Complex Manifolds 2022; 9:138–169 DE GRUYTER

Research Article **Open Access**

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On the characterization of rational homotopy types and Chern classes of closed almost complex manifolds

https://doi.org/10.1515/coma-2021-0133
Received September 19, 2021; accepted April 21, 2022

Abstract: We give an exposition of Sullivan's theorem on realizing rational homotopy types by closed smooth manifolds, including a discussion of the necessary rational homotopy and surgery theory, adapted to the realization problem for almost complex manifolds: namely, we give a characterization of the possible simply connected rational homotopy types, along with a choice of rational Chern classes and fundamental class, realized by simply connected closed almost complex manifolds in real dimensions six and greater. As a consequence, beyond demonstrating that rational homotopy types of closed almost complex manifolds are plenty, we observe that the realizability of a simply connected rational homotopy type by a simply connected closed almost complex manifold depends only on its cohomology ring. We conclude with some computations and examples.

Keywords: almost complex manifolds, rational homotopy theory, Chern classes

MSC: 32Q60, 55P62, 57N65, 57R65

1 The homotopy types of closed manifolds: background and history

In the 1930's, Hassler Whitney's pioneering work on manifolds, bundles, and cohomology marked the birth of differential topology [18]. In the same article giving the modern definition of a smooth manifold [49], Whitney showed how every manifold can be embedded in Euclidean space. These embeddings naturally equip manifolds with normal bundles, and Whitney early on saw the need for a general theory of vector bundles beyond the tangent bundle [48]. His investigation of the obstructions to linearly independent sections of vector bundles, a problem concurrently considered on the tangent bundle by Eduard Stiefel in his thesis [34], initiated the study of characteristic classes.

It was known to Whitney that all vector bundles were pulled back from Grassmannians with their tautological bundles. Lev S. Pontryagin [28] studied the homology of these universal spaces, identifying the generators of the integral lattice in rational (co)homology now known as Pontryagin classes. Shiing-Shen Chern conducted a similar study on complex manifolds [5], defining what became known as the Chern classes of the tangent bundle, using the Schubert cell decomposition of the complex Grassmannians; later Wu Wenjun [52] would generalize this notion in his thesis to arbitrary complex vector bundles .

Pontryagin observed that by considering maps of spheres into the one-point compactification of the universal trivial bundle over a point, one can identify the homotopy groups of spheres with equivalence classes of stably framed manifolds up to what is now known as framed cobordism [27]. Later, René Thom [46] built on

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this construction and applied it to all closed smooth manifolds, developing and employing transversality arguments to classify smooth manifolds up to cobordism by calculating the homotopy groups of the one-point compactification of the universal bundle over the Grassmannian.

In the late 1950's and early 1960's, Michel Kervaire and John Milnor introduced surgery, a procedure of removing from a manifold embedded spheres with trivial normal bundle, and used it to determine the finite abelian groups of smooth structures on homotopy spheres [15], in terms of Bernoulli numbers and homotopy groups of spheres, in dimensions 5 and above. Andrew Wallace [47] independently introduced surgery in the United Kingdom under the name "constructive cobordisms": applying a surgery to a manifold produces a cobordism to the resulting manifold, and any cobordism can be realized by a finite number of surgeries.

After Stephen Smale proved the generalized Poincaré conjecture in dimensions five and higher by establishing the h–cobordism theorem [32], the work of Kervaire and Milnor could be formulated as classifying the smooth structures on piecewise–linear spheres S^n, for any $n \geq 1$. Extending this work, Sergei Novikov in the Soviet Union addressed the problem of classifying smooth structures on simply connected manifolds in dimensions 5 and greater, in terms of vector bundles over their homotopy types and the homotopy groups of the one-point compactification of their normal bundles when embedded in a high–dimensional Euclidean space [24]. William Browder [3] in the United States independently did the same, along with characterizing in similar terms as [15] and [24] which homotopy types were realized by closed smooth manifolds in dimensions 5 and greater. This made use of Spivak's normal spherical fibration [33] characterizing Poincaré duality spaces, a notion earlier identified by Browder in his study of finite complexes admitting a continuous multiplication with unit. Motivated by Hilbert's 5^{th} problem on characterizing Lie groups as locally Euclidean locally compact groups [9], Browder asked if these complexes with a unital multiplication were realized by smooth manifolds.

Dennis Sullivan in his thesis [41] reformulated the stories of Kervaire–Milnor, Novikov, and Browder without choosing the normal bundle, instead classifying all the simply connected closed manifolds, piecewise–linear or smooth, in a homotopy type via obstruction theory. The obstructions in the piecewise–linear theory lay in a calculable homotopy type with fourfold periodic homotopy groups $0, \mathbb{Z}_2, 0, \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}, \ldots$. The homotopy groups in the smooth theory are still unknown, though the theory itself can be reduced to stable homotopy using the Adams conjecture, provable using the Frobenius automorphism from algebraic geometry (a possibility first voiced by Daniel Quillen [29]). Understanding these results and the utility of localizing homotopy types motivated Sullivan's 1970 "MIT notes" on localization, periodicity, and Galois symmetry [42] (see also [43]).

Upon tensoring homotopy types and maps by the rationals, the piecewise–linear and smooth obstruction theories become equivalent. The homotopy theory of rationalized simply connected spaces was shown by Quillen to be encoded algebraically in differential graded Lie algebras in his seminal "Rational Homotopy Theory" [30]. Motivated by this theory, and influenced by Whitney's treatment of differential forms on arbitrary complexes [51], Sullivan described a theory of computable algebraic models for rational homotopy types in terms of differential graded algebras of differential forms in his "Infinitesimal Computations in Topology" [44].

Here, we will give an exposition, with an emphasis on the basic surgery-theoretic techniques, of the details and necessary theory to understand and prove a theorem formulated in [44], and accompanied by a sketch proof as an illustration of the developed techniques, on the realization of simply connected rational homotopy types by closed smooth manifolds [44, Theorem 13.2]. However, our account has the adapatation of this result to the realization by closed *almost complex* manifolds as its ultimate focus, whose formulation and proof, using the enclosed tools, was left to the reader [44, Remark p.322]. We give a characterization of the possible simply connected rational homotopy types realized by closed simply connected almost complex manifolds in dimensions six and greater; in the process we characterize those realized by closed simply connected stably almost complex in dimensions five and greater.

In the last two sections we will observe some consequences of the main Theorem 2.4 and carry out some computations; beyond demonstrating that rational homotopy types of almost complex manifolds are plenty, we observe that the realizability of a simply connected rational homotopy type by a simply connected almost complex manifold depends only on the cohomology ring. We contrast this with the case of rational homotopy types realized by compact complex manifolds satisfying the $\partial\bar{\partial}$–lemma (such as Kähler manifolds), where all the higher multiplications in the associated C_∞ structure on the cohomology necessarily vanish; in this sense one can think of the rational homotopy types of $\partial\bar{\partial}$–manifolds as the free objects on their underlying cohomology algebra. In the almost complex case, for the dimensions not excluded, no further restriction is placed on the higher operations in the associated C_∞ structure beyond the requirement that the cohomology algebra with its binary multiplication satisfies Poincaré duality. One can wonder whether non–$\partial\bar{\partial}$ compact complex manifolds generally lie somewhere strictly between these two extremes.

Acknowledgements: I would thank my PhD advisor Dennis Sullivan at Stony Brook University for suggesting the current project. I am indebted to Zhixu Su for her elaboration of Sullivan's smooth realization theorem concerning obtaining a degree one map of a manifold to the space A considered later, along the lines of Barge's proof of the theorem [2, Théorème 8.2.2] following Sullivan's 1974 Orsay lectures, and for her clear formulation of the result as given in [39], [38]; our exposition in Section 5 parallels hers, with the necessary modifications made to adapt to the almost complex case, most importantly the passage to BSU in case $c_1 = 0$ in order to circumvent fundamental group issues. The Macaulay2 package "SymmetricPolynomials" [19] was of great help in calculations performed in Sections 6 and 7. I am grateful to the Max Planck Institute for Mathematics in Bonn, where the final phases of this work were performed. I further thank the referee for numerous helpful comments improving the exposition.

2 Preliminaries on almost complex manifolds and rational spaces, and statement of main theorem

In this section we review the basic notions needed to state the main Theorem 2.4.

2.1 Almost complex and stably almost complex manifolds

An *almost complex manifold* is a smooth manifold M equipped with a (smooth) endomorphism J of its tangent bundle, such that $J \circ J = -\mathrm{Id}$. This endomorphism is called an *almost complex structure* on M. In general we refer to such an endomorphism on a vector bundle E as a *complex structure on the vector bundle*. An almost complex structure induces an orientation on M; we say an *oriented* manifold is almost complex if it admits an almost complex structure inducing the given orientation.

A *stable almost complex structure* on M is a complex structure on its stable tangent bundle, i.e. on $TM \oplus \mathbb{R}^k$ for some positive k, where $\mathbb{R}^k$ denotes a trivial real k–plane bundle.

On a compact manifold M, isomorphism classes of complex vector bundles of rank n are in bijective correspondence with homotopy classes of maps $M \to BU(n)$ to the classifying space of the unitary group (or equivalently, to the classifying space $BGL(n, \mathbb{C})$ of the general linear group; the inclusion of groups $U(n) \hookrightarrow GL(n, \mathbb{C})$ induces a homotopy equivalence between $BU(n)$ and $BGL(n, \mathbb{C})$). A given complex vector bundle is obtained, up to isomorphism, by pulling back the tautological n–plane bundle over $BU(n)$.

The integral cohomology ring of $BU(n)$ is a polynomial algebra on the Chern classes $c_1, c_2, \dots$ of the tautological bundle; pulling these back via the map classifying a given complex vector bundle E gives the Chern classes $c_i(E) \in H^{2i}(M; \mathbb{Z})$ of the bundle. If the vector bundle in question is the tangent (or stable tangent) bundle of M, we often denote its Chern classes by $c_i(M)$ for simplicity, and refer to them as the Chern classes of the (stably) almost complex manifold. The map $BU(n) \to BU(n+1)$ classifying the sum of the tautological bundle over $BU(n)$ with a trivial complex line bundle induces an isomorphism on cohomology in degrees $\leq 2n$, and the Chern classes $c_1, \dots, c_n$ of the tautological bundle over $BU(n+1)$ pull back to those of the tautological bundle over $BU(n)$. Iterating this process, one is led to consider the homotopy colimit BU

of this sequence of maps. Homotopy classes of maps to BU correspond to *stable* complex vector bundles up to equivalence.

We refer the reader to [22] for details of the above, in particular for both concrete and axiomatic treatments of Chern classes, which were originally introduced by Chern and are the primary obstructions to the existence of tuples of linearly independent sections. We will use that the top Chern class c_n of a complex n–plane is the Euler class of the underlying real bundle, i.e. the primary (and only, over a $2n$–dimensional cell complex) obstruction to the existence of a section.

It is useful to consider (stably) almost complex manifolds up to *complex cobordism*, whereby we say two closed stably almost complex manifolds M_1, M_2 (with their induced orientations) of dimension n are complex cobordant if there is a stably almost complex manifold W whose boundary is the disjoint union of M_1 and M_2, and the induced stable almost complex structures agree in an appropriate sense with those on M_1 and M_2. We refer the reader to the definitive [37], and to [26] for a quick and precise introduction. Complex cobordism classes form a graded ring under disjoint union and Cartesian product, denoted Ω^U.

2.2 Chern numbers and congruences

On a closed (stably) almost complex manifold M of real dimension $2n$, one can consider the *Chern numbers* $\langle c_{i_1}(M)c_{i_2}(M)\cdots c_{i_r}(M), [M]\rangle$, where $2i_1 + 2i_2 + \cdots + 2i_r = 2n$. Here $[M]$ denotes the fundamental class of M in $H_{2n}(M;\mathbb{Z})$, and $\langle -, -\rangle$ denotes the pairing between cohomology and homology. For simplicity let us denote by $c_I(M)$ the class $c_{i_1}(M)c_{i_2}(M)\cdots c_{i_r}$, where $I = \{i_1, i_2, \ldots, i_r\}$ is a multi-index. The Chern numbers are complex cobordism invariants, as can be seen from Stokes' theorem after mapping the Chern classes into de Rham cohomology with complex coefficients and representing them by differential forms.

The Chern numbers $\langle c_I(M), [M]\rangle$ of a closed stably almost complex manifold are integers, which furthermore satisfy certain congruence conditions. Namely, consider the stable tangent bundle as a complex vector bundle, which is classified by a map $M \xrightarrow{\tau_M} BU$. We can consider the element $\tau_{M\star}[M] \in H_\star(BU;\mathbb{Q})$. Note that if M is complex cobordant to N via W, then $\tau_{M\star}[M] = \tau_{N\star}[N]$. Indeed, $0 = \tau_{W\star}[\partial W] = \tau_{M\star}[M] - \tau_{N\star}[N]$. Thus we obtain a map from complex cobordism Ω^U to the rational homology of the classifying space $H_\star(BU;\mathbb{Q})$. Stong characterized the image of this map in the following way:

Theorem 2.1. *(Stong, [35]) A class $\alpha \in H_\star(BU;\mathbb{Q})$ is in the image of $\Omega^U_\star \to H_\star(BU;\mathbb{Q})$ if and only if $\langle z\,\mathrm{Td}(c_i), \alpha\rangle \in \mathbb{Z}$ for every z in the integer polynomial ring generated by the elementary symmetric polynomials e_i in the variables $e^{x_i} - 1$, where x_i are the Chern roots of the universal Chern class in BU (i.e. formally we have $c = \prod_i(1 + x_i)$, where c is the total Chern class in $H^\star(BU;\mathbb{Z})$).*

Any considered class α will be of some finite degree and so all sums considered for the elements z are finite. The term $\mathrm{Td}(c_i)$ is the Todd genus, $\mathrm{Td}(c_i) = 1 + \frac{c_1}{2} + \frac{c_1^2+c_2}{12} + \frac{c_1c_2}{24} + \cdots$. Now the mentioned congruences among Chern numbers from above, which we will refer to as the *Stong congruences*, follow from $\langle c_I(M), [M]\rangle = \langle \tau_M^\star c_I, [M]\rangle = \langle c_I, \tau_{M\star}[M]\rangle = \langle c_I, \alpha\rangle$. Note that for degree reasons, this reduces to *finitely many conditions*. One can think of these congruences as coming from the Atiyah–Singer index theorem; namely $\int_M \mathrm{ch}(E)\,\mathrm{Td}(M)$ must be an integer for every complex vector bundle $E \to M$.

Almost complex manifolds M with $c_1(M) = 0$ admit a (further) lift of structure group to the special unitary group $SU(n)$. As in the case of $U(n)$, we have a classifying space $BSU(n)$ for such vector bundles, and a corresponding cobordism ring Ω^{SU}. The integral cohomology ring of $BSU(n)$ is the polynomial ring on the Chern classes $c_2, c_3, \ldots, c_n$ of the tautological bundle over $BSU(n)$.

If a given (stably) almost complex manifold M has $c_1(M) = 0$ in integral cohomology, and its dimension is congruent to 4 mod 8, then a further set of congruences holds among its Chern numbers, according to Stong's description of the image of the map $\Omega^{SU} \xrightarrow{\tau} H_\star(BSU;\mathbb{Q})$ [36]. Before stating it, let us recall that real vector bundles E have integral characteristic classes called Pontryagin classes $p_i \in H^{4i}(M;\mathbb{Z})$, defined by $p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C})$. If E is already a complex vector bundle, then we have the equality

$$1 - p_1(E) + p_2(E) - \cdots = (1 - c_1(E) + c_2(E) - \cdots)(1 + c_1(E) + c_2(E) + \cdots).$$

Again we refer to [22] for further discussion and details.

Now for the further Stong congruences, we have the following:

Theorem 2.2. *(Stong, [36]) Let M be a stably almost complex manifold of (real) dimension congruent to 4 mod 8, with* $c_1(M) = 0$. *Denoting by* e_i^p *the elementary symmetric polynomials in the variables* $e^{x_j} + e^{-x_j} - 2$, *where the* x_j *are given by formally writing* $1 + p_1 + p_2 + \cdots = \prod_j (1 + x_j^2)$, *we have*

$$\langle z \cdot \hat{A}(M), [M] \rangle \in 2\mathbb{Z} \text{ for every } z \in \mathbb{Z}[e_1^p, e_2^p, \ldots].$$

Here $\hat{A}(M)$ *denotes the* $\hat{A}$ *polynomial,* $\hat{A} = 1 - \frac{p_1}{24} + \frac{1}{5760}\left(7p_1^2 - 4p_2\right) + \cdots$.

Since one can express the Pontryagin classes in terms of Chern classes, these further congruences are also conditions on $c_2, c_3, \ldots$. Together with the previous congruences in Theorem 2.1, this determines the image of $\Omega^{SU} \xrightarrow{\tau} H_\star(BSU; \mathbb{Q})$ [36, Theorem 1]. In dimensions not congruent to 4 mod 8, the congruences in Theorem 2.1 already describe the image of $\Omega_\star^{SU} \xrightarrow{\tau} H_\star(BSU; \mathbb{Q})$. We refer the reader to Sections 6 and 7 for some explicit calculations of these congruences.

2.3 Rational spaces and rationalization

We say a simply connected space is *rational* if all of its reduced integer homology groups (or equivalently, homotopy groups) are isomorphic as abelian groups to rational vector spaces. We will only consider *finite type* spaces, i.e. spaces for which $H_i(X; \mathbb{Q})$ is finite–dimensional for every i.

A map between two simply connected spaces is a *rational homotopy equivalence* if it induces an isomorphism on homology groups with rational coefficients (or equivalently, on the homotopy groups tensored with the rationals). Spaces X and Y are *rationally homotopy equivalent* if there is a zig-zag of rational homotopy equivalences $X \leftarrow Z_1 \rightarrow Z_2 \leftarrow \cdots \leftarrow Z_k \rightarrow Y$ between them. Note that a rational homotopy equivalence between rational spaces is a homotopy equivalence.

For every simply connected (or more generally, nilpotent) space X, there is a rational space $X_\mathbb{Q}$ and a rational homotopy equivalence $X \xrightarrow{f} X_\mathbb{Q}$; the space $X_\mathbb{Q}$ is unique up to homotopy equivalence; we call f a *rationalization*. To rationalize spheres, one takes the homotopy colimit of the diagram $S^n \xrightarrow{2} S^n \xrightarrow{3} S^n \xrightarrow{4} S^n \xrightarrow{5} \cdots$, i.e. one forms a sequence of cylinders $S^n \times [0, 1]$ and glues the appropriate ends via a degree k self-map of the sphere. The inclusion of S^n as, say, the leftmost end is a rationalization.

To rationalize a nilpotent space (with the homotopy type of a cell complex), we note that we can build the space inductively by starting with a wedge of spheres, and then repeatedly taking the mapping cone of a map from a sphere into the previous stage. We can rationalize the spheres involved, inducing a sequence of mapping cones whose final stage will be the rationalization of our space.

Working with the rationalizations of spaces up to rational homotopy equivalence facilitates computation, as such spaces can be faithfully encoded in nilpotent graded-commutative differential algebras [44]. Algebraic properties and constructions on these nilpotent algebras correspond to geometric phenomena [44, §11]. This is particularly effective when considering smooth manifolds, where this differential algebra capturing the rational homotopy type of the space is, upon tensoring with the reals, a connected nilpotent replacement of the de Rham algebra of forms [6, Corollary 3.4].

A key property of closed manifolds that survives to their rationalizations is Poincaré duality on the rational cohomology:

Definition 2.3. A (not necessarily rational) connected space X is said to satisfy *rational Poincaré duality* if there is an index n such that $H_n(X; \mathbb{Q}) \cong \mathbb{Q}$, and the pairing $H^k(X; \mathbb{Q}) \otimes H^{n-k}(X; \mathbb{Q}) \to \mathbb{Q}$ given by $\alpha \otimes \beta \mapsto \langle \alpha\beta, [X] \rangle$ is non-degenerate, where $[X]$ is any non-zero element in $H_n(X; \mathbb{Q})$. We call n the *formal dimension* of X. If a choice of $[X]$ is fixed, we refer to it as the *fundamental class* of X.

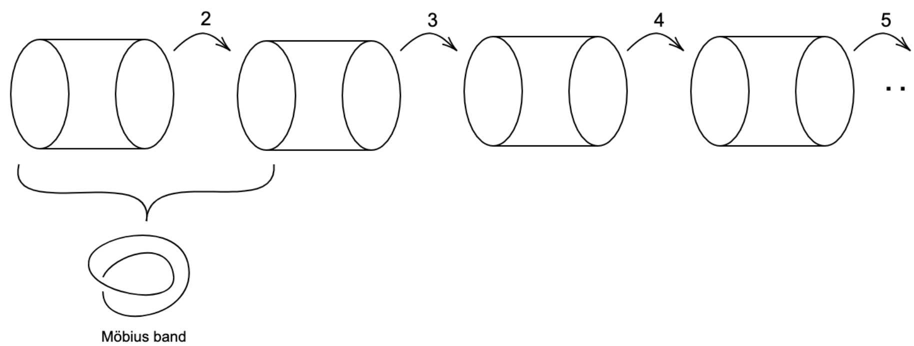


Figure 2.1: A rational circle [8, Lemma 7.5]. Note how the circle representing the left end of the leftmost cylinder may be arbitrarily divided in the fundamental group, by pushing it to the right an appropriate number of times (and multiplying if necessary). For example, to divide by three we may push it two cylinders across (seeing how to divide by six) and multiply by two. Hence the fundamental group of this construction is $\mathbb{Q}$. For higher dimensional spheres mapping in, note that by compactness any map will land in a finite stage of the construction, which deformation retracts onto a circle, and is hence nullhomotopic. Hence $\pi_{\geq 2} = 0$.

This pairing being non-degenerate is equivalent to the cap product $[X] \cap -$ being an isomorphism $H^k(X;\mathbb{Q}) \to H_{n-k}(X;\mathbb{Q})$ (see e.g. [4, Proposition I.2.1]).

Though the Chern classes and Pontryagin classes of a stably almost complex manifold lie in integral cohomology, one can of course consider their image in rational cohomology, which we will also denote by $c_i(M)$ and $p_i(M)$. We refer to these as the rational Chern and Pontryagin classes of the manifold.

2.4 The signature of closed smooth manifolds

On a closed manifold of dimension $4k$, or more generally a rational Poincaré duality space X of formal dimension $4k$ with a choice of fundamental class, we have an induced symmetric bilinear form $H^{2k}(X;\mathbb{Q}) \otimes H^{2k}(X;\mathbb{Q}) \to \mathbb{Q}$. One can represent this pairing by a symmetric matrix, and consider the number of positive eigenvalues minus the number of negative eigenvalues. This integer is an invariant of the rational Poincaré duality space with its choice of fundamental class, called its *signature* and denoted $\sigma(X)$.

From work of Hirzebruch and Thom, the signature of a closed oriented manifold is an oriented cobordism invariant which can be computed by evaluating a universal rational polynomial in its Pontryagin classes, known as Hirzebruch's L–genus (or L–polynomial). The first few terms are given by $1 + \frac{p_1}{3} + \frac{7p_2 - p_1^2}{45} + \cdots$. Recall that the rational Chern classes of any stable almost complex structure on M determine the Pontryagin classes of M by $p_i = (-1)^i \sum_j (-1)^j c_j c_{i-j}$. So, we may speak of the L–genus evaluated on Chern classes, with the understanding that first the Pontryagin classes are to be formed.

On a closed $4k$–manifold M, the pairing $H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \to \mathbb{Q}$ is the rationalization of a unimodular integral pairing on $H^{n/2}(M;\mathbb{Z})$, which is a non-trivial condition. From the theory of symmetric bilinear forms [21, §IV.2.6], this condition is equivalent to the pairing on $H^{2k}(M;\mathbb{Q})$ being equivalent over $\mathbb{Q}$ to one of the form $y_1^2 + y_2^2 + \cdots + y_r^2 - y_{r+1}^2 - \cdots - y_s^2$, i.e. there is a rational basis for which the corresponding matrix is diagonal with only ± 1 on the diagonal.

2.5 Statement of main theorem

We can now state our main result:

Theorem 2.4. *Let X be a formally n–dimensional simply connected rational space of finite type satisfying rational Poincaré duality, $n \geq 5$, and let $[X] \in H_n(X;\mathbb{Q})$ be a non-zero element. Furthermore, let $c_i \in H^{2i}(X;\mathbb{Q})$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ be cohomology classes. Then we have:*

1. *If n is odd, there is a closed simply connected stably almost complex n–manifold M and a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_\star[M] = [X]$ and $c_i(TM) = f^\star(c_i)$.*
2. *If $n \equiv 2 \bmod 4$, then there is a closed simply connected stably almost complex manifold M and a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_\star[M] = [X]$ and $c_i(TM) = f^\star(c_i)$ if the numbers $\langle c_{i_1}c_{i_2}\cdots c_{i_r}, [X]\rangle$ are integers that satisfy the Stong congruences of a stably almost complex manifold for any partition $\{i_1,\dots,i_r\}$ of $n/2$: that is, denoting by e_i the elementary symmetric polynomials in the variables $e^{x_j}-1$, where the x_j are given by formally writing $1+c_1+c_2+\cdots = \prod_j(1+x_j)$, we have*

$$\langle z\cdot \mathrm{Td}(X), [X]\rangle \in \mathbb{Z} \text{ for every } z \in \mathbb{Z}[e_1, e_2, \dots].$$

   Here $\mathrm{Td}(X)$ denotes the Todd polynomial evaluated on $c_1, c_2, \dots$.
3. *If $n \equiv 0 \bmod 4$, then there is a closed simply connected stably almost complex manifold M and a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_\star[M] = [X]$ and $c_i(TM) = f^\star(c_i)$ if*
   - *the quadratic form on $H^{\frac{n}{2}}(X;\mathbb{Q})$ given by $q(\alpha,\beta) = \langle \alpha\beta, [X]\rangle$ is equivalent over $\mathbb{Q}$ to one of the form $\sum_i \pm y_i^2$,*
   - *if we define $p_i = (-1)^i \sum_j (-1)^j c_j c_{i-j}$, then $\langle L(p_1,\dots,p_{n/4}), [X]\rangle = \sigma(X)$, where L is Hirzebruch's L–polynomial,*
   - *the numbers $\langle c_{i_1}c_{i_2}\cdots c_{i_r}, [X]\rangle$ are integers that satisfy the Stong congruences of a stably almost complex manifold described above,*
   - *if $c_1 = 0$ and $n \equiv 4 \bmod 8$, the numbers $\langle p_{i_1}p_{i_2}\cdots p_{i_r}, [X]\rangle$ are integers that satisfy a further set of Stong congruences: denoting by e_i^p the elementary symmetric polynomials in the variables $e^{x_j}+e^{-x_j}-2$, where the x_j are given by formally writing $1+p_1+p_2+\cdots = \prod_j(1+x_j^2)$, we require*

$$\langle z\cdot \hat{A}(X), [X]\rangle \in 2\mathbb{Z} \text{ for every } z \in \mathbb{Z}[e_1^p, e_2^p, \dots].$$

     Here $\hat{A}(X)$ denotes the $\hat{A}$ polynomial evaluated on $p_1, p_2, \dots$. Note that the above are conditions on $c_1, c_2, \dots$, as they determine $p_1, p_2, \dots$.

If n is even and $\langle c_{n/2}, [X]\rangle$ equals the Euler characteristic of X, and the conditions of (2) or (3) are satisfied, then the stable almost complex structure on the obtained manifold M is induced by an almost complex structure (in particular, the almost complex structure also has $f^\star(c_i)$ as its Chern classes).

Remark 2.5. *Notice that if additionally $c_1 = 0$ in any of the cases above, then the first Chern class of the resulting (stably) almost complex manifold M vanishes in* integral *cohomology, since $H^2(M;\mathbb{Z})$ is torsion-free.*

3 Necessary conditions for realization by a closed almost complex manifold

3.1 The realization problem.

We aim to describe the simply connected rational homotopy types realizable by closed almost complex manifolds, along with the rational Chern classes they may carry. To be more precise, we make the following definition:

Definition 3.1. For a simply connected rational space X, we say a closed n–manifold M *realizes* X if there is a rational homotopy equivalence $M \xrightarrow{f} X$.

Note that the existence of such a map implies that X and M have isomorphic rational cohomology rings, and so X satisfies rational Poincaré duality, with formal dimension n, and furthermore comes with a preferred fundamental class $f_\star[M]$. If M is almost complex, then X also carries a natural choice of rational “Chern classes” $f^{\star^{-1}}(c_i(M))$.

We state the realization problem, that Theorem 2.4 addresses, as:

Question 3.2. *Given a simply connected rational space X with prescribed elements $c_i \in H^{2i}(X;\mathbb{Q})$ and $0 \neq [X] \in H_n(X;\mathbb{Q})$, is there a closed almost complex manifold M and a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_\star[M] = [X]$ and $f^\star(c_i) = c_i(M)$?*

We choose to incorporate the fundamental class $[X] \in H_n(X;\mathbb{Q})$ as part of the given data, since this facilitates the calculation of the Chern numbers of the realizing manifold M by

$$\langle c_I(M), [M]\rangle = \langle f^\star(c_I), [M]\rangle = \langle c_I, f_\star[M]\rangle = \langle c_I, [X]\rangle.$$

Note also that in the case of formal dimension $n = 4k$, the bilinear form on $H^{2k}(M;\mathbb{Q})$, discussed in Section 2.4, is isometric to the one on $H^{2k}(X;\mathbb{Q})$

3.2 Necessary conditions for realization.

Let us now consider the implications on $(X, c_i, [X])$ in the case of a positive answer to Question 3.2:

(i) Since a closed manifold has finitely generated homology, we see that $H_\star(X;\mathbb{Q})$ must be finite dimensional.

(ii) X must satisfy rational Poincaré duality. Furthermore, the formal dimension n must be even (as almost complex manifolds are even–dimensional).

(iii) Since the Chern numbers $\langle c_I(M), [M]\rangle$ of any realizing manifold will satisfy the Stong congruences, so will the “Chern numbers” $\langle c_I(X), [X]\rangle$.

(iv) If the formal dimension n is furthermore divisible by four, then the symmetric bilinear form on $H^{n/2}(X;\mathbb{Q})$ induced by Poincaré duality. This pairing on a realizing manifold M is equivalent over $\mathbb{Q}$ to one of the form $y_1^2 + y_2^2 + \cdots + y_r^2 - y_{r+1}^2 - \cdots - y_s^2$ (see Section 2.4).

(v) Likewise, if the formal dimension n is divisible by four, the signature of the pairing on $H^{n/2}(X;\mathbb{Q})$ must be equal to the L–genus evaluated on $[X]$. Indeed, for a realizing manifold M, we would have $\sigma(M) = \sigma(X)$ since the bilinear forms on $H^{n/2}(-;\mathbb{Q})$ are isometric, and we would have

$$\begin{aligned}\langle L(p_1, \dots, p_{n/4}), [X]\rangle &= \langle L(p_1, \dots, p_{n/4}), f_\star[M]\rangle = \langle f^\star L(p_1, \dots, p_{n/4}), [M]\rangle \\ &= \langle L(p_1(M), \dots, p_{n/4}(M)), [M]\rangle = \sigma(M) = \sigma(X).\end{aligned}$$

(vi) We must have $\langle c_{n/2}(X), [X]\rangle = \chi(X)$, where $\chi(X)$ is the Euler characteristic. Indeed, on a closed almost complex manifold of real dimension $2n$, we have $\langle c_n(M), [M]\rangle = \chi(M)$, and if M is rationally homotopy equivalent to X, then $\chi(M) = \chi(X)$.

3.3 Sufficiency of the conditions in high dimension: ingredients of the proof

We will confirm that in formal dimensions $n \geq 6$ (i.e. $n \geq 5$ if we only want *stably* almost complex manifolds), these necessary conditions are in fact sufficient. The proof proceeds in two stages:

Stage 1 We form a simply connected space A with a rational homotopy equivalence $A \xrightarrow{g} X$ to our rational space, such that there is a complex vector bundle ξ on A whose Chern classes are $g^\star(\bar{c}_i)$. Here $\bar{c}_i$ denote the cohomology classes determined by $(1 + c_1 + c_2 + \cdots)(1 + \bar{c}_1 + \bar{c}_2 + \cdots) = 1$. Note that for large enough i, we have $\bar{c}_i = 0$ for degree reasons. We then find a closed manifold M and a map $M \xrightarrow{f} A$ such that

$f_\star[M] = g_\star^{-1}[X]$, and such that the stable normal bundle ν of M is $f^*\xi$. By *the stable normal bundle* we mean the normal bundle to M embedded in a large–dimensional sphere; if the dimension of the sphere is large enough, any two embeddings are isotopic and hence their normal bundles are isomorphic as real vector bundles. The stable normal bundle of M then inherits a complex structure from ξ, giving M the structure of a stably almost complex manifold. It is at this stage that make use of property (iii) above, in conjunction with the Pontryagin–Thom construction.

Stage 2 Once we have a map $M \to A$ covered by a map from ν to ξ as above, we perform *normal surgery* to obtain a new, simply connected, manifold M' mapping to A satisfying the properties in (1), which is furthermore a rational homotopy equivalence. To achieve this we make use of properties (i), (ii), (iv), (v). One then calculates that the Chern classes of the stable tangent bundle of M' (i.e. the sum of TM' and a trivial real bundle, with its induced almost complex structure) are the pullback of the classes c_i on X by the composition $M' \to A \to X$. We then use property (vi) to conclude that the stable almost complex structure on M' is induced by an almost complex structure.

The purpose of Stage 1 is to obtain a space which is for the purposes of rational homotopy theory just as good as our original rational space X, but is furthermore equipped with a complex bundle over it with appropriate rational Chern classes.

4 Normal surgery to a rational homotopy equivalence

We will begin with Stage 2, as it is here that surgery theory, in essence the main aspect of the proof, comes into play. This section is for the most part a review of parts of [4], with the exposition adapted to our needs (i.e. to the rational setting), certain pertinent details elaborated on, and others that would derail the continuity simply referred to. This section is fully expository, with the aim of providing the reader the key points needed to understand half of Sullivan's argument in [44, Theorem 13.2]. We emphasize that this part of the argument, except for Section 4.7, *does not* make use of the presence of an (almost) complex structure, and so the discussion will focus on smooth manifolds and bundles.

We assume we have a map $M \xrightarrow{f} A$, where M is some closed smooth manifold, and A is as in the description of Stage 1 above. That is, A is a simply connected space satisfying Poincaré duality on its rational cohomology, with fundamental class $[A]$, of the same formal dimension as M; furthermore, $f_\star[M] = [A]$ and f is covered by a bundle map $\nu \to \xi$ which is a fiberwise isomorphism, where ν is the stable normal bundle of M, and ξ is a real vector bundle over A; we refer to this as a *normal map*.

$$\begin{array}{ccc} \nu & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & A \end{array}$$

In our setting of interest, ν will be the pullback of ξ; this property will be preserved under the process of normal surgery which we now discuss. Since ξ will have an almost complex structure, the bundle ν will inherit one. For now, though, we treat ξ only as a real vector bundle, and we keep in mind for later that there is an operator J on it with $J^2 = -\mathrm{Id}$ (equivalently, its structure group lifts to the general linear group over $\mathbb{C}$).

Definition 4.1. By a *surgery* on a manifold M^n we refer to the process of removing the interior of the image of an embedding $S^p \times D^{n-p} \xrightarrow{\varphi} M$, and attaching a $D^{p+1} \times S^{n-p-1}$ identically along the boundary (note that $\partial(S^p \times D^{n-p}) = \partial(D^{p+1} \times S^{n-p-1}) = S^p \times S^{n-p-1}$), obtaining a new manifold M'. Such a process defines (after a canonical "smoothing" procedure [4, p.83]) a manifold with boundary, the *trace of the surgery* W_φ by taking $M \times [0, 1]$ with $D^{p+1} \times D^{n-p}$ attached along its boundary to the boundary of $M \times \{1\}$ with the interior of $S^p \times D^{n-p}$ removed.

Note that W_φ is a cobordism between M and M'. It has its own stable normal bundle which restricts to that of M and M' on its boundary.

Definition 4.2. Given a normal map $M \xrightarrow{f} A$, we will refer to an extension of this map to a normal map $W_\varphi \xrightarrow{F} A$ as a *normal cobordism*, obtained by performing *normal surgery*. We denote the normal map on the other end of the surgery by $M' \xrightarrow{f'} A$.

Since our surgery procedure will go through many steps, to keep notation simple, we adopt the following convention: the manifold denoted M' (and the map $M' \xrightarrow{f'} A$) which is obtained at the end of one step, will be denoted $M \xrightarrow{f} A$ in the next step.

Note that if $M \xrightarrow{f} A$ is of degree one, i.e. $f_\star[M] = [A]$, then f' is degree one as well, as we have $0 = [\partial W] = [M] - [M']$, so $F_\star[M] = F_\star[M']$, i.e. $f'_\star[M'] = f_\star[M] = [A]$.

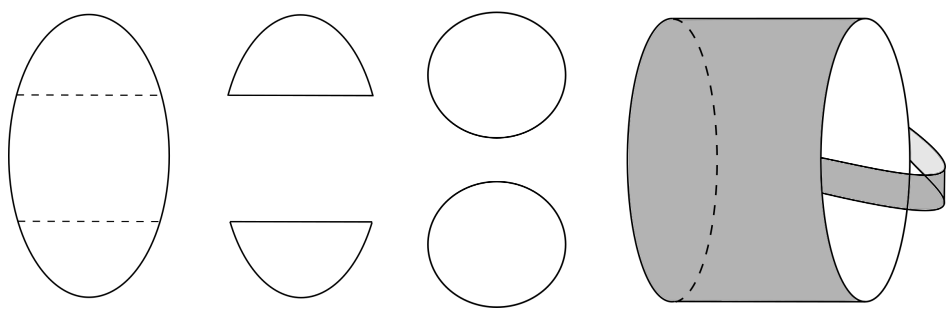


Figure 4.1: Surgery on an embedded $S^0 \times D^1$ in the circle, and its trace. Note how the trace deformation retracts onto a circle with an interval attached.

An important property of the trace is that it deformation retracts onto M with a D^{p+1} attached along the image S^p of the embedding φ; see [4, Theorem IV.1.3], and Figure (2) for an illustration. From here it follows that our normal map $M \xrightarrow{f} A$ extends to a normal map $W_\varphi \to A$ if f extends over this attached D^{p+1} and the map of bundles extends to ω restricted to D^{p+1}, where ω is the stable normal bundle of W_φ [4, Proposition IV.1.4].

Now, the approach to surgering the normal map $M \xrightarrow{f} A$ to a normal map $M' \to A$ which is a rational homotopy equivalence will be the following: we consider the exact sequence

$$\pi_{p+1}(M) \otimes \mathbb{Q} \to \pi_{p+1}(A) \otimes \mathbb{Q} \to \pi_{p+1}(f) \otimes \mathbb{Q} \to \pi_p(M) \otimes \mathbb{Q} \to \pi_p(A) \otimes \mathbb{Q} \to \cdots .$$

This is the rationalization of the long exact sequence in homotopy groups of a pair (recall, $\mathbb{Q}$ is a flat $\mathbb{Z}$–module); one often writes $\pi_p(A, M)$ for $\pi_p(f)$. (Generally $\pi_2(f)$ is not an abelian group, and $\pi_1(f)$ is not a group, so we will first perform a surgery to bring ourselves into a situation where $\pi_2(f)$ is abelian and $\pi_1(f) = 0$.) Elements of $\pi_{p+1}(f)$ are represented by maps $S^p \to M$ which extend to a map $D^{p+1} \to A$, i.e. diagrams of the form

$$\begin{array}{ccc} S^p & \longrightarrow & M \\ \downarrow & & \downarrow f \\ D^{p+1} & \longrightarrow & A \end{array}$$

The idea will be to inductively perform normal surgery on embedded p–dimensional spheres, so that the map $M' \xrightarrow{f'} A$ from the result of the surgery (i.e. the “right end” of the trace) will have smaller–dimensional $\pi_{p+1}(f') \otimes \mathbb{Q}$ than the map on the “left end” of the trace, while satisfying that $\pi_{\le p}(f') \otimes \mathbb{Q} = 0$ if the same was true on the left end of the trace.

Suppose we are given an element in $\pi_{p+1}(f)$, represented by a map $S^p \xrightarrow{\varphi} M$ which extends over $D^{p+1} \to A$. When can we perform a normal surgery on this $S^p \xrightarrow{\varphi} M$? We need the following three conditions to be satisfied:

- S^p must be embedded in M.
- The normal bundle to S^p in M must be trivial, giving us an embedded $S^p \times D^{n-p}$ in M.
- The normal bundle map from M to A must extend over D^{p+1}.

If the dimension of the sphere p is strictly smaller than half the dimension n of our manifold M, then φ can be modified by a homotopy to an embedding, by Whitney's "weak" embedding theorem [49, III], i.e. a general position argument. This can also be done if $p = \frac{n}{2}$ by Whitney's "strong" embedding theorem [50, 8–12], which we will discuss later. As for the next two bullet points, we have the following: Since we are able to extend f over D^{p+1} (since φ represents an element in $\pi_{p+1}(f)$), then the composition of φ with this extension is a nullhomotopic map to A, and hence the normal bundle ν restricted to S^p is trivial, *with an induced trivialization*. Picturing $M \cup_\varphi D^{p+1}$ as embedded in a large disk (one can think of Figure 2.1), the normal bundle to D^{p+1} is trivial, and hence extending our bundle map is equivalent to extending the trivialization of ν on S^p to all of D^{p+1} (as, up to homotopy, all of these points will be mapped to a single point in A). That is, our trivialization on S^p gives us a map from S^p to the Stiefel manifold $\mathrm{St}(k, k+n-p)$ of k-frames in $\mathbb{R}^{k+n-p}$, where k is the (real) rank of the stable normal bundle; this map must therefore be nullhomotopic.

In fact, this element in $\pi_p(\mathrm{St}(k, k+n-p))$ is the unique obstruction to extending our normal map [4, Theorem IV.1.6]. If we can indeed extend our trivialization over D^{p+1}, then the orthogonal complement to the normal bundle of D^{p+1} it determines, when restricted to S^p, gives a trivialization of the normal bundle of S^p in M.

Luckily, we have the following:

Proposition 4.3. *(see [4, Theorem IV.1.12]) For $k \geq 2$, the homotopy group $\pi_{<n-p}(\mathrm{St}(k, k+n-p))$ is trivial.*

We thus have:

Corollary 4.4. *For an embedded sphere $S^p \xrightarrow{\varphi} M$ of dimension p strictly below half the dimension of our manifold, the above described obstruction will vanish as it lies in a trivial group. That is, we can perform normal surgery on this embedded sphere.*

It follows in particular from here that the normal bundle of any embedded S^p in M, such that ν restricted to S^p is trivial, is trivial as soon as $p < \frac{n}{2}$. Note that this follows alternatively from the identity $\nu_{S^p,M} \oplus \nu|_{S^p} \cong \nu_{S^p}$, i.e. the sum of the normal bundle to S^p in M with the stable normal bundle of M restricted to M, equals the stable normal bundle of S^p. As S^p is stably parallelizable, the right-hand side is trivial, so the triviality of $\nu_{S^p,M}$ implies the triviality of $\nu|_{S^p}$.

In the case of even dimension n, when $p = \frac{n}{2}$, the corresponding homotopy group is $\mathbb{Z}_2$ if p is odd, and $\mathbb{Z}$ if p is even [4, Theorem IV.1.12]; this will become relevant in Section 4.3 and onward.

4.1 The effect of surgery on homotopy groups.

Now we consider what effect a surgery on a representative φ of $\pi_{p+1}(f)$ has on the homotopy groups of the manifold M and the map f. As the trace W_φ deformation retracts onto $M \cup_\varphi D^{p+1}$, we see that the inclusion $M \hookrightarrow W_\varphi$ induces an isomorphism on $\pi_{<p}$ and the class that φ represents in $\pi_p(M)$ maps to zero. To relate the homotopy groups of M to those of the manifold M' at the other end of the trace, we notice the following symmetry in the surgery process: *since M' is obtained from M by a surgery on an embedded $S^p \times D^{n-p}$, we have that M is obtained from M' by a surgery on an embedded $D^{p+1} \times S^{n-p-1}$*. Furthermore, the trace of the "backwards" surgery is the same as W_φ; see again Figure 2.1 for an illustration. From here we have that W_φ

deformation retracts onto M' with a D^{n-p} attached. So, looking at the inclusions $M \hookrightarrow W_\varphi \hookleftarrow M'$, some consideration of indices shows:

Proposition 4.5. *([4, Theorem IV.1.5]) If $p < \frac{n-1}{2}$, then $\pi_{<p}(M') \cong \pi_{<p}(M)$ and that $\pi_p(M')$ is isomorphic to the quotient of $\pi_p(M)$ by the $\pi_1(M)$–module generated by the image of φ in $\pi_p(M)$.*

Let us now apply normal surgery to obtain a normal map $M' \to A$ from a simply connected manifold, so that we may speak freely of tensoring with the rationals. First we achieve connectedness: by the Pontryagin–Thom construction which is used in Stage 1 to obtain our starting manifold M, we see that M is a compact subset of some sphere, and hence has finitely many connected components. Pick two points lying in different connected components. Note that this is an embedding $S^0 \hookrightarrow M$. Now by the above discussion, we may perform normal surgery on this embedding (indeed, A is connected and hence the image of this S^0 is connected by a path in A), and after finitely many such surgeries we obtain a connected manifold. The effect of the surgery is forming the connected sum of the two considered connected components along small disks around each point. As for the fundamental group, choose a finite generating set, and represent each element in the set by a smooth embedded loop. Since A is simply connected, the image of this loop is nullhomotopic. As before we can thus perform normal surgery on successively on each loop, with the effect that the fundamental group gets smaller after each surgery by the previous paragraph (here we use that our manifold has dimension ≥ 5, though 4 would suffice); after finitely many surgeries we have a simply connected manifold:

Proposition 4.6. *As $n \geq 5$, we can normally surger $M \to A$ to a normal map from a simply connected (and, in particular, connected) manifold $M' \to A$.*

Taking an element in $\pi_{p+1}(f)$ represented by an embedding $S^p \xrightarrow{\varphi} M$ with trivial normal bundle in M, such that f restricted to the image of the embedding extends over D^{p+1}, and denoting the extension of f over the trace W_φ by F, we see that $\pi_p(F)$ is the quotient of $\pi_{p+1}(f)$ by the $\pi_1(M)$–module generated by our element in $\pi_{p+1}(f)$ [4, Lemma IV.1.14]. In particular, $\dim \pi_{p+1}(F) \otimes \mathbb{Q} < \dim \pi_{p+1}(f) \otimes \mathbb{Q}$.

Now consider the following diagram of long exact sequences,

$$\begin{array}{ccccccccccccc}
\cdots & \to & \pi_{p+1}(f) & \to & \pi_p(M) & \to & \pi_p(A) & \to & \pi_p(f) & \to & \pi_{p-1}(M) & \to & \pi_{p-1}(A) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \pi_{p+1}(F) & \to & \pi_p(W_\varphi) & \to & \pi_p(A) & \to & \pi_p(F) & \to & \pi_{p-1}(W_\varphi) & \to & \pi_{p-1}(A) \to \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \to & \pi_{p+1}(f') & \to & \pi_p(M') & \to & \pi_p(A) & \to & \pi_p(f') & \to & \pi_{p-1}(M') & \to & \pi_{p-1}(A) \to \cdots
\end{array}$$

induced by the diagram

$$\begin{array}{ccc}
M & & \\
\downarrow & \overset{f}{\searrow} & \\
W_\varphi & \xrightarrow{F} & A \\
\uparrow & \underset{f'}{\nearrow} & \\
M' & &
\end{array}$$

Since $\pi_{\leq p-1}(M) \to \pi_{\leq p-1}(W_\varphi)$ are isomorphisms and $\pi_p(M) \to \pi_p(W_\varphi)$ is surjective, by the five lemma we have that $\pi_{\leq p}(f) \to \pi_{\leq p}(F)$ are isomorphisms. If furthermore $p < \frac{n-1}{2}$, then recall that W_φ is obtained from M' by performing surgery on a $n-p-1$–dimensional sphere. Since $n-1-p > n-1-\frac{n-1}{2} = \frac{n-1}{2} > p$, we have $n-p-1 \geq p+1$ and so by the previous sentence, replacing p by $n-p-1$, we have in particular that $\pi_{\leq p+1}(f') \to \pi_{\leq p+1}(F)$ are isomorphisms. Tensoring the above ladder of long exact sequences with $\mathbb{Q}$, we obtain in particular the following:

Proposition 4.7. *We have $\pi_{\leq p}(f') \otimes \mathbb{Q} \cong \pi_{\leq p}(f) \otimes \mathbb{Q}$ and $\dim \pi_{p+1}(f') \otimes \mathbb{Q} < \dim \pi_{p+1}(f) \otimes \mathbb{Q}$.*

Note that we cannot draw this conclusion if $p \geq \frac{n-1}{2}$.

4.2 Surgery below middle degree

Now we proceed to kill relative homotopy groups inductively. Suppose $M \to A$ is a normal map with M simply connected; then $\pi_1(f)$ is trivial and $\pi_2(f)$ is abelian. Note that $\pi_*(f) \otimes \mathbb{Q}$ is finite dimensional in every degree, since $\pi_*(M)$ and $\pi_*(A)$ are. Hence we may choose a finite basis of $\pi_2(f) \otimes \mathbb{Q}$, and scale each element if necessary so that it is in the image of the rationalization map $\pi_2(f) \to \pi_2(f) \otimes \mathbb{Q}$. Let us assume $n \geq 6$ for the moment, so that $2 < \frac{n-1}{2}$. We can choose representatives of these basis elements, given by maps of S^2 into M that extend over D^3 to A; we may choose this map to be a smooth embedding (by first finding a smooth representative of the map, and then using the Whitney embedding theorem). The obstruction to doing normal surgery vanishes, since the appropriate homotopy group of the Stiefel manifold vanishes for such p (recall this also implies the triviality of the normal bundle to S^2, allowing us to perform surgery on an embedded $S^2 \times D^{n-2}$). After applying finitely many such surgeries (one for each basis element), we have obtained a normal map $M' \xrightarrow{f'} A$, where M' is still simply connected, but now $\pi_2(f') \otimes \mathbb{Q} = 0$ as well.

We move on to $\pi_3(f')$, and so on; the largest p that this procedure works is $p = \lfloor n/2 \rfloor - 1$, where $\lfloor n/2 \rfloor$ denotes the floor function. Indeed, if $n = 2m$, then $p < \frac{n-1}{2}$ gives $p \leq m - 1$, i.e. $p \leq \frac{n}{2} - 1$; if $n = 2m + 1$, then $p < \frac{n-1}{2}$ gives $p \leq m - 1 = \frac{n}{2} - 1$. So we finally obtain a normal map $M' \xrightarrow{f'} A$ from a simply connected manifold M' such that $\pi_{\leq \lfloor n/2 \rfloor}(f') \otimes \mathbb{Q} = 0$ (recall that at each stage $p + 1$, we have that the vanishing of $\pi_{\leq p}(f) \otimes \mathbb{Q}$ implies the vanishing of $\pi_{\leq p}(f') \otimes \mathbb{Q}$, along with a decrease in dimension of $\pi_{p+1} \otimes \mathbb{Q}$). As we will note later, the homotopy groups of the space A we are working with, though finite dimension after tensoring with $\mathbb{Q}$, are *not finitely generated*, and so we can not hope to also kill the torsion in $\pi_*(f)$ with finitely many surgeries.

4.3 Surgery in middle degree, and employing rational Poincaré duality

We may thus assume $M \xrightarrow{f} A$ satisfies $\pi_{\leq \lfloor n/2 \rfloor}(f) \otimes \mathbb{Q}$ (along with M being simply connected); we now must deal with $\pi_{\lfloor n/2 \rfloor + 1}(f)$. The dimension of a representative sphere in an element of this group is half the dimension of our manifold if n is even, and just below $\frac{n}{2}$ if n is odd. The obstruction to performing normal surgery lies in a trivial group if n is odd, and lies in $\mathbb{Z}_2$ or $\mathbb{Z}$ if n is even. It is at this point that we finally make use of rational Poincaré duality on M and A (note that everything up to this point works without this assumption)

Rational Poincaré duality gives us the following:

- First of all, we have:

  Lemma 4.8. *If $\pi_{\leq \lfloor n/2 \rfloor + 1}(f) \otimes \mathbb{Q} = 0$, then the map $M \xrightarrow{f} A$ is a rational homotopy equivalence*

  As a consequence, there is no need to consider $\pi_{> \lfloor n/2 \rfloor + 1}(f) \otimes \mathbb{Q}$, which would be very complicated.

  Proof. We have $\pi_{\leq \lfloor n/2 \rfloor + 1}(f) \otimes \mathbb{Q} = 0$ implies that the induced maps $\pi_{\leq \lfloor n/2 \rfloor}(M) \otimes \mathbb{Q} \to \pi_{\leq \lfloor n/2 \rfloor}(A) \otimes \mathbb{Q}$ are isomorphisms, and hence, since M and A are simply connected, the maps on homology $H_{\leq \lfloor n/2 \rfloor}(M; \mathbb{Q}) \to H_{\leq \lfloor n/2 \rfloor}(M; \mathbb{Q})$ are isomorphisms by the (rationalized) Hurewicz theorem.
  Generally, a non-zero degree map of rational Poincaré duality spaces is surjective on homology in all degrees. Indeed, equivalently the dual map on cohomology is injective; suppose some $a \in H^*(A; \mathbb{Q})$ is such that $f^* a = 0$. Take a' such that $\langle aa', [A] \rangle = 1$. Then on the one hand, we must have $\langle f^*(aa'), [M] \rangle = \langle aa', f_*[M] \rangle = \langle aa', \deg(f)[A] \rangle = \deg(f)$, while $\langle f^*(aa'), [M] \rangle = \langle (f^* a)(f^* a'), [M] \rangle = 0$.)
  Now we see that above half the dimension, in each degree the map $H_*(M; \mathbb{Q}) \to H_*(A; \mathbb{Q})$ must be an isomorphism as well, as it is a surjection between spaces of equal dimension (by our conclusion up to half the dimension). □

- It enables us to study the problem of killing $\pi_{\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}$: first observe that if $\pi_1(f) = 0$ and $\pi_{\le\lfloor n/2\rfloor}(f)\otimes\mathbb{Q}=0$, then by the rational version of the relative Hurewicz theorem, we have $\pi_{\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}\cong H_{\lfloor n/2\rfloor+1}(f;\mathbb{Q})$. This latter group is isomorphic to the kernel of $H_{\lfloor n/2\rfloor}(M;\mathbb{Q})\to H_{\lfloor n/2\rfloor}(A;\mathbb{Q})$, as seen from the long exact sequence

$$\cdots\to H_{\lfloor n/2\rfloor+1}(A;\mathbb{Q})\to H_{\lfloor n/2\rfloor+1}(f;\mathbb{Q})\to H_{\lfloor n/2\rfloor}(M;\mathbb{Q})\to H_{\lfloor n/2\rfloor}(A;\mathbb{Q})\to\cdots,$$

  which splits by surjectivity of the maps $H_\star(M;\mathbb{Q})\to H_\star(A;\mathbb{Q})$ discussed in the previous point.
  Hence, with A a rational Poincaré duality space, we may think of this whole surgery procedure as "killing the kernel of f". This viewpoint was not strictly necessary up to this final stage of the surgery procedure.

4.4 The Whitney embedding theorem in dimension $n/2$

To perform surgery on representatives of elements in $\pi_{\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}$ (that are in the image of $\pi_{\lfloor n/2\rfloor+1}(f)$), we must first make sure that every map of an $\lfloor n/2\rfloor$–dimensional sphere to M is homotopic to an embedding. If n is odd, this is guaranteed by the version of the Whitney embedding theorem used before. If n is even, we must use a stronger version of the embedding theorem, proven also by Whitney several years later [50].

First of all, by the "weak" Whitney embedding theorem [49, §III], our map can be approximated by a smooth immersion whose only singular points are transverse double points (i.e. points whose preimage consists of exactly two points). If the sphere is even–dimensional, each such double point carries a sign of ±1 corresponding to whether the orientation on the tangent space in the ambient manifold obtained from adding the pushforwards of the two tangent planes on the sphere agrees with the ambient orientation or not. Given two double points of opposite sign, if the dimension of the sphere is even, or any two double points if it is odd, one can connect these points by two distinct arcs, forming a closed loop. Since our ambient manifold M is simply connected, there is a two–disk whose boundary is this loop, and which intersects the image of the sphere only on its boundary, transversally. To ensure that this disk itself has no self–intersection, we recall our assumption that M is of dimension ≥ 5 and apply the weak Whitney embedding theorem again. *It is at this point that dimension 4 must be omitted from our overarching discussion.*

Then, Whitney shows (with an argument now known as the "Whitney trick") that, using this disk, one can find a homotopy of the original map of the sphere *through immersions* to a smooth map without the two double points considered. If the number of double points of the original map was even, and the number of $+1$ double points was equal to the number of -1 double points if the dimension of the sphere is even, then one applies this argument to obtain a homotopy through immersions of the original map to an embedding. Details of this argument can be found in [50, §§8–12]. Now, if the number of double points was not even, or double points of one sign were more numerous, then an additional argument is employed, also due to Whitney. One may choose a small coordinate ball of the domain sphere in which the map from the sphere to M is an embedding, and find a homotopy of the map which is constant outside of the interior of the ball, to one that has one double point in the interior, with sign $+1$ or -1 of our choosing if the sphere is even–dimensional [50, §2]. (Note that this homotopy will *not* be a homotopy through immersions.) One then arranges the number and sign of double points to allow for repeated application of the Whitney trick, to find a homotopy of the original map of the sphere to an embedding.

We also refer the reader to [31] for a nice account of the Whitney trick.

Now that we can choose an embedded sphere to represent our element in $\pi_{\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}$, we consider the obstruction to performing normal surgery. If n is odd, the corresponding homotopy group of the Stiefel manifold vanishes. If $n\equiv 2 \bmod 4$, the obstruction lies in $\mathbb{Z}_2$. If the obstruction for our choice of map $S^{n/2}\to M$ is non-zero, we precompose it with the degree two self-map $S^{n/2}\xrightarrow{2}S^{n/2}$ of the sphere. We can then find a homotopy of the composition $S^{n/2}\xrightarrow{2}S^{n/2}\to M$ to a smooth embedding, for which the obstruction now vanishes as it is twice the original obstruction class. Killing this new class suffices for our purposes, as we only aim to obtain a rational homotopy equivalence.

If $n\equiv 0 \bmod 4$, denote the homology class represented by our map $S^{n/2}\to M$ (i.e. the pushforward of the fundamental class of the sphere) by x. Then the obstruction to performing normal surgery can be identified

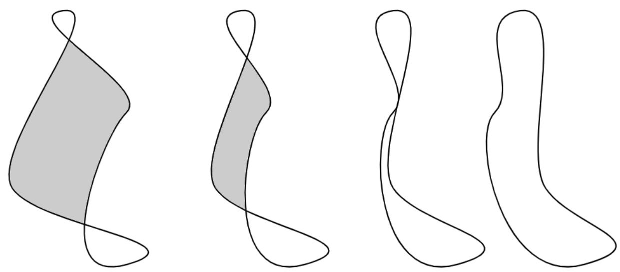


Figure 4.2: An illustration of the Whitney trick.

with $\langle \mathrm{pd}(x)\mathrm{pd}(x), [M]\rangle$, where $\mathrm{pd}(x)$ denotes the cohomology class which is Poincaré dual to x; see [4, pp. 108–111].

Let us now focus on the case of $n \equiv 0 \bmod 4$, and observe that classes in $H_{\lfloor n/2\rfloor+1}(f;\mathbb{Q}) \cong \ker f_\star$ have vanishing surgery obstruction. After this, we will discuss the effect on rational homology of performing surgery on a sphere of dimension $\frac{n}{2}$ if n is even, or $\frac{n-1}{2}$ if n is odd.

4.5 The pairing on homology in dimensions 0 mod 4 and the surgery obstruction

First observe that since $A \xrightarrow{h} X$ is a rational homotopy equivalence, A satisfies rational Poincaré duality with respect to the fundamental class $[A] = g_\star^{-1}[X]$.

Definition 4.9. For a homology class $x \in H_\star(A;\mathbb{Q})$, we denote by $\mathrm{pd}(x)$ the unique cohomology class such that $[A] \cap \mathrm{pd}(x) = x$.

We consider the pairing $H_{n/2}(A;\mathbb{Q}) \otimes H_{n/2}(A;\mathbb{Q}) \to \mathbb{Q}$ given by $x \cdot y = \langle \mathrm{pd}(x)\mathrm{pd}(y), [A]\rangle$. We note the following:

Lemma 4.10. *Cap product with $[A]$ provides an isometry from the pairing on $H^{n/2}(A;\mathbb{Q})$ to the above pairing on $H_{n/2}(A;\mathbb{Q})$.*

Proof. For cohomology classes $x', y' \in H^{n/2}(A;\mathbb{Q})$, their pairing is given by $\langle x'y', [A]\rangle$, and the pairing of $[A]\cap x', [A]\cap y' \in H_{n/2}(A;\mathbb{Q})$ is given by $\langle \mathrm{pd}([A]\cap x')\mathrm{pd}([A]\cap y'), [A]\rangle = \langle x'y', [A]\rangle$. We note that for homology classes $x, y \in H_{n/2}(A;\mathbb{Q})$, we have $x \cdot y = \langle \mathrm{pd}(x), y\rangle$, since $x \cdot y = \langle \mathrm{pd}(x)\mathrm{pd}(y), [A]\rangle = \langle \mathrm{pd}(x), [A] \cap \mathrm{pd}(y)\rangle = \langle \mathrm{pd}(x), y\rangle$ (see e.g. [4, Proposition I.1.1]). □

4.6 Splitting of the pairing

Given our degree one map $M \xrightarrow{f} A$, we now observe:

Lemma 4.11. *([2, pp.477–478]) The pairing on $H_{n/2}(M;\mathbb{Q})$ splits into a summand isometric to the pairing on $H_{n/2}(A;\mathbb{Q})$ along with the kernel, which will consist of summands isometric to the pairing on $H_{n/2}(S^{n/2}\times S^{n/2};\mathbb{Q})$ which has the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.*

A representing $S^{n/2}$ in the latter summands will thus have vanishing normal surgery obstruction.

Proof. To see that the pairing on $H_{n/2}(M;\mathbb{Q})$ splits, we note that the map $H_{n/2}(M;\mathbb{Q}) \xrightarrow{f_\star} H_{n/2}(A;\mathbb{Q})$ admits a section $H_{n/2}(A;\mathbb{Q}) \xrightarrow{\alpha} H_{n/2}(M;\mathbb{Q})$. Indeed, defining α by $\alpha(a) = [M] \cap (f^\star \mathrm{pd}(a))$, we have

$$f_\star\alpha(a) = f_\star([M] \cap (f^\star \mathrm{pd}(a))) = (f_\star[M]) \cap \mathrm{pd}(a) = [A] \cap \mathrm{pd}(a) = a.$$

This section provides an isometry from $H_{n/2}(A;\mathbb{Q})$ onto its image in $H_{n/2}(M;\mathbb{Q})$, and this image is the orthogonal complement in $H_{n/2}(M;\mathbb{Q})$ to $\ker f_\star$ in degree $n/2$. We refer the reader to [2, p.477 f.] for details. In particular, we obtain that the pairing on $H_{n/2}(M;\mathbb{Q})$ restricted to $\ker f_\star$ is non-degenerate as well [2, Corollaire 2.4.4]. □

4.7 Signature of M

Recall that our realization problem started with a simply connected rational space X with a choice of rational cohomology classes $c_i(X)$. In Stage 1, we find a simply connected space A with a rational homotopy equivalence $A \xrightarrow{g} X$ such that A has a complex vector bundle over it with Chern classes $g^\star \bar{c}_i(X)$; it is this vector bundle with respect to which we have been performing our normal surgery. Here $\bar{c}_i(X)$ denote the unique classes solving the equation $(1 + c_1(X) + c_2(X) + \cdots)(1 + \bar{c}_1(X) + \bar{c}_2(X) + \cdots)$. The classes $g^\star \bar{c}_i(X)$ pull back to be the Chern classes of the almost complex structure on the stable normal bundle to M, while the classes $g^\star c_i(X)$ pull back to those of the induced almost complex structure on the stable tangent bundle. The Pontryagin classes $p_i(M)$ of M are determined by these Chern classes, by the universal equation $1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots)$. From here we see that the rational "Pontryagin classes" of X, so formed from the classes $c_i(X)$ on X, pull back via the composition $M \xrightarrow{f} A \xrightarrow{g} X$ to the Pontryagin classes of the (stable) tangent bundle of M. By construction, $f_\star[M] = [A] = g_\star^{-1}[X]$, i.e. $g_\star f_\star[M] = [X]$. We have $\langle L(p_i(M)), [M]\rangle_M = \langle f^\star L(p_i(X)), [M]\rangle_M = \langle L(p_i(X)), [X]\rangle_X$, where the latter quantity is *by assumption* the signature of the pairing on $H^{n/2}(X;\mathbb{Q})$ (and hence of the pairing on $H_{n/2}(X;\mathbb{Q})$), and $\langle L(p_i(M)), [M]\rangle_M$ is the signature of the pairing on $H^{n/2}(M;\mathbb{Q})$ (and hence of the pairing on $H_{n/2}(M;\mathbb{Q})$) by the Hirzebruch signature theorem. Note that the pairing on $H^{n/2}(A;\mathbb{Q})$ is equivalent to the pairing on $H^{n/2}(X;\mathbb{Q})$, with the isometry given by $g^\star$. Since α is an isometry onto a direct summand of the form on $H_{n/2}(M;\mathbb{Q})$, we conclude:

Proposition 4.12. *The signature of the pairing on $H_{n/2}(M;\mathbb{Q})$ is equal to the signature of the pairing on $H_{n/2}(X;\mathbb{Q})$ plus the signature of the pairing on* $\ker f_\star$.

Combined with the previous calculation, this yields:

Corollary 4.13. *The signature of the pairing on* $\ker f_\star$ *is zero.*

4.8 The kernel pairing is equivalent to a sum of hyperbolic forms: the Witt cancellation theorem

We now determine the form of this pairing on $\ker f_\star$, using the following form of the Witt cancellation theorem [21, §I.4.4]:

Theorem 4.14 (Witt). *Suppose B_1,B_2,B_3 are non-degenerate symmetric bilinear forms over $\mathbb{Q}$ (or any field of characteristic not equal to 2). If the form $B_1 \oplus B_2$ is equivalent to the form $B_1 \oplus B_3$, then B_2 is equivalent to B_3.*

We apply the Witt cancellation theorem in the following way: First of all, note that by assumption (corresponding to the the necessary condition (iv)), the form on $H_{n/2}(X;\mathbb{Q})$ is equivalent to $\sum_{i=1}^{r} \pm y_i^2$ for some r. Denote the isometric image of this form under α by B_1. Let B_2 be the bilinear form on $\ker f_\star$; denote the dimension of $\ker f_\star$ by s. Now, we know by previous considerations that $B_1 \oplus B_2$ is equivalent to the pairing on $H_{n/2}(M;\mathbb{Q})$, which is equivalent to one of the form $\sum_{i=1}^{r+s} \pm y_i^2$ since, M being a closed manifold, it is induced by a unimodular pairing over the integers [21, §IV.2.6]. Let B_3 be the form $\sum_{i=r+1}^{r+s} \pm y_i^2$, i.e. the last s summands of the pairing on M. Then $B_1 \oplus B_2$ is equivalent to $B_1 \oplus B_3$, and so B_2 is equivalent to $\sum_{i=r+1}^{r+s} \pm y_i^2$. Since the signature of B_2 is zero, we see that s is even, and we may relabel the basis elements so that B_2 is of the form

$$(z_1^2 - z_2^2) + (z_3^2 - z_4^2) + \cdots = (z_1 - z_2)(z_1 + z_2) + (z_3 - z_4)(z_3 + z_4) + \cdots$$

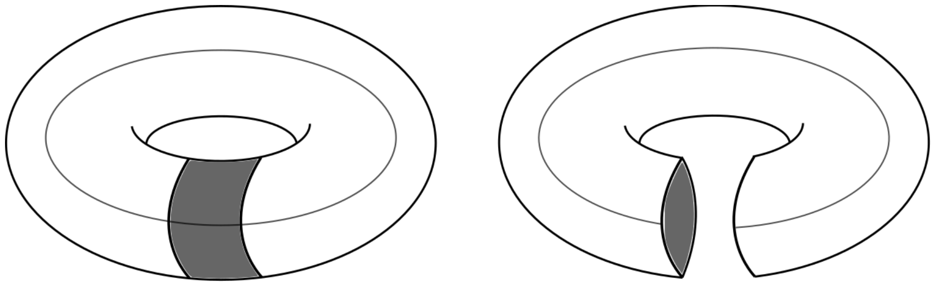


Figure 4.3: Surgery in degree $\lfloor \frac{n}{2} \rfloor$ on a non-zero element in rational homology kills the homology class and its dual under the homology pairing.

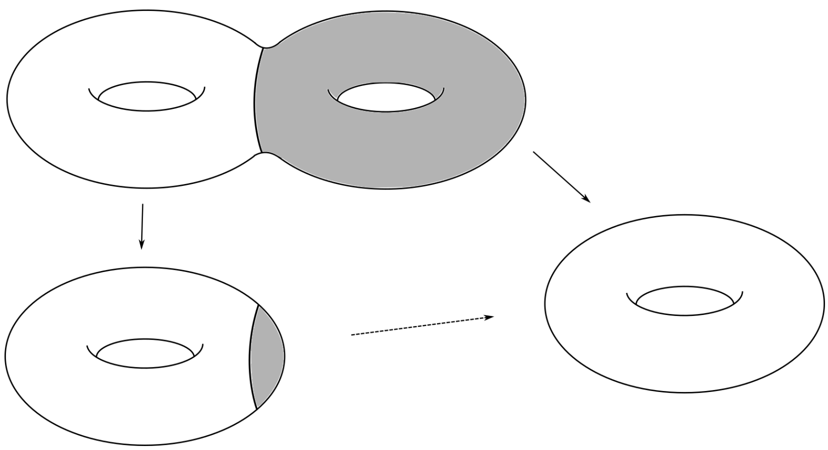


Figure 4.4: Surgery killing the kernel on homology of a degree one map.

Notice that each $(z_i - z_{i+1})(z_i + z_{i+1})$, in new basis elements $\tilde{z}_i = \frac{1}{2}(z_i - z_{i+1})$, $\tilde{z}_{i+1} = z_i + z_{i+1}$, is represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We conclude that B_2 is equivalent to a pairing of the form $\bigoplus_{i=1}^{s/2} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We may thus represent multiples of homology classes x in the kernel of $f_\star$ in degree $n/2$ by embedded spheres for which $x \cdot x = 0$, i.e. for which the obstruction to performing normal surgery vanishes. Next we will see that each such surgery in fact gets rid of two homology classes, removing one summand of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ from the pairing on the kernel.

4.9 The effect of middle–degree surgery on homology

Now we will consider the effect that surgery on embedded $\ell = \lfloor n/2 \rfloor$–spheres with trivial normal bundle, $\ell \geq 2$, has on homology. Let $S^\ell \times D^{n-\ell} \xrightarrow{\varphi} M$ be an embedding, where $\varphi|_{S^\ell}$ represents a homology class x that is non-zero in $H_\ell(M; \mathbb{Q})$, and denote by M_0 the manifold with boundary obtained by removing from M the interior of the image of φ. The result of the surgery M' will be M_0 with a $D^{\ell+1} \times S^{n-\ell-1}$ attached along the boundary.

Following an argument of Kervaire–Milnor [15], one uses the Thom isomorphism on the normal bundle to our embedded sphere, excision, and crucially Poincaré duality to show that certain parts of the long exact sequence in relative homology for the pairs (M, M_0) and (M', M_0) split (it is the existence of these isomorphisms and long exact sequences that makes the problem tractable in homology as opposed to homotopy groups). The result is that if n is even, middle–dimensional surgery lowers the rank of the middle degree homology by two (and leaves the other ranks unchanged), while if n is odd, the homology right above and below the middle decreases in rank by one; the classes being killed are x and its dual in the homology pairing. Details can be found in [4, pp. 97–99, Theorem IV.2.15]; see Figure 4.3 for an illustrative example.

4.10 Conclusion of Stage 2: obtaining a rational homotopy equivalence

Using Stage 1 of the proof, discussed below, we first find a closed n–manifold with a degree one normal map to A. Applying normal surgery below dimension ℓ, where $n = 2\ell$ or $n = 2\ell + 1$, we can then find a simply connected manifold M with a degree one normal map $M \xrightarrow{f} A$ such that $\pi_{\leq\ell}(f) \otimes \mathbb{Q} = 0$. Then applying the above discussion on middle–dimensional surgery, we find a manifold M' and a degree one normal map $M' \xrightarrow{f'} A$ such that $\pi_{\leq\ell+1}(f) \otimes \mathbb{Q} = 0$. Indeed, since $H_i(M';\mathbb{Q}) \cong H_i(M;\mathbb{Q})$ for $i \neq \ell$ the map f' still satisfies $\pi_{\leq\ell}(f) \otimes \mathbb{Q} = 0$ since it is surjective on rational homology as we saw, and through a sequence of surgeries we achieve that the kernel on $H_k(-;\mathbb{Q})$ is trivial; this is enough to conclude that f' is a rational homotopy equivalence.

Remark 4.15. *In [3], one will see the discussion of surgering a normal map to a homotopy equivalence as that of killing the kernel on homology for $M \to A$. We followed [4] and decided to postpone equating killing the kernel on homology with surgering to a homotopy equivalence, since this equivalence requires the assumption that A is a Poincaré duality complex. However, one can see that even without having A satisfy Poincaré duality, much of the discussion applies, as we have already remarked: we can still surger our map $M \to A$ to be a (rational) homotopy equivalence up to right below the middle degree. Applying this to the map classifying the stable normal bundle of a manifold with some additional structure (such as spin, string, almost complex) we obtain statements such as: in sufficiently large dimension, every spin manifold is spin cobordant to a 3–connected one (since $BSpin$ is 3–connected), every string manifold is string cobordant to a 7–connected one (since $BString$ is 7–connected), etc.*

5 Obtaining a closed manifold with a degree one normal map

Now we go through Stage 1 of the construction. Recall, given a simply connected rational space X satisfying rational Poincaré duality, of formal dimension n, with fundamental class $[X]$ and cohomology classes $c_i(X)$, our goal is to obtain a closed (simply connected) almost complex manifold M with a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_\star[M] = [X]$ and $f^\star c_i(X) = c_i(M)$.

First, we find an intermediate simply connected space A, with a rational homotopy equivalence $A \xrightarrow{g} X$, such that A comes equipped with a complex vector bundle ξ over it whose Chern classes $c_i(\xi)$ satisfy $(1 + g^\star c_1(X) + g^\star c_2(X) + \cdots)(1 + c_1(\xi) + c_2(\xi) + \cdots) = 1$.

Recall that, for any N (which will in the sequel be large compared to n), the integral cohomology ring of the classifying space $BU(N)$ is given by $\mathbb{Z}[c_1, c_2, \ldots, c_N]$, where the c_i are the Chern classes of the tautological bundle γ over $BU(N)$. Hence the rational cohomology ring of $BU(N)$ is $\mathbb{Q}[c_1, c_2, \ldots, c_N]$. Since a rational cohomology class of degree $2i$ is determined by (the homotopy class of) a map to $K(\mathbb{Q}, 2i)$ (analogously to the above, by pulling back a natural generator of $H^{2i}(K(\mathbb{Q}, 2i);\mathbb{Q})$), we have a map $BU(N) \to K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots K(\mathbb{Q}, 2N)$ given by the cohomology class $(c_1, c_2, \ldots, c_N)$. We will also denote the corresponding generators of the cohomology ring of $K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots K(\mathbb{Q}, 2N)$ by $c_1, c_2, \ldots, c_N$.

The rational cohomology ring of $K(\mathbb{Q}, 2i)$ is the polynomial algebra on one generator in degree $2i$ (see e.g. [8, p.55], using that $K(\mathbb{Q}, 2i)$ is the rationalization of $K(\mathbb{Z}, 2i)$). From here we see that our map $BU(N) \to K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots K(\mathbb{Q}, 2N)$ induces an isomorphism on rational cohomology, and hence on rational homology; since both spaces are simply connected, this is a rationalization map. From now on we write $BU(N)_\mathbb{Q}$ for $K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots K(\mathbb{Q}, 2N)$.

One can consider the classes $\bar{c}_i$ on $BU(N)$ determined by the equation

$$(1 + c_1 + c_2 + \cdots + c_N)(1 + \bar{c}_1 + \bar{c}_2 + \cdots) = 1.$$

There will be non-zero $\bar{c}_i$ of arbitrarily large degree, but notice, by solving equations inductively by degree, that $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_N$ generate the cohomology of $BU(N)$. The terms $\bar{c}_{\geq N+1}$ will be polynomials in the $\bar{c}_{\leq N}$. Hence the map $BU(N) \xrightarrow{\nu} BU(N)_\mathbb{Q}$ given by $(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_N)$ is a rationalization as well.

Now we consider the map $X \xrightarrow{c^X} BU(N)_{\mathbb{Q}}$ given by $(c_1(X), c_2(X), \ldots)$. (Here we assume that N is greater than the formal dimension; for the surgery step, Stage 2, we needed N, which will be the rank of the stable normal bundle, to be much larger than the formal dimension. From now on we take this to be the case.) Consider the homotopy fiber product of the maps c^X and v:

$$\begin{array}{ccc} A & \xrightarrow{u} & BU(N) \\ \downarrow g & & \downarrow v \\ X & \xrightarrow{c^X} & BU(N)_{\mathbb{Q}} \end{array} \tag{5.1}$$

This diagram is commutative up to homotopy. It is the space A with the complex vector bundle $\xi = u^*\gamma$ which we wish to use in our discussion in Stage 2. We list the properties we require of A and the above diagram:

- A should be a simply connected space.
- The map g should be a rational homotopy equivalence.
- There should be a degree one map from some closed manifold M to A such that the stable normal bundle of M is the pullback of ξ.

Note that the third point only makes sense after we have verified the second; the fundamental class of A we take will be $[A] = g_*^{-1}[X]$.

5.1 Fundamental group of A

An issue we face now is that, as constructed, A need not be simply connected. Indeed, denote the homotopy fiber of $BU(N) \xrightarrow{v} BU(N)_{\mathbb{Q}}$ by F. The long exact sequence in homotopy groups tells us the following sequence is exact:

$$\pi_3(BU(N)_{\mathbb{Q}}) \to \pi_2(F) \to \pi_2(BU(N)) \to \pi_2(BU(N)_{\mathbb{Q}}) \to \pi_1(F) \to \pi_1(BU),$$

i.e. since $N \geq 2$ (and hence the listed homotopy groups of $BU(N)$ are stable),

$$0 \to \pi_2(F) \to \mathbb{Z} \to \mathbb{Q} \to \pi_1(F) \to 0$$

is exact. The map $\mathbb{Z} \to \mathbb{Q}$ is injective since it is induced by rationalization, so we conclude that $\pi_1(F)$ is the abelian group $\mathbb{Q}/\mathbb{Z}$. Now consider the induced map of long exact sequences in homotopy groups associated to the above homotopy fiber product diagram:

$$\begin{array}{ccccc} \pi_2(A) & \longrightarrow & \pi_2(X) & \longrightarrow & \pi_1(F) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \pi_2(BU(N)) & \longrightarrow & \pi_2(BU(N)_{\mathbb{Q}}) & \longrightarrow & \pi_1(F) \end{array}$$

Since the map $\pi_2(BU(N)_{\mathbb{Q}}) \to \pi_1(F)$ is surjective (since $\pi_1(BU(N)) = 0$), we see that $\pi_2(X) \to \pi_1(F)$ is surjective if and only if $\pi_2(X) \to \pi_2(BU(N)_{\mathbb{Q}}) \cong \mathbb{Q}$ is surjective. By the Hurewicz theorem this is equivalent to the map $H_2(X; \mathbb{Z}) \to H_2(BU(N)_{\mathbb{Q}}; \mathbb{Z})$ being surjective. Since both spaces are rational and $H_2(BU(N)_{\mathbb{Q}}; \mathbb{Z}) \cong \mathbb{Q}$, this is equivalent to $H^2(BU(N)_{\mathbb{Q}}; \mathbb{Q}) \xrightarrow{(c^X)^*} H^2(X; \mathbb{Q})$ being non-zero; i.e. to $c_1(X)$ being a non-zero element in $H^2(X; \mathbb{Q})$. Since $\pi_1(X) = 0$, this is furthermore equivalent to $\pi_1(A) = 0$. In summary, we have:

Proposition 5.1. *The space A as defined in (5.1) is simply connected if and only if $c_1 \neq 0$.*

So, if $c_1(X) \neq 0$, we have ensured the first point above (i.e. that A be simply connected). If $c_1(X) = 0$, we will have to make a modification to our setup in order to proceed. Recall that complex rank N vector bundles with vanishing first integral Chern class are classified by maps to $BSU(N)$, where $SU(N)$ is the special unitary group. The integral cohomology of $BSU(N)$ is given by $H^*(BSU(N);\mathbb{Z}) \cong \mathbb{Z}[c_2, c_3, \ldots, c_N]$, and so $K(\mathbb{Q},4) \times K(\mathbb{Q},6) \times \cdots \times K(\mathbb{Q},2N)$ is a rationalization of $BSU(N)$, which we denote by $BSU(N)_{\mathbb{Q}}$. As above, we have a map $BSU(N) \xrightarrow{v} BSU(N)_{\mathbb{Q}}$ (where now $\bar{c}_1 = 0$), and we can consider the homotopy fiber product

$$\begin{array}{ccc} A & \xrightarrow{u} & BSU(N) \\ \big\downarrow g & & \big\downarrow v \\ X & \xrightarrow{c^X} & BSU(N)_{\mathbb{Q}} \end{array} \tag{5.2}$$

where $c^X = (c_2(X), c_3(X), \ldots)$. Since $\pi_2(BSU(N)) = 0$, the homotopy fiber of $BSU(N) \xrightarrow{v} BSU(N)_{\mathbb{Q}}$ is simply connected, and so we have $\pi_1(A) = 0$.

In either case, since the homotopy fiber of v has trivial rational homotopy groups, the map $A \xrightarrow{g} X$ is a map of simply connected spaces inducing an isomorphism on rational homotopy groups, i.e. it is a rational homotopy equivalence, so the second point above is satisfied:

Proposition 5.2. *The map $A \xrightarrow{g} X$ is a rational homotopy equivalence between simply connected spaces, where in the $c_1(X) \neq 0$ case, A is defined as in (5.1), while in the $c_1(X) = 0$ case, A is defined as in (5.2).*

5.2 Finding a degree one normal map

As for the third point above, i.e. finding a degree one map from some closed manifold M to A (where A has fundamental class $g_\star^{-1}[X]$)) such that the stable normal bundle of M is the pullback of ξ, we will have to take into consideration the two distinct cases of $c_1(X) \neq 0$ and $c_1(X) = 0$.

Consider now the tautological complex rank N bundle γ over $BU(N)$, or over $BSU(N)$ if $c_1(X) = 0$. Denote by $\xi = g^\star\gamma$ the pullback bundle over A. We consider the Thom spaces $\mathrm{Th}(\gamma)$ and $\mathrm{Th}(\xi)$ of these bundles, i.e. we consider the underlying real vector bundle, choose a metric on the fibers, take the unit disc bundle, and collapse the boundary to a point. Equivalently, we can obtain the Thom space by taking the mapping cone of the projection from the sphere bundle of our vector bundle to the base space. Any map $S^{n+2N} \to \mathrm{Th}(\xi)$ is homotopic to one whose preimage of $A \subset \mathrm{Th}(\xi)$ is a smooth n–dimensional submanifold M of S^{n+2N}, see [4, p.33]; the normal bundle of M in S^{n+2N}, i.e. the stable normal bundle of M, is the pullback of ξ by this map.

Remark 5.3. *One will see that [4, p.33] assumes the analogue of our space A to be a finite complex; if this is satisfied, we embed this finite complex into some Euclidean space and thicken it to a manifold [3]. Our A will not be a finite complex, but we can do the following: first, find a cell complex A' with a weak homotopy equivalence $A' \to A$, and pull ξ back via this map. Then we consider maps of spheres into the Thom space of this bundle. We choose a cell decomposition of the Thom space that extends that of A'; then our given map of a sphere into the Thom space, being compact, intersects only finitely many cells of A' (if the map misses A' completely, it is nullhomotopic, and hence homotopic to a constant map landing in A'). The Thom space of our bundle restricted to these finitely many cells naturally sits inside the Thom space of the bundle over A'.*

We thus obtain a normal map $M \xrightarrow{f} A$. However, the degree of this map remains unknown to us at this point. The class $f_\star[M]$ in integer homology is obtained by taking the image of the Hurewicz homomorphism applied to the homotopy class of $S^{n+2N} \to \mathrm{Th}(\xi)$, followed by cap product with the Thom class of ξ; see [4, p.39]. We compose this map further with the rationalization on homology:

Definition 5.4. We refer to the composition $\pi_{n+2N}(\mathrm{Th}(\xi)) \xrightarrow{ht_\xi} H_n(A;\mathbb{Q})$, of the Hurewicz map followed by the Thom isomorphism and rationalization, as the *Hurewicz–Thom map*.

Hence, our goal is to show that $[A] = g_\star^{-1}[X]$ is in the image of the Hurewicz–Thom map, since this will provide us with a degree one normal map $M \to A$.

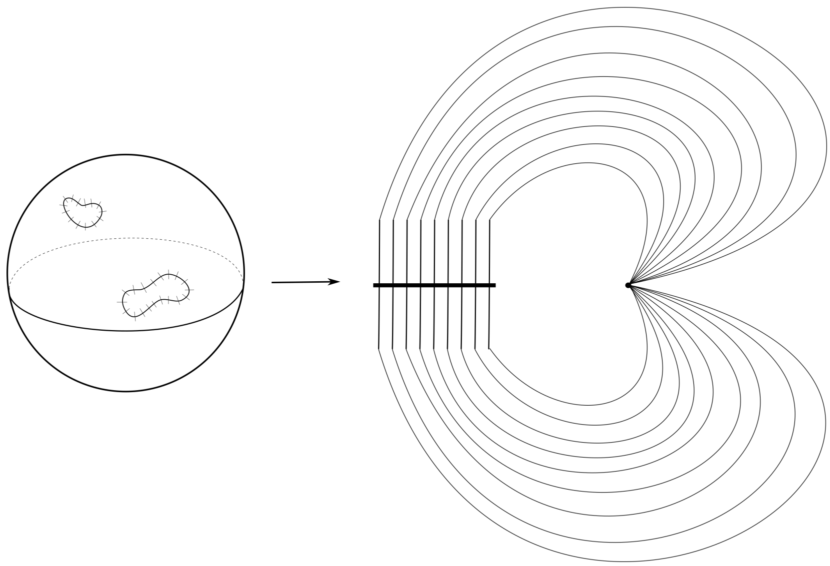


Figure 5.1: The Pontryagin–Thom construction.

The idea is to show that the diagram

$$\begin{array}{ccc} \pi_{n+2N}(\mathrm{Th}(\xi)) & \longrightarrow & \pi_{n+2N}(\mathrm{Th}(\gamma)) \\ \downarrow{\scriptstyle ht_\xi} & & {\scriptstyle ht_\gamma}\downarrow \\ H_n(A;\mathbb{Q}) & \xrightarrow{u_*} & H_n(BU(N);\mathbb{Q}) \end{array}$$

is a fiber product diagram of abelian groups. In $H_n(A;\mathbb{Q})$ we take the class $[A] = g_*^{-1}[X]$. By the assumption on the "Chern numbers" satisfying the Stong congruences, the class $u_*[A]$ will be in the image of ht_γ, and hence $[A]$ will be in the image of ht_ξ.

To conclude that the above diagram is a pullback, first note that rationalizing the sphere bundle $S(\xi) \to A$ gives a fiber bundle over $A_\mathbb{Q} = X$ whose fibers are rational spheres. Denote this bundle by $S(\xi)_\mathbb{Q} \to X$. We can do the same for $S(\gamma) \to BU(N)$ (or $S(\gamma) \to BSU(N)$; we will write $BU(N)$ for simplicity of notation from now on), and we can form the "Thom spaces" of these rational sphere bundles by taking the respective mapping cones; the induced map of long exact sequences in homology, together with the five lemma, shows that the induced map of the Thom space to the "Thom space" of the rational sphere bundles is a rationalization (these spaces are all simply connected).

We can now consider the following diagram, cf. [38, p.21]:

$$\begin{array}{ccccccc} S(\xi) & & & \longrightarrow & & & S(\gamma) \\ & \searrow & & & & \swarrow & \\ \downarrow & & A & \xrightarrow{u} & BU(N) & & \downarrow \\ & & & \searrow \quad \mathrm{Th}(\xi) \longrightarrow \mathrm{Th}(\gamma) \quad \swarrow & & & \\ & & {\scriptstyle g}\downarrow & \downarrow \qquad\qquad \downarrow & \downarrow{\scriptstyle v} & & \\ & & & \nearrow \quad \mathrm{Th}(\xi)_\mathbb{Q} \longrightarrow \mathrm{Th}(\gamma)_\mathbb{Q} \quad \nwarrow & & & \\ \downarrow & & X & \xrightarrow{c^X} & BU(N)_\mathbb{Q} & & \downarrow \\ & \nearrow & & & & \nwarrow & \\ S(\xi)_\mathbb{Q} & & & \longrightarrow & & & S(\gamma)_\mathbb{Q} \end{array} \tag{5.3}$$

The maps in this diagram are induced by the universal properties of (co)fibrations and rationalization; namely, for the latter, given a rationalization map $Y \xrightarrow{\rho} Y_\mathbb{Q}$ and a map $Y \xrightarrow{f} Z_\mathbb{Q}$ to another rational space, there is a map $Y_\mathbb{Q} \xrightarrow{f_\mathbb{Q}} Z_\mathbb{Q}$, unique up to homotopy, such that f is homotopic to $f_\mathbb{Q} \circ \rho$. This can be seen from obstruction theory; the argument is very typical of rational homotopy theory, so we include it here: the obstructions to extending f over the map ρ lie in $H^\star(Y_{\mathbb{Q}}, Y; \pi_{\star-1}(Z_{\mathbb{Q}}))$, where $(Y_{\mathbb{Q}}, Y)$ denotes the mapping cone of ρ (i.e. we convert ρ into a inclusion and consider the corresponding pair of spaces). Since ρ is a rationalization, the pair $(Y_{\mathbb{Q}}, Y)$ has only torsion in its homotopy groups. Since $Z_{\mathbb{Q}}$ is rational, the groups $H^\star(Y_{\mathbb{Q}}, Y; \pi_{\star-1}(Z_{\mathbb{Q}}))$ vanish. Likewise, the obstructions to uniqueness of the extension, which lie in $H^\star(Y_{\mathbb{Q}}, Y; \pi_\star(Z_{\mathbb{Q}}))$ vanish.

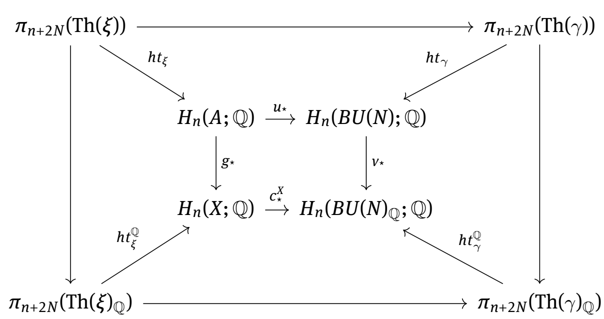
Generally, given a (homotopy) commutative square of spaces where the vertical maps are rationalizations, the square is a homotopy pullback (i.e. a homotopy fiber product) if and only if it is a homotopy pushout [45, Lemma 6.1]. Using this equivalence, one then argues the following:

Proposition 5.5. *(cf. [38]) The innermost square involving Thom spaces in (5.3) is a homotopy pullback.*

This follows from observing that this square is a homotopy pushout of two homotopy pushout (equivalently here, homotopy pullback) squares. We refer the reader to [38] (and to [20] for some further details).

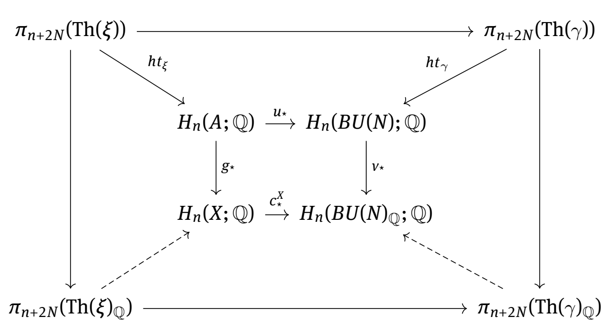
5.3 Verifying that the prescribed fundamental class is hit

With this in hand, we now consider the diagram (again cf. [38])



(5.4)

where the upper diagonals are the corresponding Hurewicz–Thom maps, and the lower diagonals the induced maps on rationalizations. More precisely, we first have the following diagram of abelian groups:



The dashed arrows are the unique maps making the left square and right square commute, respectively (a map from a finitely generated abelian group G to a rational vector space factors uniquely through a given rationalization $G \to G \otimes \mathbb{Q}$). With some diagram chasing together with this uniqueness property of the factorization through a rationalization on the level of abelian groups, we conclude that the bottom square commutes.

Lemma 5.6. *The lower diagonals $ht_\xi^{\mathbb{Q}}$ and $ht_\gamma^{\mathbb{Q}}$ in (5.4) are isomorphisms.*

Proof. Indeed, since we are taking N to be large with respect to n, the rationalized Hurewicz map $\pi_{n+2N}(\mathrm{Th}(\xi)) \otimes \mathbb{Q} \to H_{n+2N}(\mathrm{Th}(\xi);\mathbb{Q})$ is an isomorphism; this follows from the fact that $\mathrm{Th}(\xi)$ is a simply connected space whose first non-trivial rational homology group is in degree $2N$ (by the Thom isomorphism theorem) and the rational Hurewicz theorem. (A direct way to see this would be through employing rational homotopy theoretic minimal models [44]: the minimal model of $\mathrm{Th}(\xi)$ has no generators below degree $2N$, and so any non-trivial element between degrees $2N$ and $4N-1$ must be a linear combination of generators. Since the differential contains no linear terms, elements in degree $\leq 4N-2$ must be closed; in particular, $H^\star(\mathrm{Th}(\xi);\mathbb{Q})$ is spanned by closed generators of the minimal model, which is equivalent to the (dual) rationalized Hurewicz homomorphism being an isomorphism. Note that elements in degree $4N-1$ must be linear in the generators, but may not be closed; this gives the surjectivity part of the rational Hurewicz theorem.)

Then, the map $H_{n+2N}(\mathrm{Th}(\xi);\mathbb{Q}) \to H_n(A;\mathbb{Q})$ is an isomorphism by the Thom isomorphism theorem, and hence the composition $\pi_{n+2N}(\mathrm{Th}(\xi))\otimes\mathbb{Q} \to H_{n+2N}(\mathrm{Th}(\xi);\mathbb{Q}) \to H_n(A;\mathbb{Q})$ is an isomorphism, giving that $ht_\xi^{\mathbb{Q}}$ is an isomorphism (by tensoring the left-most square in (5.4) with $\mathbb{Q}$); likewise for $ht_\gamma^{\mathbb{Q}}$. □

Proposition 5.7. *The class $[A] \in H_n(A;\mathbb{Q})$ is in the image of ht_ξ, In particular, $[X] \in H_n(X;\mathbb{Q})$ is in the image of $g_\star ht_\xi$. Therefore, there is a closed stably almost complex manifold M with a normal map to A such that $f_\star[M] = [A]$.*

Proof. Consider the element $c_\star^X[X] \in H_n(BU(N)_{\mathbb{Q}};\mathbb{Q})$. We first show that this class is in the image of the map $v_\star ht_\gamma$.

As a first case, we show that $[X]$ is in the image of $g_\star ht_\xi$ in the case of n odd. In this case $c_\star^X[X] = 0$ since $H^{\mathrm{odd}}(BU(N)_{\mathbb{Q}};\mathbb{Q}) = 0$. Then $0 \in \pi_{n+2N}\mathrm{Th}(\gamma)$ will map to $c_\star^X[X]$ under $v_\star ht_\gamma$. Since $ht_\gamma^{\mathbb{Q}}$ is an isomorphism, it follows that $(ht_\xi^{\mathbb{Q}})^{-1}[X] \in \pi_{n+2N}\mathrm{Th}(\xi)_{\mathbb{Q}}$ and $0 \in \pi_{n+2N}\mathrm{Th}(\gamma)$ map to the same element in $\pi_{n+2N}\mathrm{Th}(\gamma)_{\mathbb{Q}}$ (namely $(ht_\gamma^{\mathbb{Q}})^{-1}c_\star^X[X]$).

Now, as discussed, the diagram of Thom spaces is a homotopy pullback square, and so we have an induced Mayer–Vietoris long exact sequence in homotopy groups,

$$\cdots \xrightarrow{\partial} \pi_\star(\mathrm{Th}(\xi)) \xrightarrow{(\hat{u}_\star,\hat{g}_\star)} \pi_\star(\mathrm{Th}(\gamma)) \oplus \pi_\star(\mathrm{Th}(\xi)_{\mathbb{Q}}) \xrightarrow{\hat{v}_\star - \widehat{c^X}_\star} \pi_\star(\mathrm{Th}(\gamma)_{\mathbb{Q}}) \xrightarrow{\partial} \pi_{\star-1}(\mathrm{Th}(\xi)) \to \cdots$$

where $\hat{u}, \hat{g}, \hat{v}, \widehat{c^X}$ denote the induced maps on Thom spaces. From here it follows that there is an element $\beta \in \pi_{n+2N}(\mathrm{Th}(\xi))$ that maps to $(ht_\xi^{\mathbb{Q}})^{-1}[X]$ and 0 respectively. Then $g_\star ht_\xi(\beta) = [X]$ as desired.

Now suppose that $[X]$ is even, so $c_\star^X[X]$ is not necessarily zero. We now take into consideration condition (iii) from the beginning, namely that the "Chern numbers" on X are integers satisfying the Stong congruences. If we are in the case of $c_1(X) = 0$, then the Stong congruences are strictly stronger in dimensions $n \equiv 4 \bmod 8$ (i.e. the description of the image of $\Omega^{SU} \xrightarrow{\tau} H_\star(BSU;\mathbb{Q})$ involves more congruences than those describing the image of $\Omega^U \xrightarrow{\tau} H_\star(BU;\mathbb{Q})$). In either case, suppose the Stong congruences are satisfied. This means there is some stably almost complex manifold Y (with vanishing first Chern class integrally if we are in the $c_1(X) = 0$ case) such that $\langle c_{i_1}(X)c_{i_2}(X)\cdots c_{i_r}(X), [X]\rangle = \langle c_{i_1}(Y)c_{i_2}(Y)\cdots c_{i_r}(Y), [Y]\rangle$ for all tuples $(i_1, i_2, \ldots, i_r)$ whose total degree is n. For simplicity, let us denote e.g. $c_{i_1}(X)c_{i_2}(X)\cdots c_{i_r}(X)$ by $c_\alpha(X)$.

Note,

$$\langle c_\alpha(X), [X]\rangle = \langle (c^X)^\star(c_\alpha), [X]\rangle = \langle c_\alpha, (c^X)_\star[X]\rangle.$$

On the other hand, consider the map $Y \xrightarrow{v_Y} BU$ (or, to BSU), classifying the stable normal bundle of Y, i.e. $v_Y^\star\gamma$ is the stable normal bundle of Y (with its complex structure). By the Pontryagin–Thom construction, there is an element $\beta_Y \in \pi_{n+2N}(\mathrm{Th}(\gamma))$ such that $ht_\gamma(\beta_Y) = v_\star[Y]$ (namely, Y is constructed by taking the preimage of BU or BSU under a suitable representative of the homotopy element, and as a consequence the induced map from Y to BU or BSU classifies the stable normal bundle Y). Here we consider $v_\star[Y]$ as an element in rational homology. Since v_Y pulls back the universal Chern classes c_i to the Chern classes of the stable normal bundle of Y, it follows that v_Y pulls back the classes $\bar{c}_i$ to the Chern classes of (the stable tangent bundle of) Y. So, we have

$$\langle c_\alpha(Y), [Y]\rangle = \langle v^\star\bar{c}_\alpha, [Y]\rangle = \langle \bar{c}_\alpha, (v_Y)_\star[Y]\rangle = \langle \bar{c}_\alpha, ht_\gamma(\beta_Y)\rangle.$$

Furthermore, since we are only considering Chern classes up to degree n, we have $\bar{c}_i = v^\star c_i$, and hence

$$\langle \bar{c}_\alpha, ht_\gamma(\beta_Y)\rangle = \langle v^\star(c_\alpha), ht_\gamma(\beta_Y)\rangle = \langle c_\alpha, v_\star(ht_\gamma(\beta_Y))\rangle.$$

In conclusion, $\langle c_\alpha, (c^X)_*[X]\rangle = \langle c_\alpha, \nu_*(ht_\gamma(\beta_Y))\rangle$ for all α, and since the c_α span the vector space $H^n(BU_\mathbb{Q};\mathbb{Q})$, we conclude that $(c^X)_*[X] = \nu_*(ht_\gamma(\beta_Y))$. Hence, there is an element $\beta \in \pi_{n+2N}(\mathrm{Th}(\xi))$ that maps to $(ht_\xi^\mathbb{Q})^{-1}[X]$, and $g_* ht_\xi(\beta) = [X]$. □

5.4 Stable almost complex structure on the resulting manifold and its Chern classes

Now that we have obtained a manifold and a degree one normal map to A, we go through Stage 2 of the proof to obtain a degree one normal map $M \xrightarrow{f} A$ which is a also a rational homotopy equivalence, and thus the composition $M \xrightarrow{f} A \xrightarrow{g} X$ is a rational homotopy equivalence.

We pull back the complex structure from the vector bundle γ to ξ and then to the stable normal bundle ν_M (a normal map gives a real bundle isomorphism between ν_M and $f^*\xi$, and so we can transport the complex structure from $f^*\xi$ to ν_M.) By construction, the fundamental class $[M]$ is determined by the orientation of the stable normal bundle (as a real bundle) in the Pontryagin–Thom construction (see e.g. [3, Lemma 2]), and hence the complex structure we are equipping the stable normal bundle with induces this same orientation. This can also be seen tautologically from the diagram

$$M \to A \to BU(N) \to BSO(2N),$$

where the map $BU(N) \to BSO(2N)$ classifies the real vector bundle underlying the tautological bundle γ. The stable normal bundle to M, as a real vector bundle, is by the Pontryagin–Thom construction classified by this composition, and hence it lifts to a complex vector bundle by looking at $M \to A \to BU(N)$.

We record the following well-known lemma:

Lemma 5.8. *A complex structure on the stable normal bundle of a manifold determines a complex structure on the stable tangent bundle. In particular, we have a stably almost complex structure on M.*

Proof. The stable normal bundle as a complex vector bundle is classified by a map $M \to Gr_\mathbb{C}(N, N')$ of complex N-planes in $\mathbb{C}^{N'}$, for some large N'. With the standard Hermitian inner product on $\mathbb{C}^{N'}$, we have a diffeomorphism $Gr_\mathbb{C}(N, N') \xrightarrow{\perp} Gr_\mathbb{C}(N'-N, N')$ sending a plane to its orthogonal complement. The composition $M \to Gr_\mathbb{C}(N, N') \xrightarrow{\perp} Gr_\mathbb{C}(N'-N, N')$ gives a complex structure on the stable tangent bundle to M, as seen from the commutative diagram

$$\begin{array}{ccc} Gr_\mathbb{C}(N, N') & \xrightarrow{\perp} & Gr_\mathbb{C}(N'-N, N') \\ \downarrow & & \downarrow \\ Gr_\mathbb{R}(2N, 2N') & \xrightarrow{\perp} & Gr_\mathbb{R}(2N'-2N, 2N') \end{array}$$

where the map $\perp$ between real Grassmannians sends a real plane to its orthogonal complement with respect to the standard Euclidean inner product on $\mathbb{R}^{2N'}$. We see that the total Chern classes of the stable normal bundle to M and the stable tangent bundle (with this complex structure) multiply to the unit class. □

Now we calculate the Chern classes of this stable almost complex structure. We have the following diagram:

$$\begin{array}{ccccc} \nu_M & \longrightarrow & \xi & \longrightarrow & \gamma \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{f} & A & \xrightarrow{u} & BU(N) \\ & & \downarrow g & & \downarrow \nu \\ & & X & \xrightarrow{c^X} & BU(N)_\mathbb{Q} \end{array}$$

Lemma 5.9. *The Chern classes of this complex structure on the stable tangent bundle of M satisfy $c_i(M) = (gf)^* c_i(X)$.*

Proof. Indeed, we have

$$\begin{aligned} c_i(M) &= \bar{c}_i(\nu_M) = f^\star \bar{c}_i(\xi) = f^\star \bar{c}_i(u^\star \gamma) = f^\star u^\star \bar{c}_i(\gamma) = (uf)^\star \bar{c}_i(\gamma) \\ &= (uf)^\star v^\star c_i = (vuf)^\star c_i = (c^X uf)^\star c_i = f^\star g^\star (c^X)^\star c_i = (gf)^\star c_i(X), \end{aligned}$$

where we used that $i \leq \frac{n}{2}$ in order to have $\bar{c}_i(\gamma) = v^\star c_i$. □

Now recall our necessary condition (vi) in order for this stable almost complex structure to be induced by an almost complex structure: we must have $\langle c_n(M), [M] \rangle = \chi(M)$, by the obstruction-theoretic definition of the top Chern class. Recall, since

$$\langle c_n(M), [M] \rangle = \chi(M) = \langle (gf)^\star c_n(X), [M] \rangle = \langle c_n(X), (gf)_\star [M] \rangle = \langle c_n(X), [X] \rangle,$$

this is equivalent to $\langle c_n(X), [X] \rangle = \chi(X)$, since M and X have the same Euler characteristic by virtue of being rationally homotopy equivalent. Conversely, we have:

Proposition 5.10. *If $\langle c_n(X), [X] \rangle = \chi(X)$, then the stable almost complex structure on M is induced by an almost complex structure.*

Proof. Indeed, by a classical result (see [14, Corollary 3] for details), the top Chern class evaluating to the Euler characteristic is a necessary *and sufficient* condition for reducing to a genuine almost complex structure on M. More precisely, if M is a stably almost complex manifold with $\langle c_n(M), [M] \rangle = \chi(M)$, then there is an almost complex structure on M, giving a lift $M \to BU(n)$ of the tangent bundle map $M \to BSO(2n)$, such that the composite map $M \to BU(n) \to BU$ is homotopic, through lifts of the map $M \to BSO$ classifying the stable tangent bundle, to the map $M \to BU$ provided by the original stably almost complex structure. □

This concludes the proof of Theorem 2.4.

Remark 5.11. *Consider again the diagram*

$$\begin{array}{ccc} \pi_{n+2N}(\mathrm{Th}(\xi)) & \longrightarrow & \pi_{n+2N}(\mathrm{Th}(\gamma)) \\ {\scriptstyle ht_\xi} \searrow & & \swarrow {\scriptstyle ht_\gamma} \\ H_n(A;\mathbb{Q}) & \xrightarrow{u_\star} & H_n(BU(N);\mathbb{Q}) \end{array}$$

Choosing an element of $\pi_{n+2N}\mathrm{Th}(\xi)$ gives us a fundamental class $[A]$ in $H_n(A;\mathbb{Q})$ by looking at its image under the Hurewicz–Thom map, and this fundamental class will the image of a stably almost complex manifold by applying the Pontryagin–Thom construction to our chosen homotopy element in the Thom space. Indeed, by construction (i.e. commutativity of the diagram) the image of $[A]$ in the homology of $BU(N)$ will land in the lattice described by Stong. The issue here is that we do not have control of what exactly the Chern numbers of our manifold will be, so we do not know what the top Chern class will evaluate to, and, in the case of dimension divisible by four, whether the signature will be computed correctly.

Remark 5.12. *Let us comment further on the case of $c_1(X) = 0$. For dimension n not congruent to 4 mod 8, suppose we do not replace BU by BSU; we would be led to do so if we wanted to characterize all (not necessarily simply connected) stably almost complex manifolds with a rational homology equivalence to our given rational homotopy type. The homotopy pullback of $X \xrightarrow{c^X} BU(N)_{\mathbb{Q}}$ and $BU(N) \xrightarrow{v} BU(N)_{\mathbb{Q}}$ gives us the map $A \xrightarrow{g} X$. From the long exact sequence in homotopy groups, since the homotopy fiber of v has trivial rational homotopy groups, we see that $A \xrightarrow{g} X$ is an isomorphism on rational homotopy groups.*

The fundamental group of A is $\mathbb{Q}/\mathbb{Z}$. Now, as before, we can obtain a stably almost complex manifold M with a degree one normal map to A. Since A is not simply connected as it was before, we cannot simply surger M down to a simply connected manifold. However, noticing that all commutators in the finitely presented group $\pi_1(M)$ become trivial when mapped over to A, we can surger M down to a manifold with abelian fundamental *group. Since this is now a finitely generated abelian group, we can identify the infinite cyclic summands in the group; the generators of these groups, mapped to A, become torsion, and hence some multiple of the generator in each of the infinite cyclic summands can be surgered out. We end up with M whose fundamental group is finite and abelian; in particular, the map on fundamental groups to A is a rational isomorphism, since both groups are torsion. We can further perform surgery to make the map $\pi_1(M) \to \pi_1(A)$ injective. Then $\pi_2(f)$ is an abelian group. Since it is rationally homotopy equivalent to X, A has degree-wise finite dimensional rational homotopy groups, and this surgered M does as well, since it admits a finite cover which is a simply connected closed manifold.*

We can then perform surgery as before, getting rid of $\pi_*(f) \otimes \mathbb{Q}$ up to the middle degree; however, it is in middle degree that the following difficulty arises: we have no guarantee that the relative Hurewicz map will give an isomorphism between the homotopy group of f and the homology of the kernel, an identification that was crucial earlier. One could hope that relative Hurewicz would hold (rationally) if the pair (A, M) were nilpotent, but whether this is the case is not clear.

6 Consequences and remarks

Notice that all the necessary conditions (i)–(vi) for realization by a closed almost complex manifold were cohomological, and that c_1 of a simply connected manifold is trivial integrally if it is trivial rationally. Since in dimensions ≥ 6 these conditions were also sufficient for realization by a simply connected closed almost complex manifold, we have:

Corollary 6.1. The realizability of a simply connected rational homotopy type by a simply connected closed almost complex manifold depends only on its cohomology ring.

Proof. It only remains to check dimensions ≤ 4, the only non-trivial case being dimension 4. Any simply connected manifold in this dimension has the rational homotopy type of $k\mathbb{CP}^2\#\ell\overline{\mathbb{CP}^2}$. By Hirzebruch's congruence, if M is a closed almost complex manifold with the rational homotopy type of $k\mathbb{CP}^2\#\ell\overline{\mathbb{CP}^2}$, then k is odd. On the other hand, for $k = 2k' + 1$ odd, the manifold $k\mathbb{CP}^2\#\ell\overline{\mathbb{CP}^2}$ admits an almost complex structure by Wu's criterion. Indeed, choosing $c = (3, 1, 3, 1, \ldots, 3; 1, 1, \ldots, 1) \in H^2(k\mathbb{CP}^2\#\ell\overline{\mathbb{CP}^2}; \mathbb{Z})$ (where the ; is placed between positions k and $k+1$), we have that c reduces mod 2 to the second Stiefel–Whitney class, and that c^2 evaluates to $10k' + 9 - \ell = 2\chi + 3\sigma$. □

We remark that there might exist non-simply connected almost complex manifolds with $c_1 = 0$ rationally not satisfying the stronger set of SU congruences, and hence to obtain the above equivalence in dimensions congruent to 4 mod 8, we restrict to simply connected manifolds.

Remark 6.2. *A simply connected rational homotopy type is determined, up to homotopy equivalence, by a minimal C_∞–algebra structure on its cohomology (extending the given multiplication), up to isomorphism [13]. That is, for a given graded-commutative algebra (H, m_2), where m_2 is the multiplication, we have*

$$\{\text{rational spaces } X \text{ with } H^*X \cong (H, m_2)\}/\sim \;\equiv\; \{C_\infty \text{ structures } (H, m_2, m_3, m_4, \ldots)\}/\sim .$$

The realizability of a simply connected rational space by a simply connected closed almost complex manifold is insensitive to the higher operations $m_{\geq 3}$. Contrast this with the case of compact complex manifolds which satisfy the $\partial\bar{\partial}$–lemma (for example, compact Kähler manifolds), where among all rational homotopy types realizing a given cohomology algebra, at most one of them, namely the formal one, is realized by such a manifold [6].

In particular, for every simply connected almost complex manifold, there is a formal almost complex manifold with the same rational cohomology ring (with the same Chern numbers, so the two manifolds are furthermore complex cobordant). We remark that in dimensions 2 and 4 every closed simply connected manifold is formal. It is perhaps the other direction that is more interesting: knowing that a formal rational homotopy type can be re-

alized by a closed almost complex manifold implies that any rational homotopy type with the same cohomology ring can also be realized.

An easy consequence of Theorem 2.4 that demonstrates the abundance of rational homotopy types of closed almost complex manifolds is the following:

Corollary 6.3. Any simply connected rational space satisfying Poincaré duality of formal dimension $4k+2$, with Euler characteristic zero, is realized by a closed almost complex manifold.

Proof. Since the Euler characteristic vanishes, one can choose all rational Chern classes to be trivial, and make any choice of fundamental class, to satisfy the conditions of Theorem 2.4. □

In the case of formal dimension $4k$, note that the property from the necessary condition (iv), i.e. the middle–degree pairing being of the form $\sum_i \pm x_i^2$, is indepedent of the choice of fundamental class; see Remark 6.7 for details. Thus, choosing any fundamental class, and all rational Chern classes to be zero, we also have the following:

Corollary 6.4. Any simply connected rational space satisfying Poincaré duality of formal dimension $4k$, with Euler characteristic zero and signature zero, whose middle–degree pairing is of the form $\sum_i \pm x_i^2$, is realized by a closed almost complex manifold.

Example 6.5. For reference, we list the congruences among Chern numbers for stably almost complex manifolds of dimension ≤ 10. The congruences in dimension ≤ 8 were listed by Hirzebruch in [10]. We implicitly assume the given class is paired with the fundamental class of the stably almost complex manifold.

- Dimension 2: $c_1 \in 2\mathbb{Z}$
- Dimension 4: $c_1^2 + c_2 \in 12\mathbb{Z}$
- Dimension 6: $c_1^3 \in 2\mathbb{Z}$, $c_3 \in 2\mathbb{Z}$, $c_1c_2 \in 24\mathbb{Z}$
- Dimension 8:

$$\begin{aligned} 2c_1^4 + c_1^2c_2 &\in 12\mathbb{Z}, \\ c_1c_3 - 2c_4 &\in 4\mathbb{Z}, \\ -c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4 &\in 720\mathbb{Z} \end{aligned}$$

- Dimension 10:

$$\begin{aligned} c_1c_4 + c_5 &\in 12\mathbb{Z}, \\ 4c_1^3c_2 + 8c_1^2c_3 + c_1c_4 + 9c_5 &\in 24\mathbb{Z}, \\ 15c_1^5 - 5c_1^3c_2 + 12c_1c_2^2 + 8c_1^2c_3 - 8c_1c_4 &\in 24\mathbb{Z}, \\ c_1^5 + c_1^3c_2 + 6c_1^2c_3 &\in 12\mathbb{Z}, \\ -c_1^3c_2 + 3c_1c_2^2 + c_1^2c_3 - c_1c_4 &\in 1440\mathbb{Z}. \end{aligned}$$

From the congruences listed above, we see that any simply connected rational space satisfying Poincaré duality in dimension 6 is realized by an almost complex manifold. Indeed, we can choose $c_1 = 0$, $c_2 = 0$, and the fundamental class and c_3 so that c_3 evaluates to the Euler characteristic. The congruences in dimension 6 require c_3 to be even, but this will be automatically satisfied as the Euler characteristic of a $4k+2$–dimensional Poincaré duality algebra is even. A simply connected rational space satisfying rational Poincaré duality of formal dimension 6 is formal [23], that is, its rational homotopy type is determined by its cohomology algebra. From here, by degree reasons we see that any such rational homotopy type will be of the form $M\#N$, where M is a simply connected 6–manifold with $b_3 = 0$, and N is a connected sum of some number of copies of $S^3 \times S^3$. We remark that even for a small value of b_2 there are many rational homotopy types with $b_3 = 0$ and this b_2; for example the real homotopy type of $\mathbb{CP}^3\#\mathbb{CP}^3$ contains infinitely many rational homotopy types, i.e. rational cohomology algebras; see [16, Example 3.5]. (For $b_2 \leq 1$ and $b_3 = 0$ there is only one rational homotopy type.)

Remark 6.6. *On the cohomology of a 6–dimensional simply connected rational Poincaré duality space, duality gives us an isomorphism $H^2 \cong (H^4)^\vee$, and so the product $H^2 \otimes H^2 \to H^4$ is given by a symmetric trilinear form $H^2 \otimes H^2 \otimes H^2 \to \mathbb{Q}$. This trilinear form determines the cohomology algebra of our space, and hence its rational homotopy type. In the case of $\dim H^2 = 3$, such trilinear forms correspond to (rational) cubic plane curves; the abundance and structure of such curves, paired with choices of rational Chern classes, suggests this may be an interesting line of further study, cf. [44, Example p.322].*

In all even dimensions $n \geq 8$ there are examples of simply connected rational homotopy types not realized by almost complex manifolds; indeed one can take the rationalized spheres $S^n_{\mathbb{Q}}$ (see [1, Theorem 2.2], adapting a famous observation of Borel and Serre to the rational setting).

In dimension 10, we see from the congruences in Example 6.5 that any simply connected rational homotopy type satisfying Poincaré duality, with Euler characteristic divisible by 24, is realized by an almost complex manifold ,by setting the lower Chern classes to be zero. In all dimensions of the form $4k + 2$, we see that the only obstruction to realizability is a finite divisibility constraint on the Euler characteristic.

One can also ask about the realizability of *real* homotopy types by closed almost complex manifolds. Unfortunately, an immediate problem presents itself in this case as $H^*(K(\mathbb{R}, n); \mathbb{R})$ is *not* the free graded-commutative algebra on one generator [25]. In fact, $H_\star(K(\mathbb{R}, n); \mathbb{R})$ has uncountable dimension. It is the case that $H^*(K(\mathbb{R}, n); \mathbb{R})$ is the free graded-commutative algebra on one generator if one interprets the former in the context of *continuous cohomology*, but we do not pursue this here. Of course, for a given real homotopy type to be realized by a closed manifold, there must be a rational commutative differential graded algebra which reproduces the given real homotopy type upon tensoring with the reals. If we have such a rational cdga, we may take the corresponding rational homotopy type and attempt to realize it by a manifold as usual; however, the problem of detecting whether there exists such a rational cdga for a given real homotopy type seems quite delicate.

Remark 6.7. *In Sullivan's original formulation of the realization theorem for closed smooth manifolds [44, Theorem 13.2], one sees that the Stong congruences (for BSO; they are non-trivial only in dimensions of the form $n = 4k$) are not mentioned in the signature 0 case. If the quadratic form on $H^{n/2}(X; \mathbb{Q})$ given by $\alpha \otimes \beta \mapsto \langle \alpha\beta, [X]\rangle$ is equivalent over $\mathbb{Q}$ to one of the form $\sum_i \pm x_i^2$ for some choice of fundamental class $[X] \in H_n(X; \mathbb{Q})$, then it will be of this form for any other non-zero choice of $[X]' \in H_n(X; \mathbb{Q})$. Indeed, since the signature is zero, by assumption we can write the intersection form with respect to $[X]$ as $\sum_i x_i^2 - y_i^2$. Scaling the fundamental class by a rational changes this into $\sum_i \frac{p}{q} x_i^2 - \frac{p}{q} y_i^2$, which is the same as $\sum_i ((1 + \frac{p}{4q})x + (1 - \frac{p}{4q})y)^2 - ((1 - \frac{p}{4q})x + (1 + \frac{p}{4q})y)^2$. In particular, we may scale the fundamental class until all of the Stong congruences are satisfied. We cannot do the same in the almost complex realization problem in the signature 0 case, as our choice of top Chern class is tethered to the fundamental class by the requirement $\langle c_{2k}, [X]\rangle = \chi(X)$.*

7 Examples

We now give some concrete calculations concerning the realization of certain rational homotopy types by closed almost complex manifolds. As in Example 6.5, we implicitly pair top degree classes with the fundamental class.

7.1 Rational connected sums of quaternionic projective planes

Using the results of [7], one can calculate that $k\mathbb{HP}^2 \# \ell\overline{\mathbb{HP}^2}$ (with its standard smooth structure) admits an almost complex structure if and only if $(k, l) = (4n + 3, 2n + 1)$ for some n. Let us see what happens in the rational case; i.e. we consider 8–manifolds M with $H^*(M;\mathbb{Q}) \cong H^*(k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2};\mathbb{Q})$. We refer to such a manifold as a *rational* $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$.

We will use the Chern number congruences for stably almost complex 8–manifolds in what follows (see Example 6.5):

$$-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4 \in 720\mathbb{Z},$$
$$c_1^2c_2 + 2c_1^4 \in 12\mathbb{Z},$$
$$-2c_4 + c_1c_3 \in 4\mathbb{Z},$$

which in our case trivially becomes

$$-c_4 + 3c_2^2 \in 720\mathbb{Z}, \text{ and } c_4 \text{ is even.}$$

Now, suppose we have a rational $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ that admits an almost complex structure. Then $\sigma = k-\ell$ and $\chi = 2+k+\ell$, so from Hirzebruch's relation $\sigma \equiv \chi \bmod 4$ in dimension 8 [11, p.777], we have $k-\ell \equiv 2+k+\ell \bmod 4$, i.e. $2\ell \equiv 2 \bmod 4$, i.e. ℓ is odd. Since $k + \ell + 2 = \chi = c_4$ must be even, we conclude that k is odd as well.

We show that the above observation on when $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ admits almost complex structures does not carry over to the rational setting. Let us consider as a first example $k = \ell = 23$. Then c_4 must evaluate to $\chi = 48$, and $\sigma = 0$. We can write c_2 as $c_2 = \sum_{i=1}^{23} x_i + \sum_{i=1}^{23} y_i^2$, where the x_i and y_i are degree 4 classes such that $\langle x_i^2, \mu\rangle = 1$ and $\langle y_i^2, \mu\rangle = -1$ for an appropriate choice of fundamental class μ, and $x_ix_j = x_iy_j = y_iy_j = 0$ for all $i \neq j$. (The variables x_i correspond to the degree 4 generators in the $\mathbb{HP}^2$ summands, while the y_i correspond to $\overline{\mathbb{HP}^2}$.) The signature equation in terms of Chern classes is $\frac{1}{45}(3c_2^2 + 14c_4) = 0$, i.e. $c_2^2 = -224$. We see that the Stong congruences are satisfied for this c_2^2 and c_4. Indeed, c_4 is even and $-c_4 + 3c_2^2 = 0$. It only remains to check that one can solve for c_2. Taking $c_2 = 4y_1 + 8y_2 + 12y_3$, we have $c_2^2 = -224$.

By the almost complex realization theorem we conclude that there is an almost complex manifold realizing this data.

We now observe that the above fits into a more general solution; let $k, \ell \geq 0$ be arbitrary. The signature of a rational $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ is $k - \ell$ and the Euler characteristic is $2 + k + \ell$. Besides the Euler characteristic being even, we must have $3c_2^2 + 14c_4 = 45(k - \ell)$ and $-c_4 + 3c_2^2 \in 720\mathbb{Z}$. Let us write this as

$$3c_2^2 = 31k - 59\ell - 28,$$
$$3c_2^2 = 720m + k + \ell + 2.$$

From $31k-59\ell-28 = 720m+k+\ell+2$ we have $k = 2\ell+1+24m$. In particular, k is odd, so we write $k = 2n+1$. Then $\ell = n-12m$, and $c_2^2 = 236m+n+1$. Since $2+k+\ell$ must be even (by the Euler characteristic requirement), we have that ℓ is odd as well, i.e. n is odd; we write $n = 2u + 1$.

So, the solutions are $(k, \ell, c_2^2) = (4u + 3, 2u + 1 - 12m, 2u + 2 + 236m)$; since we require $k, \ell \geq 0$, we have $u \geq 0$ and $2u + 1 \geq 12m$. Fixing k and ℓ, i.e. u and m, we see that the problem of obtaining an almost complex rational $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ comes down to finding a class c_2 such that $c_2^2 = 2u + 2 + 236m$. We use the same notation for generators of the cohomology as in the case of $k = \ell = 23$ considered above. If $m = 0$, and $u = 0$, we have $(k, \ell, c_2^2) = (3, 1, 2)$, and we can take $c_2 = x_1 + x_2$, which satisfies $c_2^2 = 2$. If $m = 0$ and $u > 0$, then $k \geq 4$, and we may solve for c_2 using Lagrange's four–square theorem since $c_2^2 > 0$. If $m > 0$, then $u \geq 6$ and so $k \geq 4$, and we may again apply Lagrange's four–square theorem to solve for c_2 since $c_2^2 > 0$. If $m < 0$, then $\ell \geq 4$, and so if $c_2^2 \leq 0$ we can solve for c_2. If $m < 0$ and $c_2^2 > 0$, then $2u + 2 > -236m$, so in particular $k = 4u + 3 \geq 4$, and we can solve for c_2 again.

In conclusion, we have the following:

Proposition 7.1. *There is a closed almost complex manifold with the rational cohomology ring of* $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ *if and only if* $(k, l) = (4u + 3, 2u + 1 + 12m)$ *with* $k, l \geq 0$.

For $m \neq 0$ any such obtained manifold is consequently not of the same oriented diffeomorphism type as $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ with its standard smooth structure. The previous observation above with $k = \ell = 23$ is obtained by taking $u = 5$, $m = 1$. We refer the reader to the upcoming [40] for general results on almost complex manifolds with Betti numbers concentrated in middle degree.

Remark 7.2.
- *Generally, the coefficients along the x_i and y_i could be taken to be rational numbers, not necessarily integers. However, an integer is a sum of rational squares if and only if it is a sum of integer squares, so we are reduced to considering integer coefficients regardless.*
- *Above we used Hirzebruch's relation [11, p.777] that on a closed almost complex $4n$–manifold, we have $\chi \equiv (-1)^n \sigma \bmod 4$. Since every even–dimensional stably almost complex manifold is complex cobordant to an almost complex manifold [14, Corollary 5], and the Chern numbers and signature are complex cobordism invariants, this shows us that Hirzebruch's relation is the restriction to almost complex manifolds of a general congruence $c_{2n} \equiv (-1)^n \sigma \bmod 4$ for stably almost complex manifolds. Since this is a relation between Chern numbers, it is implied by the Stong congruences.*
  For example, in the case of almost complex 4–manifolds, we have $\chi + \sigma \equiv 0 \bmod 0$. This follows from the integrality of the Todd genus, $c_1^2 + c_2 \in 12\mathbb{Z}$, combined with $3\sigma = p_1 = c_1^2 - 2c_2$. Indeed, expressing c_1^2 in two ways gives us $12k - c_2 = 3\sigma + 2c_2$ for some integer k, i.e. $3(\sigma + c_2) = 12k$, whence $\sigma + c_2 \equiv 0 \bmod 4$.
- *Let M be an almost complex rational $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ constructed by the above procedure (so that $c_1 = 0$ integrally). By [7, Corollary 7], if $\overline{M}$ were to admit an almost complex structure, i.e. one inducing the opposite orientation on M, then $\chi(M) = 0$. This shows, for example, that a rational $23\mathbb{HP}^2\#23\overline{\mathbb{HP}^2}$ obtained as above does not admit an orientation-reversing diffeomorphism (even though the notation might suggest so).*

7.2 An almost complex rational $\mathbb{HP}^3$

We show the following:

Theorem 7.3. *There exists a closed simply connected almost complex 12–manifold with the rational homotopy type of $\mathbb{HP}^3$.*

A classical result of Massey [17] states that no $\mathbb{HP}^n$ admits an almost complex structure; a rational $\mathbb{HP}^1 = S^4$ does not admit an almost complex structure by a quick signature argument, and a rational $\mathbb{HP}^2$ does not admit an almost complex structure by a calculation with the Stong congruences. In general, the only dimension in which there exists a closed almost complex manifold with sum of (rational) Betti numbers 3 is dimension 4, by Zhixu Su [40] and Jiahao Hu's [12] independent resolutions of the case of dimensions equal to a power of two left open in [1].

Proof. Take the rational algebra $\mathbb{Q}[x]/(x^4)$, where x is of degree 4, and take any rational space with this as its cohomology, e.g. an $S^{15}_{\mathbb{Q}}$ bundle over $K(\mathbb{Q}, 4)$ where the degree 15 generator in S^{15} kills the fourth power of the generator in $K(\mathbb{Q}, 4)$ in the Serre spectral sequence. Choose the fundamental class so that x^3 evaluates to 1. Note that necessarily $c_1 = 0$ rationally, and so we will have to satisfy two sets of congruences in order to produce an almost complex manifold with this rational cohomology algebra. By [44, p.317, (v)] or [23], any closed manifold with this rational cohomology is formal, and in particular rationally homotopy equivalent to $\mathbb{HP}^3$.

Taking into consideration that necessarily $c_1 = c_3 = c_5 = 0$ rationally, the first set of SU congruences, coming from the condition $z \cdot Td \in \mathbb{Z}$ for all $z \in \mathbb{Z}[e_1, e_2, \ldots]$ (recall Theorem 2.1), come down to

$$\begin{aligned} 10c_2^3 - 9c_2c_4 + 2c_6 &\in 60480\mathbb{Z}, \\ c_2c_4 + 2c_6 &\in 240\mathbb{Z}, \\ -c_2^3 + 4c_2c_4 &\in 12\mathbb{Z}, \\ c_2^3 - 16c_2c_4 &\in 12\mathbb{Z}, \\ c_6 &\in 4\mathbb{Z}. \end{aligned}$$

As before, it is understood that the products of Chern classes above have been paired with the fundamental class.

The second set of SU congruences, coming from the condition $w \cdot \hat{A}(p_i) \in 2\mathbb{Z}$ for all $w \in \mathbb{Z}[e_1^p, e_2^p, \ldots]$ (recall Theorem 2.2), when translated into Chern classes, gives us

$$\begin{aligned} \tfrac{1}{6048}c_2^3 - \tfrac{1}{6720}c_2c_4 + \tfrac{1}{30240}c_6 &\in 2\mathbb{Z}, \\ -\tfrac{1}{120}c_2c_4 - \tfrac{1}{60}c_6 &\in 2\mathbb{Z}, \\ -\tfrac{1}{3}c_2^3 + \tfrac{4}{3}c_2c_4 &\in 2\mathbb{Z}, \\ -\tfrac{1}{12}c_2^3 + \tfrac{1}{3}c_2c_4 + \tfrac{1}{2}c_6 &\in 2\mathbb{Z}, \end{aligned}$$

The signature being 0 gives us, from the L–polynomial,

$$5c_2^3 - 36c_2c_4 - 68c_6 = 0.$$

Now, $c_2 = ax$ for some rational number a. Since $\langle a^3x^3, [X]\rangle = a^3$ must be an integer, a must be an integer. Also, $c_4 = bx^2$ for some rational number b; note that it does not follow that b is an integer, but let us look for $b \in \mathbb{Z}$ regardless. Then, since $\chi = 4$, we have $c_4 = 4$, and simplifying the above congruences gives us the follow system of Diophantine equations:

$$\begin{aligned} -a^3 + 4ab &\in 24\mathbb{Z}, \\ ab + 8 &\in 1920\mathbb{Z}, \\ 5a^3 - 36ab &= 248. \end{aligned}$$

This system has a solution of $a = -2$, $b = 4$, and hence by Theorem 2.4 we obtain the desired almost complex manifold. □

Conflict of interest: I hereby declare no conflict of interest.

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