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Boundary and Eisenstein cohomology of $G_2(\mathbb{Z})$

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Abstract
In this article, Eisenstein cohomology of the arithmetic group $G_2(\mathbb{Z})$ with coefficients in any finite dimensional highest weight irreducible representation has been determined. This is accomplished by studying the cohomology of the boundary of the Borel–Serre compactification.

Keywords: G_2, Borel–Serre compactification, Boundary and Eisenstein cohomology

1 Introduction

Let G be a semisimple algebraic group defined over $\mathbb{Q}$, $K_\infty \subset \mathrm{G}(\mathbb{R})$ be a maximal compact subgroup and $\mathrm{S} = \mathrm{G}(\mathbb{R})/K_\infty$ be the corresponding symmetric space. If $\Gamma \subset \mathrm{G}(\mathbb{Q})$ is an arithmetic subgroup then every representation $(\rho, \mathcal{M})$ of G defines, in a natural way, a sheaf $\widetilde{\mathcal{M}}$ on the locally symmetric space $\mathrm{S}_\Gamma = \Gamma\backslash\mathrm{S}$. Since S is contractible, the locally symmetric space S_Γ is homotopic to the Eilenberg–MacLane space $K(\Gamma, 1)$ for Γ, hence one has the isomorphism

$$H^\bullet(\Gamma, \mathcal{M}) \cong H^\bullet(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}), \tag{1}$$

for details see Chapter 7 of [3]. Note that, throughout the paper, we use $H^\bullet$ to represent the full cohomology group, namely, $H^\bullet(\Gamma, \mathcal{M}) = \oplus_q H^q(\Gamma, \mathcal{M})$. On the other hand, let $\overline{\mathrm{S}}_\Gamma$ denote the Borel–Serre compactification of S_Γ, then the inclusion $i : \mathrm{S}_\Gamma \hookrightarrow \overline{\mathrm{S}}_\Gamma$, which is a homotopy equivalence, induces a canonical isomorphism in the cohomology

$$H^\bullet(\overline{\mathrm{S}}_\Gamma, i_*(\widetilde{\mathcal{M}})) \cong H^\bullet(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}), \tag{2}$$

where i_* denotes the direct image functor defined by i. On the other hand, let $\partial\mathrm{S}_\Gamma = \overline{\mathrm{S}}_\Gamma \backslash \mathrm{S}_\Gamma$ and $j : \partial\mathrm{S}_\Gamma \to \overline{\mathrm{S}}_\Gamma$ be the closed embedding. Then, the following exact sequence of sheaves

$$0 \to i_!(\widetilde{\mathcal{M}}) \to i_*(\widetilde{\mathcal{M}}) \to j_*(\widetilde{\mathcal{M}}) \to 0$$

gives rise to a long exact sequence of cohomology groups associated to S_Γ,

$$\cdots \to H^q_c(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}) \to H^q(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}) \xrightarrow{r} H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}) \to \cdots.$$

The cohomology group $H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$ will be called the *boundary cohomology of* Γ *with coefficients in* $\mathcal{M}$. Then, the *Eisenstein cohomology of* Γ *with coefficients in* $\mathcal{M}$, denoted by $H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$, is defined to be the image of the map r in the following exact sequence

$$0 \to H^q_!(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}) \to H^q(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}) \xrightarrow{r} H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}) \to 0,$$

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where $H^q_!(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$ is the kernel of the restriction map r.

The study of Eisenstein cohomology was initiated by Harder [6], and he discovered that Eisenstein cohomology is fundamentally related to several important topics in number theory, e.g., special values of L-functions, extension of motives, to simply mention a few. See Harder's ICM report [8] for more details on the relation of Eisenstein cohomology with other topics. The interested reader is also referred to [10] for recent advances on the subject. Although lots of work have been done, our understanding of Eisenstein cohomology is still far from complete.

Let G_2 be the Chevalley group defined over $\mathbb{Z}$ of type G_2. The main purpose of this article is to determine the boundary and Eisenstein cohomology of the arithmetic group $G_2(\mathbb{Z})$ with coefficients in any finite dimensional highest weight representation $\mathcal{M}$ of G_2. Let us now state the main results obtained in this article.

- Theorem 9, where the boundary cohomology with coefficients in every finite dimensional highest weight representation is described.
- Theorem 10, where the Eisenstein cohomology with coefficients in every finite dimensional highest weight representation is described.

1.1 Related results

The Eisenstein cohomology of arithmetic subgroups of $G_2(\mathbb{Q})$ with trivial coefficients has been previously studied in [13]. The basic setting of this article is more restrictive, that is, we only consider the cohomology of $G_2(\mathbb{Z})$, but we provide complete results for the boundary and Eisenstein cohomology of $G_2(\mathbb{Z})$ with coefficients in any finite dimensional highest weight representation.

On the other hand, it is worthwhile to mention the article [1], where the boundary and Eisenstein cohomology of $\mathrm{SL}_3(\mathbb{Z})$ with coefficients in any finite dimensional highest weight representation is determined by using Euler characteristic. Unfortunately, their method, being elementary but tricky, does not work here in the case of $G_2(\mathbb{Z})$ for dimensional reasons.

1.2 Methodology

Let us first discuss the computation of the boundary cohomology. The Borel–Serre boundary $\partial\mathrm{S}_\Gamma$ is a finite union of the boundary components parametrized by certain conjugacy classes of parabolic subgroups. Then, the cohomology of $\partial\mathrm{S}_\Gamma$ can be computed by using a spectral sequence whose first page consists of the cohomology of each boundary component, which will be computed by using the classical results of the cohomology of $\mathrm{GL}_2(\mathbb{Z})$ and the theorem of Kostant.

For the computation of Eisenstein cohomology, we follow closely the work of Harder in [9]. In particular, our method is constructive and involves the theory of Eisenstein series, intertwining operators, $(\mathfrak{g}, K_\infty)$-cohomology and L-functions. Indeed, it is well known that the Eisenstein cohomology $H^\bullet_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$ spans a maximal isotropic subspace of the boundary cohomology $H^\bullet(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$ with respect to the Poincaré duality. In particular, we know that $\dim H^\bullet_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$ is exactly half of $\dim H^\bullet(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$. Hence, we are done if we manage to construct enough classes in $H^\bullet_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$. Starting from cohomology classes in $H^\bullet(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$, we can construct cohomology classes in $H^\bullet(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}})$ by evaluating the corresponding Eisenstein series at the special point, and then we get non-trivial Eisenstein cohomology classes by restricting these classes back to the boundary. Subtlety appears when the corresponding Eisenstein series is not holomorphic at the special point, or equivalently, the corresponding L-function is not holomorphic at the special point. We establish our results by further exploring the Hecke action and Poincaré duality, as well as a detailed study of the corresponding $(\mathfrak{g}, K_\infty)$-cohomology.

1.3 Overview of the article

We quickly summarize the content of each section of the article. In Sect. 2, we review the basic structure of the group G_2 and the highest weight representations. In Sect. 3 we compute the cohomology of each boundary components, and in Sect. 4 we compute the boundary cohomology. In Sect. 5, we achieve our main goal by determining the Eisenstein cohomology of $G_2(\mathbb{Z})$ for every finite dimensional highest weight representation.

2 Preliminaries

This section quickly reviews the basic properties of G_2 and familiarize the reader with the notations to be used throughout the article. We discuss the corresponding locally symmetric space, Weyl group, the associated spectral sequence and Kostant representatives of the standard parabolic subgroups.

2.1 Structure theory

Let G_2 be the Chevalley group defined over $\mathbb{Z}$ of type G_2 and Φ be the corresponding root system. Let us fix a maximal $\mathbb{Q}$-split torus T and a Borel subgroup B that contains T. The set of simple roots associated to B is denoted by $\Delta = \{\alpha_1, \alpha_2\}$ with α_1 and α_2 the short and long simple roots, respectively. The Weyl group $\mathcal{W}$ of Φ is isomorphic to the dihedral group D_6. The fundamental weights associated to this root system are given by $\gamma_1 = 2\alpha_1 + \alpha_2$ and $\gamma_2 = 3\alpha_1 + 2\alpha_2$.

Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{g}_2$ and $\mathfrak{t} \subset \mathfrak{g}$ be the Lie subalgebra associated to T. Let $\Phi = \Phi^+ \cup \Phi^-$ be the corresponding root system. We know that

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Finally, we write $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha = 5\alpha_1 + 3\alpha_2$.

Recall that a $\mathbb{Q}$-parabolic subgroup is called *standard* if it contains the Borel subgroup B. Let $\{\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2\}$ be the set of standard $\mathbb{Q}$-parabolic subgroups, where P_1 (resp. P_2) denotes the maximal $\mathbb{Q}$-parabolic subgroup corresponding to the simple roots α_1 (resp. α_2) and $\mathrm{P}_0 = \mathrm{B}$. Thus,

$$\mathfrak{p}_1 = \mathfrak{u}_{-\alpha_2} \oplus \mathfrak{t} \oplus_{\alpha \in \Phi^+} \mathfrak{u}_\alpha, \text{ and } \mathfrak{p}_2 = \mathfrak{u}_{-\alpha_1} \oplus \mathfrak{t} \oplus_{\alpha \in \Phi^+} \mathfrak{u}_\alpha,$$

where $\mathfrak{p}_i$ denotes the Lie algebra of P_i and $\mathfrak{u}_\alpha$ denotes the root space corresponding to α. Note that the minimal $\mathbb{Q}$-parabolic P_0 is simply the group $\mathrm{P}_1 \cap \mathrm{P}_2$. Therefore, the corresponding Levi quotients are given by

$$\mathrm{M}_0 = \mathbb{G}_m^2, \quad \mathrm{M}_1 = \mathrm{GL}_2 \quad \text{and} \quad \mathrm{M}_2 = \mathrm{GL}_2.$$

Let us choose and fix a maximal compact subgroup $K_\infty \subset G_2(\mathbb{R})$. It is well known that it can be identified with $SO_4(\mathbb{R})$. From now on throughout the article, let $\mathrm{S} = G_2(\mathbb{R})/K_\infty$, Γ be the arithmetic group $G_2(\mathbb{Z})$ and $\mathrm{S}_\Gamma = \Gamma \backslash \mathrm{S}$.

2.2 Γ-conjugacy classes of $\mathbb{Q}$-parabolic subgroups

In this subsection, we determine the Γ-conjugacy classes of $\mathbb{Q}$-parabolic subgroups $\mathcal{P}_{\mathbb{Q}}(G_2, G_2(\mathbb{Z}))$, which should be well known. But as we don't know of a proper reference, a proof is given below.

Lemma 1 $\mathcal{P}_{\mathbb{Q}}(G_2, G_2(\mathbb{Z})) = \{\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2\}$.

Proof We have to show that for each standard parabolic P, $G_2(\mathbb{Z})$ acts transitively on $G_2/\mathrm{P}(\mathbb{Q})$. Since G_2 is semisimple algebraic group split over $\mathbb{Q}$, by strong approximation, we have

$$G_2(\mathbb{Q}) \prod_p G_2(\mathbb{Z}_p) = G_2(\mathbb{A}_f),$$

which implies that $G_2(\mathbb{Q})/G_2(\mathbb{Z}) = G_2(\mathbb{A}_f)/\prod_p G_2(\mathbb{Z}_p)$. Moreover, as $\mathbb{Q}$ is of class number 1, we have

$$\mathrm{P}(\mathbb{Q}) \prod_p \mathrm{P}(\mathbb{Z}_p) = \mathrm{P}(\mathbb{Q}_p).$$

According to the Iwasawa decomposition for p-adic groups, $\mathrm{P}(\mathbb{Q}_p)G_2(\mathbb{Z}_p) = G_2(\mathbb{Q}_p)$. Consequently, we have

$$\mathrm{P}(\mathbb{Q}) \prod_p G_2(\mathbb{Z}_p) = G_2(\mathbb{A}_f).$$

Hence $\mathrm{P}(\mathbb{Q})$ acts transitively on $G_2(\mathbb{Q})/G_2(\mathbb{Z})$, which implies $G_2(\mathbb{Z})$ acts transitively on $G_2(\mathbb{Q})/\mathrm{P}(\mathbb{Q})$.

On the other hand, we have the following sequence of Galois cohomology,

$$1 \to \mathrm{P}(\mathbb{Q}) \to G_2(\mathbb{Q}) \to G_2/\mathrm{P}(\mathbb{Q}) \to H^1(\mathbb{Q}, \mathrm{P}) \to H^1(\mathbb{Q}, G_2) \to \cdots.$$

Note that P is the semi-direct product of its unipotent radical and its Levi subgroup, which is isomorphic to either $\mathbb{G}_m^2$ or GL_2. According to Hilbert's Theorem 90 (see, e.g. [21, Chapter III]), we have $H^1(\mathbb{Q}, \mathrm{P}) = 1$. Thus, $G_2/\mathrm{P}(\mathbb{Q}) = G_2(\mathbb{Q})/\mathrm{P}(\mathbb{Q})$, from which the lemma follows. □

2.3 Irreducible representations

The fundamental weights associated to Φ^+ are given by $\gamma_1 = 2\alpha_1+\alpha_2$ and $\gamma_2 = 3\alpha_1+2\alpha_2$. Thus, irreducible finite dimensional representations of G_2 are determined by their highest weights, which in this case are the linear functionals of the form $m_1\gamma_1 + m_2\gamma_2$ with m_1, m_2 nonnegative integers. For any $\lambda = m_1\gamma_1 + m_2\gamma_2$, we set $\mathcal{M}_\lambda$ to be the representation defined over $\mathbb{Q}$ with highest weight λ.

2.4 Kostant representatives

It is known that the Weyl group $\mathcal{W} = \mathcal{W}(\Phi)$ is the dihedral group D_6 given by 12 elements. They are listed in the first column of Table 1 and described in the second column as a product of simple reflections s_1 and s_2, associated to the simple roots α_1 and α_2, respectively. Then, we have

$$s_1(\alpha_1) = -\alpha_1,\ s_1(\alpha_2) = 3\alpha_1 + \alpha_2 \quad \text{and} \quad s_2(\alpha_1) = \alpha_1 + \alpha_2,\ s_2(\alpha_2) = -\alpha_2. \tag{3}$$

In the third column we make a note of their lengths, and in the last column we describe the element $w \cdot \lambda = w(\lambda + \rho) - \rho$, where the pair (a, b) denotes the element $a\alpha_1 + b\alpha_2 \in \mathfrak{t}^*$.

Table 1 The Weyl Group of G_2

Label	w	$\ell(w)$	$w \cdot \lambda$
w_1	1	0	$(2m_1 + 3m_2,\ m_1 + 2m_2)$
w_2	s_1	1	$(m_1 + 3m_2 - 1,\ m_1 + 2m_2)$
w_3	s_2	1	$(2m_1 + 3m_2,\ m_1 + m_2 - 1)$
w_4	s_1s_2	2	$(m_1 - 4,\ m_1 + m_2 - 1)$
w_5	s_2s_1	2	$(m_1 + 3m_2 - 1,\ m_2 - 2)$
w_6	$s_1s_2s_1$	3	$(-m_1 - 6,\ m_2 - 2)$
w_7	$s_2s_1s_2$	3	$(m_1 - 4,\ -m_2 - 4)$
w_8	$s_1s_2s_1s_2$	4	$(-m_1 - 3m_2 - 9,\ -m_2 - 4)$
w_9	$s_2s_1s_2s_1$	4	$(-m_1 - 6,\ -m_1 - m_2 - 5)$
w_{10}	$s_1s_2s_1s_2s_1$	5	$(-2m_1 - 3m_2 - 10,\ -m_1 - m_2 - 5)$
w_{11}	$s_2s_1s_2s_1s_2$	5	$(-m_1 - 3m_2 - 9,\ -m_1 - 2m_2 - 6)$
w_{12}	$s_1s_2s_1s_2s_1s_2$	6	$(-2m_1 - 3m_2 - 10,\ -m_1 - 2m_2 - 6)$

The Weyl group acts naturally on the set of roots. For each $i \in \{0, 1, 2\}$, let $\Delta(\mathrm{u}_i)$ denote the set consisting of every root whose corresponding root space is contained in the Lie algebra u_i of the unipotent radical of P_i. The set of Weyl representatives $\mathcal{W}^{\mathrm{P}_i} \subset \mathcal{W}$ associated to the parabolic subgroup P_i (see [12]) is defined by

$$\mathcal{W}^{\mathrm{P}_i} = \left\{ w \in \mathcal{W} : w(\Phi^-) \cap \Phi^+ \subset \Delta(\mathrm{u}_i) \right\}.$$

Clearly $\mathcal{W}^{\mathrm{P}_0} = \mathcal{W}$ and, by using the table, one can see that

$$\mathcal{W}^{\mathrm{P}_1} = \{1, s_1, s_1s_2, s_1s_2s_1, s_1s_2s_1s_2, s_1s_2s_1s_2s_1\} \tag{4}$$

$$\mathcal{W}^{\mathrm{P}_2} = \{1, s_2, s_2s_1, s_2s_1s_2, s_2s_1s_2s_1, s_2s_1s_2s_1s_2\}\ . \tag{5}$$

2.4.1 Kostant representatives for minimal parabolic P_0

$$\begin{aligned}
w_1 \cdot \lambda\ &= m_1\gamma_1 + m_2\gamma_2 \\
w_2 \cdot \lambda\ &= (-m_1 - 2)\gamma_1 + (m_1 + m_2 + 1)\gamma_2 \\
w_3 \cdot \lambda\ &= (m_1 + 3m_2 + 3)\gamma_1 + (-m_2 - 2)\gamma_2 \\
w_4 \cdot \lambda\ &= (-m_1 - 3m_2 - 5)\gamma_1 + (m_1 + 2m_2 + 2)\gamma_2 \\
w_5 \cdot \lambda\ &= (2m_1 + 3m_2 + 4)\gamma_1 + (-m_1 - m_2 - 3)\gamma_2 \\
w_6 \cdot \lambda\ &= (-2m_1 - 3m_2 - 6)\gamma_1 + (m_1 + 2m_2 + 2)\gamma_2 \\
w_7 \cdot \lambda\ &= (2m_1 + 3m_2 + 4)\gamma_1 + (-m_1 - 2m_2 - 4)\gamma_2 \\
w_8 \cdot \lambda\ &= (-2m_1 - 3m_2 - 6)\gamma_1 + (m_1 + m_2 + 1)\gamma_2 \\
w_9 \cdot \lambda\ &= (m_1 + 3m_2 + 3)\gamma_1 + (-m_1 - 2m_2 - 4)\gamma_2 \\
w_{10} \cdot \lambda\ &= (-m_1 - 3m_2 - 5)\gamma_1 + m_2\gamma_2 \\
w_{11} \cdot \lambda\ &= m_1\gamma_1 + (-m_1 - m_2 - 3)\gamma_2 \\
w_{12} \cdot \lambda\ &= (-m_1 - 2)\gamma_1 + (-m_2 - 2)\gamma_2
\end{aligned}$$

2.4.2 Kostant representatives for maximal parabolic P_1

Here, we take

$$\gamma^{M_1} = \frac{1}{2}\alpha_2 = -\frac{3}{2}\gamma_1 + \gamma_2 \quad \text{and} \quad \kappa^{M_1} = \frac{1}{2}\gamma_1$$
$$w_1 \cdot \lambda = m_2\gamma^{M_1} + (2m_1 + 3m_2)\kappa^{M_1}$$
$$w_2 \cdot \lambda = (m_1 + m_2 + 1)\gamma^{M_1} + (m_1 + 3m_2 - 1)\kappa^{M_1}$$
$$w_4 \cdot \lambda = (m_1 + 2m_2 + 2)\gamma^{M_1} + (m_1 - 4)\kappa^{M_1}$$
$$w_6 \cdot \lambda = (m_1 + 2m_2 + 2)\gamma^{M_1} + (-m_1 - 6)\kappa^{M_1}$$
$$w_8 \cdot \lambda = (m_1 + m_2 + 1)\gamma^{M_1} + (-m_1 - 3m_2 - 9)\kappa^{M_1}$$
$$w_{10} \cdot \lambda = m_2\gamma^{M_1} + (-2m_1 - 3m_2 - 10)\kappa^{M_1}$$

2.4.3 Kostant representatives for maximal parabolic P_2

Here, we take

$$\gamma^{M_2} = \frac{1}{2}\alpha_1 = \gamma_1 - \frac{1}{2}\gamma_2 \quad \text{and} \quad \kappa^{M_2} = \frac{1}{2}\gamma_2.$$
$$w_1 \cdot \lambda = m_1\gamma^{M_2} + (m_1 + 2m_2)\kappa^{M_2}$$
$$w_3 \cdot \lambda = (m_1 + 3m_2 + 3)\gamma^{M_2} + (m_1 + m_2 - 1)\kappa^{M_2}$$
$$w_5 \cdot \lambda = (2m_1 + 3m_2 + 4)\gamma^{M_2} + (m_2 - 2)\kappa^{M_2}$$
$$w_7 \cdot \lambda = (2m_1 + 3m_2 + 4)\gamma^{M_2} + (-m_2 - 4)\kappa^{M_2}$$
$$w_9 \cdot \lambda = (m_1 + 3m_2 + 3)\gamma^{M_2} + (-m_1 - m_2 - 5)\kappa^{M_2}$$
$$w_{11} \cdot \lambda = m_1\gamma^{M_2} + (-m_1 - 2m_2 - 6)\kappa^{M_2}$$

For $i = 1, 2$, we also write

$$w \cdot \lambda = a_i(\lambda, w)\gamma^{M_i} + b_i(\lambda, w)\kappa^{M_i}. \tag{6}$$

The symmetry of the coefficients can be explained by the following lemma.

Lemma 2 *Let* P *be a parabolic subgroup of* G_2 *with Levi subgroup* M, A_P *be the central torus of* M *and* S_P *be the unique maximal torus of the semisimple part of* M *contained in* T. *Let* N_P *denote the unipotent radical of* P *in* G_2 *and* w_{G_2} *(resp.* w_M*) be the longest Weyl element in* $\mathcal{W}$ *(resp.* $\mathcal{W}_M$*). Then, the following are true.*

(a) *The map* $w \mapsto w' := w_M w w_{G_2}$ *defines an involution on* $\mathcal{W}^P$ *and* $\ell(w) + \ell(w') = \dim N_P$.
(b) $w'(\lambda + \rho) - \rho|_{A_P} = w(\lambda + \rho) - \rho|_{A_P}$.
(c) $w'(\lambda + \rho) - \rho|_{S_P} + w'(\lambda + \rho) - \rho|_{S_P} = -2\rho|_{S_P}$.

Proof The proof is the same as in Schwermer [20, Section 4.2] by noticing the fact that the Weyl group $\mathcal{W}$ is self-dual, namely $w_{G_2} = -1$. □

2.5 Boundary of the Borel–Serre compactification

Let $\mathcal{P}_{\mathbb{Q}}(G, \Gamma)$ be the equivalence class of parabolic subgroups defined over $\mathbb{Q}$, where two parabolic subgroups are equivalent if and only if they are conjugate under action of Γ. In general, the boundary of the Borel–Serre compactification $\partial S_\Gamma = \overline{S}_\Gamma \setminus S_\Gamma$ is a finite union of the boundary components ∂_P for $P \in \mathcal{P}_{\mathbb{Q}}(G, \Gamma)$, i.e.

$$\partial S_\Gamma = \bigcup_{P \in \mathcal{P}_{\mathbb{Q}}(G,\Gamma)} \partial_P, \tag{7}$$

where

$$\partial_{\mathrm{P}} = (\Gamma \cap \mathrm{P}(\mathbb{R}))\backslash \mathrm{P}(\mathbb{R})/(\mathrm{P}(\mathbb{R}) \cap K_\infty).$$

This decomposition (7) determines a spectral sequence in cohomology abutting to the cohomology of the boundary

$$E_1^{p,q} = \bigoplus_{prk(\mathrm{P})=p+1} H^q(\partial_{\mathrm{P}}, \widetilde{\mathcal{M}}_\lambda) \Rightarrow H^{p+q}(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) \quad (8)$$

where $prk(\mathrm{P})$ denotes the parabolic rank of P (the dimension of the maximal $\mathbb{Q}$-split torus in the center of the Levi quotient M of P).

In our case, as the $\mathbb{Q}$-rank of G is 2, this spectral sequence is simplified to a long exact sequence (of Mayer–Vietoris) in cohomology of the following form

$$\cdots \to H^q(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) \to H^q(\partial_{\mathrm{P}_1}, \widetilde{\mathcal{M}}_\lambda) \oplus H^q(\partial_{\mathrm{P}_2}, \widetilde{\mathcal{M}}_\lambda) \to H^q(\partial_{\mathrm{P}_0}, \widetilde{\mathcal{M}}_\lambda) \to \cdots \quad (9)$$

by noticing Lemma 1. We will use this long exact sequence to describe the cohomology of the boundary of the Borel–Serre compactification.

2.6 The theorem of Kostant

Let $\mathrm{P} = \mathrm{MN}$ be the decomposition into its Levi subgroup M and unipotent radical N. Then, ∂_{P} is a fiber bundle over

$$\mathrm{S}_\Gamma^{\mathrm{M}} = (\Gamma \cap \mathrm{M}(\mathbb{R}))\backslash \mathrm{M}(\mathbb{R})/(\mathrm{M}(\mathbb{R}) \cap K_\infty)$$

with fibers isomorphic to $\mathrm{N}_\Gamma := (\Gamma \cap \mathrm{N}(\mathbb{R}))\backslash \mathrm{N}(\mathbb{R})$. Hence, we have

$$H^\bullet(\partial_{\mathrm{P}}, \widetilde{\mathcal{M}}_\lambda) = H^\bullet(\mathrm{S}_\Gamma^{\mathrm{M}}, H^\bullet(\mathrm{N}_\Gamma, \widetilde{\mathcal{M}}_\lambda)),$$

here we use $H^\bullet(\mathrm{N}_\Gamma, \widetilde{\mathcal{M}}_\lambda)$ to denote the corresponding sheaf by abuse of notation. By the theorem of Nomizu [15], we know

$$H^\bullet(\mathrm{N}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = H^\bullet(\mathfrak{n}, \mathcal{M}_\lambda).$$

By the theorem of Kostant [12], we get

$$H^q(\mathfrak{n}, \mathcal{M}_\lambda) = \bigoplus_{w \in \mathcal{W}^{\mathrm{P}}:\ell(w)=q} \mathcal{N}_{w\cdot\lambda},$$

where $\mathcal{N}_{w\cdot\lambda}$ denotes the irreducible representation of M with highest weight $w \cdot \lambda$. In conclusion, we get

$$H^q(\partial_{\mathrm{P}}, \widetilde{\mathcal{M}}_\lambda) = \bigoplus_{w \in \mathcal{W}^{\mathrm{P}}} H^{q-\ell(w)}(\mathrm{S}_\Gamma^{\mathrm{M}}, \widetilde{\mathcal{N}}_{w\cdot\lambda}).$$

Note that, when there is no ambiguity, we also use the old notation $\mathcal{M}_{w\cdot\lambda}$ to denote $\mathcal{N}_{w\cdot\lambda}$.

2.7 Cohomological dimension

For any discrete group H, set the *virtual cohomological dimension* of H, denoted as $\mathrm{vcd}H$, to be

$$\mathrm{vcd}H = \min\{\mathrm{cd}H' : [H : H'] < \infty\},$$

where $\mathrm{cd}H'$ refers to the cohomological dimension of H'. Now, using the compactification they had introduced, Borel and Serre showed in [2] that for any semisimple group G and its arithmetic subgroup H:

$$\mathrm{vcd}H = \dim \mathrm{G} - \dim K - \mathrm{rank}_{\mathbb{Q}}\mathrm{G},$$

where K is the maximal compact subgroup of $\mathrm{G}(\mathbb{R})$. In particular, $\dim G_2 = 14$, $\dim \mathrm{SO}_4(\mathbb{R}) = 6$ and $\mathrm{rank}_{\mathbb{Q}} G_2 = 2$, thus we have $\mathrm{vcd}G_2(\mathbb{Z}) = 6$. As a consequence, $H^q(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}) = 0$ for all $q > 6$ for any coefficient system $\mathcal{M}$ of Γ.

Remark 3 It is interesting to note that, for any $\mathbb{Q}$-split semisimple group G, we have

$$\mathrm{vcdG}(\mathbb{Z}) = \dim \mathrm{N} = \max_{w\in\mathcal{W}} \ell(w),$$

where N is the unipotent radical of any Borel subgroup and $\ell(w)$ is the length of w. Indeed, the first equality follows from the Iwasawa decomposition, while the second one follows from theory of Weyl groups. Note that this does not hold in general. For example: when G is $\mathbb{Q}$-anisotropic, $\mathrm{vcdG}(\mathbb{Z})$ equals to $\dim \mathrm{G} - \dim K$, which is not the maximal length of Weyl elements.

3 Cohomology of the boundary components

The cohomology of the boundary is obtained by using a spectral sequence whose terms are expressed by the cohomology of the faces associated to each standard parabolic subgroup. In this section, we establish, for each standard parabolic P and irreducible representation $\mathcal{M}_\nu$ of the Levi subgroup $\mathrm{M} \subset \mathrm{P}$ with highest weight ν, a condition to be satisfied in order to have non-trivial cohomology $H^\bullet(\mathrm{S}_\Gamma^\mathrm{M}, \widetilde{\mathcal{M}}_\nu)$. Here, $\widetilde{\mathcal{M}}_\nu$ is the sheaf on $\mathrm{S}_\Gamma^\mathrm{M}$ given by $\mathcal{M}_\nu$.

3.1 Minimal parabolic subgroup

We analyze the parity condition imposed to the face associated to the minimal parabolic ∂_{P_0}. As mentioned, the Levi subgroup of P_0 is the two-dimensional torus $\mathrm{M}_0 \cong \mathbb{G}_m^2$. The elements lying in $\Xi := \mathrm{M}_0(\mathbb{Z}) \cap \mathrm{K}_\infty$ must act trivially on the representation $\mathcal{M}_\nu$ in order to have nonzero cohomology. By using this fact, one can deduce the following

Lemma 4 *Let ν be given by $m_1'\gamma_1 + m_2'\gamma_2$. If m_1' or m_2' is odd then the corresponding local system $\widetilde{\mathcal{M}}_\nu$ on $\mathrm{S}_\Gamma^{\mathrm{M}_0}$ is cohomologically trivial, that is $H^\bullet(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_\nu) = 0$.*

Proof According to [10, Prop 4.3], we have $H^\bullet(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_\nu) = H^\bullet(\widetilde{\mathrm{S}_\Gamma^{\mathrm{M}_0}}, \widetilde{\mathcal{M}}_\nu)^\Xi$, where $\widetilde{\mathrm{S}_\Gamma^{\mathrm{M}_0}}$ denotes the locally symmetric spaces associated to M_0. As $\widetilde{\mathrm{S}_\Gamma^{\mathrm{M}_0}}$ is simply a point, all the higher cohomology vanishes. It is clear that $\Xi \cong (\mathbb{Z}/2\mathbb{Z})^2$ and for each $\xi \in \Xi$ the action on $\mathcal{M}$ is given by $\nu(\xi) \in \{-1, 1\}$. Thus, if there exists $\xi \in \Xi$ with $\nu(\xi) = -1$, then $H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_\nu) = 0$. On the other hand, since γ_1, γ_2 forms an integral basis for $X^*(\mathrm{M}_0) := \mathrm{Hom}(\mathrm{M}_0, \mathbb{G}_\mathrm{m})$, there exists $\xi \in \Xi$ with $\nu(\xi) = -1$ if m_1' or m_2' is odd. This completes the proof. □

Note that every ν will be of the form $w \cdot \lambda$ for $w \in \mathcal{W}$. We denote by $\overline{\mathcal{W}}^0$ the set of Weyl elements w such that $w \cdot \lambda$ do not satisfy the condition of Lemma 4.

Remark 5 For notational convenience, we simply use ∂_i to denote the boundary face ∂_{P_i} associated to the parabolic subgroup P_i and the arithmetic group Γ for $i \in \{0, 1, 2\}$.

3.1.1 Cohomology groups of ∂_0

In this case $H^q(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) = 0$ for every $q \geq 1$. The Weyl group $\mathcal{W}^{\mathrm{P}_0} = \mathcal{W}$ and the lengths of its elements are between 0 and 6 as shown in Table 1 above. Thus,

$$\begin{aligned} H^q(\partial_0, \widetilde{\mathcal{M}}_\lambda) &= \bigoplus_{w\in\mathcal{W}^{\mathrm{P}_0}} H^{q-\ell(w)}(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) \\ &= \bigoplus_{w\in\mathcal{W}^{\mathrm{P}_0}:\ell(w)=q} H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w\cdot\lambda}), \end{aligned}$$

Therefore, for $0 \le q \le 6$,

$$\begin{aligned}
H^0(\partial_0, \widetilde{\mathcal{M}}_\lambda) &= H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_\lambda)\\
H^1(\partial_0, \widetilde{\mathcal{M}}_\lambda) &= H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_3\cdot\lambda})\\
H^2(\partial_0, \widetilde{\mathcal{M}}_\lambda) &= H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda})\\
H^3(\partial_0, \widetilde{\mathcal{M}}_\lambda) &= H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda})\\
H^4(\partial_0, \widetilde{\mathcal{M}}_\lambda) &= H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda})\\
H^5(\partial_0, \widetilde{\mathcal{M}}_\lambda) &= H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda})\\
H^6(\partial_0, \widetilde{\mathcal{M}}_\lambda) &= H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_{12}\cdot\lambda})
\end{aligned}$$

and for every $q \ge 7$, the cohomology groups $H^q(\partial_0, \widetilde{\mathcal{M}}_\lambda) = 0$.

3.2 Maximal parabolic subgroups

In this section, we study the parity conditions for the maximal parabolics. Let $i \in \{1, 2\}$, then $\mathrm{M_i} \cong \mathrm{GL}_2$ and in this setting, $K_\infty \cap \mathrm{M}_i(\mathbb{R}) = \mathrm{O}_2(\mathbb{R})$ is the orthogonal group and $\Gamma_{\mathrm{M_i}} = \mathrm{GL}_2(\mathbb{Z})$. Therefore,

$$\mathrm{S}_\Gamma^{\mathrm{M_i}} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R})/\mathrm{O}(2)\mathbb{R}^\times_{>0}.$$

We also consider the following double cover of $\mathrm{S}_\Gamma^{\mathrm{M_i}}$,

$$\widetilde{\mathrm{S}_\Gamma^{\mathrm{M_i}}} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R})/\mathrm{SO}(2)\mathbb{R}^\times_{>0},$$

which is isomorphic to the locally symmetric space associated to $\mathrm{M_i}$.

Let i be 1 or 2. Recall from (6) that, for $w \in \mathcal{W}^{\mathrm{P}_i}$, $w \cdot \lambda = a_i(\lambda, w)\gamma^{\mathrm{M}_i} + b_i(\lambda, w)\kappa^{\mathrm{M}_i}$.

Lemma 6 *If $a_i(\lambda, w)$ is odd, or equivalently, $b_i(\lambda, w)$ is odd as they are congruent modulo 2, we have $H^\bullet(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) = 0$. Moreover, if $a_i(\lambda, w) = 0$ and $b_i(\lambda, w)/2$ is odd, then $H^\bullet(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) = 0$.*

Proof The proof here follows the proof of Lemma 4 closely. According to [10, Prop 4.3], we have $H^\bullet(\mathrm{S}_\Gamma^{\mathrm{M_i}}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) = H^\bullet(\widetilde{\mathrm{S}_\Gamma^{\mathrm{M_i}}}, \widetilde{\mathcal{M}}_{w\cdot\lambda})^{\Xi_i}$, where $\Xi_i = \mathrm{M}_i(\mathbb{Z}) \cap K_\infty$. Note that M_i can be identified with GL_2, and $K_\infty \cap \mathrm{M}_i(\mathbb{R})$ equals to $\mathrm{O}_2(\mathbb{R})$, hence Ξ_i can be identified as $\mathrm{GL}_2(\mathbb{Z}) \cap \mathrm{O}_2(\mathbb{R})$. For the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Xi_i$, the action on $H^\bullet(\widetilde{\mathrm{S}_\Gamma^{\mathrm{M_i}}}, \widetilde{\mathcal{M}}_\nu)$ is given by $(-1)^{a_i(\lambda,w)}$, hence we get the first conclusion. On the other hand, consider the element $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \Xi_i$, whose action on $H^\bullet(\widetilde{\mathrm{S}_\Gamma^{\mathrm{M_i}}}, \widetilde{\mathcal{M}}_\nu)$ is given by $(-1)^{b_i(\lambda,w)/2}$ when $a_i(\lambda, w) = 0$. This completes the proof. □

Note that, as a representation of $\mathrm{M}_i \cong \mathrm{GL}_2 (i = 1, 2)$,

$$\mathcal{M}_{w\cdot\lambda} \cong Sym^{a_i(\lambda,w)} V \otimes Det^{\frac{b_i(\lambda,w)-a_i(\lambda,w)}{2}}, \tag{10}$$

where V denotes the standard representation of GL_2. Let $\mathrm{B} \subset \mathrm{M}_i$ be a standard Borel subgroup and N be its unipotent radical with $\mathfrak{n}$ its Lie algebra. Then for $i = 1, 2$, we have the following exact sequence,

$$\begin{aligned}
&H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, H^0(\mathfrak{n}, \mathcal{M}_{w\cdot\lambda})) \hookrightarrow H^1_c(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \mathcal{M}_{w\cdot\lambda}) \to H^1(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \mathcal{M}_{w\cdot\lambda})\\
&\quad \twoheadrightarrow H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, H^1(\mathfrak{n}, \mathcal{M}_{w\cdot\lambda})).
\end{aligned}$$

Here, by abuse of notation, we use $H^0(\mathfrak{n}, \mathcal{M}_{w\cdot\lambda})$ and $H^1(\mathfrak{n}, \mathcal{M}_{w\cdot\lambda})$ to denote the corresponding sheaves. In view of Lemma 4 and (10), we get

$$H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, H^1(\mathfrak{n}, \mathcal{M}_{w\cdot\lambda})) = 0, \quad \text{if} \quad \frac{b_i(\lambda, w) - a_i(\lambda, w)}{2} \equiv 0\,(\text{mod } 2),$$

$$H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, H^0(\mathfrak{n}, \mathcal{M}_{w\cdot\lambda})) = 0, \quad \text{if} \quad \frac{b_i(\lambda, w) - a_i(\lambda, w)}{2} \equiv 1\,(\text{mod } 2).$$

Consequently, we have

$$H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \mathcal{M}_{w\cdot\lambda}) = H^1(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \mathcal{M}_{w\cdot\lambda}), \quad \text{if} \quad \frac{b_i(\lambda, w) - a_i(\lambda, w)}{2} \equiv 0\,(\text{mod } 2), \tag{11}$$

$$H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \mathcal{M}_{w\cdot\lambda}) = H_c^1(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \mathcal{M}_{w\cdot\lambda}), \quad \text{if} \quad \frac{b_i(\lambda, w) - a_i(\lambda, w)}{2} \equiv 1\,(\text{mod } 2). \tag{12}$$

This fact will be extensively used in the next section for the computation of boundary cohomology.

Throughout the paper, we set

$$S_{m+2} = H_!^1(\mathrm{GL}_2(\mathbb{Z}), Sym^m V) \cong H_!^1(\mathrm{GL}_2(\mathbb{Z}), Sym^m V \otimes Det). \tag{13}$$

It is well known that $H_!^\bullet(\mathrm{GL}_2(\mathbb{Z}), \mathcal{M}) = H_{cusp}^\bullet(\mathrm{GL}_2(\mathbb{Z}), \mathcal{M})$ for finite dimensional representation $\mathcal{M}$, hence it is safe to replace "!" by "*cusp*" in (13). Consequently, by Eichler–Shimura isomorphism, the space S_{m+2} can be identified with the space of holomorphic cusp forms of weight $m + 2$.

Remark 7 For notational convenience, in what follows we will denote the set of Weyl elements for which $H^\bullet(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) \neq 0$ by $\overline{\mathcal{W}}^i$.

In the following subsections, we make note of the cohomology groups associated to the boundary components ∂_1 and ∂_2 which will be used in the computations involved to determine the boundary cohomology in the next section.

3.2.1 Cohomology of ∂_1

In this case, the Levi M_1 is isomorphic to GL_2 and therefore $H^q(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) = 0$ for every $q \geq 2$. The set $\mathcal{W}^{\mathrm{P}_1} = \{w_1, w_2, w_4, w_6, w_8, w_{10}\}$ where the length of elements are, respectively, 0, 1, 2, 3, 4, 5. Thus,

$$\begin{aligned} H^q(\partial_1, \widetilde{\mathcal{M}}_\lambda) &= \bigoplus_{w \in \mathcal{W}^{\mathrm{P}_1}} H^{q-\ell(w)}(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) \\ &= H^q(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda) \oplus H^{q-1}(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^{q-2}(\mathrm{S}^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \\ &\quad \oplus H^{q-3}(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^{q-4}(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^{q-5}(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}). \end{aligned} \tag{14}$$

Therefore, for $0 \leq q \leq 6$,

$$\begin{aligned} H^0(\partial_1, \widetilde{\mathcal{M}}_\lambda) &= H^0(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda) \\ H^1(\partial_1, \widetilde{\mathcal{M}}_\lambda) &= H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \\ H^2(\partial_1, \widetilde{\mathcal{M}}_\lambda) &= H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \\ H^3(\partial_1, \widetilde{\mathcal{M}}_\lambda) &= H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \\ H^4(\partial_1, \widetilde{\mathcal{M}}_\lambda) &= H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \\ H^5(\partial_1, \widetilde{\mathcal{M}}_\lambda) &= H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \\ H^6(\partial_1, \widetilde{\mathcal{M}}_\lambda) &= H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \end{aligned}$$

and for every $q \geq 7$, the cohomology groups $H^q(\partial_1, \widetilde{\mathcal{M}}_\lambda) = 0$.

3.2.2 Cohomology of ∂_2

In this case, the Levi M_2 is isomorphic to GL_2 as well and therefore $H^q(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) = 0$ for every $q \geq 2$. The Weyl group $W^{P_2} = \{w_1, w_3, w_5, w_7, w_9, w_{11}\}$ where the length of elements are, respectively, 0, 1, 2, 3, 4, 5. Thus,

$$\begin{aligned} H^q(\partial_2, \widetilde{\mathcal{M}}_\lambda) &= \bigoplus_{w\in W^{P_2}} H^{q-\ell(w)}(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) \\ &= H^q(S^{M_2}, \widetilde{\mathcal{M}}_\lambda) \oplus H^{q-1}(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \oplus H^{q-2}(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \\ &\quad \oplus H^{q-3}(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \oplus H^{q-4}(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \oplus H^{q-5}(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}). \end{aligned} \tag{15}$$

Therefore, for $0 \leq q \leq 6$,

$$\begin{aligned} H^0(\partial_2, \widetilde{\mathcal{M}}_\lambda) &= H^0(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_\lambda) \\ H^1(\partial_2, \widetilde{\mathcal{M}}_\lambda) &= H^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_\lambda) \oplus H^0(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \\ H^2(\partial_2, \widetilde{\mathcal{M}}_\lambda) &= H^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \oplus H^0(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \\ H^3(\partial_2, \widetilde{\mathcal{M}}_\lambda) &= H^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \oplus H^0(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \\ H^4(\partial_2, \widetilde{\mathcal{M}}_\lambda) &= H^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \oplus H^0(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \\ H^5(\partial_2, \widetilde{\mathcal{M}}_\lambda) &= H^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \oplus H^0(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}) \\ H^6(\partial_2, \widetilde{\mathcal{M}}_\lambda) &= H^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}) \end{aligned}$$

and for every $q \geq 7$, the cohomology groups $H^q(\partial_2, \widetilde{\mathcal{M}}_\lambda) = 0$.

4 Boundary cohomology

In this section, we discuss the cohomology of the boundary by giving a complete description of the spectral sequence. The covering of the boundary of the Borel–Serre compactification defines a spectral sequence in cohomology.

$$E_1^{p,q} = \bigoplus_{prk(P)=(p+1)} H^q(\partial_P, \widetilde{\mathcal{M}}_\lambda) \Rightarrow H^{p+q}(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda)$$

and the nonzero terms of $E_1^{p,q}$ are for

$$(p, q) \in \left\{(i, n) | 0 \leq i \leq 1, 0 \leq n \leq 6\right\}. \tag{16}$$

More precisely,

$$\begin{aligned} E_1^{0,q} &= \bigoplus_{i=1}^{2} H^q(\partial_i, \widetilde{\mathcal{M}}_\lambda) \\ &= \bigoplus_{i=1}^{2} \left[\bigoplus_{w\in\mathcal{W}^{P_i}} H^{q-\ell(w)}(S_\Gamma^{M_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda}) \right], \\ E_1^{1,q} &= H^q(\partial_0, \widetilde{\mathcal{M}}_\lambda) \\ &= \bigoplus_{w\in\mathcal{W}^{P_0}:\ell(w)=q} H^0(S_\Gamma^{M_0}, \widetilde{\mathcal{M}}_{w\cdot\lambda}). \end{aligned} \tag{17}$$

Since G_2 is of rank two, the spectral sequence has only two columns namely $E_1^{0,q}$, $E_1^{1,q}$ and to study the boundary cohomology, the task reduces to analyzing the following morphisms

$$E_1^{0,q} \xrightarrow{d_1^{0,q}} E_1^{1,q} \tag{18}$$

where $d_1^{0,q}$ is the differential map and the higher differentials vanish. One has

$$E_2^{0,q} := Ker(d_1^{0,q}) \quad \text{and} \quad E_2^{1,q} := Coker(d_1^{0,q}).$$

In addition, due to being in a rank 2 situation, the spectral sequence degenerates in degree 2. Therefore, we can use the fact that

$$0 \longrightarrow E_2^{1,q-1} \longrightarrow H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow E_2^{0,q} \longrightarrow 0 \quad . \tag{19}$$

In what follows, we have to treat nine different cases separately, namely when m_1 (resp. m_2) is zero, nonzero even, and odd. This is due to the parity conditions established in Sect. 3 which has major influence on the cohomology.

4.1 Case 1: $m_1 = 0$ and $m_2 = 0$ (trivial coefficient system)

Following Lemma 4 and Lemma 6 from Sect. 3, we get

$$\overline{\mathcal{W}}^0 = \{w_1, w_6, w_7, w_{12}\}, \quad \overline{\mathcal{W}}^1 = \{w_1, w_4, w_6\} \quad \text{and} \quad \overline{\mathcal{W}}^2 = \{w_1, w_5, w_7\}.$$

By using (17), we record the values of $E_1^{0,q}$ and $E_1^{1,q}$ for the distinct values of q below. Note that following (16) we know that for $q \geq 7$, $E_1^{i,q} = 0$ for $i = 1, 2$.

$$E_1^{0,q} = \begin{cases} H^0(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{e\cdot\lambda}) \oplus H^0(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{e\cdot\lambda}), & q = 0 \\ H^1(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}), & q = 3 \\ H^1(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}), & q = 4 \\ 0, & \text{otherwise} \end{cases} , \tag{20}$$

and

$$E_1^{1,q} = \begin{cases} H^0(S_\Gamma^{M_0}, \widetilde{\mathcal{M}}_{e\cdot\lambda}), & q = 0 \\ H^0(S_\Gamma^{M_0}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^0(S_\Gamma^{M_0}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}), & q = 3 \\ H^0(S_\Gamma^{M_0}, \widetilde{\mathcal{M}}_{w_{12}\cdot\lambda}), & q = 6 \\ 0, & \text{otherwise} \end{cases} . \tag{21}$$

We now make a thorough analysis of (18) to get the complete description of the spaces $E_2^{0,q}$ and $E_2^{1,q}$ which will give us the cohomology $H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda)$. We begin with $q = 0$.

4.1.1 At the level $q = 0$

Observe that the short exact sequence (19) reduces to

$$0 \longrightarrow H^0(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow E_2^{0,0} \longrightarrow 0 \quad .$$

To compute $E_2^{0,0}$, consider the differential $d_1^{0,0} : E_1^{0,0} \to E_1^{1,0}$. Following (20) and (21), we have $d_1^{0,0} : \mathbb{Q} \oplus \mathbb{Q} \longrightarrow \mathbb{Q}$ and we know that the differential $d_1^{0,0}$ is surjective (see [6]). Therefore

$$E_2^{0,0} := Ker(d_1^{0,0}) = \mathbb{Q} \quad \text{and} \quad E_2^{1,0} := Coker(d_1^{0,0}) = 0. \tag{22}$$

Hence, we get

$$H^0(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \mathbb{Q}.$$

4.1.2 At the level $q = 1$

Following (22), in this case, our short exact sequence (19) reduces to

$$0 \longrightarrow H^1(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow E_2^{0,1} \longrightarrow 0 \quad ,$$

and we need to compute $E_2^{0,1}$. Consider the differential $d_1^{0,1} : E_1^{0,1} \longrightarrow E_1^{1,1}$ and following (20) and (21), we observe that $d_1^{0,1}$ is map between zero spaces. Therefore, we obtain

$$E_2^{0,1} = 0 \qquad \text{and} \qquad E_2^{1,1} = 0.$$

As a result, we get

$$H^1(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0.$$

4.1.3 At the level $q = 2$

Following the similar process as in level $q = 1$, we get

$$E_2^{0,2} = 0 \qquad \text{and} \qquad E_2^{1,2} = 0. \tag{23}$$

This results into

$$H^2(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0.$$

4.1.4 At the level $q = 3$

Following (23), in this case, the short exact sequence (19) reduces to

$$0 \longrightarrow H^3(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow E_2^{0,3} \longrightarrow 0 \quad ,$$

and we need to compute $E_2^{0,3}$. In view of (12), consider the differential $d_1^{0,3} : E_1^{0,3} \longrightarrow E_1^{1,3}$ and following (20) and (21), we have

$$E_2^{0,3} = H_!^1(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H_!^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}), \text{ and}$$
$$Im(d_1^{0,3}) = H_{Eis}^1(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H_{Eis}^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \cong \mathbb{Q} \oplus \mathbb{Q}.$$

Therefore,

$$E_2^{0,3} = 0 \qquad \text{and} \qquad E_2^{1,3} = 0. \tag{24}$$

This gives us

$$H^3(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0.$$

4.1.5 At the level $q = 4$

Following (24), in this case, the short exact sequence (19) reduces to

$$0 \longrightarrow H^4(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow E_2^{0,4} \longrightarrow 0 \quad ,$$

and we need to compute $E_2^{0,4}$. Consider the differential $d_1^{0,4} : E_1^{0,4} \longrightarrow E_1^{1,4}$ and following (20) and (21), we have $E_2^{0,4} = H_!^1(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H_!^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda})$. Since $Im(d_1^{0,4}) = H_{Eis}^1(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H_{Eis}^1(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) = \{0\}$ by (11). Therefore,

$$E_2^{0,4} = 0 \qquad \text{and} \qquad E_2^{1,4} = 0. \tag{25}$$

and we get

$$H^4(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0.$$

4.1.6 At the level $q = 5$

Following (25), in this case, the short exact sequence (19) reduces to

$$0 \longrightarrow H^4(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow E_2^{0,5} \longrightarrow 0 \quad ,$$

and we need to compute $E_2^{0,5}$. Consider the differential $d_1^{0,5} : E_1^{0,5} \longrightarrow E_1^{1,5}$ and following (20) and (21), we have $d_1^{0,5} : 0 \longrightarrow 0$. Therefore,

$$E_2^{0,5} = 0 \qquad \text{and} \qquad E_2^{1,5} = 0\,. \tag{26}$$

and we get

$$H^5(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0\,.$$

4.1.7 At the level $q = 6$

Following (26), in this case, the short exact sequence (19) reduces to

$$0 \longrightarrow H^6(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow E_2^{0,6} \longrightarrow 0 \quad ,$$

and we need to compute $E_2^{0,6}$. Consider the differential $d_1^{0,6} : E_1^{0,6} \longrightarrow E_1^{1,6}$ and following (20) and (21), we have $d_1^{0,6} : 0 \longrightarrow \mathbb{Q}$. Therefore,

$$E_2^{0,6} = 0 \qquad \text{and} \qquad E_2^{1,6} = \mathbb{Q}\,. \tag{27}$$

and we get

$$H^6(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0\,.$$

4.1.8 At the level $q = 7$

Following (27), in this case, the short exact sequence (19) reduces to

$$0 \longrightarrow \mathbb{Q} \longrightarrow H^7(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow E_2^{0,7} \longrightarrow 0 \quad ,$$

and we need to compute $E_2^{0,7}$. Consider the differential $d_1^{0,7} : E_1^{0,7} \longrightarrow E_1^{1,7}$ and following (20) and (21), we have $d_1^{0,7} : 0 \longrightarrow 0$. Therefore,

$$E_2^{0,7} = 0 \qquad \text{and} \qquad E_2^{1,7} = 0\,.$$

and we get

$$H^7(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \mathbb{Q}\,.$$

Hence, we can summarize the above discussion as follows :

$$H^q(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} \mathbb{Q}, & q = 0, 7 \\ 0, & \text{otherwise} \end{cases}.$$

4.2 Case 2: $m_1 = 0, m_2 \neq 0, m_2$ even

Following Lemma 4 and Lemma 6 from Sect. 3, we get

$$\overline{\mathcal{W}}^0 = \{w_1, w_6, w_7, w_{12}\}, \quad \overline{\mathcal{W}}^1 = \{w_1, w_4, w_6, w_{10}\} \quad \text{and} \quad \overline{\mathcal{W}}^2 = \{w_1, w_5, w_7\}\,.$$

By using (17), we record the values of $E_1^{0,q}$ and $E_1^{1,q}$ for the distinct values of q below. Note that following (16) we know that for $q \geq 7, E_1^{i,q} = 0$ for $i = 1, 2$.

$$E_1^{0,q} = \begin{cases} H^0(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{e\cdot\lambda}), & q = 0 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda), & q = 1 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}), & q = 3 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}), & q = 4 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}), & q = 6 \\ 0, & \text{otherwise} \end{cases},$$

and

$$E_1^{1,q} = \begin{cases} H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{e\cdot\lambda}), & q=0 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}), & q=3 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_{12}\cdot\lambda}), & q=6 \\ 0, & \text{otherwise} \end{cases}.$$

Having a thorough analysis of (18) as in previous section, we get the complete description of the spaces $E_2^{0,q}$ and $E_2^{1,q}$ which will give us the cohomology $H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda)$ described as follows :

$$H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda) \cong S_{m_2+2}, & q=1 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \cong S_{2m_2+4} \oplus S_{3m_2+6}, & q=3 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \cong S_{2m_2+4} \oplus S_{3m_2+6}, & q=4 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \cong S_{m_2+2}, & q=6 \\ 0, & \text{otherwise} \end{cases}.$$

Here, S_k is defined as in (13). We conclude the above discussion as follows:

$$H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_2+2}, & q=1,6 \\ S_{2m_2+4} \oplus S_{3m_2+6}, & q=3,4 \\ 0, & \text{otherwise} \end{cases}.$$

Remark 8 The deduction of boundary cohomology in remaining cases is completely analogous to the cases considered in Sects. 4.1 and 4.2. Hence, we simply state the final formulas of $E_1^{0,q}$, $E_1^{1,q}$ and $H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda)$, along with the Weyl representatives $\overline{\mathcal{W}}^0$, $\overline{\mathcal{W}}^1$ and $\overline{\mathcal{W}}^2$ for all the remaining seven cases.

4.3 Case 3: $m_1 = 0$, m_2 odd

$$\overline{\mathcal{W}}^0 = \{w_1, w_4, w_9, w_{11}\}, \quad \overline{\mathcal{W}}^1 = \{w_2, w_4, w_6, w_8\} \quad \text{and} \quad \overline{\mathcal{W}}^2 = \{w_3, w_9, w_{11}\}.$$

$$E_1^{0,q} = \begin{cases} H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}), & q=2 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}), & q=3 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}), & q=4 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}), & q=5 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}), & q=6 \\ 0, & \text{otherwise} \end{cases},$$

$$E_1^{1,q} = \begin{cases} H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \cong \mathbb{Q}, & q=1 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \cong \mathbb{Q}, & q=2 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \cong \mathbb{Q}, & q=4 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}) \cong \mathbb{Q}, & q=5 \\ 0, & \text{otherwise} \end{cases}.$$

$$H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \oplus \mathbb{Q} \cong S_{m_2+3} \oplus S_{3m_2+5} \oplus \mathbb{Q}, & q=2 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \cong S_{2m_2+4}, & q=3 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \cong S_{2m_2+4}, & q=4 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \oplus \mathbb{Q} \cong S_{m_2+3} \oplus S_{3m_2+5} \oplus \mathbb{Q}, & q=5 \\ 0, & \text{otherwise} \end{cases}.$$

4.4 Case 4: $m_1 \neq 0$ even and $m_2 = 0$

$$\overline{\mathcal{W}}^0 = \{w_1, w_6, w_7, w_{12}\}, \quad \overline{\mathcal{W}}^1 = \{w_1, w_4, w_6\} \quad \text{and} \quad \overline{\mathcal{W}}^2 = \{w_1, w_5, w_7, w_{11}\}.$$

$$E_1^{0,q} = \begin{cases} H^0(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda), & q=0 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_\lambda), & q=1 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}), & q=3 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}), & q=4 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}), & q=6 \\ 0, & \text{otherwise} \end{cases},$$

$$E_1^{1,q} = \begin{cases} H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_\lambda) \cong \mathbb{Q}, & q=0 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \cong \mathbb{Q}, & q=3 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_{12}\cdot\lambda}) \cong \mathbb{Q}, & q=6 \\ 0, & \text{otherwise} \end{cases}.$$

$$H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_\lambda) \cong S_{m_1+2}, & q=1 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \cong S_{m_1+4} \oplus S_{2m_1+6}, & q=3 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \cong S_{m_1+4} \oplus S_{2m_1+6}, & q=4 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}) \cong S_{m_1+2}, & q=6 \\ 0, & \text{otherwise} \end{cases}.$$

4.5 Case 5: $m_1(\neq 0)$ even, $m_2(\neq 0)$ even

$$\overline{\mathcal{W}}^0 = \{w_1, w_6, w_7, w_{12}\}, \quad \overline{\mathcal{W}}^1 = \{w_1, w_4, w_6, w_{10}\} \quad \text{and}$$
$$\overline{\mathcal{W}}^2 = \{w_1, w_5, w_7, w_{11}\}.$$

$$E_1^{0,q} = \begin{cases} H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_\lambda), & q=1 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}), & q=3 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}), & q=4 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}), & q=6 \\ 0, & \text{otherwise} \end{cases},$$

$$E_1^{1,q} = \begin{cases} H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_\lambda) \cong \mathbb{Q}, & q=0 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \cong \mathbb{Q}, & q=3 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_{12}\cdot\lambda}) \cong \mathbb{Q}, & q=6 \\ 0, & \text{otherwise} \end{cases}.$$

$$H^q(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_\lambda) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_\lambda) \oplus \mathbb{Q} \cong S_{m_2+2} \oplus S_{m_1+2} \oplus \mathbb{Q}, & q=1 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \cong S_{m_1+2m_2+4} \oplus S_{2m_1+3m_2+6}, & q=3 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \cong S_{m_1+2m_2+4} \oplus S_{2m_1+3m_2+6}, & q=4 \\ H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \oplus H_!^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}) \oplus \mathbb{Q} \cong S_{m_2+2} \oplus S_{m_1+2} \oplus \mathbb{Q}, & q=6 \\ 0, & \text{otherwise} \end{cases}.$$

4.6 Case 6: $m_1(\neq 0)$ even, m_2 odd

$$\overline{\mathcal{W}}^0 = \{w_2, w_4, w_9, w_{11}\}, \quad \overline{\mathcal{W}}^1 = \{w_2, w_4, w_6, w_8\}$$
$$\text{and} \quad \overline{\mathcal{W}}^2 = \{w_1, w_3, w_9, w_{11}\}.$$

$$E_1^{0,q} = \begin{cases} H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_\lambda), & q=1 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}), & q=2 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}), & q=3 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}), & q=4 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}), & q=5 \\ H^1(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}), & q=6 \\ 0, & \text{otherwise} \end{cases},$$

$$E_1^{1,q} = \begin{cases} H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}), & q=1 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}), & q=2 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}), & q=4 \\ H^0(\mathrm{S}_\Gamma^{\mathrm{M}_0}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}) \cong \mathbb{Q}, & q=5 \\ 0, & \text{otherwise} \end{cases}.$$

$$H^q(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_\lambda) \cong S_{m_1+2}, & q=1 \\ H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \cong S_{m_1+m_2+3} \oplus S_{m_1+3m_2+5}, & q=2 \\ H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_4\cdot\lambda}) \cong S_{m_1+2m_2+4}, & q=3 \\ H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_6\cdot\lambda}) \cong S_{m_1+2m_2+4}, & q=4 \\ H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \cong S_{m_1+m_2+3} \oplus S_{m_1+3m_2+5}, & q=5 \\ H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda}) \cong S_{m_1+2}, & q=6 \\ 0, & \text{otherwise} \end{cases}.$$

4.7 Case 7: m_1 odd, $m_2 = 0$

$$\overline{\mathcal{W}}^0 = \{w_3, w_5, w_8, w_{10}\}, \quad \overline{\mathcal{W}}^2 = \{w_3, w_5, w_7, w_9\} \quad \text{and} \quad \overline{\mathcal{W}}^1 = \{w_2, w_8, w_{10}\}.$$

$$E_1^{0,q} = \begin{cases} H^1(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^1(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}), & q=2 \\ H^1(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}), & q=3 \\ H^1(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}), & q=5 \\ H^1(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^0(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \oplus H^1(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}), & q=5 \\ H^1(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}), & q=6 \\ 0, & \text{otherwise} \end{cases},$$

$$E_1^{1,q} = \begin{cases} H^0(\mathrm{S}^{\mathrm{M}_0}_\Gamma, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \cong \mathbb{Q}, & q=1 \\ H^0(\mathrm{S}^{\mathrm{M}_0}_\Gamma, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \cong \mathbb{Q}, & q=2 \\ H^0(\mathrm{S}^{\mathrm{M}_0}_\Gamma, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \cong \mathbb{Q}, & q=4 \\ H^0(\mathrm{S}^{\mathrm{M}_0}_\Gamma, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \cong \mathbb{Q}, & q=5 \\ 0, & \text{otherwise} \end{cases}.$$

$$H^q(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \oplus \mathbb{Q} \cong S_{m_1+3} \oplus S_{m_1+5} \oplus \mathbb{Q}, & q=2 \\ H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \cong S_{2m_1+6}, & q=3 \\ H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \cong S_{2m_1+6}, & q=4 \\ H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \oplus \mathbb{Q} \cong S_{m_1+3} \oplus S_{m_1+5} \oplus \mathbb{Q}, & q=5 \\ 0, & \text{otherwise} \end{cases}.$$

4.8 Case 8: m_1 odd, $m_2 (\neq 0)$ even

$$\overline{\mathcal{W}}^0 = \{w_3, w_5, w_8, w_{10}\}, \quad \overline{\mathcal{W}}^1 = \{w_1, w_2, w_8, w_{10}\}$$
$$\text{and} \quad \overline{\mathcal{W}}^2 = \{w_3, w_5, w_7, w_9\}.$$

$$E_1^{0,q} = \begin{cases} H^1(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_\lambda), & q=1 \\ H^1(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^1(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}), & q=2 \\ H^1(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}), & q=3 \\ H^1(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}), & q=4 \\ H^1(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^1(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}), & q=5 \\ H^1(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}), & q=6 \\ 0, & \text{otherwise} \end{cases},$$

$$E_1^{1,q} = \begin{cases} H^0(\mathrm{S}^{\mathrm{M}_0}_\Gamma, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \cong \mathbb{Q}, & q=1 \\ H^0(\mathrm{S}^{\mathrm{M}_0}_\Gamma, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \cong \mathbb{Q}, & q=2 \\ H^0(\mathrm{S}^{\mathrm{M}_0}_\Gamma, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \cong \mathbb{Q}, & q=4 \\ H^0(\mathrm{S}^{\mathrm{M}_0}_\Gamma, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \cong \mathbb{Q}, & q=5 \\ 0, & \text{otherwise} \end{cases}.$$

$$H^q(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_\lambda) \cong S_{m_2+2}, & q=1 \\ H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_2\cdot\lambda}) \oplus H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_3\cdot\lambda}) \cong S_{m_1+m_2+3} \oplus S_{m_1+3m_2+5}, & q=2 \\ H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_5\cdot\lambda}) \cong S_{2m_1+3m_2+6}, & q=3 \\ H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_7\cdot\lambda}) \cong S_{2m_1+3m_2+6}, & q=4 \\ H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_8\cdot\lambda}) \oplus H^1_!(\mathrm{S}^{\mathrm{M}_2}_\Gamma, \widetilde{\mathcal{M}}_{w_9\cdot\lambda}) \cong S_{m_1+m_2+3} \oplus S_{m_1+3m_2+5}, & q=5 \\ H^1_!(\mathrm{S}^{\mathrm{M}_1}_\Gamma, \widetilde{\mathcal{M}}_{w_{10}\cdot\lambda}) \cong S_{m_2+2}, & q=6 \\ 0, & \text{otherwise} \end{cases}.$$

4.9 Case 9: m_1 odd, m_2 odd

By checking the parity conditions for standard parabolics, following Lemmas 4 and 6, we see that $\overline{\mathcal{W}}^i = \emptyset$ for $i = 0, 1, 2$. This simply implies that

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0, \qquad \forall q.$$

4.10 A summary of the boundary cohomology of $G_2(\mathbb{Z})$

We now end this section by summarizing the results obtained about the boundary cohomology above in the form of following theorem which will be our base for further exploration on Eisenstein cohomology in Sect. 5.

Theorem 9 *The boundary cohomology of the locally symmetric space* S_Γ *of the arithmetic group* $\Gamma := G_2(\mathbb{Z})$ *with respect to the coefficients in any highest weight representation* $\mathcal{M}_\lambda$, *with* $\lambda = m_1\lambda_1 + m_2\lambda_2$, *is described as follows.*

(1) *Case 1 :* $m_1 = 0 = m_2$.

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} \mathbb{Q}, & q = 0, 7 \\ 0, & \text{otherwise} \end{cases}.$$

(2) *Case 2 :* $m_1 = 0$, $m_2 (\neq 0)$ *even.*

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_2+2}, & q = 1, 6 \\ S_{2m_2+4} \oplus S_{3m_2+6}, & q = 3, 4 \\ 0, & \text{otherwise} \end{cases}.$$

(3) *Case 3 :* $m_1 = 0$, m_2 *odd.*

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_2+3} \oplus S_{3m_2+5} \oplus \mathbb{Q}, & q = 2, 5 \\ S_{2m_2+4}, & q = 3, 4 \\ 0, & \text{otherwise} \end{cases}.$$

(4) *Case 4 :* $m_1 (\neq 0)$ *even,* $m_2 = 0$.

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_1+2}, & q = 1, 6 \\ S_{m_1+4} \oplus S_{2m_1+6}, & q = 3, 4 \\ 0, & \text{otherwise} \end{cases}.$$

(5) *Case 5 :* $m_1 (\neq 0)$ *even,* $m_2 (\neq 0)$ *even.*

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_2+2} \oplus S_{m_1+2} \oplus \mathbb{Q}, & q = 1, 6 \\ S_{m_1+2m_2+4} \oplus S_{2m_1+3m_2+6}, & q = 3, 4 \\ 0, & \text{otherwise} \end{cases}.$$

(6) *Case 6 :* $m_1 (\neq 0)$ *even,* m_2 *odd.*

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_1+2}, & q = 1, 6 \\ S_{m_1+m_2+3} \oplus S_{m_1+3m_2+5}, & q = 2, 5 \\ S_{m_1+2m_2+4}, & q = 3, 4 \\ 0, & \text{otherwise} \end{cases}.$$

(7) *Case 7 :* m_1 *odd,* $m_2 = 0$.

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_1+3} \oplus S_{m_1+5} \oplus \mathbb{Q}, & q = 2, 5 \\ S_{2m_1+6}, & q = 3, 4 \\ 0, & \text{otherwise} \end{cases}.$$

(8) *Case 8 : m_1 odd, $m_2(\neq 0)$ even.*

$$H^q(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_2+2}, & q = 1, 6 \\ S_{m_1+m_2+3} \oplus S_{m_1+3m_2+5}, & q = 2, 5 \\ S_{2m_1+3m_2+6}, & q = 3, 4 \\ 0, & \text{otherwise} \end{cases} .$$

(9) *Case 9 : m_1 odd, m_2 odd.*

$$H^q(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0, \qquad \forall q .$$

5 Eisenstein cohomology

In this section, by using the information obtained about the boundary cohomology of $\Gamma := G_2(\mathbb{Z})$, we will determine the Eisenstein cohomology with coefficients in $\mathcal{M}_\lambda$. Let us recall that, at any degree q, the Eisenstein cohomology $H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda)$ is, by definition, the image of the restriction map $r : H^q(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) \longrightarrow H^q(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda)$.

5.1 Main result on the Eisenstein cohomology of $G_2(\mathbb{Z})$

The following is one of the main results of this article that gives both the dimension of the Eisenstein cohomology together with its sources—the corresponding parabolic subgroups.

Indeed, it is clear from the definition that the Eisenstein cohomology $H^\bullet_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda)$ is defined over $\mathbb{Q}$ as $\widetilde{\mathcal{M}}_\lambda$ is defined over $\mathbb{Q}$. But in the theorem stated below, we will consider $H^\bullet_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$ instead in certain cases. The reason is that, our method yields a basis of $H^\bullet_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$, but since the method is transcendental, the basis we get is not necessarily defined over $\mathbb{Q}$. Let Σ_k be the canonical basis of normalized eigenfunctions of S_k. For $i = 1, 2$, we set $k_i(\lambda, w) = a_i(\lambda, w) + 2$, where $a_i(\lambda, w)$ is the constant defined in (6). Then by the Eichler–Shimura isomorphism, for $i = 1, 2$, we have

$$H^1_!(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda} \otimes \mathbb{C}) = \bigoplus_{\psi \in \Sigma_{k_i(\lambda,w)}} H^1_!(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda} \otimes \mathbb{C})(\psi),$$

where the $\mathbb{C}$-vector spaces $H^1_!(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda} \otimes \mathbb{C})(\psi)$ are of dimension 1. Set

$$\mathcal{Y}_k = \{\psi \in \Sigma_k : L(1/2, \pi^\psi) \neq 0\}.$$

Theorem 10 (1) *Case 1 : $m_1 = 0 = m_2$.*

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} \mathbb{Q}, & q = 0 \\ 0, & \text{otherwise} \end{cases} .$$

(2) *Case 2 : $m_1 = 0$, $m_2(\neq 0)$ even.*

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{2m_2+4} \oplus S_{3m_2+6}, & q = 4 \\ S_{m_2+2}, & q = 6 \\ 0, & \text{otherwise} \end{cases} .$$

(3) *Case 3 : $m_1 = 0$, m_2 odd.*

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{2m_2+4} & q = 4 \\ S_{3m_2+5} \oplus S_{m_2+3} \oplus \mathbb{Q}, & q = 5 \\ 0, & \text{otherwise} \end{cases} .$$

(4) *Case 4 :* $m_1(\neq 0)$ *even,* $m_2 = 0$.

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_1+4} \oplus S_{2m_1+6}, & q = 4 \\ S_{m_1+2}, & q = 6 \\ 0, & \text{otherwise} \end{cases}.$$

(5) *Case 5 :* $m_1(\neq 0)$ *even,* $m_2(\neq 0)$ *even.*

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_1+2m_2+4} \oplus S_{2m_1+3m_2+6}, & q = 4 \\ S_{m_2+2} \oplus S_{m_1+2} \oplus \mathbb{Q}, & q = 6 \\ 0, & \text{otherwise} \end{cases}.$$

(6) *Case 6 :* $m_1(\neq 0)$ *even,* m_2 *odd.*

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{m_1+2m_2+4}, & q = 4 \\ S_{m_1+m_2+3} \oplus S_{m_1+3m_2+5}, & q = 5 \\ S_{m_1+2}, & q = 6 \\ 0, & \text{otherwise} \end{cases}.$$

(7) *Case 7 :* m_1 *odd,* $m_2 = 0$.

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) = \begin{cases} \bigoplus\limits_{\psi \in \mathcal{Y}_{2m_1+6}} H^1_!(\mathrm{S}^{\mathrm{M}_i}_\Gamma, \widetilde{\mathcal{M}}_{w\cdot\lambda} \otimes \mathbb{C})(\psi), & q = 3 \\ \bigoplus\limits_{\psi \notin \mathcal{Y}_{2m_1+6}} H^1_!(\mathrm{S}^{\mathrm{M}_i}_\Gamma, \widetilde{\mathcal{M}}_{w\cdot\lambda} \otimes \mathbb{C})(\psi), & q = 4 \\ S_{m_1+3} \oplus S_{m_1+5} \oplus \mathbb{C}, & q = 5 \\ 0, & \text{otherwise} \end{cases}.$$

(8) *Case 8 :* m_1 *odd,* $m_2(\neq 0)$ *even.*

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = \begin{cases} S_{2m_1+3m_2+6}, & q = 4 \\ S_{m_1+m_2+3} \oplus S_{m_1+3m_2+5}, & q = 5 \\ S_{m_2+2}, & q = 6 \\ 0, & \text{otherwise} \end{cases}.$$

(9) *Case 9 :* m_1 *odd,* m_2 *odd.*

$$H^q_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0, \qquad \forall q\,.$$

Now, the proof of Theorem 10 will occupy the rest of the paper, and will follow in several steps. The proof closely follows the strategy developed in [9] (see also [13], [17] and [18]). Since our method is transcendental, we will consider the module $\widetilde{\mathcal{M}}_\lambda \otimes \mathbb{C}$, from now on we will simply write it as $\widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}$.

5.2 General strategy

We now briefly describe the strategy which will be carried out in detail in the rest of this section. As mentioned above, our approach relies on the fact that the Eisenstein cohomology spans a maximal isotropic subspace of the boundary cohomology with respect to the Poincaré dual pairing (see Theorem 11). Indeed, certain cohomology classes are constructed using the theory of Eisenstein series, so we are done if the cohomology classes thus constructed spans a maximal isotropic subspace.

More precisely, let ω be a harmonic differential form that represents certain cohomology class in the component $H^q_!(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})(i = 0, 1, 2)$ of the boundary cohomology. By mimicking the construction of Eisenstein series, we get a family of differential forms $E(\omega, \theta)$ on S_Γ, where θ is a certain special parameter, to be discussed in Sect. 5.4. If $E(\omega, \theta)$ is holomorphic at a certain θ_ω, we get a non-trivial harmonic form $E(\omega, \theta_\omega)$ that represents a certain cohomology class in $H^q(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. By restricting the harmonic form back to the boundary, we get a non-trivial Eisenstein cohomology class in $H^q_{Eis}(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. This geometric formulation is closely related to the theory of $(\mathfrak{g}, K_\infty)$-cohomology and the classical Eisenstein series. In particular, the restriction of the cohomology class $E(\omega, \theta_\omega)$ to the boundary can be computed using the constant term of Eisenstein series. On the other hand, if the differential form $E(\omega, \theta)$ has a simple pole at θ_ω (or along some hyperplane that contains θ_ω), by taking residue, we still get a differential form $E'(\omega, \theta_\omega)$, whose restriction to the boundary also gives a certain Eisenstein cohomology class.

The study of Eisenstein cohomology is basically divided into two parts. In the first part, we study the Eisenstein cohomology classes that come from maximal boundary components, that is, those constructed from the cohomology classes in $H^\bullet_!(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ with $i = 1, 2$. In the second part, we study the Eisenstein cohomology classes that come from the minimal boundary component, that is, those constructed from the cohomology classes in $H^\bullet_!(\partial_0, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$.

5.3 Poincaré duality

For simplicity, we write

$$H^q_!(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) := H^q_!(\partial_1, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \oplus H^q_!(\partial_2, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}),$$

According to the Manin–Drinfeld principle [14], the Hecke eigenvalues associated to the space $H^q_!(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ are different from those associated to the remaining part of $H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. Hence, there is a canonical Hecke equivariant section from $H^q_!(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ to $H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. In particular, we can safely regard $H^q_!(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ as a subspace of $H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. Moreover, we set

$$H^q_{!,Eis}(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) := H^q_{Eis}(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \cap H^q_!(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}).$$

We shall need the following theorem from [10, Proposition 6.1].

Theorem 11 *Under the Poincaré dual pairing* $\langle \cdot, \cdot \rangle$

$$H^q(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \times H^{7-q}(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \to \mathbb{C}, \quad H^q_!(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \times H^{7-q}_!(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \to \mathbb{C},$$

we have

$$H^q_{Eis}(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^{7-q}_{Eis}(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})^\perp, \quad H^q_{!,Eis}(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^{7-q}_{!,Eis}(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})^\perp.$$

In particular, the Eisenstein cohomology is a maximal isotropic subspace of the boundary cohomology under the Poincaré duality.

Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) be the ring of adeles (resp. finite adeles) of $\mathbb{Q}$ and $K_f = \prod_p G_2(\mathbb{Z}_p)$. It is clear that the Poincaré dual pairings are Hecke equivariant, hence Theorem 11 can be further refined by considering the Hecke action. Let $\mathcal{H}_{K_f}$ be the spherical Hecke algebra of G_2. Now let $i = 1, 2$. The inner cohomology $H^\bullet_!(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$, considered as a $\mathcal{H}_{K_f}$ module, can be decomposed as

$$H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = \bigoplus_{w\in\mathcal{W}^{\mathrm{P}_i}} H_!^{1+\ell(w)}(S_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda,\mathbb{C}})$$
$$= \bigoplus_{w\in\mathcal{W}^{\mathrm{P}_i}} \bigoplus_{\pi=\pi_\infty\otimes\pi_f} m_0(\pi) H^{1+\ell(w)}(\mathfrak{m}_i, K_\infty^{\mathrm{M}_i}, \pi_\infty\otimes\mathcal{M}_{w\cdot\lambda,\mathbb{C}})(\pi_f).$$

Here, π denotes a cuspidal automorphic representation of $\mathrm{M}_i(\mathbb{A})$ with unramified π_f and $m_0(\pi)$ denotes the multiplicity of π. A cohomology class in $H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ is said to be of type (π, w) if it comes from the summand $H^{1+\ell(w)}(\mathfrak{m}_i, K^{\mathrm{M}_i}, \pi_\infty\otimes\mathcal{M}_{w\cdot\lambda,\mathbb{C}})(\pi_f)$ in the above decomposition.

Now let $\beta_1, \beta_2 \in H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ be the cohomology classes of type (π_1, w_1) and (π_2, w_2), respectively. Recall that the Poincaré dual pairing is also $\mathcal{H}_{K_f}$ equivariant, hence if $\pi_{1,f} \neq \pi_{2,f}$, or equivalently, $\pi_1 \neq \pi_2$ by strong multiplicity one, then $\langle\beta_1, \beta_2\rangle = 0$. On the other hand, for dimensional reasons, $\langle\beta_1, \beta_2\rangle \neq 0$ only when $\ell(w_1) + \ell(w_2) = 5$, which is equivalent to saying that w_1 is mapped to w_2 under the involution introduced in Lemma 2. In conclusion, we get the following lemma.

Lemma 12 *Let $i = 1, 2$ and $\beta_1, \beta_2 \in H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ be cohomology classes of type (π_1, w_1) and (π_2, w_2), respectively. Then, $\langle\beta_1, \beta_2\rangle$ is nonzero only if $\pi_1 = \pi_2$ and $w_1 = w_2'$.*

Let $\mathcal{W}_>^{\mathrm{P}_i} = \{w \in \mathcal{W}^{\mathrm{P}_i} : \ell(w) \geq \ell(w')\}$. In view of this lemma, we may regroup $H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ using the Weyl elements $w \in \mathcal{W}_>^{\mathrm{P}_i}$ as follows:

$$H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = \bigoplus_{w\in\mathcal{W}_>^{\mathrm{P}_i}} \bigoplus_{\psi\in\Sigma_{k_i(\lambda,w)}} H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}, w),$$

with

$$H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}, w) := H_!^{1+\ell(w')}(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w'\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f})$$
$$\oplus H_!^{1+\ell(w)}(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}),$$

where π^ψ denotes the automorphic representation associated to the Hecke eigenform ψ. By the multiplicity one theorem, $\dim H_!^{1+\ell(w)}(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}) = 1$ for any (π_f^ψ, w). Hence, by combining Theorem 11 and Lemma 12, we get

Proposition 13 *The Eisenstein cohomology $H_{!,Eis}^\bullet(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ decomposes as*

$$H_{!,Eis}^\bullet(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H_{!,Eis}^\bullet(\partial_1, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \oplus H_{!,Eis}^\bullet(\partial_2, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}),$$

where

$$H_{!,Eis}^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = \bigoplus_{w\in\mathcal{W}_>^{\mathrm{P}_i}} \bigoplus_{\psi\in\Sigma_{k_i(\lambda,w)}} H_{!,Eis}^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}, w),$$

with $H_{!,Eis}^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}, w)$ equals

$$\text{either} \quad H_!^{1+\ell(w')}(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w'\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}) \quad \text{or} \quad H_!^{1+\ell(w)}(\mathrm{S}_\Gamma^{\mathrm{M}_i}, \widetilde{\mathcal{M}}_{w\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}).$$

5.4 Eisenstein forms

Let $\omega \in \Omega^*(\partial_i, \widetilde{\mathcal{M}}_\lambda)(i = 0, 1, 2)$ be a differential form on the boundary component ∂_i and $\widetilde{\omega} \in \Omega^*(\Gamma_{\mathrm{P}_i}\backslash \mathrm{S}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ be the pull-back of ω along the projection of $\Gamma_{\mathrm{P}}\backslash\mathrm{S} = \partial_i \times A_{\mathrm{P}_i}$ to the first factor. For any $\theta \in \mathfrak{b}_i := \mathfrak{a}^*_{\mathrm{P}_i} \otimes \mathbb{C}$, set

$$\omega_\theta = \widetilde{\omega} \times a^{\theta+\rho_i} \in \Omega^*(\Gamma_{\mathrm{P}_i}\backslash \mathrm{S}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}),$$

where $\rho_i := \rho|_{\mathfrak{b}_i}$ Then we define the corresponding Eisenstein form as

$$E(\omega, \theta) = \sum_{\gamma\in\Gamma/\Gamma_{\mathrm{P}_i}} \omega_\theta \circ \gamma \in \Omega^*(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}).$$

It is well known that the series $E(\omega, \theta)$ converges in certain region and admits an analytic continuation to a meromorphic function on $\mathfrak{b}_i$.

5.5 Constant terms and intertwining operators

To proceed, we consider the representation theoretic reformulation of the Eisenstein forms. The complex of smooth forms $\Omega^*(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ can be computed as

$$\begin{aligned}\Omega^*(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) &= C^*(\mathfrak{g}, K_\infty; C^\infty(\Gamma\backslash G_2(\mathbb{R})) \otimes \mathcal{M}_\lambda)\\ &= C^*(\mathfrak{g}, K_\infty; C^\infty(G_2(\mathbb{Q})\backslash G_2(\mathbb{A})) \otimes \mathcal{M}_\lambda)^{K_f}.\end{aligned}$$

Consequently, we have

$$H^\bullet(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^\bullet(\mathfrak{g}, K_\infty; C^\infty(G_2(\mathbb{Q})\backslash G_2(\mathbb{A})) \otimes \mathcal{M}_\lambda)^{K_f}$$

Now let P_i be a standard parabolic, $\pi = \pi_\infty \otimes \pi_f$ be an automorphic representation of $\mathrm{M}_i(\mathbb{A})$ and let $\theta \in \mathfrak{b}_i$ be a parameter. For $\psi_\theta \in V(\theta, \pi) := \mathrm{Ind}^{G_2}_{\mathrm{P}_i}\pi \otimes \mathbb{C}^{\theta+\rho_i}$ defined as

$$\mathrm{Ind}^{G_2}_{\mathrm{P}_i}\pi \otimes \mathbb{C}^{\theta+\rho_i} = \left\{f \in C^\infty(G_2(\mathbb{A}), H_\pi) : f(pg) = \pi(p)p^{\theta+\rho_i} f(g) \quad \text{for all } p \in \mathrm{P}_i(\mathbb{A})\right\},$$

where H_π is the representation space of π, define the Eisenstein series as

$$E_{\mathrm{P}_i}(\theta, \pi, \psi_\theta)(g) = \sum_{\gamma\in \mathrm{P}_i(\mathbb{Q})\backslash G_2(\mathbb{Q})} \psi_\theta(\gamma g).$$

For θ from a certain region, the Eisenstein series defined above converges absolutely, hence defines an intertwining operator

$$\mathscr{E}_\theta : V(\theta, \pi) \to C^\infty(G_2(\mathbb{Q})\backslash G_2(\mathbb{A})),$$

in this region. The Eisenstein series has a meromorphic continuation to $\mathfrak{b}_i$, hence defines a meromorphic continuation of the corresponding intertwining operator. When the Eisenstein series has a simple pole at θ, by taking the derivative, we get an intertwining operator

$$\mathscr{E}'_\theta : V(\theta, \pi) \to C^\infty(G_2(\mathbb{Q})\backslash G_2(\mathbb{A})),$$

which is no longer an embedding in general. It is clear that, for a closed form $\omega \in \Omega^*(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ that represents certain cohomology class of type (π, w), the intertwining operator $\mathscr{E}$ is holomorphic (resp. has a simple pole) at θ if and only if the corresponding Eisenstein form $E(\omega, \theta)$ is holomorphic (resp. has a simple pole) at θ.

To determine whether $\mathscr{E}$ is holomorphic at certain point or not, it suffices to study the constant terms, which is a map

$$CT : C^\infty(\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})) \to C^\infty(\mathrm{N}_i(\mathbb{A})\mathrm{P}_i(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})), f \mapsto \int_{\mathrm{N}_i(\mathbb{Q})\backslash \mathrm{N}_i(\mathbb{A})} f(ng)\mathrm{d}n,$$

where N_i is the unipotent radical of P_i. First, we consider the case when the parabolic subgroup is maximal. Let $\mathrm{Q} = \mathrm{M_Q N_Q}$ be another standard parabolic subgroup. Then, the constant term along Q is defined as

$$E_{\mathrm{Q}}(\theta, \pi, \psi_\theta)(g) = \int_{\mathrm{N_Q}(\mathbb{Q})\backslash \mathrm{N_Q}(\mathbb{A})} E(\theta, \pi, \psi_\theta)(ng)\mathrm{d}n.$$

Since the maximal parabolic subgroups of G_2 are self-conjugate, the constant term is non-trivial only when $\mathrm{Q} = \mathrm{P}_i$, where we have

$$E_{\mathrm{P}_i}(\theta, \pi, \psi_\theta)(g) = \psi_\theta(g) + M(\theta, \pi, w_{\mathrm{P}_i})\psi_\theta(g),$$

where w_{P_i} is the longest element in $\mathcal{W}^{\mathrm{P}_i}$ and $M(\theta, \pi, w_{\mathrm{P}_i})$ is a global intertwining operator from $V(\theta, \pi)$ to $V(-\theta, \pi)$. The global intertwining operator $M(\theta, \pi, w_{\mathrm{P}_i})$ is a product of local intertwining operator

$$M(\theta, \pi, w_{\mathrm{P}_i}) = A(\theta, \pi_\infty, w_{\mathrm{P}_i}) \otimes A(\theta, \pi_f, w_{\mathrm{P}_i}), \text{ where } A(\theta, \pi_f, w_{\mathrm{P}_i}) = \otimes_p A(\theta, \pi_p, w_{\mathrm{P}_i}).$$

Note that if π_p is unramified, the induced representation $V(\theta, \pi_p)$ is also unramified. Works of Langlands and Gindikin–Karpelevich, see for example [4], give a description of the local intertwining operator on $G_2(\mathbb{Z}_p)$-invariant vectors. As a consequence, we have the following description of the global intertwining operator.

Lemma 14 (1) *Let* $i = 1$ *and* $\theta = z\gamma_1 \in \mathfrak{b}_1$. *Then,*

$$M(\theta, \pi, w_{\mathrm{P}_1}) = c_1(\theta, \pi)A(\theta, \pi_\infty, w_{\mathrm{P}_1}) \otimes A'(\theta, \pi_f, w_{\mathrm{P}_1})$$

where $A'(\theta, \pi_f, w_{\mathrm{P}_i})$ *is the intertwining operator from* $V(\theta, \pi_f) := \otimes_p V(\theta, \pi_p)$ *to* $V(-\theta, \pi_f)$ *that sends a normalized* K_f*-invariant vector in* $V(\theta, \pi_f)$ *to a normalized* K_f*-invariant vector in* $V(-\theta, \pi_f)$ *and*

$$c_1(\theta, \pi) = \frac{L(z, Sym^3\pi)}{L(z+1, Sym^3\pi)} \frac{\zeta(2z)}{\zeta(2z+1)}. \tag{28}$$

(2) *Let* $i = 2$ *and* $\theta = z\gamma_2 \in \mathfrak{b}_2$. *Then,*

$$M(\theta, \pi, w_{\mathrm{P}_2}) = c_2(\theta, \pi)A(\theta, \pi_\infty, w_{\mathrm{P}_2}) \otimes A'(\theta, \pi_f, w_{\mathrm{P}_2})$$

where $A'(\theta, \pi_f, w_{\mathrm{P}_2})$ *is the intertwining operator from* $V(\theta, \pi_f)$ *to* $V(-\theta, \pi_f)$ *that sends a normalized* K_f*-invariant vector in* $V(\theta, \pi_f)$ *to a normalized* K_f*-invariant vector in* $V(-\theta, \pi_f)$ *and*

$$c_2(\theta, \pi) = \frac{L(z, \pi)}{L(z+1, \pi)} \frac{\zeta(2z)}{\zeta(2z+1)} \frac{L(3z, \pi)}{L(3z+1, \pi)}. \tag{29}$$

Proof Recall that the factor of the intertwining operator for G_2 is given by the adjoint action of the L-group of the Levi component of the unipotent radical. For maximal parabolic subgroup, the Levi is isomorphic to GL_2. Hence, the L-group is isomorphic to $\mathrm{GL}_2(\mathbb{C})$. Let $V \cong \mathbb{C}^2$ be the standard representation of $\mathrm{GL}_2(\mathbb{C})$ of dimension 2. As determined in [4], the adjoint action of $\mathrm{GL}_2(\mathbb{C})$ on ${}^L\mathfrak{n}_1$ and the adjoint action of $\mathrm{GL}_2(\mathbb{C})$ on ${}^L\mathfrak{n}_2$ decompose as

$${}^L\mathfrak{n}_1 = Sym^3 V \otimes (\wedge^2 V)^{-1} \oplus \wedge^2 V \quad {}^L\mathfrak{n}_2 = V \otimes \wedge^2 V \oplus V \oplus \wedge^2 V.$$

Since π is unramified, the factor for P_1 is given by (28) and the factor for P_2 is given by (29). ◻

Now let $i = 0$. Here, we consider the special case when $\pi = \mathbb{C}$ is the trivial representation. For $\theta = z_1\gamma_1 + z_2\gamma_2 \in \mathfrak{b}_0$, set $V(\theta) = V(\theta, \mathbb{C})$. Then, the constant term can be computed as

$$E_{\mathrm{P}_0}(\theta, \psi_\theta)(g) = \sum_{w \in \mathcal{W}} M(\theta, w)(\psi_\theta),$$

where $M(\theta, w)$ denotes a global intertwining functor from $V(\theta)$ to $V(w \cdot \theta)$. Again, the global intertwining operator is a product of local intertwining operators

$$M(\theta, \pi, w) = A(\theta, \pi_\infty, w) \otimes A(\theta, \pi_f, w), \text{ where } A(\theta, \pi_f, w) = \otimes_p A(\theta, \pi_p, w).$$

Lemma 15 *Let $i = 0$ and $\theta = z_1\gamma_1 + z_2\gamma_2$. Then*

$$M(\theta, \pi, w) = c_0(\theta, w) A(\theta, \pi_\infty, w) \otimes A'(\theta, \pi_f, w)$$

where $A'(\theta, \pi_f, w)$ is the intertwining operator from $V(\theta, \pi_f) := \otimes_p V(\theta, \pi_p)$ to $V(w \cdot \theta, \pi_f)$ that sends a normalized K_f-invariant vector in $V(\theta, \pi_f)$ to a normalized K_f-invariant vector in $V(w \cdot \theta, \pi_f)$ and

$$c_0(\theta, w) = \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}\alpha \in -\Phi^+}} \frac{\zeta(\langle \alpha, \gamma_1 \rangle (z_1 + 1) + \langle \alpha, \gamma_2 \rangle (z_2 + 1) - 1)}{\zeta(\langle \alpha, \gamma_1 \rangle (z_1 + 1) + \langle \alpha, \gamma_2 \rangle (z_2 + 1))}.$$

Proof This follows from direct computation, for a quick reference see [9, p. 159] and [7, Section 1.2.4] for the details. □

5.6 The inner part of the Eisenstein cohomology

Now we are ready to determine the space $H^{\bullet}_{!,Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. We begin with the following lemma.

Lemma 16 *Let $\lambda = m_1\gamma_1 + m_2\gamma_2$ and set $\theta^i_{\lambda,w} := -w(\lambda + \rho)|_{\mathfrak{b}_i} (i = 1, 2)$.*

(1) *The constant term $c_1(\theta, \pi)$ has a simple pole at $\theta^1_{\lambda,w}$ if $w = w_6$, $m_1 = 0$ and $L(1/2, \pi)$ is nonzero and is holomorphic at $\theta^1_{\lambda,w}$ otherwise.*
(2) *The constant term $c_2(\theta, \pi)$ has a simple pole at $\theta^2_{\lambda,w}$ if $w = w_7$, $m_1 = 0$ and $L(1/2, Sym^3\pi)$ is nonzero and is holomorphic at $\theta^1_{\lambda,w}$ otherwise.*

Proof As $\rho = \gamma^{\mathrm{M}_1} + 5\kappa^{\mathrm{M}_1}$, we have

$$-w(\lambda + \rho)|_{\mathfrak{b}_1} = -\frac{1}{2}(t_i(w, \lambda) + 5)\gamma_1.$$

Note that for $\theta = z\gamma_1$,

$$c_2(\theta, \pi) = \frac{L(z, Sym^3\pi)}{L(z + 1, Sym^3\pi)} \frac{\zeta(2z)}{\zeta(2z + 1)}.$$

Since the automorphic representations π considered here are all unramified, the corresponding central character is trivial, hence π is not monomial. Then, according to [11], the L-function $L(z, Sym^3\pi)$ is entire. Hence, in view of Sect. 2.4, for $c_2(\theta, \pi)$ to have pole it is necessary to have $w = w_7$, $m_2 = 0$. The possible pole comes from the simple pole of the zeta function at $2z = 1$. But the simple pole may be canceled by a possible zero of $L(z, Sym^3\pi)$ at $z = 1/2$. This shows the part (2).

As $\rho = \gamma^{M_2} + 3\kappa^{M_2}$, we have $-w(\lambda+\rho)|_{\mathfrak{b}_2} = -\frac{1}{2}(t_i(w,\lambda)+3)\gamma_2$. Note that, for $\theta = z\gamma_2$,

$$c_1(\theta,\pi) = \frac{L(z,\pi)}{L(z+1,\pi)} \frac{\zeta(2z)}{\zeta(2z+1)} \frac{L(3z,\pi)}{L(3z+1,\pi)}.$$

It is well known that $L(z,\pi)$ appearing here are holomorphic and nonzero at z when $\Re z > 1$. Hence in view of Sect. 2.4, for $c_1(\theta,\pi)$ to have pole, it is necessary to have $w = w_6$, $m_1 = 0$. The possible pole comes from the simple pole of the zeta function at $2z = 1$. But the simple pole may be canceled by a possible zero of $L(z,\pi)$ at $z = 1/2$. This shows the part (1). □

We shall need the following theorem.

Theorem 17 *Let $i = 1, 2$, π be a cuspidal automorphic representation of $M_i(\mathbb{A})$ and $w \in \mathcal{W}^{P_i}_{>}$. Let $\beta \in H_!^{1+\ell(w)}(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ be a cohomology class of type (π, w) and $\omega \in \Omega^*(\partial_i, \widetilde{\mathcal{M}}_\lambda)$ be a closed harmonic form that represents β.*

1. *If the Eisenstein series $E(\omega,\theta)$ is holomorphic at $\theta^i_{\lambda,w}$, then $E(\omega,\theta^i_{\lambda,w}) \in \Omega^*(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ is a closed form such that the restriction of its cohomology class to the boundary $r([E(\omega,\theta^i_{\lambda,w})]) \in H_!^{1+\ell(w)}(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ is non-trivial and of type (π, w).*
2. *If the Eisenstein series $E(\omega,\theta)$ has a simple pole at $\theta^i_{\lambda,w}$, then the residue $E'(\omega,\theta^i_{\lambda,w}) \in \Omega^*(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ is a closed form such that its restriction to the boundary $r([E'(\omega,\theta^i_{\lambda,w})]) \in H_!^{1+\ell(w')}(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ is non-trivial and of type (π, w').*

Proof When the group G is of $\mathbb{Q}$-rank 1, the corresponding statements are proved in [5]. The same proof works when the cohomology classes come from the maximal boundary, hence works for this theorem as well. We also refer [17] for details. □

The subspace of $H_!^\bullet(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ spanned by the cohomology classes of the form $r([E(\omega,\theta^i_{\lambda,w})])$ appearing in part (1) (resp. $r([E'(\omega,\theta^i_{\lambda,w})])$ appearing in part (2)) of Theorem 17 is denoted by $H^\bullet_{reg}(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ (resp. $H^\bullet_{res}(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$). Then, we have the natural inclusion

$$\bigoplus_{i=1,2} \left(H^\bullet_{reg}(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \oplus H^\bullet_{res}(\partial_i, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \right) \subset H^\bullet_{!,Eis}(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}). \tag{30}$$

By combining Proposition 13, Lemma 16 and Theorem 17, the above inclusion (30) is indeed an equality. Moreover, we conclude the following.

Proposition 18 *Let $\lambda = m_1\gamma_1 + m_2\gamma_2$, $w \in \mathcal{W}^{P_i}_{>}(i = 1, 2)$ and $\psi \in \Sigma_{k_i(\lambda,w)}$. Then*

$$H^\bullet_{!,Eis}(\partial_1, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}, w) = \begin{cases} H_!^{1+\ell(w')}(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{w'\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}) & \text{if } \substack{w=w_6, m_1=0, \text{and} \\ L(1/2, Sym^3\pi)\neq 0.} \\ H_!^{1+\ell(w)}(S_\Gamma^{M_1}, \widetilde{\mathcal{M}}_{w\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}) & \text{otherwise.} \end{cases} \tag{31}$$

$$H^\bullet_{!,Eis}(\partial_2, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}, w) = \begin{cases} H_!^{1+\ell(w')}(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w'\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}) & \text{if } \substack{w=w_7, m_2=0, \text{and} \\ L(1/2,\pi)\neq 0.} \\ H_!^{1+\ell(w)}(S_\Gamma^{M_2}, \widetilde{\mathcal{M}}_{w\cdot\lambda,\mathbb{C}})((\pi_f^\psi)^{K_f}) & \text{otherwise.} \end{cases} \tag{32}$$

Now, we want to invoke the following result.

Lemma 19 *Let π be a regular automorphic representation of GL_2 everywhere unramified of weight k. Then, we have*

1. $L(1/2,\pi) = 0$ *if* $k = 2 \pmod 4$.
2. $L(1/2, Sym^3\pi) = 0$.

Proof By the functional equation, it suffices to show that $\epsilon(1/2, \pi) = -1$. Since π is unramified everywhere, $\epsilon(1/2, \pi) = \epsilon(1/2, \pi_\infty) = (-1)^{k/2}$. This completes the proof of the first assertion.

For the second assertion, we use the following argument suggested by the referee. Clearly, we have $L(s, \pi \otimes \pi \otimes \pi) = L(s, Sym^3\pi)L(s, \pi)^2$. Consequently, we have $\epsilon(1/2, \pi \otimes \pi \otimes \pi) = \epsilon(1/2, Sym^3\pi)\epsilon(1/2, \pi)^2$. In view of [16, Thm. 2], we have $\epsilon(1/2, \pi\otimes\pi\otimes\pi) = -1$, hence $\epsilon(1/2, Sym^3\pi) = -1$. This implies that $L(1/2, Sym^3\pi) = 0$ as required. □

Combining Proposition 18 and Lemma 19, we arrive at

Proposition 20 *Let* $\lambda = m_1\gamma_1 + m_2\gamma_2$, $w \in \mathcal{W}^{\mathrm{P}_i}_{>}(i = 1, 2)$ *and* $\psi \in \Sigma_{k_i(\lambda,w)}$. *Then*

$$H^{\bullet}_{!,Eis}(\partial_1, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})((\pi_f^{\psi})^{K_f}, w) = H_!^{1+\ell(w)}(\mathrm{S}_\Gamma^{\mathrm{M}_1}, \widetilde{\mathcal{M}}_{w\cdot\lambda,\mathbb{C}})((\pi_f^{\psi})^{K_f}) \tag{33}$$

$$H^{\bullet}_{!,Eis}(\partial_2, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})((\pi_f^{\psi})^{K_f}, w) = H_!^{1+\ell(w)}(\mathrm{S}_\Gamma^{\mathrm{M}_2}, \widetilde{\mathcal{M}}_{w\cdot\lambda,\mathbb{C}})((\pi_f^{\psi})^{K_f}) \tag{34}$$

Now, since the inclusion (30) is an equality, if we assume that $H^{\bullet}(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \cong H_!^{\bullet}(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$, we find that $H^{\bullet}_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \cong H^{\bullet}_{!,Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. Now, following Theorem 9 we know that for the cases 2,4,6 and 8, $H^{\bullet}(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \cong H_!^{\bullet}(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$, and therefore by combining the information achieved in Proposition 13 and Proposition 18 we have now determined $H^{\bullet}_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ for the cases 2,4,6 and 8 as described in Theorem 10 and we describe these spaces explicitly in Sect. 5.8. Hence, we are left to treat the cases 1, 3, 5 and 7 of Theorem 9.

5.7 The boundary part of the Eisenstein cohomology

In this section, we determine the Eisenstein cohomology classes that come from the minimal boundary ∂_0. As a consequence, we determine $H^{\bullet}_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ for all the cases left. Throughout this subsection, we assume that $H^{\bullet}(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \neq H_!^{\bullet}(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$, or equivalently, we are considering the cases 1, 3, 5 and 7 of Theorem 9.

Let β be a cohomology class in $H^6(\widetilde{\partial}_0, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$[1], and $\omega \in \Omega^6(\widetilde{\partial}_0, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ be a closed harmonic form that represents β. Recall that, as a module of the spherical Hecke algebra of T, we have

$$H^6(\widetilde{\partial}_0, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^0(\widetilde{\mathrm{S}_\Gamma^{\mathrm{M}_0}}, H^6(\mathrm{N}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})) = \mathbb{C}^{-\lambda-2\rho}.$$

The overall idea for the construction of the Eisenstein cohomology classes is the same as before. If the Eisenstein form $E(\omega, \theta)$ is holomorphic at $\theta_\lambda := \lambda + \rho$, then $E(\omega, \theta_\lambda) \in \Omega^6(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ is a closed form such that the restriction of its cohomology class to the boundary is non-trivial, see [19, Theorem 7.2]. Otherwise, we need to take residues of the Eisenstein form and compute their restriction to the boundary using the constant term. As before, we denote the subspace of $H^{\bullet}(\partial\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ spanned by the Eisenstein cohomology classes that come from the restriction of the Eisenstein forms (resp. residues of the Eisenstein forms) by $H^{\bullet}_{B,reg}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ (resp, $H^{\bullet}_{B,res}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$).

For simplicity, set

$$H^{\bullet}_{B,Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^{\bullet}_{B,reg}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \oplus H^{\bullet}_{B,res}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}).$$

Then we have the natural inclusion

$$H^{\bullet}_{B,Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \oplus H^{\bullet}_{!,Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \subset H^{\bullet}_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}). \tag{35}$$

[1] Here, $\widetilde{\partial}_0$ denotes the cover of ∂_0, which is easily seen to be isomorphic to the unipotent radical N.

According to Theorem 11, the Eisenstein cohomology $H^{\bullet}_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ is a maximal isotropic subspace of the boundary cohomology under the Poincaré duality. In particular,

$$\dim H^{\bullet}_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = \frac{1}{2} \dim H^{\bullet}(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}).$$

On the other hand, according to Theorem 9, we always have

$$\dim H^{\bullet}(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) - \dim H^{\bullet}_{!}(\partial \mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = 2,$$

for the cases studied in this subsection. Hence to determine $H^{\bullet}_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$, it suffices to fill $H^{\bullet}_{B,Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ with one non-trivial cohomology class.

The following proposition is the main result of this subsection

Proposition 21 *Let the notations be as in Theorem 9.*

1. *In case 1, we have* $H^{\bullet}_{Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^{0}_{B,res}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \cong \mathbb{C}$.
2. *In case 3 and 7, we have* $H^{\bullet}_{B,Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^{5}_{B,res}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \cong \mathbb{C}$.
3. *In case 5, we have* $H^{\bullet}_{B,Eis}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^{6}_{B,reg}(\mathrm{S}_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \cong \mathbb{C}$.

Proof The proof will be based on the computation of $(\mathfrak{g}, K_\infty)$-cohomology of certain induced modules. To begin, let us start with the proof of part (2). Without loss of generality, we may assume that $\lambda = m_2\gamma_2$ for some $m_2 > 0$.

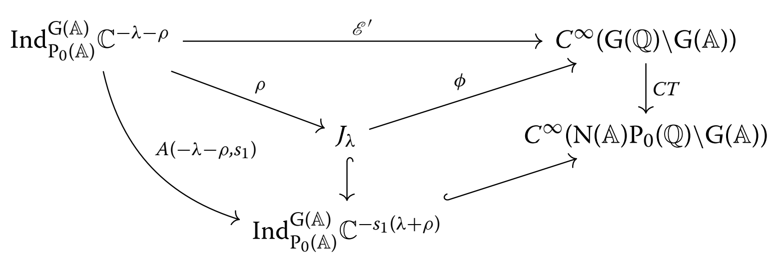
Recall that

$$c_0(\theta, w) = \prod_{\substack{\alpha\in\Phi^+ \\ w^{-1}\alpha\in-\Phi^+}} \frac{\zeta(\langle\alpha,\gamma_1\rangle(z_1+1) + \langle\alpha,\gamma_2\rangle(z_2+1) - 1)}{\zeta(\langle\alpha,\gamma_1\rangle(z_1+1) + \langle\alpha,\gamma_2\rangle(z_2+1))}.$$

Hence, in the case of $w = w_2(= s_1)$ the constant term $c_0(\theta, w)$ has a simple pole along $z_1 = 0$. Hence, the corresponding Eisenstein series has a simple pole along the line $z_1 = 0$. By taking the residue along the line $z_1 = 0$, we get an intertwining operator

$$\mathscr{E}' : \quad \mathrm{Ind}_{\mathrm{P}_0}^{G_2}\mathbb{C}^{-\lambda-\rho} \to C^\infty(G_2(\mathbb{Q})\backslash G_2(\mathbb{A})).$$

Moreover, we have the following diagram.



Note that,

$$J_\lambda = \mathrm{Ind}_{\mathrm{P}_1(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}\mathbb{C}^{-s_1(\lambda+\rho)},$$

and the intertwining operator ρ is simply the induction of the intertwining operator

$$\mathrm{Ind}_{\mathrm{B}(\mathbb{A})}^{\mathrm{M}_1(\mathbb{A})}\mathbb{C}^{-\lambda-\rho} \longrightarrow \mathbb{C}^{-s_1(\lambda+\rho)},$$

where B denotes the corresponding Borel subgroup of M_1. Namely, ρ is the map

$$\mathrm{Ind}_{\mathrm{P}_1(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}\mathrm{Ind}_{\mathrm{B}(\mathbb{A})}^{\mathrm{M}_1(\mathbb{A})}\mathbb{C}^{-\lambda-\rho} \longrightarrow \mathrm{Ind}_{\mathrm{P}_1(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}\mathbb{C}^{-s_1(\lambda+\rho)}.$$

By taking the cohomology of the map ϕ, we get

$$H^5(\mathfrak{g}, K_\infty, J_\lambda \otimes \mathcal{M}_\lambda)^{K_f} \xrightarrow{\phi} H^5(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}).$$

Moreover, the map ϕ fits into the following diagram.

$$\begin{array}{ccccc} H^5(\mathfrak{g}, K_\infty, J_\lambda \otimes \mathcal{M}_\lambda)^{K_f} & & \xrightarrow{\quad A \quad} & & H^5(\mathfrak{g}, K_\infty, \mathrm{Ind}_{P_0(\mathbb{A})}^{G(\mathbb{A})}\mathbb{C}^{-s_1(\lambda+\rho)} \otimes \mathcal{M}_\lambda)^{K_f} \\ \downarrow \phi & & & & \downarrow \psi \\ H^5(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) & \xrightarrow{r^5} & H^5(\partial S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) & \xrightarrow{r'} & H^0(S_\Gamma^{M_0}, \widetilde{\mathcal{M}}_{w_{11}\cdot\lambda,\mathbb{C}}) \end{array}$$

According to the cohomology of the induced modules [3, Chapter III], the $(\mathfrak{g}, K_\infty)$ cohomology of $J_\lambda \otimes \mathcal{M}_\lambda$, $\mathrm{Ind}_{P_0(\mathbb{A})}^{G(\mathbb{A})}\mathbb{C}^{-s_1(\lambda+\rho)}$, and the quotient module is trivial when $q < 5$ and

$$\dim H^5(\mathfrak{g}, K_\infty, J_\lambda \otimes \mathcal{M}_\lambda)^{K_f} = \dim H^5(\mathfrak{g}, K_\infty, \mathrm{Ind}_{P_0(\mathbb{A})}^{G(\mathbb{A})}\mathbb{C}^{-s_1(\lambda+\rho)} \otimes \mathcal{M}_\lambda)^{K_f} = 1.$$

Hence, the map A is injective, and as both the source and the target has dimension 1, it is an isomorphism. On the other hand, it is well known that the map ψ is an isomorphism. Consequently, the map $\psi \circ A$ is an isomorphism. This implies that both the map $r' \circ r^5$ is surjective. This completes the proof of part (2).

For the proof of part (1) and part (3), the same strategy applies and the proofs are indeed easier. In part (1), the constant term, hence the Eisenstein series, has a double pole at $z_1 = 0, z_2 = 0$. By taking successive residues, the Langlands quotient we get is the constant representation, which provides non-trivial Eisenstein cohomology classes $H^0_{B,res}(S_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. While in case 5, the corresponding Eisenstein series is holomorphic at the special point θ_λ. Hence, part (3) can be proved by just taking cohomology of the map $\mathscr{E}$, see [19] for more general cases. □

5.8 Proof of Theorem 10

Now, by combining Proposition 20 with Proposition 21, Theorem 10 can be verified through a case-by-case study. □

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Data availability

All data generated or analyzed during this study are included in this published article [and its supplementary information files].

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