for the relevant background of this theorem.

As a result, we have $|\Gamma_n(R)| \sim \frac{R^{4n-3}}{(4n-3)\log R}$ on each connected component of the strata. But for fixed n , there exists only finite number of such components. Therefore we get $|\Gamma_n(R)| \sim \frac{R^{4n-3}}{(4n-3)\log R}$ without restricting to the components. Note that since n is fixed, we can just say $|\Gamma_n(R)| \sim \frac{R^{4n-3}}{\log R}$.

As a direct corollary, we have:

Corollary 3. $|\mathcal{D}_n(R)| \lesssim \frac{R^{4n-3}}{\log R}$.

In the exactly same way, we can obtain an analogue of Corollary 3 for $|\mathcal{D}'_n(R)|$ from the following theorem which is due to Eskin and Mirzakhani [4]:

Theorem 4 (Theorem 1.1, [4]). *The number of geodesics in the moduli space of genus n surface of length at most $\log(R) \sim \frac{R^{6n-6}}{(6n-6)\log(R)}$.*

And as a direct corollary, just like Corollary 3, we have:

Corollary 5. $|\mathcal{D}'_n(R)| \lesssim \frac{R^{6n-6}}{\log R}$.

We remark that this does not imply $|\mathcal{D}_n(R)| \sim \frac{R^{4n-3}}{\log R}$ or $|\mathcal{D}'_n(R)| \sim \frac{R^{6n-6}}{\log R}$, since each element in $\mathcal{D}_n(R)$ or $\mathcal{D}'_n(R)$ may correspond to a lot of different closed geodesics in the moduli space.

Now we study how $|\mathcal{B}_n(R)|$ grows. Let $P_n(R)$ be the set of Perron polynomials of degree n with roots no larger than R , ($x > 1$ is Perron if it is the root of a monic irreducible polynomial with integer coefficients, so that the other roots of the polynomial are (strictly) less than x in absolute value).

Lemma 6. $|P_n(R)| \sim R^{n(n+1)/2}$.

Proof. $|P_n(R)| \lesssim R^{n(n+1)/2}$: Due to Vieta's formula the absolute value of the coefficient of x^k of a monic, degree n polynomial with all roots no larger than R is $\lesssim R^{n-k}$. Hence, the total number of such polynomials must be $\lesssim \prod_k R^{n-k} = R^{n(n+1)/2}$.

$R^{n(n+1)/2} \lesssim |P_n(R)|$: Let a_k be the coefficient of x^k in a degree n monic polynomial (so $a_n = 1$). By Rouché's theorem, when

$$(1) \quad 1 > \left| \frac{a_0}{R^n} \right| + \left| \frac{a_1}{R^{n-1}} \right| + \cdots + \left| \frac{a_{n-1}}{R} \right|$$

$$(2) \quad \left(\frac{1}{2} \right)^{n-1} \left| \frac{a_{n-1}}{R} \right| > \left| \frac{a_0}{R^n} \right| + \cdots + \left| \frac{a_{n-2}}{R^2} \right| \left(\frac{1}{2} \right)^{n-2} + \left(\frac{1}{2} \right)^n$$

and

$$(3) \quad \left(\frac{1}{3} \right)^{n-1} \left| \frac{a_{n-1}}{R} \right| > \left| \frac{a_0}{R^n} \right| + \cdots + \left| \frac{a_{n-2}}{R^2} \right| \left(\frac{1}{3} \right)^{n-2} + \left(\frac{1}{3} \right)^n$$

one root λ of this polynomial has magnitude between $R/2$ and R while all other roots have magnitude smaller than $R/3$. Hence λ must be real. If $\lambda < 0$ one can multiply $(-1)^{n-k}$ to a_k to get a polynomial with a root $-\lambda$. Hence, half of those polynomials have a leading positive real root, so they are in $P_n(R)$.

The inequalities (1), (2) and (3) are satisfied if and only if the point $(a_0/R^n, \dots, a_{n-1}/R)$ lies in a non-empty open subset $U \subset [-1, 1]^n$. The number of such points as $R \rightarrow \infty$ converges to the volume of this open subset divided by the co-volume of the lattice $\mathbb{Z}/R^n \times \mathbb{Z}/R^{n-1} \times \cdots \times \mathbb{Z}/R$, and the co-volume of this lattice is $R^{-n(n+1)/2}$. □

Lemma 7. $\lim_{R \rightarrow \infty} \frac{|\{\text{reducible elements in } P_n(R)\}|}{|P_n(R)|} = 0$

Proof. Because any reducible monic integer polynomial can be written as the product of two monic integer polynomials of lower degree, we have:

$$|\{\text{reducible elements in } P_n(R)\}| \leq \sum_k |b_k(R)| |b_{n-k}(R)|$$

where $b_k(R)$ is the set of monic polynomials with roots bounded by R . The first part of the proof of the previous lemma implies that $|b_k(R)| \lesssim R^{k(k+1)/2}$, hence $|\{\text{reducible elements in } P_n(R)\}| \sim o(R^{n(n+1)/2})$. □

Let $\mathcal{P}_n(R)$ be the set of Perron numbers of degree n no larger than R . They have one-one correspondence with irreducible elements in $P_n(R)$ which by Lemma 7 constitute almost all of $P_n(R)$ asymptotically. Hence by Lemma 6, $|\mathcal{P}_n(R)| \sim R^{n(n+1)/2}$.

Lemma 8. $|\mathcal{B}_n(R)| \sim R^{n(n+1)/2}$.

Proof. When $x > 1$, $1/x < 1$. Hence, $x \in \mathcal{B}_n(R)$ implies that $x + 1/x \in \mathcal{P}_n(R+1)$, hence $|\mathcal{B}_n(R)| \lesssim (R+1)^{n(n+1)/2} \sim R^{n(n+1)/2}$.

On the other hand, from the proof of Lemma 6 and the discussion above, the number of Perron numbers of degree n which lie between R and $R/2$ and have all conjugates smaller than $R/3$ is $\sim R^{n(n+1)/2}$. When R is sufficiently large, each such Perron number y correspond to a bi-Perron number x by the relation $x + 1/x = y$. And in fact x is an algebraic unit (so it is in $\mathcal{B}_n(R)$). Note that both roots of the polynomial $x^2 - yx + 1$ are algebraic integers, since it is a monic polynomial with algebraic integer coefficients. Furthermore, the product of these roots is 1, so they must be algebraic units. Hence $\mathcal{B}_n(R) \gtrsim R^{n(n+1)/2}$. \square

Now we are ready to prove our main theorem.

Proof of Theorem 1. By Corollary 3 and Lemma 8, we get

$$\frac{|\mathcal{D}_m(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} \lesssim \frac{R^{4m-3}}{\log R \cdot R^{n(n+1)/2}}.$$

The right-hand side goes to 0 as R goes to $+\infty$ as long as we have $4m-3 \leq n(n+1)/2$. When $m = n$, this inequality is satisfied if and only if $n \geq 6$ (recall that n is always assumed to be at least 2), which proved part (1). Part (2) follows from the same argument above but using Corollary 5 instead of Corollary 3. \square

Recall that Γ_n is the set of all closed geodesics in the moduli space of area one abelian differentials on the surface S_n , and $\Gamma_n(R)$ is the subset of Γ_n which consists of the closed geodesics of length no larger than $\log R$. For each $\gamma \in \Gamma_n$, let m_γ be the number $|\{\gamma' \in \Gamma_n : \ell(\gamma') = \ell(\gamma)\}|$.

We remark that if m_γ were uniformly bounded, one could have obtained Theorem 1 (1) for $n \geq 2$ instead of $n \geq 6$ using Theorem 1 (1) of [10]. But at least in the low genus cases, this is not true. As an application of our argument, we obtain the following theorem.

Theorem 9. *Suppose $n \leq 5$. For any positive integer k , the set $\{\gamma \in \Gamma_n : m_\gamma \geq k\}$ is typical, i.e.,*

$$\lim_{R \rightarrow \infty} \frac{|\{\gamma \in \Gamma_n(R) : m_\gamma \geq k\}|}{|\Gamma_n(R)|} \rightarrow 1.$$

Proof. Suppose not. Then for some k , we have

$$\limsup \frac{|\{\gamma \in \Gamma_n(R) : m_\gamma < k\}|}{|\Gamma_n(R)|} > 0.$$

But this implies that

$$\limsup \frac{|\mathcal{D}_n(R)|}{|\Gamma_n(R)|} \geq \limsup \frac{\frac{1}{k} |\{\gamma \in \Gamma_n(R) : m_\gamma < k\}|}{|\Gamma_n(R)|} > 0.$$

On the other hand, we know that $\mathcal{D}_n(R) \subset \mathcal{B}_n(R)$. As a consequence, $\lim \frac{|\mathcal{B}_n(R)|}{|\Gamma_n(R)|} \geq \limsup \frac{|\mathcal{D}_n(R)|}{|\Gamma_n(R)|}$. By Corollary 3 and Lemma 8, we get

$$\frac{|\mathcal{B}_n(R)|}{|\Gamma_n(R)|} \sim \frac{R^{n(n+1)/2} \log R}{R^{4n-3}} \rightarrow 0, \text{ for } n \leq 5,$$

a contradiction. □

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