Pramana–J. Phys. () :
DOI

Results on Hilfer Fractional Switched Dynamical System with Non-Instantaneous Impulses

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MS received 19 February 2022; revised 04 April 2022; accepted 07 April 2022

Abstract. This manuscript concerns the existence, uniqueness, Ulam's Hyer (UH) stability, and total controllability results for the Hilfer fractional switched impulsive systems in the finite-dimensional spaces. Mainly, this manuscript can be divided into three parts. In the first part, we examine the existence of a unique solution. In the second part, we establish the UH stability results, and in the third segment, we study the total controllability results. The main outcomes are acquired by utilizing the nonlinear analysis, fractional calculus, Mittag-Leffler function and Banach contraction principle. Finally, we have given two examples to validate the obtained analytical results.

Keywords. Existence, Stability, Controllability, Fractional impulsive differential equations.

PACS Nos 04.20.Ex; 02.30.Yy; 45.10.Hj; 87.19.rp

1. Introduction

There are many physical phenomena of science and engineering, for example, control theory, neural networks, population dynamics, mechanical systems and biological systems in which the states of the system change rapidly at some moments by some external effects, these changes are called the impulsive effects in the system. Recently, differential equations with impulsive effects have attracted significant attention because of huge applications in different areas of engineering and science, for example, networked control systems, ecology, population dynamics, biotechnology and so on [1, 2]. In the literature, the impulsive systems are comprehensively characterized into two classes; the first is the instantaneous impulsive systems where the sudden changes occur in the system for a small portion of time, for example in shocks, natural disasters and heart pulsate [3, 4, 5]. Second is the non-instantaneous impulsive systems where the length of such unexpected changes continues throughout a little timespan. For further study on non-instantaneous impulses, one can go through [6, 7, 8, 9, 10, 11].

The theory of fractional differential equations is an advanced and more generalized version of differential equation theory. Over the most recent twenty years, fractional calculus has attracted numerous physicists, engineers, mathematicians and notable contributions have been made to both applications and theory of fractional differential equations [12, 13]. However, the applications of fractional calculus and their outcomes vary as much as the definitions of fractional derivatives and integrals such as Riesz-Caputo, Grunwald-Letnikov, Caputo, Riemann-Liouville, Caputo-Fabrizio, Hadamard, Weyl, Chen, and so on. For the fundamental study of fractional systems, one can go through [14, 15, 16] and references therein. More recently, Hilfer [17] introduced a new fractional derivative by including both Caputo fractional derivative and Riemann-Liouville knows as Hilfer fractional derivative. This definition made a significant challenge to its realization but soon it discovered its way into many applications of engineering and science, for example, mechanical engineering and thermal science. In the last few years, many authors considered the Hilfer fractional differential equations and investigated many results such as the existence of solutions, data dependency and stability results

[18, 19, 20, 21, 22, 23, 24].

The concept of controllability was given by Kalman in 1960 and soon became an active area of examination by various researchers. Many problems of control theory, for example, stabilizability, optimal control and pole-assignment problems may be examined under the assumption that the system is controllable. The concept of controllability denotes the ability to move the state of the dynamical control system from an initial state to the desired final state by using a suitable control function. Recently, the issue of controllability for different kinds of dynamical systems of an integer as well as fractional order has been broadly investigated by numerous researchers, for instance please see [25, 26, 27, 28, 29, 30, 31, 32] and the references cited therein. Furthermore, in [33], the author considered Hilfer fractional differential equations and investigated the controllability results by applying the Mönch fixed point method, semigroup theory and measures of noncompactness. In [34], the authors investigated the controllability results of Hilfer fractional neutral differential systems by using the fixed point theorem and measures of noncompactness. In [35], the authors investigated the existence and approximate controllability results for Hilfer fractional differential equations. In [36], the authors studied the approximate controllability of semilinear Hilfer fractional differential inclusions with instantaneous impulses by applying the fixed point method, multivalued analysis and semigroup theory. In [37], the authors established the controllability results of Hilfer fractional dynamic inclusions with the nonlocal and non-instantaneous impulsive conditions by applying the semigroup theory, fixed point method and multivalued analysis.

On the other side, various systems encountered in practice involve a coupling between continuous dynamics and discrete events. Dynamic systems in which these two types of dynamics coincide and cooperate are generally called hybrid dynamical systems. Switched systems represent a class of hybrid dynamical systems. A switched system is a dynamic system consisting of a family of continuous-time subsystems along with a switching rule that determines the switching among subsystems. Mathematically, these subsystems are generally described by a collection of differential equations or differences indexed. For instance, the following phenomena give rise to switching behavior: dynamics of a vehicle changing unexpectedly because of wheels bolting and opening on ice; airplane entering, intersection and leaving an air traffic control area; biological cells developing and separating; a thermostat turning the heat on and off; a valve or a power switch opening and closing [38, 39]. In the last few years, controllability results of switched dynamical systems with and without impulses have been examined by numerous authors, see for example [40, 41, 42] and the references cited therein. However, the above mentioned results cannot be easily extended to the case of Hilfer fractional switched dynamical systems with non-instantaneous impulses.

In practicality, there is no impulse that happens instantaneously rather it is non-instantaneous howsoever the season of the event is little. For example, in many biological real problems, the introduction of a drug or a vaccine in the bloodstream is a gradual process, since it starts abruptly but remains active for a finite time interval, then one is forced to consider the drug or vaccine as a non-instantaneous impulse [6, 10]; in the model of dam pollution, the main cause of dam pollution is the polluted river enters the dam which takes some time to reach the middle region of the dam. Since the introduction of the river water into the dam and the consequent absorption of the dam water are gradual and continuous processes so that non-instantaneous impulses take place [22]. Henceforth, it is beneficial to concentrate on a class of differential equations with non-instantaneous impulses. Motivated by the above facts, in this manuscript, we study the existence of a unique solution and UH type stability analysis of Hilfer fractional switched differential equation with the non-instantaneous impulsive condition of the following form:

$$\begin{aligned}
&D^{\varrho,\vartheta}_{\vartheta_i^+}\mathsf{y}(t) = \Lambda_{\sigma(t)}\mathsf{y}(t) + \mathcal{P}_{\sigma(t)}\left(t, \mathsf{y}(t)\right), \ t \in (\vartheta_i, t_{i+1}],\\
&\qquad i = 0, 1, \ldots, \jmath,\\
&\mathsf{y}(t) = \mathcal{G}_{\sigma(t)}(t, \mathsf{y}(t_i^-)), \ t \in (t_i, \vartheta_i], \ i = 1, 2, \ldots, \jmath, \qquad (1.1)\\
&I^{1-\varpi}_{0^+}\mathsf{y}(0) = \mathsf{y}_0, \ I^{1-\varpi}_{\vartheta_i^+}\mathsf{y}(\vartheta_i^+) = \mathcal{G}_{\sigma(t)}(\vartheta_i, \mathsf{y}(t_i^-))
\end{aligned}$$

and for the controllability results, we consider the following switched impulsive system:

$$\begin{aligned}
&D^{\varrho,\vartheta}_{\vartheta_i^+}\mathsf{y}(t) = \Lambda_{\sigma(t)}\mathsf{y}(t) + C_{\sigma(t)}\mathsf{v}(t) + \mathcal{P}_{\sigma(t)}\left(t, \mathsf{y}(t)\right),\\
&\qquad t \in (\vartheta_i, t_{i+1}], \ i = 0, 1, \ldots, \jmath,\\
&\mathsf{y}(t) = \mathcal{G}_{\sigma(t)}(t, \mathsf{y}(t_i^-)), \ t \in (t_i, \vartheta_i], \ i = 1, 2, \ldots, \jmath, \qquad (1.2)\\
&I^{1-\varpi}_{0^+}\mathsf{y}(0) = \mathsf{y}_0, \ I^{1-\varpi}_{\vartheta_i^+}\mathsf{y}(\vartheta_i^+) = \mathcal{G}_{\sigma(t)}(\vartheta_i, \mathsf{y}(t_i^-)),
\end{aligned}$$

where $D^{\varrho,\vartheta}_{\vartheta_i^+}$ denotes the left-sided Hilfer fractional derivative with lower limit at ϑ_i of the type $\varrho \in [0, 1]$ and order $\vartheta \in (0, 1)$. $\varpi = \varrho + \vartheta - \varrho\vartheta$. $\mathsf{y} \in \mathbb{R}^n$ is the state variable, $I = [0, T], T > 0$. ϑ_i and t_i satisfy the relation $0 = t_0 = \vartheta_0 < t_1 < \vartheta_1 < t_2 < \ldots < \vartheta_\jmath < t_{\jmath+1} = T$, $\mathsf{y}(t_i^+) = \lim_{h\to 0^+} \mathsf{y}(t_i + h)$ and $\mathsf{y}(t_i^-) = \lim_{h\to 0^+} \mathsf{y}(t_i - h)$ denote the right and left limit of $\mathsf{y}(t)$ at $t = t_i$ respectively, $\Lambda_{\sigma(t)}$ and $C_{\sigma(t)}$ are some matrices of order $n \times n$

and $n \times m$ respectively, $\mathsf{v} \in \mathbb{R}^m$ is the control function, $\mathcal{P}_{\sigma(t)}$ and $\mathcal{G}_{\sigma(t)}$ are some given functions.

The switching signal $\sigma : I \mapsto \{0, 1, \ldots, \jmath\}$ is assumed to be known. It only changes its values at switching times t_i. That is to say,

$$\sigma(t) = i, \ t_i \le t < t_{i+1}, \ i = 0, 1, \ldots, \jmath.$$

Therefore, by applying the above switching law in switched systems (1.1) and (1.2), we get the following systems

$$\begin{aligned} &D^{\varrho,\vartheta}_{\vartheta_i^+}\mathsf{y}(t) = \Lambda_i \mathsf{y}(t) + \mathcal{P}_i(t, \mathsf{y}(t)), \ t \in (\vartheta_i, t_{i+1}], \\ &\qquad\qquad i = 0, 1, \ldots, \jmath, \\ &\mathsf{y}(t) = \mathcal{G}_i(t, \mathsf{y}(t_i^-)), \ t \in (t_i, \vartheta_i], \ i = 1, 2, \ldots, \jmath, \\ &I^{1-\varpi}_{0^+}\mathsf{y}(0) = \mathsf{y}_0, \ I^{1-\varpi}_{\vartheta_i^+}\mathsf{y}(\vartheta_i^+) = \mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-)) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &D^{\varrho,\vartheta}_{\vartheta_i^+}\mathsf{y}(t) = \Lambda_i \mathsf{y}(t) + C_i \mathsf{v}(t) + \mathcal{P}_i(t, \mathsf{y}(t)), \ t \in (\vartheta_i, t_{i+1}], \\ &\qquad\qquad i = 0, 1, \ldots, \jmath, \\ &\mathsf{y}(t) = \mathcal{G}_i(t, \mathsf{y}(t_i^-)), \ t \in (t_i, \vartheta_i], \ i = 1, 2, \ldots, \jmath, \\ &I^{1-\varpi}_{0^+}\mathsf{y}(0) = \mathsf{y}_0, \ I^{1-\varpi}_{\vartheta_i^+}\mathsf{y}(\vartheta_i^+) = \mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-)), \end{aligned} \tag{1.4}$$

respectively. From now onwards, we will study the switched impulsive systems (1.3) and (1.4).

The main contributions can be highlighted as follows.

- We consider a class of switched Hilfer dynamic equation with non-instantaneous impulses.
- We investigate the existence of unique solution and Ulam's Hyer stability results for the considered system.
- Also, we studied the controllability results by introducing a new class of control function which control the system at the final time of the interval as well as at each of the impulse points, i.e., we studied the total controllability results.
- We used the fractional calculus, Mittag-Leffler function and fixed point theorem to study these results.
- Two simulated examples are given to illustrate the obtained analytical results.

The rest of the paper is formulated as follows: In Section 2., we give some basic definitions, notations and important lemmas. In Section 3. and Section 4., we examine the existence of a unique solution and UH stability analysis of the system (1.3), respectively. Section 5., is devoted to the study of the controllability results for the system (1.4). In the last Section 6., we give an example to show validity of the theoretical results.

2. Preliminaries and Definitions

Below we introduce some basic definitions, notations, lemmas and important results which are often used throughout the manuscript. Let $\mathbb{R}^n$ be the space of $n-$dimensional column vectors $\mathsf{y} = \mathrm{col}(\mathsf{y}_1, \mathsf{y}_2, \ldots, \mathsf{y}_n)$ with a norm $\|\cdot\|$. $C(I, \mathbb{R}^n)$ denotes the Banach space of all continuous functions $\mathcal{P} : I \to \mathbb{R}^n$ with the norm $\|\mathcal{P}\| = \sup_{t \in I} \|\mathcal{P}(t)\|$.

We define the Banach space of all piecewise continuous functions $PC_{1-\varpi}(I, \mathbb{R}^n) = \{\mathsf{y} : (t - t_i)^{1-\varpi}\mathsf{y}(t) \in C((t_i, t_{i+1}], \mathbb{R}^n), \ i = 0, 1, \ldots, \jmath$ and there exists $\mathsf{y}(t_i^-)$ and $\mathsf{y}(t_i^+), \ i = 1, 2, \ldots, \jmath$, with $\mathsf{y}(t_i^-) = \mathsf{y}(t_i)\}$ with the norm $\|\mathsf{y}\|_{PC_{1-\varpi}} = \sup_{t \in [a,b]} (t - a)^{1-\varpi}\|\mathsf{y}(t)\|$.

Definition 1 *[15] Let $f : [a, \infty) \to \mathbb{R}$ be a function. Then, the fractional Riemann-Liouville integral of f of order $p > 0$ with lower limit a is given by*

$$I^p_{a^+} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t - \varsigma)^{p-1} f(\varsigma) d\varsigma, \ t > a,$$

provided R.H.S of the above equation is point-wise defined on $[a, \infty)$. Here, $\Gamma(\cdot)$ denotes the usual Gamma function.

Definition 2 *[15] Let $f : [a, \infty) \to \mathbb{R}$ be a function. Then, the fractional Riemann-Liouville derivative of f of order $p > 0$ is defined by*

$$D^p_{a^+} f(t) = \frac{1}{\Gamma(n - p)} \frac{d^n}{dt^n} \int_a^t (t - \varsigma)^{n-1-p} f(\varsigma) d\varsigma, \ t > a,$$

where $n - 1 < p < n$.

Definition 3 *[15] Let $f : [a, \infty) \to \mathbb{R}$ be a function. Then, the Caputo fractional derivative of f of order $p > 0$ is defined by*

$${}^cD^p_{a^+} f(t) = D^p_{a^+}\left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \ t > a$$

where $n - 1 < p < n$.

Definition 4 *[17] Let $f : [a, \infty) \to \mathbb{R}$ be a function. Then, the generalized Riemann-Liouville fractional derivative (or Hilfer derivative) of f with the type $0 \le \varrho \le 1$ and order $0 < \vartheta < 1$ with lower limit a is defined by*

$$D^{\varrho,\vartheta}_{a^+} f(t) = (I^{\varrho(1-\vartheta)}_{a^+} \frac{d}{dt} (I^{(1-\varpi)}_{a^+} f))(t), \ \varpi = \varrho + \vartheta - \varrho\vartheta,$$

provided that the expression on the R.H.S. exists.

Definition 5 *[15] The Mittag-Leffler function is defined as*

$$E_{\varrho,\vartheta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\varrho+\vartheta)},\ z\in\mathbb{C},\ \varrho,\vartheta>0.$$

Also, the Laplace transform of Mittag-Leffler function is given by

$$\mathcal{L}\{t^{\varrho-1}E_{\varrho,\vartheta}(\pm at^{\varrho})\}(s)=\frac{s^{\varrho-\vartheta}}{s^{\varrho}\mp a}.$$

Definition 6 *[15] The Mittag-Leffler function for a matrix Λ of order $n\times n$ is defined as*

$$E_{\varrho,\vartheta}(\Lambda)=\sum_{k=0}^{\infty}\frac{\Lambda^k}{\Gamma(k\varrho+\vartheta)},\ z\in\mathbb{C},\ \varrho,\vartheta>0.$$

Also, the Laplace transform of matrix valued Mittag-Leffler function is given by

$$\mathcal{L}\{t^{\varrho-1}E_{\varrho,\vartheta}(\pm\Lambda t^{\varrho})\}(s)=\frac{s^{\varrho-\vartheta}}{s^{\varrho}\mp\Lambda}.$$

For the further study on fractional calculus, one can go through the books [14, 15].

Lemma 1 *Let Λ be a $n\times n$ matrix and $\mathcal{P}\in C(I,\mathbb{R}^n)$ be a function. Then, the solution of the following Hilfer fractional system*

$$\begin{aligned}&D_{0^+}^{\varrho,\vartheta}\mathrm{y}(t)=\Lambda\mathrm{y}(t)+\mathcal{P}(t),\ t\in(0,T],\\ &I_{0^+}^{1-\varpi}\mathrm{y}(0)=\mathrm{y}_0,\ \varpi=\varrho+\vartheta-\varrho\vartheta,\end{aligned}\tag{2.5}$$

is

$$\mathrm{y}(t)=t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda t^{\vartheta})\mathrm{y}_0+\int_0^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda(t-\varsigma)^{\vartheta})\mathcal{P}(\varsigma)d\varsigma$$

for all $t\in(0,T]$.

Proof: The above system (2.5) is equivalent to the following equation

$$\mathrm{y}(t)=\frac{\mathrm{y}_0}{\Gamma(\varpi)}t^{\varpi-1}+\frac{1}{\Gamma(\vartheta)}\int_0^t(t-\varsigma)^{\vartheta-1}\Lambda\mathrm{y}(\varsigma)d\varsigma+\frac{1}{\Gamma(\vartheta)}\int_0^t(t-\varsigma)^{\vartheta-1}\mathcal{P}(\varsigma)d\varsigma.$$

Now, by applying the Laplace transform in the above equation on both sides, we get

$$\hat{\mathrm{y}}(s)=\frac{1}{\lambda^{\varpi}}\mathrm{y}_0+\frac{1}{\lambda^{\vartheta}}\Lambda\hat{\mathrm{y}}(s)+\frac{1}{\lambda^{\vartheta}}\hat{\mathcal{P}}(s).$$

Now, apply the inverse Laplace transform in the above equation on both sides, we get

$$\mathrm{y}(t)=t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda t^{\vartheta})\mathrm{y}_0+\int_0^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda(t-\varsigma)^{\vartheta})\mathcal{P}(\varsigma)d\varsigma$$

for all $t\in(0,T]$. ■

In the next definition, by using the Lemma 1, we give the solution of the switched impulsive system (1.3).

Definition 7 *A function $\mathrm{y}\in PC_{1-\varpi}(I,\mathbb{R}^n)$ is a solution of the system (1.3), if y satisfies*

(i) $I_{0^+}^{1-\varpi}\mathrm{y}(0)=\mathrm{y}_0$ *and* $I_{\vartheta_i^+}^{1-\varpi}\mathrm{y}(\vartheta_i^+)=\mathcal{G}_i(\vartheta_i,\mathrm{y}(t_i^-))$,

(ii) $\mathrm{y}(t)=\mathcal{G}_i(t,\mathrm{y}(t_i^-)),\ t\in(t_i,\vartheta_i],\ i=1,2,\ldots,\jmath$

and

$$\mathrm{y}(t)=t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t^{\vartheta})\mathrm{y}_0+\int_0^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma,\mathrm{y}(\varsigma))d\varsigma$$

for all $t\in(0,t_1]$ and

$$\mathrm{y}(t)=(t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i,\mathrm{y}(t_i^-))+\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma,\mathrm{y}(\varsigma))d\varsigma$$

for all $t\in(\vartheta_i,t_{i+1}],\ i=1,2,\ldots,\jmath$.

The following assumptions are required to establish the main results:

(Z1): The maps $\mathcal{P}_i:T_i\times\mathbb{R}^n\to\mathbb{R}^n, T_i=[\vartheta_i,t_{i+1}],\ i=0,1,\ldots,\jmath$, are continuous. Also, there exists a number $L_{\mathcal{P}}>0$ such that

$$\|\mathcal{P}_i(t,\mathrm{y}_1)-\mathcal{P}_i(t,\mathrm{y}_2)\|\le L_{\mathcal{P}}\|\mathrm{y}_1-\mathrm{y}_1\|$$

for all $\mathrm{y}_1,\mathrm{y}_2\in\mathbb{R}^n$ and $t\in T_i$.

(Z2): The maps $\mathcal{G}_i:J_i\times\mathbb{R}^n\to\mathbb{R}^n, J_i=[t_i,\vartheta_i],\ i=1,2,\ldots,\jmath$, are continuous. Also, there exists a number $L_{\mathcal{G}}>0$ such that

$$\|\mathcal{G}_i(t,\mathrm{y}_1)-\mathcal{G}_i(t,\mathrm{y}_2)\|\le L_{\mathcal{G}}\|\mathrm{y}_1-\mathrm{y}_2\|$$

for all $\mathrm{y}_1,\mathrm{y}_2\in\mathbb{R}^n$ and $t\in J_i$.

We set

$c_1=\max_{i=0,1,\ldots,\jmath}\sup_{t\in I}\|E_{\vartheta,\varpi}(\Lambda_it^{\vartheta})\|$; $c_2=\max_{i=0,1,\ldots,\jmath}\sup_{t\in I}\|E_{\vartheta,\vartheta}(\Lambda_i(T-t)^{\vartheta})\|$; $\sup_{t\in I}\|\mathcal{P}_i(t,0)\|\le M_{\mathcal{P}}$; $\sup_{t\in I}\|\mathcal{G}_i(t,0)\|\le M_{\mathcal{G}}$; $\mathcal{N}_0=c_1\|\mathrm{y}_0\|+\frac{c_2M_{\mathcal{P}}t_1^{\vartheta+1-\varpi}}{\vartheta}$; $\mathcal{N}_i=c_1M_{\mathcal{G}}+\frac{c_2M_{\mathcal{P}}t_{i+1}^{\vartheta+1-\varpi}}{\vartheta}$, $i=1,2,\ldots,\jmath$; $\mathcal{Q}_0=t_1^{\vartheta}c_2L_{\mathcal{P}}B(\varpi,\vartheta)$; $\mathcal{Q}_i=c_1L_{\mathcal{G}}t_{i+1}^{\varpi-1}+t_{i+1}^{\vartheta}c_2L_{\mathcal{P}}B(\varpi,\vartheta)$, $i=1,2,\ldots,\jmath$.

(Z3): $L_{\Xi_1}<1$, where $L_{\Xi_1}=\max\{\max_{0\le i\le\jmath}\mathcal{Q}_i,\ L_{\mathcal{G}}\}$.

3. Existence Result

In this section, we establish the existence of a unique solution for the system (1.3) by using the Banach fixed point theorem.

Theorem 1 *If the assumptions (Z1), (Z2) and (Z3) fulfilled. Then, the system (1.3) has a unique solution.*

Proof: For a positive constant δ_1, we define a subset $\mathcal{D}_1 \subseteq PC_{1-\varpi}(I, \mathbb{R}^n)$ such that

$$\mathcal{D}_1 = \{\mathsf{y} \in PC_{1-\varpi}(I, \mathbb{R}^n) : \|\mathsf{y}\|_{PC_{1-\varpi}} \leq \delta_1\},$$

where

$$\delta_1 = \max\left(\max_{0\leq i\leq J} \frac{\mathcal{N}_i}{1-Q_i}, \frac{(\vartheta_i - t_i)^{1-\varpi} M_{\mathcal{G}}}{1-L_{\mathcal{G}}}\right).$$

Define an operator $\Xi_1 : \mathcal{D}_1 \to \mathcal{D}_1$ as

$$(\Xi_1\mathsf{y})(t) = t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0 t^{\vartheta})\mathsf{y}_0 + \int_0^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma, \mathsf{y}(\varsigma))d\varsigma, \quad \forall\, t \in (0, t_1],$$

$$(\Xi_1\mathsf{y})(t) = \mathcal{G}_i(t, \mathsf{y}(t_i^-)), \ \forall\, t \in (t_i, \vartheta_i], \ i = 1, 2, \ldots, J,$$

$$(\Xi_1\mathsf{y})(t) = (t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-)) + \int_{\vartheta_i}^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma, \mathsf{y}(\varsigma))d\varsigma, \quad \forall\, t \in (\vartheta_i, t_{i+1}], i = 1, 2, \ldots, J.$$

For the better readability, we split the proof into the following two steps:

Step 1: We shall show that Ξ_1 maps $\mathcal{D}_1$ into $\mathcal{D}_1$. For any $t \in (0, t_1]$ and $\mathsf{y} \in \mathcal{D}_1$,

$$\begin{aligned}
&t^{1-\varpi}\|(\Xi_1\mathsf{y})(t)\| \\
&\leq \|E_{\vartheta,\varpi}(\Lambda_0 t^{\vartheta})\mathsf{y}_0\| \\
&\quad + t^{1-\varpi}\int_0^t (t-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma, \mathsf{y}(\varsigma))\|d\varsigma \\
&\leq c_1\|\mathsf{y}_0\| + t^{1-\varpi}c_2 L_{\mathcal{P}}\int_0^t (t-\varsigma)^{\vartheta-1}\|\mathsf{y}(\varsigma)\|d\varsigma \\
&\quad + t^{1-\varpi}c_2 M_{\mathcal{P}}\int_0^t (t-\varsigma)^{\vartheta-1}d\varsigma \\
&\leq c_1\|\mathsf{y}_0\| + t^{\vartheta}c_2 L_{\mathcal{P}}B(\varpi, \vartheta)\delta_1 + \frac{c_2 M_{\mathcal{P}} t^{\vartheta+1-\varpi}}{\vartheta} \\
&\leq \mathcal{N}_0 + Q_0\delta_1 \leq \delta_1,
\end{aligned} \tag{3.6}$$

where we use

$$\begin{aligned}
&\int_a^t (t-\varsigma)^{\vartheta-1}\|\mathsf{y}(\varsigma)\|d\varsigma \\
&\leq \left(\int_a^t (t-\varsigma)^{\vartheta-1}(\varsigma-a)^{\varpi-1}d\varsigma\right)\|\mathsf{y}\|_{PC_{1-\varpi}} \\
&= (t-a)^{\vartheta+\varpi-1}B(\varpi, \vartheta)\|\mathsf{y}\|_{PC_{1-\varpi}}
\end{aligned}$$

and $B(\cdot, \cdot)$ denotes the usual beta function. Now, for any $\mathsf{y} \in \mathcal{D}_1$ and $t \in (t_i, \vartheta_i]$, $i = 1, 2, \ldots, J$,

$$\begin{aligned}
(t-t_i)^{1-\varpi}\|(\Xi_1\mathsf{y})(t)\| &\leq (t-t_i)^{1-\varpi}\|\mathcal{G}_i(t, \mathsf{y}(t_i^-))\| \\
&\leq L_{\mathcal{G}}\delta_1 + (\vartheta_i - t_i)^{1-\varpi}M_{\mathcal{G}} \\
&\leq \delta_1.
\end{aligned} \tag{3.7}$$

Similarly, for any $\mathsf{y} \in \mathcal{D}_1$ and $t \in (\vartheta_i, t_{i+1}], i = 1, 2, \ldots, J$,

$$\begin{aligned}
&(t-\vartheta_i)^{1-\varpi}\|(\Xi_1\mathsf{y})(t)\| \\
&\leq c_1\|\mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-))\| \\
&\quad + (t-\vartheta_i)^{1-\varpi}c_2 L_{\mathcal{P}}\int_{\vartheta_i}^t (t-\varsigma)^{\vartheta-1}\|\mathsf{y}(\varsigma)\|d\varsigma \\
&\quad + (t-\vartheta_i)^{1-\varpi}c_2 M_{\mathcal{P}}\int_{\vartheta_i}^t (t-\varsigma)^{\vartheta-1}d\varsigma \\
&\leq c_1 M_{\mathcal{G}} + c_1 L_{\mathcal{G}}(t-\vartheta_i)^{\varpi-1}\delta_1 + (t-\vartheta_i)^{\vartheta}c_2 L_{\mathcal{P}}B(\varpi, \vartheta)\delta_1 \\
&\quad + \frac{c_2 M_{\mathcal{P}}(t-\vartheta_i)^{\vartheta+1-\varpi}}{\vartheta} \\
&\leq \mathcal{N}_i + Q_i\delta_1 \leq \delta_1.
\end{aligned} \tag{3.8}$$

Now, using the inequalities (3.6), (3.7) and (3.8), we get,

$$\|\Xi_1\mathsf{y}\|_{PC_{1-\varpi}} \leq \delta_1, \ \forall\, t \in I.$$

Hence, Ξ_1 maps $\mathcal{D}_1$ into $\mathcal{D}_1$.

Step 2: Here, we show that Ξ_1 is a strict contracting operator. Now, for any $\mathsf{y}, \mathsf{z} \in \mathcal{D}_1$ and $t \in (0, t_1]$,

$$\begin{aligned}
&t^{1-\varpi}\|(\Xi_1\mathsf{y})(t) - (\Xi_1\mathsf{z})(t)\| \\
&\leq t^{1-\varpi}\int_0^t (t-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})(\mathcal{P}_0(\varsigma, \mathsf{y}(\varsigma)) \\
&\quad - \mathcal{P}_0(\varsigma, \mathsf{z}(\varsigma)))\|d\varsigma \\
&\leq t^{1-\varpi}c_2 L_{\mathcal{P}}\int_0^t (t-\varsigma)^{\vartheta-1}\|\mathsf{y}(\varsigma) - \mathsf{z}(\varsigma)\|d\varsigma \\
&\leq t^{\vartheta}c_2 L_{\mathcal{P}}B(\varpi, \vartheta)\|\mathsf{y} - \mathsf{z}\|_{PC_{1-\varpi}} \\
&\leq Q_0\|\mathsf{y} - \mathsf{z}\|_{PC_{1-\varpi}}.
\end{aligned} \tag{3.9}$$

Also, for any $\mathsf{y}, \mathsf{z} \in \mathcal{D}_1$ and $t \in (t_i, \vartheta_i], i = 1, 2, \ldots, J$,

$$\begin{aligned}
&(t-t_i)^{1-\varpi}\|(\Xi_1\mathsf{y})(t) - (\Xi_1\mathsf{z})(t)\| \\
&\leq (t-t_i)^{1-\varpi}\|\mathcal{G}_i(t, \mathsf{y}(t_i^-)) - \mathcal{G}_i(t, \mathsf{z}(t_i^-))\| \\
&\leq L_{\mathcal{G}}\|\mathsf{y} - \mathsf{z}\|_{PC_{1-\varpi}}.
\end{aligned} \tag{3.10}$$

Similarly, for any $\mathsf{y}, \mathsf{z} \in \mathcal{D}_1$ and $t \in (\vartheta_i, t_{i+1}]$, $i = 1, 2, \ldots, J$,

$$\begin{aligned}
&(t-\vartheta_i)^{1-\varpi}\|(\Xi_1\mathsf{y})(t) - (\Xi_1\mathsf{z})(t)\| \\
&\leq \|E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\|\|\mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-)) - \mathcal{G}_i(\vartheta_i, \mathsf{z}(t_i^-))\|
\end{aligned}$$

$$
\begin{aligned}
&+ (t-\vartheta_i)^{1-\varpi}\int_{\vartheta_i}^{t}(t-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\|\\
&\quad\times\|\mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma))-\mathcal{P}_i(\varsigma,\mathsf{z}(\varsigma))\|d\varsigma\\
&\le c_1L_{\mathcal{G}}\|\mathsf{y}(t_i^-)-\mathsf{z}(t_i^-)\|\\
&+ c_2L_{\mathcal{P}}(t-\vartheta_i)^{1-\varpi}\int_{\vartheta_i}^{t}(t-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\|\\
&\quad\times\|\mathsf{y}(\varsigma))-\mathsf{z}(\varsigma)\|d\varsigma\\
&\le c_1L_{\mathcal{G}}(t-\vartheta_i)^{\varpi-1}\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}}\\
&+ c_2L_{\mathcal{P}}(t-\vartheta_i)^{\vartheta}B(\varpi,\vartheta)\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}}\\
&\le Q_i\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}}.
\end{aligned}\tag{3.11}
$$

Therefore, by using the inequalities (3.9), (3.10) and (3.11), we get

$$\|\Xi_1\mathsf{y}-\Xi_1\mathsf{z}\|_{PC_{1-\varpi}}\le L_{\Xi_1}\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}},\ \forall\, t\in I.$$

Hence, from assumption (Z3), Ξ_1 is a contracting operator.

Thus, by collecting the step 1 and step 2, one can easily see that the operator Ξ_1 fulfilled all the requirements of Banach contraction principle. Henceforth, system (1.3) has a unique solution. ∎

4. Ulam's Hyer Stability

This section is devoted to the examination of UH stability of the switched system (1.3).

For $\epsilon>0$, consider the following inequality

$$
\begin{cases}
\|D^{\varrho,\vartheta}_{\vartheta_i^+}\mathsf{z}(t)-\Lambda_i\mathsf{z}(t)-\mathcal{P}_i(t,\mathsf{z}(t))\|\le\epsilon,\ t\in(\vartheta_i,t_{i+1}],\\
\quad i=0,1,\ldots,\jmath,\\
\|\mathsf{z}(t)-\mathcal{G}_i(t,\mathsf{z}(t_i^-))\|\le\epsilon,\ t\in(t_i,\vartheta_i],\ i=1,2,\ldots,\jmath,
\end{cases}\tag{4.12}
$$

Definition 8 *[5]. Equation (1.3) is UH stable if there exists a positive constant $H_{(L_{\mathcal{P}},L_{\mathcal{G}})}$ such that for $\epsilon>0$ and for any solution z of inequality (4.12), there exists a unique solution y of system (1.3) satisfies*

$$\|\mathsf{z}(t)-\mathsf{y}(t)\|\le H_{(L_{\mathcal{P}},L_{\mathcal{G}})}\epsilon,\ \forall\, t\in I.$$

Definition 9 *[5]. Equation (1.3) is Generalized UH stable if there exist $\mathcal{H}_{(L_{\mathcal{P}},L_{\mathcal{G}})}\in C(\mathbb{R}^+,\mathbb{R}^+),\mathcal{H}_{(L_{\mathcal{P}},L_{\mathcal{G}})}(0)=0$ such that for any solution z of inequalities (4.12), there exists a unique solution y of system (1.3) satisfies*

$$\|\mathsf{z}(t)-\mathsf{y}(t)\|\le\mathcal{H}_{(L_{\mathcal{P}},L_{\mathcal{G}})}(\epsilon),\ \forall\, t\in I.$$

Remark 1 *Definition 8 $\Longrightarrow$ Definition 9.*

Remark 2 *A function $\mathsf{z}\in PC_{1-\varpi}(I,\mathbb{R}^n)$ is a solution of inequality (4.12) iff there is a sequence G_i, $i=1,2,\ldots,\jmath$ and $\mathsf{G}\in PC_{1-\varpi}(I,\mathbb{R}^n)$ such that*

(a) $\|\mathsf{G}(t)\|\le\epsilon,\forall\, t\in(\vartheta_i,t_{i+1}],\ i=0,1,\ldots,\jmath$ and $\|\mathsf{G}_i\|\le\epsilon,\ \forall\, i=1,2,\ldots,\jmath.$

(b) $D^{\varrho,\vartheta}_{\vartheta_i^+}\mathsf{z}(t)=\Lambda_i\mathsf{z}(t)+\mathcal{P}_i(t,\mathsf{z}(t))+\mathsf{G}(t),\ t\in(\vartheta_i,t_{i+1}],\ i=0,1,\ldots,\jmath.$

(c) $\mathsf{z}(t)=\mathcal{G}_i(t,\mathsf{z}(t_i^-))+\mathsf{G}_i,\ t\in(t_i,\vartheta_i],\ i=1,2,\ldots,\jmath.$

From the above remark, we get

$$
\begin{cases}
D^{\varrho,\vartheta}_{\vartheta_i^+}\mathsf{z}(t)=\Lambda_i\mathsf{z}(t)+\mathcal{P}_i(t,\mathsf{z}(t))+\mathsf{G}(t),\ t\in(\vartheta_i,t_{i+1}],\\
\quad i=0,1,\ldots,\jmath,\\
\mathsf{z}(t)=\mathcal{G}_i(t,\mathsf{z}(t_i^-))+\mathsf{G}_i,\ t\in(t_i,\vartheta_i],\ i=1,2,\ldots,\jmath.
\end{cases}
$$

From Definition 7, the solution z with $I^{1-\varpi}_{0^+}\mathsf{z}(0)=\mathsf{y}_0$, $I^{1-\varpi}_{\vartheta_i^+}\mathsf{z}(\vartheta_i^+)=\mathcal{G}_i(\vartheta_i,\mathsf{z}(t_i^-))+\mathsf{G}_i$ of the above system is defined as

$$
\begin{aligned}
\mathsf{z}(t)&=t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t^{\vartheta})\mathsf{y}_0\\
&\quad+\int_0^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})(\mathcal{P}_0(\varsigma,\mathsf{z}(\varsigma))\\
&\quad+\mathsf{G}(\varsigma))d\varsigma,\ \forall\, t\in(0,t_1],\\
\mathsf{z}(t)&=\mathcal{G}_i(t,\mathsf{z}(t_i^-))+\mathsf{G}_i,\ t\in(t_i,s_i],\ i=1,2,\ldots,\jmath,\\
\mathsf{z}(t)&=(t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})(\mathcal{G}_i(\vartheta_i,\mathsf{z}(t_i^-))+\mathsf{G}_i)\\
&\quad+\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})(\mathcal{P}_i(\varsigma,\mathsf{z}(\varsigma))\\
&\quad+\mathsf{G}(\varsigma))d\varsigma,\ \forall\, t\in(\vartheta_i,t_{i+1}],\ i=1,2,\ldots,\jmath.
\end{aligned}
$$

Therefore, for any $t\in(0,t_1]$,

$$
\begin{aligned}
&\Bigg\|\mathsf{z}(t)-t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t^{\vartheta})\mathsf{y}_0\\
&-\int_0^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma,\mathsf{z}(\varsigma))d\varsigma\Bigg\|\\
&\le\left\|\int_0^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathsf{G}(\varsigma)d\varsigma\right\|\le\frac{c_2t_1^{\vartheta}\epsilon}{\vartheta}.
\end{aligned}
$$

Also, for any $t\in(\vartheta_i,t_{i+1}],\ i=1,2,\ldots,\jmath,$

$$
\begin{aligned}
&\Bigg\|\mathsf{z}(t)-(t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i,\mathsf{z}(t_i^-))\\
&\quad-\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma,\mathsf{z}(\varsigma))d\varsigma\Bigg\|\\
&\quad\le\|(t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathsf{G}_i\|\\
&\quad\quad+\left\|\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\mathsf{G}(\varsigma)d\varsigma\right\|\\
&\quad\le c_1(t_{i+1}-\vartheta_i)^{\varpi-1}\epsilon+\frac{c_2t_{i+1}^{\vartheta}\epsilon}{\vartheta}.
\end{aligned}
$$

Similarly, for $t\in(t_i,\vartheta_i],\ i=1,2,\ldots,\jmath$, we get $\|\mathsf{z}(t)-\mathcal{G}_i(t,\mathsf{z}(t_i^-))\|\le\epsilon.$

Theorem 2 *If the assumptions (Z1), (Z2) and (Z3) fulfilled. Then, the system (1.3) is UH stable.*

Proof: Let $\mathsf{z} \in PC_{1-\varpi}(I, \mathbb{R}^n)$ be the solution of inequality (4.12) and $\mathsf{y} \in PC_{1-\varpi}(I, \mathbb{R}^n)$ be a unique solution of the system (1.3). Then, from Definition 7, we have $I_{0^+}^{1-\varpi}\mathsf{y}(0) = \mathsf{y}_0$, $I_{\vartheta_i^+}^{1-\varpi}\mathsf{y}(\vartheta_i^+) = \mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-))$, $\mathsf{y}(t) = \mathcal{G}_i(t, \mathsf{y}(t_i^-))$, $t \in (t_i, \vartheta_i]$, $i = 1, 2, \dots, \jmath$ and

$$\begin{aligned}\mathsf{y}(t) &= t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0 t^{\vartheta})\mathsf{y}_0 \\ &\quad + \int_0^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma, \mathsf{y}(\varsigma))d\varsigma\end{aligned}$$

for all $t \in (0, t_1]$ and

$$\begin{aligned}\mathsf{y}(t) &= (t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-)) \\ &\quad + \int_{\vartheta_i}^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma, \mathsf{y}(\varsigma))d\varsigma\end{aligned}$$

for all $t \in (\vartheta_i, t_{i+1}]$, $i = 1, 2, \dots, \jmath$.

Now, for any $t \in (0, t_1]$, we have

$$\begin{aligned}&t^{1-\varpi}\|\mathsf{z}(t) - \mathsf{y}(t)\| \\ &\leq t^{1-\varpi}\Big\|\mathsf{z}(t) - t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0 t^{\vartheta})\mathsf{y}_0 \\ &\quad - \int_0^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma, \mathsf{z}(\varsigma))d\varsigma\Big\| \\ &\quad + t^{1-\varpi}\Big\|\int_0^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})(\mathcal{P}_0(\varsigma, \mathsf{z}(\varsigma)) \\ &\quad - \mathcal{P}_0(\varsigma, \mathsf{y}(\varsigma))d\varsigma\Big\| \\ &\leq \frac{c_2 t_1^{1+\vartheta-\varpi}\epsilon}{\vartheta} + t^{1-\varpi}L_{\mathcal{P}}c_2\int_0^t (t-\varsigma)^{\vartheta-1}\|\mathsf{z}(\varsigma) - \mathsf{y}(\varsigma)\|d\varsigma \\ &\leq \frac{c_2 t_1^{1+\vartheta-\varpi}\epsilon}{\vartheta} + Q_0\|\mathsf{z} - \mathsf{y}\|_{PC_{1-\varpi}}.\end{aligned} \tag{4.13}$$

Also, for any $t \in (\vartheta_i, t_{i+1}]$, $i = 1, 2, \dots, \jmath$, we have

$$\begin{aligned}&(t-\vartheta_i)^{1-\varpi}\|\mathsf{z}(t) - \mathsf{y}(t)\| \\ &\leq (t-\vartheta_i)^{1-\varpi}\Big\|\mathsf{z}(t) \\ &\quad - (t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-)) \\ &\quad - \int_{\vartheta_i}^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma, \mathsf{y}(\varsigma))d\varsigma\Big\| \\ &\leq c_1\epsilon + \frac{c_2 t_{i+1}^{1+\vartheta-\varpi}\epsilon}{\vartheta} + \|E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i, \mathsf{z}(t_i^-)) \\ &\quad - E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-))\| \\ &\quad + (t-\vartheta_i)^{1-\varpi}\|\int_{\vartheta_i}^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta}) \\ &\quad \times (\mathcal{P}_i(\varsigma, \mathsf{z}(\varsigma)) - \mathcal{P}_i(\varsigma, \mathsf{y}(\varsigma)))d\varsigma\| \\ &\leq c_1\epsilon + \frac{c_2 t_{i+1}^{1+\vartheta-\varpi}\epsilon}{\vartheta} + Q_i\|\mathsf{z} - \mathsf{y}\|_{PC_{1-\varpi}}.\end{aligned} \tag{4.14}$$

Similarly, for any $t \in (t_i, \vartheta_i]$, $i = 1, 2, \dots, \jmath$, we have

$$\begin{aligned}(t-t_i)^{1-\varpi}\|\mathsf{z}(t) - \mathsf{y}(t)\| &= (t-t_i)^{1-\varpi}\|\mathsf{z}(t) - \mathcal{G}_i(t, \mathsf{y}(t_i^-))\| \\ &\leq t_{i+1}^{1-\varpi}\epsilon + L_{\mathcal{G}}\|\mathsf{z} - \mathsf{y}\|_{PC_{1-\varpi}}.\end{aligned} \tag{4.15}$$

Now, by using the inequalities (4.13), (4.14) and (4.15), for all $t \in I$, we get

$$\begin{aligned}\|\mathsf{z} - \mathsf{y}\|_{PC_{1-\varpi}} &\leq t_{i+1}^{1-\varpi}\left(1 + c_1 t_{i+1}^{\varpi-1} + \frac{c_2 t_{i+1}^{\vartheta}}{\vartheta}\right) \\ &\quad + L_{\Xi_1}\|\mathsf{z} - \mathsf{y}\|_{PC_{1-\varpi}}\epsilon,\end{aligned}$$

which immediately gives

$$\|\mathsf{z} - \mathsf{y}\|_{PC_{1-\varpi}} \leq H_{(L_{\mathcal{P}}, L_{\mathcal{G}})}\epsilon,$$

where $H_{(L_{\mathcal{P}}, L_{\mathcal{G}})} = \dfrac{t_{i+1}^{1-\varpi}}{1 - L_{\Xi_1}}\left(1 + c_1 t_{i+1}^{\varpi-1} + \dfrac{c_2 t_{i+1}^{\vartheta}}{\vartheta}\right)$. Hence, the system (1.3) is UH stable. Furthermore, if we set $\mathcal{H}_{(L_{\mathcal{P}}, L_{\mathcal{G}})}(\epsilon) = H_{(L_{\mathcal{P}}, L_{\mathcal{G}})}\epsilon$, $\mathcal{H}_{(L_{\mathcal{P}}, L_{\mathcal{G}})}(0) = 0$, then the system (1.3) is GUH stable. ■

5. Controllability Results

In this segment, we establish the total controllability results for the switched impulsive control system (1.4) by applying the Banach contraction principle.

Definition 10 *Switched control system (1.4) is controllable on $[0, T]$, if for every $\mathsf{y}_0, \mathsf{y}_T \in \mathbb{R}^n$, there exists a function $\mathsf{v} \in L^2([0, T], \mathbb{R}^m)$ such that the solution of (1.4) satisfies $\mathsf{y}(0) = \mathsf{y}_0$ and $\mathsf{y}(T) = \mathsf{y}_T$.*

Definition 11 *Switched control system (1.4) is totally controllable on $[0, T]$, if it is controllable on $(0, t_1]$ and $(\vartheta_i, t_{i+1}]$, $i = 1, 2, \dots, \jmath$, i.e., for every $\mathsf{y}_0, \mathsf{y}_{t_{i+1}} \in \mathbb{R}^n, i = 0, 1, \dots, \jmath$, there exists a function $\mathsf{v} \in L^2([0, T], \mathbb{R}^m)$ such that the solution of (1.4) satisfies $\mathsf{y}(0) = \mathsf{y}_0$ and $\mathsf{y}(t_{i+1}) = \mathsf{y}_{t_{i+1}}$, $i = 0, 1, \dots, \jmath$.*

Remark 3 *Definition 11 $\implies$ Definition 10.*

Definition 12 *A function $\mathsf{y} \in PC_{1-\varpi}(I, \mathbb{R}^n)$ is a solution of the switched impulsive control system (1.4), if x satisfies*

(i) $I_{0^+}^{1-\varpi}\mathsf{y}(0) = \mathsf{y}_0$ *and* $I_{\vartheta_i^+}^{1-\varpi}\mathsf{y}(\vartheta_i^+) = \mathcal{G}_i(\vartheta_i, \mathsf{y}(t_i^-))$,

(ii) $\mathsf{y}(t) = \mathcal{G}_i(t, \mathsf{y}(t_i^-))$, $t \in (t_i, \vartheta_i]$, $i = 1, 2, \dots, \jmath$

and

$$y(t) = t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0 t^{\vartheta})y_0 + \int_0^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma, y(\varsigma))d\varsigma + \int_0^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{C}_0 v(\varsigma)d\varsigma \quad (5.16)$$

for all $t \in (0, t_1]$,

$$y(t) = (t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i, y(t_i^-)) + \int_{\vartheta_i}^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma, y(\varsigma))d\varsigma + \int_{\vartheta_i}^t (t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\mathcal{C}_i v(\varsigma)d\varsigma \quad (5.17)$$

for all $t \in (\vartheta_i, t_{i+1}],\ i = 1, 2, \dots, \jmath$.

Next, we define the controllability Grammian type matrices as follows:

$$\mathcal{Z}_{\vartheta_i}^{t_{i+1}} = \int_{\vartheta_i}^{t_{i+1}} E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\varsigma)^{\vartheta})\mathcal{C}_i\mathcal{C}_i^* \times E_{\vartheta,\vartheta}(\Lambda_i^*(t_{i+1}-\varsigma)^{\vartheta})d\varsigma,\ i = 0, 1, \dots, \jmath. \quad (5.18)$$

(Z4): The matrices $\mathcal{Z}_{\vartheta_i}^{t_{i+1}}$, $i = 0, 1, \dots, \jmath$, defined by (5.18) are invertible. Further, there exist some positive constants $M_{\mathcal{Z}}^i, i = 0, 1, \dots, \jmath$, such that $\|(\mathcal{Z}_{\vartheta_i}^{t_{i+1}})^{-1}\| \le M_{\mathcal{Z}}^i$.
Also, there exists a positive constant M_C such that for $i = 0, 1, \dots, \jmath$, $\|\mathcal{C}_i\| \le M_C$.

We set $c_3 = \max_{i=0,1,\dots,\jmath}\sup_{t\in I}(T-t)^{1-\vartheta}\|\mathcal{C}_i^* E_{\vartheta,\vartheta}(\Lambda_i^*(T-t)^{\vartheta})\|$; $K_i = \dfrac{c_2c_3M_CM_{\mathcal{Z}}^i t_{i+1}^{\vartheta}}{\vartheta}, i = 0, 1, \dots, \jmath$; $\mathcal{M}_i = \mathcal{N}_i + K_i((t_{i+1})^{1-\varpi}\|y_{t_{i+1}}\| + \mathcal{N}_i)$; $\mathcal{R}_i = \mathcal{Q}_i(1+K_i)$, $i = 0, 1, \dots, \jmath$.

(Z5): $L_{\Xi_2} < 1$, where $L_{\Xi_2} = \max\{\max_{0\le i\le \jmath}\mathcal{R}_i,\ L_{\mathcal{G}}\}$.

Lemma 2 *If the assumptions (Z1), (Z2) and (Z4) fulfilled. Then, the required control function for the system (1.4) has an estimate* $\|v(t)\| \le M_v^0, \forall\ t \in (0, t_1]$, *where*

$$M_v^0 = c_3M_{\mathcal{Z}}^0\Big[\|y_{t_1}\| + c_1t_1^{\varpi-1}\|y_0\| + \frac{c_2M_{\mathcal{P}}t_1^{\vartheta}}{\vartheta} + c_2L_{\mathcal{P}}t_1^{\vartheta+\varpi-1}B(\varpi,\vartheta)\|y\|_{PC_{1-\varpi}}\Big].$$

Proof: For $t \in (0, t_1]$, define the control function as

$$v(t) = (t_1-t)^{1-\vartheta}\mathcal{C}_0^*E_{\vartheta,\vartheta}(\Lambda_0^*(t_1-t)^{\vartheta})(\mathcal{Z}_0^{t_1})^{-1}[y_{t_1} - \int_0^{t_1}(t_1-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t_1-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma, y(\varsigma))d\varsigma - t_1^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t_1^{\vartheta})y_0]. \quad (5.19)$$

Now, by putting $t = t_1$ in the solution $y(t)$ of the system (1.4) on $(0, t_1]$, we get

$$\begin{aligned}
y(t_1) &= t_1^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t_1^{\vartheta})y_0 + \int_0^{t_1}(t_1-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t_1-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma, y(\varsigma))d\varsigma \\
&\quad + \int_0^{t_1}E_{\vartheta,\vartheta}(\Lambda_0(t_1-\varsigma)^{\vartheta})\mathcal{C}_0\mathcal{C}_0^*E_{\vartheta,\vartheta}(\Lambda_0^*(t_1-t)^{\vartheta}) \times (\mathcal{Z}_0^{t_1})^{-1}\Big[y_{t_1} - t_1^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t_1^{\vartheta})y_0 \\
&\quad - \int_0^{t_1}(t_1-\tau)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t_1-\tau)^{\vartheta})\mathcal{P}_0(\tau, y(\tau))d\tau\Big]d\varsigma \\
&= t_1^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t_1^{\vartheta})y_0 + \int_0^{t_1}(t_1-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t_1-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma, y(\varsigma))d\varsigma \\
&\quad + \mathcal{Z}_0^{t_1}(\mathcal{Z}_0^{t_1})^{-1}\Big[y_{t_1} - t_1^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t_1^{\vartheta})y_0 - \int_0^{t_1}(t_1-\tau)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t_1-\tau)^{\vartheta})\mathcal{P}_0(\tau, y(\tau))d\tau\Big] \\
&= y_{t_1}.
\end{aligned}$$

Therefore, control function (5.19) is suitable for $t \in (0, t_1]$. Furthermore,

$$\begin{aligned}
\|v(t)\| &\le \|(t_1-t)^{1-\vartheta}\mathcal{C}_0^*E_{\vartheta,\vartheta}(\Lambda_0^*(t_1-t)^{\vartheta})(\mathcal{Z}_0^{t_1})^{-1}\|\Big[\|y_{t_1}\| + \int_0^{t_1}(t_1-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_0(t_1-\varsigma)^{\vartheta}) \\
&\quad \times \mathcal{P}_0(\varsigma, y(\varsigma))\|d\varsigma + \|t_1^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t_1^{\vartheta})y_0\|\Big] \\
&\le c_3M_{\mathcal{Z}}^0\Big[\|y_{t_1}\| + c_1t_1^{\varpi-1}\|y_0\| + c_2\int_0^{t_1}(t_1-\varsigma)^{\vartheta-1}\|\mathcal{P}_0(\varsigma, y(\varsigma))\|d\varsigma\Big] \\
&\le c_3M_{\mathcal{Z}}^0\Big[\|y_{t_1}\| + c_1t_1^{\varpi-1}\|y_0\| + \frac{c_2M_{\mathcal{P}}t_1^{\vartheta}}{\vartheta} + c_2L_{\mathcal{P}}t_1^{\vartheta+\varpi-1}B(\varpi,\vartheta)\|y\|_{PC_{1-\varpi}}\Big] \\
&= M_v^0.
\end{aligned}$$

■

Lemma 3 *If the assumptions (Z1), (Z2) and (Z4) fulfilled. Then, the required control function for the system (1.4) has an estimate* $\|v(t)\| \le M_v^i,\ \forall\ t \in (\vartheta_i, t_{i+1}],\ i = 1, 2, \dots, \jmath$ *where*

$$M_v^i = c_3M_{\mathcal{Z}}^i\Big[\|y_{t_{i+1}}\| + c_1t_{i+1}^{\varpi-1}L_{\mathcal{G}}\|y(t_i^-)\| + c_1t_{i+1}^{\varpi-1}M_{\mathcal{G}} + \frac{c_2M_{\mathcal{P}}t_{i+1}^{\vartheta}}{\vartheta} + c_2L_{\mathcal{P}}t_{i+1}^{\vartheta+\varpi-1}B(\varpi,\vartheta)\|y\|_{PC_{1-\varpi}}\Big].$$

Proof: For $t \in (\vartheta_i, t_{i+1}]$, $i = 1,2,\ldots,\jmath$, define the control function by

$$\begin{aligned}\mathsf{v}(t) &= (t_{i+1}-t)^{1-\vartheta}C_i^* E_{\vartheta,\vartheta}(\Lambda_i^*(t_{i+1}-t)^{\vartheta})(\mathcal{Z}_{\vartheta_i}^{t_{i+1}})^{-1}\Big[\mathsf{y}_{t_{i+1}} \\ &\quad - (t_{i+1}-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t_{i+1}-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-)) \\ &\quad - \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\varsigma)^{\vartheta}) \\ &\quad \times \mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma))d\varsigma\Big]. \qquad (5.20)\end{aligned}$$

Now, by putting $t = t_{i+1}$ in the solution $\mathsf{y}(t)$ of the system (1.4) on $(\vartheta_i, t_{i+1}]$, $i = 1,2,\ldots,\jmath$, we get

$$\begin{aligned}&\mathsf{y}(t_{i+1}) \\ &= (t_{i+1}-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t_{i+1}-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-)) \\ &\quad + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma))d\varsigma \\ &\quad + \int_{\vartheta_i}^{t_{i+1}}E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\varsigma)^{\vartheta})C_iC_i^*E_{\vartheta,\vartheta}(\Lambda_i^*(t_{i+1}-t)^{\vartheta}) \\ &\quad \times (\mathcal{Z}_{\vartheta_i}^{t_{i+1}})^{-1}\Big[\mathsf{y}_{t_{i+1}} - (t_{i+1}-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t_{i+1}-\vartheta_i)^{\vartheta}) \\ &\quad \times \mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-)) - \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\varsigma)^{\vartheta}) \\ &\quad \times \mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma))d\varsigma\Big]d\varsigma \\ &= (t_{i+1}-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t_{i+1}-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-)) \\ &\quad + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma))d\varsigma \\ &\quad + \mathcal{Z}_{\vartheta_i}^{t_{i+1}}(\mathcal{Z}_{\vartheta_i}^{t_{i+1}})^{-1}\Big[\mathsf{y}_{t_{i+1}} - (t_{i+1}-\vartheta_i)^{\varpi-1} \\ &\quad \times E_{\vartheta,\varpi}(\Lambda_i(t_{i+1}-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-)) \\ &\quad - \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma))d\varsigma\Big] \\ &= \mathsf{y}_{t_{i+1}}.\end{aligned}$$

Therefore, control function (5.20) is suitable for $t \in (\vartheta_i, t_{i+1}]$, $i = 1,2,\ldots,\jmath$. Furthermore,

$$\begin{aligned}\|\mathsf{v}(t)\| &\leq \|(t_{i+1}-t)^{1-\vartheta}C_i^*E_{\vartheta,\vartheta}(\Lambda_i^*(t_{i+1}-t)^{\vartheta})(\mathcal{Z}_{\vartheta_i}^{t_{i+1}})^{-1}\| \\ &\quad \times \Big[\|\mathsf{y}_{t_{i+1}}\| + \|(t_{i+1}-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t_{i+1}-\vartheta_i)^{\vartheta}) \\ &\quad \times \mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-))\| + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\vartheta-1} \\ &\quad \times \|E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\varsigma)^{\vartheta})\mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma))\|d\varsigma\Big] \\ &\leq c_3M_{\mathcal{Z}}^i\Big[\|\mathsf{y}_{t_{i+1}}\| + (t_{i+1}-\vartheta_i)^{\varpi-1}c_1(L_{\mathcal{G}}\|\mathsf{y}(t_i^-)\| + M_{\mathcal{G}}) \\ &\quad + \frac{c_2M_{\mathcal{P}}t_{i+1}^{\vartheta}}{\vartheta} + c_2L_{\mathcal{P}}t_{i+1}^{\vartheta+\varpi-1}B(\varpi,\vartheta)\|\mathsf{y}\|_{PC_{1-\varpi}}\Big] \\ &\leq M_{\mathsf{v}}^i.\end{aligned}$$

■

Theorem 3 *If the assumptions (Z1), (Z2), (Z4) and (Z5) fulfilled. Then, the system (1.4) is totally controllable.*

Proof: For a positive constant δ_2, we define a subset $\mathcal{D}_2 \subseteq PC_{1-\varpi}(I,\mathbb{R}^n)$ such that

$$\mathcal{D}_2 = \{\mathsf{y} \in PC_{1-\varpi}(I,\mathbb{R}^n) : \|\mathsf{y}\|_{PC_{1-\varpi}} \leq \delta_2\},$$

where

$$\delta_2 = \max\left(\max_{0\leq i\leq \jmath}\frac{\mathcal{M}_i}{1-\mathcal{R}_i}, \frac{(\vartheta_i - t_i)^{1-\varpi}M_{\mathcal{G}}}{1-L_{\mathcal{G}}}\right).$$

Define an operator $\Xi_2 : \mathcal{D}_2 \to \mathcal{D}_2$ as

$$\begin{aligned}(\Xi_2\mathsf{y})(t) &= t^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_0t^{\vartheta})\mathsf{y}_0 \\ &\quad + \int_0^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta}) \\ &\quad \times(\mathcal{P}_0(\varsigma,\mathsf{y}(\varsigma)) + C_0\mathsf{v}(\varsigma))d\varsigma,\ \forall\, t\in(0,t_1], \\ (\Xi_2\mathsf{y})(t) &= \mathcal{G}_i(t,\mathsf{y}(t_i^-)),\ \forall\, t\in(t_i,\vartheta_i],\ i=1,2,\ldots,\jmath, \\ (\Xi_2\mathsf{y})(t) &= (t-\vartheta_i)^{\varpi-1}E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-)) \\ &\quad + \int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta}) \\ &\quad \times(\mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma)) + C_i\mathsf{v}(\varsigma))d\varsigma, \\ &\qquad \forall\, t\in(\vartheta_i,t_{i+1}],\ i=1,2,\ldots,\jmath.\end{aligned}$$

For the better readability, we split the proof into the following two steps:
Step 1: We show that Ξ_2 maps $\mathcal{D}_2$ into $\mathcal{D}_2$. Now, for any $t \in (0,t_1]$ and $\mathsf{y} \in \mathcal{D}_2$, we have

$$\begin{aligned}&t^{1-\varpi}\|(\Xi_2\mathsf{y})(t)\| \\ &\leq c_1\|\mathsf{y}_0\| + t^{1-\varpi}c_2M_{\mathcal{P}}\int_0^t(t-\varsigma)^{\vartheta-1}d\varsigma \\ &\quad + t^{1-\varpi}c_2L_{\mathcal{P}}\int_0^t(t-\varsigma)^{\vartheta-1}\|\mathsf{y}(\varsigma)\|d\varsigma \\ &\quad + t^{1-\varpi}c_2M_C\int_0^t(t-\varsigma)^{\vartheta-1}c_3M_{\mathcal{Z}}^0\Big[\|\mathsf{y}_{t_1}\| + c_1t_1^{\varpi-1}\|\mathsf{y}_0\| \\ &\quad + \frac{c_2M_{\mathcal{P}}t_1^{\vartheta}}{\vartheta} + c_2L_{\mathcal{P}}t_1^{\vartheta+\varpi-1}B(\varpi,\vartheta)\|\mathsf{y}\|_{PC_{1-\varpi}}\Big]d\varsigma\end{aligned}$$

$$\leq c_1\|\mathsf{y}_0\| + t^{\vartheta}c_2L_{\mathcal{P}}B(\varpi,\vartheta)\delta_2 + \frac{c_2M_{\mathcal{P}}t^{\vartheta+1-\varpi}}{\vartheta}$$
$$+\frac{t^{1-\varpi+\vartheta}c_2c_3M_CM^0_{\mathcal{Z}}}{\vartheta}\Big[\|\mathsf{y}_{t_1}\| + c_1t_1^{\varpi-1}\|\mathsf{y}_0\|$$
$$+\frac{c_2M_{\mathcal{P}}t_1^{\vartheta}}{\vartheta} + c_2L_{\mathcal{P}}t_1^{\vartheta+\varpi-1}B(\varpi,\vartheta)\delta_2\Big]$$
$$\leq \mathcal{N}_0 + \mathcal{Q}_0\delta_2 + K_0(t^{1-\varpi}\|\mathsf{y}_{t_1}\| + \mathcal{N}_0 + \mathcal{Q}_0\delta_2)$$
$$\leq \mathcal{M}_0 + \mathcal{R}_0\delta_2 \leq \delta_2. \tag{5.21}$$

Now, for any $\mathsf{y}\in\mathcal{D}_2$ and $t\in(t_i,\vartheta_i],\ i=1,2,\dots,\jmath$, we have

$$(t-t_i)^{1-\varpi}\|(\Xi_2\mathsf{y})(t)\| \leq (t-t_i)^{1-\varpi}\|\mathcal{G}_i(t,\mathsf{y}(t_i^-))\|$$
$$\leq L_{\mathcal{G}}\delta_2 + (\vartheta_i-t_i)^{1-\varpi}M_{\mathcal{G}}$$
$$\leq \delta_2. \tag{5.22}$$

Similarly, for any $\mathsf{y}\in\mathcal{D}_2$ and $t\in(\vartheta_i,t_{i+1}], i = 1,2,\dots,\jmath$, we have

$$(t-\vartheta_i)^{1-\varpi}\|(\Xi_2\mathsf{y})(t)\|$$
$$\leq \|E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-))\|$$
$$+(t-\vartheta_i)^{1-\varpi}\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})$$
$$\times(\mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma)) + C_i\mathsf{v}(\varsigma))\|d\varsigma$$
$$\leq c_1\|\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-))\|$$
$$+(t-\vartheta_i)^{1-\varpi}c_2L_{\mathcal{P}}\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}\|\mathsf{y}(\varsigma))\|d\varsigma$$
$$+(t-\vartheta_i)^{1-\varpi}c_2M_{\mathcal{P}}\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}d\varsigma$$
$$+(t-\vartheta_i)^{1-\varpi}c_2c_3M_CM^i_{\mathcal{Z}}\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}\Big[\|\mathsf{y}_{t_{i+1}}\|$$
$$+c_1t_{i+1}^{\varpi-1}L_{\mathcal{G}}\|\mathsf{y}(t_i^-)\| + c_1t_{i+1}^{\varpi-1}M_{\mathcal{G}} + \frac{c_2M_{\mathcal{P}}t_{i+1}^{\vartheta}}{\vartheta}$$
$$+c_2L_{\mathcal{P}}t_{i+1}^{\vartheta+\varpi-1}B(\varpi,\vartheta)\|\mathsf{y}\|_{PC_{1-\varpi}}\Big]d\varsigma$$
$$\leq c_1M_{\mathcal{G}} + c_1L_{\mathcal{G}}(t-\vartheta_i)^{\varpi-1}\delta_2 + (t-\vartheta_i)^{\vartheta}c_2L_{\mathcal{P}}B(\varpi,\vartheta)\delta_2$$
$$+\frac{c_2M_{\mathcal{P}}(t-\vartheta_i)^{\vartheta+1-\varpi}}{\vartheta} + \frac{c_2c_3M^i_{\mathcal{Z}}M_Ct_{i+1}^{\vartheta}}{\vartheta}$$
$$\times\Big((t-\vartheta_i)^{1-\varpi}\|\mathsf{y}_{t_{i+1}}\| + c_1M_{\mathcal{G}} + c_1L_{\mathcal{G}}(t-\vartheta_i)^{1-\varpi}\delta_2$$
$$+(t-\vartheta_i)^{\vartheta}c_2L_{\mathcal{P}}B(\varpi,\vartheta)\delta_2 + \frac{c_2M_{\mathcal{P}}t_{i+1}^{\vartheta+1-\varpi}}{\vartheta}\Big)$$
$$\leq \mathcal{N}_i + \mathcal{Q}_i\delta_2 + K_i((t-\vartheta_i)^{1-\varpi}\|\mathsf{y}_{t_{i+1}}\| + \mathcal{N}_i + \mathcal{Q}_i\delta_2)$$
$$\leq \mathcal{M}_i + \mathcal{R}_i\delta_2 \leq \delta_2. \tag{5.23}$$

From the inequalities (5.21), (5.22) and (5.23), for $t\in I$, we get

$$\|\Xi_2\mathsf{y}\|_{PC_{1-\varpi}} \leq \delta_2.$$

Hence, Ξ_2 maps $\mathcal{D}_2$ into $\mathcal{D}_2$.

Step 2: Here, we show that Ξ_2 is a contracting operator. For any $\mathsf{y},\mathsf{z}\in\mathcal{D}_2$ and $t\in(0,t_1]$, we have

$$t^{1-\varpi}\|(\Xi_2\mathsf{y})(t) - (\Xi_2\mathsf{z})(t)\|$$
$$\leq t^{1-\varpi}\int_0^t(t-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})\mathcal{P}_0(\varsigma,\mathsf{y}(\varsigma))$$
$$-\mathcal{P}_0(\varsigma,\mathsf{z}(\varsigma))\|d\varsigma + t^{1-\varpi}\int_0^t(t-\varsigma)^{\vartheta-1}$$
$$\times\|E_{\vartheta,\vartheta}(\Lambda_0(t-\varsigma)^{\vartheta})C_0(t_1-\varsigma)^{1-\vartheta}C_0^*E_{\vartheta,\vartheta}(\Lambda_0^*(t_1-\varsigma)^{\vartheta})$$
$$\times\|(\mathcal{Z}_0^{t_1})^{-1}\Big[\int_0^{t_1}(t_1-\tau)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_0(t_1-\tau)^{\vartheta})\|$$
$$\times\|\mathcal{P}_0(\tau,\mathsf{y}(\tau)) - \mathcal{P}_0(\tau,\mathsf{z}(\tau))d\tau\Big]d\varsigma$$
$$\leq t^{1-\varpi}c_2L_{\mathcal{P}}\int_0^t(t-\varsigma)^{\vartheta-1}\|\mathsf{y}(\varsigma)-\mathsf{z}(\varsigma)\|d\varsigma$$
$$+c_2^2c_3M_CM^0_{\mathcal{Z}}L_{\mathcal{P}}t^{1-\varpi}\int_0^t(t-\varsigma)^{\vartheta-1}$$
$$\times\Big[\int_0^{t_1}(t_1-\tau)^{\vartheta-1}\|\mathsf{y}(\tau)-\mathsf{z}(\tau)\|d\tau\Big]d\varsigma$$
$$\leq t^{\vartheta}c_2L_{\mathcal{P}}B(\varpi,\vartheta)\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}}$$
$$+\frac{c_2^2c_3M_CM^0_{\mathcal{Z}}L_{\mathcal{P}}t^{\vartheta}t_1^{\vartheta}L_{\mathcal{P}}B(\varpi,\vartheta)}{\vartheta}\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}}$$
$$\leq \mathcal{Q}_0(1+K_0)\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}} \leq \mathcal{R}_0\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}}. \tag{5.24}$$

Also, for any $\mathsf{y},\mathsf{z}\in\mathcal{D}_2$ and $t\in(t_i,\vartheta_i], i = 1,2,\dots,\jmath$, we have

$$(t-t_i)^{1-\varpi}\|(\Xi_2\mathsf{y})(t) - (\Xi_2\mathsf{z})(t)\|$$
$$\leq (t-t_i)^{1-\varpi}\|\mathcal{G}_i(t,\mathsf{y}(t_i^-)) - \mathcal{G}_i(t,\mathsf{z}(t_i^-))\|$$
$$\leq L_{\mathcal{G}}\|\mathsf{y}-\mathsf{z}\|_{PC_{1-\varpi}}. \tag{5.25}$$

Similarly, for any $\mathsf{y},\mathsf{z}\in\mathcal{D}_2$ and $t\in(\vartheta_i,t_{i+1}]$, $i = 1,2,\dots,\jmath$, we have

$$(t-\vartheta_i)^{1-\varpi}\|(\Xi_2\mathsf{y})(t) - (\Xi_2\mathsf{z})(t)\|$$
$$\leq \|E_{\vartheta,\varpi}(\Lambda_i(t-\vartheta_i)^{\vartheta})\|\|\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-)) - \mathcal{G}_i(\vartheta_i,\mathsf{z}(t_i^-))\|$$
$$+(t-\vartheta_i)^{1-\varpi}\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\|$$
$$\times\|\mathcal{P}_i(\varsigma,\mathsf{y}(\varsigma)) - \mathcal{P}_i(\varsigma,\mathsf{z}(\varsigma))\|d\varsigma$$
$$+(t-\vartheta_i)^{1-\varpi}\int_{\vartheta_i}^t(t-\varsigma)^{\vartheta-1}\|E_{\vartheta,\vartheta}(\Lambda_i(t-\varsigma)^{\vartheta})\|$$
$$\times\|C_i\|\|(t_{i+1}-\varsigma)^{1-\vartheta}C_i^*E_{\vartheta,\vartheta}(\Lambda_i^*(t_{i+1}-\varsigma)^{\vartheta})\|$$
$$\times\|(\mathcal{Z}_{\vartheta_i}^{t_{i+1}})^{-1}\|\Big[(t_{i+1}-\vartheta_i)^{\varpi-1}\|E_{\vartheta,\varpi}(\Lambda_i(t_{i+1}-\vartheta_i)^{\vartheta})\|$$
$$\times\|\mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-)) - \mathcal{G}_i(\vartheta_i,\mathsf{y}(t_i^-))\|$$

$$
\begin{aligned}
&\quad + \int_{\vartheta_i}^{t_{i+1}} (t_{i+1}-\tau)^{\vartheta-1} \| E_{\vartheta,\vartheta}(\Lambda_i(t_{i+1}-\tau)^{\vartheta}) \| \\
&\quad \times \| \mathcal{P}_i(\tau, \mathsf{y}(\tau)) - \mathcal{P}_i(\tau, \mathsf{z}(\tau)) \| d\tau \Big] d\varsigma \\
&\le c_1 L_{\mathcal{G}} (t_{i+1}-\vartheta_i)^{\varpi-1} \| \mathsf{y}-\mathsf{z} \|_{PC_{1-\varpi}} \\
&\quad + c_2 L_{\mathcal{P}} (t-\vartheta_i)^{\vartheta} B(\varpi,\vartheta) \| \mathsf{y}-\mathsf{z} \|_{PC_{1-\varpi}} \\
&\quad + \frac{c_2 c_3 M_{\mathcal{Z}}^{i} M_{C} t_{i+1}^{\vartheta}}{\vartheta} \Big(c_1 L_{\mathcal{G}} (t_{i+1}-\vartheta_i)^{\varpi-1} \| \mathsf{y}-\mathsf{z} \|_{PC_{1-\varpi}} \\
&\quad + c_2 L_{\mathcal{P}} (t_{i+1}-\vartheta_i)^{\vartheta} B(\varpi,\vartheta) \| \mathsf{y}-\mathsf{z} \|_{PC_{1-\varpi}} \Big) \\
&\le \mathcal{Q}_i \| \mathsf{y}-\mathsf{z} \|_{PC_{1-\varpi}} + K_i \mathcal{Q}_i \| \mathsf{y}-\mathsf{z} \|_{PC_{1-\varpi}} \\
&\le \mathcal{R}_i \| \mathsf{y}-\mathsf{z} \|_{PC_{1-\varpi}}. \qquad (5.26)
\end{aligned}
$$

Therefore, from the inequalities (5.24), (5.25) and (5.26), for any $t \in I$, we have

$$\| \Xi_2 \mathsf{y} - \Xi_2 \mathsf{z} \|_{PC_{1-\varpi}} \le L_{\Xi_2} \| \mathsf{y}-\mathsf{z} \|_{PC_{1-\varpi}}.$$

Hence, from assumption (Z5), Ξ_2 is a contracting operator.

Therefore, from step 1 and step 2, one can see that the operator Ξ_2 fulfilled all the conditions of Banach contraction principle. Hence, the system (1.4) is totally controllable on I. ∎

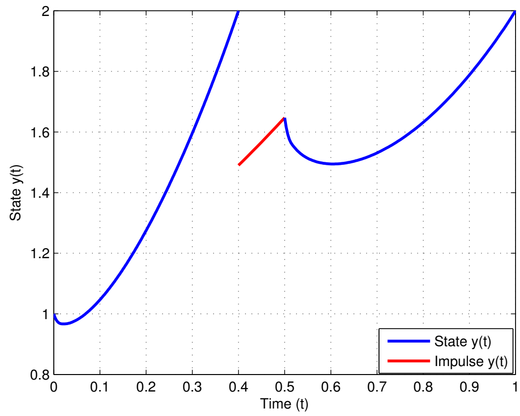
6. Examples

Example 1 *We consider the following switched impulsive control system in the space* $\mathbb{R}$

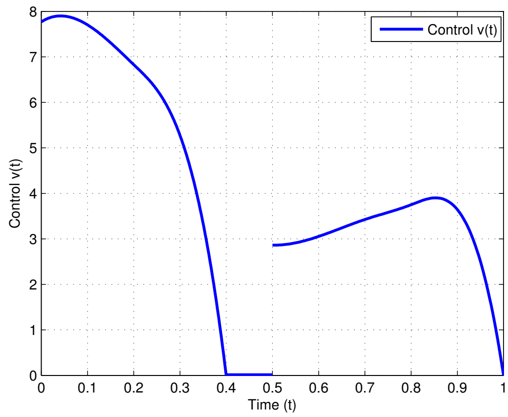
$$
\begin{aligned}
&D_{0^+}^{0.6,0.5} \mathsf{y}(t) = -0.3\mathsf{y}(t) + \frac{\sin(\mathsf{y}(t))}{30e^{(t+1)}} + \mathsf{v}(t), \ t \in (0, 0.4], \\
&D_{0.5^+}^{0.6,0.5} \mathsf{y}(t) = -0.4\mathsf{y}(t) + \frac{\cos(\mathsf{y}(t))}{2e^{(t^2+4)}} + e^{2t} + \mathsf{v}(t), \\
&\qquad\qquad t \in (0.5, 1], \qquad (6.27) \\
&\mathsf{y}(t) = \frac{(t+1)^2 \cos(\mathsf{y}(0.4^-))}{25e^{(t+2)}} + e^t, \ t \in (0.4, 0.5], \\
&I_{0^+}^{1-\varpi} \mathsf{y}(0) = 1, \ I_{0.^+}^{1-\varpi} \mathsf{y}(0.5^+) = \frac{(0.5+1)^2 \cos(\mathsf{y}(0.4^-))}{25e^{(0.5+2)}}.
\end{aligned}
$$

The system (6.27) can be written in the form of (1.4), where $\Lambda_0 = -0.3$, $\Lambda_1 = -0.4$, $C_0 = 1$, $C_1 = 1$, $t_0 = 0$, $t_1 = 0.4$, $\vartheta_1 = 0.5$, $t_2 = T = 1$, $\jmath = 1$, $\varrho = 0.6$, $\vartheta = 0.5$, $\mathsf{y}_0 = 1$,

$$
\begin{aligned}
&\mathcal{P}_0 = \frac{\sin(\mathsf{y}(t))}{30e^{(t+1)}}, \ \mathcal{P}_1 = \frac{\cos(\mathsf{y}(t))}{2e^{(t^2+4)}} + e^{2t}, \\
&\mathcal{G}_1(t, \mathsf{y}(t)) = \frac{(t+1)^2 \cos(\mathsf{y}(t))}{25e^{(t+2)}} + e^t.
\end{aligned}
$$



(a) State trajectory of the system (6.27) when $x(t_1) = 2$ and $x(T) = 2$



(b) Trajectory of the control function $\mathsf{v}(t)$

Figure 1: (a) shows the controlled trajectory of the system (6.27), (b) shows the trajectory of the control function for the system (6.27)

We choose the final target points as $\mathsf{y}(t_1) = 2$ and $\mathsf{y}(T) = 2$. Clearly, we can see that the conditions (Z1) and (Z2) fulfilled. Also, one can easily calculate

$$
\begin{aligned}
\mathcal{Q}_0 &= t_1^{\vartheta} c_2 L_{\mathcal{P}} B(\varpi,\vartheta) = 0.0087 \\
\mathcal{Q}_1 &= c_1 L_{\mathcal{G}} t_2^{\varpi-1} + t_2^{\vartheta} c_2 L_{\mathcal{P}} B(\varpi,\vartheta) = 0.0176.
\end{aligned}
$$

Thus, assumption (Z3) hold. Therefore, all the conditions of Theorem 1 and 2 satisfied and hence the system (6.27) has a HU stable unique solution. Furthermore, to apply the Theorem 3, it remains to check the assumptions (Z4) and (Z5). After some calculations, we get

$$
\begin{aligned}
\mathcal{Z}_0^{t_1} &= \int_0^{t_1} E_{\vartheta,\vartheta}(\Lambda_0(t_1-\varsigma)^{\vartheta}) C_0 C_0^* E_{\vartheta,\vartheta}(\Lambda_0^*(t_1-\varsigma)^{\vartheta}) d\varsigma \\
&= 0.0835,
\end{aligned}
$$

$$\mathcal{Z}_{\vartheta_1}^{t_2} = \int_{\vartheta_1}^{t_2} E_{\vartheta,\vartheta}(\Lambda_1(t_2-\varsigma)^{\vartheta})C_1C_1^{*}E_{\vartheta,\vartheta}(\Lambda_1^{*}(t_2-\varsigma)^{\vartheta})d\varsigma$$
$$= 0.0864$$

and

$$\mathcal{R}_0 = \mathcal{Q}_0(1+K_0) = 0.0309, \mathcal{R}_1 = \mathcal{Q}_1(1+K_1) = 0.0967.$$

Hence, $L_{\Xi_2} = \max\{\mathcal{R}_0, \mathcal{R}_1, L_{\mathcal{G}}\} = 0.0967 < 1$. Thus, all the assumptions of the Theorem 3 fulfilled. Hence, the control system (6.27) is totally controllable on $[0,1]$. The controlled state trajectory of the system (6.27) is shown in the Figure 1a and the control function is shown in the Figure 1b. Also, the CPU run time for different time intervals is given in the Table 1.

Table 1: CPU time for different time intervals

Time step	Intervals	CPU time (sec)
0.01	$[0, t_1]$	1.0096e+03
0.01	$[t_1, s_1]$	0.250
0.01	$[s_1, t_2]$	1.2301e+03

Example 2 *We consider the following switched impulsive control system in the space* $\mathbb{R}^2$

$$D_{0^+}^{0.8,0.3}y_1(t) = -0.1y_1(t) + 0.2y_2(t) + \frac{t^3(5+|y_1(t)|)}{90e^{t+7}(1+|y_1(t)|)} + \frac{e^t}{2},\ t \in (0, 0.5],$$
$$D_{0^+}^{0.8,0.3}y_2(t) = 0.1y_1(t) + 0.25y_2(t) + v_2(t) + \frac{\sin(y_2(t))}{30(3+2t^2)e^{t+7}},\ t \in (0, 0.5],$$
$$D_{0.7^+}^{0.8,0.3}y_1(t) = 0.15y_1(t) + 0.3y_2(t) + v_1(t) + \frac{(1+t)^2(7+|y_2(t)|)}{85e^{t+8}(5+|y_2(t)|)} + e^{t^2},\ t \in (0.7, 1],$$
$$D_{0.7^+}^{0.8,0.3}y_2(t) = -0.5y_2(t) + v_2(t) + \frac{t^2y_1(t)}{90e^{t^2+8}},\ t \in (0.7, 1], \tag{6.28}$$
$$y_1(t) = \frac{t\cos(y_1(0.5^-))}{25(5+t)e^{t+8}} + \sin(t)e^t,$$
$$y_2(t) = \frac{\sin(t)y_2(0.5^-)}{50e^{t+9}} + \frac{\cos(t)}{e^{t+7}},\ t \in (0.5, 0.7],$$
$$I_{0.7^+}^{1-\varpi}y_1(0.7^+) = \frac{0.7\cos(y_1(0.5^-))}{25(5+0.7)e^{0.7+8}} + \sin(0.7)e^{0.7},$$
$$I_{0.7^+}^{1-\varpi}y_2(0.7^+) = \frac{\sin(0.7)y_2(0.5^-)}{50e^{0.7+9}} + \frac{\cos(0.7)}{e^{0.7}},$$
$$I_{0^+}^{1-\varpi}y_1(0) = 1,\ I_{0^+}^{1-\varpi}y_2(0) = 2,$$

The system (6.28) can be written in the form of (1.4), where $t_0 = 0$, $t_1 = 0.5$, $\vartheta_1 = 0.7$, $t_2 = T = 1$, $\jmath = 1$, $\varrho = 0.8$, $\vartheta = 0.3$,

$$\mathrm{y}(t) = \begin{bmatrix} \mathrm{y}_1(t) \\ \mathrm{y}_2(t) \end{bmatrix},\ \mathrm{y}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \Lambda_0 = \frac{1}{10}\begin{bmatrix} -1 & 2 \\ 1 & 2.5 \end{bmatrix},$$
$$\Lambda_1 = \frac{1}{10}\begin{bmatrix} 1.5 & 3 \\ 0 & -5 \end{bmatrix},\ C_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\mathrm{v}(t) = \begin{bmatrix} \mathrm{v}_1(t) \\ \mathrm{v}_2(t) \end{bmatrix}, \mathcal{P}_0(t,\mathrm{y}(t)) = \begin{bmatrix} \mathcal{P}_{01}(t,\mathrm{y}(t)) \\ \mathcal{P}_{02}(t,\mathrm{y}(t)) \end{bmatrix},$$
$$\mathcal{P}_1(t,\mathrm{y}(t)) = \begin{bmatrix} \mathcal{P}_{11}(t,\mathrm{y}(t)) \\ \mathcal{P}_{12}(t,\mathrm{y}(t)) \end{bmatrix},$$
$$\mathcal{G}_1(t,\mathrm{y}(t)) = \begin{bmatrix} \mathcal{G}_{11}(t,\mathrm{y}(t)) \\ \mathcal{G}_{21}(t,\mathrm{y}(t)) \end{bmatrix},$$

with

$$\mathcal{P}_{01}(t,\mathrm{y}(t)) = \frac{t^3(5+|\mathrm{y}_1(t)|)}{90e^{t+7}(1+|\mathrm{y}_1(t)|)} + \frac{e^t}{2},$$
$$\mathcal{P}_{02}(t,\mathrm{y}(t)) = \frac{\sin(\mathrm{y}_2(t))}{30(3+2t^2)e^{t+7}},$$
$$\mathcal{P}_{11}(t,\mathrm{y}(t)) = \frac{(1+t)^2(7+|\mathrm{y}_2(t)|)}{85e^{t+8}(5+|\mathrm{y}_2(t)|)} + e^{t^2},$$
$$\mathcal{P}_{11}(t,\mathrm{y}(t)) = \frac{t^2\mathrm{y}_1(t)}{90e^{t^2+8}},$$
$$\mathcal{G}_{11}(t,\mathrm{y}(t)) = \frac{t\cos(\mathrm{y}_1(t))}{25(5+t)e^{t+8}} + \sin(t)e^t,$$
$$\mathcal{G}_{21}(t,\mathrm{y}(t)) = \frac{\sin(t)\mathrm{y}_2(t)}{50e^{t+9}} + \frac{\cos(t)}{e^t}.$$

We choose the final target points as $\mathrm{y}(t_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathrm{y}(T) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Clearly, we can see that the assumptions (Z1) and (Z2) hold. Also, one can easily calculate

$$\mathcal{Q}_0 = t_1^{\vartheta}c_2L_{\mathcal{P}}B(\varpi,\vartheta) = 1.9550\times 10^{-05}$$
$$\mathcal{Q}_1 = c_1L_{\mathcal{G}}t_2^{\varpi-1} + t_2^{\vartheta}c_2L_{\mathcal{P}}B(\varpi,\vartheta) = 2.6885\times 10^{-05}.$$

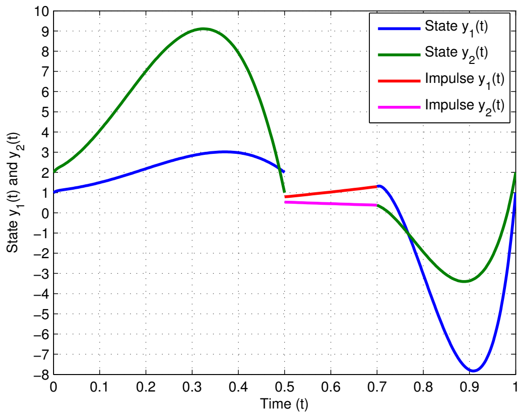
Thus, assumption (Z3) hold. Therefore, all the conditions of Theorem 1 and 2 satisfied and hence the system (6.28) has a HU stable unique solution. Now, to apply the Theorem 3, it remains to check the conditions (Z4) and (Z5). After some calculations, we get

$$\mathcal{Z}_0^{t_1} = \begin{bmatrix} 0.0054 & 0.0247 \\ 0.0247 & 0.1162 \end{bmatrix}, \mathcal{Z}_{\vartheta_1}^{t_2} = \begin{bmatrix} 0.0701 & 0.0297 \\ 0.0297 & 0.0130 \end{bmatrix}$$

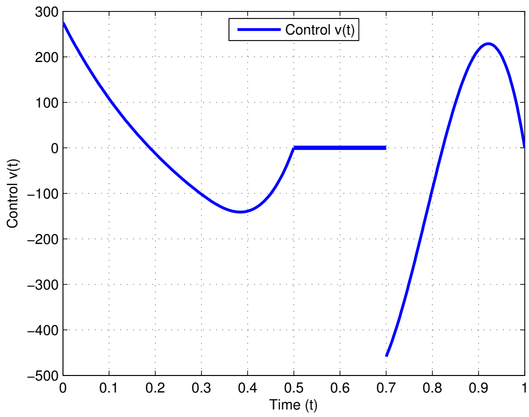
and

$$\mathcal{R}_0 = \mathcal{Q}_0(1+K_0) = 0.1423, \mathcal{R}_1 = \mathcal{Q}_1(1+K_1) = 0.1091.$$

Hence, $L_{\Xi_2} = \max\{\mathcal{R}_0, \mathcal{R}_1, L_{\mathcal{G}}\} = 0.1423 < 1$. Thus, all the assumptions of the Theorem 3 fulfilled. Hence,



(a) State trajectory of the system (6.28) when $x(t_1) = [2\ 1]^*$ and $x(T) = [1\ 2]^*$



(b) Trajectory of the control function $\mathsf{v}(t)$

Figure 2: (a) shows the controlled trajectory of the system (6.28), (b) shows the trajectory of the control function for the system (6.28)

the switched impulsive control system (6.28) is totally controllable on $[0, 1]$. The controlled state trajectory of the system (6.28) is shown in the Figure 2a and the control function is shown in the Figure 2b. Also, the CPU run time for different time intervals is given in the Table 2.

Conclusion

In this article, we have successfully investigated the existence, uniqueness, UH stability, and total controllability results of Hilfer fractional switched dynamical system with non-instantaneous jump. More precisely, we established the existence of a unique solution and UH stability of the system (1.3) by using the Banach contraction principle, fractional calculus and Mittag Leffler function. Further, some sufficient conditions are investigated to guarantee that system (1.4) is totally controllable. Finally, we have presented some numerical examples to validate the effectiveness of the obtained analytical outcomes. The stochastic differential equations play an important role in many fields of science, therefore, in the future, one can use the technique of this manuscript to establish the controllability results for the nonlinear Hilfer fractional switched impulsive dynamic systems with stochastic effects.

Table 2: CPU time for different time intervals

Time step	Intervals	CPU time (sec)
0.01	$[0, t_1]$	1.3246e+03
0.01	$[t_1, s_1]$	0.5781
0.01	$[s_1, t_2]$	1.4069e+03

Acknowledgement

The authors would like to express their sincere thanks to the associate editor and anonymous reviewers for their constructive comments and suggestions to improve the quality of this manuscript.

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