# Snowmass White Paper: Gravitational Waves and Scattering Amplitudes

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ABSTRACT: We review recent progress and future prospects for harnessing powerful tools from theoretical high-energy physics, such as scattering amplitudes and effective field theory, to develop a precise and systematically improvable framework for calculating gravitational-wave signals from binary systems composed of black holes and/or neutron stars. This effort aims to provide state-of-the-art predictions that will enable high-precision measurements at future gravitational-wave detectors. In turn, applying the tools of quantum field theory in this new arena will uncover theoretical structures that can transform our understanding of basic phenomena and lead to new tools that will further the cycle of innovation. While still in a nascent stage, this research direction has already derived new analytic results in general relativity, and promises to advance the development of highly accurate waveform models for ever more sensitive detectors.

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# Contents

# 1 Executive Summary

The ambitious future of gravitational-wave (GW) science calls for invigorating the theoretical framework for precision calculations of GW signals, thus galvanizing a new approach that harnesses the cutting-edge tools of theoretical high-energy physics such as on-shell methods, double copy, advanced multiloop integration, and effective field theory (EFT). These are the engines that drive modern calculations of scattering amplitudes in particle theory, and integrating them together for application to GWs has recently led to new results in the perturbative, analytic solution of the two-body problem in general relativity. Theorists developing waveform models for the LIGO-Virgo-KAGRA (LVK) collaboration [1–3] have performed initial studies [4, 5] of these early results, see Figure 1, and have strongly encouraged further developments. Indeed, if these calculations are pushed to higher orders, and are extended to include all physical effects (i.e., spins and tides), they can be used, in combination with other analytic methods [6–11][1] and with numerical-relativity (NR) simulations [16–19], to provide highly accurate waveform models of binary systems composed of black holes and/or neutron stars. Another aspect of this program that has drawn significant interest from

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[1]See also [12–15] for recent reviews.
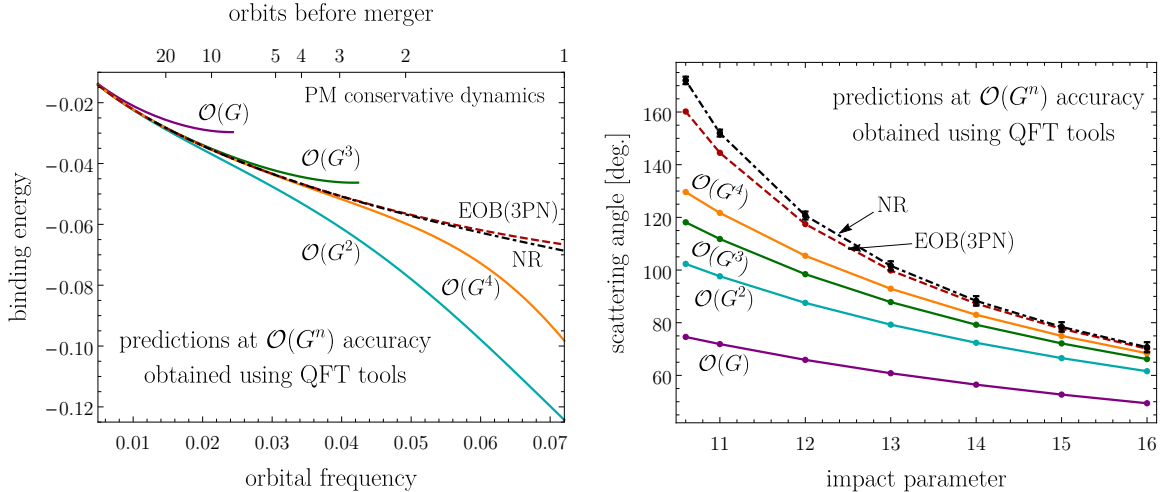
**Figure 1**. **(left)** The binding energy (in units of the reduced mass) versus the orbital frequency of an equal-mass non-spinning binary black hole following an adiabatic quasi-circular orbit towards merger. The horizontal axis is also given as the number of orbits before merger. **(right)** The scattering angle versus impact parameter of two equal-mass non-spinning black holes following hyperbolic trajectories with initial relative velocity $v = 0.4$. These plots are adapted from [5], where the authors compared predictions for post-Minkowskian (PM) conservative dynamics obtained using QFT tools (solid lines of increasing accuracy in Newton's constant $G$, i.e., in the PM approximation) with NR [28, 29] and effective-one-body (EOB) results, here at third post-Newtonian (PN) order, which are the benchmarks for building waveform models in the LVK collaboration.

both the high-energy physics and general relativity communities is the exploration of theoretical structures that emerge in the classical limit of scattering amplitudes.

This new research direction in theoretical high-energy physics is an opportunity to deploy the classic and modern tools of quantum field theory (QFT) in a new arena, thereby impacting an important experimental frontier and uncovering rich theoretical structure that can lead to new tools. The program is in a nascent stage, and significant progress will come in the next several years, building towards the vision that QFT tools will advance the computation of gravitational waveforms. In particular, they will address the need for high precision in upcoming LVK runs, in space-based detectors such as LISA [20], and in future ground-based detectors such as LIGO-India [21], Cosmic Explorer [22] and Einstein Telescope [23]. High-precision waveform models will be crucial for maximizing the discovery potential and extracting the best science with GW observations of ever more sensitive future detectors [20, 24–27].

## 2 Introduction

The emergence of GW science [30–32] has already transformed multiple domains of astronomy, cosmology, and particle physics, yet this represents only a small fraction of its future potential [20, 26]. Space- and ground-based observatories of the coming decade will map out and characterize millions of merger events per year with sensitivity well beyond that of current LIGO/Virgo facilities (see, e.g., [33–37]). One of the key challenges will be advancing the theoretical modeling of compact binary coalescences to produce accurate gravitational waveforms.

Theoretical modeling of GW sources is challenging due to multiple physical scales that are nonlinearly coupled through general relativity. Notably, these complications — multiple scales and nonlinearity — are exactly what drove breakthroughs in QFT in the last few decades, leading to the modern scattering amplitudes program. Scattering amplitudes have revealed mathematical structures in gauge theory and gravity, leading to new physical insights, efficient methods for computation, and the seeds for even bolder ideas.

The Parke-Taylor formula [38] reduced pages of Feynman calculus to a half-line expression describing gluon scattering, famously heralding the enormous value of understanding theoretical structures lying at the heart of scattering amplitudes. In recent decades, major advances have been driven by two parallel developments. First are new methods that formulate QFT without explicit quantum fields, thus focusing on physical quantities. These "on-shell methods", reinvigorated by twistor string ideas [39–42], have become efficient mainstream tools for tree-level [43] and loop-level [44–47] calculations in gauge and gravity theories; see [48–52] for reviews. Second, is a radically new perspective on gravity: gravitational scattering amplitudes $\mathcal{M}_{\text{gravity}}$ can be realized as a "double copy" of gauge theory amplitudes $\mathcal{M}_{\text{gauge}}$ [53, 54],

$$\mathcal{M}_{\text{gauge}} \times \mathcal{M}_{\text{gauge}} \sim \mathcal{M}_{\text{gravity}}. \tag{2.1}$$

This structure builds on the relation between tree-level open and closed string scattering [55], its field theory limit [56], and structures gleaned in explicit higher-loop calculations [57, 58]. It extends vanilla examples of related gauge and gravity theories to a veritable web of theories that share common building blocks, and has been applied to the exploration of a number of new directions, such as the ultraviolet properties of supergravity theories up to five loops [59–63], and the nonperturbative structure of classical solutions of Einstein's equations with sources such as Schwarzschild [64]; see [65] for a recent review and also the dedicated Snowmass White Paper [66].

In the past few years there has been a flurry of activity in applying both on-shell methods and the double copy, in combination with advanced multiloop integration

$$G(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)$$
$$G^2(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)$$
$$G^3(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)$$
$$G^4(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)$$
$$G^5(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)$$
$$G^6(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)$$
$$G^7(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)$$

**Figure 2**. Map of perturbative corrections to Newton's potential, where $G$ is Newton's constant and $v$ is the relative velocity of the binary constituents. New results through $O(G^4)$ were recently obtained using QFT tools (red box). They are valid to all orders in velocity, and overlap with the state-of-the-art from the PN expansion (dark triangle) and the contributions required by future detectors (light triangle) (see, e.g., [33–37]).

techniques and EFT, to develop new tools for state-of-the-art predictions of GW signals. This development was encouraged by the general relativity community [67], and has led to a number of new results in general relativity; for example see Figures 1 and 2.

The new approach based on tools from theoretical high-energy physics aims to complement, and has greatly benefited from, decades of successful work using traditional approaches to solve the relativistic two-body problem, including the post-Newtonian (PN) approximation [68–76], the gravitational self-force formalism [77, 78], the effective–one-body (EOB) formalism [10, 11], the nonrelativistic general-relativity (NRGR) formalism [79], the post-Minkowskian (PM) approximation [67, 80–88], and NR [16–18] (see, e.g., [12–15, 89] for recent reviews). In particular, the seminal work [79] introduced nonrelativistic EFT ideas from particle physics to the worldline approach to binary dynamics, and has led to a number of landmark results; for recent developments see the dedicated Snowmass White Paper [90] on this subject.

The tools of theoretical particle physics have been honed in a wide variety of intricate quantum calculations such as the Higgs gluon-fusion cross section at $N^3LO$ [91], NNLO corrections to $e^+e^-$ event shapes [92], cusp anomalous dimension at four loops [93], electron $g - 2$ at four loops [94], $\mathcal{N} = 8$ supergravity four-point amplitude at five loops [95], and $\mathcal{N} = 4$ super-Yang-Mills four-point amplitude at six loops [96]. Leveraging these tools for GW physics yields several advantages. First, the structure of perturbation theory is vastly simplified by special relativity, on-shell methods, and the
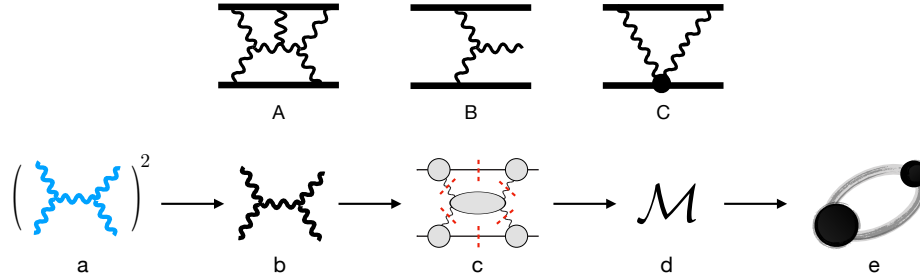
**Figure 3**. **(top)** Classical binary dynamics is encoded in scattering amplitudes of massive particles (thick lines) interacting through gravitons (wavy lines): (A) four-point scattering encodes higher-order corrections to conservative binary dynamics, (B) five-point scattering encodes radiative effects due to graviton emission, and (C) higher-dimension operators (solid circle) encode tidal deformation of neutron stars. Spinning black holes can be described by higher-spin representations in QFT. **(bottom)** A sample calculational pipeline using the tools of theoretical high-energy physics: Starting from tree-level gauge theory amplitudes (a), corresponding gravitational amplitudes (b) are obtained using the double-copy. These are then fused into loop amplitudes (c) using generalized unitarity. The integrated amplitude (d) is obtained using advanced multiloop integration methods developed in high-energy physics, in combination with EFT. The amplitude can then be mapped, using a variety of methods, to the EOB Hamiltonian used for producing waveforms (e). The classical limit is applied at every stage, leading to vast simplifications.

double copy, leading to compact expressions that make theoretical structures manifest. Second, the technology and deep knowledge base for integration of loops in QFT, which have seen decades of development for collider physics applications, are directly transferred, including integration-by-parts systems [97–99] and differential equations [100–106]. Finally, EFT efficiently and systematically targets contributions to various processes in the classical limit. Figure 3 illustrates these tools in action.

Aside from providing state-of-the-art predictions, the new approach using particle physics tools aims to explore theoretical structures that emerge in the classical limit of scattering amplitudes. These structures may be familiar from particle physics, but are not manifest in traditional approaches to the two-body problem in general relativity, or they may be rooted in the classical regime and are yet to be explored through the lens of QFT. Examples include universality in the high-energy limit, the interplay between conservative and dissipative effects, nonperturbative connections to classical solutions, and perturbation theory in curved backgrounds.

The upshot is that scattering amplitudes are highly effective tools for understanding and precise modeling of GW sources, in ways similar to their application for interactions of fundamental particles. The recent progress and future goals of this program revolve

around three objectives, broadly defined:

- Provide state-of-the-art predictions for the dynamics and gravitational radiation of compact binaries composed of black holes and neutron stars, which can be used, in combination with other analytic methods and NR simulations, to produce highly accurate waveforms. This includes modeling spin, tidal, and radiative effects.

- Develop the mathematical and physical tools of theoretical high-energy physics for application to GWs. This involves tailoring existing tools to improve scalability, and formulating new tools and approaches to the binary problem.

- Explore the theoretical landscape of scattering amplitudes in the classical regime. Examples include the universality of high-energy scattering, nonperturbative relations between scattering amplitudes and classical dynamics, and connections to exact spacetime geometries.

In the rest of this white paper, we will discuss the classical limit, and then describe each of the three topics above, summarizing recent progress. Then we will describe future directions.

## 3 Classical Limit

The correspondence principle states that classical physics emerges from the quantum theory in the limit of macroscopic conserved charges such as masses, electric charges, spins, orbital angular momenta, etc. Perturbations around such a configuration, which are subleading in the large charges, can be systematically included and have the natural interpretation of quantum corrections.[2]

For the application to GWs, the classical limit of scattering amplitudes is defined by two properties that distinguish compact binaries from their quantum counterparts:

- Bound compact objects have large angular momentum $J \gg \hbar$, as opposed to $J \sim \hbar \equiv 1$ for quantum bound states.

- Compact objects, such as black holes or neutron stars, have large gravitational charges $M_\odot/M_{\mathrm{Planck}} \sim 10^{38}$, as opposed to $e/Q_{\mathrm{Planck}} \sim 10^{-1}$ for electric charges of elementary particles. Here $M_\odot$ and $e$ denote the solar mass and electron charge, while $M_{\mathrm{Planck}}$ and $Q_{\mathrm{Planck}}$ are the Planck mass and charge, respectively.

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[2]This perspective was used to great effect in string theory tests of the asymptotic Bethe ansatz for the anomalous dimensions of single-trace operators in $\mathcal{N} = 4$ super-Yang-Mills theory, see e.g. [107–109] and [110] for a review.

In other words, for a system of two gravitationally interacting spinless bodies with masses $m_1$ and $m_2$, the classical regime emerges in the limit $J \gg \hbar$ and $m_1, m_2 \gg M_{\text{Planck}}$. This limit also coincides with the limit in which the de Broglie wave length $\lambda$ of the particles is much smaller than the particles' separation $|\boldsymbol{b}|$, which is conjugate to the momentum transfer $\boldsymbol{q}$. Thus, the kinematic regime of classical physics is when the momentum transfer is much smaller than the incoming momenta, $|\boldsymbol{p}| \gg |\boldsymbol{q}|$, similar to the Regge limit. Other charges that may characterize classical particles also have a similar scaling in the classical limit; for example, the spin $S$ and finite size $R$ scale as $|\boldsymbol{q}|R \sim \mathcal{O}(1)$, $|\boldsymbol{q}||\boldsymbol{S}| \sim m_{1,2}$. These scalings are consistent with the classical nature of Newton's potential, $|\boldsymbol{b}| \gg Gm$, and fixes the general form of four-point amplitudes as well as of generating functions of observables.

Another perspective on the classical limit was taken in [111], which together with the associated framework is usually referred to as the KMOC formalism. Quantum mechanical computation of observables yields the classical result in the correspondence limit which is arranged in two steps: (1) a suitable restoration of Planck's constant $\hbar$, which effectively plays a role similar to the momentum transfer $\boldsymbol{q}$, and expansion at small $\hbar$, and (2) incorporation from the outset that from a quantum-mechanical perspective, particles are described by wavepackets having finite width in both position- and momentum-space rather than plane waves. This width must be negligible in the classical limit, leading to inequalities which constrain the parameters of the scattering, analogous to those coming from the large-$J$ limit.

One implication of the classical limit is that loop amplitudes involving massive particles may contain classical contributions. The classical limit can be taken at the earliest stages of calculations, yielding vast simplifications prior to integration. This enables the calculation of loop amplitudes that would otherwise be beyond the reach of current technology if evaluated including quantum effects. Properties of amplitudes in an expansion around the classical limit were previously unexplored systematically. The application to GW physics provides the motivation to fill this gap, and have led to interesting results [112, 113].

## 4   State-of-the-Art Predictions

Waveform models for the inspiral stage of a binary system are built from the conservative and dissipative two-body dynamics. Therefore a direct path for amplitudes methods to have an impact is to compute state-of-the-art conservative two-body potentials, including the effects of spin, and tidal deformation for modeling neutron stars [12, 20, 22, 23]. For instance, the general relativity community had urged ampli-

tudes experts to compute the $\mathcal{O}(G^3)$ contributions to conservative binary dynamics for spinless compact objects [67].

Potentials are naturally encoded in four-point scattering amplitudes, and can be extracted from the latter via a variety of methods such as an amplitude matching calculation [114], direct mapping to an EOB parameterization [115, 116], using the eikonal phase as a generating functional [117–120], and analytic continuation [121–124]. However, at $\mathcal{O}(G^4)$, the conservative two-body Hamiltonian is no longer universal to both bound and unbound trajectories. The effect of back-scattered radiation on the conservative dynamics — in particular the so-called tail effects [125–127] — leads to nonlocal-in-time contributions to the Hamiltonian that depend on the specific trajectory. Understanding the mapping between bound and unbound orbits in the presence of radiation is an important open problem that needs to be solved to maximally leverage scattering amplitudes for deriving classical bound state dynamics.

**Higher-Order Corrections to Conservative Binary Dynamics.** Conservative binary dynamics at $\mathcal{O}(G)$ and $\mathcal{O}(G^2)$ and to all orders in velocity[3] were derived from scattering amplitudes in Ref. [114]. While these results were known in general relativity using classical methods, the method employed in Ref. [114] laid a basic path for extending to higher orders using scattering amplitudes.

Conservative binary dynamics at $\mathcal{O}(G^3)$ was computed in Refs. [128, 129] using amplitudes techniques, and has since been verified using a variety of methods (see, e.g., [130–132]). The result of Refs. [128, 129] for conservative binary dynamics has been extended to include dissipative effects in a number of studies [133–142].

Conservative binary dynamics at $\mathcal{O}(G^4)$ was computed in Refs. [143, 144] using amplitudes techniques. Radiative contributions to the conservative dynamics first appear at $\mathcal{O}(G^4)$, and complicates the analysis due to difficulties in precisely defining conservative processes in the presence of radiation. This represents one of the main obstacles for advancing the theoretical modeling of GW sources to higher orders. Interestingly, some of the issues are sidestepped in the QFT-based approach where everything follows from well-established properties of scattering amplitudes. In particular, four-point scattering amplitudes should fully capture two-body conservative dynamics for hyperbolic orbits and is equivalent to using time-symmetric graviton propagators [88, 145, 146].

The result of Refs. [143, 144] reproduces the sixth-order PN result in Ref. [147]. Moreover, due to Lorentz invariance and dimensional analysis, the result exhibits a simple mass dependence, confirming arguments made in [115]. This mass dependence puts

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[3]The $\mathcal{O}(G^n)$ result to all orders in velocity is also referred to as the $n$-th order post-Minkowskian result or $n$PM result. In this White Paper we will often simply refer to this as the $\mathcal{O}(G^n)$ result, following the conventional nomenclature of perturbative orders in particle physics.
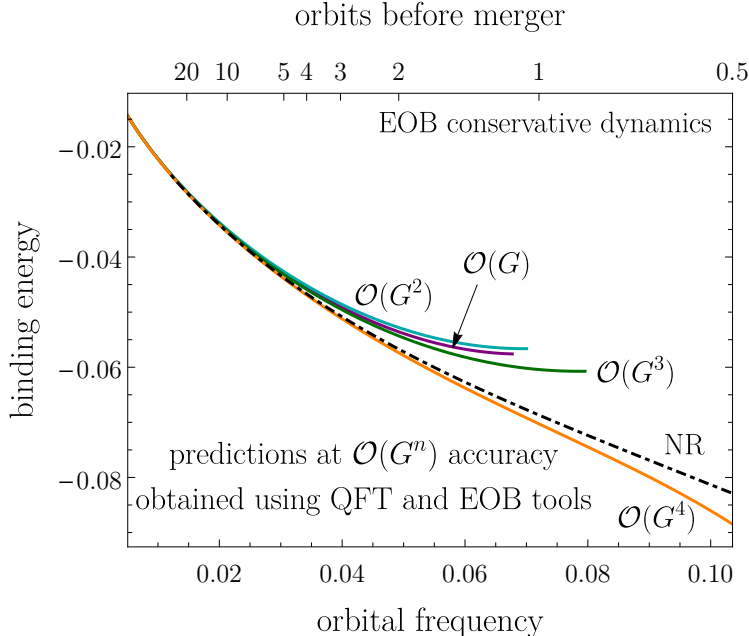
**Figure 4**. The binding energy (in units of the reduced mass) versus the orbital frequency of an equal-mass non-spinning binary black hole following an adiabatic quasi-circular orbit towards merger. The horizontal axis is also given as the number of orbits before merger. The plot is adapted from [5], where the authors compared predictions for post-Minkowskian (PM) conservative dynamics obtained using QFT tools (solid lines of increasing accuracy in Newton's constant $G$, i.e., in the PM approximation) to NR results [29]. In contrast to Figure 1, the predictions shown here are obtained by incorporating the QFT results into the EOB formalism, resulting in better agreement with NR towards merger.

strong constraints on radiative contributions to conservative binary dynamics [147]. For instance, the result of Refs. [143, 144] also agrees with the fifth-order PN result of Ref. [148] up to a single term that does not have the expected mass dependence. The origin of such contributions requires further study. The result of Refs. [143, 144] to all orders in velocity has also been partially verified [149, 150]. Initial studies of these results were performed by theorists developing waveform models for the LVK collaboration with encouraging conclusions; see Figure 1. Quite interestingly, the results derived from amplitudes and EFT can be included in the EOB framework in a way that improves the comparison with NR towards merger; see Figure 4. While these results at 4PM order are in an early stage with assumptions [4] that require further investigation,

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[4]The results at 4PM order, i.e. at $\mathcal{O}(G^4)$, have been obtained from the Hamiltonian computed for scattering orbits in [143, 144, 149, 150].

the key message is that results from scattering amplitudes are promising and further developments are strongly encouraged.

**Spin Effects.** From a theoretical perspective, inclusion of the spin of the binary components in amplitudes methods faces the difficulty that fields of arbitrary spins are needed while no-go theorems [151–156] show that under certain assumptions such field theories have unphysical features. Five proposals [157–161] have been put forth to construct classical spin-dependent two-body Hamiltonians. Their results are consistent with each other, and also with results obtained through standard general-relativity methods, when the latter are available.

The stress tensor of a Kerr black hole [162] was obtained from higher-spin interactions in [163] and [158], the latter also obtaining the stress tensor of more general spinning compact objects [164]. The all-orders-in-spin $\mathcal{O}(G)$ and quartic-in-spin $\mathcal{O}(G^2)$ scattering angle for an aligned-spin configuration (i.e., spins that are aligned or anti-aligned with the orbital angular momentum) was found in [165]. The $\mathcal{O}(G^2)$ Hamiltonians expected to describe Kerr black hole binaries for general spin configurations through fourth power of the spin were derived in [158, 166–169], and a general direct relation between amplitudes and observables was conjectured in [158]. The change in momentum and in the spin at $\mathcal{O}(G^3)$ to quadratic order in spin have been computed in [170], together with the radiation-reaction effects due to the radiated momentum at 2PM order [171].

Proceeding past fourth order in spin at $\mathcal{O}(G^2)$ had been an important open problem for standard general-relativity methods. The neutron star Hamiltonian quintic in spin was predicted in [172] for arbitrary spin orientation, and the Kerr scattering amplitude to eighth order in spin in [173]. Refs. [172, 173] proposed that the Kerr black hole corresponds to a particular subclass of spin structures that can appear in a general scattering amplitude, which was explained in [172] through a shift symmetry of the spin vector reminiscent of reparametrization invariance in heavy-particle EFT [174, 175].

Relatedly, the dynamics of spin is technically very similar to the dynamics of color in classical (infrared-free) Yang-Mills theories; using similar techniques, the 2PM Hamiltonian for color-charged matter was found in [176, 177].

**Tidal Effects.** Recent detections of GWs from the merger of neutron stars already constrain the equation of state of matter at nuclear densities [31, 178], and more accurate measurements at future detectors strongly motivate calculations at higher precision [33–37]. Similar to traditional approaches using classical methods [79, 132, 179], the approach based on scattering amplitudes uses higher-dimension operators to model the rigidity of the body and the susceptibility of its shape to change in response to a

tidal potential [180]. The $\mathcal{O}(G^3)$ contributions from the leading tidal operators were computed using scattering amplitudes in [180], and was verified using a worldline approach in [181]. Other amplitudes-based calculations of tidal effects have been pursued in Refs. [182–185], including results for infinite classes of tidal operators as well as all orders in $G$ results in the probe limit.

Aside from calculating predictions for tidal effects for neutron stars, tidal effects for black holes have also been the subject of many recent studies within the particle physics community [186–188]. Black holes have vanishing static, conservative tidal responses, and a symmetry explanation for this has recently been put forth [189, 190].

**Beyond General Relativity.** Two-body Hamiltonians capturing specific models of physics beyond general relativity have also been computed using scattering amplitudes methods [180, 184, 191–196].

**Radiative and Absorptive Effects.** As mentioned above, waveform models rely on both the conservative and dissipative dynamics. Calculations of radiative effects have so far focused on hyperbolic scattering trajectories, in particular the scattering angle [133, 135–142], as well as the loss of linear [139, 140, 197] and angular momentum [134, 147, 171, 198–201], and the energy flux from higher-order tail effects [202, 203]. A proposal for deriving radiation-reaction forces from scattering amplitudes was recently given in [201].

So far, absorptive effects have not been incorporated in a QFT framework, see [204] for a worldline EFT approach.

# 5   Theoretical Structures

In this section we give an overview of mathematical and physical structures that are relevant in the classical regime of scattering amplitudes. Some of these structures, such as nonperturbative properties of amplitudes and their direct connections to classical solutions, were exposed through explicit calculations of different observables at higher orders. Others, such as the eikonal phase and universality of high-energy scattering, have been studied in the context of theoretical high energy physics but only recently applied for GWs. We highlight examples where theoretical structures bring insight to the phenomenology of GWs, and can be leveraged to develop calculational tools.

**High Energy Limit.** Even for classical scattering, the high energy (ultra-relativistic) limit exposes interesting structures such as the interplay between exclusive and inclusive observables [205–207], connections to soft graviton theorems [141], and universality among gravitational theories with and without supersymmetry [106, 118, 133].

For example, the four-point classical potential scattering amplitude at $\mathcal{O}(G^3)$ develops a singularity in the high-energy limit, $s \to \infty$:

$$\mathcal{M} \to -8\pi G^3 s^2 \log(-t) \log\left(\frac{m_1 m_2}{s}\right) , \qquad (5.1)$$

where $s$ and $t$ are Mandelstam variables, and $m_i$ are the masses of the scattering black holes. On general grounds, such singularities cannot exist in complete amplitudes [208]. The finite inclusive observable, including contributions from radiation modes, can be derived using soft graviton theorems to describe graviton emission [141] or by a linear response analysis [134]. Moreover, the double logarithmic structure can be understood from the Regge limit, and is given by the cusp anomalous dimension [209], or by the heavy-heavy current anomalous dimension in Heavy Quark Effective Theory [210].

These connections illustrate that amplitudes in simplifying limits, even in supersymmetric versions of gravity, can carry useful information about classical binary dynamics. More generally, there is a wealth of knowledge on four- and five-point amplitudes in theoretical high energy physics, and many of the tools and physical insights can be transplanted to classical binary dynamics. For example, quantum electrodynamics (QED) as well as supersymmetric cousins of gravity, such as $\mathcal{N} = 8$ supergravity, offer controlled laboratory settings for sharpening tools and dissecting basic phenomena.

**Eikonal Phase.** It has long been known that in the classical limit the eikonal phase provides a good description of elastic four-point amplitudes in gauge and gravity theories, which effectively exponentiate as a consequence of unitarity [211–217]

$$i\mathcal{M} = e^{i\delta} - 1 . \qquad (5.2)$$

It is moreover expected that the dominant contributions at high energies to the eikonal $\delta$ come from the exchange of the highest-spin state in the theory [118, 119, 214], and therefore that it is universal in gravitational theories [218].

Modern evidence for these properties was obtained in recent work on massive theories [106, 133, 136], as well as massless theories with various amounts of supersymmetry, including general relativity [219]. This led to the idea that inclusion of all contributions from exchanged gravitons with momenta of the order of the transferred momentum is a necessary (and possibly sufficient) condition for universality of the eikonal in the ultrarelativistic limit and for a smooth interpolation to the nonrelativistic limit [133]. A self-contained treatment of the real and imaginary parts of the eikonal in the entire soft region at $\mathcal{O}(G^3)$ in massive $\mathcal{N} = 8$ and in general relativity was presented in [135] and confirmed in [220], giving another demonstration of the universality properties.

The eikonal form of the S matrix is quite similar to that of the amplitude-radial action relation (5.3), but they differ in the definition of the iteration terms [143]. Similar

to the radial action, the eikonal is a generating function of scattering observables [221], such as the scattering angle or the time delay. See Sec. 6 for further discussion.

**Eikonal with Spin.** The unitarity argument for the exponentiation of elastic amplitudes holds also for spinning external states. This was demonstrated explicitly for spin-1/2 particles in [222] and for arbitrary spin in the classical limit from the perspective of minimization of the spin variance in [112].

The eikonal continues to be a generating function for observables when scattering spinning particles. This was verified explicitly through $\mathcal{O}(S_1 S_2)$ in [158] where the change in momentum (also referred to as the "impulse") and the change in spin (also referred to as the "spin kick") obtained from the eikonal were compared with the results of Hamilton's equations. A conjecture was also put forth [158], connecting eikonal and scattering observables to all orders in spin. This relation was successfully tested through fourth order in spin [166, 169] at $\mathcal{O}(G^2)$. It is quite interesting and surprising that a single function can capture the complicated three-dimensional dynamics of scattering of spinning particles.

**Nonperturbative Structures.** EFT methods can be used to connect scattering amplitudes and classical binary dynamics through a matching calculation between gravity and a low energy EFT that describes particles interacting through an effective potential. An outcome of this matching calculation is a beautiful relation between the classical limit of the scattering amplitude $\mathcal{M}$ in impact parameter space and the radial action $I_r$ of Hamilton-Jacobi theory:

$$i\mathcal{M} = e^{iI_r} - 1. \tag{5.3}$$

This means that the scattering amplitude directly determines the radial action $I_r$, which in turn determines orbital trajectories. This equation was first derived in [143], and is dubbed the "amplitude-action relation". The exponential form of the amplitude-action relation is reminiscent of the eikonal phase (5.2), and has an important practical implication for calculations: a large class of integrals come from exponentiation of lower-order contributions and can be systematically dropped prior to integration. The amplitude-action relation was studied to all orders in perturbation theory for a probe in the Schwarzschild or pure-NUT gravitational backgrounds as well as for a probe interacting with point-charges and monopoles in [223].

The amplitude-action relation in Eq. (5.3) is an example of remarkable nonperturbative structures that appear in the classical limit. Another example is the following relation between the classical scattering amplitude in position space $\mathcal{M}(r)$ and the local
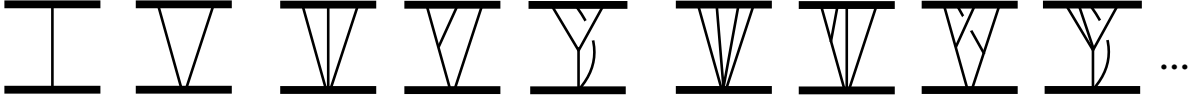
**Figure 5**. The resummation of these diagrams to all orders in $G$ yields the amplitude for a probe particle in a Schwarzschild background.

center-of-mass momentum in a hyperbolic orbit,

$$\mathcal{M}(r) = \frac{p(r)^2 - p(\infty)^2}{2E}\,. \tag{5.4}$$

Here $p(r)^2$ is the squared center-of-mass momentum at position $r$, and $E$ is the total center-of-mass energy. This was first noticed and used in [128, 129], and further developed and formalized in [122] (where it was dubbed "impetus formula") and also [224]. This relation can be used to derive amplitudes to all orders in $G$. For instance, the diagrams in Figure 5 describe a probe particle in a Schwarzschild background, and can be resummed by determining $p(r)$ from geodesic motion. One can also derive nonperturbative amplitudes involving higher-dimensional operators that describe tidal effects [185].

Another nonperturbative relation builds on the Newman-Janis [225] relation between the Schwarzschild and Kerr solutions of Einstein's equations. This relation, employing a certain complex shift, has been used in [226] to obtain the impulse for spinning particles from that of spinless particles at leading post-Minkowskian order. It remains an intriguing open question whether the relation holds at higher orders and how to exploit it [227].

**Gravitational Self-Force.** Extreme-mass-ratio inspirals are binary systems consisting of compact bodies with masses $m_1$ and $m_2$ with $m_2 \gg m_1$. The limiting case is described by a probe particle orbiting in a background spacetime such as Schwarzschild or Kerr. Beyond this, the particle interacts with its own gravitational field, giving rise to an effective "self-force", which is computed as an expansion in $m_1/m_2$ but to all orders in $G$. The gravitational self-force for generic bound geodesics in Schwarzschild and Kerr spacetimes were found in [228] and [229], respectively. The precision of LISA will require the second-order self force in the conservative and dissipative dynamics, i.e., corrections of $\mathcal{O}(m_1^2/m_2^2)$ and all orders in $G$, which is not completely solved. However, see Ref. [230] for significant recent progress.

Interestingly, scattering amplitudes have revealed a connection between perturbative corrections to binary dynamics and self-force corrections. The mass dependence of the $n$-loop scattering amplitude in the classical limit follows simply from dimensional analysis, and implies that it can probe the $\mathcal{O}(G^{n+1})$ contribution to the $\lfloor \frac{n}{2} \rfloor$-th

order self-force correction. In other words, the $n$-loop scattering amplitude contains contributions of the form $\sim G^{n+1}(m_1/m_2)^{\lfloor \frac{n}{2} \rfloor}$. This structure is leveraged in a powerful new method, dubbed "Tutti-Frutti", for extracting perturbative corrections to binary dynamics from self-force calculations [115, 231–236]. This mass dependence, which ultimately traces back to Lorentz invariance, also has strong implications for the radiative contribution to conservative dynamics [115]. In particular, it implies nontrivial cancellations among contributions from source multipole moments and imposes strong consistency checks on perturbative calculations. Notably, there are now numerical and theoretical efforts to extend gravitational self-force calculations to hyperbolic trajectories in order to make contact with results from scattering amplitudes; see, e.g., [198].

**Coherent States and Waveforms.** The gravitational waveform during a scattering event can itself be directly computed from amplitudes [237]. At lowest perturbative order it is given by an integral of a five-point tree scattering amplitude. The relation to the intuitive picture, in which a GW consists of a large number of gravitons, is interesting: GWs are described by coherent states (with very large occupation numbers) of the gravitational field. Gravitons building up this state are emitted independently [112, 113], and each emission is described by the same five-point amplitude.

Coherent states also play a role in understanding the emergence of classical spin and color from QFT [158, 176, 238], and in understanding the emergence of classical physics from quantum mechanics quite generally [239].

Eikonal methods have been generalized to include such coherent outgoing radiation [112], building on earlier work of [240] and [237]. In particular, Ref. [112] argued that the eikonal is effectively extended by a coherent state operator, which creates arbitrarily many outgoing massless particles and that the nonperturbative radiation field is described by the waveshape parameter defining the coherent state, which in turn is determined by the five-point amplitude.

**Soft Gravitons.** Recently it has become clear that there is a beautiful relation between the soft limit in quantum theories and memory effects in classical dynamics [241]. This can be understood in the KMOC formalism (see Sec. 6) by studying the radiated momentum. In the long-wavelength limit, the scattering amplitude involved in the radiation simplifies as a soft factor times a lower point amplitude, recovering the impulse [242]. Classically, this impulse is the "memory" of the step-change in the field described by its very low frequency Fourier components. More generally, the KMOC formalism reveals a rich interplay between classical physics, soft or low-frequency radiation, and scattering amplitudes [243–247].

**Analytic Continuation and Time Nonlocality.** The integrability of the two-body equations of motion with Newton's potential guarantees that their solutions are uniquely specified by integrals of motion – the total energy and the orbital angular momentum. Hyperbolic and elliptic motions are mapped into each other by analytic continuation of the boundary conditions. While integrability is not known to exist for two-body conservative Hamiltonians even at 2PM order [248], Refs. [121–124] argued that bound state observables can be obtained through a suitable analytic continuation in energy and in angular momentum from scattering observables. This procedure was dubbed "Boundary-to-Bound" or "B2B" and formalizes the fact that such Hamiltonians can be constructed by matching, e.g. the scattering angle, and then subsequently used for bound motion by changing the boundary conditions. Physically, this can be understood as a consequence of the time locality of the potential generated by potential-region gravitons.

However, this approach fails for conservative radiation-reaction effects, beginning at $\mathcal{O}(G^4)$ with the tail effects [124]. Given a scattering angle, it is always possible to construct a local Hamiltonian that reproduces it. Fundamentally however, because it captures effects of radiation modes propagating over long periods before being reabsorbed by the binary, the "off-shell" Hamiltonian has both an instantaneous component and a non-local in time one [75]. While the analytic continuation of the local and universal (logarithmic) part of the nonlocal Hamiltonian is straightforward, obtaining the non-universal part of the bound Hamiltonian from the unbound one is an important open problem.

**Exploring Structure in Simpler Theories.** Even in the classical limit gravitational interactions are complicated. Before proceeding to full-fledged gravitational calculations, it is useful to test ideas and tools and search for theoretical structures in much simpler settings such as gauge theory. Since color in the classical limit becomes essentially abelian, it suffices to study QED. Calculations designed to explore some of the subtleties that appear at $\mathcal{O}(G^3)$ in gravitational calculations were carried out in gauge theories, both using scattering amplitudes [176, 249], and classical gravity methods [250, 251].

Gravitational theories with additional symmetries, like supersymmetry are another possibility. As discussed above, results obtained in supergravity theories confirmed universality properties of the eikonal function and led to an understanding of the role of radiative corrections in classical scattering.

**Relations Between Amplitude Fragments.** The essential difference between classical and quantum measurements is the variance: it is nonzero for quantum measure-

ments and must become negligible in the classical limit. Defining measurements in terms of expectation values of operators, Ref. [112] showed that the condition of zero variance (effectively requiring factorization of expectation values of products of operators) implies an infinite set of relations between different amplitude "fragments" (that is different orders in an amplitude's expansion in momentum transfer) with different numbers of loops and legs. For four-point amplitudes they are the relations required for eikonal exponentiation. At five points, higher-loop fragments are related to lower-loop five-point and four-point amplitude fragments. Among the implications of the zero-variance condition are detailed predictions for the momentum transfer dependence of amplitudes and a novel understanding of the eikonal in the presence of outgoing radiation mentioned above.

# 6 Developing Tools

In this section we describe the tools that enable state-of-the-art calculations, such as those summarized in Sec. 4, as well as, recently developed tools that were inspired by new theoretical structures or by the challenges of pushing the cutting edge. These tools are forged by streamlining existing calculations, testing on toy theories such as QED or supersymmetric cousins of gravity, and direct forays into new calculations in Einstein's gravity. We highlight synergies between scattering amplitudes and other particle physics tools such as EFT and advanced multiloop integration techniques, as well as, with methods in general relativity such as gravitational self-force and the worldline formalism.

**Effective Field Theory.** Compact binaries share many of the essential characteristics of bound states of elementary particles such as positronium, hydrogen, or quarkonia. Therefore, the nonrelativistic EFT techniques developed for QED and QCD [210, 252–257] are well-suited for application to compact binaries. This is underscored by the many landmark calculations that were enabled by the NRGR framework [79], see the dedicated Snowmass White Paper [90] for recent developments on this subject. These physical systems are characterized by scales that define the following modes or regions:

$$
\begin{array}{|c|c|}
\hline
\text{hard} & (m, m) \\
\hline
\text{soft} & \left(\frac{mv}{J}, \frac{mv}{J}\right) \\
\hline
\text{potential} & \left(\frac{mv^2}{J}, \frac{mv}{J}\right) \\
\hline
\text{radiation} & \left(\frac{mv^2}{J}, \frac{mv^2}{J}\right) \\
\hline
\end{array}
\tag{6.1}
$$

Here $(E, p)$ denotes the scalings of the energy and three-momentum of the modes, $m$ is the mass of the binary constituents, $v$ is the relative velocity, and $J$ is the angular

momentum. As discussed in Sec. 3, the classical limit corresponds to the limit of large angular momentum.

The new approach based on scattering amplitudes adapts these EFT tools, and closely related methods such as the method of regions [255], to efficiently extract classical contributions from the various modes described above. In particular, power-counting and factorization allows contributions from potential and radiation modes to be systematically and separately considered in scattering amplitudes, and then re-summed to all orders in the velocity $v$ using differential equations.

Similar to applications in QED and QCD [256], another basic role of EFT is to connect scattering amplitudes and classical binary dynamics through a matching calculation between gravity and a low energy EFT that describes particles interacting through an effective potential [114, 258, 259]. For the case of conservative dynamics, the basic framework has now been applied and extended for application at higher orders [114, 128, 129, 143, 144], for including spin and tidal effects [158, 166, 180, 182–185], and for gravitational theories with supersymmetry and in arbitrary dimensions [106, 248, 260]. In Ref. [143], this EFT matching procedure was used to derive the amplitude-action relation (5.3).

**Advanced Multiloop Integration.** The quest for high-precision calculations in particle physics, e.g. for colliders and for theoretical explorations of various field theories, has led to the development of advanced technology for integration of loop contributions. This can be transplanted for application to GWs, where the evaluation of integrals is a challenge common to all field-theory–based approaches, including traditional approaches in general relativity.

Integration by parts reduction as implemented in automated programs, such as AIR [261], FIRE [99, 262], Kira [263], Reduze [264], and LiteRED [265, 266], can be used to obtain a set of master integrals, which are then evaluated using the method of differential equations [100–103], perhaps in canonical form [104, 105]. The results are given in terms of multiple polylogarithms, their elliptic generalizations and perhaps more complicated functions. A judicious choice of variables is crucial [106, 267]. Integrals that appear in general-relativity–based approaches to PN calculations are also amenable to these methods, as was demonstrated in [268, 269]. These methods mesh well with the expansion around the classical limit, and extraction of potential and radiation contributions as implemented through the method of regions [255].

**Generating Functions.** The radial action $I_r$, defined as the integral of the radial momentum along the trajectory, contains the entire classical central-field dynamics. Observables such as the scattering angle, redshift, and time delay, can be derived

from the radial action via thermodynamic-type relations, e.g. $dI_r = \frac{\theta}{2\pi}dL + \tau dE + \sum_a \langle z_a \rangle dm_a$, where $\theta$ is the scattering angle, $L$ is the orbital angular momentum, $\tau$ is the time delay, and $\langle z \rangle$ is the redshift. Ref. [143] demonstrated a direct relation (5.3) between elastic four-point scattering amplitudes and the radial action for the scattering trajectory of the two particles. The radial action can thus be directly obtained from particular contributions to the four-point scattering amplitude [143, 270]. Ref. [137] argued that the radial action constructed this way is related to the WKB approximation to the scattering process, which establishes a connection with the eikonal exponentiation of amplitudes (5.2). The amplitude-action relation has also been used to explore non-perturbative results in the probe limit [223].

From a field theory point of view, the eikonal function provides another generating function for observables. Indeed, the kinematics that selects the classical part of a four-point amplitude, $s \gg t = -q^2$, is the same as the Regge kinematics, so properties of amplitudes in this regime, such as its exponentiation in terms of the eikonal function, can be used to gain insight into classical physics. The original framework has been extended to capture the exponentiation of amplitudes of spinning particles and of five-point amplitudes, under suitable assumptions regarding the coherence of the emitted gravitons [112]. While structurally similar, the radial action and the eikonal functions differ in the details of the definition of the exponential or, equivalently, in the definition of the iteration terms.

**KMOC – A Formalism for Observables from Amplitudes.** Another approach to extracting classical physics from amplitudes focuses on determining physical observables which are well defined in both the classical and quantum theories. This observables-based approach was first discussed in [111], and is sometimes referred to as the KMOC formalism. It is a quantum-first treatment of observables, which incorporates key aspects of classical physics from the beginning. By arranging a system to be under the purview of the correspondence principle, then quantum effects are negligible and it must be the case that the quantum mechanical computation of the observables will yield the classical result.

In the KMOC formalism observables are evaluated as expectation values of operators in the quantum field theory, $\langle \psi | \mathcal{O} | \psi \rangle$. Scattering amplitudes enter through time evolution which, given an initial state $|\psi\rangle$ in the far past, produces a final state $S |\psi\rangle$ in the far future, and scattering amplitudes are the matrix elements of $S$. This strategy is somewhat reminiscent to QCD event shapes [271–277], such as the energy or charge flow [278] and energy-energy correlators [271, 272].

A whole range of observables can be computed using the KMOC formalism. A particularly simple example is the impulse. The impulse is the change in the expecta-

tion value of the momentum operator $\mathbb{P}^\mu$ of the quantum field describing the massive particle and is closely related to the classical potential. There is a direct connection between the impulse and the scattering angle: once the final momentum is known, it is straightforward to determine the direction of motion relative to the initial momentum. On the other hand if the scattering angle is known the final momentum can be reconstructed, assuming no energy is radiated away.

The amount of momentum radiated is itself another simple observable, arising as the expectation value of the momentum operator $\mathbb{K}^\mu$ of whatever field is transporting the momentum. For example, in gravitational scattering, it is the momentum operator of the gravitational field. Because conservation of momentum is built into the underlying quantum field theory, the KMOC formalism naturally incorporates the effects of radiation reaction. It also has natural generalizations that include the spin of particles [163, 226, 279] and capture the dynamics of color in the context of classical (infrared-free) Yang-Mills theories [176, 177].

The gravitational waveform emitted during a scattering event follows directly from amplitudes [237]; the observable of interest is the Riemann curvature operator $\mathbb{R}_{\mu\nu\rho\sigma}(x)$ in linearized gravity. The linearized approximation is appropriate as the GWs travel over extremely large spatial distances and retains only the leading term in inverse distance. The waveform itself is an integral of an appropriate component of the expectation $\langle\psi|\,S^\dagger \mathbb{R}_{\mu\nu\rho\sigma}(x)S\,|\psi\rangle$. At lowest perturbative order, it is an integral of a five-point tree scattering amplitude. The relation with the intuitive picture in which a wave is composed of a large number of gravitons is established by describing the wave as a coherent state of the gravitational field. Interestingly, vanishing variance in the classical limit [112] implies that correlators of products of operators, $_{\mathrm{out}}\langle\psi|\mathcal{O}_1\mathcal{O}_2|\psi\rangle_{\mathrm{out}}$ factorize as $_{\mathrm{out}}\langle\psi|\mathcal{O}_1|\psi\rangle_{\mathrm{out}} {}_{\mathrm{out}}\langle\psi|\mathcal{O}_2|\psi\rangle_{\mathrm{out}}$; they nevertheless contain new classical information as demonstrated in [201].

**Amplitude Building-Blocks in the Classical Limit.** Through generalized unitarity, tree-level amplitudes are the building blocks for all higher-order calculations. The application of scattering amplitudes to GWs has motivated reorienting the structure of these building blocks to take advantage of simplifications in the classical limit. A number of approaches are based on heavy-particle effective theory [280], and the double-copy in such theories [138, 281, 282], which build on ongoing efforts to understand the kinematic algebra behind color/kinematics duality, see e.g. [283, 284]. Other approaches focus on reorganizing the soft expansion to reduce the number of master integrals by combining terms related by permutations of graviton legs [220].

Another approach uses generalized unitarity to first construct gauge-theory amplitudes and then applies the double-copy to obtain the corresponding gravitational

amplitudes [285]. Various methods are available to project out the dilaton and axion [286, 287] that naturally appear in this approach. It will be interesting to explore how to take the classical limit before the double-copy, and use gauge theory tree-level amplitudes adapted for this purpose.

Many developments focused on amplitudes for higher-spin particles. The amplitudes obtained in the classical limit avoid the strong constraints imposed by no-go theorems [151–156]. Ref. [157] constructed three-point amplitudes for arbitrary-spin massive particles through standard amplitudes methods and, by demanding good high-energy properties, identified the so-called "minimal amplitudes" which correspond to the linear response of Kerr black holes to an external gravitational field [162]. Using Lagrangian methods, Ref. [158, 172] constructed an EFT that described the classical interactions of higher-spin fields at large impact parameter, $|\boldsymbol{b}| \gg |\boldsymbol{p}|^{-1}$, which includes arbitrary spin-induced multipoles and whose three-point amplitudes exhibit a double-copy relation to higher-spin fields coupled with gluons. The heavy-particle approach in Ref. [280] was extended for higher-spin fields with double-copy properties in [160, 173, 288]. For suitable values of the parameters, the three-point amplitudes following from these Lagrangians reproduce the Kerr stress tensor [162]. A gravitational Compton amplitude for spin-5/2 massive particles that is free of spurious poles was found in [159], demonstrating that the methods employed can produce consistent higher-point amplitudes for higher-spin particles. Another construction [289, 290] makes use of the scattering equation formalism to provide diagrammatic and recursive tools for finding covariant expressions for $D$-dimensional tree-level $n$-point amplitudes with pairs of spinning massive particles.

**Synergy with Traditional Approaches.** The new approach based on the tools of theoretical high-energy physics has benefited immensely from traditional methods, in terms of both conceptual guidance from existing frameworks and practical guidance from explicit results. In turn, the emergence of new results derived from scattering amplitudes has spurred new developments within traditional methods, as well as, hybrid approaches that meld traditional methods and particle-physics tools. Such an extensive exchange between various approaches in general relativity and particle physics has led to the rapid advance of recent years, and will continue to drive breakthroughs in the future.

For instance, one of the main tools for describing binary dynamics within the PN approach to general relativity is the worldline approach [79, 89, 291–297]. Recently, inspired by amplitudes methods, this has been extended to a PM framework [132] to derive results to all orders in velocity. Calculations in this approach share many similarities with amplitudes-based methods, such as the use of integration technology

imported from particle physics. Other worldline-based approaches having aspects of scattering amplitudes have also been recently developed for calculations of tail effects in a nonrelativistic EFT for a binary [203] and of PM observables [170, 171, 298], including spin [161, 170, 299] and radiative [170, 199] effects.

As discussed in Sec. 5, new results from scattering amplitudes have exposed a number of theoretical structures. One feature that was hidden in traditional non-relativistic calculations was the simple mass dependence of scattering amplitudes, or equivalently the radial action and scattering angle. Indeed, it was noticed [300], and then rigorously proved [115], that the scattering angle exhibits a particular dependence in the symmetric mass ratio $m_1 m_2/(m_1 + m_2)^2$, leading to the so-called "good mass polynomiality" rule. The latter is the basis of the "Tutti-Frutti" method [115, 231–233], which establishes the overlap between the self-force expansion and the PM expansion. This powerful method has derived a number of new results, which have in turn provided important guidance for new amplitudes-based calculations. The "Tutti-Frutti" method has also been extended for systems with spins [234–236].

The B2B map is another instance in which patterns revealed from explicit ampli-tudes calculations [128, 129] were combined with the application of analytic continua-tion [121] to develop efficient new tools [122–124].

# 7 Future directions and challenges

In this section we summarize several broad directions that are as essential for long-term progress and fruitful synergy between particle and GW physics.

**New Perspectives on Observables.** Scattering amplitudes are naturally defined by data at asymptotic infinity and can offer insights on this class of observables in general relativity, which can be subtle due to general coordinate invariance. Of par-ticular significance are questions regarding radiative contributions, which are not only of theoretical interest, but also of practical importance for obtaining high-precision waveforms.

One open question is: how are observables in bound and unbound orbits generally related? In the absence of radiative effects, both types of dynamics can be straightfor-wardly derived from the same Hamiltonian and are related by analytic continuation. However, this is no longer true in the presence of radiative effects, which introduce correlations over arbitrary time intervals, e.g., due to tail effects [125–127]. This leads to a nonlocal-in-time Hamiltonian that depends on the particular trajectory [75] (see also [301] for a more recent perspective). This non-universality prevents a simple ana-lytic continuation [124].

It is imperative to address this question in order to fully leverage scattering amplitudes, which are naturally associated with unbound motion, for describing the dynamics of bound compact objects. A complete framework for connecting bound and unbound trajectories will not only apply to conservative dynamics but also to the radiated energy, momentum, and angular momentum, and perhaps to the waveforms themselves. Interestingly, there is evidence for double copy in GW emission in both bound and unbound systems [302–304].

Relatedly, it would be useful to carefully formulate important concepts such as conservative vs. dissipative dynamics, as well as inclusive vs. exclusive or IR safe vs. unsafe observables. For instance, recent differences in the conservative dynamics at $\mathcal{O}(G^4)$ are due to difficulties in precisely defining conservative dynamics in the presence of radiative effects. To this end, it may be fruitful to compute new observables, beyond those traditionally considered in binary dynamics, that can serve as toy models for understanding these issues. Such new observables may also be of phenomenological interest. For example, it would be interesting to detect waveforms from hyperbolic encounters. Although such events are expected to be rare and have a short-duration signal, their theoretical predictions are well-understood and can be obtained directly with scattering amplitude methods without need of analytic continuation.

Observables and unexpected structures may be further revealed by formulations of gravitational interactions without direct reference to geometry, in analogy with structure exposed by formulations of quantum field theory without quantum fields. The classical double copy [64] provides an example, as do recent attempts to define the Kerr black hole from a QFT perspective [172, 173]. While currently perturbative, such theoretical tools may help fully exploit GW observations in the quest to understand gravitating compact objects and black-hole horizons.

**Still-Higher Perturbative Orders.** The precision demands of future experiments [20–23, 305] are quite challenging since, depending on the source, GWs from binary systems will be observed with a signal-to-noise ratio that is one or two orders of magnitude higher than with current detectors [1–3]. To keep the modeling systematics below the expected statistical errors [33–37], the accuracy of current state-of-the-art waveform models [306–319], which are built from combining analytic and numerical relativity, needs to be improved by, say, two orders of magnitude. This would require up to $\mathcal{O}(G^7)$ and/or $\mathcal{O}(v^{12})$, as shown in Figure 2 for the potential. Moreover, higher-order calculations are needed for the dissipative sector, the multipolar waveforms, and other physical effects such as spins, tides, and eccentricity.

While amplitude-based methods are powerful, achieving this will still require a significant build-up of technology over several years of cycling between performing state-

of-the-art calculations and developing new tools to solve bottlenecks. For example, current pipelines can be significantly improved by streamlining the construction of classical scattering amplitudes and efficiently removing iteration terms that are already determined by lower-order calculations. In the same spirit, it will be important to exploit the nontrivial interplay between different graph topologies that lead to vast simplifications in the integrand, as demonstrated in [217] and further explored in the context of GW physics in [138, 281]. Similarly, it will be interesting to understand and utilize the consequences of the infinite number of relationships between multi-point, multi-loop amplitudes in the classical limit [112], which suggest that various terms in higher-point and/or higher loop amplitudes can be predicted from lower loop, lower point amplitudes. It also would be interesting to examine whether this phenomenon could be useful in contexts outside of GW physics, for example in collider experiments.

Apart from applications to precision GW physics, higher-order calculations also bring new perspectives on known phenomena. For example, the calculation of conservative binary dynamics at $\mathcal{O}(G^5)$ including radiative effects can give a better understanding of the role of GW memory in binary dynamics.

Higher-order calculations contain a wealth of information about the analytic structure of the two-body Hamiltonian and of observables. Up to rational functions, they are given by the nontrivial loop Feynman integrals that can appear at a given order in perturbation theory. A classification of the functionally independent integrals can directly constrain observables from a general knowledge of their analytic properties or their behavior in various limits. Moreover, these functions can be matched to NR simulations, which, for the time being, provide the most accurate description of the final moments of a binary inspiral. This fitting procedure may offer further analytic insight into this period of the binary's evolution and, conversely, on analytic approaches to nonperturbative properties of gravitational interactions.

**Integration Challenges.** Evaluation of higher-loop Feynman-type integrals is a challenge shared by QFT applications to both particle physics and to gravitational and GW physics.

The integration-by-parts (IBP) reduction [97–99, 263–266, 320] used to reduce the expanded Feynman integrals to basis of integrals, referred to as master integrals, becomes computationally demanding as the number of loops increase. Cutting edge developments that increase the efficiency of IBP reduction [321–329], perhaps with further improvements related to the specific structure of the integrals that appear in the classical limit, will be vital for field theory-based approaches to reach high perturbative orders in the post-Minkowskian and the post-Newtonian expansion demanded by future observations. Further improvements may come from the development of an algorithmic

choice of master integral basis tailored for the two-body general-relativity problem.

The master integrals relevant for the two-body problem depend only on the relative velocity of the two bodies and can be evaluated through a system of linear first-order ODEs [100–103] in this velocity. Expansion in the static limit, $v \to 0$, yields the integrals that appear in the post-Newtonian expansion and the leading terms serve as boundary conditions for the ODE system. The differential equations themselves simplify when a "canonical basis" of master integrals is chosen [104, 105]. New algorithms for finding such canonical bases, improving upon those in [330–339], and the study of the special functions that solve the resulting differential equations [340–345] will be important for further progress, see also the dedicated Snowmass White Paper [346]. For specific practical applications it will also be profitable to explore other methods for evaluating master integrals, such as Mellin-Barnes representations [347], direct Feynman parameter integration [343, 348], and difference equations from dimensional recurrence [349–351]. These methods apply equally well to conservative calculations and to the calculation of observables involving outgoing gravitational radiation, such as the total energy radiated during a scattering event. The collider method of reverse unitarity [352–355] allows systematic evaluation of the phase space integrals appearing in the KMOC formalism with techniques similar to those used for loop integrals, as demonstrated in [139, 140].

**Higher-Order Structure, Resummation, and Beyond Perturbation Theory.** Exact results in interacting theories, even in the classical regime, are rare. Patterns and structure exposed by explicit higher perturbative orders may hold the key to understanding the structure of perturbation theory and eventually lead to its resummation, as illustrated e.g. by the BDS/ABDK conjectures [356, 357] for the resummation of the planar MHV amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. This is an important theoretical justification, which complements the practical one emphasized earlier, for further in-depth higher-order exploration. The complete velocity dependence gives us access to the analytic structure of the Hamiltonian – and of the observables derived from it – and thus can inform on the structure at even higher orders, possibly providing sufficient information to resum the entire perturbation theory.

A proof of principle is the conjectured expression relating scattering observables of spinning bodies to the eikonal of the corresponding scattering amplitude [158]. While expected from the motion of spinless bodies, the existence a single function – the eikonal, or the radial action – that potentially captures all conservative classical observables of such processes is surprising. Tests of such conjectures and the identification of novel, bolder ones may exploit special configurations of particles, new perturbative expansions which access different regions of parameter space, etc.

When present, symmetries are powerful means to extrapolate fixed-order properties of observables. The double copy hints at a relationship between the asymptotic symmetry group of Yang-Mills theory and of general relativity. Since the expectation value of the field itself can be computed (e.g., using scattering amplitudes and the KMOC formalism), it may be possible to test this idea directly. Quantum mechanical deformations of the symmetry group, relating for example to infrared divergences, should be accessible by studying appropriate observables and may reveal all-order structures.

Another path to exact results proceeds through enhancements of perturbative calculations. For instance, the EOB approach [10, 11] provides a framework that can incorporate not only perturbative results from PN, PM and gravitational self-force but also exact results available in the probe limit. In general, it is critical that information from distinct regions of parameter space and from different physical phenomena be included in the same formalism. The amount of physics contained in all-order results is remarkable and includes, among others, possible analytic access to horizon formation in the binary coalescence.

Yet another systematic strategy is the gravitational self-force approach [15, 77, 78], in which one expands in the mass ratio of the binary. It makes contact with QFT in curved space, which we will return to shortly. It is a challenge for the future to develop the tools necessary to carry out such calculations with QFT methods.

Apart from having been instrumental for driving our understanding of gravity, iconic classical solutions of general relativity such as the Schwarzschild and Kerr solutions also are at the foundation of the EOB theory and of the self-force approach. Scattering amplitudes and the KMOC formalism applied to three-point amplitudes [358] can access these via analytic continuation either to complex momenta or to space-times with an exotic signature. This led, for example, to a new understanding of the Newman-Janis shift [226] which relates the Kerr solution in a very simple way to the Schwarzschild solution. Extending this observation beyond leading order should establish a connection between scattering amplitudes and higher-order terms in the self-force expansion to all orders in Newton's constant, beyond the "good mass polynomiality" already manifest in perturbative scattering amplitudes and associated observables. It could also be interesting to examine this connection in higher dimensions where the space of known classical solutions if far richer than in four dimensions.

**Many-Body Dynamics from Amplitudes.** While there is a large effort directed towards understanding compact binaries, systems of three or more massive bodies also occur in our Universe and may source GWs detectable in the future. These systems may probe scenarios of sequential and hierarchical mergers [359–362], and have qualitatively different dynamics [363], including chaotic dynamics with interesting effects

at relatively high velocities [364]. The current state-of-the-art Hamiltonian is the 2PN order [365–367] obtained through general-relativity methods. A formal 2PM expression was obtained in [368] using the worldline formalism and can be used to generate higher-order PN terms of the form $G^2 v^{2n}$.

Scattering amplitude methods used for two-body dynamics can also be applied for $N$-body dynamics, and exploring this may uncover novel features of gravitational interactions. This will require extending tools for constructing $2N$-point matter amplitudes, identifying the classical limit, evaluation of integrals, etc. Further assumptions about the hierarchy of scales in $N$-body dynamics establish connections with multi-Regge kinematics in particle physics.

**Ringdown from Amplitudes.** The focus of the recent effort to understand and explore binary coalescence with particle physics methods has been the inspiral phase of the process, which can be treated perturbatively around flat space. The GW signal contains information about the masses and spins of the constituents, the shape of the orbit (eccentricity and mean anomaly), and in late stages also about their tidal deformability. For binary black holes, the final phase of a merger, known as the ringdown, appears as a superposition of quasi-normal modes of the merged remnant [8, 369]. The frequency and decay time of each mode are determined in general relativity by the remnant's mass and spin, while their relative amplitude contains information about the properties of the binary's components and geometry, including the spin orientations (see, e.g., [370–372]).

This poses an interesting challenge and an equally interesting opportunity: to develop field theory methods to describe the emission of gravitons from the remnant of a binary merger. A naive expectation is that the relevant framework is QFT in a time-dependent curved space which is perhaps a classical double copy and that the relevant observables are final-state correlation functions akin to cosmological correlators. While still in its infancy, see e.g. [373–383] and the Snowmass White Paper [384], the generalization of the amplitudes program to curved space together with the gauge theory realization of classical solutions of general relativity [64, 385–390] may offer a new perspective on the ringdown phase, as well as, new means to explore analytically the correlation between initial parameters and intrinsic properties of the remnant, including its tidal deformability, etc. Clearly, a systematic development of these methods will have wider applications, establishing connections to other areas of physics, such as gauge/string duality. Curved space methods will also make contact with the self-force approach to the two-body problem [15, 77, 78], which can be interpreted as the back-reacted propagation of a light massive particle in the curved space generated by a heavy massive particle, thus providing another path to analytic resummation of the

two-body Hamiltonian or two-body observables. See [391] for a related application of the KMOC formalism.

**Probes of General Relativity, Quantum Gravity and New Physics.** Quantum gravity remains one of the biggest mysteries of fundamental physics of this century. While indirect evidence for it abound, its precise formulation is elusive, and there is currently no observed deviations from predictions of classical general relativity in GW observations [392]. Future more precise observations with also a much larger number of events will provide a means for more efficient probes of physics beyond general relativity [393], or even of quantum gravity and horizon-scale physics (e.g., [394–397]); see also the Snowmass White Paper [27] on fundamental physics and beyond the standard model. More accurate waveforms, covering long-time evolution, as well as various possibilities for new physics, is a step towards this goal, as are the identification of new observables tailored for such physics. A complementary approach is the development of model-independent studies, covering both classical extensions of general relativity and its quantization. From an EFT perspective there is of course a certain overlap between them, so identifying methods to lift this degeneracy will be important.

Further developments in amplitude methods, e.g. extending the work in [398–401] to systematically include and classify all possible higher-dimension/higher-derivative operators that can appear as counterterms in gravitational theories, including those discussed using standard GR methods in e.g. [402–404], will aid both model-specific and model-independent studies of GW signals, both with regard to the inspiral and the ringdown phase, see also Snowmass White Paper [405] on UV constraints and IR physics.


# 8    Coda

In its short history, the program to apply particle physics methods to gravitational physics in general and gravitational-wave physics in particular has achieved remarkable success by uncovering rich theoretical structures, developing powerful new tools, and producing predictions for future precision gravitational-wave detectors. Close synergy with the EFT and gravitational-wave communities has been essential, and provided crucial guidance regarding the theoretical needs for waveform modeling and the necessary tools to achieve them.

This white paper provides a snapshot of the current status of this vibrant field and of its progress to date. There are many open questions and avenues to explore, ranging from the very formal to the very practical. Some can be addressed with existing methods whose full potential is not yet realized, while others require novel ideas that

may even open up completely unexpected directions that will further our knowledge of gravitational interactions, binary dynamics, black-hole physics and perhaps even quantum field theory.

## Acknowledgements

## References

[1] LIGO Scientific collaboration, *Advanced LIGO*, *Class. Quant. Grav.* **32** (2015) 074001 [1411.4547].

[2] VIRGO collaboration, *Advanced Virgo: a second-generation interferometric gravitational wave detector*, *Class. Quant. Grav.* **32** (2015) 024001 [1408.3978].

[3] KAGRA collaboration, *Overview of KAGRA: Calibration, detector characterization, physical environmental monitors, and the geophysics interferometer*, *PTEP* **2021** (2021) 05A102 [2009.09305].

[4] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent and J. Vines, *Energetics of two-body Hamiltonians in post-Minkowskian gravity*, *Phys. Rev. D* **99** (2019) 104004 [1901.07102].

[5] M. Khalil, A. Buonanno, J. Steinhoff and J. Vines, *In preparation*, .

[6] T. Regge and J. A. Wheeler, *Stability of a Schwarzschild singularity*, *Phys. Rev.* **108** (1957) 1063.

[7] F. J. Zerilli, *Effective potential for even parity Regge-Wheeler gravitational perturbation equations*, *Phys. Rev. Lett.* **24** (1970) 737.

[8] C. V. Vishveshwara, *Stability of the schwarzschild metric*, *Phys. Rev. D* **1** (1970) 2870.

[9] S. A. Teukolsky, *Perturbations of a rotating black hole. 1. Fundamental equations for gravitational electromagnetic and neutrino field perturbations*, *Astrophys. J.* **185** (1973) 635.

[10] A. Buonanno and T. Damour, *Effective one-body approach to general relativistic two-body dynamics*, *Phys. Rev. D* **59** (1999) 084006 [gr-qc/9811091].

[11] A. Buonanno and T. Damour, *Transition from inspiral to plunge in binary black hole coalescences*, *Phys. Rev. D* **62** (2000) 064015 [`gr-qc/0001013`].

[12] L. Blanchet, *Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries*, *Living Rev. Rel.* **17** (2014) 2 [`1310.1528`].

[13] R. A. Porto, *The effective field theorist's approach to gravitational dynamics*, *Phys. Rept.* **633** (2016) 1 [`1601.04914`].

[14] G. Schäfer and P. Jaranowski, *Hamiltonian formulation of general relativity and post-Newtonian dynamics of compact binaries*, *Living Rev. Rel.* **21** (2018) 7 [`1805.07240`].

[15] L. Barack and A. Pound, *Self-force and radiation reaction in general relativity*, *Rept. Prog. Phys.* **82** (2019) 016904 [`1805.10385`].

[16] F. Pretorius, *Evolution of binary black hole spacetimes*, *Phys. Rev. Lett.* **95** (2005) 121101 [`gr-qc/0507014`].

[17] M. Campanelli, C. O. Lousto, P. Marronetti and Y. Zlochower, *Accurate evolutions of orbiting black-hole binaries without excision*, *Phys. Rev. Lett.* **96** (2006) 111101 [`gr-qc/0511048`].

[18] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz and J. van Meter, *Gravitational wave extraction from an inspiraling configuration of merging black holes*, *Phys. Rev. Lett.* **96** (2006) 111102 [`gr-qc/0511103`].

[19] F. Foucart, P. Laguna, G. Lovelace, D. Radice and H. Witek, *Snowmass2021 Cosmic Frontier White Paper: Numerical relativity for next-generation gravitational-wave probes of fundamental physics*, `2203.08139`.

[20] LISA collaboration, *Laser Interferometer Space Antenna*, `1702.00786`.

[21] M. Saleem et al., *The science case for LIGO-India*, *Class. Quant. Grav.* **39** (2022) 025004 [`2105.01716`].

[22] D. Reitze et al., *Cosmic Explorer: The U.S. Contribution to Gravitational-Wave Astronomy beyond LIGO*, *Bull. Am. Astron. Soc.* **51** (2019) 035 [`1907.04833`].

[23] M. Punturo et al., *The Einstein Telescope: A third-generation gravitational wave observatory*, *Class. Quant. Grav.* **27** (2010) 194002.

[24] B. S. Sathyaprakash et al., *Extreme Gravity and Fundamental Physics*, `1903.09221`.

[25] M. Maggiore et al., *Science Case for the Einstein Telescope*, *JCAP* **03** (2020) 050 [`1912.02622`].

[26] V. Kalogera et al., *The Next Generation Global Gravitational Wave Observatory: The Science Book*, `2111.06990`.

[27] E. Berti, V. Cardoso, Z. Haiman, D. E. Holz, E. Mottola, S. Mukherjee et al.,
*Snowmass2021 Cosmic Frontier White Paper: Fundamental Physics and Beyond the
Standard Model*, in *2022 Snowmass Summer Study*, 3, 2022, 2203.06240.

[28] T. Damour, F. Guercilena, I. Hinder, S. Hopper, A. Nagar and L. Rezzolla,
*Strong-Field Scattering of Two Black Holes: Numerics Versus Analytics*, *Phys. Rev.
D* **89** (2014) 081503 [1402.7307].

[29] S. Ossokine, T. Dietrich, E. Foley, R. Katebi and G. Lovelace, *Assessing the
Energetics of Spinning Binary Black Hole Systems*, *Phys. Rev. D* **98** (2018) 104057
[1712.06533].

[30] LIGO Scientific, Virgo collaboration, *Observation of Gravitational Waves from a
Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102 [1602.03837].

[31] LIGO Scientific, Virgo collaboration, *GW170817: Observation of Gravitational
Waves from a Binary Neutron Star Inspiral*, *Phys. Rev. Lett.* **119** (2017) 161101
[1710.05832].

[32] LIGO Scientific, KAGRA, VIRGO collaboration, *Observation of Gravitational
Waves from Two Neutron Star–Black Hole Coalescences*, *Astrophys. J. Lett.* **915**
(2021) L5 [2106.15163].

[33] M. Favata, *Systematic parameter errors in inspiraling neutron star binaries*, *Phys.
Rev. Lett.* **112** (2014) 101101 [1310.8288].

[34] A. Samajdar and T. Dietrich, *Waveform systematics for binary neutron star
gravitational wave signals: effects of the point-particle baseline and tidal descriptions*,
*Phys. Rev. D* **98** (2018) 124030 [1810.03936].

[35] M. Pürrer and C.-J. Haster, *Gravitational waveform accuracy requirements for future
ground-based detectors*, *Phys. Rev. Res.* **2** (2020) 023151 [1912.10055].

[36] Y. Huang, C.-J. Haster, S. Vitale, V. Varma, F. Foucart and S. Biscoveanu, *Statistical
and systematic uncertainties in extracting the source properties of neutron star - black
hole binaries with gravitational waves*, *Phys. Rev. D* **103** (2021) 083001 [2005.11850].

[37] R. Gamba, M. Breschi, S. Bernuzzi, M. Agathos and A. Nagar, *Waveform systematics
in the gravitational-wave inference of tidal parameters and equation of state from
binary neutron star signals*, *Phys. Rev. D* **103** (2021) 124015 [2009.08467].

[38] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering*, *Phys. Rev. Lett.*
**56** (1986) 2459.

[39] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun.
Math. Phys.* **252** (2004) 189 [hep-th/0312171].

[40] R. Roiban, M. Spradlin and A. Volovich, *On the tree level S matrix of Yang-Mills
theory*, *Phys. Rev. D* **70** (2004) 026009 [hep-th/0403190].

[41] S. Gukov, L. Motl and A. Neitzke, *Equivalence of twistor prescriptions for superYang-Mills*, *Adv. Theor. Math. Phys.* **11** (2007) 199 [`hep-th/0404085`].

[42] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [`hep-th/0403047`].

[43] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [`hep-th/0501052`].

[44] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys. B* **425** (1994) 217 [`hep-ph/9403226`].

[45] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B435** (1995) 59 [`hep-ph/9409265`].

[46] Z. Bern, L. J. Dixon and D. A. Kosower, *One-loop amplitudes for $e^+e^-$ to four partons*, *Nucl. Phys.* **B513** (1998) 3 [`hep-ph/9708239`].

[47] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $\mathcal{N} = 4$ super-Yang-Mills*, *Nucl. Phys.* **B725** (2005) 275 [`hep-th/0412103`].

[48] Z. Bern, L. J. Dixon and D. A. Kosower, *Progress in one loop QCD computations*, *Ann. Rev. Nucl. Part. Sci.* **46** (1996) 109 [`hep-ph/9602280`].

[49] H. Elvang and Y.-t. Huang, *Scattering amplitudes*, `1308.1697`.

[50] H. Elvang and Y.-t. Huang, *Scattering Amplitudes in Gauge Theory and Gravity*. Cambridge University Press, 4, 2015.

[51] J. J. M. Carrasco and H. Johansson, *Generic multiloop methods and application to $\mathcal{N} = 4$ super-Yang-Mills*, *J. Phys.* **A44** (2011) 454004 [`1103.3298`].

[52] Z. Bern and Y.-t. Huang, *Basics of generalized unitarity*, *J. Phys.* **A44** (2011) 454003 [`1103.1869`].

[53] Z. Bern, J. J. M. Carrasco and H. Johansson, *New relations for gauge-theory amplitudes*, *Phys. Rev.* **D78** (2008) 085011 [`0805.3993`].

[54] Z. Bern, J. J. M. Carrasco and H. Johansson, *Perturbative quantum gravity as a double copy of gauge theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [`1004.0476`].

[55] H. Kawai, D. C. Lewellen and S. H. H. Tye, *A relation between tree amplitudes of closed and open strings*, *Nucl. Phys.* **B269** (1986) 1.

[56] Z. Bern, D. C. Dunbar and T. Shimada, *String based methods in perturbative gravity*, *Phys. Lett. B* **312** (1993) 277 [`hep-th/9307001`].

[57] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, *Manifest ultraviolet behavior for the three-loop four-point amplitude of $\mathcal{N} = 8$ supergravity*, *Phys. Rev.* **D78** (2008) 105019 [`0808.4112`].

[58] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, *The Ultraviolet Behavior of N=8 Supergravity at Four Loops*, *Phys. Rev. Lett.* **103** (2009) 081301 [`0905.2326`].

[59] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, *Three-loop superfiniteness of $\mathcal{N} = 8$ supergravity*, *Phys. Rev. Lett.* **98** (2007) 161303 [`hep-th/0702112`].

[60] Z. Bern, S. Davies, T. Dennen and Y.-t. Huang, *Absence of three-loop four-point divergences in $\mathcal{N} = 4$ supergravity*, *Phys. Rev. Lett.* **108** (2012) 201301 [`1202.3423`].

[61] Z. Bern, S. Davies, T. Dennen, A. V. Smirnov and V. A. Smirnov, *Ultraviolet properties of $\mathcal{N} = 4$ supergravity at four loops*, *Phys. Rev. Lett.* **111** (2013) 231302 [`1309.2498`].

[62] Z. Bern, S. Davies and T. Dennen, *Enhanced ultraviolet cancellations in $\mathcal{N} = 5$ supergravity at four loops*, *Phys. Rev.* **D90** (2014) 105011 [`1409.3089`].

[63] Z. Bern, J. J. Carrasco, W.-M. Chen, A. Edison, H. Johansson, J. Parra-Martinez et al., *Ultraviolet properties of $\mathcal{N} = 8$ supergravity at five loops*, *Phys. Rev.* **D98** (2018) 086021 [`1804.09311`].

[64] R. Monteiro, D. O'Connell and C. D. White, *Black holes and the double copy*, *JHEP* **12** (2014) 056 [`1410.0239`].

[65] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, `1909.01358`.

[66] e. John Joseph Carrasco, *Snowmass 2021 Whitepaper: the Double Copy and its Applications*, `2203.ikjlm`.

[67] T. Damour, *High-energy gravitational scattering and the general relativistic two-body problem*, *Phys. Rev. D* **97** (2018) 044038 [`1710.10599`].

[68] A. Einstein, L. Infeld and B. Hoffmann, *The Gravitational equations and the problem of motion*, *Annals Math.* **39** (1938) 65.

[69] A. Einstein and L. Infeld, *The Gravitational equations and the problem of motion. 2.*, *Annals Math.* **41** (1940) 455.

[70] T. Ohta, H. Okamura, T. Kimura and K. Hiida, *Physically acceptable solution of einstein's equation for many-body system*, *Prog. Theor. Phys.* **50** (1973) 492.

[71] P. Jaranowski and G. Schaefer, *Third postNewtonian higher order ADM Hamilton dynamics for two-body point mass systems*, *Phys. Rev. D* **57** (1998) 7274 [`gr-qc/9712075`].

[72] T. Damour, P. Jaranowski and G. Schaefer, *Dynamical invariants for general*

relativistic two-body systems at the third postNewtonian approximation, *Phys. Rev. D* **62** (2000) 044024 [`gr-qc/9912092`].

[73] L. Blanchet and G. Faye, *Equations of motion of point particle binaries at the third postNewtonian order*, *Phys. Lett. A* **271** (2000) 58 [`gr-qc/0004009`].

[74] T. Damour, P. Jaranowski and G. Schaefer, *Dimensional regularization of the gravitational interaction of point masses*, *Phys. Lett. B* **513** (2001) 147 [`gr-qc/0105038`].

[75] T. Damour, P. Jaranowski and G. Schäfer, *Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems*, *Phys. Rev. D* **89** (2014) 064058 [`1401.4548`].

[76] P. Jaranowski and G. Schäfer, *Derivation of local-in-time fourth post-Newtonian ADM Hamiltonian for spinless compact binaries*, *Phys. Rev. D* **92** (2015) 124043 [`1508.01016`].

[77] Y. Mino, M. Sasaki and T. Tanaka, *Gravitational radiation reaction to a particle motion*, *Phys. Rev. D* **55** (1997) 3457 [`gr-qc/9606018`].

[78] T. C. Quinn and R. M. Wald, *An Axiomatic approach to electromagnetic and gravitational radiation reaction of particles in curved space-time*, *Phys. Rev. D* **56** (1997) 3381 [`gr-qc/9610053`].

[79] W. D. Goldberger and I. Z. Rothstein, *An Effective field theory of gravity for extended objects*, *Phys. Rev. D* **73** (2006) 104029 [`hep-th/0409156`].

[80] B. Bertotti, *On gravitational motion*, *Nuovo Cim.* **4** (1956) 898.

[81] R. P. Kerr, *The Lorentz-covariant approximation method in general relativity I*, *Nuovo Cim.* **13** (1959) 469.

[82] B. Bertotti and J. Plebanski, *Theory of gravitational perturbations in the fast motion approximation*, *Annals Phys.* **11** (1960) 169.

[83] M. Portilla, *MOMENTUM AND ANGULAR MOMENTUM OF TWO GRAVITATING PARTICLES*, *J. Phys. A* **12** (1979) 1075.

[84] K. Westpfahl and M. Goller, *GRAVITATIONAL SCATTERING OF TWO RELATIVISTIC PARTICLES IN POSTLINEAR APPROXIMATION*, *Lett. Nuovo Cim.* **26** (1979) 573.

[85] M. Portilla, *SCATTERING OF TWO GRAVITATING PARTICLES: CLASSICAL APPROACH*, *J. Phys. A* **13** (1980) 3677.

[86] L. Bel, T. Damour, N. Deruelle, J. Ibanez and J. Martin, *Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity*, *Gen. Rel. Grav.* **13** (1981) 963.

[87] K. Westpfahl, *High-Speed Scattering of Charged and Uncharged Particles in General Relativity*, *Fortsch. Phys.* **33** (1985) 417.

[88] T. Damour, *Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory*, *Phys. Rev. D* **94** (2016) 104015 [1609.00354].

[89] M. Levi, *Effective Field Theories of Post-Newtonian Gravity: A comprehensive review*, *Rept. Prog. Phys.* **83** (2020) 075901 [1807.01699].

[90] e. Walter Goldberger, *Snowmass 2021 Whitepaper: EFT of Gravity and NRGR*, 2203.ikjlm.

[91] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog and B. Mistlberger, *Higgs Boson Gluon-Fusion Production in QCD at Three Loops*, *Phys. Rev. Lett.* **114** (2015) 212001 [1503.06056].

[92] A. Gehrmann-De Ridder, T. Gehrmann, E. W. N. Glover and G. Heinrich, *NNLO corrections to event shapes in e+ e- annihilation*, *JHEP* **12** (2007) 094 [0711.4711].

[93] J. M. Henn, G. P. Korchemsky and B. Mistlberger, *The full four-loop cusp anomalous dimension in $\mathcal{N} = 4$ super Yang-Mills and QCD*, *JHEP* **04** (2020) 018 [1911.10174].

[94] S. Laporta, *High-precision calculation of the 4-loop contribution to the electron g-2 in QED*, *Phys. Lett. B* **772** (2017) 232 [1704.06996].

[95] Z. Bern, J. J. Carrasco, W.-M. Chen, A. Edison, H. Johansson, J. Parra-Martinez et al., *Ultraviolet Properties of $\mathcal{N} = 8$ Supergravity at Five Loops*, *Phys. Rev. D* **98** (2018) 086021 [1804.09311].

[96] J. J. M. Carrasco, A. Edison and H. Johansson, *Maximal Super-Yang-Mills at Six Loops via Novel Integrand Bootstrap*, 2112.05178.

[97] K. G. Chetyrkin and F. V. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops*, *Nucl. Phys. B* **192** (1981) 159.

[98] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys. A* **15** (2000) 5087 [hep-ph/0102033].

[99] A. V. Smirnov, *Algorithm FIRE – Feynman Integral REduction*, *JHEP* **10** (2008) 107 [0807.3243].

[100] A. V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett.* **B254** (1991) 158.

[101] Z. Bern, L. J. Dixon and D. A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B412** (1994) 751 [hep-ph/9306240].

[102] E. Remiddi, *Differential equations for Feynman graph amplitudes*, *Nuovo Cim.* **A110** (1997) 1435 [hep-th/9711188].

[103] T. Gehrmann and E. Remiddi, *Differential equations for two loop four point functions*, *Nucl. Phys.* **B580** (2000) 485 [`hep-ph/9912329`].

[104] J. M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601 [`1304.1806`].

[105] J. M. Henn, A. V. Smirnov and V. A. Smirnov, *Evaluating single-scale and/or non-planar diagrams by differential equations*, *JHEP* **03** (2014) 088 [`1312.2588`].

[106] J. Parra-Martinez, M. S. Ruf and M. Zeng, *Extremal black hole scattering at $\mathcal{O}(G^3)$: graviton dominance, eikonal exponentiation, and differential equations*, *JHEP* **11** (2020) 023 [`2005.04236`].

[107] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *A Semiclassical limit of the gauge / string correspondence*, *Nucl. Phys. B* **636** (2002) 99 [`hep-th/0204051`].

[108] M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, *Large spin limit of AdS(5) x S**5 string theory and low-energy expansion of ferromagnetic spin chains*, *Nucl. Phys. B* **692** (2004) 3 [`hep-th/0403120`].

[109] S. Giombi, R. Ricci, R. Roiban, A. A. Tseytlin and C. Vergu, *Quantum AdS(5) x S5 superstring in the AdS light-cone gauge*, *JHEP* **03** (2010) 003 [`0912.5105`].

[110] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3 [`1012.3982`].

[111] D. A. Kosower, B. Maybee and D. O'Connell, *Amplitudes, observables, and classical scattering*, *JHEP* **02** (2019) 137 [`1811.10950`].

[112] A. Cristofoli, R. Gonzo, N. Moynihan, D. O'Connell, A. Ross, M. Sergola et al., *The Uncertainty Principle and Classical Amplitudes*, `2112.07556`.

[113] R. Britto, R. Gonzo and G. R. Jehu, *Graviton particle statistics and coherent states from classical scattering amplitudes*, `2112.07036`.

[114] C. Cheung, I. Z. Rothstein and M. P. Solon, *From scattering amplitudes to classical potentials in the post-Minkowskian expansion*, *Phys. Rev. Lett.* **121** (2018) 251101 [`1808.02489`].

[115] T. Damour, *Classical and quantum scattering in post-Minkowskian gravity*, *Phys. Rev. D* **102** (2020) 024060 [`1912.02139`].

[116] P. H. Damgaard and P. Vanhove, *Remodeling the effective one-body formalism in post-Minkowskian gravity*, *Phys. Rev. D* **104** (2021) 104029 [`2108.11248`].

[117] D. Amati, M. Ciafaloni and G. Veneziano, *Superstring Collisions at Planckian Energies*, *Phys. Lett. B* **197** (1987) 81.

[118] G. 't Hooft, *Graviton Dominance in Ultrahigh-Energy Scattering*, *Phys. Lett. B* **198** (1987) 61.

[119] I. J. Muzinich and M. Soldate, *High-Energy Unitarity of Gravitation and Strings*, *Phys. Rev. D* **37** (1988) 359.

[120] D. Amati, M. Ciafaloni and G. Veneziano, *Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions*, *Int. J. Mod. Phys. A* **3** (1988) 1615.

[121] G. Cho, A. Gopakumar, M. Haney and H. M. Lee, *Gravitational waves from compact binaries in post-Newtonian accurate hyperbolic orbits*, *Phys. Rev. D* **98** (2018) 024039 [1807.02380].

[122] G. Kälin and R. A. Porto, *From Boundary Data to Bound States*, *JHEP* **01** (2020) 072 [1910.03008].

[123] G. Kälin and R. A. Porto, *From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist)*, *JHEP* **02** (2020) 120 [1911.09130].

[124] G. Cho, G. Kälin and R. A. Porto, *From Boundary Data to Bound States III: Radiative Effects*, 2112.03976.

[125] K. S. Thorne, *Multipole Expansions of Gravitational Radiation*, *Rev. Mod. Phys.* **52** (1980) 299.

[126] L. Blanchet and T. Damour, *Tail Transported Temporal Correlations in the Dynamics of a Gravitating System*, *Phys. Rev. D* **37** (1988) 1410.

[127] L. Blanchet and G. Schaefer, *Gravitational wave tails and binary star systems*, *Class. Quant. Grav.* **10** (1993) 2699.

[128] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, *Scattering amplitudes and the conservative Hamiltonian for binary systems at third post-Minkowskian order*, *Phys. Rev. Lett.* **122** (2019) 201603 [1901.04424].

[129] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, *Black hole binary dynamics from the double copy and effective theory*, 1908.01493.

[130] J. Blümlein, A. Maier, P. Marquard and G. Schäfer, *Testing binary dynamics in gravity at the sixth post-Newtonian level*, *Phys. Lett. B* **807** (2020) 135496 [2003.07145].

[131] C. Cheung and M. P. Solon, *Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams*, *JHEP* **06** (2020) 144 [2003.08351].

[132] G. Kälin and R. A. Porto, *Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics*, *JHEP* **11** (2020) 106 [2006.01184].

[133] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, *Universality of ultra-relativistic gravitational scattering*, *Phys. Lett. B* **811** (2020) 135924 [2008.12743].

[134] T. Damour, *Radiative contribution to classical gravitational scattering at the third order in G*, *Phys. Rev. D* **102** (2020) 124008 [`2010.01641`].

[135] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, *The eikonal approach to gravitational scattering and radiation at $\mathcal{O}(G^3)$*, *JHEP* **07** (2021) 169 [`2104.03256`].

[136] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté and P. Vanhove, *The amplitude for classical gravitational scattering at third Post-Minkowskian order*, *JHEP* **08** (2021) 172 [`2105.05218`].

[137] P. H. Damgaard, L. Plante and P. Vanhove, *On an exponential representation of the gravitational S-matrix*, *JHEP* **11** (2021) 213 [`2107.12891`].

[138] A. Brandhuber, G. Chen, G. Travaglini and C. Wen, *Classical gravitational scattering from a gauge-invariant double copy*, *JHEP* **10** (2021) 118 [`2108.04216`].

[139] E. Herrmann, J. Parra-Martinez, M. S. Ruf and M. Zeng, *Gravitational Bremsstrahlung from Reverse Unitarity*, *Phys. Rev. Lett.* **126** (2021) 201602 [`2101.07255`].

[140] E. Herrmann, J. Parra-Martinez, M. S. Ruf and M. Zeng, *Radiative classical gravitational observables at $\mathcal{O}(G^3)$ from scattering amplitudes*, *JHEP* **10** (2021) 148 [`2104.03957`].

[141] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, *Radiation Reaction from Soft Theorems*, *Phys. Lett. B* **818** (2021) 136379 [`2101.05772`].

[142] C. Heissenberg, *Infrared divergences and the eikonal exponentiation*, *Phys. Rev. D* **104** (2021) 046016 [`2105.04594`].

[143] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon et al., *Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$*, *Phys. Rev. Lett.* **126** (2021) 171601 [`2101.07254`].

[144] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon et al., *Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at $O(G^4)$*, `2112.10750`.

[145] J. A. Wheeler and R. P. Feynman, *Classical electrodynamics in terms of direct interparticle action*, *Rev. Mod. Phys.* **21** (1949) 425.

[146] T. Damour and G. Esposito-Farese, *Testing gravity to second postNewtonian order: A Field theory approach*, *Phys. Rev. D* **53** (1996) 5541 [`gr-qc/9506063`].

[147] D. Bini, T. Damour and A. Geralico, *Radiative contributions to gravitational scattering*, *Phys. Rev. D* **104** (2021) 084031 [`2107.08896`].

[148] J. Blümlein, A. Maier, P. Marquard and G. Schäfer, *The fifth-order post-Newtonian*

*Hamiltonian dynamics of two-body systems from an effective field theory approach*, 2110.13822.

[149] C. Dlapa, G. Kälin, Z. Liu and R. A. Porto, *Dynamics of Binary Systems to Fourth Post-Minkowskian Order from the Effective Field Theory Approach*, 2106.08276.

[150] C. Dlapa, G. Kälin, Z. Liu and R. A. Porto, *Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-eccentricity Expansion*, 2112.11296.

[151] A. Shamaly and A. Z. Capri, *Propagation of interacting fields*, *Annals Phys.* **74** (1972) 503.

[152] M. Hortacsu, *Demonstration of noncausality for the rarita-schwinger equation*, *Phys. Rev. D* **9** (1974) 928.

[153] S. Deser, V. Pascalutsa and A. Waldron, *Massive spin 3/2 electrodynamics*, *Phys. Rev. D* **62** (2000) 105031 [hep-th/0003011].

[154] M. Porrati and R. Rahman, *Intrinsic Cutoff and Acausality for Massive Spin 2 Fields Coupled to Electromagnetism*, *Nucl. Phys. B* **801** (2008) 174 [0801.2581].

[155] X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling*, *JHEP* **02** (2016) 020 [1407.5597].

[156] N. Afkhami-Jeddi, S. Kundu and A. Tajdini, *A Bound on Massive Higher Spin Particles*, *JHEP* **04** (2019) 056 [1811.01952].

[157] N. Arkani-Hamed, T.-C. Huang and Y.-t. Huang, *Scattering amplitudes for all masses and spins*, *JHEP* **11** (2021) 070 [1709.04891].

[158] Z. Bern, A. Luna, R. Roiban, C.-H. Shen and M. Zeng, *Spinning black hole binary dynamics, scattering amplitudes, and effective field theory*, *Phys. Rev. D* **104** (2021) 065014 [2005.03071].

[159] M. Chiodaroli, H. Johansson and P. Pichini, *Compton Black-Hole Scattering for $s \leq 5/2$*, 2107.14779.

[160] R. Aoude, K. Haddad and A. Helset, *On-shell heavy particle effective theories*, *JHEP* **05** (2020) 051 [2001.09164].

[161] G. U. Jakobsen, G. Mogull, J. Plefka and J. Steinhoff, *SUSY in the sky with gravitons*, *JHEP* **01** (2022) 027 [2109.04465].

[162] J. Vines, *Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings*, *Class. Quant. Grav.* **35** (2018) 084002 [1709.06016].

[163] A. Guevara, A. Ochirov and J. Vines, *Black-hole scattering with general spin directions from minimal-coupling amplitudes*, *Phys. Rev. D* **100** (2019) 104024 [`1906.10071`].

[164] M. Levi and J. Steinhoff, *Spinning gravitating objects in the effective field theory in the post-Newtonian scheme*, *JHEP* **09** (2015) 219 [`1501.04956`].

[165] A. Guevara, A. Ochirov and J. Vines, *Scattering of Spinning Black Holes from Exponentiated Soft Factors*, *JHEP* **09** (2019) 056 [`1812.06895`].

[166] D. Kosmopoulos and A. Luna, *Quadratic-in-spin Hamiltonian at $\mathcal{O}(G^2)$ from scattering amplitudes*, *JHEP* **07** (2021) 037 [`2102.10137`].

[167] M.-Z. Chung, Y.-T. Huang and J.-W. Kim, *Classical potential for general spinning bodies*, *JHEP* **09** (2020) 074 [`1908.08463`].

[168] M.-Z. Chung, Y.-t. Huang, J.-W. Kim and S. Lee, *Complete Hamiltonian for spinning binary systems at first post-Minkowskian order*, *JHEP* **05** (2020) 105 [`2003.06600`].

[169] W.-M. Chen, M.-Z. Chung, Y.-t. Huang and J.-W. Kim, *The 2PM Hamiltonian for binary Kerr to quartic in spin*, `2111.13639`.

[170] G. U. Jakobsen and G. Mogull, *Conservative and radiative dynamics of spinning bodies at third post-Minkowskian order using worldline quantum field theory*, `2201.07778`.

[171] G. U. Jakobsen, G. Mogull, J. Plefka and J. Steinhoff, *Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory*, *Phys. Rev. Lett.* **126** (2021) 201103 [`2101.12688`].

[172] Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban and F. Teng, *Binary Dynamics Through the Fifth Power of Spin at $\mathcal{O}(G^2)$*, `2203.06202`.

[173] R. Aoude, K. Haddad and A. Helset, *Searching for Kerr in the 2PM amplitude*, `2203.06197`.

[174] M. E. Luke and A. V. Manohar, *Reparametrization invariance constraints on heavy particle effective field theories*, *Phys. Lett. B* **286** (1992) 348 [`hep-ph/9205228`].

[175] J. Heinonen, R. J. Hill and M. P. Solon, *Lorentz invariance in heavy particle effective theories*, *Phys. Rev. D* **86** (2012) 094020 [`1208.0601`].

[176] L. de la Cruz, B. Maybee, D. O'Connell and A. Ross, *Classical Yang-Mills observables from amplitudes*, *JHEP* **12** (2020) 076 [`2009.03842`].

[177] L. de la Cruz, A. Luna and T. Scheopner, *Yang-Mills observables: from KMOC to eikonal through EFT*, *JHEP* **01** (2022) 045 [`2108.02178`].

[178] LIGO SCIENTIFIC, VIRGO collaboration, *GW190425: Observation of a Compact*

*Binary Coalescence with Total Mass* $\sim 3.4M_\odot$, *Astrophys. J. Lett.* **892** (2020) L3 [2001.01761].

[179] D. Bini, T. Damour and A. Geralico, *Scattering of tidally interacting bodies in post-Minkowskian gravity*, *Phys. Rev. D* **101** (2020) 044039 [2001.00352].

[180] C. Cheung and M. P. Solon, *Tidal Effects in the Post-Minkowskian Expansion*, *Phys. Rev. Lett.* **125** (2020) 191601 [2006.06665].

[181] G. Kälin, Z. Liu and R. A. Porto, *Conservative Tidal Effects in Compact Binary Systems to Next-to-Leading Post-Minkowskian Order*, *Phys. Rev. D* **102** (2020) 124025 [2008.06047].

[182] K. Haddad and A. Helset, *Tidal effects in quantum field theory*, *JHEP* **12** (2020) 024 [2008.04920].

[183] R. Aoude, K. Haddad and A. Helset, *Tidal effects for spinning particles*, *JHEP* **03** (2021) 097 [2012.05256].

[184] Z. Bern, J. Parra-Martinez, R. Roiban, E. Sawyer and C.-H. Shen, *Leading Nonlinear Tidal Effects and Scattering Amplitudes*, *JHEP* **05** (2021) 188 [2010.08559].

[185] C. Cheung, N. Shah and M. P. Solon, *Mining the Geodesic Equation for Scattering Data*, *Phys. Rev. D* **103** (2021) 024030 [2010.08568].

[186] H. S. Chia, *Tidal deformation and dissipation of rotating black holes*, *Phys. Rev. D* **104** (2021) 024013 [2010.07300].

[187] L. Hui, A. Joyce, R. Penco, L. Santoni and A. R. Solomon, *Static response and Love numbers of Schwarzschild black holes*, *JCAP* **04** (2021) 052 [2010.00593].

[188] P. Charalambous, S. Dubovsky and M. M. Ivanov, *On the Vanishing of Love Numbers for Kerr Black Holes*, *JHEP* **05** (2021) 038 [2102.08917].

[189] P. Charalambous, S. Dubovsky and M. M. Ivanov, *Hidden Symmetry of Vanishing Love Numbers*, *Phys. Rev. Lett.* **127** (2021) 101101 [2103.01234].

[190] L. Hui, A. Joyce, R. Penco, L. Santoni and A. R. Solomon, *Ladder symmetries of black holes. Implications for love numbers and no-hair theorems*, *JCAP* **01** (2022) 032 [2105.01069].

[191] W. T. Emond and N. Moynihan, *Scattering Amplitudes, Black Holes and Leading Singularities in Cubic Theories of Gravity*, *JHEP* **12** (2019) 019 [1905.08213].

[192] A. Cristofoli, *Post-Minkowskian Hamiltonians in Modified Theories of Gravity*, *Phys. Lett. B* **800** (2020) 135095 [1906.05209].

[193] M. Accettulli Huber, A. Brandhuber, S. De Angelis and G. Travaglini, *Note on the absence of $R^2$ corrections to Newton's potential*, *Phys. Rev. D* **101** (2020) 046011 [1911.10108].

[194] M. Accettulli Huber, A. Brandhuber, S. De Angelis and G. Travaglini, *From amplitudes to gravitational radiation with cubic interactions and tidal effects*, *Phys. Rev. D* **103** (2021) 045015 [2012.06548].

[195] M. Accettulli Huber, A. Brandhuber, S. De Angelis and G. Travaglini, *Eikonal phase matrix, deflection angle and time delay in effective field theories of gravity*, *Phys. Rev. D* **102** (2020) 046014 [2006.02375].

[196] M. Carrillo-González, C. de Rham and A. J. Tolley, *Scattering amplitudes for binary systems beyond GR*, *JHEP* **11** (2021) 087 [2107.11384].

[197] M. M. Riva and F. Vernizzi, *Radiated momentum in the post-Minkowskian worldline approach via reverse unitarity*, *JHEP* **11** (2021) 228 [2110.10140].

[198] S. E. Gralla and K. Lobo, *Self-force effects in post-Minkowskian scattering*, 2110.08681.

[199] G. U. Jakobsen, G. Mogull, J. Plefka and J. Steinhoff, *Gravitational Bremsstrahlung and Hidden Supersymmetry of Spinning Bodies*, *Phys. Rev. Lett.* **128** (2022) 011101 [2106.10256].

[200] S. Mougiakakos, M. M. Riva and F. Vernizzi, *Gravitational Bremsstrahlung in the post-Minkowskian effective field theory*, *Phys. Rev. D* **104** (2021) 024041 [2102.08339].

[201] A. V. Manohar, A. K. Ridgway and C.-H. Shen, *Radiated Angular Momentum and Dissipative Effects in Classical Scattering*, 2203.04283.

[202] D. Bini and A. Geralico, *Higher-order tail contributions to the energy and angular momentum fluxes in a two-body scattering process*, *Phys. Rev. D* **104** (2021) 104020 [2108.05445].

[203] A. Edison and M. Levi, *A tale of tails through generalized unitarity*, 2202.04674.

[204] W. D. Goldberger and I. Z. Rothstein, *An Effective Field Theory of Quantum Mechanical Black Hole Horizons*, *JHEP* **04** (2020) 056 [1912.13435].

[205] F. Bloch and A. Nordsieck, *Note on the Radiation Field of the electron*, *Phys. Rev.* **52** (1937) 54.

[206] T. Kinoshita, *Mass singularities of Feynman amplitudes*, *J. Math. Phys.* **3** (1962) 650.

[207] T. D. Lee and M. Nauenberg, *Degenerate Systems and Mass Singularities*, *Phys. Rev.* **133** (1964) B1549.

[208] R. Akhoury, R. Saotome and G. Sterman, *Collinear and Soft Divergences in Perturbative Quantum Gravity*, *Phys. Rev. D* **84** (2011) 104040 [1109.0270].

[209] J. M. Henn and T. Huber, *Systematics of the cusp anomalous dimension*, *JHEP* **11** (2012) 058 [1207.2161].

[210] A. V. Manohar and M. B. Wise, *Heavy quark physics*, vol. 10. 2000.

[211] M. Levy and J. Sucher, *Eikonal approximation in quantum field theory*, *Phys. Rev.* **186** (1969) 1656.

[212] H. D. I. Abarbanel and C. Itzykson, *Relativistic eikonal expansion*, *Phys. Rev. Lett.* **23** (1969) 53.

[213] H. Cheng and T. T. Wu, *High-energy elastic scattering in quantum electrodynamics*, *Phys. Rev. Lett.* **22** (1969) 666.

[214] D. Amati, M. Ciafaloni and G. Veneziano, *Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions*, *Nucl. Phys. B* **347** (1990) 550.

[215] D. N. Kabat and M. Ortiz, *Eikonal quantum gravity and Planckian scattering*, *Nucl. Phys. B* **388** (1992) 570 [`hep-th/9203082`].

[216] R. Saotome and R. Akhoury, *Relationship Between Gravity and Gauge Scattering in the High Energy Limit*, *JHEP* **01** (2013) 123 [`1210.8111`].

[217] R. Akhoury, R. Saotome and G. Sterman, *High Energy Scattering in Perturbative Quantum Gravity at Next to Leading Power*, *Phys. Rev. D* **103** (2021) 064036 [`1308.5204`].

[218] A. Bellini, M. Ademollo and M. Ciafaloni, *Superstring one loop and gravitino contributions to Planckian scattering*, *Nucl. Phys. B* **393** (1993) 79 [`hep-th/9207113`].

[219] Z. Bern, H. Ita, J. Parra-Martinez and M. S. Ruf, *Universality in the classical limit of massless gravitational scattering*, *Phys. Rev. Lett.* **125** (2020) 031601 [`2002.02459`].

[220] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté and P. Vanhove, *Classical gravity from loop amplitudes*, *Phys. Rev. D* **104** (2021) 026009 [`2104.04510`].

[221] D. Amati, M. Ciafaloni and G. Veneziano, *Towards an S-matrix description of gravitational collapse*, *JHEP* **02** (2008) 049 [`0712.1209`].

[222] K. Haddad, *Exponentiation of the leading eikonal phase with spin*, *Phys. Rev. D* **105** (2022) 026004 [`2109.04427`].

[223] U. Kol, D. O'connell and O. Telem, *The Radial Action from Probe Amplitudes to All Orders*, `2109.12092`.

[224] N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, *Post-Minkowskian Scattering Angle in Einstein Gravity*, *JHEP* **08** (2020) 038 [`1910.09366`].

[225] E. T. Newman and A. I. Janis, *Note on the Kerr spinning particle metric*, *J. Math. Phys.* **6** (1965) 915.

[226] N. Arkani-Hamed, Y.-t. Huang and D. O'Connell, *Kerr black holes as elementary particles*, *JHEP* **01** (2020) 046 [`1906.10100`].

[227] A. Guevara, B. Maybee, A. Ochirov, D. O'connell and J. Vines, *A worldsheet for Kerr*, *JHEP* **03** (2021) 201 [2012.11570].

[228] L. Barack and N. Sago, *Gravitational self-force on a particle in eccentric orbit around a Schwarzschild black hole*, *Phys. Rev. D* **81** (2010) 084021 [1002.2386].

[229] M. van de Meent, *Gravitational self-force on generic bound geodesics in Kerr spacetime*, *Phys. Rev. D* **97** (2018) 104033 [1711.09607].

[230] A. Pound, B. Wardell, N. Warburton and J. Miller, *Second-Order Self-Force Calculation of Gravitational Binding Energy in Compact Binaries*, *Phys. Rev. Lett.* **124** (2020) 021101 [1908.07419].

[231] D. Bini, T. Damour and A. Geralico, *Novel approach to binary dynamics: application to the fifth post-Newtonian level*, *Phys. Rev. Lett.* **123** (2019) 231104 [1909.02375].

[232] D. Bini, T. Damour and A. Geralico, *Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders*, *Phys. Rev. D* **102** (2020) 024062 [2003.11891].

[233] D. Bini, T. Damour and A. Geralico, *Radiative contributions to gravitational scattering*, *Phys. Rev. D* **104** (2021) 084031 [2107.08896].

[234] A. Antonelli, C. Kavanagh, M. Khalil, J. Steinhoff and J. Vines, *Gravitational spin-orbit coupling through third-subleading post-Newtonian order: from first-order self-force to arbitrary mass ratios*, *Phys. Rev. Lett.* **125** (2020) 011103 [2003.11391].

[235] A. Antonelli, C. Kavanagh, M. Khalil, J. Steinhoff and J. Vines, *Gravitational spin-orbit and aligned spin$_1$-spin$_2$ couplings through third-subleading post-Newtonian orders*, *Phys. Rev. D* **102** (2020) 124024 [2010.02018].

[236] M. Khalil, *Gravitational spin-orbit dynamics at the fifth-and-a-half post-Newtonian order*, *Phys. Rev. D* **104** (2021) 124015 [2110.12813].

[237] A. Cristofoli, R. Gonzo, D. A. Kosower and D. O'Connell, *Waveforms from Amplitudes*, 2107.10193.

[238] R. Aoude and A. Ochirov, *Classical observables from coherent-spin amplitudes*, *JHEP* **10** (2021) 008 [2108.01649].

[239] L. G. Yaffe, *Large n Limits as Classical Mechanics*, *Rev. Mod. Phys.* **54** (1982) 407.

[240] M. Ciafaloni, D. Colferai and G. Veneziano, *Infrared features of gravitational scattering and radiation in the eikonal approach*, *Phys. Rev. D* **99** (2019) 066008 [1812.08137].

[241] A. Strominger and A. Zhiboedov, *Gravitational Memory, BMS Supertranslations and Soft Theorems*, *JHEP* **01** (2016) 086 [1411.5745].

[242] Y. F. Bautista and A. Guevara, *From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems*, 1903.12419.

[243] A. Laddha and A. Sen, *Gravity Waves from Soft Theorem in General Dimensions*, *JHEP* **09** (2018) 105 [`1801.07719`].

[244] A. Laddha and A. Sen, *Logarithmic Terms in the Soft Expansion in Four Dimensions*, *JHEP* **10** (2018) 056 [`1804.09193`].

[245] B. Sahoo and A. Sen, *Classical and Quantum Results on Logarithmic Terms in the Soft Theorem in Four Dimensions*, *JHEP* **02** (2019) 086 [`1808.03288`].

[246] A. Manu, D. Ghosh, A. Laddha and P. V. Athira, *Soft radiation from scattering amplitudes revisited*, *JHEP* **05** (2021) 056 [`2007.02077`].

[247] Y. F. Bautista and A. Laddha, *Soft Constraints on KMOC Formalism*, `2111.11642`.

[248] S. Caron-Huot and Z. Zahraee, *Integrability of Black Hole Orbits in Maximal Supergravity*, *JHEP* **07** (2019) 179 [`1810.04694`].

[249] Z. Bern, J. P. Gatica, E. Herrmann, A. Luna and M. Zeng, *Scalar QED as a toy model for higher-order effects in classical gravitational scattering*, `2112.12243`.

[250] M. V. S. Saketh, J. Vines, J. Steinhoff and A. Buonanno, *Conservative and radiative dynamics in classical relativistic scattering and bound systems*, `2109.05994`.

[251] A. Buonanno, *Reduction of the two-body dynamics to a one-body description in classical electrodynamics*, *Phys. Rev. D* **62** (2000) 104022 [`hep-th/0004042`].

[252] W. E. Caswell and G. P. Lepage, *Effective Lagrangians for Bound State Problems in QED, QCD, and Other Field Theories*, *Phys. Lett. B* **167** (1986) 437.

[253] N. Isgur and M. B. Wise, *Weak Decays of Heavy Mesons in the Static Quark Approximation*, *Phys. Lett. B* **232** (1989) 113.

[254] G. T. Bodwin, E. Braaten and G. P. Lepage, *Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium*, *Phys. Rev. D* **51** (1995) 1125 [`hep-ph/9407339`].

[255] M. Beneke and V. A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, *Nucl. Phys. B* **522** (1998) 321 [`hep-ph/9711391`].

[256] A. V. Manohar and I. W. Stewart, *The QCD heavy quark potential to order v**2: One loop matching conditions*, *Phys. Rev. D* **62** (2000) 074015 [`hep-ph/0003032`].

[257] A. V. Manohar and I. W. Stewart, *The Zero-Bin and Mode Factorization in Quantum Field Theory*, *Phys. Rev. D* **76** (2007) 074002 [`hep-ph/0605001`].

[258] D. Neill and I. Z. Rothstein, *Classical Space-Times from the S Matrix*, *Nucl. Phys. B* **877** (2013) 177 [`1304.7263`].

[259] V. Vaidya, *Gravitational spin Hamiltonians from the S matrix*, *Phys. Rev. D* **91** (2015) 024017 [`1410.5348`].

[260] A. Cristofoli, P. H. Damgaard, P. Di Vecchia and C. Heissenberg, *Second-order Post-Minkowskian scattering in arbitrary dimensions*, *JHEP* **07** (2020) 122 [2003.10274].

[261] C. Anastasiou and A. Lazopoulos, *Automatic integral reduction for higher order perturbative calculations*, *JHEP* **07** (2004) 046 [hep-ph/0404258].

[262] A. V. Smirnov and V. A. Smirnov, *FIRE4, LiteRed and accompanying tools to solve integration by parts relations*, *Comput. Phys. Commun.* **184** (2013) 2820 [1302.5885].

[263] P. Maierhoefer, J. Usovitsch and P. Uwer, *Kira - A Feynman Integral Reduction Program*, 1705.05610.

[264] C. Studerus, *Reduze-Feynman Integral Reduction in C++*, *Comput. Phys. Commun.* **181** (2010) 1293 [0912.2546].

[265] R. N. Lee, *Presenting LiteRed: a tool for the Loop InTEgrals REDuction*, 1212.2685.

[266] R. N. Lee, *LiteRed 1.4: a powerful tool for reduction of multiloop integrals*, *J. Phys. Conf. Ser.* **523** (2014) 012059 [1310.1145].

[267] P. V. Landshoff and J. C. Polkinghorne, *Iterations of regge cuts*, *Phys. Rev.* **181** (1969) 1989.

[268] D. Bini, T. Damour, A. Geralico, S. Laporta and P. Mastrolia, *Gravitational dynamics at $O(G^6)$: perturbative gravitational scattering meets experimental mathematics*, 2008.09389.

[269] D. Bini, T. Damour, A. Geralico, S. Laporta and P. Mastrolia, *Gravitational scattering at the seventh order in G: nonlocal contribution at the sixth post-Newtonian accuracy*, *Phys. Rev. D* **103** (2021) 044038 [2012.12918].

[270] N. E. J. Bjerrum-Bohr, L. Plante and P. Vanhove, *Post-Minkowskian Radial Action from Soft Limits and Velocity Cuts*, 2111.02976.

[271] C. L. Basham, L. S. Brown, S. D. Ellis and S. T. Love, *Energy Correlations in electron - Positron Annihilation: Testing QCD*, *Phys. Rev. Lett.* **41** (1978) 1585.

[272] C. L. Basham, L. S. Brown, S. D. Ellis and S. T. Love, *Energy Correlations in electron-Positron Annihilation in Quantum Chromodynamics: Asymptotically Free Perturbation Theory*, *Phys. Rev. D* **19** (1979) 2018.

[273] D. G. Richards, W. J. Stirling and S. D. Ellis, *Energy-energy Correlations to Second Order in Quantum Chromodynamics*, *Nucl. Phys. B* **229** (1983) 317.

[274] G. P. Korchemsky and G. F. Sterman, *Power corrections to event shapes and factorization*, *Nucl. Phys. B* **555** (1999) 335 [hep-ph/9902341].

[275] Y. L. Dokshitzer, G. Marchesini and B. R. Webber, *Nonperturbative effects in the energy energy correlation*, *JHEP* **07** (1999) 012 [hep-ph/9905339].

[276] C. Lee and G. F. Sterman, *Universality of nonperturbative effects in event shapes*, *eConf* **C0601121** (2006) A001 [`hep-ph/0603066`].

[277] C. W. Bauer, S. P. Fleming, C. Lee and G. F. Sterman, *Factorization of e+e- Event Shape Distributions with Hadronic Final States in Soft Collinear Effective Theory*, *Phys. Rev. D* **78** (2008) 034027 [`0801.4569`].

[278] A. V. Belitsky, G. P. Korchemsky and G. F. Sterman, *Energy flow in QCD and event shape functions*, *Phys. Lett. B* **515** (2001) 297 [`hep-ph/0106308`].

[279] B. Maybee, D. O'Connell and J. Vines, *Observables and amplitudes for spinning particles and black holes*, *JHEP* **12** (2019) 156 [`1906.09260`].

[280] P. H. Damgaard, K. Haddad and A. Helset, *Heavy Black Hole Effective Theory*, *JHEP* **11** (2019) 070 [`1908.10308`].

[281] A. Brandhuber, G. Chen, G. Travaglini and C. Wen, *A new gauge-invariant double copy for heavy-mass effective theory*, *JHEP* **07** (2021) 047 [`2104.11206`].

[282] A. Brandhuber, G. Chen, H. Johansson, G. Travaglini and C. Wen, *Kinematic Hopf Algebra for BCJ Numerators in Heavy-Mass Effective Field Theory and Yang-Mills Theory*, `2111.15649`.

[283] G. Chen, H. Johansson, F. Teng and T. Wang, *On the kinematic algebra for BCJ numerators beyond the MHV sector*, *JHEP* **11** (2019) 055 [`1906.10683`].

[284] G. Chen, H. Johansson, F. Teng and T. Wang, *Next-to-MHV Yang-Mills kinematic algebra*, *JHEP* **10** (2021) 042 [`2104.12726`].

[285] J. J. M. Carrasco and I. A. Vazquez-Holm, *Loop-Level Double-Copy for Massive Quantum Particles*, *Phys. Rev. D* **103** (2021) 045002 [`2010.13435`].

[286] H. Johansson and A. Ochirov, *Pure Gravities via Color-Kinematics Duality for Fundamental Matter*, *JHEP* **11** (2015) 046 [`1407.4772`].

[287] A. Luna, I. Nicholson, D. O'Connell and C. D. White, *Inelastic Black Hole Scattering from Charged Scalar Amplitudes*, *JHEP* **03** (2018) 044 [`1711.03901`].

[288] K. Haddad and A. Helset, *The double copy for heavy particles*, *Phys. Rev. Lett.* **125** (2020) 181603 [`2005.13897`].

[289] N. E. J. Bjerrum-Bohr, T. V. Brown and H. Gomez, *Scattering of Gravitons and Spinning Massive States from Compact Numerators*, *JHEP* **04** (2021) 234 [`2011.10556`].

[290] N. E. J. Bjerrum-Bohr, A. Cristofoli, P. H. Damgaard and H. Gomez, *Scalar-Graviton Amplitudes*, *JHEP* **11** (2019) 148 [`1908.09755`].

[291] S. Foffa, R. A. Porto, I. Rothstein and R. Sturani, *Conservative dynamics of binary*

systems to fourth Post-Newtonian order in the EFT approach II: Renormalized Lagrangian, *Phys. Rev. D* **100** (2019) 024048 [`1903.05118`].

[292] S. Foffa and R. Sturani, *Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach I: Regularized Lagrangian, Phys. Rev. D* **100** (2019) 024047 [`1903.05113`].

[293] J. Blümlein, A. Maier and P. Marquard, *Five-Loop Static Contribution to the Gravitational Interaction Potential of Two Point Masses, Phys. Lett. B* **800** (2020) 135100 [`1902.11180`].

[294] S. Foffa, P. Mastrolia, R. Sturani, C. Sturm and W. J. Torres Bobadilla, *Static two-body potential at fifth post-Newtonian order, Phys. Rev. Lett.* **122** (2019) 241605 [`1902.10571`].

[295] S. Foffa and R. Sturani, *Effective field theory calculation of conservative binary dynamics at third post-Newtonian order, Phys. Rev. D* **84** (2011) 044031 [`1104.1122`].

[296] J. B. Gilmore and A. Ross, *Effective field theory calculation of second post-Newtonian binary dynamics, Phys. Rev. D* **78** (2008) 124021 [`0810.1328`].

[297] B. Kol and M. Smolkin, *Non-Relativistic Gravitation: From Newton to Einstein and Back, Class. Quant. Grav.* **25** (2008) 145011 [`0712.4116`].

[298] G. Mogull, J. Plefka and J. Steinhoff, *Classical black hole scattering from a worldline quantum field theory, JHEP* **02** (2021) 048 [`2010.02865`].

[299] G. Cho, B. Pardo and R. A. Porto, *Gravitational radiation from inspiralling compact objects: Spin-spin effects completed at the next-to-leading post-Newtonian order, Phys. Rev. D* **104** (2021) 024037 [`2103.14612`].

[300] J. Vines, J. Steinhoff and A. Buonanno, *Spinning-black-hole scattering and the test-black-hole limit at second post-Minkowskian order, Phys. Rev. D* **99** (2019) 064054 [`1812.00956`].

[301] S. Foffa and R. Sturani, *Tail terms in gravitational radiation reaction via effective field theory, Phys. Rev. D* **87** (2013) 044056 [`1111.5488`].

[302] W. D. Goldberger and A. K. Ridgway, *Radiation and the classical double copy for color charges, Phys. Rev. D* **95** (2017) 125010 [`1611.03493`].

[303] W. D. Goldberger and A. K. Ridgway, *Bound states and the classical double copy, Phys. Rev. D* **97** (2018) 085019 [`1711.09493`].

[304] C.-H. Shen, *Gravitational Radiation from Color-Kinematics Duality, JHEP* **11** (2018) 162 [`1806.07388`].

[305] KAGRA, LIGO Scientific, Virgo, VIRGO collaboration, *Prospects for observing*

and localizing gravitational-wave transients with Advanced LIGO, Advanced Virgo and KAGRA, *Living Rev. Rel.* **21** (2018) 3 [`1304.0670`].

[306] T. Hinderer, A. Taracchini, F. Foucart, A. Buonanno, J. Steinhoff et al., *Effects of neutron-star dynamic tides on gravitational waveforms within the effective-one-body approach*, *Phys. Rev. Lett.* **116** (2016) 181101 [`1602.00599`].

[307] T. Dietrich, S. Bernuzzi and W. Tichy, *Closed-form tidal approximants for binary neutron star gravitational waveforms constructed from high-resolution numerical relativity simulations*, *Phys. Rev. D* **96** (2017) 121501 [`1706.02969`].

[308] A. Nagar, F. Messina, P. Rettegno, D. Bini, T. Damour, A. Geralico et al., *Nonlinear-in-spin effects in effective-one-body waveform models of spin-aligned, inspiralling, neutron star binaries*, *Phys. Rev. D* **99** (2019) 044007 [`1812.07923`].

[309] A. Nagar et al., *Time-domain effective-one-body gravitational waveforms for coalescing compact binaries with nonprecessing spins, tides and self-spin effects*, *Phys. Rev. D* **98** (2018) 104052 [`1806.01772`].

[310] R. Cotesta, A. Buonanno, A. Bohé, A. Taracchini, I. Hinder and S. Ossokine, *Enriching the Symphony of Gravitational Waves from Binary Black Holes by Tuning Higher Harmonics*, *Phys. Rev. D* **98** (2018) 084028 [`1803.10701`].

[311] V. Varma, S. E. Field, M. A. Scheel, J. Blackman, L. E. Kidder and H. P. Pfeiffer, *Surrogate model of hybridized numerical relativity binary black hole waveforms*, *Phys. Rev. D* **99** (2019) 064045 [`1812.07865`].

[312] T. Dietrich, A. Samajdar, S. Khan, N. K. Johnson-McDaniel, R. Dudi and W. Tichy, *Improving the NRTidal model for binary neutron star systems*, *Phys. Rev. D* **100** (2019) 044003 [`1905.06011`].

[313] V. Varma, S. E. Field, M. A. Scheel, J. Blackman, D. Gerosa, L. C. Stein et al., *Surrogate models for precessing binary black hole simulations with unequal masses*, *Phys. Rev. Research.* **1** (2019) 033015 [`1905.09300`].

[314] S. Ossokine, A. Buonanno, S. Marsat, R. Cotesta, S. Babak et al., *Multipolar Effective-One-Body Waveforms for Precessing Binary Black Holes: Construction and Validation*, *Phys. Rev. D* **102** (2020) 044055 [`2004.09442`].

[315] G. Pratten et al., *Computationally efficient models for the dominant and subdominant harmonic modes of precessing binary black holes*, *Phys. Rev. D* **103** (2021) 104056 [`2004.06503`].

[316] J. E. Thompson, E. Fauchon-Jones, S. Khan, E. Nitoglia, F. Pannarale, T. Dietrich et al., *Modeling the gravitational wave signature of neutron star black hole coalescences*, *Phys. Rev. D* **101** (2020) 124059 [`2002.08383`].

[317] A. Matas, T. Dietrich, A. Buonanno, T. Hinderer, M. Puerrer et al., *Aligned-spin*

neutron-star–black-hole waveform model based on the effective-one-body approach and numerical-relativity simulations, *Phys. Rev. D* **102** (2020) 043023 [`2004.10001`].

[318] H. Estellés, M. Colleoni, C. García-Quirós, S. Husa, D. Keitel, M. Mateu-Lucena et al., *New twists in compact binary waveform modelling: a fast time domain model for precession*, `2105.05872`.

[319] R. Gamba, S. Akçay, S. Bernuzzi and J. Williams, *Effective-one-body waveforms for precessing coalescing compact binaries with post-Newtonian Twist*, `2111.03675`.

[320] S. Laporta and E. Remiddi, *The Analytical value of the electron $(g-2)$ at order $\alpha^3$ in QED*, *Phys. Lett.* **B379** (1996) 283 [`hep-ph/9602417`].

[321] J. Gluza, K. Kajda and D. A. Kosower, *Towards a Basis for Planar Two-Loop Integrals*, *Phys. Rev. D* **83** (2011) 045012 [`1009.0472`].

[322] R. M. Schabinger, *A New Algorithm For The Generation Of Unitarity-Compatible Integration By Parts Relations*, *JHEP* **01** (2012) 077 [`1111.4220`].

[323] H. Ita, *Two-loop Integrand Decomposition into Master Integrals and Surface Terms*, *Phys. Rev. D* **94** (2016) 116015 [`1510.05626`].

[324] A. Georgoudis, K. J. Larsen and Y. Zhang, *Azurite: An algebraic geometry based package for finding bases of loop integrals*, *Comput. Phys. Commun.* **221** (2017) 203 [`1612.04252`].

[325] H. Ita, *Towards a Numerical Unitarity Approach for Two-loop Amplitudes in QCD*, *PoS* **LL2016** (2016) 080 [`1607.00705`].

[326] Y. Zhang, *Lecture Notes on Multi-loop Integral Reduction and Applied Algebraic Geometry*, 2016, `1612.02249`, https://inspirehep.net/record/1502112/files/arXiv:1612.02249.pdf.

[327] Z. Bern, M. Enciso, H. Ita and M. Zeng, *Dual Conformal Symmetry, Integration-by-Parts Reduction, Differential Equations and the Nonplanar Sector*, *Phys. Rev. D* **96** (2017) 096017 [`1709.06055`].

[328] P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, *JHEP* **02** (2019) 139 [`1810.03818`].

[329] H. Frellesvig, F. Gasparotto, M. K. Mandal, P. Mastrolia, L. Mattiazzi and S. Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*, *Phys. Rev. Lett.* **123** (2019) 201602 [`1907.02000`].

[330] R. N. Lee, *Reducing differential equations for multiloop master integrals*, *JHEP* **04** (2015) 108 [`1411.0911`].

[331] M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert et al., *Magnus and Dyson Series for Master Integrals*, *JHEP* **03** (2014) 082 [`1401.2979`].

[332] J. M. Henn, *Lectures on differential equations for Feynman integrals*, *J. Phys. A* **48** (2015) 153001 [`1412.2296`].

[333] O. Gituliar and V. Magerya, *Fuchsia and master integrals for splitting functions from differential equations in QCD*, *PoS* **LL2016** (2016) 030 [`1607.00759`].

[334] O. Gituliar and V. Magerya, *Fuchsia: a tool for reducing differential equations for Feynman master integrals to epsilon form*, *Comput. Phys. Commun.* **219** (2017) 329 [`1701.04269`].

[335] M. Prausa, *epsilon: A tool to find a canonical basis of master integrals*, *Comput. Phys. Commun.* **219** (2017) 361 [`1701.00725`].

[336] D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang and S. Zoia, *All Master Integrals for Three-Jet Production at Next-to-Next-to-Leading Order*, *Phys. Rev. Lett.* **123** (2019) 041603 [`1812.11160`].

[337] C. Dlapa, J. Henn and K. Yan, *Deriving canonical differential equations for Feynman integrals from a single uniform weight integral*, *JHEP* **05** (2020) 025 [`2002.02340`].

[338] J. Chen, X. Jiang, X. Xu and L. L. Yang, *Constructing canonical Feynman integrals with intersection theory*, *Phys. Lett. B* **814** (2021) 136085 [`2008.03045`].

[339] C. Dlapa, X. Li and Y. Zhang, *Leading singularities in Baikov representation and Feynman integrals with uniform transcendental weight*, *JHEP* **07** (2021) 227 [`2103.04638`].

[340] F. C. S. Brown, *On the periods of some Feynman integrals*, `0910.0114`.

[341] F. Brown and K. Yeats, *Spanning forest polynomials and the transcendental weight of Feynman graphs*, *Commun. Math. Phys.* **301** (2011) 357 [`0910.5429`].

[342] F. C. S. Brown and A. Levin, *Multiple Elliptic Polylogarithms*, `1110.6917`.

[343] E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, *Comput. Phys. Commun.* **188** (2015) 148 [`1403.3385`].

[344] J. Broedel, C. Duhr, F. Dulat and L. Tancredi, *Elliptic polylogarithms and iterated integrals on elliptic curves. Part I: general formalism*, *JHEP* **05** (2018) 093 [`1712.07089`].

[345] J. L. Bourjaily, A. J. McLeod, M. von Hippel and M. Wilhelm, *Bounded Collection of Feynman Integral Calabi-Yau Geometries*, *Phys. Rev. Lett.* **122** (2019) 031601 [`1810.07689`].

[346] J. L. Bourjaily et al., *Functions Beyond Multiple Polylogarithms for Precision Collider Physics*, 3, 2022, `2203.07088`.

[347] V. A. Smirnov, *Evaluating Feynman integrals*, *Springer Tracts Mod. Phys.* **211** (2004) 1.

[348] J. L. Bourjaily, A. J. McLeod, M. von Hippel and M. Wilhelm, *Rationalizing Loop Integration*, *JHEP* **08** (2018) 184 [`1805.10281`].

[349] O. V. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, *Phys. Rev. D* **54** (1996) 6479 [`hep-th/9606018`].

[350] R. N. Lee, *Space-time dimensionality D as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to D*, *Nucl. Phys. B* **830** (2010) 474 [`0911.0252`].

[351] R. N. Lee and V. A. Smirnov, *The Dimensional Recurrence and Analyticity Method for Multicomponent Master Integrals: Using Unitarity Cuts to Construct Homogeneous Solutions*, *JHEP* **12** (2012) 104 [`1209.0339`].

[352] C. Anastasiou and K. Melnikov, *Higgs boson production at hadron colliders in NNLO QCD*, *Nucl. Phys. B* **646** (2002) 220 [`hep-ph/0207004`].

[353] C. Anastasiou, L. J. Dixon and K. Melnikov, *NLO Higgs boson rapidity distributions at hadron colliders*, *Nucl. Phys. B Proc. Suppl.* **116** (2003) 193 [`hep-ph/0211141`].

[354] C. Anastasiou, L. J. Dixon, K. Melnikov and F. Petriello, *Dilepton rapidity distribution in the Drell-Yan process at NNLO in QCD*, *Phys. Rev. Lett.* **91** (2003) 182002 [`hep-ph/0306192`].

[355] C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, F. Herzog and B. Mistlberger, *Soft expansion of double-real-virtual corrections to Higgs production at $N^3LO$*, *JHEP* **08** (2015) 051 [`1505.04110`].

[356] C. Anastasiou, Z. Bern, L. J. Dixon and D. A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [`hep-th/0309040`].

[357] Z. Bern, L. J. Dixon and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev. D* **72** (2005) 085001 [`hep-th/0505205`].

[358] R. Monteiro, D. O'Connell, D. Peinador Veiga and M. Sergola, *Classical solutions and their double copy in split signature*, *JHEP* **05** (2021) 268 [`2012.11190`].

[359] S. Naoz, B. Kocsis, A. Loeb and N. Yunes, *Resonant Post-Newtonian Eccentricity Excitation in Hierarchical Three-body Systems*, *Astrophys. J.* **773** (2013) 187 [`1206.4316`].

[360] H. Lim and C. L. Rodriguez, *Relativistic three-body effects in hierarchical triples*, *Phys. Rev. D* **102** (2020) 064033 [`2001.03654`].

[361] M. A. S. Martinez et al., *Black Hole Mergers from Hierarchical Triples in Dense Star Clusters*, *Astrophys. J.* **903** (2020) 67 [`2009.08468`].

[362] G. Fragione, M. A. S. Martinez, K. Kremer, S. Chatterjee, C. L. Rodriguez, C. S. Ye et al., *Demographics of triple systems in dense star clusters*, *Astrophys. J.* **900** (2020) 16 [2007.11605].

[363] T. C. N. Boekholt, A. Moerman and S. F. P. Zwart, *Relativistic Pythagorean three-body problem*, *Phys. Rev. D* **104** (2021) 083020 [2109.07013].

[364] S. F. P. Zwart, T. C. N. Boekholt, E. Por, A. S. Hamers and S. L. W. McMillan, *Chaos in self-gravitating many-body systems: Lyapunov time dependence of N and the influence of general relativity*, 2109.11012.

[365] G. Schäfer, *Three-body hamiltonian in general relativity*, *Physics Letters A* **123** (1987) 336.

[366] Y.-Z. Chu, *The n-body problem in General Relativity up to the second post-Newtonian order from perturbative field theory*, *Phys. Rev. D* **79** (2009) 044031 [0812.0012].

[367] C. M. Will, *Incorporating post-Newtonian effects in N-body dynamics*, *Phys. Rev. D* **89** (2014) 044043 [1312.1289].

[368] F. Loebbert, J. Plefka, C. Shi and T. Wang, *Three-body effective potential in general relativity at second post-Minkowskian order and resulting post-Newtonian contributions*, *Phys. Rev. D* **103** (2021) 064010 [2012.14224].

[369] W. H. Press, *Long Wave Trains of Gravitational Waves from a Vibrating Black Hole*, *Astrophys. J. Lett.* **170** (1971) L105.

[370] I. Kamaretsos, M. Hannam, S. Husa and B. S. Sathyaprakash, *Black-hole hair loss: learning about binary progenitors from ringdown signals*, *Phys. Rev. D* **85** (2012) 024018 [1107.0854].

[371] S. A. Hughes, A. Apte, G. Khanna and H. Lim, *Learning about black hole binaries from their ringdown spectra*, *Phys. Rev. Lett.* **123** (2019) 161101 [1901.05900].

[372] X. Li, L. Sun, R. K. L. Lo, E. Payne and Y. Chen, *Angular emission patterns of remnant black holes*, *Phys. Rev. D* **105** (2022) 024016 [2110.03116].

[373] T. Adamo, E. Casali, L. Mason and S. Nekovar, *Amplitudes on plane waves from ambitwistor strings*, *JHEP* **11** (2017) 160 [1708.09249].

[374] T. Adamo, E. Casali, L. Mason and S. Nekovar, *Plane wave backgrounds and colour-kinematics duality*, *JHEP* **02** (2019) 198 [1810.05115].

[375] T. Adamo and A. Ilderton, *Classical and quantum double copy of back-reaction*, *JHEP* **09** (2020) 200 [2005.05807].

[376] T. Adamo, L. Mason and A. Sharma, *Gluon scattering on self-dual radiative gauge fields*, 2010.14996.

[377] T. Adamo, A. Cristofoli and P. Tourkine, *Eikonal amplitudes from curved backgrounds*, 2112.09113.

[378] P. Diwakar, A. Herderschee, R. Roiban and F. Teng, *BCJ amplitude relations for Anti-de Sitter boundary correlators in embedding space*, *JHEP* **10** (2021) 141 [2106.10822].

[379] C. Cheung, J. Parra-Martinez and A. Sivaramakrishnan, *On-shell Correlators and Color-Kinematics Duality in Curved Symmetric Spacetimes*, 2201.05147.

[380] A. Herderschee, R. Roiban and F. Teng, *On the Differential Representation and Color-Kinematics Duality of AdS Boundary Correlators*, 2201.05067.

[381] N. Arkani-Hamed, D. Baumann, H. Lee and G. L. Pimentel, *The Cosmological Bootstrap: Inflationary Correlators from Symmetries and Singularities*, *JHEP* **04** (2020) 105 [1811.00024].

[382] D. Baumann, C. Duaso Pueyo, A. Joyce, H. Lee and G. L. Pimentel, *The Cosmological Bootstrap: Spinning Correlators from Symmetries and Factorization*, *SciPost Phys.* **11** (2021) 071 [2005.04234].

[383] P. Benincasa, *From the flat-space S-matrix to the Wavefunction of the Universe*, 1811.02515.

[384] D. Baumann, D. Green, A. Joyce, E. Pajer, G. L. Pimentel, C. Sleight et al., *Snowmass White Paper: The Cosmological Bootstrap*, in *2022 Snowmass Summer Study*, 3, 2022, 2203.08121.

[385] A. Luna, R. Monteiro, D. O'Connell and C. D. White, *The classical double copy for Taub–NUT spacetime*, *Phys. Lett.* **B750** (2015) 272 [1507.01869].

[386] A. Luna, R. Monteiro, I. Nicholson, D. O'Connell and C. D. White, *The double copy: Bremsstrahlung and accelerating black holes*, *JHEP* **06** (2016) 023 [1603.05737].

[387] A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O'Connell, N. Westerberg et al., *Perturbative spacetimes from Yang-Mills theory*, *JHEP* **04** (2017) 069 [1611.07508].

[388] M. Carrillo González, B. Melcher, K. Ratliff, S. Watson and C. D. White, *The classical double copy in three spacetime dimensions*, *JHEP* **07** (2019) 167 [1904.11001].

[389] M. Carrillo González, R. Penco and M. Trodden, *Radiation of scalar modes and the classical double copy*, *JHEP* **11** (2018) 065 [1809.04611].

[390] M. Carrillo-González, R. Penco and M. Trodden, *The classical double copy in maximally symmetric spacetimes*, *JHEP* **04** (2018) 028 [1711.01296].

[391] T. Adamo, A. Cristofoli and A. Ilderton, *Classical physics from amplitudes on curved backgrounds*, 2203.13785.

[392] LIGO Scientific, VIRGO, KAGRA collaboration, *Tests of General Relativity with GWTC-3*, 2112.06861.

[393] L. Barack et al., *Black holes, gravitational waves and fundamental physics: a roadmap*, *Class. Quant. Grav.* **36** (2019) 143001 [1806.05195].

[394] S. D. Mathur, *The Fuzzball proposal for black holes: An Elementary review*, *Fortsch. Phys.* **53** (2005) 793 [hep-th/0502050].

[395] S. D. Mathur, *Fuzzballs and the information paradox: A Summary and conjectures*, 0810.4525.

[396] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, *Black Holes: Complementarity or Firewalls?*, *JHEP* **02** (2013) 062 [1207.3123].

[397] A. Almheiri, D. Marolf, J. Polchinski, D. Stanford and J. Sully, *An Apologia for Firewalls*, *JHEP* **09** (2013) 018 [1304.6483].

[398] J. J. M. Carrasco, L. Rodina and S. Zekioglu, *Composing effective prediction at five points*, *JHEP* **06** (2021) 169 [2104.08370].

[399] Q. Bonnefoy, G. Durieux, C. Grojean, C. S. Machado and J. Roosmale Nepveu, *The seeds of EFT double copy*, 2112.11453.

[400] H.-H. Chi, H. Elvang, A. Herderschee, C. R. T. Jones and S. Paranjape, *Generalizations of the Double-Copy: the KLT Bootstrap*, 2106.12600.

[401] S. Endlich, V. Gorbenko, J. Huang and L. Senatore, *An effective formalism for testing extensions to General Relativity with gravitational waves*, *JHEP* **09** (2017) 122 [1704.01590].

[402] C. de Rham, J. Francfort and J. Zhang, *Black Hole Gravitational Waves in the Effective Field Theory of Gravity*, *Phys. Rev. D* **102** (2020) 024079 [2005.13923].

[403] P. A. Cano, K. Fransen and T. Hertog, *Ringing of rotating black holes in higher-derivative gravity*, *Phys. Rev. D* **102** (2020) 044047 [2005.03671].

[404] P. A. Cano, K. Fransen, T. Hertog and S. Maenaut, *Gravitational ringing of rotating black holes in higher-derivative gravity*, *Phys. Rev. D* **105** (2022) 024064 [2110.11378].

[405] C. de Rham, S. Kundu, M. Reece, A. J. Tolley and S.-Y. Zhou, *Snowmass White Paper: UV Constraints on IR Physics*, in *2022 Snowmass Summer Study*, 3, 2022, 2203.06805.