Controllability of Switched Hilfer Neutral Fractional Dynamic Systems with Impulses

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Abstract: The aim of this work is to investigate the controllability of a class of switched Hilfer neutral fractional systems with non-instantaneous impulses in the finite-dimensional spaces. We construct a new class of control function that controls the system at the final time of the time-interval and controls the system at each of the impulsive points i.e., we give the so-called total controllability results. Also, we extend these results to the corresponding integro-system. We mainly use the fixed point theorem, Laplace transformation, Mittag-Leffler function, Gramian type matrices, and fractional calculus to establish these results. In the end, we provide a simulated example to verify the obtained analytical results.

Keywords: Switched systems; Controllability; Mittag-Leffler function; Fractional differential equations.
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1 Introduction

The theory of fractional calculus evolves from the differential and integral operators of arbitrary order. Nowadays, it has attracted many mathematicians, physicists, and engineers. There is also with a growing number of applications in signal processing, control theory, biomedicine, viscoelasticity, electrochemistry, physics, etc (please see [2, 40, 25] and the references therein). However, the applications of fractional calculus and their outcomes have change as much as the definitions of fractional derivatives and integrals, such as Riemann-Liouville, Caputo, Caputo-Fabrizio, Riesz-Caputo, Grunwald-Letnikov, and so on. For the basic study of fractional calculus, one can go through [38, 39, 30] and references therein. Recently, Hilfer [21] introduced a new fractional derivative, known as Hilfer fractional derivative of the form $D_{0^+}^{\alpha,\beta}$, where α is the order, and β is the type. The type β allows one to interpolate between the Riemann-Liouville derivative $\beta = 0$), and the so-called Caputo-Liouville derivative ($\beta = 1$). Therefore, the results obtained from Hilfer fractional derivative extend and generalize the existing results of Riemann-Liouville or Caputo-Liouville fractional derivative.

Hilfer fractional models are studied in many applications of engineering and science, for example, in mechanical engineering and thermal science [14]. Very recentely, many authors investigated the Hilfer fractional differential equations and studied the various dynamic behaviours, such as the existence of solution, stability, data dependence, and control problems, see [13, 17, 7, 24, 47, 18, 54, 55, 48, 33, 15] and the cited references therein.

Controllability is an important aspect of mathematical control theory which was introduced by Kalman in 1963 [22]. The concept of controllability denotes the ability to move the state of the dynamical control system from an initial state to the desired final state by using a suitable control function. In the last few years, many authors have studied the controllability problem for ordinary as well as fractional dynamic systems, see for instance [4, 16, 6, 5, 46, 41] and the cited references therein. Furthermore, few authors have studied the controllability problem for the Hilfer fractional systems [23, 10, 43, 36, 49, 37, 56].

On the other hand, the various systems encountered in practice involve a coupling between continuous dynamics and discrete events. These types of systems can be studied in terms of switched dynamical systems. A switched system is a dynamic system consisting of a family of continuous-time subsystems along with a switching rule that determines the switching among subsystems. Mathematically, these subsystems are generally described by a collection of differential equations or differences indexed. For instance, the following phenomena give rise to switching behavior: dynamics of a vehicle changing unexpectedly because of wheels bolting and opening on ice; an airplane entering, intersection and leaving an air traffic control area; biological cells developing and separating; a thermostat turning the heat on and off; a valve or a power switch opening and closing [31, 32].

It is well known that many physical systems in engineering, biology, physics, and information science, have some sudden changes in their states. Such sudden changes are called the impulsive effect in the systems, and the corresponding systems are called the instantaneous impulsive systems [29, 42]. Recently, in 2013, Hernández and O'Regan [20], introduced a new class of impulsive systems, known as non-instantaneous impulsive systems in which the sudden changes stay active for a finite time interval. In practicality, there is no impulse that occurs instantaneously rather it is non-instantaneous howsoever the time of occurrence is small. For example, in some real biological medical problems, the introduction of a drug or a vaccine in the bloodstream is a gradual process. Then one is forced to consider the drug or vaccine as a non-instantaneous impulse since it starts abruptly but remains active for a finite time interval [20, 3]; in dam pollution models, the main cause of dam pollution is the polluted river, that enters the dam and takes some time to reach the middle region of the dam. Since the introduction of the river water into the dam and the consequent absorption of the dam water are gradual and continuous processes, so that the non-instantaneous impulses take place [11]. For further studies on non-instantaneous impulsive systems, see [12, 19, 34, 8, 35, 52, 51] and the cited references therein.

Recently, some authors have reported a few controllability results for the ordinary as well as fractional dynamic system with non-instantaneous impulses, see for instance [50, 44, 45, 28, 27, 26] and the cited references therein. Also, some authors established these results for the Hilfer type fractional dynamic systems with non-instantaneous impulses. Particularly, in [53], the authors investigated the controllability of Hilfer fractional dynamic inclusions with nonlocal and non-instantaneous impulsive conditions in Banach spaces by using the fixed point technique along with measures of noncompactness. In [1], the authors studied the problem of approximate controllability for the Hilfer fractional neutral stochastic integrodifferential equations with fractional Brownian motion and non-instantaneous impulses by using the Sadovskii's fixed point theorem and fractional power of operators. In [9], the authors studied the problem of approximate controllability for Hilfer fractional differential inclusions with non-instantaneous impulsive hybrid systems on weighted spaces by using the family of fractional resolvent operators, Laplace transformation, and a hybrid fixed point theorem for three operators of the Schaefer's type.

The above-mentioned works on controllability of Hilfer fractional systems and non-instantaneous impulsive systems cannot be easily extended to the case of switched Hilfer fractional dynamical systems with non-instantaneous impulses. From the authors' points of view, there is no work reported which investigated the total controllability of a class of switched Hilfer neutral fractional systems. Therefore, considering the importance of switched systems and non-instantaneous impulses, we fill this gap by establishing the total controllability results for a class of switched neutral systems with Hilfer fractional derivative and non-instantaneous impulses in the finite-dimensional spaces.

The primary contribution and advantage of this paper can be foreground as follows:

- We consider a new class of switched Hilfer neutral fractional systems with non-instantaneous impulses in the finite-dimensional spaces.
- We define a new piecewise control function for the proposed system and examine the total controllability result, where we control the system not only at the final point of the interval but also at every impulse point of the interval.
- Also, we establish the total controllability results for the considered problem with the integral term.
- We apply the fixed point technique, Mittag-Leffler function and Gramian type matrices to establish these results.

- We illustrate the general applicability of the obtained analytical results by giving a simulated numerical example.

The rest of the paper is formulated as follows: In Section 2, the statement of the problem is given. In Section 3, we give some basic definitions, notations, and important lemmas. In Section 4, we investigate the total controllability of the considered problem. In Section 5, we extend the results of the section 4 to the integro dynamic system. In the last section, we give a numerical example with simulation to show the effectiveness of the obtained theoretical results.

2 Statement of the Problem

We consider the following switched impulsive control system:

$$\begin{aligned} D^{\alpha,\beta}_{\vartheta_i^+}[x(t)-\Upsilon_{\sigma(t)}(t,x_a(t))] &= \mathcal{A}_{\sigma(t)}[x(t)-\Upsilon_{\sigma(t)}(t,x_a(t))]+\Psi_{\sigma(t)}(t,x_b(t))+\mathcal{B}_{\sigma(t)}u(t),\ t\in\cup_{i=0}^{m}(\vartheta_i,t_{i+1}],\\ x(t) &= \mathcal{J}_{\sigma(t)}(t,x(t_i^-)),\ t\in(t_i,\vartheta_i],\ i=1,2,...,m,\\ I^{1-\gamma}_{0^+}x(0^+) &= x_0,\ I^{1-\gamma}_{\vartheta_i^+}x(\vartheta_i^+)=\mathcal{J}_{\sigma(t)}(\vartheta_i,x(t_i^-)),\ \gamma=\alpha+\beta-\alpha\beta, \end{aligned} \tag{2.1}$$

where $x\in\mathbb{R}^n$ is the state variable; $D^{\alpha,\beta}_{\vartheta_i^+}$ denotes the Hilfer fractional derivative with lower limit at ϑ_i of the order $\alpha\in(0,1)$ and type $\beta\in[0,1]$; $a,b:I\to I$ are some delay functions such that $a(t),b(t)\le t$; $x_a(t)=x(a(t))$ and $x_b(t)=x(b(t))$; $\sigma:I\to\{0,1,...,m\}$ is some switching law; ϑ_i and t_i are some arbitrary points which satisfy the relation $0=t_0=\vartheta_0<t_1<\vartheta_1<t_2<...<\vartheta_m<t_{m+1}=T$; $x(t_i^+)=\lim_{h\to0^+}x(t_i+h)$ and $x(t_i^-)=\lim_{h\to0^+}x(t_i-h)$ denote the right and left limit of $x(t)$ at $t=t_i$ respectively;

$I^{1-\gamma}_{\vartheta_i^+}$ represent the left-sided Riemann–Liouville integral of order $1-\gamma$ with lower limit at ϑ_i, and $I^{1-\gamma}_{\vartheta_i^+}x(\vartheta_i^+)=\lim_{t\to\vartheta_i^+}I^{1-\gamma}_{\vartheta_i^+}x(t)$;

$\mathcal{A}_{\sigma(t)}$ and $\mathcal{B}_{\sigma(t)}$ are some matrices of order $n\times n$ and $n\times m$ respectively; $u\in\mathbb{R}^m$ is the control function, $\Upsilon_{\sigma(t)},\Psi_{\sigma(t)},\mathcal{J}_{\sigma(t)}$ are some given functions which satisfies some conditions to be specified later.

In this manuscript, the switching signal σ is assumed to be known and satisfies the minimal dwell time condition. It only changes its values at switching times t_i. The discrete state $\sigma(t)\in\{0,1,...,m\}$ determines the actual system dynamics among the possible operating modes which corresponds to a specific instance of $\mathcal{A}_i,\mathcal{B}_i,\Upsilon_i,\Psi_i$ and $\mathcal{J}_i$. That is to say,

$$\sigma(t)=i,\ t\in[t_i,t_{i+1}),\ i=0,1,...,m. \tag{2.2}$$

Consequently, using the above switching law in system (2.1), we get the following switched impulsive control system

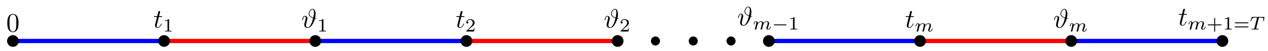
$$\begin{aligned} D^{\alpha,\beta}_{\vartheta_i^+}[x(t)-\Upsilon_i(t,x_a(t))] &= \mathcal{A}_i[x(t)-\Upsilon_i(t,x_a(t))]+\Psi_i(t,x_b(t))+\mathcal{B}_iu(t),\ t\in\cup_{i=0}^{m}(\vartheta_i,t_{i+1}],\\ x(t) &= \mathcal{J}_i(t,x(t_i^-)),\ t\in(t_i,\vartheta_i],\ i=1,2,...,m,\\ I^{1-\gamma}_{0^+}x(0^+) &= x_0,\ I^{1-\gamma}_{\vartheta_i^+}x(\vartheta_i^+)=\mathcal{J}_i(\vartheta_i,x(t_i^-)),\ \gamma=\alpha+\beta-\alpha\beta \end{aligned} \tag{2.3}$$

and hence, under the switching law (2.2), the dynamic behaviours of switched control system (2.1) and the switched control system (2.3) are same.

Remark 2.1. *Here, we are giving a brief description of the problem (2.3).*

- *$x(t)$ satisfies the first equation of the problem (2.3) when $t\in(0,t_1]$.*
- *$x(t)$ is given by the second equation of the problem (2.3) when $t\in(t_1,\vartheta_1]$.*
- *$x(t)$ satisfies the first equation of the problem (2.3) when $t\in(\vartheta_1,t_2]$.*
- *After repeating this process, $x(t)$ satisfies the first equation of the problem (2.3) on the interval $(\vartheta_i,t_{i+1}]$ and $x(t)$ is given by the second equation of the problem (2.3) on the interval $(t_i,\vartheta_i]$, .*

Graphically, this means that the solution $x(t)$ satisfies the first equation of the problem (2.3) on the blue intervals $(\vartheta_i,t_{i+1}],\ i=0,1,\cdots,m$ and the second equation of the problem (2.3) on the red intervals $(t_i,\vartheta_i],\ i=1,2,\cdots,m$.



3 Preliminaries and Assumptions

Below we introduce some basic definitions, notations, lemmas and important results which are often used throughout the manuscript.

Important notations: $\mathbb{R}^n$ denotes the space of $n-$dimensional column vectors $x = \mathrm{col}(x_1, x_2, ..., x_n)$ with a norm $\|\cdot\|$; $I = [0, T], T > 0$; $\Gamma(\cdot)$ and $B(\cdot,\cdot)$ denote the usual Gamma and Beta function, respectively; Superscript $*$ denotes the matrix transpose of a matrix.

$C(I, \mathbb{R}^n)$ denotes the Banach space of all continuous functions $f : I \to \mathbb{R}^n$ with the norm $\|f\| = \sup_{t\in I} \|f(t)\|$. We define the Banach space of all piecewise continuous functions $PC = PC_\gamma(I, \mathbb{R}^n) = \{x : (t - t_i)^{1-\gamma}x(t) \in C((t_i, t_{i+1}], \mathbb{R}^n),\ i = 0, 1, ..., m$ and there exists $x(t_i^-)$ and $x(t_i^+),\ i = 1, 2, ..., m,$ with $x(t_i^-) = x(t_i)\}$ with the norm $\|x\|_\gamma = \sup_{t\in[0,T]} t^{1-\gamma}\|x(t)\|$.

Next, for a function $f : [a, \infty) \to \mathbb{R}$, we define the following definitions:

Definition 3.1 ([39]). *The fractional Riemann-Liouville integral of f of order $p > 0$ with lower limit a is given by*

$$I^p_{a^+}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t - \varsigma)^{p-1}f(\varsigma)d\varsigma,\ t > a,$$

provided R.H.S of the above equation is point-wise defined on $[a, \infty)$.

Definition 3.2 ([39]). *The fractional Riemann-Liouville derivative of f of order $p > 0$ is defined by*

$$D^p_{a^+}f(t) = \frac{1}{\Gamma(n - p)}\frac{d^n}{dt^n}\int_a^t (t - \varsigma)^{n-1-p}f(\varsigma)d\varsigma,\ t > a,\ n - 1 < p < n.$$

Definition 3.3 ([39]). *The Caputo fractional derivative of f of order $p > 0$ is defined by*

$${}^cD^p_{a^+}f(t) = D^p_{a^+}\left[f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right],\ t > a,\ n - 1 < p < n.$$

Definition 3.4 ([21]). *The generalized Riemann-Liouville fractional derivative (or Hilfer derivative) of f with the order $0 < \alpha < 1$ and type $0 \le \beta \le 1$ with lower limit a is defined by*

$$D^{\alpha,\beta}_{a^+}f(t) = (I^{\beta(1-\alpha)}_{a^+}\frac{d}{dt}(I^{(1-\gamma)}_{a^+}f))(t),\ \gamma = \alpha + \beta - \alpha\beta,$$

provided that the expression on the R.H.S. exists.

From the above definition, we have the following remark.

Remark 3.5. *(i) When $\beta = 0$, $\alpha \in (0, 1)$ and $a = 0$, the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative:*

$$D^{\alpha,0}_{0^+} = \frac{d}{dt}(I^{(1-\alpha)}_{0^+}f)(t) = D^\alpha_{0^+}f(t).$$

(ii) When $\beta = 1$, $\alpha \in (0, 1)$ and $a = 0$, the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative:

$$D^{\alpha,1}_{0^+} = (I^{(1-\alpha)}_{a^+}\frac{d}{dt}f)(t) = {}^cD^\alpha_{0^+}f(t).$$

Next, we define some basics of *Mittag-Leffler* functions.

The Mittag-Leffler function is defined as

$$E_{p,q}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(kp+q)},\ z\in\mathbb{C},\ p,q>0$$

and the Laplace transform is given by

$$\mathcal{L}\{t^{p-1}E_{p,q}(\pm at^p)\}(s)=\frac{s^{p-q}}{s^p\mp a}.$$

The Mittag-Leffler function for a matrix A of order $n\times n$ is defined as

$$E_{p,q}(A)=\sum_{k=0}^{\infty}\frac{A^k}{\Gamma(kp+q)},\ z\in\mathbb{C},\ p,q>0$$

and the Laplace transform is given by

$$\mathcal{L}\{t^{p-1}E_{p,q}(\pm At^p)\}(s)=\frac{s^{p-q}}{s^p\mp A}.$$

For more details on fractional calculus, please see the books [38, 39].

Next, we give an important lemma.

Lemma 3.6. *Let $\mathcal{A}$ be a $n\times n$ matrix and $\Upsilon\in C(I,\mathbb{R}^n)$ be a function. Then, the solution of the following Hilfer fractional system*

$$\begin{aligned} &D_{0^+}^{\alpha,\beta}x(t)=\mathcal{A}x(t)+f(t)\,,\ \alpha\in(0,1),\ \beta\in[0,1],\ t\in(0,T],\\ &I_{0^+}^{1-\gamma}x(0^+)=x_0,\ \gamma=\alpha+\beta-\alpha\beta, \end{aligned}\tag{3.4}$$

is

$$x(t)=t^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}t^\alpha)x_0+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}(t-\varsigma)^\alpha)f(\varsigma)d\varsigma \text{ for all } t\in(0,T].$$

Proof. Taking the Laplace transform on both sides of the above equation, we get

$$\begin{aligned} &s^\alpha X(s)-s^{\beta(\alpha-1)}(D_{0^+}^{(1-\beta)(\alpha-1),0}x)(0^+)=\mathcal{A}X(s)+F(s)\\ &s^\alpha X(s)-\mathcal{A}X(s)=s^{\beta(\alpha-1)}x_0+F(s)\\ &X(s)=\frac{s^{\beta(\alpha-1)}}{s^\alpha\mathcal{I}-\mathcal{A}}x_0+\frac{F(s)}{s^\alpha\mathcal{I}-\mathcal{A}}, \end{aligned}$$

where $\mathcal{I}$ is the identity matrix. Now, taking the inverse Laplace transform to both sides of the last expression, we get

$$\mathcal{L}^{-1}\{X(s)\}(t)=\mathcal{L}^{-1}\left\{\frac{s^{\beta(\alpha-1)}}{s^\alpha\mathcal{I}-\mathcal{A}}\right\}(t)x_0+\mathcal{L}^{-1}\left\{\frac{F(s)}{s^\alpha\mathcal{I}-\mathcal{A}}\right\}(t)$$

Finally substituting the Laplace transformation of Mittag-Leffler function and Laplace convolution operator, we get

$$\begin{aligned} x(t)&=t^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}t^\alpha)x_0+\Upsilon(t)*t^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}t^\alpha)\\ &=t^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}t^\alpha)x_0+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}(t-\varsigma)^\alpha)f(\varsigma)d\varsigma. \end{aligned}$$

□

Remark 3.7. *One could notice that if $\beta=1$, in the equation (3.4), then it has the well known solution*

$$x(t)=E_\alpha(\mathcal{A}t^\alpha)x_0+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}(t-\varsigma)^\alpha)f(\varsigma)d\varsigma \text{ for all } t\in(0,T],$$

where $E_\alpha(x)=E_{\alpha,1}(x)$ denotes the ordinary Mittag-Leffler function.

Now, by using the Lemma 3.6, we can say a function $x \in PC$ is a solution of the system (2.3), if x satisfies

(i) $I_{0^+}^{1-\gamma}x(0^+) = x_0$ and $I_{\vartheta_i{}^+}^{1-\gamma}x(\vartheta_i^+) = \mathcal{J}_i(\vartheta_i, x(t_i^-))$,

(ii) $x(t) = \mathcal{J}_i(t, x(t_i^-)),\ t \in (t_i, \vartheta_i],\ i = 1, 2, ..., m$

and the following equations

$$x(t) = t^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t^{\alpha})[x_0 - \Upsilon_0(0, x_0)] + \Upsilon_0(t, x_a(t)) + \int_0^t (t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^{\alpha})\Psi_0(\varsigma, x_b(\varsigma))d\varsigma + \int_0^t (t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^{\alpha})\mathcal{B}_0 u(\varsigma)d\varsigma$$

for $t \in (0, t_1]$ and

$$x(t) = (t-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t-\vartheta_i)^{\alpha})[\mathcal{J}_i(\vartheta_i, x(t_i^-)) - \Upsilon_i(\vartheta_i, x_a(\vartheta_i))] + \Upsilon_i(t, x_a(t)) + \int_{\vartheta_i}^t (t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\Psi_i(\varsigma, x_b(\varsigma))d\varsigma + \int_{\vartheta_i}^t (t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\mathcal{B}_i u(\varsigma)d\varsigma$$

for $t \in (\vartheta_i, t_{i+1}],\ i = 1, 2, ..., m$.

Next, we give some assumptions which are required to establish the main results of this paper as follows:

(H1): $\Upsilon_i, \Psi_i : T_i \times \mathbb{R}^n \to \mathbb{R}^n, T_i = [\vartheta_i, t_{i+1}],\ i = 0, 1, ..., m$, are continuous and satisfy

$$\|\Upsilon_i(t, x) - \Upsilon_i(t, y)\| \le L_{\Upsilon}\|x - y\| \text{ and } \|\Psi_i(t, x) - \Psi_i(t, y)\| \le L_{\Psi}\|x - y\|,$$

for all $x, y \in \mathbb{R}^n$ and $t \in T_i$, where L_{Υ} and L_{Ψ} are some positive numbers.

(H2): $\mathcal{J}_i : J_i \times \mathbb{R}^n \to \mathbb{R}^n, J_i = [t_i, \vartheta_i],\ i = 1, 2, ..., m$, are continuous and satisfy

$$\|\mathcal{J}_i(t, x) - \mathcal{J}_i(t, y)\| \le L_{\mathcal{J}}\|x - y\|,$$

for all $x, y \in \mathbb{R}^n$ and $t \in J_i$, where $L_{\mathcal{J}}$ is a positive constant.

(H3): The matrices

$$\mathcal{G}_{\vartheta_i}^{t_{i+1}} = \int_{\vartheta_i}^{t_{i+1}} E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\mathcal{B}_i\mathcal{B}_i^* E_{\alpha,\alpha}(\mathcal{A}_i^*(t_{i+1}-\varsigma)^{\alpha})d\varsigma,\ i = 0, 1, ..., m. \tag{3.5}$$

are invertible and there exist some positive constants $M_{\mathcal{G}}^i, i = 0, 1, ..., m$, such that $\|(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\| \le M_{\mathcal{G}}^i$. Also, there exists a positive constant $M_{\mathcal{B}}$ such that for $i = 0, 1, ..., m$, $\|\mathcal{B}_i\| \le M_{\mathcal{B}}$.

Now onwards, throughout the manuscript, we set

$K_1 = \max_{i=0,1,...,m}\sup_{t\in I}\|E_{\alpha,\gamma}(\mathcal{A}_i t^{\alpha})\|$; $K_2 = \max_{i=0,1,...,m}\sup_{t\in I}\|E_{\alpha,\alpha}(\mathcal{A}_i(T-t)^{\alpha})\|$;

$K_3 = \max_{i=0,1,...,m}\sup_{t\in I}(T-t)^{1-\alpha}\|\mathcal{B}_i^* E_{\alpha,\alpha}(\mathcal{A}_i^*(T-t)^{\alpha})\|$; $M_{\Upsilon} = \max_{i=0,1,...,m}\sup_{t\in I}\|\Upsilon_i(t, 0)\|$;

$M_{\Psi} = \max_{i=0,1,...,m}\sup_{t\in I}\|\Psi_i(t, 0)\|$; $M_{\mathcal{J}} = \max_{i=1,...,m}\sup_{t\in I}\|\mathcal{J}_i(t, 0)\|$;

$S_i = \dfrac{K_2K_3M_{\mathcal{B}}M_{\mathcal{G}}^i t_{i+1}^{\alpha}}{\alpha}, i = 0, 1, ..., m$; $\mathcal{N}_0 = K_1(\|x_0\| + \|\Upsilon_0(0, x_0)\|) + t_1^{1-\gamma}M_{\Upsilon} + \dfrac{K_2M_{\Psi}t_1^{\alpha+1-\gamma}}{\alpha}$;

$\mathcal{N}_i = K_1(M_{\mathcal{J}} + M_{\Upsilon}) + t_{i+1}^{1-\gamma}M_{\Upsilon} + \dfrac{K_2M_{\Psi}t_{i+1}^{\alpha+1-\gamma}}{\alpha},\ i = 1, 2, ..., m$; $\mathcal{Q}_0 = L_{\Upsilon} + t_1^{\alpha}K_2L_{\Psi}B(\gamma, \alpha)$;

$\mathcal{Q}_i = t_{i+1}^{\gamma-1}K_1(L_{\mathcal{J}} + L_{\Upsilon}) + L_{\Upsilon} + t_{i+1}^{\alpha}K_2L_{\Psi}B(\gamma, \alpha),\ i = 1, 2, ..., m$;

$\mathcal{M}_i = \mathcal{N}_i + S_i(t_{i+1}^{1-\gamma}\|x_{t_{i+1}}\| + \mathcal{N}_i)$; $\mathcal{R}_i = \mathcal{Q}_i(1 + S_i),\ i = 0, 1, ..., m$;

$L_{F_1} = \max(\max_{0\le i\le m}\mathcal{R}_i,\ L_{\mathcal{J}})$.

4 Controllability Results

Here, we establish the total controllability results for the switched control system (2.3) by using the Banach contraction principle.

Before giving the main results, we give some important definitions as follows.

Definition 4.1. *Switched control system (2.3) is controllable on I, if for every $x_0, x_T \in \mathbb{R}^n$, there exists a control function $u \in L^2(I, \mathbb{R}^m)$*
such that the solution of (2.3) satisfies $I_{0^+}^{1-\gamma}x(0^+) = x_0$ and $x(T) = x_T$.

Definition 4.2. *Switched control system (2.3) is totally controllable on I, if it is controllable on $(0, t_1]$ and $(\vartheta_i, t_{i+1}]$, $i = 1, 2, ..., m$, i.e., for every $x_0, x_{t_{i+1}} \in \mathbb{R}^n, i = 0, 1, ..., m$,*
there exists a control function $u \in L^2(I, \mathbb{R}^m)$
such that the solution of (2.3) satisfies $I_{0^+}^{1-\gamma}x(0^+) = x_0$ and $x(t_{i+1}) = x_{t_{i+1}}$, $i = 0, 1, ..., m$.

Remark 4.3. *From Definition 4.1 and Definition 4.2, it is clear that Definition 4.2 implies Definition 4.1.*

Now, we provide some important lemmas.

Lemma 4.4. *Let the assumptions (H1)–(H3) hold, then the control function*

$$\begin{aligned} u(t) = (t_1 - t)^{1-\alpha}\mathcal{B}_0^* E_{\alpha,\alpha}(\mathcal{A}_0^*(t_1 - t)^\alpha)(\mathcal{G}_0^{t_1})^{-1}\Big[x_{t_1} - t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0, x_0)) \\ - \Upsilon_0(t_1, x_a(t_1)) - \int_0^{t_1}(t_1 - \varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1 - \varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))d\varsigma\Big], \quad t \in (0, t_1]. \end{aligned} \tag{4.6}$$

steers the state of the control system (2.3) from x_0 to x_{t_1} at time $t = t_1$. Also, the estimate of control function $u(t)$ is $\|u(t)\| \le M_u^0$ for all $t \in (0, t_1]$, where

$$M_u^0 = K_3 M_{\mathcal{G}}^0\Big[\|x_{t_1}\| + t_1^{\gamma-1}\mathcal{N}_0 + L_\Upsilon \sup_{t\in[0,t_1]}\|x(t)\| + K_2 L_\Psi t_1^{\alpha+\gamma-1}B(\gamma, \alpha)\|x\|_\gamma\Big].$$

Proof. By putting $t = t_1$ in the solution $x(t)$ of the system (2.3) on $(0, t_1]$, we get

$$\begin{aligned} x(t_1) &= t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0, x_0)) + \Upsilon_0(t_1, x_a(t_1)) + \int_0^{t_1}(t_1 - \varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1 - \varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))d\varsigma \\ &\quad + \int_0^{t_1}(t_1 - \varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1 - \varsigma)^\alpha)\mathcal{B}_0 u(\varsigma)d\varsigma \\ &= t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0, x_0)) + \Upsilon_0(t_1, x_a(t_1)) + \int_0^{t_1}(t_1 - \varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1 - \varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))d\varsigma \\ &\quad + \int_0^{t_1} E_{\alpha,\alpha}(\mathcal{A}_0(t_1 - \varsigma)^\alpha)\mathcal{B}_0\mathcal{B}_0^* E_{\alpha,\alpha}(\mathcal{A}_0^*(t_1 - t)^\alpha)(\mathcal{G}_0^{t_1})^{-1}\Big[x_{t_1} - t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0, x_0)) \\ &\quad - \Upsilon_0(t_1, x_a(t_1)) - \int_0^{t_1}(t_1 - \tau)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1 - \tau)^\alpha)\Psi_0(\tau, x_b(\tau))d\tau\Big]d\varsigma \\ &= t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0, x_0)) + \Upsilon_0(t_1, x_a(t_1)) + \int_0^{t_1}(t_1 - \varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1 - \varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))d\varsigma \\ &\quad + \mathcal{G}_0^{t_1}(\mathcal{G}_0^{t_1})^{-1}\Big[x_{t_1} - t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0, x_0)) - \Upsilon_0(t_1, x_a(t_1)) \\ &\quad - \int_0^{t_1}(t_1 - \varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1 - \varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))d\varsigma\Big] \\ &= x_{t_1}. \end{aligned}$$

Therefore, control function (4.6) is suitable for $t \in (0, t_1]$. Furthermore,

$$\begin{aligned}
\|u(t)\| &\leq \|(t_1-t)^{1-\alpha}\mathcal{B}_0^* E_{\alpha,\alpha}(\mathcal{A}_0^*(t_1-t)^\alpha)(\mathcal{G}_0^{t_1})^{-1}\| \Big[\|x_{t_1}\| + \|t^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0,x_0))\| \\
&\quad + \|\Upsilon_0(t_1, x_a(t_1))\| + \int_0^{t_1}(t_1-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))\|d\varsigma\Big] \\
&\leq K_3 M_{\mathcal{G}}^0\Big[\|x_{t_1}\| + t_1^{\gamma-1}K_1(\|x_0\| + \|\Upsilon_0(0,x_0)\|\|) + M_\Upsilon + L_\Upsilon\|x(t_1)\| + K_2 M_\Psi\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}d\varsigma \\
&\quad + K_2 L_\Upsilon\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}\|x(\varsigma)\|d\varsigma\Big] \\
&\leq K_3 M_{\mathcal{G}}^0\Big[\|x_{t_1}\| + t_1^{\gamma-1}K_1(\|x_0\| + \|\Upsilon_0(0,x_0)\|\|) + M_\Upsilon + L_\Upsilon\sup_{t\in[0,t_1]}\|x(t)\| + \frac{K_2 M_\Psi t_1^\alpha}{\alpha} \\
&\quad + K_2 L_\Psi t_1^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_\gamma\Big] \\
&= M_u^0,
\end{aligned}$$

where we use

$$\begin{aligned}
\int_a^t (t-\varsigma)^{\alpha-1}\|x(\varsigma)\|d\varsigma &\leq \left(\int_a^t (t-\varsigma)^{\alpha-1}(\varsigma-a)^{\gamma-1}d\varsigma\right)\|x\|_\gamma \\
&= (t-a)^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_\gamma.
\end{aligned}$$

□

Lemma 4.5. *Let the assumptions (H1)–(H3) hold, then the control*

$$\begin{aligned}
u(t) &= (t_{i+1}-t)^{1-\alpha}\mathcal{B}_i^* E_{\alpha,\alpha}(\mathcal{A}_i^*(t_{i+1}-t)^\alpha)(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\Big[x_{t_{i+1}} - (t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)[\mathcal{J}_i(\vartheta_i, x(t_i^-)) \\
&\quad - \Upsilon_i(\vartheta_i, x_a(\vartheta_i))] - \Upsilon_i(t_{i+1}, x_a(t_{i+1})) - \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\Psi_i(\varsigma, x_b(\varsigma))d\varsigma\Big], \qquad (4.7)
\end{aligned}$$

for $t \in (\vartheta_i, t_{i+1}], i = 1, 2, ..., m$, steers the state of the control system (2.3) from x_0 to $x_{t_{i+1}}$ at time $t = t_{i+1}$. Also, the estimate of control function $u(t)$ is $\|u(t)\| \leq M_u^i$ for all $t \in (\vartheta_i, t_{i+1}], i = 1, 2, ..., m$, where

$$M_u^i = K_3 M_{\mathcal{G}}^i\left[\|x_{t_{i+1}}\| + t_{i+1}^{\gamma-1}\mathcal{N}_i + (t_{i+1}^{\gamma-1}K_1(L_{\mathcal{J}} + L_\Upsilon) + L_\Upsilon)\sup_{t\in[\vartheta_i,t_{i+1}]}\|x(t)\| + K_2 L_\Psi t_{i+1}^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_\gamma\right].$$

Proof. By putting $t = t_{i+1}$ in the solution $x(t)$ of the system (2.3) on $(\vartheta_i, t_{i+1}], i = 1, 2, ..., m$, we get

$$\begin{aligned}
x(t_{i+1}) &= (t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)[\mathcal{J}_i(\vartheta_i, x(t_i^-)) - \Upsilon_i(\vartheta_i, x_a(\vartheta_i))] + \Upsilon_i(t_{i+1}, x_a(t_{i+1})) \\
&\quad + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\Psi_i(\varsigma, x_b(\varsigma))d\varsigma + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\mathcal{B}_i u(\varsigma)d\varsigma \\
&= (t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)[\mathcal{J}_i(\vartheta_i, x(t_i^-)) - \Upsilon_i(\vartheta_i, x_a(\vartheta_i))] + \Upsilon_i(t_{i+1}, x_a(t_{i+1})) \\
&\quad + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\Psi_i(\varsigma, x_b(\varsigma))d\varsigma + \int_{\vartheta_i}^{t_{i+1}}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\mathcal{B}_i \\
&\quad \times \mathcal{B}_i^* E_{\alpha,\alpha}(\mathcal{A}_i^*(t_{i+1}-t)^\alpha)(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\Big[x_{t_{i+1}} - (t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)[\mathcal{J}_i(\vartheta_i, x(t_i^-)) \\
&\quad - \Upsilon_i(\vartheta_i, x_a(\vartheta_i))] - \Upsilon_i(t_{i+1}, x_a(t_{i+1})) - \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\tau)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\tau)^\alpha)\Psi_i(\tau, x_b(\tau))d\tau\Big]d\varsigma
\end{aligned}$$

$$
\begin{aligned}
&= (t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^{\alpha})[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]+\Upsilon_i(t_{i+1},x_a(t_{i+1}))\\
&\quad+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\Psi_i(\varsigma,x_b(\varsigma))d\varsigma+\mathcal{G}_{\vartheta_i}^{t_{i+1}}(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\Big[x_{t_{i+1}}\\
&\quad-(t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^{\alpha})[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]-\Upsilon_i(t_{i+1},x_a(t_{i+1}))\\
&\quad-\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\Psi_i(\varsigma,x_b(\varsigma))d\varsigma\Big]\\
&= x_{t_{i+1}}.
\end{aligned}
$$

Therefore, control function (4.7) is suitable for $(\vartheta_i,t_{i+1}], i=1,2,...,m$. Furthermore,

$$
\begin{aligned}
\|u(t)\| &\le \|(t_{i+1}-t)^{1-\alpha}\mathcal{B}_i^*E_{\alpha,\alpha}(\mathcal{A}_i^*(t_{i+1}-t)^{\alpha})(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\|\Big[\|x_{t_{i+1}}\|+\|(t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^{\alpha})\|[\|\mathcal{J}_i(\vartheta_i,x(t_i^-))\|\\
&\quad+\|\Upsilon_i(\vartheta_i,x_a(\vartheta_i))\|]+\|\Upsilon_i(t_{i+1},x_a(t_{i+1}))\|+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\|\|\Psi_i(\varsigma,x_b(\varsigma))\|d\varsigma\Big]\\
&\le K_3M_{\mathcal{G}}^i\Big[\|x_{t_{i+1}}\|+t_{i+1}^{\gamma-1}K_1(M_{\mathcal{J}}+L_{\mathcal{J}}\|x(t_i^-)\|+M_{\Upsilon}+L_{\Upsilon}\|x_a(\vartheta_i)\|)+M_{\Upsilon}+L_{\Upsilon}\|x(t_{i+1})\|\\
&\quad+K_2M_{\Psi}\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}d\varsigma+K_2L_{\Upsilon}\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}\|x(\varsigma)\|d\varsigma\Big]\\
&\le K_3M_{\mathcal{G}}^i\Big[\|x_{t_{i+1}}\|+t_{i+1}^{\gamma-1}K_1(M_{\mathcal{J}}+M_{\Upsilon})+t_{i+1}^{\gamma-1}K_1(L_{\mathcal{J}}+L_{\Upsilon})\sup_{t\in[\vartheta_i,t_{i+1}]}\|x(t)\|+M_{\Upsilon}+L_{\Upsilon}\sup_{t\in[\vartheta_i,t_{i+1}]}\|x(t)\|\\
&\quad+\frac{K_2M_{\Psi}t_{i+1}^{\alpha}}{\alpha}+K_2L_{\Psi}t_{i+1}^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_{\gamma}\Big]\\
&= M_u^i.
\end{aligned}
$$

□

Theorem 4.6. *Let the assumptions (H1)–(H3) hold, then the control system (2.3) is totally controllable on I, provided*

$$
L_{\digamma_1}<1. \tag{4.8}
$$

Proof. Consider a subset $\Omega_1\subseteq PC$ such that

$$
\Omega_1=\{x\in PC:\|x\|_{\gamma}\le\omega_1\},
$$

where

$$
\omega_1=\max\left(\max_{0\le i\le m}\frac{\mathcal{M}_i}{1-\mathcal{R}_i},\ \frac{(\vartheta_i-t_i)^{1-\gamma}M_{\Psi}}{1-L_{\mathcal{J}}}\right).
$$

Now, we define an operator $\digamma_1:\Omega_1\to\Omega_1$ as

$$
\begin{aligned}
(\digamma_1x)(t)&=t^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0t^{\alpha})[x_0-\Upsilon_0(0,x_0)]+\Upsilon_0(t,x_a(t))+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^{\alpha})\Psi_0(\varsigma,x_b(\varsigma))d\varsigma\\
&\qquad+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^{\alpha})\mathcal{B}_0u(\varsigma)d\varsigma,\ t\in(0,t_1],\\
(\digamma_1x)(t)&=\Psi_i(t,x(t_i^-)),\ t\in(t_i,\vartheta_i],\ i=1,2,...,m,\\
(\digamma_1x)(t)&=(t-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t-\vartheta_i)^{\alpha})[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]+\Upsilon_i(t,x_a(t))\\
&\qquad+\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\Psi_i(\varsigma,x_b(\varsigma))d\varsigma+\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\mathcal{B}_iu(\varsigma)d\varsigma,\\
&\qquad t\in(\vartheta_i,t_{i+1}],\ i=1,2,...,m,
\end{aligned}
$$

where $u(t)$ is given by the equations (4.6) and (4.7) in the intervals $(0,t_1]$ and $(\vartheta_i,t_{i+1}], i=1,2,...,m$, respectively. It is clear from the Lemma 4.4 and Lemma 4.5, $x(t)$ satisfies $x(t_1)=x_{t_1}$ and $x(t_{i+1})=x_{t_{i+1}}, i=1,2,...,m$. Thus, to proof the controllability of the switched control system (2.3), it remains to show that the operator $\digamma_1$ has a fixed point. For the simplicity, we split the proof into the following two main steps:

Step 1: We shall show that $\digamma_1$ maps Ω_1 into Ω_1. Now, for any $t\in(0,t_1]$ and $x\in\Omega_1$, we have

$$\begin{aligned}
t^{1-\gamma}\|(\digamma_1x)(t)\| &\leq \|E_{\alpha,\gamma}(\mathcal{A}_0t^{\alpha})[x_0-\Upsilon_0(0,x_0)]\|+t^{1-\gamma}\|\Upsilon_0(t,x_a(t))\|\\
&\quad+t^{1-\gamma}\int_0^t(t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^{\alpha})\Psi_0(\varsigma,x_b(\varsigma))\|d\varsigma\\
&\quad+t^{1-\gamma}\int_0^t(t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^{\alpha})\mathcal{B}_0u(\varsigma)\|d\varsigma\\
&\leq K_1[\|x_0\|+\|\Upsilon_0(0,x_0)\|]+t^{1-\gamma}M_f+t^{1-\gamma}L_{\Upsilon}\|x_a(t)\|+t^{1-\gamma}K_2M_{\Psi}\int_0^t(t-\varsigma)^{\alpha-1}d\varsigma\\
&\quad+t^{1-\gamma}K_2L_{\Psi}\int_0^t(t-\varsigma)^{\alpha-1}\|x_b(\varsigma)\|d\varsigma+t^{1-\gamma}K_2M_{\mathcal{B}}\int_0^t(t-\varsigma)^{\alpha-1}K_3M_{\mathcal{G}}^0\Big[\|x_{t_1}\|\\
&\quad+t^{\gamma-1}\mathcal{N}_0+L_{\Upsilon}\sup_{t\in[0,t_1]}\|x(t)\|+K_2L_{\Psi}t_1^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_{\gamma}\Big]d\varsigma\\
&\leq K_1[\|x_0\|+\|\Upsilon_0(0,x_0)\|]+t^{1-\gamma}M_f+L_{\Upsilon}\omega_1+\frac{t^{\alpha+1-\gamma}K_2M_{\Psi}}{\alpha}\\
&\quad+t^{\alpha}K_2L_{\Psi}\omega_1B(\gamma,\alpha)+\frac{t^{\alpha}K_2M_{\mathcal{B}}K_3M_{\mathcal{G}}^0}{\alpha}\left[t^{1-\gamma}\|x_{t_1}\|+\mathcal{N}_0+L_{\Upsilon}\omega_1+K_2L_{\Psi}t_1^{\alpha}B(\gamma,\alpha)\omega_1\right]\\
&\leq\mathcal{N}_0+\mathcal{Q}_0\omega_1+S_0(t_1^{1-\gamma}\|x_{t_1}\|+\mathcal{N}_0+\mathcal{Q}_0\omega_1)\\
&\leq\mathcal{M}_0+\mathcal{R}_0\omega_1\leq\omega_1.
\end{aligned}\tag{4.9}$$

Now, for any $x\in\Omega_1$ and $t\in(t_i,\vartheta_i],\ i=1,2,...,m$, we have

$$\begin{aligned}
(t-t_i)^{1-\gamma}\|(\digamma_1x)(t)\| &\leq (t-t_i)^{1-\gamma}\|\mathcal{J}_i(t,x(t_i^-))\|\\
&\leq L_{\mathcal{J}}\omega_1+(\vartheta_i-t_i)^{1-\gamma}M_{\mathcal{J}}\leq\omega_1.
\end{aligned}\tag{4.10}$$

Similarly, for any $x\in\Omega_1$ and $t\in(\vartheta_i,t_{i+1}], i=1,2,...,m$, we have

$$\begin{aligned}
(t-\vartheta_i)^{1-\gamma}\|(\digamma_1x)(t)\| &\leq \|E_{\alpha,\gamma}(\mathcal{A}_i(t-\vartheta_i)^{\alpha})[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]\|+(t-\vartheta_i)^{1-\gamma}\|\Upsilon_i(t,x_a(t))\|\\
&\quad+(t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\Psi_i(\varsigma,x_b(\varsigma))\|d\varsigma\\
&\quad+(t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\mathcal{B}_iu(\varsigma)\|d\varsigma\\
&\leq K_1[M_{\mathcal{J}}+L_{\mathcal{J}}\|x(t_i^-)\|+M_{\Upsilon}+L_{\Upsilon}\|x_a(\vartheta_i)\|]+(t-\vartheta_i)^{1-\gamma}M_f\\
&\quad+(t-\vartheta_i)^{1-\gamma}L_{\Upsilon}\|x_a(t)\|+(t-\vartheta_i)^{1-\gamma}K_2L_{\Psi}\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\|x_b(\varsigma))\|d\varsigma\\
&\quad+(t-\vartheta_i)^{1-\gamma}K_2M_{\Upsilon}\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}d\varsigma+(t-\vartheta_i)^{1-\gamma}K_2K_3M_{\mathcal{B}}M_{\mathcal{G}}^i\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\Big[\|x_{t_{i+1}}\|\\
&\quad+t_{i+1}^{\gamma-1}\mathcal{N}_i+(t_{i+1}^{\gamma-1}K_1(L_{\mathcal{J}}+L_{\Upsilon})+L_{\Upsilon})\sup_{t\in[\vartheta_i,t_{i+1}]}\|x(t)\|+K_2L_{\Psi}t_{i+1}^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_{\gamma}\Big]d\varsigma\\
&\leq K_1[M_{\mathcal{J}}+M_{\Upsilon}]+(t-\vartheta_i)^{\gamma-1}K_1(L_{\mathcal{J}}+L_{\Upsilon})\omega_1+(t-\vartheta_i)^{1-\gamma}M_f\\
&\quad+L_{\Upsilon}\omega_1+(t-\vartheta_i)^{\alpha}K_2L_{\Psi}B(\gamma,\alpha)\omega_1+\frac{(t-\vartheta_i)^{\alpha+1-\gamma}K_2M_{\Psi}}{\alpha}+\frac{(t-\vartheta_i)^{\alpha}K_2K_3M_{\mathcal{B}}M_{\mathcal{G}}^i}{\alpha}\\
&\quad\times\Big[(t-\vartheta_i)^{1-\gamma}\|x_{t_{i+1}}\|+\mathcal{N}_i+(t_{i+1}^{\gamma-1}K_1(L_{\mathcal{J}}+L_{\Upsilon})+L_{\Upsilon})\omega_1+K_2L_{\Psi}t_{i+1}^{\alpha}B(\gamma,\alpha)\omega_1\Big]
\end{aligned}$$

$$\leq \mathcal{N}_i + \mathcal{Q}_i\omega_1 + S_i(t_{i+1}^{1-\gamma}\|x_{t_{i+1}}\| + \mathcal{N}_i + \mathcal{Q}_i\omega_1)$$
$$\leq \mathcal{M}_i + \mathcal{R}_i\omega_1 \leq \omega_1. \tag{4.11}$$

From the inequalities (4.9), (4.10) and (4.11), for $t \in I$, we get

$$\|\digamma_1 x\|_\gamma \leq \omega_1.$$

Hence, $\digamma_1$ maps Ω_1 into Ω_1.
Step 2: Here, we show that $\digamma_1$ is a contracting operator. For any $x, y \in \Omega_1$ and $t \in (0, t_1]$, we have

$$\begin{aligned}
&t^{1-\gamma}\|(\digamma_1 x)(t) - (\digamma_1 y)(t)\| \\
&\quad \leq t^{1-\gamma}\|\Upsilon_0(t, x_a(t)) - \Upsilon_0(t, y_a(t))\| \\
&\qquad + t^{1-\gamma}\int_0^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma)) - \Psi_0(\varsigma, y_b(\varsigma))\|d\varsigma \\
&\qquad + t^{1-\gamma}\int_0^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\mathcal{B}_0(t_1-\varsigma)^{1-\alpha}\mathcal{B}_0^* E_{\alpha,\alpha}(\mathcal{A}_0^*(t_1-\varsigma)^\alpha)(\mathcal{G}_0^{t_1})^{-1}\| \\
&\qquad \times \left[\|\Upsilon_0(t_1, x_a(t_1)) - \Upsilon_0(t_1, y_a(t_1))\| + \int_0^{t_1}(t_1-\tau)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\tau)^\alpha)\|\|\Psi_0(\tau, x_b(\tau)) - \Psi_0(\tau, y_b(\tau))\|d\tau\right]d\varsigma \\
&\quad \leq t^{1-\gamma}L_\Upsilon\|x_a(t) - y_a(t)\| + t^{1-\gamma}K_2L_\Psi\int_0^t (t-\varsigma)^{\alpha-1}\|x_b(\varsigma) - y_b(\varsigma)\|d\varsigma \\
&\qquad + K_2K_3M_\mathcal{B}M_\mathcal{G}^0 t^{1-\gamma}\int_0^t (t-\varsigma)^{\alpha-1}\left[L_\Upsilon\|x_a(t_1) - y_a(t_1)\| + K_2L_\Psi\int_0^{t_1}(t_1-\tau)^{\alpha-1}\|x_b(\tau) - y_b(\tau)\|d\tau\right]d\varsigma \\
&\quad \leq L_\Upsilon\|x-y\|_\gamma + t^\alpha K_2L_\Psi B(\gamma,\alpha)\|x-y\|_\gamma + \frac{K_2K_3M_\mathcal{B}M_\mathcal{G}^0 t^\alpha}{\alpha}[L_\Upsilon + t_1^\alpha K_2L_\Psi B(\gamma,\alpha)]\|x-y\|_\gamma \\
&\quad \leq \mathcal{Q}_0(1+S_0)\|x-y\|_\gamma \\
&\quad \leq \mathcal{R}_0\|x-y\|_\gamma.
\end{aligned} \tag{4.12}$$

Also, for any $x, y \in \Omega_1$ and $t \in (t_i, \vartheta_i], i = 1, 2, ..., m$, we have

$$\begin{aligned}
(t-t_i)^{1-\gamma}\|(\digamma_1 x)(t) - (\digamma_1 y)(t)\| &\leq (t-t_i)^{1-\gamma}\|\mathcal{J}_i(t, x(t_i^-)) - \mathcal{J}_i(t, y(t_i^-))\| \\
&\leq L_\mathcal{J}\|x-y\|_\gamma.
\end{aligned} \tag{4.13}$$

Similarly, for any $x, y \in \Omega_1$ and $t \in (\vartheta_i, t_{i+1}], i = 1, 2, ..., m$, we have

$$\begin{aligned}
&(t-\vartheta_i)^{1-\gamma}\|(\digamma_1 x)(t) - (\digamma_1 y)(t)\| \\
&\qquad \leq \|E_{\alpha,\gamma}(\mathcal{A}_i(t-\vartheta_i)^\alpha)\|[\|\mathcal{J}_i(\vartheta_i, x(t_i^-)) - \mathcal{J}_i(\vartheta_i, y(t_i^-))\| \\
&\qquad\quad + \|\Upsilon_i(\vartheta_i, x_a(\vartheta_i)) - \Upsilon_i(\vartheta_i, y_a(\vartheta_i))\|] + (t-\vartheta_i)^{1-\gamma}\|\Upsilon_i(t, x_a(t)) - \Upsilon_i(t, y_a(t))\| \\
&\qquad\quad + (t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^\alpha)\|\|\Psi_i(\varsigma, x_b(\varsigma)) - \Psi_i(\varsigma, y_b(\varsigma))\|d\varsigma \\
&\qquad\quad + (t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^\alpha)\|\|\mathcal{B}_i\|\|(t_{i+1}-\varsigma)^{1-\alpha}\mathcal{B}_i^* E_{\alpha,\alpha}(\mathcal{A}_i^*(t_{i+1}-\varsigma)^\alpha)\| \\
&\qquad\quad \times \|(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\|\Big[\|(t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)\|[\|\mathcal{J}_i(\vartheta_i, x(t_i^-)) - \mathcal{J}_i(\vartheta_i, x(t_i^-))\| \\
&\qquad\quad + \|\Upsilon_i(\vartheta_i, x_a(\vartheta_i)) - \Upsilon_i(\vartheta_i, y_a(\vartheta_i))\|\|\|] + \|\Upsilon_i(t_1, x_a(t_1)) - \Upsilon_i(t_1, y_a(t_1))\| \\
&\qquad\quad + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\tau)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\tau)^\alpha)\|\|\Psi_i(\tau, x_b(\tau)) - \Psi_i(\tau, y_b(\tau))\|d\tau\Big]d\varsigma
\end{aligned}$$

$$\le K_1(L_{\mathcal{J}}\|x(t_i^-)-y(t_i^-)\|+L_\Upsilon\|x_a(\vartheta_i)-y_a(\vartheta_i)\|)+(t-\vartheta_i)^{1-\gamma}L_\Upsilon\|x_a(t)-y_a(t)\|$$
$$+K_2L_\Psi(t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\|x(\varsigma))-y(\varsigma)\|d\varsigma$$
$$+K_2K_3M_{\mathcal{B}}M_{\mathcal{G}}^i(t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\Big[(t_{i+1}-\vartheta_i)^{\gamma-1}K_1(L_{\mathcal{J}}\|x(t_i^-)-y(t_i^-)\|+L_\Upsilon\|x_a(\vartheta_i)-y_a(\vartheta_i)\|)$$
$$+L_\Upsilon\|x_a(t_1)-y_a(t_1)\|+K_2L_\Psi\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\tau)^{\alpha-1}\|x(\tau)-y(\tau)\|d\tau\Big]d\varsigma$$
$$\le(t-\vartheta_i)^{\gamma-1}K_1(L_{\mathcal{J}}+L_\Upsilon)\|x-y\|_\gamma+L_\Upsilon\|x-y\|_\gamma+K_2L_\Psi(t-\vartheta_i)^\alpha B(\gamma,\alpha)\|x-y\|_\gamma+\frac{K_2K_3M_{\mathcal{G}}^iM_{\mathcal{B}}t_{i+1}^\alpha}{\alpha}$$
$$\times\Big((t_{i+1}-\vartheta_i)^{\gamma-1}K_1(L_\Psi+L_\Upsilon)\|x-y\|_\gamma+L_\Upsilon\|x-y\|_\gamma+K_2L_\Psi(t_{i+1}-\vartheta_i)^\alpha B(\gamma,\alpha)\|x-y\|_\gamma\Big)$$
$$\le\mathcal{Q}_i\|x-y\|_\gamma+S_i\mathcal{Q}_i\|x-y\|_\gamma$$
$$\le\mathcal{R}_i\|x-y\|_\gamma. \tag{4.14}$$

Therefore, from the inequalities (4.12), (4.13) and (4.14), for any $t\in I$, we have

$$\|\digamma_1x-\digamma_1y\|_\gamma\le L_{\digamma_1}\|x-y\|_\gamma.$$

Hence, from the inequality (4.8), $\digamma_1$ is a contracting operator.

Therefore, from the step 1 and step 2, we can conclude that the operator $\digamma_1$ satisfies all the conditions of Banach fixed point theorem and hence, the control system (2.3) has a unique solution. Subsequently, the control system (2.3) is totally controllable on I. Also, from the Remark 4.3, the control system (2.3) is controllable on I. □

5 Controllability of Integro-differential Equation

Here, we establish the total controllability of control system (2.3) with the integral term as follows:

$$\begin{aligned}D_{\vartheta_i^+}^{\alpha,\beta}[x(t)-\Upsilon_i(t,x_a(t))]&=\mathcal{A}_i[x(t)-\Upsilon_i(t,x_a(t))]+\Psi_i(t,x_b(t))+\mathcal{B}_iu(t)+\int_{\vartheta_i}^t k(t,\varsigma)\mathcal{Z}_i(\varsigma,x(\varsigma))d\varsigma,\ t\in\cup_{i=0}^m(\vartheta_i,t_{i+1}],\\ x(t)&=\mathcal{J}_i(t,x(t_i^-)),\ t\in(t_i,\vartheta_i],\ i=1,2,...,m,\\ I_{0^+}^{1-\gamma}x(0^+)=x_0,\ I_{\vartheta_i^+}^{1-\gamma}x(\vartheta_i^+)&=\mathcal{J}_i(\vartheta_i,x(t_i^-)),\ \gamma=\alpha+\beta-\alpha\beta.\end{aligned}\tag{5.15}$$

Definition 5.1. *A function $x\in PC$ is a solution of the system (2.3), if x satisfies*

(i) $I_{0^+}^{1-\gamma}x(0^+)=x_0$ and $I_{\vartheta_i^+}^{1-\gamma}x(\vartheta_i^+)=\mathcal{J}_i(\vartheta_i,x(t_i^-))$,

(ii) $x(t)=\mathcal{J}_i(t,x(t_i^-)),\ t\in(t_i,\vartheta_i],\ i=1,2,...,m$

and the following equations

$$x(t)=t^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0t^\alpha)[x_0-\Upsilon_0(0,x_0)]+\Upsilon_0(t,x_a(t))+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\Psi_0(\varsigma,x_b(\varsigma))d\varsigma$$
$$+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\mathcal{B}_0u(\varsigma)d\varsigma+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\int_0^\varsigma k(\varsigma,\tau)\mathcal{Z}_0(\tau,x(\tau))d\tau d\varsigma$$

for $t\in(0,t_1]$ and

$$x(t)=(t-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t-\vartheta_i)^\alpha)[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]+\Upsilon_i(t,x_a(t))$$
$$+\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^\alpha)\Psi_i(\varsigma,x_b(\varsigma))d\varsigma+\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^\alpha)\mathcal{B}_iu(\varsigma)d\varsigma$$
$$+\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^\alpha)\int_{\vartheta_i}^\varsigma k(\varsigma,\tau)\mathcal{Z}_i(\tau,x(\tau))d\tau d\varsigma$$

for $t\in(\vartheta_i,t_{i+1}],\ i=1,2,...,m.$

We impose the following extra assumptions to establish the main results for the integro control system (5.15).

(H4): $k : T_i \times T_i \to \mathbb{R}$ is continuous and there exists a positive constant M_k such that $\int_{\vartheta_i}^{t} |k(t,\varsigma)|d\varsigma \leq M_k$ for $i = 0, 1, ..., m$.

(H5): $\mathcal{Z}_i : T_i \times \mathbb{R}^n \to \mathbb{R}^n,\ i = 1, 2, ..., m$, are continuous and satisfy

$$\|\mathcal{Z}_i(t,x) - \mathcal{Z}_i(t,y)\| \leq L_{\mathcal{Z}}\|x-y\|,$$

for all $x, y \in \mathbb{R}^n$ and $t \in T_i$, where $L_{\mathcal{Z}}$ is a positive constant.

We set

$\sup_{t\in I}\|\mathcal{Z}_i(t,0)\| \leq M_{\mathcal{Z}};\ \overline{\mathcal{N}_0} = t_1^{1-\gamma}\left(t_1^{\gamma-1}K_1(\|x_0\| + \|\Upsilon_0(0,x_0)\|) + M_\Upsilon + \dfrac{K_2 t_1^{\alpha}(M_\Psi + M_k M_{\mathcal{Z}})}{\alpha}\right);$

$\overline{\mathcal{N}_i} = t_{i+1}^{1-\gamma}\left(t_{i+1}^{\gamma-1}K_1(M_{\mathcal{J}} + M_\Upsilon) + M_f + \dfrac{K_2 t_{i+1}^{\alpha}(M_\Psi + M_k M_{\mathcal{Z}})}{\alpha}\right),\ i = 1, 2, ..., m;$

$\overline{\mathcal{Q}_0} = L_\Upsilon + t_1^{\alpha}K_2 B(\gamma,\alpha)(L_\Psi + M_k L_{\mathcal{Z}});$

$\overline{\mathcal{Q}_i} = t_{i+1}^{1-\gamma}K_1(L_{\mathcal{J}} + L_\Upsilon) + L_\Upsilon + t_{i+1}^{\alpha}K_2 B(\gamma,\alpha)(L_\Psi + M_k L_{\mathcal{Z}}),\ i = 1, 2, ..., m;$

$\overline{\mathcal{M}_i} = \overline{\mathcal{N}_i} + S_i((t_{i+1})^{1-\gamma}\|x_{t_{i+1}}\| + \overline{\mathcal{N}_i});\ \overline{\mathcal{R}_i} = \overline{\mathcal{Q}_i}(1 + S_i),\ i = 0, 1, ..., m;$

$L_{F_2} = \max\left(\max_{0\leq i\leq m}\overline{\mathcal{R}_i},\ L_{\mathcal{J}}\right).$

Lemma 5.2. *Let the assumptions (H1)–(H5) hold, then the control function*

$$\begin{aligned}u(t) = (t_1 - t)^{1-\alpha}\mathcal{B}_0^* E_{\alpha,\alpha}(\mathcal{A}_0^*(t_1 - t)^\alpha)(\mathcal{G}_0^{t_1})^{-1}\Bigg[&x_{t_1} - t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0,x_0)) \\ &- \Upsilon_0(t_1, x_a(t_1)) - \int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))d\varsigma \\ &- \int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\int_0^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_0(\tau, x(\tau))d\tau d\varsigma\Bigg],\quad t \in (0, t_1]. \end{aligned} \tag{5.16}$$

steers the state of the control system (5.15) *from* x_0 *to* x_{t_1} *at time* $t = t_1$. *Also, the estimate of control function* $u(t)$ *is* $\|u(t)\| \leq \overline{M_u^0}, \forall\ t \in (0, t_1]$, *where*

$$\overline{M_u^0} = K_3 M_{\mathcal{G}}^0\left[\|x_{t_1}\| + t_1^{\gamma-1}\overline{\mathcal{N}_0} + L_\Upsilon \sup_{t\in[0,t_1]}\|x(t)\| + K_2 t_1^{\alpha+\gamma-1}B(\gamma,\alpha)(L_\Psi + M_k L_{\mathcal{Z}})\|x\|_\gamma\right].$$

Proof. By putting $t = t_1$ in the solution $x(t)$ of the system (5.15) on $(0, t_1]$, we get

$$\begin{aligned}x(t_1) &= t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)[x_0 - \Upsilon_0(0,x_0)] + \Upsilon_0(t_1, x_a(t_1)) + \int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))d\varsigma \\ &\quad + \int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\int_0^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_0(\tau,x(\tau))d\tau d\varsigma + \int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\mathcal{B}_0 u(\varsigma)d\varsigma \\ &= t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0,x_0)) + \Upsilon_0(t_1, x_a(t_1)) + \int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\Psi_0(\varsigma, x_b(\varsigma))d\varsigma \\ &\quad + \int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\int_0^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_0(\tau,x(\tau))d\tau d\varsigma + \int_0^{t_1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\mathcal{B}_0 \\ &\quad \times \mathcal{B}_0^* E_{\alpha,\alpha}(\mathcal{A}_0^*(t_1-t)^\alpha)(\mathcal{G}_0^{t_1})^{-1}\Bigg[x_{t_1} - t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^\alpha)(x_0 - \Upsilon_0(0,x_0)) - \Upsilon_0(t_1, x_a(t_1)) \\ &\quad - \int_0^{t_1}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\tau)^\alpha)\Psi_0(\tau, x_b(\tau))d\tau \\ &\quad - \int_0^{t_1}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\tau)^\alpha)\int_0^{\tau}k(\tau,s)\mathcal{Z}_0(s,x(s))dsd\tau\Bigg]d\varsigma\end{aligned}$$

$$
\begin{aligned}
&= t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^{\alpha})[x_0-\Upsilon_0(0,x_0)]+\Upsilon_0(t_1,x_a(t_1))+\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^{\alpha})\Psi_0(\varsigma,x_b(\varsigma))d\varsigma\\
&\quad+\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^{\alpha})\int_0^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_0(\tau,x(\tau))d\tau d\varsigma\\
&\quad+\mathcal{G}_0^{t_1}(\mathcal{G}_0^{t_1})^{-1}\Big[x_{t_1}-t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^{\alpha})(x_0-\Upsilon_0(0,x_0))-\Upsilon_0(t_1,x_a(t_1))-\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^{\alpha})\\
&\quad\times\Psi_0(\varsigma,x_b(\varsigma))d\varsigma-\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^{\alpha})\int_0^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_0(\tau,x(\tau))d\tau d\varsigma\Big]\\
&= x_{t_1}.
\end{aligned}
$$

Therefore, control function (5.16) is suitable for $t\in(0,t_1]$. Furthermore,

$$
\begin{aligned}
\|u(t)\| &\le \|(t_1-t)^{1-\alpha}\mathcal{B}_0^{*}E_{\alpha,\alpha}(\mathcal{A}_0^{*}(t_1-t)^{\alpha})(\mathcal{G}_0^{t_1})^{-1}\|\Big[\|x_{t_1}\|+\|t_1^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0 t_1^{\alpha})(x_0-\Upsilon_0(0,x_0))\|\\
&\quad+\|\Upsilon_0(t_1,x_a(t_1))\|+\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^{\alpha})\Psi_0(\varsigma,x_b(\varsigma))\|d\varsigma\\
&\quad+\int_0^{t_1}(t_1-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^{\alpha})\|\int_0^{\varsigma}|k(\varsigma,\tau)|\|\mathcal{Z}_0(\tau,x(\tau))\|d\tau d\varsigma\Big]\\
&\le K_3M_{\mathcal{G}}^0\Big[\|x_{t_1}\|+t_1^{\gamma-1}K_1(\|x_0\|+\|\Upsilon_0(0,x_0)\|)+M_{\Upsilon}+L_{\Upsilon}\sup_{t\in[0,t_1]}\|x(t)\|+\frac{K_2M_{\Psi}t_1^{\alpha}}{\alpha}\\
&\quad+K_2L_{\Psi}t_1^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_{\gamma}+\frac{K_2M_kM_{\mathcal{Z}}t_1^{\alpha}}{\alpha}+K_2L_{\mathcal{Z}}M_kt_1^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_{\gamma}\Big]\\
&=\overline{M_u^0}.
\end{aligned}
$$

□

Lemma 5.3. *Let the assumptions (H1)–(H5) hold, then the control*

$$
\begin{aligned}
u(t)&=(t_{i+1}-t)^{1-\alpha}\mathcal{B}_i^{*}E_{\alpha,\alpha}(\mathcal{A}_i^{*}(t_{i+1}-t)^{\alpha})(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\Big[x_{t_{i+1}}-(t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^{\alpha})[\mathcal{J}_i(\vartheta_i,x(t_i^-))\\
&\quad-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]-\Upsilon_i(t_{i+1},x_a(t_{i+1}))-\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\Psi_i(\varsigma,x_b(\varsigma))d\varsigma\\
&\quad-\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\int_{\vartheta_i}^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_i(\tau,x(\tau))d\tau d\varsigma\Big],
\end{aligned}
\tag{5.17}
$$

for $t\in(\vartheta_i,t_{i+1}]$, $i=1,2,...,m$, steers the state of the control system (5.15) from x_0 to $x_{t_{i+1}}$ at time $t=t_{i+1}$. Also, the estimate of control function $u(t)$ is $\|u(t)\|\le\overline{M_u^i}$, $\forall\ t\in(\vartheta_i,t_{i+1}]$, $i=1,2,...,m$, where

$$
\begin{aligned}
\overline{M_u^i}&=K_3M_{\mathcal{G}}^i[\|x_{t_{i+1}}\|+t_{i+1}^{\gamma-1}\overline{\mathcal{N}_i}+(t_{i+1}^{\gamma-1}K_1(L_{\mathcal{J}}+L_{\Upsilon})+L_{\Upsilon})\sup_{t\in[\vartheta_i,t_{i+1}]}\|x(t)\|\\
&\quad+K_2t_{i+1}^{\alpha+\gamma-1}B(\gamma,\alpha)(L_{\Psi}+M_kL_{\mathcal{Z}})\|x\|_{\gamma}].
\end{aligned}
$$

Proof. By putting $t=t_{i+1}$ in the solution $x(t)$ of the system (5.15) on $(\vartheta_i,t_{i+1}]$, $i=1,2,...,m$, we get

$$
\begin{aligned}
x(t_{i+1})&=(t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^{\alpha})[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]+\Upsilon_i(t_{i+1},x_a(t_{i+1}))\\
&\quad+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\Psi_i(\varsigma,x_b(\varsigma))d\varsigma+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\mathcal{B}_iu(\varsigma)d\varsigma\\
&\quad+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^{\alpha})\int_{\vartheta_i}^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_i(\tau,x(\tau))d\tau d\varsigma
\end{aligned}
$$

$$
\begin{aligned}
&= (t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]+\Upsilon_i(t_{i+1},x_a(t_{i+1}))\\
&\quad+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\Psi_i(\varsigma,x_b(\varsigma))d\varsigma\\
&\quad+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\int_{\vartheta_i}^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_i(\tau,x(\tau))d\tau d\varsigma\\
&\quad+\int_{\vartheta_i}^{t_{i+1}}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\mathcal{B}_i\mathcal{B}_i^*E_{\alpha,\alpha}(\mathcal{A}_i^*(t_{i+1}-t)^\alpha)(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\Big[x_{t_{i+1}}-(t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)\\
&\quad\times[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]-\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\tau)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\tau)^\alpha)\Psi_i(\tau,x_b(\tau))d\tau\\
&\quad-\Upsilon_i(t_{i+1},x_a(t_{i+1}))-\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\tau)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\tau)^\alpha)\int_{\vartheta_i}^{\tau}k(\tau,s)\mathcal{Z}_i(s,x(s))dsd\tau\Big]d\varsigma\\
&= x_{t_{i+1}}.
\end{aligned}
$$

Therefore, control function (5.17) is suitable for $(\vartheta_i,t_{i+1}], i=1,2,...,m$. Furthermore,

$$
\begin{aligned}
\|u(t)\| &\le \|(t_{i+1}-t)^{1-\alpha}\mathcal{B}_i^*E_{\alpha,\alpha}(\mathcal{A}_i^*(t_{i+1}-t)^\alpha)(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\|\Big[\|x_{t_{i+1}}\|+\|t_{i+1}^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)\|[\|\mathcal{J}_i(\vartheta_i,x(t_i^-))\|\\
&\qquad+\|\Upsilon_i(\vartheta_i,x_a(\vartheta_i))\|]+\|\Upsilon_i(t_{i+1},x_a(t_{i+1}))\|+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\|\|\Psi_i(\varsigma,x_b(\varsigma))\|d\varsigma\\
&\qquad+\int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\varsigma)^\alpha)\|\int_{\vartheta_i}^{\varsigma}|k(\varsigma,\tau)|\|\mathcal{Z}_i(\tau,x(\tau))\|d\tau d\varsigma\Big]\\
&\le K_3M_{\mathcal{G}}^i\Big[\|x_{t_{i+1}}\|+t_{i+1}^{\gamma-1}K_1(M_{\mathcal{J}}+M_\Upsilon)+t_{i+1}^{\gamma-1}K_1(L_{\mathcal{J}}+L_\Upsilon)\sup_{t\in[\vartheta_i,t_{i+1}]}\|x(t)\|+L_\Upsilon\sup_{t\in[\vartheta_i,t_{i+1}]}\|x(t_{i+1})\|\\
&\qquad+M_\Upsilon+\frac{K_2M_\Psi t_{i+1}^\alpha}{\alpha}+K_2L_\Psi t_{i+1}^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_\gamma++\frac{K_2M_kM_{\mathcal{Z}}t_{i+1}^\alpha}{\alpha}+K_2M_kL_{\mathcal{Z}}t_{i+1}^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_\gamma\Big]\\
&=\overline{M_u^i}.
\end{aligned}
$$

□

Theorem 5.4. *Let the assumptions (H1)–(H5) hold, then the control system (5.15) is totally controllable on I, provided*

$$L_{F_2}<1. \tag{5.18}$$

Proof. Consider a subset $\Omega_2\subseteq PC$ such that

$$\Omega_2=\{x\in PC:\|x\|_\gamma\le\omega_2\},$$

where

$$\omega_2=\max\left(\max_{0\le i\le m}\frac{\overline{\mathcal{M}_i}}{1-\overline{\mathcal{R}_i}},\ \frac{(\vartheta_i-t_i)^{1-\gamma}M_\Psi}{1-L_{\mathcal{J}}}\right).$$

Now, we define an operator $F_2:\Omega_2\to\Omega_2$ as

$$
\begin{aligned}
(F_2x)(t)&=t^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_0t^\alpha)[x_0-\Upsilon_0(0,x_0)]+\Upsilon_0(t,x_a(t))+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\Psi_0(\varsigma,x_b(\varsigma))d\varsigma\\
&\quad+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\mathcal{B}_0u(\varsigma)d\varsigma\\
&\quad+\int_0^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\int_0^\varsigma k(\varsigma,\tau)\mathcal{Z}_0(\tau,x(\tau))d\tau d\varsigma,\ t\in(0,t_1],\\
(F_2x)(t)&=\Psi_i(t,x(t_i^-)),\ \forall\,t\in(t_i,\vartheta_i],\ i=1,2,...,m,
\end{aligned}
$$

$$\begin{aligned}(\digamma_2x)(t)&=(t-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t-\vartheta_i)^{\alpha})[\mathcal{J}_i(\vartheta_i,x(t_i^-))-\Upsilon_i(\vartheta_i,x_a(\vartheta_i))]+\Upsilon_i(t,x_a(t))\\&+\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\Psi_i(\varsigma,x_b(\varsigma))d\varsigma+\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\mathcal{B}_iu(\varsigma)d\varsigma\\&+\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^{\alpha})\int_{\vartheta_i}^{\varsigma}k(\varsigma,\tau)\mathcal{Z}_i(\tau,x(\tau))d\tau d\varsigma,\ t\in(\vartheta_i,t_{i+1}],\ i=1,2,...,m,\end{aligned}$$

where $u(t)$ is given by the equations (5.16) and (5.17) in the intervals $(0,t_1]$ and $(\vartheta_i,t_{i+1}],i=1,2,...,m$, respectively. It is clear from the Lemma 5.2 and Lemma 5.3, $x(t)$ satisfies $x(t_1)=x_{t_1}$ and $x(t_{i+1})=x_{t_{i+1}},i=1,2,...,m$. Thus, to proof the controllability of the switched control system (5.15), it remains to show that the operator $\digamma_2$ has a fixed point. Let, for any $t\in(0,t_1]$ and $x\in\Omega_2$, we have

$$\begin{aligned}t^{1-\gamma}\|(\digamma_2x)(t)\|&\le K_1[\|x_0\|+\|\Upsilon_0(0,x_0)\|]+t^{1-\gamma}M_f+t^{1-\gamma}L_\Upsilon\|x_a(t)\|+t^{1-\gamma}K_2M_\Psi\int_0^t(t-\varsigma)^{\alpha-1}d\varsigma\\&+t^{1-\gamma}K_2L_\Psi\int_0^t(t-\varsigma)^{\alpha-1}\|x_b(\varsigma)\|d\varsigma+t^{1-\gamma}K_2\int_0^t(t-\varsigma)^{\alpha-1}\int_0^{\varsigma}|k(\varsigma,\tau)|M_{\mathcal{Z}}d\tau d\varsigma\\&+t^{1-\gamma}K_2\int_0^t(t-\varsigma)^{\alpha-1}\int_0^{\varsigma}|k(\varsigma,\tau)|L_{\mathcal{Z}}\|x(\tau)\|d\tau d\varsigma+t^{1-\gamma}K_2M_{\mathcal{B}}\int_0^t(t-\varsigma)^{\alpha-1}K_3M_{\mathcal{G}}^0\\&\times\left[\|x_{t_1}\|+t^{\gamma-1}\overline{\mathcal{N}_0}+L_\Upsilon\sup_{t\in[0,t_1]}\|x(t)\|+K_2t_1^{\alpha+\gamma-1}B(\gamma,\alpha)(L_\Psi+M_kL_{\mathcal{Z}})\|x\|_\gamma\right]d\varsigma\\&\le\overline{\mathcal{M}_0}+\overline{\mathcal{R}_0}\omega_2\le\omega_2.\end{aligned}\tag{5.19}$$

Now, for any $x\in\Omega_2$ and $t\in(t_i,\vartheta_i],\ i=1,2,...,m$, we have

$$(t-t_i)^{1-\gamma}\|(\digamma_2x)(t)\|\le\omega_2.\tag{5.20}$$

Similarly, for any $x\in\Omega_2$ and $t\in(\vartheta_i,t_{i+1}],i=1,2,...,m$, we have

$$\begin{aligned}(t-\vartheta_i)^{1-\gamma}\|(\digamma_2x)(t)\|&\le K_1[M_{\mathcal{J}}+\|x(t_i^-)\|+M_\Upsilon+L_\Upsilon\|x_a(\vartheta_i)\|]+(t-\vartheta_i)^{1-\gamma}M_f\\&+(t-\vartheta_i)^{1-\gamma}L_\Upsilon\|x_a(t)\|+(t-\vartheta_i)^{1-\gamma}K_2L_\Psi\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\|x_b(\varsigma))\|d\varsigma\\&+(t-\vartheta_i)^{1-\gamma}K_2M_\Upsilon\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}d\varsigma+(t-\vartheta_i)^{1-\gamma}K_2\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\int_{\vartheta_i}^{\varsigma}|k(\varsigma,\tau)|M_{\mathcal{Z}}d\tau d\varsigma\\&+(t-\vartheta_i)^{1-\gamma}K_2\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\int_{\vartheta_i}^{\varsigma}|k(\varsigma,\tau)|L_{\mathcal{Z}}\|x(\tau)\|d\tau d\varsigma\\&+(t-\vartheta_i)^{1-\gamma}K_2K_3M_{\mathcal{B}}M_{\mathcal{G}}^i\int_{\vartheta_i}^t(t-\varsigma)^{\alpha-1}\Big[\|x_{t_{i+1}}\|+t_{i+1}^{\gamma-1}\overline{\mathcal{N}_i}\\&+(t_{i+1}^{\gamma-1}K_1(L_{\mathcal{J}}+L_\Upsilon)+L_\Upsilon)\sup_{t\in[\vartheta_i,t_{i+1}]}\|x(t)\|+K_2t_{i+1}^{\alpha+\gamma-1}B(\gamma,\alpha)(L_\Psi+M_kL_{\mathcal{Z}})\|x\|_\gamma\Big]d\varsigma\\&\le\overline{\mathcal{M}_i}+\overline{\mathcal{R}_i}\omega_2\le\omega_2.\end{aligned}\tag{5.21}$$

From the inequalities (5.19), (5.20) and (5.21), for $t\in I$, we get

$$\|\digamma_2x\|_\gamma\le\omega_2.$$

Hence, $\digamma_2$ maps Ω_2 into Ω_2. Next, we show that $\digamma_2$ is a contracting operator. For any $x,y\in\Omega_2$ and $t\in(0,t_1]$, we have

$$
\begin{aligned}
&t^{1-\gamma}\|(F_2x)(t)-(F_2y)(t)\| \\
&\quad\le t^{1-\gamma}\|\Upsilon_0(t,x_a(t))-\Upsilon_0(t,y_a(t))\| \\
&\qquad + t^{1-\gamma}\int_0^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\Psi_0(\varsigma,x_b(\varsigma))-\Psi_0(\varsigma,y_b(\varsigma))\|d\varsigma \\
&\qquad + t^{1-\gamma}\int_0^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\|\int_0^\varsigma |k(\varsigma,\tau)|\|\mathcal{Z}_0(\tau,x(\tau))-\mathcal{Z}_0(\tau,y(\tau))\|d\tau d\varsigma \\
&\qquad + t^{1-\gamma}\int_0^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t-\varsigma)^\alpha)\mathcal{B}_0(t_1-\varsigma)^{1-\alpha}\mathcal{B}_0^*E_{\alpha,\alpha}(\mathcal{A}_0^*(t_1-\varsigma)^\alpha)(\mathcal{G}_0^{t_1})^{-1}\| \\
&\qquad \times\Big[\|\Upsilon_0(t_1,x_a(t_1))-\Upsilon_0(t_1,y_a(t_1))\|+\int_0^{t_1}(t_1-\tau)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\tau)^\alpha)\|\|\Psi_0(\tau,x_b(\tau))-\Psi_0(\tau,y_b(\tau)) \\
&\qquad + \int_0^{t_1}(t_1-r)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_0(t_1-r)^\alpha)\|\int_0^r |k(r,\tau)|\|\mathcal{Z}_0(\tau,x(\tau))-\mathcal{Z}_0(\tau,y(\tau))\|d\tau dr\Big]d\varsigma \\
&\quad\le \overline{\mathcal{Q}_0}(1+S_0)\|x-y\|_\gamma \le \overline{\mathcal{R}_0}\|x-y\|_\gamma.
\end{aligned}
\tag{5.22}
$$

Also, for any $x,y\in\Omega_2$ and $t\in(t_i,\vartheta_i], i=1,2,...,m$, we have

$$
(t-t_i)^{1-\gamma}\|(F_2x)(t)-(F_2y)(t)\|\le L_{\mathcal{J}}\|x-y\|_\gamma. \tag{5.23}
$$

Similarly, for any $x,y\in\Omega_2$ and $t\in(\vartheta_i,t_{i+1}], i=1,2,...,m$, we have

$$
\begin{aligned}
&(t-\vartheta_i)^{1-\gamma}\|(F_2x)(t)-(F_2y)(t)\| \\
&\quad\le \|E_{\alpha,\gamma}(\mathcal{A}_i(t-\vartheta_i)^\alpha)\|[\|\mathcal{J}_i(\vartheta_i,x(t_i^-))-\mathcal{J}_i(\vartheta_i,y(t_i^-))\| \\
&\qquad + \|\Upsilon_i(\vartheta_i,x_a(\vartheta_i))-\Upsilon_i(\vartheta_i,y_a(\vartheta_i))\|\|\|]+(t-\vartheta_i)^{1-\gamma}\|\Upsilon_i(t,x_a(t))-\Upsilon_i(t,y_a(t))\|\|\| \\
&\qquad + (t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^\alpha)\|[\|\Psi_i(\varsigma,x_b(\varsigma))-\Psi_i(\varsigma,y_b(\varsigma))\|d\varsigma \\
&\qquad + (t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^\alpha)\|\int_{\vartheta_i}^\varsigma |k(\varsigma,\tau)|\|\mathcal{Z}_i(\tau,x(\tau))-\mathcal{Z}_i(\tau,y(\tau))\|d\tau d\varsigma \\
&\qquad + (t-\vartheta_i)^{1-\gamma}\int_{\vartheta_i}^t (t-\varsigma)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t-\varsigma)^\alpha)\|\|\mathcal{B}_i\|\|(t_{i+1}-\varsigma)^{1-\alpha}\mathcal{B}_i^*E_{\alpha,\alpha}(\mathcal{A}_i^*(t_{i+1}-\varsigma)^\alpha)\| \\
&\qquad \times\|(\mathcal{G}_{\vartheta_i}^{t_{i+1}})^{-1}\|\Big[\|(t_{i+1}-\vartheta_i)^{\gamma-1}E_{\alpha,\gamma}(\mathcal{A}_i(t_{i+1}-\vartheta_i)^\alpha)\|[\|\mathcal{J}_i(\vartheta_i,x(t_i^-))-\mathcal{J}_i(\vartheta_i,x(t_i^-))\| \\
&\qquad + \|\Upsilon_i(\vartheta_i,x_a(\vartheta_i))-\Upsilon_i(\vartheta_i,y_a(\vartheta_i))\|\|\|]+\|\Upsilon_i(t_1,x_a(t_1))-\Upsilon_i(t_1,y_a(t_1))\| \\
&\qquad + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-\tau)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-\tau)^\alpha)\|\|\Psi_i(\tau,x_b(\tau))-\Psi_i(\tau,y_b(\tau))\|d\tau \\
&\qquad + \int_{\vartheta_i}^{t_{i+1}}(t_{i+1}-r)^{\alpha-1}\|E_{\alpha,\alpha}(\mathcal{A}_i(t_{i+1}-r)^\alpha)\|\int_{\vartheta_i}^r |k(r,\tau)|\|\mathcal{Z}_i(\tau,x(\tau))-\mathcal{Z}_i(\tau,y(\tau))\|d\tau dr\Big]d\varsigma \\
&\quad\le \overline{\mathcal{Q}_i}(1+S_i)\|x-y\|_\gamma\le\overline{\mathcal{R}_i}\|x-y\|_\gamma.
\end{aligned}
\tag{5.24}
$$

Therefore, from the inequalities (5.22), (5.23) and (5.24), for any $t\in I$, we have

$$
\|F_2x-F_2y\|_\gamma\le L_{F_2}\|x-y\|_\gamma.
$$

Hence, from the inequality (5.18), F_2 is a contracting operator. Therefore, F_2 satisfies all the conditions of Banach fixed point theorem and hence, the control system (5.15) has a unique solution. Subsequently, the control system (5.15)) is totally controllable on I. Also, from the Remark 4.3, the control system (5.15) is controllable on I. □

6 Example

Example 6.1. *We consider the following switched impulsive control system in the space* $\mathbb{R}^2$

$$
\begin{aligned}
D^{0.8,0.7}_{0^+}\left[x_1(t)-\frac{t(3+|x_1(t/2)|)}{45e^5(1+|x_1(t/2)|)}-e^t\right] &= -\left[x_1(t)-\frac{t(3+|x_1(t/2)|)}{45e^5(1+|x_1(t/2)|)}-e^t\right]+0.2\left[x_2-\frac{t\sin(x_2(t/2))}{48e^5}\right]\\
&\quad+\frac{t^2x_1(2t/3)}{35e^6}+\frac{e^t}{2}+u_1(t),\ t\in(0,0.4],\\
D^{0.8,0.7}_{0^+}\left[x_2(t)-\frac{t\sin(x_2(t/2))}{48e^5}\right] &= 0.2\left[x_1(t)-\frac{t(3+|x_1(t/2)|)}{45e^5(1+|x_1(t/2)|)}-e^t\right]-0.1\left[x_2(t)-\frac{t\sin(x_2(t/2))}{48e^5}\right]\\
&\quad+\frac{t\cos(x_2(2t/3))}{35(1+t)e^{t+6}}+\sin(t)+u_2(t),\ t\in(0,0.4],\\
D^{0.8,0.7}_{0.6^+}\left[x_1(t)-\frac{t(3+|x_1(t/2)|)}{45e^{t+5}(1+|x_1(t/3)|)}-\frac{1}{e^t}\right] &= -2\left[x_1(t)-\frac{t(3+|x_1(t/2)|)}{45e^{t+5}(1+|x_1(t/3)|)}-\frac{1}{e^t}\right]+0.3\left[x_2(t)-\frac{t\sin(x_2(t/2))}{24(2+t)e^{t+5}}\right]\\
&\quad+\frac{(1+t^2)x_1(2t/3)}{35e^{t+6}}+e^t+u_1(t),\ t\in(0.6,1],\\
D^{0.8,0.7}_{0.6^+}\left[x_2(t)-\frac{t\sin(x_2(t/2))}{24(2+t)e^{t+5}}\right] &= 0.3\left[x_1(t)-\frac{t(3+|x_1(t/2)|)}{45e^{t+5}(1+|x_1(t/3)|)}-\frac{1}{e^t}\right]-0.5\left[x_2(t)-\frac{t\sin(x_2(t/2))}{24(2+t)e^{t+5}}\right]\\
&\quad+\frac{t\cos(x_2(2t/3))}{35(1+t)e^{t+6}}+\sin(t)e^{t^2}+u_2(t),\ t\in(0.6,1],\\
x_1(t)=\frac{\cos(t)x_1(0.4^-)}{25e^{t+4}}+\frac{\sin(t)}{e^t},\ & x_2(t)=\frac{\sin(t)x_2(0.4^-)}{30e^{t+4}}+\frac{\cos(t)}{e^t},\ t\in(0.4,0.6],\\
I^{1-\gamma}_{0.6^+}x_1(0.6^+)=\frac{\cos(0.6)x_1(0.4^-)}{25e^{0.6+4}}+\frac{\sin(0.6)}{e^{0.6}},\ & I^{1-\gamma}_{0.6^+}x_2(0.6^+)=\frac{\sin(0.6)x_2(0.4^-)}{30e^{0.6+4}}+\frac{\cos(0.6)}{e^{0.6}},\\
I^{1-\gamma}_{0^+}x_1(0^+)=2,\ & I^{1-\gamma}_{0^+}x_2(0^+)=3,
\end{aligned}\tag{6.25}
$$

The system (6.25) can be written in the form of (2.3), where $t_0=0, t_1=0.4, \vartheta_1=0.6, t_2=T=1, m=1, \alpha=0.8, \beta=0.7, a(t)=t/2, b(t)=2t/3$,

$$
x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix},\ x_0=\begin{bmatrix}2\\3\end{bmatrix},\ \mathcal{A}_0=\begin{bmatrix}-1.0&0.2\\0.2&-0.1\end{bmatrix},\ \mathcal{A}_1=\begin{bmatrix}-2&0.3\\0.3&-0.5\end{bmatrix},\ \mathcal{B}_0=\begin{bmatrix}1\\1\end{bmatrix},\mathcal{B}_1=\begin{bmatrix}1\\1\end{bmatrix},
$$
$$
u(t)=\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix},\Upsilon_0(t,x_a(t))=\begin{bmatrix}\Upsilon_{01}(t,x_a(t))\\\Upsilon_{02}(t,x_a(t))\end{bmatrix},\ \Upsilon_1(t,x(t))=\begin{bmatrix}\Upsilon_{11}(t,x(t))\\\Upsilon_{12}(t,x(t))\end{bmatrix},\ \Psi_0(t,x(t))=\begin{bmatrix}\Psi_{01}(t,x(t))\\\Psi_{02}(t,x(t))\end{bmatrix},
$$
$$
\Psi_1(t,x(t))=\begin{bmatrix}\Psi_{11}(t,x(t))\\\Psi_{12}(t,x(t))\end{bmatrix},\mathcal{J}_1(t,x(t))=\begin{bmatrix}\mathcal{J}_{11}(t,x(t))\\\mathcal{J}_{21}(t,x(t))\end{bmatrix},
$$

with

$$
\Upsilon_{01}(t,x_a(t))=\frac{t(3+|x_1(a(t))|)}{45e^5(1+|x_1(a(t))|)}+e^t,\ \Upsilon_{02}(t,x_a(t))=\frac{t\sin(x_2(a(t)))}{48e^5},
$$
$$
\Upsilon_{11}(t,x_a(t))=\frac{t(3+|x_1(a(t))|)}{45e^{t+5}(1+|x_1(a(t))|)}+e^{-t},\ \Upsilon_{12}(t,x_a(t))=\frac{t\sin(x_2(a(t)))}{24(2+t)e^{t+5}}
$$

$$
\Psi_{i1}(t,x_b(t))=\frac{(i+t^2)x_1(b(t))}{35e^{it+6}}+\frac{(1+i)e^t}{2},\ \Psi_{i2}(t,x(t))=\frac{t\cos(x_2(b(t)))}{35(1+t)e^{t+6}}+\sin(t)e^{it^2},\ i=0,1,
$$
$$
\mathcal{J}_{11}(t,x(t))=\frac{\cos(t)x_2(t)}{45e^{t+8}}+\frac{\sin(t)}{e^t},\ \mathcal{J}_{21}(t,x(t))=\frac{\sin(t)x_2(t)}{50e^{t+8}}+\frac{\cos(t)}{e^t}.
$$

We choose the final target points as $x(t_1)=\begin{bmatrix}3\\2\end{bmatrix}$ and $x(T)=\begin{bmatrix}2\\3\end{bmatrix}$. Clearly, we can see that the assumptions (H1) and (H2) hold. Now, after some calculations, we get

$$\mathcal{G}_0^{t_1} = \int_0^{t_1} E_{\alpha,\alpha}(\mathcal{A}_0(t_1-\varsigma)^\alpha)\mathcal{B}_0\mathcal{B}_0^* E_{\alpha,\alpha}(\mathcal{A}_0^*(t_1-\varsigma)^\alpha)d\varsigma = \begin{bmatrix} 0.1817 & 0.2348 \\ 0.2348 & 0.3089 \end{bmatrix},$$

$$\mathcal{G}_{\vartheta_1}^{t_2} = \int_{\vartheta_1}^{t_2} E_{\alpha,\alpha}(\mathcal{A}_1(t_2-\varsigma)^\alpha)\mathcal{B}_1\mathcal{B}_1^* E_{\alpha,\alpha}(\mathcal{A}_1^*(t_2-\varsigma)^\alpha)d\varsigma = \begin{bmatrix} 0.1158 & 0.1652 \\ 0.1652 & 0.2452 \end{bmatrix}$$

and

$$\mathcal{R}_0 = \mathcal{Q}_0(1+S_0) = 0.1322, \mathcal{R}_1 = \mathcal{Q}_1(1+S_1) = 0.6842.$$

Hence,

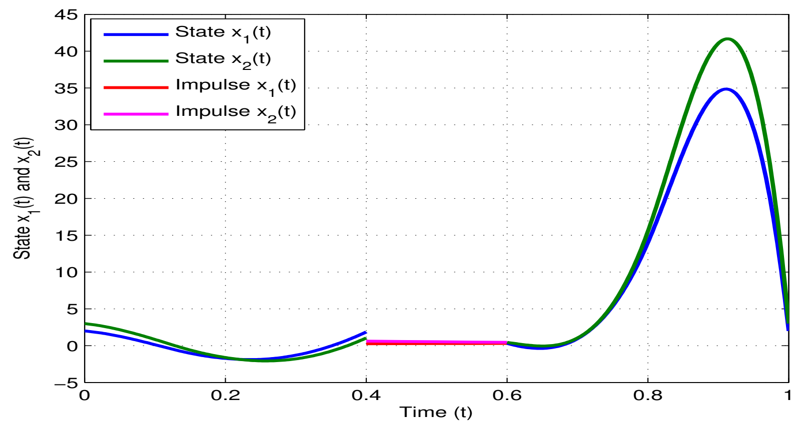


Figure 1: State trajectory of the system (6.25) when $x(t_1) = [3\ 2]^*, x(T) = [2\ 3]^*$.

$$L_{F_1} = \max\{\mathcal{R}_0, \mathcal{R}_1, L_{\mathcal{J}}\} = 0.6842 < 1.$$

Thus, all the assumptions of the Theorem 4.6 fulfilled. Hence, the switched control system (6.25) is totally controllable on $[0, 1]$. The controlled state trajectory of the system (6.25) is shown in the Figure 1 and the control function is shown in the Figure 2.

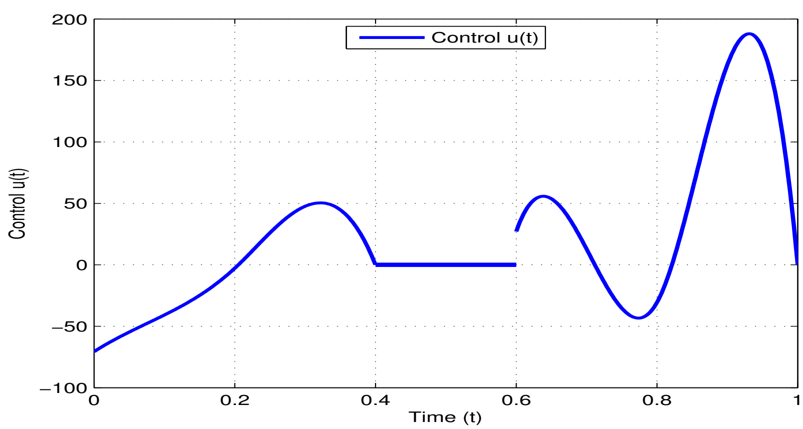


Figure 2: Trajectory of the control function $u(t)$.

Conclusion

We have successfully established the total controllability results of Hilfer fractional switched dynamical system with non-instantaneous impulses. Also, we have extended these results to the corresponding integro system. We used the Banach fixed point principle, Mittag-Leffler functions, fractional calculus, and Gramian type matrices to establish these results. Finally, we have given a numerical example with simulation to validate the obtained analytical outcomes. As further directions, the developed methodology can be used to establish the controllability results of fractional stochastic differential equations with impulses.

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References

[1] H. M. Ahmed, M. M. El-Borai, A. O. El Bab, and M. E. Ramadan. Approximate controllability of noninstantaneous impulsive hilfer fractional integrodifferential equations with fractional brownian motion. *Boundary Value Problems*, 2020(1):1–25, 2020.

[2] R. L. Bagley and P. Torvik. A theoretical basis for the application of fractional calculus to viscoelasticity. *Journal of Rheology*, 27(3):201–210, 1983.

[3] L. Bai, J. J. Nieto, and J. M. Uzal. On a delayed epidemic model with non-instantaneous impulses. *Communications on Pure & Applied Analysis*, 19(4):1915, 2020.

[4] K. Balachandran and J. Dauer. Controllability of nonlinear systems in banach spaces: a survey. *Journal of Optimization Theory and Applications*, 115(1):7–28, 2002.

[5] K. Balachandran, V. Govindaraj, L. Rodriguez-Germá, and J. J. Trujillo. Controllability results for nonlinear fractional-order dynamical systems. *Journal of Optimization Theory and Applications*, 156(1):33–44, 2013.

[6] K. Balachandran, J. Park, and J. Trujillo. Controllability of nonlinear fractional dynamical systems. *Nonlinear Analysis: Theory, Methods & Applications*, 75(4):1919–1926, 2012.

[7] S. P. Bhairat. Existence and continuation of solutions of hilfer fractional differential equations. *Journal of Mathematical Modeling*, 7(1):1–20, 2019.

[8] J. Borah and S. N. Bora. Existence of mild solution of a class of nonlocal fractional order differential equation with not instantaneous impulses. *Fractional Calculus and Applied Analysis*, 22(2):495–508, 2019.

[9] A. Boudjerida and D. Seba. Approximate controllability of hybrid hilfer fractional differential inclusions with non-instantaneous impulses. *Chaos, Solitons & Fractals*, 150:111125, 2021.

[10] C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, and K. S. Nisar. A discussion on the approximate controllability of hilfer fractional neutral stochastic integro-differential systems. *Chaos, Solitons & Fractals*, 142:110472, 2021.

[11] N. Durga and P. Muthukumar. Optimal control of sobolev-type stochastic hilfer fractional non-instantaneous impulsive differential inclusion involving poisson jumps and clarke subdifferential. *IET Control Theory & Applications*, 14(6):887–899, 2019.

[12] M. Fečkan and J. Wang. A general class of impulsive evolution equations. *Topological Methods in Nonlinear Analysis*, 46(2):915–933, 2015.

[13] K. M. Furati, M. D. Kassim, et al. Existence and uniqueness for a problem involving hilfer fractional derivative. *Computers & Mathematics with Applications*, 64(6):1616–1626, 2012.

[14] E. Gerolymatou, I. Vardoulakis, and R. Hilfer. Modelling infiltration by means of a nonlinear fractional diffusion model. *Journal of Physics D: Applied Physics*, 39(18):4104, 2006.

[15] H. Gou and Y. Li. Existence and approximate controllability of hilfer fractional evolution equations in banach spaces. *Journal of Applied Analysis & Computation*, 11(6):2895–2920, 2021.

[16] V. Govindaraj and R. K. George. Controllability of fractional dynamical systems: A functional analytic approach. *Mathematical Control & Related Fields*, 7(4):537, 2017.

[17] H. Gu and J. J. Trujillo. Existence of mild solution for evolution equation with hilfer fractional derivative. *Applied Mathematics and Computation*, 257:344–354, 2015.

[18] A. Harrat, J. J. Nieto, and A. Debbouche. Solvability and optimal controls of impulsive hilfer fractional delay evolution inclusions with clarke subdifferential. *Journal of Computational and Applied Mathematics*, 344:725–737, 2018.

[19] E. Hernández. Abstract impulsive differential equations without predefined time impulses. *Journal of Mathematical Analysis and Applications*, 491(1):124288, 2020.

[20] E. Hernández and D. O'Regan. On a new class of abstract impulsive differential equations. *Proceedings of the American Mathematical Society*, 141(5):1641–1649, 2013.

[21] R. Hilfer. *Applications of fractional calculus in physics*. World scientific, 2000.

[22] R. E. Kalman et al. Contributions to the theory of optimal control. *Bol. soc. mat. mexicana*, 5(2):102–119, 1960.

[23] K. Kavitha, V. Vijayakumar, and R. Udhayakumar. Results on controllability of hilfer fractional neutral differential equations with infinite delay via measures of noncompactness. *Chaos, Solitons & Fractals*, 139:110035, 2020.

[24] K. Kavitha, V. Vijayakumar, R. Udhayakumar, and K. S. Nisar. Results on the existence of hilfer fractional neutral evolution equations with infinite delay via measures of noncompactness. *Mathematical Methods in the Applied Sciences*, 44(2):1438–1455, 2021.

[25] V. V. Kulish and J. L. Lage. Application of fractional calculus to fluid mechanics. *J. Fluids Eng.*, 124(3):803–806, 2002.

[26] V. Kumar and M. Malik. Controllability results of fractional integro-differential equation with non-instantaneous impulses on time scales. *IMA Journal of Mathematical Control and Information*, 38(1):211–231, 2021.

[27] V. Kumar, M. Malik, and A. Debbouche. Stability and controllability analysis of fractional damped differential system with non-instantaneous impulses. *Applied Mathematics and Computation*, 391:125633, 2021.

[28] V. Kumar, M. Malik, and A. Debbouche. Total controllability of neutral fractional differential equation with non-instantaneous impulsive effects. *Journal of Computational and Applied Mathematics*, 383:113158, 2021.

[29] V. Lakshmikantham, P. S. Simeonov, et al. *Theory of impulsive differential equations*, volume 6. World scientific, 1989.

[30] V. Lakshmikantham and A. S. Vatsala. Basic theory of fractional differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 69(8):2677–2682, 2008.

[31] Z. Li, Y. Soh, and C. Wen. *Switched and impulsive systems: Analysis, design and applications*, volume 313. Springer Science & Business Media, 2005.

[32] D. Liberzon. *Switching in systems and control*. Springer Science & Business Media, 2003.

[33] X. Liu. On the finite approximate controllability for hilfer fractional evolution systems with nonlocal conditions. *Open Mathematics*, 18(1):529–539, 2020.

[34] D. Luo and Z. Luo. Existence and finite-time stability of solutions for a class of nonlinear fractional differential equations with time-varying delays and non-instantaneous impulses. *Advances in Difference Equations*, 2019(1):1–21, 2019.

[35] D. Luo and Z. Luo. Existence of solutions for fractional differential inclusions with initial value condition and non-instantaneous impulses. *Filomat*, 33(17):5499–5510, 2019.

[36] J. Lv and X. Yang. Approximate controllability of hilfer fractional differential equations. *Mathematical Methods in the Applied Sciences*, 43(1):242–254, 2020.

[37] J. Lv and X. Yang. Approximate controllability of hilfer fractional differential equations. *Mathematical Methods in the Applied Sciences*, 43(1):242–254, 2020.

[38] K. S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*. Wiley, 1993.

[39] I. Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.

[40] Y. A. Rossikhin and M. V. Shitikova. Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results. *Applied Mechanics Reviews*, 63(1), 2010.

[41] R. Sakthivel, N. I. Mahmudov, and J. J. Nieto. Controllability for a class of fractional-order neutral evolution control systems. *Applied Mathematics and Computation*, 218(20):10334–10340, 2012.

[42] A. M. Samoilenko and N. Perestyuk. *Impulsive differential equations*. World Scientific, 1995.

[43] V. Singh. Controllability of hilfer fractional differential systems with non-dense domain. *Numerical Functional Analysis and Optimization*, 40(13):1572–1592, 2019.

[44] B. Sundaravadivoo. Controllability analysis of nonlinear fractional order differential systems with state delay and non-instantaneous impulsive effects. *Discrete & Continuous Dynamical Systems-S*, 13(9):2561, 2020.

[45] B. S. Vadivoo, R. Raja, J. Cao, G. Rajchakit, and A. R. Seadawy. Controllability criteria of fractional differential dynamical systems with non-instantaneous impulses. *IMA Journal of Mathematical Control and Information*, 37(3):777–793, 2020.

[46] N. Valliammal, C. Ravichandran, and J. H. Park. On the controllability of fractional neutral integrodifferential delay equations with nonlocal conditions. *Mathematical Methods in the Applied Sciences*, 40(14):5044–5055, 2017.

[47] D. Vivek, K. Kanagarajan, and E. Elsayed. Some existence and stability results for hilfer-fractional implicit differential equations with nonlocal conditions. *Mediterranean Journal of Mathematics*, 15(1):1–21, 2018.

[48] D. Vivek, K. Kanagarajan, and E. Elsayed. Some existence and stability results for hilfer-fractional implicit differential equations with nonlocal conditions. *Mediterranean Journal of Mathematics*, 15(1):1–21, 2018.

[49] J. Wang, A. Ibrahim, and D. O'Regan. Finite approximate controllability of hilfer fractional semilinear differential equations. *Miskolc Mathematical Notes*, 21(1):489–507, 2020.

[50] J. Wang, A. G. Ibrahim, M. Fečkan, and Y. Zhou. Controllability of fractional non-instantaneous impulsive differential inclusions without compactness. *IMA Journal of Mathematical Control and Information*, 36(2):443–460, 2019.

[51] J. Wang, A. G. Ibrahim, and D. O'Regan. Hilfer-type fractional differential switched inclusions with noninstantaneous impulsive and nonlocal conditions. *Nonlinear Analysis: Modelling and Control*, 23(6):921–941, 2018.

[52] J. Wang, A. G. Ibrahim, and D. O'Regan. Global attracting solutions to hilfer fractional differential inclusions of sobolev type with noninstantaneous impulses and nonlocal conditions. *Nonlinear Analysis: Modelling and Control*, 24(5):775–803, 2019.

[53] J. Wang, G. Ibrahim, and D. O'Regan. Controllability of hilfer fractional noninstantaneous impulsive semilinear differential inclusions with nonlocal conditions. *Nonlinear Analysis: Modelling and Control*, 24(6):958–984, 2019.

[54] J. Wang and Y. Zhang. Nonlocal initial value problems for differential equations with hilfer fractional derivative. *Applied Mathematics and Computation*, 266:850–859, 2015.

[55] M. Yang and Q. Wang. Existence of mild solutions for a class of hilfer fractional evolution equations with nonlocal conditions. *Fractional Calculus and Applied Analysis*, 20(3):679–705, 2017.

[56] M. Yang and Q.-R. Wang. Approximate controllability of hilfer fractional differential inclusions with nonlocal conditions. *Mathematical Methods in the Applied Sciences*, 40(4):1126–1138, 2017.