# First-order thermodynamics of scalar-tensor cosmology

Serena Giardino[a]*, Valerio Faraoni[b]†, and Andrea Giusti[c]‡

[a] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute) Callinstraße 38, 30167 Hannover, Germany*

[b] *Department of Physics & Astronomy, Bishop's University 2600 College Street, Sherbrooke, Québec, Canada J1M 1Z7*

[c] *Institute for Theoretical Physics, ETH Zurich Wolfgang-Pauli-Strasse 27, 8093 Zurich, Switzerland*

16th February 2022

**Abstract**

A new thermodynamics of scalar-tensor gravity is applied to spatially homogeneous and isotropic cosmologies in this class of theories and tested on analytical solutions. A forever-expanding universe approaches the Einstein "state of equilibrium" with zero effective temperature at late times and departs from it near spacetime singularities. "Cooling" by expansion and "heating" by singularities compete near the Big Rip, where it is found that the effective temperature diverges in the case of a conformally coupled scalar field.

## 1 Introduction

The fascinating connection between gravity and thermodynamics, first suggested in the context of black holes, has been put on a firmer footing by Jacobson's derivation of the Einstein field equations of General Relativity (GR) as an equation of state, based only on thermodynamical considerations [1]. This seminal work has deep implications for the nature of gravity and has inspired a large body of literature, defining the so-called "thermodynamics of spacetime". Most interestingly, this picture suggests that classical gravity could be non-fundamental in nature and could represent an emergent phenomenon.

In another intriguing ramification of this work, the field equations of metric $f(\mathcal{R})$ gravity were recovered using only thermodynamical considerations [2]. This result opened up the possibility that a "thermodynamics of gravitational theories" could exist, in which GR represents an equilibrium state while modifications such as $f(\mathcal{R})$ gravity correspond to dissipative non-equilibrium states. More generally, any gravitational theory containing dynamical degrees of freedom in addition to the two massless spin-two modes of GR would correspond to an "excited" state. Through a dissipation

*E-mail: serena.giardino@aei.mpg.de
†E-mail: vfaraoni@ubishops.ca
‡E-mail: agiusti@phys.ethz.ch

process, gravity could tend towards a state of thermodynamical equilibrium such as GR. In spite of the interest sparked by this work, neither the equation describing how the equilibrium state is reached, nor the order parameter ruling this process have been identified.

Recently, it has been shown in [3] (expanding on [4]) that the contribution of the scalar field $\phi$ to the field equations of scalar-tensor gravity can be described as an effective relativistic dissipative fluid. This is found by simply rewriting the field equations and does not entail extra assumptions. This result opened up the possibility of applying Eckart's first-order thermodynamics [5] to the effective imperfect $\phi$-fluid with the goal of extracting its relevant thermodynamical quantities [6, 7]. It was therefore possible to derive not only explicit expressions for the heat current density, the "temperature of modified gravity", the viscosity coefficients, and the entropy density, but also to find a diffusion equation describing how the equilibrium state is approached, all without requiring extra assumptions, unlike in Jacobson's thermodynamics. While Eckart's theory suffers from important shortcomings such as causality violation and instabilities, it is widely used as a simple model of relativistic thermodynamics. For our purposes, it represents a first step towards the study of the thermodynamics of modified gravity, yet it is promising, especially since there was no reason to expect *a priori* that one could derive such physical quantities by formally identifying the effective fluid with a thermodynamical system.

More specifically, it was found [7] that the product between effective temperature $\mathcal{T}$ and thermal conductivity $\mathcal{K}$ is positive-definite (which was not granted *a priori*), the shear viscosity $\eta$ is negative (which could allow the entropy density $s$ to decrease but is consistent with the non-isolated and effective nature of the fluid), while the GR equilibrium state corresponds to $\mathcal{K}\mathcal{T} = 0$. This new thermodynamical formalism has also been applied to Horndeski gravity [8]. It was discovered that the formalism does not work for the most general Horndeski theory, but only when there are no terms in the Horndeski action which violate the equality between the propagation speeds of light and of gravitational waves. This is even more remarkable in light of the recent very stringent constraints on the speed of gravitational waves [9].

Cosmology is a fruitful arena for the study of extended theories of gravity [10]: although deviations from GR on small scales may be extremely small at present, the situation might have been different on cosmological scales in the past or may be different in the future. The main motivation for formulating a scalar-tensor theory in the first place also originated in cosmology: the new theory would explicitly incorporate Mach's principle and, due to the variability of the gravitational coupling, the distribution of matter on cosmological scales could affect local gravity [11]. Currently, the main motivation for studying modified gravity arises from the need to explain the accelerated expansion of the present universe without an *ad hoc* dark energy [12, 13]. Moreover, scalar-tensor gravity is widely used in the inflationary paradigm of the early universe.

Interestingly, the idea of scalar-tensor theories relaxing towards GR in a cosmological setting has been explored in [14,15] (albeit with a very different scope from that of our work). The authors found that, during the matter-dominated era, the expansion of the universe drives the scalar field toward a state where scalar-tensor gravity becomes effectively indistinguishable from GR: the expected present deviations from GR would therefore be small, but not unmeasurably so, which has since been corroborated further.

In the present work, we extend the results of Refs. [6, 7] to cosmology by studying Friedmann-Lemaître-Robertson Walker (FLRW) universes in scalar-tensor gravity. Secs. 2 and 3 introduce the necessary background and the basic formalism based on Eckart's thermodynamics, respectively. We analyse general FLRW universes in Sec. 4 and, in Sec. 5, we test ideas of the thermodynamical formalism using analytical solutions of scalar-tensor cosmology. We follow the notation of Ref. [16].

## 2 Generalities

The Jordan frame scalar-tensor action is

$$S_{\rm ST} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi \mathcal{R} - \frac{\omega(\phi)}{\phi} \nabla^c\phi \nabla_c\phi - V(\phi) \right] + S^{(\rm m)} \,, \tag{2.1}$$

where $\mathcal{R}$ is the Ricci scalar, the Brans-Dicke scalar $\phi > 0$ is approximately the inverse of the effective gravitational coupling $G_{\rm eff}$, $\omega(\phi)$ is the "Brans-Dicke coupling", $V(\phi)$ is a potential for the scalar field, and $S^{(\rm m)} = \int d^4x \sqrt{-g}\, \mathcal{L}^{(\rm m)}$ is the matter action.

The Jordan frame field equations are [17–20]

$$\begin{aligned} G_{ab} &\equiv \mathcal{R}_{ab} - \frac{1}{2}\, g_{ab} \mathcal{R} = \frac{8\pi}{\phi}\, T_{ab}^{(\rm m)} + \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2}\, g_{ab} \nabla_c \phi \nabla^c \phi \right) \\ &\quad + \frac{1}{\phi} \left( \nabla_a \nabla_b \phi - g_{ab} \Box \phi \right) - \frac{V}{2\phi}\, g_{ab} \,, \end{aligned} \tag{2.2}$$

$$\Box \phi = \frac{1}{2\omega+3} \left( \frac{8\pi T^{(\rm m)}}{\phi} + \phi\, V_{,\phi} - 2V - \omega_{,\phi} \nabla^c \phi \nabla_c \phi \right) \,, \tag{2.3}$$

where $\mathcal{R}_{ab}$ is the Ricci tensor, $T^{(\rm m)} \equiv g^{ab} T_{ab}^{(\rm m)}$ is the trace of the matter stress-energy tensor $T_{ab}^{(\rm m)}$, and $\omega_{,\phi} \equiv d\omega/d\phi$, $V_{,\phi} \equiv dV/d\phi$. The effective stress-energy tensor of the Brans-Dicke-like field that one reads off the right-hand side of Eq. (2.2) is

$$8\pi T_{ab}^{(\phi)} = \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2}\, g_{ab} \nabla^c \phi \nabla_c \phi \right) + \frac{1}{\phi} \left( \nabla_a \nabla_b \phi - g_{ab} \Box \phi \right) - \frac{V}{2\phi}\, g_{ab} \,. \tag{2.4}$$

$T_{ab}^{(\phi)}$ has the form of an imperfect fluid energy-momentum tensor [3, 4],

$$T_{ab} = \rho\, u_a u_b + q_a u_b + q_b u_a + \Pi_{ab} \,, \tag{2.5}$$

where the effective energy density, heat flux density, stress tensor, isotropic pressure, and anisotropic stresses (the trace-free part $\pi_{ab}$ of the stress tensor $\Pi_{ab}$) in the comoving frame are

$$\rho = T_{ab} u^a u^b \,, \tag{2.6}$$

$$q_a = -T_{cd}\, u^c {h_a}^d \,, \tag{2.7}$$

$$\Pi_{ab} = P h_{ab} + \pi_{ab} = T_{cd}\, {h_a}^c\, {h_b}^d \,, \tag{2.8}$$

$$P = \frac{1}{3}\, g^{ab} \Pi_{ab} = \frac{1}{3}\, h^{ab} T_{ab} \,, \tag{2.9}$$

$$\pi_{ab} = \Pi_{ab} - P h_{ab} \,, \tag{2.10}$$

respectively (see [3, 6–8] for details). The heat flux density is purely spatial when $\dot{u}^a u_a = 0$ and the stress tensor is always purely spatial,

$$q_c u^c = 0 \tag{2.11}$$

and

$$\Pi_{ab}u^b = \pi_{ab}u^b = \Pi_{ab}u^a = \pi_{ab}u^a = 0\,, \qquad \pi^a{}_a = 0\,. \tag{2.12}$$

Consider a FLRW universe described by the line element

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega_{(2)}^2\right) \tag{2.13}$$

in comoving coordinates $(t, r, \vartheta, \varphi)$, where $d\Omega_{(2)}^2 \equiv d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the line element on the unit 2-sphere, $K$ is the curvature index, and $a(t)$ is the scale factor [16]. Because of spatial homogeneity and isotropy one has that $\phi = \phi(t)$, the heat flux density $q_a^{(\phi)} = 0$, and the anisotropic stresses $\pi_{ab}^{(\phi)} = 0$. This implies that also the shear viscosity vanishes; however, it still makes sense to consider bulk viscosity, which is isotropic. Thus, the only two non-vanishing contributions to (2.5) are

$$8\pi\rho^{(\phi)} = 8\pi T_{ab}^{(\phi)}u^a u^b = -\frac{\omega}{2\phi^2}\,\nabla^e\phi\nabla_e\phi + \frac{V}{2\phi} + \frac{1}{\phi}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{2.14}$$

$$8\pi P^{(\phi)} = \frac{8\pi}{3}\,h^{ab}T_{ab}^{(\phi)} = -\frac{\omega}{2\phi^2}\,\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} - \frac{1}{3\phi}\left(2\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi}\right). \tag{2.15}$$

Since $\phi = \phi(t)$, then

$$\nabla_a\phi = \delta_a^0\,\dot\phi\,, \qquad \nabla^a\phi = g^{0a}\,\dot\phi\,, \qquad 2X = -\nabla^e\phi\nabla_e\phi = \dot\phi^2\,. \tag{2.16}$$

Furthermore, one has that

$$\Box\phi = -\left(\ddot\phi + 3H\dot\phi\right), \tag{2.17}$$

$$\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi = \left(\nabla^0\phi\right)^2\left(\partial_a\partial_b\phi - \Gamma_{00}^0\partial_0\phi\right) = \ddot\phi\,\dot\phi^2 \tag{2.18}$$

where, for the latter, we have used the fact that $\Gamma_{00}^0 = 0$. From (2.14) and (2.15) we can therefore infer that

$$8\pi\rho^{(\phi)} = \frac{\omega\,\dot\phi^2}{2\phi^2} + \frac{V}{2\phi} - 3H\,\frac{\dot\phi}{\phi}\,, \tag{2.19}$$

$$8\pi P^{(\phi)} = \frac{\omega\,\dot\phi^2}{2\phi^2} - \frac{V}{2\phi} + \frac{\ddot\phi}{\phi} + 2H\,\frac{\dot\phi}{\phi}\,. \tag{2.20}$$

Moving on to the kinematic properties of the $\phi$-fluid [21], assuming a scalar field $\phi$ strictly monotonic in $t$, the vector field

$$v^a \equiv \frac{\nabla^a\phi}{\sqrt{2X}} = g^{a0}\,\mathrm{Sign}(\dot\phi) = (-\mathrm{Sign}(\dot\phi), \mathbf{0}) \tag{2.21}$$

is timelike, though it is not necessarily future-directed. Therefore, we define the 4-velocity of the comoving observer as

$$u^a = -\mathrm{Sign}(\dot\phi)\,v^a\,, \tag{2.22}$$

so that $u^a$ is a timelike, future-directed vector field with $u^a u_a = -1$.

The $3+1$ splitting of spacetime is obtained by identifying the Riemannian metric of the 3-space orthogonal to the 4-velocity of the comoving observers with

$$h_{ab} \equiv g_{ab} + u_a u_b = g_{ab} + v_a v_b \,. \tag{2.23}$$

${h_a}^b$ is the projection operator on this 3-space,

$$h_{ab} u^a = h_{ab} u^b = 0 \,, \tag{2.24}$$

$$h^a{}_b \, h^b{}_c = h^a{}_c \,, \qquad h^a{}_a = 3 \,, \tag{2.25}$$

and the effective fluid four-acceleration is $\dot{u}^a \equiv u^b \nabla_b u^a$, however in this case it is $\dot{u}^a = 0$.

The projection of the velocity gradient onto the 3-space of the comoving observers is

$$V_{ab} \equiv {h_a}^c \, {h_b}^d \, \nabla_d u_c \,; \tag{2.26}$$

it splits as

$$V_{ab} = \Theta_{ab} + \omega_{ab} = \sigma_{ab} + \frac{\Theta}{3} \, h_{ab} + \omega_{ab} \,, \tag{2.27}$$

with $\Theta_{ab} = V_{(ab)}$ the expansion tensor (the symmetric part of $V_{ab}$) with trace $\Theta \equiv \Theta^c{}_c = \nabla^c u_c$. The vorticity tensor $\omega_{ab} = V_{[ab]}$ is its antisymmetric part, which vanishes identically because the 4-velocity $u^a$ is derived from a gradient, while the trace-free shear tensor

$$\sigma_{ab} \equiv \Theta_{ab} - \frac{\Theta}{3} \, h_{ab} \tag{2.28}$$

also vanishes because of spatial homogeneity and isotropy. Furthermore, for FLRW geometries the expansion scalar reduces to $\Theta = 3H$.

## 3 Eckart's thermodynamics of scalar-tensor gravity

In Eckart's thermodynamics [5, 22–24], the quantities describing dissipation include the viscous pressure $P_{\text{vis}}$, the heat current density $q^c$, and the anisotropic stresses $\pi_{ab}$. They are related to the expansion $\Theta$, temperature $\mathcal{T}$, and shear tensor $\sigma_{ab}$ by the constitutive equations [5]

$$P_{\text{vis}} = -\zeta \, \Theta \,, \tag{3.1}$$

$$q_a^{(\phi)} = -\mathcal{K} \left( h_{ab} \nabla^b \mathcal{T} + \mathcal{T} \dot{u}_a \right) \,, \tag{3.2}$$

$$\pi_{ab}^{(\phi)} = -2\eta \, \sigma_{ab} \,, \tag{3.3}$$

where $\zeta$ is the bulk viscosity, $\mathcal{K}$ is the thermal conductivity, and $\eta$ is the shear viscosity. Comparing the expressions of the acceleration $\dot{u}_a$ and the heat flux density $q_a^{(\phi)}$ in scalar-tensor gravity in [3] leads to

$$q_a^{(\phi)} = -\frac{\sqrt{-\nabla^c \phi \nabla_c \phi}}{8\pi\phi} \, \dot{u}_a \,. \tag{3.4}$$

In the previous works [6–8], the comparison between Eckart's generalized Fourier law and the heat flux density contained in the effective scalar field fluid led to the identification of thermal conductivity times temperature with

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi} \tag{3.5}$$

in the general scalar-tensor theory (2.1).

# 4 Eckart's thermodynamics of scalar-tensor gravity in FLRW

Going back to the effective pressure (2.20) of the $\phi$-fluid, one can now use the equation of motion (2.3) of the Brans-Dicke-like scalar field to eliminate $\ddot{\phi}$. We use the Hubble function $H \equiv \dot{a}/a$ and denote differentiation with respect to the comoving time $t$ with an overdot. Substituting

$$\frac{\ddot{\phi}}{\phi} = -\frac{3H\dot{\phi}}{\phi} - \frac{8\pi T^{(\mathrm{m})}}{(2\omega+3)\phi^2} + \frac{2V - \phi V_{,\phi}}{(2\omega+3)\phi} - \frac{\dot{\phi}^2\,\omega_{,\phi}}{(2\omega+3)\phi} \tag{4.1}$$

into Eq. (2.20) yields

$$\begin{aligned} P^{(\phi)} &= P_{\mathrm{non-visc}} + P_{\mathrm{visc}} \\ &= \frac{1}{8\pi}\left[\frac{(2\omega+3)\omega - 2\phi\,\omega_{,\phi}}{2(2\omega+3)}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{4V - 2\phi V_{,\phi} - (2\omega+3)V}{2(2\omega+3)\phi} - \frac{8\pi T^{(\mathrm{m})}}{(2\omega+3)\phi^2}\right] \\ &\quad -\frac{H\dot{\phi}}{8\pi\phi}\,. \end{aligned} \tag{4.2}$$

According to Eckart's constitutive relation (3.1), the viscous pressure is

$$P_{\mathrm{visc}} = -3\zeta H, \tag{4.3}$$

which leads to the straightforward identification of the bulk viscosity coefficient

$$\zeta = \frac{\dot{\phi}}{24\pi\phi}\,. \tag{4.4}$$

One wonders whether the splitting of the pressure into a non-viscous part $P_{\mathrm{non-visc}}$ and a viscous part $-3\zeta H$ could be performed differently, leading to a different result for $P_{\mathrm{visc}}$, in which we are interested here. We argue that the identification performed is the only natural one.

Consider, for the sake of simplicity, vacuum scalar-tensor gravity in a spatially flat ($K = 0$) FLRW universe. The variables appearing in the field equations equations are the scale factor $a(t)$ and the scalar field $\phi(t)$. While the acceleration equation for $a(t)$ and the Klein-Gordon-like equation for $\phi(t)$ are of second order, when $K = 0$ the scale factor only appears in the combination $H(t) \equiv \dot{a}/a$. The dynamical variables are, therefore, $H(t)$ and $\phi(t)$ and the phase space reduces to the three-dimensional space $\left(H, \phi, \dot{\phi}\right)$ [25]. Furthermore, the orbits of the solutions are forced to lie on a two-dimensional submanifold of this space identified by the first order Hamiltonian constraint, which is analogous to an energy constraint in point particle dynamics [25]. The right-hand side of Eq. (4.2) contains only the phase space variables $H, \phi$, and $\dot{\phi}$ (in addition to the functions $V(\phi), \omega(\phi)$), while only the last term contains $H = \Theta/3$. It is natural to identify the viscous pressure with this term only, and any attempt to split $P^{(\phi)}$ differently into viscous and non-viscous parts would be contrived.

In a spatially curved ($K \neq 0$) FLRW universe, one cannot eliminate $a(t)$ in terms of $H(t)$ and the phase space variables are $\left(a, \dot{a}, \phi, \dot{\phi}\right)$; again, the Hamiltonian constraint forces the orbits of the solutions to lie in a 3-dimensional subspace but the previous argument still applies because, again, in the right-hand side of Eq. (4.2) only the last term depends on the scale factor $a(t)$ and the splitting performed is the only natural one (doing otherwise would require to add and subtract terms containing the scale factor or its derivatives, which would be completely arbitrary and unmotivated).

An immediate consequence of Eq. (4.4) is that GR, obtained for $\phi =$ const., corresponds to zero viscosity $\zeta$ and can still be regarded as a state of equilibrium. Increasing $\phi$ corresponds to decreasing strength of gravity $G_{\text{eff}} = 1/\phi$ and to increasing bulk viscosity coefficient, going away from the GR equilibrium state. *Vice-versa*, decreasing $\phi$ (with increasing gravitational coupling) leads to the GR equilibrium state and to the decrease of bulk viscosity dissipation.

Since the heat flux density $q_a^{(\phi)}$ vanishes identically in FLRW universes by virtue of spatial isotropy, Eq. (3.5) and the concept of effective temperature of scalar-tensor gravity lose meaning. However, Eq. (3.5) is deduced in the general theory without reference to particular geometries and one may want to regard this temperature as a general concept holding even in FLRW spacetimes. The possibility of considering the heat flux as a timelike vector aligned with the four-velocity of comoving observers would preserve the spatial homogeneity and isotropy of FLRW spaces. In this case, Eckart's Eq. (3.4) would hold only for a timelike four-acceleration of the fluid, which is the case for FLRW spacetimes sourced by a perfect fluid. However, dealing with a timelike heat current density would require an extension of the formalism used in [3, 4] which is beyond the scope of this work.

Admittedly, ours is a rather generic argument but, if one assumes Eq. (3.5) to hold, then it reduces to

$$\mathcal{K}\mathcal{T} = \frac{|\dot{\phi}|}{8\pi\phi} \tag{4.5}$$

in FLRW universes, and then the bulk viscosity coefficient $\zeta = \mathcal{K}\mathcal{T}/3$ is linear in the temperature and vanishes in the GR equilibrium state together with it.

## 5 Exact FLRW solutions of scalar-tensor gravity

In order to test the thermodynamical formalism detailed in the previous sections, we now turn to studying some well-known exact FLRW solutions of scalar-tensor gravity [11], with particular attention to the simpler $K = 0$ case. The analytical solutions chosen, which are specifically solutions of Jordan frame Brans-Dicke theory, exhibit interesting features for the purposes of cosmology, once particular forms of the cosmic matter are chosen. The latter is described by a perfect fluid with stress-energy tensor and equation of state

$$T_{ab}^{(\text{m})} = \left(P^{(\text{m})} + \rho^{(\text{m})}\right) u_a u_b + P^{(\text{m})} g_{ab} \,, \tag{5.1}$$

$$P^{(\text{m})} = (\gamma - 1)\rho^{(\text{m})}, \qquad \gamma = \text{const.} \tag{5.2}$$

Most of the solutions in the following have a power-law behaviour: such solutions play a role analogous to that of the inflationary de Sitter attractor in GR.

### 5.1 O'Hanlon and Tupper solution

This solution [26] corresponds to vacuum, $V(\phi)=0$, and $\omega > -3/2$, $\omega \neq 0, -4/3$:

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{q_\pm} , \tag{5.3}$$

$$\phi(t) = \phi_0 \left(\frac{t}{t_0}\right)^{s_\pm} , \tag{5.4}$$

where

$$q_\pm = \frac{\omega}{3(\omega+1) \mp \sqrt{3(2\omega+3)}} , \tag{5.5}$$

$$s_\pm = \frac{1 \mp \sqrt{3(2\omega+3)}}{3\omega+4} , \tag{5.6}$$

satisfying $3q+s=1$. The two sets of exponents with upper or lower sign correspond to the so-called fast and slow solutions, respectively, a nomenclature tied to the behaviour of the Brans-Dicke scalar at early times [11]. This solution is endowed with a Big Bang singularity for $t \to 0$ and its limit for $\omega \to +\infty$, namely $a(t) \propto t^{1/3}$ and $\phi =$ const., does not reproduce the corresponding GR solution, which is Minkowski space (this is a well-known anomaly of the $\omega \to \infty$ limit of Brans-Dicke theory [11], which can be approached also with the effective fluid formulation of this theory [27]).

The behaviours of the scale factor and the scalar field yield

$$\frac{\dot{\phi}}{\phi} = \frac{s_\pm}{t} , \tag{5.7}$$

$$\frac{\dot{a}}{a} = \frac{q_\pm}{t} , \tag{5.8}$$

hence the viscous pressure (4.3)

$$P_{\text{visc}} = -\frac{q_\pm \, s_\pm}{8\pi t^2} \tag{5.9}$$

and the bulk viscosity coefficient

$$\zeta = \frac{s_\pm}{24\pi \, t} \tag{5.10}$$

vanish at late times, recovering the GR limit. Since $T^{(\mathrm{m})}=0$, the total pressure is

$$P^{(\phi)} = \frac{s_\pm}{8\pi t^2}\left(\frac{\omega \, s_\pm}{2} - q_\pm\right) \tag{5.11}$$

and the ratio $P_{\text{visc}}/P_{\text{non-visc}}$ is time-independent,

$$\frac{P_{\text{visc}}}{P_{\text{non-visc}}} = -\frac{2q_\pm}{\omega \, s_\pm} . \tag{5.12}$$

The product of the effective temperature and the thermal conductivity

$$\mathcal{K}\mathcal{T} = \frac{|s_\pm|}{8\pi\, t} \tag{5.13}$$

vanishes for $t \to +\infty$ similarly to the bulk viscosity coefficient, recovering the GR limit. The limit $\mathcal{K}\mathcal{T} \to +\infty$ is obtained at early times $t \to 0^+$, in accordance with the existence of a Big Bang for this solution and the hypothesis that gravity is "hot" near spacetime singularities [6, 7].

### 5.2 Brans-Dicke dust solution

This solution [17] corresponds to a pressureless dust fluid ($\gamma = 1$) and a matter-dominated universe characterised by $V(\phi) = 0$ and $\omega \neq -4/3$. The scale factor, scalar field, and matter energy density behave as

$$a(t) = a_0\, t^q\,, \tag{5.14}$$

$$\phi(t) = \phi_0\, t^s\,, \tag{5.15}$$

$$\rho^{(\mathrm{m})} = \rho_0\, t^r\,, \tag{5.16}$$

where $\rho_0 = C/a_0^3$, $C$ is an integration constant related to initial conditions, and

$$q = \frac{2(\omega+1)}{3\omega+4}\,, \tag{5.17}$$

$$s = \frac{2}{3\omega+4}\,, \tag{5.18}$$

$$r = -3q\,, \tag{5.19}$$

satisfying $3q + s = 2$.

In order to find the expressions needed for writing Eq. (4.2), we use the fact that a pressureless fluid has $T^{(\mathrm{m})} = -\rho^{(\mathrm{m})}$. The scale factor and scalar field of this solution yield

$$\frac{\dot{\phi}}{\phi} = \frac{s}{t} = \frac{2}{(3\omega+4)t}\,, \tag{5.20}$$

$$H = \frac{q}{t} = \frac{2(\omega+1)}{(3\omega+4)t}\,, \tag{5.21}$$

while the viscous pressure is

$$P_{\mathrm{visc}} = -\frac{H\dot{\phi}}{8\pi\phi} = -\frac{\omega+1}{2\pi(3\omega+4)^2t^2}\,. \tag{5.22}$$

The bulk viscosity coefficient is thus

$$\zeta = \frac{1}{12\pi(3\omega+4)t} \tag{5.23}$$

and it vanishes at late times, meaning that this cosmology approaches the GR equilibrium state.

The full expression of the effective $\phi$-fluid pressure reads

$$P^{(\phi)} = \frac{1}{8\pi}\left[\frac{\omega}{2}\left(\frac{2}{(3\omega+4)t}\right)^2 + \frac{8\pi\rho_0\, t^r}{(2\omega+3)\,\phi_0^2\, t^{2s}}\right] - \frac{\omega+1}{2\pi(3\omega+4)^2 t^2}\,. \tag{5.24}$$

The ratio $P_{\text{visc}}/P_{\text{non-visc}}$ goes to zero as $t \to +\infty$ if $s < 0$, to $-1/2$ if $s = 0$, and to $-1$ if $s > 0$. An alternative way to see this limit uses the relationship $-r + s = 2$ between the exponents, which yields

$$\frac{P_{\text{visc}}}{P_{\text{non-visc}}} \propto -\frac{1}{1+t^{r-2s+2}} = -\frac{1}{1+t^{-s}}\,. \tag{5.25}$$

The ratio tends to $-1$ as $t \to +\infty$ if $s > 0$; this choice of sign for $s$ is supported by the observational constraints on the Brans-Dicke coupling $\omega$, which provide a lower bound $\omega \gtrsim 10^3$ (for recent results, see for example [28]).

As for the temperature of scalar-tensor gravity, if one assumes Eq. (3.5) to hold even in FLRW spacetimes, one has

$$\mathcal{K}\mathcal{T} = \frac{1}{4\pi\,|3\omega+4|\,t}\,. \tag{5.26}$$

Since $\omega \neq -4/3$, $\mathcal{K}\mathcal{T} \to \infty$ at the Big Bang $t \to 0^+$, which agrees again with the hypothesis that gravity is "hot" near spacetime singularities [6,7]. The GR equilibrium state $\mathcal{K}\mathcal{T} \to 0$ is approached at late times $t \to +\infty$.

## 5.3 Nariai solution

The power-law Nariai solution [29, 30] describes a $K = 0$ FLRW universe filled by a perfect fluid, $V(\phi) = 0$, and $\omega \neq -4[3\gamma(2-\gamma)]^{-1} < 0$, given by

$$a(t) = a_0(1+\delta t)^q\,, \tag{5.27}$$

$$\phi(t) = \phi_0(1+\delta t)^s\,, \tag{5.28}$$

$$\rho^{(\mathrm{m})}(t) = \rho_0(1+\delta t)^r\,, \tag{5.29}$$

where

$$q = \frac{2[\omega(2-\gamma)+1]}{3\omega\gamma(2-\gamma)+4}\,, \tag{5.30}$$

$$s = \frac{2(4-3\gamma)}{3\omega\gamma(2-\gamma)+4}\,, \tag{5.31}$$

$$r = -3\gamma q\,. \tag{5.32}$$

Using $\alpha \equiv \dfrac{2(4-3\gamma)}{(2\omega+3)(2-\gamma)+3\gamma-4}$ and $A \equiv \dfrac{2\omega+3}{12}$, we write

$$\delta = \left(\frac{\alpha+3\gamma}{2}\right)\frac{8\pi\rho_0}{3\phi_0\left[(1+\alpha/2)^2 - A\alpha^2\right]}\,. \tag{5.33}$$

The Nariai solution contains the Brans-Dicke dust solution already discussed as a special case. Other special cases of interest include a radiative fluid and the cosmological constant.

### 5.3.1 Radiative fluid

This solution corresponds to $\gamma = 4/3$, $P^{(\mathrm{m})} = \rho^{(\mathrm{m})}/3$, $\alpha = 0$, and

$$a(t) = a_0\sqrt{1+\delta t}\,, \tag{5.34}$$

$$\phi(t) = \phi_0 = \text{const.}\,, \tag{5.35}$$

$$\rho^{(\mathrm{m})}(t) = \frac{\rho_0}{(1+\delta t)^2}\,, \tag{5.36}$$

with $\delta = \left(\dfrac{32\pi\rho_0}{3\phi_0}\right)^{1/2}$. The constant scalar field translates into $P_{\mathrm{visc}} = 0$ and $\mathcal{K}\mathcal{T} = 0$ at all times, which reproduces the GR equilibrium state. Moreover, $P_{\mathrm{non-visc}}$ also vanishes, since the first three terms in Eq. (4.2) are zero.

### 5.3.2 Cosmological constant

In this case $\gamma = 0$, $P^{(\mathrm{m})} = -\rho^{(\mathrm{m})}$, $\alpha = \dfrac{4}{2\omega+1}$, and

$$a(t) = a_0(1+\delta t)^{\omega+1/2}\,, \tag{5.37}$$

$$\phi(t) = \phi_0(1+\delta t)^2\,, \tag{5.38}$$

$$\delta = \left[\frac{32\pi\rho_0}{\phi_0}\,\frac{1}{(6\omega+5)(2\omega+3)}\right]^{1/2}\,, \tag{5.39}$$

while $\rho^{(\mathrm{m})}(t) = \rho_0$ due to Eq. (5.29). This is not the only solution describing a universe driven by a cosmological constant in scalar-tensor cosmology but it is an attractor in phase space, which makes it relevant for the extended inflationary scenario [31]. For this solution it is

$$\frac{\dot{\phi}}{\phi} = \frac{2\delta}{1+\delta t}\,, \tag{5.40}$$

$$H = \frac{\delta(\omega+1/2)}{1+\delta t}\,, \tag{5.41}$$

while the trace of the matter stress-energy tensor is $T^{(\mathrm{m})} = -4\rho^{(\mathrm{m})}$. The viscous pressure reads

$$P_{\mathrm{visc}} = -\frac{\delta^2(\omega+1/2)}{4\pi(1+\delta t)^2}\,, \tag{5.42}$$

yielding the bulk viscosity coefficient

$$\zeta = \frac{\delta}{12\pi(1+\delta t)} \tag{5.43}$$

which tends to the GR equilibrium state at late times. The total pressure is

$$P^{(\phi)} = \frac{1}{8\pi}\left[\frac{\omega}{2}\left(\frac{2\delta}{1+\delta t}\right)^2 + \frac{32\pi\rho_0}{(2\omega+3)\,\phi_0^2\,(1+\delta t)^4}\right] - \frac{\delta^2(\omega+1/2)}{4\pi(1+\delta t)^2}\,, \tag{5.44}$$

while the ratio between viscous and non-viscous pressures is

$$\frac{P_{\text{visc}}}{P_{\text{non-visc}}} = -\frac{\delta^2(2\omega+1)\,(1+\delta\,t)^2}{\frac{32\pi\rho_0}{(2\omega+3)\phi_0^2} + 2\omega\,\delta^2\,(1+\delta\,t)^2} \to -\frac{(2\omega+1)}{2\omega} \quad \text{as } \; t\to+\infty\,. \tag{5.45}$$

The product of effective temperature and thermal conductivity

$$\mathcal{KT} = \frac{|\delta|}{4\pi(1+\delta t)} \tag{5.46}$$

vanishes as $t \to +\infty$, recovering the GR equilibrium state.

### 5.4 Big Rip with conformally coupled scalar field

About twenty years ago, inspired by the first observational constraints from cosmological probes on the dark energy equation of state (which approached the boundary $w \equiv P/\rho = -1$), the possibility of a phantom equation of state parameter $w < -1$ was first explored. Such values of $w$ cannot be explained by Einstein gravity coupled minimally with a scalar field of positive energy density. A regime with $w < -1$ is associated with $\dot{H} > 0$ (superacceleration) [32], while a dark energy fluid that could exhibit superacceleration is named phantom energy or superquintessence. The phantom energy density would grow in time instead of redshifting and would quickly come to dominate all other forms of energy, leading to a scale factor diverging in a finite amount of time, reaching a peculiar end of the universe (Big Rip) in which gravitationally bound structures would be ripped apart [35, 36]. The Big Rip is not unavoidable in models with a time-dependent equation of state and could occur or not, depending on the specific model adopted.

The current observational bounds on the dark energy equation of state are more precise thanks to surveys such as Planck and $w < -1$ is no longer favoured. However, the value of $w$ still hovers around $-1$ and a Big Rip is not completely ruled out, although the closer $w$ is to $-1$, the further in the future the Big Rip would be. For example, combining Planck data with data coming from supernovae, Baryon Acoustic Oscillations and other datasets, one has $w = -1.028 \pm 0.031$ [33]. Models that could exhibit superacceleration have been studied in various contexts, including Brans-Dicke-like fields in scalar-tensor gravity. This makes such models interesting as applications of our thermodynamical formalism to analytical solutions of scalar-tensor cosmology. In the following, we consider one of the simplest superquintessence models consisting of a single scalar field $\phi$ nonminimally coupled to the Ricci curvature, with action

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{1}{8\pi} - \xi\phi^2\right)\frac{\mathcal{R}}{2} - \frac{1}{2}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(\text{m})}\,, \tag{5.47}$$

where $\xi$ is a dimensionless coupling constant. We can rewrite this action in the general scalar-tensor form (2.1) with scalar $\psi$

$$S_{\text{ST}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\psi\mathcal{R} - \frac{\omega(\psi)}{\psi}\,\nabla^c\psi\nabla_c\psi - V(\psi)\right] + S^{(\text{m})}\,, \tag{5.48}$$

where the scalar fields $\phi$ and $\psi$ are related by

$$\psi = 1 - 8\pi\xi\phi^2 \,. \tag{5.49}$$

Since we now start from the different action (5.47), we consider an expression for the effective pressure which is different from Eq. (4.2) used in the previous sections. The action (5.47) can be explicitly recast as a scalar-tensor action with a variable Brans-Dicke parameter. From [34], the expression for the pressure of the effective fluid equivalent to the nonminimally coupled scalar (in a flat FLRW universe) reads

$$P^{(\phi)} = \frac{\dot{\phi}^2}{2} - V(\phi) - \xi[4H\phi\dot{\phi} + 2\dot{\phi}^2 + 2\phi\ddot{\phi} + (2\dot{H} + 3H^2)\phi^2] \,. \tag{5.50}$$

As before, we use the equation of motion

$$\ddot{\phi} = -3H\dot{\phi} - \xi\mathcal{R}\phi - V_{,\phi} \tag{5.51}$$

to substitute for $\ddot{\phi}$, obtaining

$$P^{(\phi)} = \frac{\dot{\phi}^2}{2} - V(\phi) - \xi\left[-2H\phi\dot{\phi} + 2\dot{\phi}^2 - 2\xi\mathcal{R}\phi^2 - 2\phi V_{,\phi} + (2\dot{H} + 3H^2)\phi^2\right] \,. \tag{5.52}$$

Further use of the expression $\mathcal{R} = 6\left(\dot{H} + 2H^2\right)$ of the Ricci scalar yields

$$P^{(\phi)} = \left(\frac{1}{2} - 2\xi\right)\dot{\phi}^2 + 2\xi\phi V_{,\phi} - V + \xi\phi^2\left[2\left(6\xi - 1\right)\dot{H} + 3\left(8\xi - 1\right)H^2\right] + 2\xi H\phi\dot{\phi} \,. \tag{5.53}$$

Following [34], we consider a simple toy model with conformal coupling $\xi = 1/6$, potential $V(\phi) = \dfrac{m^2\phi^2}{2} + \lambda\phi^4$, and $\lambda < 0$.[1] Since $H(t)$ and $\phi(t)$ grow very quickly in the superacceleration regime, we consider solutions for the scale factor and scalar field that assume large values of these quantities. Such solutions have the pole-like form

$$a(t) = \frac{a_*}{|t - t_0|^{\alpha_\pm}} \tag{5.54}$$

and

$$\phi(t) = \frac{\phi_*}{|t - t_0|^{\beta_\pm}} \,, \tag{5.55}$$

where we restrict to $t < t_0$ and where $\alpha_\pm, \beta_\pm > 0$, while $t_0$, $a_*$, and $\phi_*$ are constants. If $\mu = 4\pi m^2/3$, then

$$\alpha_\pm = \frac{\pm\sqrt{-\lambda(2\mu + \lambda)} - \mu - \lambda}{\mu + 4\lambda} \,, \tag{5.56}$$

$$\beta_\pm = 1 \,, \tag{5.57}$$

$$\phi_*^\pm = \pm\frac{(1 + \alpha_\pm)}{\sqrt{-2\lambda}} \,, \tag{5.58}$$

[1] A negative potential yields a negative energy density for a minimally coupled scalar field but, since here $\xi \neq 0$, the positivity of $\rho^{(\phi)}$ is not spoiled. Negative potentials are common in supergravity.

leading to

$$H = \frac{\alpha_{\pm}}{t_0 - t}\,, \tag{5.59}$$

$$\dot{\phi} = \frac{\phi_*}{(t_0 - t)^2}\,, \tag{5.60}$$

which we substitute in Eq. (5.53). The only term containing $\Theta = 3H$ is the third one on the right-hand side of Eq. (5.53), giving the viscous pressure

$$P_{\rm visc} = 2\xi H \phi \dot{\phi} = \frac{\alpha_{\pm}\,\phi_*^2}{3\,(t_0 - t)^4} \tag{5.61}$$

and the bulk viscosity coefficient

$$\zeta = -\frac{2\xi\phi\dot{\phi}}{3} = -\frac{\phi_*^2}{9\,(t_0 - t)^3}\,. \tag{5.62}$$

An expanding universe ends its existence in the Big Rip as $t \to t_0^-$, where $\zeta$ diverges. This behaviour is interesting because, while gravity is "hot" near spacetime singularities, it is "cooled" by expansion and it is not *a priori* clear what to expect at a Big Rip singularity in which the universe expands explosively.

Substituting the scale factor and scalar field in the total pressure (5.53) yields

$$P^{(\phi)} = -\frac{m^2\phi_*^2}{6\,(t_0 - t)^2} + \frac{\phi_*^2}{3\,(t_0 - t)^4}\left[\frac{(\alpha_{\pm} + 1)^2}{2} + \lambda\phi_*^2\right]. \tag{5.63}$$

The ratio $P_{\rm visc}/P_{\rm non-visc}$ has the $t \to t_0$ limit

$$\frac{P_{\rm visc}}{P_{\rm non-visc}} = \frac{\alpha_{\pm}\phi_*^2}{-\alpha_{\pm}\phi_*^2 - \frac{m^2\phi_*^2}{2}\,(t_0 - t)^2 + \phi_*^2\left[\frac{(\alpha_{\pm}+1)^2}{2} + \lambda\phi_*^2\right]} \to \frac{2\alpha_{\pm}}{\alpha_{\pm}^2 + 1 + 2\lambda\phi_*^2} \qquad \text{as } t \to t_0^-\,. \tag{5.64}$$

Considering now the product of thermal conductivity and effective temperature, the 4-velocity and the 4-acceleration of the effective fluid (see Sec. 2) have the same form for $\phi$ and $\psi$ in the actions (5.47) and (2.1). Therefore, only the factor in front of $\dot{u}_a$ in Eq. (3.4) is different if we start from the action (5.47), and the expression for the heat flux density now reads

$$q_a^{(\phi)} = -\frac{2\,|\xi\phi|\,\sqrt{-\nabla^e\phi\nabla_e\phi}}{1 - 8\pi\xi\phi^2}\,\dot{u}_a\,, \tag{5.65}$$

so that

$$\mathcal{K}\mathcal{T} = \frac{2\,|\xi\phi|\,\sqrt{-\nabla^e\phi\nabla_e\phi}}{1 - 8\pi\xi\phi^2}\,. \tag{5.66}$$

Substituting the solution $\phi(t)$ yields

$$\mathcal{KT} = \frac{\phi_*^2}{3\,(t_0-t)^3}\left[1-\frac{4\pi\phi_*^2}{3\,(t_0-t)^2}\right]^{-1} = \frac{\phi_*^2}{(t_0-t)\left[3\,(t_0-t)^2-4\pi\phi_*^2\,(t_0-t)\right]}\,. \tag{5.67}$$

This expression diverges as $t \to t_0^-$, recovering the expected result for the approach to a singularity. $\mathcal{KT}$ diverges also at an earlier time when $\phi(t)$ approaches the critical value $\phi_c \equiv (8\pi\xi)^{-1/2}$, which is always present for $\xi > 0$ (in our case, $\phi_c = (4\pi/3)^{-1/2}$). At this critical value, the effective gravitational coupling

$$G_{\text{eff}} = \frac{G}{1-8\pi\xi\phi^2} \tag{5.68}$$

diverges and its sign changes for $|\phi| > \phi_c$, together with the sign of $\mathcal{KT}$. Indeed, for $\mathcal{KT}$ to make sense for nonminimally coupled scalar fields, it must be $|\phi| < \phi_c$, but this limitation coincides with the familiar one requiring that $G_{\text{eff}}$ be positive [11].

# 6 Conclusions

We have considered the cosmological consequences of the "thermodynamics of scalar-tensor gravity" obtained by applying Eckart's first-order thermodynamics to FLRW universes in [6–8]. In the general theory the effective stress-energy tensor of the Brans-Dicke-like scalar field $\phi$ exhibits a heat current, anisotropic stresses, shear viscosity, and bulk viscosity. In unperturbed FLRW universes the heat flux, anisotropic stressses, and shear must necessarily vanish to respect spatial isotropy, but isotropic bulk viscosity is possible. Using the only surviving constitutive relation of Eckart's theory, we identify the effective bulk viscosity coefficient in FLRW universes that solve the field equations of scalar-tensor gravity. If, in addition, the effective temperature and thermal conductivity found in the general theory still apply to FLRW universes, as seems reasonable, these expressions are fully consistent with the bulk viscosity found and with the approach to a GR equilibrium state characterized by zero "temperature of gravity".

We have then tested our results on analytical solutions of scalar-tensor cosmology. This procedure supports the previous ideas, formulated in [6, 7], that gravity approaches the GR equilibrium state of zero temperature with expansion, while it departs from it near spacetime singularities. A peculiar situation arises with Big Rip singularities in which the universe expands explosively in a pole-like singularity: here (extreme) expansion occurs simultaneously with a spacetime singularity. By analyzing an exact solution in a conformally coupled scalar field model [34], it is found that gravity still departs from the GR equilibrium state at the Big Rip.

Much needs to be done to validate and develop, or else to reject, the program of scalar-tensor thermodynamics. Future work will examine more analytical solutions of scalar-tensor cosmology, further attempting to falsify the above-mentioned ideas, and will generalize the analysis of the thermodynamics of scalar-tensor gravity to anisotropic Bianchi models, perturbed FLRW universes, and more general Horndeski cosmologies.

# Acknowledgments

A.G. is supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions (grant agreement No. 895648–CosmoDEC). The work of

A.G has also been carried out in the framework of the activities of the Italian National Group of Mathematical Physics [Gruppo Nazionale per la Fisica Matematica (GNFM), Istituto Nazionale di Alta Matematica (INdAM)]. V.F. is supported by the Natural Sciences & Engineering Research Council of Canada (Grant 2016-03803). S.G. thanks Jean-Luc Lehners at AEI Potsdam for hospitality.

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