



ISSN: (Print) (Online) Journal homepage: <https://www.tandfonline.com/loi/lagb20>

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Rudradip Biswas

To cite this article: Rudradip Biswas (2022) Injective generation of derived categories and other applications of cohomological invariants of infinite groups, *Communications in Algebra*, 50:10, 4460-4480, DOI: [10.1080/00927872.2022.2063300](https://doi.org/10.1080/00927872.2022.2063300)

To link to this article: <https://doi.org/10.1080/00927872.2022.2063300>



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Published online: 23 Apr 2022.



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Injective generation of derived categories and other applications of cohomological invariants of infinite groups

Rudradip Biswas

Max Planck Institute for Mathematics, Bonn, Germany

ABSTRACT

In the study of the representation theory of infinite groups, cohomological invariants play a very useful role. In a recent paper, we proved a number of properties regarding how these invariants interact with each other, extending the scope of some results in the literature. In this short article, we look into several ways in which the behavior of these invariants can be applied in various areas.

ARTICLE HISTORY

Received 6 August 2021
Revised 8 March 2022
Communicated by Sarah Witherspoon

KEYWORDS

Complete resolutions; derived categories; finitistic dimension; Gorenstein cohomological dimension; Kropholler's hierarchy

2020 MATHEMATICS

SUBJECT

CLASSIFICATION

Primary: 20C07; Secondary: 18G05; 20K40

In [7], we focus on the behavior of a range of cohomological invariants for infinite groups. In this paper, we look at various directions, often with not much connections between each other, in which our results from [7] can be applied.

The applications shown in this paper have been divided into two parts—Part 1 (Secs. 2 and 3) and Part 2 (Secs. 4–8). The applications shown in Part 1 have to do with devising some useful generation properties of the derived unbounded category and the stable module category of large classes of infinite groups. In Part 2, we find applications of our invariants in some general questions related to the cohomology and representation theoretic properties of infinite groups. Although there are no apparent connections between the sections in Part 2, the applicability of these invariants and related properties is the unifying theme.

We start with some background on the cohomological invariants that we will be dealing with and recalling some of the results involving them that we will be applying throughout this paper.

1. Background on cohomological invariants

Most of the invariants that we deal with are either defined over group algebras or for groups over commutative rings:

CONTACT Rudradip Biswas  biswas@mpim-bonn.mpg.de; rudradipbiswas@gmail.com  Max Planck Institute for Mathematics, Bonn, Germany.

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

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Definition 1.1. Let R be a ring. Denote by $\text{spli}(R)$ and $\text{silp}(R)$ the supremum over the projective dimension of injective R -modules and the supremum over the injective dimension of projective R -modules respectively (these invariants were first introduced in [18]).

The finitistic dimension of R , denoted $\text{fin. dim}(R)$, is defined to be the supremum over the projective dimension of all R -modules that have finite projective dimension.

When $R = A\Gamma$, where Γ is a group and A is a commutative ring, $k(A\Gamma)$ is defined to be the supremum over the projective dimension of those $A\Gamma$ -modules that have finite projective dimension when restricted to finite subgroups of Γ .

Definition 1.2. Let R be a ring. An R -module M is said to admit complete resolutions (over R) iff some high enough syzygy of it occurs as a kernel in a double infinite totally acyclic complex of R -projectives (similarly, M is said to admit weak complete resolutions over R iff some high enough syzygy of it occurs as a kernel in a doubly infinite acyclic complex of R -projectives) (see Remark 1.3 for a clarification of the term “totally acyclic”); we call M a Gorenstein projective R -module iff M occurs as a kernel in a doubly infinite totally acyclic complex of R -projectives.

The Gorenstein projective dimension of M over R , denoted $\text{Gpd}_R(M)$, is the minimal length of a resolution of Gorenstein projective R -modules admitted by M .

When $R = A\Gamma$, for some commutative ring A and some group Γ , $\text{Gcd}_A(\Gamma) := \text{Gpd}_{A\Gamma}(A)$. We say a group Γ admits complete resolutions over A iff the trivial $A\Gamma$ -module A admits complete resolutions.

Remark 1.3. (See Remark 1.7 of [7]) Following standard terminology from homological algebra, note that the phrase “totally acyclic complex” of R -modules, for any ring R , refers to an acyclic complex of R -projectives, P_* , such that $\text{Hom}_R(P_*, Q)$ is acyclic for any R -projective Q .

Also, it is easy to note that for any ring R , an R -module M admitting complete resolutions is equivalent to $\text{Gpd}_R(M) < \infty$.

The last invariant that we need to introduce is defined as the projective dimension of a particular module:

Definition 1.4. For any commutative ring A and any group Γ , denote by $B(\Gamma, A)$ the module of those functions $\Gamma \rightarrow A$ that are only allowed to take finitely many values in A . The $A\Gamma$ -module structure on $B(\Gamma, A)$ is given the following way: for any $f \in B(\Gamma, A)$, $(\gamma_1 \cdot f)(\gamma) := f(\gamma_1^{-1}\gamma)$, for all $\gamma, \gamma_1 \in \Gamma$.

Following [4], we define an $A\Gamma$ -module M to be a Benson’s cofibrant if $M \otimes_A B(\Gamma, A)$ is a projective $A\Gamma$ -module.

We first define groups of type Φ as those groups will play a crucial role in our treatment.

Definition 1.5. (made over \mathbb{Z} in [36]) For any commutative ring A , a group Γ is said to be of type Φ over A if, for any $A\Gamma$ -module M , the following two statements are equivalent.

- (a) $\text{proj. dim}_{A\Gamma} M < \infty$.
- (b) $\text{proj. dim}_{AG} M < \infty$, for all finite $G \leq \Gamma$.

We denote the class of all groups of type Φ over A by $\mathcal{F}_{\phi, A}$. When $A = \mathbb{Z}$, we write $\mathcal{F}_{\phi} := \mathcal{F}_{\phi, \mathbb{Z}}$.

Examples of groups of type Φ over all commutative rings of finite global dimension are groups of finite virtual cohomological dimension, groups acting on trees with finite stabilizers. (see [28] or [34] for more examples).

Another important class of groups comes from Kropholler’s hierarchy:

Definition 1.6. ([24]) Let \mathcal{X} be a class of groups. Define a hierarchy of groups in the following way: $H_0\mathcal{X} := \mathcal{X}$, and for any successor ordinal (like an integer) α , a group $\Gamma \in H_\alpha\mathcal{X}$ iff there exists a finite dimensional contractible CW-complex on which Γ acts by permuting the cells with all the cell stabilizers in $H_{\alpha-1}\mathcal{X}$. If α is a limit ordinal, $H_\alpha\mathcal{X} := \cup_{\beta < \alpha} H_\beta\mathcal{X}$. A group is said to be in $H\mathcal{X}$ iff it is in $H_\alpha\mathcal{X}$ for some ordinal α . Also, for any ordinal α , $H_{<\alpha}\mathcal{X} := \cup_{\beta < \alpha} H_\beta\mathcal{X}$.

The class $L\mathcal{X}$ is defined to be the class of all groups Γ such that every finitely generated subgroup of Γ is in \mathcal{X} .

Throughout this article, \mathcal{F} denotes the class of all finite groups.

Part of the following conjecture, which appears in this form in [7], was originally made over $A = \mathbb{Z}$ in [36], and one of the crucial conjectured equivalent statements comes from Conjecture 43.1 of [11].

Conjecture 1.7. (Conjecture 2.5 of [7]) *For any group Γ and any commutative ring A of finite global dimension, the following are equivalent.*

- (a) Γ is of type Φ over A .
- (b) $\text{silp}(A\Gamma) < \infty$.
- (c) $\text{spli}(A\Gamma) < \infty$.
- (d) $\text{proj. dim}_{A\Gamma} B(\Gamma, A) < \infty$.
- (e) $\text{Gcd}_A(\Gamma) < \infty$.
- (f) $\text{fin. dim}(A\Gamma) < \infty$.
- (g) $k(A\Gamma) < \infty$.

When $A = \mathbb{Z}$, we can add the condition

- (h) $\Gamma \in H_1\mathcal{F}$, where \mathcal{F} is the class of all finite groups.

Before stating one of the main results that we proved in [7], we collect below a few useful facts about the invariants introduced so far.

Theorem 1.8.

- (a) For any ring R , we always have $\text{fin. dim}(R) \leq \text{silp}(R)$ (follows from the proof of Theorem C of [12]).
- (b) $\text{Gcd}_{\mathbb{Z}}(\Gamma) = 0$ iff Γ is finite iff $\text{spli}(\mathbb{Z}\Gamma) = 1$ (Corollary 2.3 of [16] and the main result of [13]).
- (c) For any commutative ring A , $\text{Gcd}_A(\Gamma) \leq \text{Gcd}_{\mathbb{Z}}(\Gamma)$, for any group Γ .
When $R = A\Gamma$, for some group Γ and for some commutative ring A with finite global dimension t , we have the results (d) – (g):
- (d) $\text{silp}(A\Gamma), \text{spli}(\Gamma) \leq \text{Gcd}_A(\Gamma) + t$ (Corollary 1.6 of [16]).
- (e) The following are equivalent (Theorem 1.7 of [16]):
 - (i) $\text{Gcd}_A(\Gamma) < \infty$.
 - (ii) $\text{spli}(A\Gamma) = \text{silp}(A\Gamma) < \infty$.
 - (iii) $\text{Gpd}_{A\Gamma}(M) < \infty$, for all $A\Gamma$ -modules M .
- (f) If A is Noetherian, then $\text{spli}(A\Gamma) = \text{silp}(A\Gamma)$ (Theorem 4.4 of [15]).
- (g) Γ is of type Φ over A iff $k(A\Gamma) < \infty$ (Lemma 4.1 of [7]).

The following is one of our main results from [7].

Theorem 1.9. (Theorem 3.1 of [7]) *Let $\Gamma \in LH\mathcal{F}_{\phi, A}$ with A being a commutative ring of global dimension t . Then,*

$$\text{proj. dim}_{A\Gamma} B(\Gamma, A) = \text{Gcd}_A(\Gamma)$$

and

$\text{proj. dim}_{A\Gamma} B(\Gamma, A) \leq \text{fin. dim}(A\Gamma) = \text{silp}(A\Gamma) = \text{spli}(A\Gamma) = k(A\Gamma) \leq \text{proj. dim}_{A\Gamma} B(\Gamma, A) + t$

In addition to [Conjecture 1.7](#), there is the following conjecture regarding the coincidence of two important classes of modules which will come handy for us in [Sec. 3](#).

Notation 1.10. For any ring R , denote by $\text{GProj}(R)$ the class of Gorenstein projective R -modules, and for any group ring $A\Gamma$, denote by $\text{CoF}(A\Gamma)$ the class of all Benson's cofibrant $A\Gamma$ -modules.

Conjecture 1.11. ([8], made over \mathbb{Z} in [14], Conjecture 3.2 of [7]) For any group Γ and any commutative ring A of finite global dimension, $\text{GProj}(A\Gamma) = \text{CoF}(A\Gamma)$.

Using more or less the same methods as [14], we were able to prove the following result in relation to [Conjecture 1.11](#):

Theorem 1.12. (see [Theorem 3.4](#) of [7], originally from [8]) Let A be a commutative ring of finite global dimension and let $\Gamma \in \text{LH}\mathcal{F}_{\phi, A}$. Then,

- (a) Any $A\Gamma$ -module admits a complete resolution iff it admits a weak complete resolution.
- (b) $\text{GProj}(A\Gamma) = \text{CoF}(A\Gamma)$.

The way [Conjecture 1.11](#) is related to the invariants introduced earlier is through a generation property in the module category explored in [9]. We will not be using this notion of generation in this paper, but we briefly introduce it nonetheless to end this section so that the above-mentioned connection is clear.

Definition 1.13. ([Definition 3.5](#) of [9]) Let R be a ring. Let \mathcal{T} be a class of R -modules. An R -module M is generated in zero steps from \mathcal{T} iff it is in \mathcal{T} and in n steps iff there is an exact sequence $0 \rightarrow M_2 \rightarrow M_1 \rightarrow M \rightarrow 0$, where M_i is generated from \mathcal{T} in a_i steps, and $a_1 + a_2 \leq n - 1$. The class of all R -modules generated in finitely many steps from \mathcal{T} is denoted $\langle \mathcal{T} \rangle$.

In the language of [Definition 1.13](#) and in light of [Conjecture 1.11](#), one can expect that, for any group Γ and any commutative ring A of finite global dimension, $\langle \text{GProj}(A\Gamma) \rangle = \langle \text{CoF}(A\Gamma) \rangle$. A special case would be when either of $\langle \text{GProj}(A\Gamma) \rangle$ and $\langle \text{CoF}(A\Gamma) \rangle$ is the whole module category. In this regard, we were able to show the following:

Theorem 1.14. (follows from [Proposition 3.5](#) of [8]) Let A be a commutative ring of finite global dimension. Let Γ be a group such that $\text{Gcd}_A(\Gamma) = \text{proj. dim}_{A\Gamma} B(\Gamma, A)$ (this is stated as a separate conjecture in [Conjecture 1.17](#) of [7] and proved for the case $\text{proj. dim}_{A\Gamma} B(\Gamma, A) < \infty$ ([Theorem 1.18](#) of [7]) and for $\Gamma \in \text{LH}\mathcal{F}_{\phi, A}$ (by [Theorem 1.9](#))). Then, the following are equivalent.

- (a) $\langle \text{CoF}(A\Gamma) \rangle = \text{Mod}(A\Gamma)$.
- (b) $\langle \text{GProj}(A\Gamma) \rangle = \text{Mod}(A\Gamma)$.

Notation 1.15. For any ring R , we denote by $\text{Mod}(R)$ the standard abelian category of R -modules. This notation is used in [Theorem 1.14](#) above and also later.

Part 1. Applications in derived and stable categories

Derived and stable module categories are two of the very frequently arising triangulated categories in representation theory. In [6], we studied various generation properties of a range of derived categories of modules over groups in Kropholler's hierarchy ([Definition 1.6](#)); so, our handling of derived categories here in [Sec. 2](#) can be considered as providing more information about these derived categories (see [Question 2.12](#)).

Stable module categories are usually studied for finite groups, so we make clear in [Sec. 3](#) what definitions we are using for infinite groups. Upon close reading of the applications in Part 1, one can see that the properties of cohomological invariants (or in the case of the stable module categories, of a result ([Theorem 1.12](#)) closely related to the cohomological invariants) that are being used are only being applied at one or two key steps.

2. Injective generation of derived categories

A very important conjecture in the area of finite dimensional algebras is the finitistic dimension conjecture which goes as follows:

Conjecture 2.1. *Let R be a finite dimensional algebra over a field. Then, $\text{fin. dim}(R) < \infty$.*

Recently, Rickard [32] showed that proving [Conjecture 2.1](#) for a given R can be connected to a generation property of the unbounded derived category of cochain complexes of R -modules. Throughout this section, whenever we will write “complexes,” we will mean “cochain complexes.” Before we can recall Rickard’s result, we need to introduce the following definition.

Definition 2.2. Let \mathcal{T} be a triangulated category admitting arbitrary coproducts. For any class of objects \mathcal{U} in \mathcal{T} , denote by $\langle \mathcal{U} \rangle$ the smallest triangulated subcategory of \mathcal{T} containing \mathcal{U} and closed under arbitrary coproducts. In other words, $\langle \mathcal{U} \rangle$ denotes the smallest localizing subcategory of \mathcal{T} containing \mathcal{U} .

Notation 2.3. For any ring R , denote by $\mathcal{D}(\text{Mod-}R)$ and $\mathcal{D}^b(\text{Mod-}R)$, respectively, the derived unbounded category and the derived bounded category of complexes of R -modules. Denote also by $\text{Proj}(R)$ and $\text{Inj}(R)$ the class of R -projectives and R -injectives respectively. When we will consider these classes as classes of complexes in a derived category, it will be understood that we are considering the modules as complexes concentrated in degree zero.

In the language of [Definition 2.2](#), what Rickard proved was the following:

Theorem 2.4. (Theorem 4.3 of [32]) *Let R be a finite dimensional algebra over a field. Then, $\mathcal{D}(\text{Mod-}R) = \langle \text{Inj}(R) \rangle \Rightarrow \text{fin. dim}(R) < \infty$.*

It is noted by Rickard in [32] that we do not know of a finite dimensional algebra R over a field for which $\mathcal{D}(\text{Mod-}R) = \langle \text{Inj}(R) \rangle$. In general, checking generation properties for derived unbounded categories of modules over a given algebra can be easier to handle than computing the finitistic dimension of that algebra—this direction in research can be traced back to some groundbreaking work by Happel [20] in the eighties. So, in the statement of [Theorem 2.4](#), if the injective generation property and the finiteness of the finitistic dimension were actually equivalent instead of one implying the other, that would be much more convenient:

Question 2.5. *Let R be a finite dimensional algebra over a field. Then, is $\mathcal{D}(\text{Mod-}R) = \langle \text{Inj}(R) \rangle \iff \text{fin. dim}(R) < \infty$?*

Noting that for group rings, the finitistic dimension appears as one of the invariants in [Conjecture 1.7](#) and [Theorem 1.9](#), it is an interesting question to ask whether any of the implications in [Question 2.5](#) holds for group rings. Before proving our main original result in this regard, we need the following basic lemma whose proof we will omit.

Lemma 2.6. (Lemma 4.2 of [6], Proposition 2.1.f of [32]) *Let R be a ring and let \mathcal{T} be a triangulated subcategory of $\mathcal{D}(\text{Mod-}R)$ or $\mathcal{D}^b(\text{Mod-}R)$. Then, any complex, X_* , of the form $0 \rightarrow X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n \rightarrow 0$ is in \mathcal{T} if each X_i , when considered as a complex concentrated in degree zero, is in \mathcal{T} .*

Proposition 2.7. Let R be a ring and let $\mathcal{T} = \mathcal{D}^b(\text{Mod-}R)$ or $\mathcal{D}(\text{Mod-}R)$. For any class of objects \mathcal{U} in \mathcal{T} , denote by $\Delta_{\mathcal{T}}(\mathcal{U})$ the smallest triangulated subcategory of \mathcal{T} containing \mathcal{U} .

Then,

- (a) $\text{silp}(R) < \infty \Rightarrow \Delta_{\mathcal{T}}(\text{Proj}(R)) \subseteq \Delta_{\mathcal{T}}(\text{Inj}(R))$.
- (b) $\text{spli}(R) < \infty \Rightarrow \Delta_{\mathcal{T}}(\text{Inj}(R)) \subseteq \Delta_{\mathcal{T}}(\text{Proj}(R))$.

Therefore, $\text{silp}(R), \text{spli}(R) < \infty \Rightarrow \Delta_{\mathcal{T}}(\text{Inj}(R)) = \Delta_{\mathcal{T}}(\text{Proj}(R))$.

Proof. We prove (a). The proof for (b) is similar.

If $\text{silp}(R) < \infty$, every projective module, as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded complex of injectives. So, by Lemma 2.6, $\text{Proj}(R) \subseteq \Delta_{\mathcal{T}}(\text{Inj}(R))$, and therefore $\Delta_{\mathcal{T}}(\text{Proj}(R)) \subseteq \Delta_{\mathcal{T}}(\text{Inj}(R))$. \square

Remark 2.8. Note that if R does not have finite injective dimension over itself, i.e. if it is not of finite self-injective dimension, then $\Delta_{\mathcal{T}}(\text{Inj}(R)) \neq \Delta_{\mathcal{T}}(\text{Proj}(R))$, where $\mathcal{T} = \mathcal{D}^b(\text{Mod-}R)$, because R cannot be quasi-isomorphic to a bounded complex of injectives.

The following lemma shows us that the finiteness of the silp-invariant can be quite strong and useful for handling generation of the unbounded derived category in relation to Question 2.5.

Lemma 2.9. Let R be a ring such that $\text{silp}(R) < \infty$. Then,

- (a) $\mathcal{D}(\text{Mod-}R) = \langle \text{Inj}(R) \rangle$.
- (b) $\text{fin. dim}(R) < \infty$.

Proof.

- (a) It follows from Proposition 2.7.a., that $\langle \text{Proj}(R) \rangle \subseteq \langle \text{Inj}(R) \rangle$. It is standard fact that $\langle \text{Proj}(R) \rangle = \mathcal{D}(\text{Mod-}R)$ (see Proposition 2.2 of [32]), so we are done.
- (b) This follows directly from Lemma 1.8. \square

Note that if (a) \Rightarrow (b) in Conjecture 1.7, then by Lemma 2.9, $\text{fin. dim}(A\Gamma) < \infty \Rightarrow \mathcal{D}(\text{Mod-}A\Gamma) = \langle \text{Inj}(A\Gamma) \rangle$, for any group Γ and any commutative ring A of finite global dimension. However, we can get the same result for groups in $LH_{\phi, A}$:

Proposition 2.10. Let $\Gamma \in LH_{\phi, A}$, with A of finite global dimension. Then, $\text{fin. dim}(A\Gamma) < \infty \Rightarrow \mathcal{D}(\text{Mod-}A\Gamma) = \langle \text{Inj}(A\Gamma) \rangle$.

Proof. This follows directly from Theorem 1.9 which gives us that if $\text{fin. dim}(A\Gamma) < \infty$, then $\text{silp}(A\Gamma) = \text{fin. dim}(A\Gamma) < \infty$, and Lemma 2.9.a. \square

Since Proposition 2.10 forces generation results in the derived unbounded category with just the finiteness of an invariant as the hypothesis, it is relevant to state the following interesting generation property admitted by derived unbounded categories of modules over groups in Kropholler's hierarchy.

Theorem 2.11. (Theorem 4.5.a of [6]) Let $\Gamma \in H_n^{\mathcal{F}}$, for some integer n and let A be a commutative ring. Then, $\mathcal{D}(\text{Mod-}A\Gamma) = \langle I(\Gamma, \mathcal{F}) \rangle$, where $I(\Gamma, \mathcal{F})$ is the class of all modules induced up from finite subgroups of Γ .

We end this section with the following couple of questions that can be easily seen to be related to Conjecture 1.7, Proposition 2.10 and all the results discussed in this section including Theorem 2.11.

Question 2.12. (a) If Γ is an $H\mathcal{F}$ -group satisfying $\mathcal{D}(\text{Mod-}A\Gamma) = \langle \text{Inj}(A\Gamma) \rangle$, for some commutative ring A , then is $\Gamma \in H_1\mathcal{F}$? The answer is unlikely to be in the affirmative for any A , but in light of Conjecture 1.7, we can expect this to be the case when $A = \mathbb{Z}$.

(b) It follows from Lemma 2.9 and Theorem 2.11 that if Γ is an $H_1\mathcal{F}$ -group, then $\langle I(\Gamma, \mathcal{F}) \rangle = \langle \text{Inj}(A\Gamma) \rangle = \mathcal{D}(\text{Mod-}A\Gamma)$, where A is of finite global dimension. Now, does $\langle I(\Gamma, \mathcal{F}) \rangle = \langle \text{Inj}(A\Gamma) \rangle$, for all A of finite global dimension, imply that $\Gamma \in H_1\mathcal{F}$?

Also, can we find a group Γ such that for some A , $\langle I(\Gamma, \mathcal{F}) \rangle = \langle \text{Inj-}A\Gamma \rangle$ but $\langle I(\Gamma, \mathcal{F}) \rangle, \langle \text{Inj}(A\Gamma) \rangle \neq \mathcal{D}(\text{Mod-}A\Gamma)$? It follows from Theorem 2.11 that such a Γ cannot be in $H_n\mathcal{F}$ for any integer n ; whether it can still be in $H_\alpha\mathcal{F}$ for some higher ordinal α is unclear.

3. Generating stable module categories of infinite groups

It is well-known that for a finite group G and a field k , the class of finitely generated kG -modules forms a triangulated subcategory of $St.Mod(kG)$, usually denoted $st.mod(kG)$.

Since we will be dealing with stable module categories of not necessarily finite groups in this section, we briefly recall the definition that we will be using. Throughout this section, we fix a commutative ring A of finite global dimension and a group Γ of type Φ over A because the stable module category constructed in [28] applies to this class of groups.

Definition 3.1. (Section 3 of [28]) Write $\underline{\text{Mod}}(A\Gamma)$ for the quotient category of $\text{Mod}(A\Gamma)$ whose objects are the same as that of $\text{Mod}(A\Gamma)$ and for morphisms between M and N for any $M, N \in \underline{\text{Mod}}(A\Gamma)$, take

$$\text{Hom}_{\underline{\text{Mod}}}(A\Gamma)(M, N) := \text{Hom}_{\text{Mod}(A\Gamma)}(M, N) / P \text{Hom}_{\text{Mod}(A\Gamma)}(M, N)$$

where $P\text{Hom}_{\text{Mod}(A\Gamma)}(M, N)$ is the class of all morphisms $f : M \rightarrow N$ such that f is the composition of two morphisms $g : M \rightarrow P$ and $h : P \rightarrow N$ for some projective $A\Gamma$ -module P .

Now, the stable module category of $A\Gamma$ -modules, denoted $\text{Stab}(A\Gamma)$, is defined as having the same objects as $\text{Mod}(A\Gamma)$, and for any $M, N \in \text{Stab}(A\Gamma)$,

$$\text{Hom}_{\text{Stab}(A\Gamma)}(M, N) := \varinjlim_n \text{Hom}_{\underline{\text{Mod}}}(A\Gamma)(\Omega^n(M), \Omega^n(N)).$$

$\text{Stab}(A\Gamma)$ is a triangulated category with the inverse syzygy functor Ω^{-1} as the suspension functor.

Remark 3.2. Note that although in [28], Mazza and Symonds require A to be Noetherian in addition to having finite global dimension, we do not need the Noetherian condition. That is because in [28], the Noetherian condition is only used to conclude that $\text{silp}(A\Gamma) < \infty$ for $\Gamma \in \mathcal{F}_{\phi, A}$, and we know this holds without the Noetherian condition (by Theorem 1.8.g. and Theorem 1.9).

For the rest of this section, we make the extra assumption that Γ is an $LH\mathcal{F}$ -group.

Remark 3.3. Note that, in Conjecture 1.7, groups that are of type Φ over the integers are conjectured to be in $H_1\mathcal{F}$ (note that $H_1\mathcal{F} \subset LH\mathcal{F}$); $H_1\mathcal{F}$ -groups are of type Φ over any commutative ring of finite global dimension (this follows from Proposition 2.5 of [28]). We do not need to assume that Γ is in $H_1\mathcal{F}$; we can just make the weaker assumption that $\Gamma \in LH\mathcal{F}$ because the

only reason this assumption is useful is that we want to use [Theorem 3.9](#). The only groups that are known to be outside $LH\mathcal{F}$ do not admit complete resolutions over any commutative ring of finite global dimension. Groups admitting complete resolutions over any given commutative ring of finite global dimension is important to us for getting the Ω^{-1} functor. So, keeping the extra $LH\mathcal{F}$ -assumption in mind, we can just state that Γ is an $LH\mathcal{F}$ -group admitting complete resolutions over A . Such groups are of type Φ over A (see [Proposition 4.15](#)), so the Mazza-Symonds stable module category construction still works. Note that if a group is of type Φ over A , it admits complete resolutions over A (see [Remark 4.11](#)), but the reverse is not known to be true for all groups (it is true for groups in $LH\mathcal{F}_{\phi,A}$) and also it is not known if type Φ groups over any given commutative ring of finite global dimension is necessarily in $LH\mathcal{F}$.

In light of [Remark 3.3](#), let Γ be an $LH\mathcal{F}$ -group that admits complete resolutions over a commutative ring A of finite global dimension, and consider the class of all $A\Gamma$ -modules of type FP_{∞} (i.e. those modules that admit a projective resolution by finitely generated projectives), denoted $FP(A\Gamma)$. Then, we look at the smallest triangulated subcategory of $\text{Stab}(A\Gamma)$ containing $FP(A\Gamma)$, denoted $\text{stab}(A\Gamma)$, and prove a generation property admitted by it. Stable module categories of infinite groups with some additional finiteness properties on the modules were partially considered in [5], however we are not using the definitions of the stable category used in [5].

Note that the objects in $\text{stab}(A\Gamma)$ can be given an easy characterization:

Lemma 3.4. *Let \mathcal{M} be the class of all $A\Gamma$ -modules M which are eventually of type FP_{∞} in the module category, i.e. there exists a non-negative integer n such that $\Omega^n(M)$ is of type FP_{∞} . Then,*

- (a) *In $\text{Stab}(A\Gamma)$, \mathcal{M} is a triangulated subcategory of $\text{Stab}(A\Gamma)$. Note that when we consider \mathcal{M} as a class of modules in $\text{Stab}(A\Gamma)$, \mathcal{M} contains all modules that are stably isomorphic to modules which are eventually of type FP_{∞} in the module category.*
- (b) *As triangulated subcategories of $\text{Stab}(A\Gamma)$, $\mathcal{M} = \text{stab}(A\Gamma)$.*

Proof. (a) Consider \mathcal{M} as a class of modules in $\text{Stab}(A\Gamma)$. Since Ω^{-1} is the suspension functor of $\text{Stab}(A\Gamma)$, we need to show that $M \in \mathcal{M}$ forces $\Omega^{-1}(M) \in \mathcal{M}$ and that for any short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ where two of the three modules are in \mathcal{M} , the third one is in \mathcal{M} as well.

Let $M \in \mathcal{M}$. We have that $\Omega^n(M) \in \mathcal{M}$, for some n , then for $N = \Omega^{-1}(M)$, $\Omega^{n+1}(N) \in \mathcal{M}$.

Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be a short exact sequence where 2 of M_1, M_2, M_3 are eventually of type FP_{∞} . So, in the module category, two of M_1, M_2 and M_3 admit projective resolutions that are eventually of finite type, and therefore the third module does as well.

(b) Any module M of type FP_{∞} is in \mathcal{M} as $\Omega^0(M)$ is isomorphic to M in $\text{Stab}(A\Gamma)$. Therefore, the smallest triangulated subcategory of $\text{Stab}(A\Gamma)$ containing all modules that are of type FP_{∞} in the module category is contained in \mathcal{M} , i.e. $\text{stab}(A\Gamma) \subseteq \mathcal{M}$.

Now, take a module $M \in \mathcal{M}$. Then, for some non-negative n , $\Omega^n(M) \in \text{stab}(A\Gamma)$. Since $\text{stab}(A\Gamma)$ is a triangulated subcategory of $\text{Stab}(A\Gamma)$, by repeated applications of the suspension functor Ω^{-1} , we get that $\Omega^0(M) \in \text{stab}(A\Gamma)$. Thus, $M \in \text{stab}(A\Gamma)$ as M is isomorphic to $\Omega^0(M)$ in the stable category. \square

Before going forward, we need to define two classes of modules—*completely finitary* modules and *polybasic* modules. However, since we need to invoke complete Ext-groups to define these classes, we start with the definition of complete Ext-groups.

Definition 3.5. ([4]) Let R be a ring. For any two R -modules M and N , denote by $\underline{\text{Hom}}_R(M, N)$ the quotient of $\text{Hom}_R(M, N)$ by the additive subgroup consisting of $M \rightarrow N$ homomorphisms which factor through an R -projective.

Now, define complete cohomology in the following way:

$$\widehat{\text{Ext}}_R^r(M, N) := \varinjlim_i \underline{\text{Hom}}_R(\Omega^{i+r}(M), \Omega^i(N))$$

Definition 3.6. (defined over $A = \mathbb{Z}$ in Definition 2.1 of [19]) Let Γ be a group. An $A\Gamma$ -module M is called completely finitary if the functors $\widehat{\text{Ext}}_{A\Gamma}^*(M, ?)$ commute with all filtered colimit systems of coefficient modules.

Definition 3.7. (made over \mathbb{Z} in Definition 2.6 of [19]) Let Γ be a group. An $A\Gamma$ -module is said to be basic if it is of the form $U \otimes_{AG} A\Gamma$ where G is a finite subgroup of Γ and U is a completely finitary, Benson’s cofibrant AG -module. An $A\Gamma$ -module M is called polybasic if there is a filtration $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$ where each M_i/M_{i-1} is a basic $A\Gamma$ -module.

Remark 3.8. In the notations of Definition 3.6, if M is an FP_∞ module, i.e. if there is a projective resolution of M with finitely generated projectives, then by the characterization of FP_∞ modules in terms of Ext -functors, we have that the functors $\widehat{\text{Ext}}_{A\Gamma}^*(M, ?)$ commute with filtered colimits of coefficient modules, and by Result 4.1 of [24], it follows that M is completely finitary.

The following result will be crucial for us.

Theorem 3.9. (done over \mathbb{Z} in Proposition 2.13 of [19]) Let $\Gamma \in LH\mathcal{F}$. Take M to be an $A\Gamma$ -module that is both completely finitary and Benson’s cofibrant. Then M is isomorphic to the summand of a polybasic module and a projective module.

It is easy to note that the class of polybasics, as defined in Definition 3.7, allows us to just deal with basic modules and capture all polybasics by triangles in the the stable category:

Lemma 3.10. Let Γ be a group that admits complete resolutions over A . Then, any triangulated subcategory of $\text{Stab}(A\Gamma)$ containing all basic $A\Gamma$ -modules contains all polybasic $A\Gamma$ -modules.

Definition 3.11. Let \mathcal{T} be a triangulated category. A thick subcategory of \mathcal{T} is defined to be a full triangulated subcategory of \mathcal{T} , \mathcal{S} , such that given $M, N \in \mathcal{S}$, with $M \oplus N \in \mathcal{S}$, then $M, N \in \mathcal{S}$.

For any class of objects \mathcal{U} in \mathcal{T} and any object $M \in \mathcal{T}$, we say M is properly generated by \mathcal{U} in \mathcal{T} if M is in the smallest thick subcategory of \mathcal{T} containing \mathcal{U} .

Remark 3.12. We have seen generation in triangulated categories using localizing in Sec. 2. Generation using thick subcategories is also a very useful concept (one can consult [33] to see more about the theory surrounding this) in general—to be clear, in this concept, one can say a class of objects \mathcal{U} in a triangulated subcategory \mathcal{T} “generates” \mathcal{T} iff the smallest thick subcategory of \mathcal{T} containing \mathcal{U} is all of \mathcal{T} .

So, the definition of “proper” generation that we provide in Definition 3.11 is not very unnatural.

Take an $LH\mathcal{F}$ -group that admits complete resolutions over commutative ring A of finite global dimension (such a group is in $\mathcal{F}_{\phi, A}$ by Proposition 4.15). Note that if we take any FP_∞ module M , some high enough syzygy of it, say $\Omega^n(M)$, is Gorenstein projective (=Benson’s cofibrant in this case, see Theorem 1.12), and also of type FP_∞ . Recall that FP_∞ modules are completely finitary by Remark 3.8. Now, by Theorem 3.9, Lemma 3.10 and Definition 3.11, $\Omega^n(M)$ is in the

smallest thick subcategory of $\text{Stab}(A\Gamma)$ containing the basics (note that projectives are isomorphic to zero in the stable category). Like we saw in the proof of Lemma 3.4.b., it is straightforward to note that whenever $\Omega^n(M)$ is in a triangulated subcategory $\mathcal{T} \subseteq \text{Stab}(A\Gamma)$, then, by repeated application of the suspension functor Ω^{-1} , $\Omega^0(M)$ (which is isomorphic to M in $\text{Stab}(A\Gamma)$) is in \mathcal{T} . Thus, we have the following result.

Theorem 3.13. *Let Γ be an $LH\mathcal{F}$ -group that admits complete resolutions over a commutative ring A of finite global dimension, and let \mathcal{B} be the class of all basic $A\Gamma$ -modules. Then, in the language of Definition 3.11, every object in $\text{stab}(A\Gamma)$ is properly generated by \mathcal{B} in $\text{Stab}(A\Gamma)$.*

Part 2. Applications in cohomology and representation theory

In Part 2, we look at some representation theoretic applications. In Sec. 4, we explore the properties admitted by groups in Ikenaga's classes (see Definition 4.3), which is a close analogue of the main hierarchy of groups (Kropholler's hierarchy—Definition 1.6) from where we get most of our groups. To study how closely Ikenaga's classes admit similar properties as groups in Kropholler's hierarchy, after proving some of our results, we draw up a list of conjectured relations (Conjecture 4.16) and show how those conjectures interact with each other (Proposition 2.7). This is helpful because until now, such comparative study of Ikenaga's and Kropholler's classes had not been carried out in such detail.

Earlier, in Sec. 2, we dealt with the finitistic dimension of group rings. Since, over integral group rings, it is conjectured (see Conjecture 4.13) that groups whose integral group rings have finite finitistic dimension have finite dimensional models for their classifying space of proper actions (see Definition 4.12), it is a natural question to ask for similar algebraic properties of groups implying the same conclusion. In this regard, we use a result by Lück [26] and apply a result from [7] on cohomological invariants to get a new result (Proposition 5.5). In Secs. 6 and 7, we deal with very general questions on projectivity of modules and on groups with periodic cohomology respectively. Although there is no apparent connection between them, it is interesting to see how at key moments, one can invoke properties of cohomological invariants and related questions to extend the scope of some existing results from the literature. Finally, in Sec. 8, we look at some groups that are known to lie beyond Kropholler's hierarchy with the class of finite groups as the base class. We investigate whether one can prove these results for the case where the base class is the much larger class of type Φ groups (See Definition 1.5). It is worth noting that although one should expect $H\mathcal{F}_{\phi, \mathbb{Z}} = H\mathcal{F}$ (as Conjecture 1.7 claims that $H_1\mathcal{F} = \mathcal{F}_{\phi, \mathbb{Z}}$), for any arbitrary commutative ring A of finite global dimension in place of \mathbb{Z} , we do not have the same expectation. We do not however know of a concrete example of a group in $H\mathcal{F}_{\phi, A} \setminus H\mathcal{F}$ or $LH\mathcal{F}_{\phi, A} \setminus LH\mathcal{F}$, for some commutative A of finite global dimension.

4. Ikenaga's hierarchy

About 10 years before Peter Kropholler introduced his hierarchy of groups, Bruce Ikenaga used similar geometric ideas to introduce his classes of groups which we define below. We need to provide the definition of a new invariant for a group, called the *generalized cohomological dimension*, first.

Definition 4.1. (made over \mathbb{Z} in [21]) For any commutative ring A and any group Γ , define the generalized cohomological dimension of Γ with respect to A , denoted $\text{cd}_A(\Gamma)$, to be $\sup\{n \in \mathbb{Z}_{\geq 0} : \text{Ext}_{A\Gamma}^n(M, F) \neq 0, \text{ for some } A\text{-free } M \text{ and some } A\Gamma\text{-free } F\}$.

The following are some useful facts regarding the generalized cohomological dimension:

Theorem 4.2. (*Theorem 1.12 and Remark 1.13 of [7]*) *If A is a commutative ring of finite global dimension and $\text{Gcd}_A(\Gamma) < \infty$, then $\underline{\text{cd}}_A(\Gamma) = \text{Gcd}_A(\Gamma)$. Moreover, if A is Noetherian, then always $\text{Gcd}_A(\Gamma) = \underline{\text{cd}}_A(\Gamma)$.*

Ikenaga’s classes of groups were defined in the following way.

Definition 4.3. (based on Sec. 5, [21]) Let \mathcal{X} be a class of groups. Define $C_0(\mathcal{X}) := \mathcal{X}$, and a group $\Gamma \in C_n(\mathcal{X})$ iff there exists an acyclic simplicial complex X on which Γ acts by permuting the simplices such that $\Gamma_\sigma \in C_{n-1}(\mathcal{X})$, for each simplex $\sigma \in X$, where Γ_σ denotes the stabilizer of σ , and $\sup_{\sigma \in \Sigma} \{ \dim(\sigma) + \underline{\text{cd}}_{\mathbb{Z}}(\Gamma) \} < \infty$, where Σ is a set of representatives of X modulo the Γ -action.

$$C_\infty(\mathcal{X}) := \bigcup_{n \geq 0} C_n(\mathcal{X})$$

For groups in $C_\infty(\mathcal{F})$, the following was proved in [21].

Theorem 4.4. *Groups in $C_\infty(\mathcal{F})$ have finite generalized cohomological dimension over \mathbb{Z} and they admit weak complete resolutions over \mathbb{Z} .*

Although it was not noted in [21], groups in $C_\infty(\mathcal{F})$ actually admit complete resolutions, which we can show using the following result.

Lemma 4.5. (*done over \mathbb{Z} in Lemma 2.2 of [30], same proof works here*) *If Γ admits weak complete resolutions over a commutative ring A and $\text{silp}(A\Gamma) < \infty$, then Γ admits complete resolutions over A .*

Corollary 4.6. *$C_\infty(\mathcal{F})$ -groups admit complete resolutions over any commutative ring A .*

Proof. Let $\Gamma \in C_\infty(\mathcal{F})$. Then by **Theorem 4.4**, $\underline{\text{cd}}_{\mathbb{Z}}(\Gamma) < \infty$. Note that $\underline{\text{cd}}_{\mathbb{Z}}(\Gamma) = \text{Gcd}_{\mathbb{Z}}(\Gamma)$ by **Theorem 4.2**. So by **Theorem 1.8**, $\text{silp}(\mathbb{Z}\Gamma) < \infty$. So, by **Lemma 4.5** and **Theorem 4.4**, Γ admits complete resolutions over \mathbb{Z} . The result translates to all commutative rings due to **Theorem 1.8.c** and **Remark 1.3**. □

One can form Ikenaga’s classes of groups starting with the class of all groups of type Φ as the base class. Whether or not we get any groups that we do not get when we start with the class of all finite groups as the base class is part of a conjecture (See **Conjecture 4.13**) that we make later.

Remark 4.7. Since both the definitions of Ikenaga’s classes and Kropholler’s hierarchy involve a kind of iteration on the definition of a level to get to the next level, it is natural to wonder whether one can do something similar with type Φ groups by iterating Definition 1.5. It turns out we can’t as we explain below.

Let’s fix an A of finite global dimension, and call type Φ groups type Φ^1 . For all $n \geq 1$, define a group Γ to be of type Φ^n if, for any $A\Gamma$ -module M , M is of finite projective dimension as an $A\Gamma$ -module iff it is of finite projective dimension over all type Φ^{n-1} subgroups.

If Γ is type Φ^n , and M is of finite projective dimension over finite subgroups, then M is of finite projective dimension over type Φ subgroups, and by the iterative definition above, it is of finite projective dimension over type Φ^2 groups, and going on like this, it is of finite projective dimension over type Φ^{n-1} groups, from which it follows from the iterative definition above again, that $\text{proj. dim}_{A\Gamma} M < \infty$. Thus, Γ is of type Φ .

Remark 4.8. By the main result of [22], it is now known that for every integer n , $H_{n+1}\mathcal{F}$ is a strictly bigger class than $H_n\mathcal{F}$. No such result is known for Ikenaga's classes and that is why we feature this as a conjecture in [Conjecture 4.16](#).

The following are some handy connections between the classes of groups we have introduced.

Lemma 4.9.

- (a) $C_n(\mathcal{F}) := \{\Gamma \in H_n\mathcal{F} : \text{spli}(\mathbb{Z}\Gamma) < \infty\}$; $C_1(\mathcal{F}) = H_1\mathcal{F}$. (Corollary 2.6 of [30])
- (b) $C_\infty(\mathcal{F}) \subseteq \mathcal{F}_{\phi,A}$, for any A of finite global dimension.

Proof. Take A to be any commutative ring of finite global dimension. The only part of [Lemma 4.9](#) that is new is the claim that $C_\infty(\mathcal{F}) \subseteq \mathcal{F}_{\phi,A}$. Let $\Gamma \in C_\infty(\mathcal{F})$. So, by [Corollary 4.6](#), Γ admits complete resolutions over A , and therefore $\text{Gcd}_A(\Gamma) < \infty$.

Since $C_\infty(\mathcal{F}) \subseteq H\mathcal{F}$ by [Lemma 4.9.a.](#), it follows from [Theorem 1.9](#) that $k(A\Gamma) < \infty$. By [Theorem 1.8.g.](#), it now follows that Γ is of type Φ over A . \square

It is noteworthy that the operator L is quite powerful in that when applied to classes of groups like $C_\infty(\mathcal{F})$, $\mathcal{F}_{\phi,A}$ (for any A of finite global dimension) and $H_1\mathcal{F}$, it gives a strictly larger class of groups:

Proposition 4.10. For any commutative ring A of finite global dimension, $LH_1\mathcal{F} \neq H_1\mathcal{F}$; $LC_\infty(\mathcal{F}) \neq C_\infty(\mathcal{F})$; $L\mathcal{F}_{\phi,A} \neq \mathcal{F}_{\phi,A}$.

Proof. Take Γ to be a free abelian group of infinite rank. Then any finitely generated subgroup of it, say a free abelian groups of finite rank n , acts on an n -dimensional CW-complex with \mathbb{R}^n as the underlying space, and therefore $\Gamma \in LH_1\mathcal{F}$ and by [Lemma 4.9.a.](#), is in $LC_\infty(\mathcal{F})$ and $L\mathcal{F}_{\phi,A}$. Γ does not admit complete resolutions over A , so it is not in $H_1\mathcal{F}$, $C_\infty(\mathcal{F})$ or $\mathcal{F}_{\phi,A}$. \square

It follows from [Theorem 1.8.g.](#) and [Theorem 1.9](#) that for type Φ groups all of our invariants are finite and well-behaved (see [Remark 4.11](#) below).

Remark 4.11. Note that it follows from [Theorem 1.8.g.](#), [Theorem 1.9](#), [Theorem 4.2](#) and [Remark 1.3](#) that if $\Gamma \in \mathcal{F}_{\phi,A}$ with A of finite global dimension, then Γ admits complete resolutions over A and all the invariants— $\text{cd}_A(\Gamma)$, $\text{Gcd}_A(\Gamma)$, $\text{proj. dim}_{A\Gamma} B(\Gamma, A)$, $\text{silp}(A\Gamma)$, $\text{spli}(A\Gamma)$, $\text{fin. dim}(A\Gamma)$, $k(A\Gamma)$ —are finite. We are recording this here because we will be making repeated use of this in the proof of [Proposition 4.17](#).

We will be making repeated use of [Remark 4.11](#) in proving the connections between the various conjectures in [Proposition 4.9](#).

Before stating our conjectures, we need to state a close restatement of [Conjecture 1.7](#), with the difference being that we include a statement on the classifying space of proper actions. We need to define the classifying space of proper actions of a group first.

Definition 4.12. For any group Γ , $\underline{E}\Gamma$ denotes a CW-complex on which Γ acts cellularly with finite stabilizers such that for any finite subgroup G of Γ , the fixed point subcomplex $\underline{E}\Gamma^G$ is contractible. (it is known that for any group, such a complex exists)

Conjecture 4.13. Let A be a commutative ring of finite global dimension. For any group Γ , the following are equivalent.

- (a) Γ is of type Φ over A
- (b) $\text{Gcd}_A(\Gamma) < \infty$.
- (c) $\text{spli}(A\Gamma) < \infty$.
- (d) $\text{silp}(A\Gamma) < \infty$.

- (e) $\text{fin. dim}(A\Gamma) < \infty$.
- (f) $k(A\Gamma) < \infty$.

When $A = \mathbb{Z}$, we can add the following statement:

- (g) Γ admits a finite dimensional model for $\underline{E}\Gamma$.

We deal with classifying spaces later in [Proposition 4.17](#) and then in [Sec. 5](#).

Remark 4.14. [Conjecture 4.13](#) looks very similar to [Conjecture 1.7](#), except (g) of [Conjecture 1.7](#) is the statement that $\Gamma \in H_1\mathcal{F}$, but here (g) of [Conjecture 4.13](#) is the statement that Γ admits a finite dimensional model for $\underline{E}\Gamma$. Although it is clear that if Γ satisfies the latter it is definitely in $H_1\mathcal{F}$ (see [Sec. 4](#) of [\[29\]](#)), whether the converse holds is still open to conjecture (see [Conjecture 43.1](#) of [\[11\]](#)).

It seems a sensible question to ask whether one could place all groups with complete resolutions within a known hierarchical class. The following result sheds some light in that direction.

Proposition 4.15. *Let A be a commutative ring of finite global dimension. Then, any $LH\mathcal{F}_{\phi,A}$ -group that admits complete resolutions over A is in $\mathcal{F}_{\phi,A}$.*

Proof. Let Γ be a group in $LH\mathcal{F}_{\phi,A}$ that admits complete resolutions over A . Then, $\text{Gcd}_A(\Gamma) < \infty$ by [Theorem 1.9](#), and therefore by [Theorem 1.9](#), $k(A\Gamma)$ is finite. So, by [Theorem 1.8.g.](#), $\Gamma \in \mathcal{F}_{\phi,A}$. □

Whether or not [Proposition 4.15](#) can in any way be stated with the base class \mathcal{F} instead of $\mathcal{F}_{\phi,A}$, i.e. whether we can say that any $H\mathcal{F}$ -group with complete resolutions has to be in a particular level of Kropholler’s hierarchy, is an interesting question and it forms one of our conjectured statements below. In [Conjecture 4.16](#) below, most of the statements are expectations based on evidence of a lack of examples to indicate otherwise. [4.16.b.](#), for example, is a standard question to ask once all the different hierarchical classes have been defined in any hierarchy in general. The same logic applies to asking [4.16.c/d/f](#). For the following conjecture, we denote by $\text{CR}(\mathbb{Z})$ the class of all groups that admit complete resolutions over the integers.

Conjecture 4.16. *The following statements are true.*

- (a) $H\mathcal{F} \cap \text{CR}(\mathbb{Z}) = H_1\mathcal{F}$.
- (b) $C_1(\mathcal{F}) = C_2(\mathcal{F}) = \dots$
- (c) $C_\infty(\mathcal{F}) = H_1\mathcal{F}$.
- (d) $C_\infty(\mathcal{F}) = \mathcal{F}_\phi$.
- (e) $C_\infty(\mathcal{F}) = \{\Gamma : \underline{cd}_{\mathbb{Z}}(\Gamma) < \infty\}$
- (f) $\mathcal{F}_\phi = H_1\mathcal{F}$.
- (g) $C_\infty(\mathcal{F}_\phi) = C_\infty(\mathcal{F})$.
- (h) $H\mathcal{F}_\phi = H\mathcal{F}$.

In the following result, whenever we say $p_1 \xrightarrow{p_2} p_3$, or $p_1 \xleftrightarrow{p_2} p_3$, for some statements p_1, p_2, p_3 , we mean $p_1 \Rightarrow p_3$, or resp. $p_1 \Leftrightarrow p_3$, if p_2 is assumed to be true.

Proposition 4.17. *The following implications are true involving the statements of [Conjecture 4.16](#).*

- (a) $4.16.a. \Leftrightarrow 4.16.b. \Leftrightarrow 4.16.c.$
- (b) $4.16.c. \xleftrightarrow{4.16.f} 4.16.d.$
- (c) $4.16.e. \Rightarrow 4.16.d.$
- (d) $4.16.c. \xrightarrow{4.16.e} 4.16.f. \text{ and } 4.16.f. \Rightarrow 4.16.c.$

- (e) 4.16.d. ^{4.16.a.} \Rightarrow 4.16.f. and, 4.16.f. \Rightarrow 4.16.d.
 (f) 4.16.f. \Rightarrow 4.16.h. If a slightly stronger version of 4.16.h., namely $H_\omega \mathcal{F} = H_\omega \mathcal{F}_\phi$ where ω is the first infinite ordinal, is assumed to be true, then 4.16.g. holds.
 (g) If, in [Conjecture 4.13](#), when $A = \mathbb{Z}$, 4.13.b./c./d./e. \Rightarrow 4.13.g., then 4.16.a.-4.16.h. are true.

Proof. (a) Note that by [Theorem 1.8.e.](#), [Theorem 1.8.f.](#) and [Remark 1.3](#), $\text{spli}(\mathbb{Z}\Gamma) < \infty \iff \Gamma \in \text{CR}(\mathbb{Z})$. So, 4.16.a. \iff 4.16.c. by [Lemma 4.9.a.](#) 4.16.b. \iff 4.16.c. also follows from the fact that $C_1(\mathcal{F}) = H_1 \mathcal{F}$ by [Lemma 4.9.a.](#)

(b) This is obvious.

(c) We know from [Lemma 4.9.b.](#) that $C_\infty(\mathcal{F}) \subseteq \mathcal{F}_\phi$. Now, if $\Gamma \in \mathcal{F}_\phi$, $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$ by [Remark 4.11](#), and by 4.16.e., $\Gamma \in C_\infty(\mathcal{F})$.

(d) To show that 4.16.c. \Rightarrow 4.16.f. if we assume 4.16.e., note that $H_1 \mathcal{F} \subseteq \mathcal{F}_\phi$ by [Lemma 4.9](#) and if $\Gamma \in \mathcal{F}_\phi$, then $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$ again by [Remark 4.11](#), and therefore by 4.16.e., $\Gamma \in C_\infty(\mathcal{F}) = H_1 \mathcal{F}$ (the last equality is from the hypothesis 4.16.c.).

If 4.16.f. is true, then $C_\infty(\mathcal{F}) \subseteq \mathcal{F}_\phi$ (by [Lemma 4.9](#)) $= H_1 \mathcal{F}$ (by 4.16.f.). We already know courtesy of [Lemma 4.9](#) that $H_1 \mathcal{F} = C_1(\mathcal{F}) \subseteq C_\infty(\mathcal{F})$.

(e) To show that 4.16.d. \Rightarrow 4.16.f. if we assume 4.16.a., note that if $\Gamma \in \mathcal{F}_\phi$, then $\Gamma \in \text{CR}(\mathbb{Z})$ by [Remark 4.11](#), and since 4.16.d. gives us that $\Gamma \in C_\infty(\mathcal{F})$, we get from 4.16.a. and [Lemma 4.9.a.](#) that $\Gamma \in H_1 \mathcal{F}$. Again, $H_1 \mathcal{F} \subseteq \mathcal{F}_\phi$ follows from [Lemma 4.9.b.](#)

4.16.f. \Rightarrow 4.16.d. is easy to see as $H_1 \mathcal{F} = C_1(\mathcal{F})$ by [Lemma 4.9.a.](#)

(f) 4.16.f. \Rightarrow 4.16.h. is obvious as $\mathcal{F}_\phi = H_1 \mathcal{F}$ implies $H \mathcal{F}_\phi = H(H_1 \mathcal{F}) = H \mathcal{F}$. From the proof of [Lemma 4.9.a.](#), as provided in [30], it follows that $C_\infty(\mathcal{F}_\phi) = H_\omega \mathcal{F}_\phi$ (the proof in [30] is for \mathcal{F} as the base class but it translates to the case where \mathcal{F}_ϕ is the base class). Therefore, $H_\omega \mathcal{F} = H_\omega \mathcal{F}_\phi$ implies $C_\omega \mathcal{F} = C_\omega \mathcal{F}_\phi$.

(g) It follows from [Theorem 1.8.a.](#), [Theorem 1.8.d.](#) and [Theorem 1.8.f.](#) that we can streamline our hypothesis to 4.13.e. \Rightarrow 4.13.g. (we denote this statement by $(*)$). We assume $(*)$ is true.

$(*) \Rightarrow$ 4.16.c. : Now, if $\Gamma \in C_\infty(\mathcal{F})$, then $\text{fin. dim}(\mathbb{Z}\Gamma) < \infty$ by [Lemma 4.9](#) and [Remark 4.11](#), so there is a finite dimensional model for $\underline{E}\Gamma$, so clearly $\Gamma \in H_1 \mathcal{F}$ (see [Remark 4.14](#)). Thus 4.16.c. holds, and so 4.16.a.-c. hold as well by part (a) of this proposition.

$(*) \Rightarrow$ 4.16.d. : If $\Gamma \in \mathcal{F}_\phi$, $\text{fin. dim}(\mathbb{Z}\Gamma) < \infty$ by [Remark 4.11](#), and since $(*)$ holds, there is a finite dimensional model for $\underline{E}\Gamma$, therefore $\Gamma \in H_1 \mathcal{F} = C_1(\mathcal{F}) \subseteq C_\infty(\mathcal{F})$. So, 4.16.d. holds as we already know that $C_\infty(\mathcal{F}) \subseteq \mathcal{F}_\phi$ by [Lemma 4.9.b.](#)

$(*) \Rightarrow$ 4.16.e. : If Γ be a group such that $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$, then by [Theorem 4.2](#), [Theorem 1.8.d.](#) and [Theorem 1.8.a.](#), $\text{fin. dim}(\mathbb{Z}\Gamma) < \infty$, and therefore there is a finite dimensional model for $\underline{E}\Gamma$, so $\Gamma \in H_1 \mathcal{F} = C_1(\mathcal{F})$. Again note that we already know that groups in Ikenaga's classes have finite generalized cohomological dimension over the integers ([Theorem 4.4](#)).

$(*) \Rightarrow$ 4.16.f. : We already know that $H_1 \mathcal{F} \subseteq \mathcal{F}_\phi$ by [Lemma 4.9.b.](#) Now let $\Gamma \in \mathcal{F}_\phi$. Then, by [Remark 4.11](#), $\text{fin. dim}(\mathbb{Z}\Gamma) < \infty$ and by $(*)$, there exists a finite dimensional model for $\underline{E}\Gamma$ and therefore $\Gamma \in H_1 \mathcal{F}$.

Thus, $(*)$ implies 4.16.g. and 4.16.h. as well by part (f) of this proposition. \square

5. A small result on classifying spaces

As we saw in [Proposition 4.17.g.](#), the finiteness of almost any cohomological invariant for Γ implying the existence of a finite dimensional model for $\underline{E}\Gamma$ is quite strong. In this section, we show using a key result from [26] that some of the classes of groups that we have dealt with

admit finite dimensional models for their classifying space of proper actions if an additional condition is satisfied. To introduce this additional condition, we need the following definition.

Definition 5.1. For a finite group G , define the length of G , denoted $l(G)$, as the supremum over n such that there is a nested sequence $H_0 \subsetneq H_1 \subsetneq \dots \subsetneq H_n$ where each H_i is a subgroup of G .

Definition 5.2. For any integer d , a group Γ is said to be of type $b(d)$ if for every $\mathbb{Z}\Gamma$ -module M that is projective over finite subgroups, $\text{proj. dim}_{\mathbb{Z}\Gamma} M \leq d$. Γ is said to be of type $B(d)$ if, for every finite $G \leq \Gamma$, $W_\Gamma(G) := N_\Gamma(G)/G$ is of type $b(d)$.

It is easy to note that groups of type Φ over the integers are of type $b(d)$ for some $d \geq 0$:

Lemma 5.3. Let $\Gamma \in \mathcal{F}_\phi$ ($:= \mathcal{F}_{\phi, \mathbb{Z}}$, see Definition 1.5). Then, Γ is of type $b(k(\mathbb{Z}\Gamma))$.

Proof. It follows from Remark 4.11 that $k(\mathbb{Z}\Gamma) < \infty$. If M is projective over finite subgroups of Γ , then by the definition of $k(\mathbb{Z}\Gamma)$, $\text{proj. dim}_{\mathbb{Z}\Gamma} M \leq k(\mathbb{Z}\Gamma)$. □

The following is the key result from [26] that we will be using in this section.

Theorem 5.4. (Theorem 1.10 of [26]) Let Γ be a group of type $B(d)$ for some finite $d \geq 0$ and let there be a finite bound on the length of all finite subgroups of Γ . Then, Γ admits a finite dimensional model for $\underline{E}\Gamma$.

Using Theorem 5.4, we can prove the following result.

Proposition 5.5. Let $\Gamma \in LH\mathcal{F} \cap \mathcal{F}_\phi$ such that there is a bound on the length of all finite subgroups of Γ . Then, there exists a finite dimensional model for $\underline{E}\Gamma$.

Proof. From Theorem 5.4, it follows that we will be done if we show that Γ is of type $B(d)$ for some $d \geq 0$. From Remark 4.11, it follows that $\text{Gcd}_{\mathbb{Z}}(\Gamma) < \infty$.

For any finite subgroup $G \leq \Gamma$, $W_\Gamma(G)$ is in $LH\mathcal{F}$ (this follows from the fact that $H\mathcal{F}$ is Weyl group closed—see Proposition 7.1 of [25]). So, it follows from Theorem 1.9 that $k(\mathbb{Z}W_\Gamma(G)) \leq \text{Gcd}_{\mathbb{Z}}(W_\Gamma(G)) + 1 \leq \text{Gcd}_{\mathbb{Z}}(\Gamma) + 1$ (the last inequality is by Proposition 2.5 of [16]). Thus, $W_\Gamma(G)$ is of type $b(\text{Gcd}_{\mathbb{Z}}(\Gamma) + 1)$. So we have shown that Γ is of type $B(\text{Gcd}_{\mathbb{Z}}(\Gamma) + 1)$, and we are done. □

It is interesting to note that we can replace the hypothesis $\Gamma \in LH\mathcal{F} \cap \mathcal{F}_\phi$ in the statement of Proposition 5.5 by $\Gamma \in C_\infty(\mathcal{F})$.

Corollary 5.6. Let Γ be in $C_\infty(\mathcal{F})$ with a bound on the length of its finite subgroups. Then, there is a finite dimensional model for $\underline{E}\Gamma$.

Proof. This follows directly from Proposition 5.5 using Lemma 4.9. □

The hypothesis of Corollary 5.6 is weaker than that of Proposition 5.5, but we state Corollary 5.6 separately because it is an interesting question as to what the connection is between groups in Ikenaga’s hierarchy and groups admitting finite dimensional models for their classifying space of proper actions.

6. Two general questions on projectivity

It is well-known that for a finite group G , a $\mathbb{Z}G$ -module M is projective iff M is \mathbb{Z} -free and of finite projective dimension as a $\mathbb{Z}G$ -module. In [23], the following question was asked:

Question 6.1. (Question A of [23]) Fix \mathbb{Z} to be the base ring. Is it only for finite groups G that a $\mathbb{Z}G$ -module is projective iff it is \mathbb{Z} -free and of finite projective dimension as a $\mathbb{Z}G$ -module?

In making some progress on [Question 6.1](#), the following theorem was proved in [\[23\]](#).

Theorem 6.2. (*Theorem 2.4 of [23]*) *Let Γ be a group such that every \mathbb{Z} -free $\mathbb{Z}\Gamma$ -module of finite projective dimension is projective. If $\Gamma \in H\mathcal{F}$, then Γ is finite.*

One can prove the statement of [Theorem 6.2](#) replacing “ $\Gamma \in H\mathcal{F}$ ” with “ $\Gamma \in LH\mathcal{F}_\phi$ ”:

Theorem 6.3. *Let Γ be a group such that every \mathbb{Z} -free $\mathbb{Z}\Gamma$ -module of finite projective dimension is projective. If $\Gamma \in LH\mathcal{F}_\phi$, then Γ is finite.*

Proof. It is straightforward to see (also by [Proposition 2.3](#) of [\[23\]](#)) that $\text{fin. dim}(\mathbb{Z}\Gamma)$ is either 0 or 1 as the global dimension of \mathbb{Z} is 1. Since we are assuming that $\Gamma \in LH\mathcal{F}_\phi$, $\text{fin. dim}(\mathbb{Z}\Gamma) = 0$ is an absurdity because if $\text{fin. dim}(\mathbb{Z}\Gamma) = 0$, then by [Theorem 1.9](#), $\text{Gcd}_{\mathbb{Z}}(\Gamma) = 0$ and by [Theorem 1.8.b.](#), Γ is finite, but for finite Γ , by [Theorem 1.8.b.](#), $\text{spli}(\mathbb{Z}\Gamma) = 1$ and by [Theorem 1.9](#), $\text{fin. dim}(\mathbb{Z}\Gamma) = 1$, and so we have a contradiction.

Now, if $\text{fin. dim}(\mathbb{Z}\Gamma) = 1$, then by [Theorem 1.9](#), $\text{spli}(\mathbb{Z}\Gamma) = 1$, and again by [Theorem 1.8.b.](#), Γ is finite. □

The second question on projectivity that [\[23\]](#) tackles deals with stably flat modules as defined in [Definition 6.4](#). Stably flat modules arise in the study of complete cohomology for infinite groups. This is again a concept that we have not dealt with elsewhere, so we provide a definition below:

Definition 6.4. Let A be a commutative ring and let Γ be a group. An $A\Gamma$ -module N is called stably flat iff $\widehat{\text{Ext}}_{A\Gamma}^0(M, N) = 0$ for all $A\Gamma$ -modules M of type FP_∞ .

The following was proved in [\[23\]](#):

Theorem 6.5. (*Theorem 3.4 of [23]*) *Let $\Gamma \in LH\mathcal{F}$. Then, any stably flat $\mathbb{Z}\Gamma$ -module M that is also a Benson’s cofibrant is projective.*

We can prove the statement of [Theorem 6.5](#) replacing \mathbb{Z} with a coherent commutative ring A of finite global dimension and the condition “ $\Gamma \in LH\mathcal{F}$ ” with “ $\Gamma \in LH\mathcal{F}_{\phi, A}$.” To do this proof, we need two results about stably flats that come from Alcock. Alcock [\[2\]](#) considered the question of whether, for some class of infinite groups and with certain conditions on the base ring, one can provide a complete characterization of stably flat modules in terms of projective dimension and proved the following:

Theorem 6.6. (*Theorem A of [2]*) *Let A be a coherent commutative ring (i.e. every finitely generated ideal is finitely presented as a module) with finite global dimension and let $\Gamma \in H_1\mathcal{F}$. Then, for any $A\Gamma$ -module N , the following are equivalent.*

- (a) N is stably flat as an $A\Gamma$ -module.
- (b) $\text{proj. dim}_{A\Gamma} N < \infty$.

We have seen before that $H_1\mathcal{F} \subseteq \mathcal{F}_{\phi, A}$. We can now prove the statement of [Theorem 6.6](#) replacing the condition $\Gamma \in H_1\mathcal{F}$ with $\Gamma \in \mathcal{F}_{\phi, A}$:

Theorem 6.7. *Let A be a coherent commutative ring of finite global dimension and let $\Gamma \in \mathcal{F}_{\phi, A}$. Then, for any $A\Gamma$ -module N , the following are equivalent:*

- (a) N is stably flat as an $A\Gamma$ -module.
- (b) $\text{proj. dim}_{A\Gamma} N < \infty$.

Proof. (b) \Rightarrow (a). This is obvious as if $\text{proj. dim}_{A\Gamma} N < \infty$, then $\widehat{\text{Ext}}^0_{A\Gamma}(M, N) = 0$ for all M of type FP_∞ because complete cohomology vanishes on modules with finite projective dimension.

(a) \Rightarrow (b). If N is stably flat as an $A\Gamma$ -module, then by Corollary 3.4 of [2], N is stably flat as an AG -module for all finite $G \leq \Gamma$, and by Theorem A' of [2] (or even just by Theorem 6.6), $\text{proj. dim}_{AG} N < \infty$ for all finite $G \leq \Gamma$, and therefore $\text{proj. dim}_{A\Gamma} N < \infty$ as Γ is of type Φ over A . □

Theorem 6.8. *Let A be a coherent commutative ring of finite global dimension and let $\Gamma \in LH\mathcal{F}_{\phi,A}$. For any $A\Gamma$ -module M , if M is stably flat as an $A\Gamma$ -module and also a Benson’s cofibrant, then it is projective.*

Proof. First, we deal with the case when $\Gamma \in H\mathcal{F}_{\phi,A}$. We proceed by transfinite induction on the smallest ordinal α such that $\Gamma \in H_\alpha\mathcal{F}_{\phi,A}$. If $\alpha = 0$, then by Theorem 6.7, $\text{proj. dim}_{A\Gamma} M < \infty$. So, by Proposition 5.4 of [4], M is projective. Now, as our induction hypothesis, assume that the statement of the theorem holds for all $\Gamma \in H_\beta\mathcal{F}_{\phi,A}$ for all ordinals $\beta < \alpha$. If, now, $\Gamma \in H_\alpha\mathcal{F}_{\phi,A}$, then Γ acts on a finite dimensional contractible CW-complex with stabilizers in $H_{<\alpha}\mathcal{F}_{\phi,A}$, and by tensoring the augmented cellular complex with M , we get a finite length resolution of M with modules that are direct sums of modules of the form $\text{Ind}^{\Gamma'}_{\Gamma}(\text{Res}^{\Gamma'}_{\Gamma}(M))$ for some $\Gamma' \in H_{<\alpha}\mathcal{F}_{\phi,A}$ (Here Ind and Res denote the induction and restriction functors respectively). As an $A\Gamma'$ -module, $\text{Res}^{\Gamma'}_{\Gamma}M$ is stably flat by Corollary 3.4 of [2] and also Benson’s cofibrant as cofibrants remain cofibrant upon restriction to subgroups (see Remark 6.9 below), and so by our induction hypothesis, it is projective as an $A\Gamma'$ -module and $\text{Ind}^{\Gamma'}_{\Gamma}(\text{Res}^{\Gamma'}_{\Gamma}(M))$ is projective as an $A\Gamma$ -module. Therefore, M has finite projective dimension as an $A\Gamma$ -module. Again by Proposition 5.4 of [4], M is projective. This ends our proof for the case where $\Gamma \in H\mathcal{F}_{\phi,A}$.

Now, let $\Gamma \in LH\mathcal{F}_{\phi,A}$. We can assume that Γ is uncountable because if it is countable then since every countable group admits an action on a tree with finitely generated vertex and edge stabilizers (see Lemma 2.5 of [22]), it follows that $\Gamma \in H\mathcal{F}_{\phi,A}$. Assume, as an induction hypothesis, that the theorem has been proved for all groups with cardinality strictly smaller than Γ . We can express Γ as an ascending union of subgroups $\{\Gamma_\lambda : \lambda \in \Lambda\}$ where each Γ_λ is of strictly smaller cardinality than Γ . By the induction hypothesis, M is projective over each Γ_λ (note that again, to go down to subgroups here, we are using Corollary 3.4 of [2] and the fact that cofibrants remain cofibrants when restricted to subgroups), and so by Lemma 5.6 of [4], $\text{proj. dim}_{A\Gamma} M \leq 1$. And again, this means M is Gorenstein projective with finite projective dimension, so it must be projective. □

Remark 6.9. (See Lemma 4.8 of [8]) For any group Γ and any commutative ring A , take an $A\Gamma$ -module M such that $M \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma$ -module. This implies that $\text{Res}^{\Gamma'}_{\Gamma}(M \otimes_A B(\Gamma, A))$ is projective as an $A\Gamma'$ -module for any subgroup Γ' of Γ . $B(\Gamma', A)$ is a direct summand of $\text{Res}^{\Gamma'}_{\Gamma}(B(\Gamma, A))$ —this is proved in [27] for $A = \mathbb{Z}$ and it follows over any commutative ring A because $B(\Gamma, A) = B(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} A$ (see the proof of Lemma 3.4 of [4]). So, $\text{Res}^{\Gamma'}_{\Gamma}M \otimes_A B(\Gamma', A)$ is a summand of $\text{Res}^{\Gamma'}_{\Gamma}M \otimes_A \text{Res}^{\Gamma'}_{\Gamma}B(\Gamma, A) = \text{Res}^{\Gamma'}_{\Gamma}(M \otimes_A B(\Gamma, A))$ which is projective. Therefore, $\text{Res}^{\Gamma'}_{\Gamma}M$ is cofibrant as an $A\Gamma'$ -module.

Our proof of Theorem 6.8 here is quite independent of the way Theorem 6.5 is proved in [23].

7. Periodic cohomology and complete resolutions

We have seen before how for a group the property of admitting complete resolutions is quite helpful in dealing with many questions. A good indicator of whether a group admits (weak)

complete resolutions or not, is checking whether it has periodic cohomology after a finite number of steps (Proposition 3.1 of [35]). Since we didn't work much with periodic cohomology elsewhere, we provide its definition here.

Definition 7.1. (see [30, 35]) A group Γ is said to have periodic cohomology of period q after k steps iff the functors $H^i(\Gamma, ?)$ and $H^{i+q}(\Gamma, ?)$ are naturally equivalent for all $i > k$.

The following important conjecture was made for groups with periodic cohomology in [35].

Conjecture 7.2. (Conjecture A of [35]) A group Γ has periodic cohomology after some steps iff Γ admits a finite-dimensional free Γ -CW-complex, homotopy equivalent to a sphere.

Talelli settled Conjecture 7.2 for $H\mathcal{F}$ -groups in Corollary 3.5 of [35]. Almost the same proof works for $LH\mathcal{F}_\phi$ -groups.

Theorem 7.3. If $\Gamma \in LH\mathcal{F}_\phi$, Conjecture 7.2 holds true for Γ .

Proof. We assume that Γ has periodic cohomology of period q after k steps. By Proposition 3.1 of [35], Γ admits a weak complete resolution, and since $\Gamma \in LH\mathcal{F}_\phi$, this implies that Γ admits complete resolutions by Theorem 1.12, and therefore by Theorem 1.8.e. and Remark 1.3, $\text{silp}(\mathbb{Z}\Gamma) < \infty$. So, by Theorem 3.2 and Corollary 3.3 of [35], the periodicity isomorphisms are induced by the cup product in $H^q(G, \mathbb{Z})$, and as noted in [35], Adem and Smith [1] proved that Conjecture 7.2 holds when this happens. \square

8. Groups outside Kropholler's hierarchy with type Φ groups as the base class

It was shown in [24] that Thompson's group $F := \langle x_0, x_1, x_2, \dots : x_k^{-1}x_nx_k = x_{n+1}, k < n \rangle$ is not in $LH\mathcal{F}$. Using basically the same argument, we can say that F is not in $LH\mathcal{F}_\phi$, as we show here.

First, we quote the following theorem from [24] which is one of the main results of that paper.

Theorem 8.1. (Theorem A of [24]) Let \mathcal{X} be a class of groups and let A be a commutative ring. Take an $LH\mathcal{X}$ -group Γ and an $A\Gamma$ -module M . Assume that $\text{Ext}_{A\Gamma}^n(M, ?)$ commutes with direct limits for infinitely many non-negative n . Then, the following statements are equivalent.

- (a) $\text{proj. dim}_{A\Gamma} M < \infty$.
- (b) $\text{proj. dim}_{A\Gamma'} M < \infty$, for all $\Gamma' \leq \Gamma$ such that $\Gamma' \in \mathcal{X}$.

Corollary 8.2. Thompson's group F is not in $LH\mathcal{F}_\phi$.

Proof. Note that for any group Γ and any commutative ring A , an $A\Gamma$ -module M is of type FP_∞ iff the functors $\text{Ext}_{A\Gamma}^*(M, ?)$ commute with direct limits. Now take $\mathcal{X} = \mathcal{F}_\phi$ and $A = \mathbb{Z}$ in the statement of Theorem 8.1, and let M be of type FP_∞ . Then, if $\Gamma \in LH\mathcal{F}_\phi$, then $\text{proj. dim}_{\mathbb{Z}\Gamma} M < \infty$ iff $\text{proj. dim}_{\mathbb{Z}\Gamma'} M < \infty$ for all type Φ subgroups $\Gamma' \leq \Gamma$, which in turn can happen iff $\text{proj. dim}_{\mathbb{Z}G} M < \infty$ for all finite subgroups $G \leq \Gamma$ (this last bit follows from the definition of type Φ groups).

It follows from Corollary 5.4 of [10] that F is of type FP_∞ , i.e. the trivial module \mathbb{Z} is of type FP_∞ as a $\mathbb{Z}F$ -module, and it follows from Corollary 1.5 of [10] that F is torsion-free. So, the only finite subgroup of F is the trivial subgroup. It therefore follows from the preceding paragraph that if $F \in LH\mathcal{F}_\phi$, then $\text{proj. dim}_{\mathbb{Z}F} \mathbb{Z} < \infty$, i.e. F has finite cohomological dimension over \mathbb{Z} , which is not possible as F contains a free abelian group of infinite rank which has infinite cohomological dimension. \square

For a long time, since F was the most well-known group outside $LH\mathcal{F}$, the most common way to show a group was not in $H\mathcal{F}$ or $LH\mathcal{F}$ was to show that it had a subgroup isomorphic to F . In [3], the authors introduce a different set of methods that give examples of groups outside Kropholler’s hierarchy. We quote below one of the main theorems of [3].

Theorem 8.3. (part of Theorem 1.1 of [3]) *There exists an infinite finitely generated group Q which cannot act on any finite dimensional CW-complex without a global fixed point. For any countable group C , Q can be chosen so that Q is simple, has Kazhdan’s property (T) and contains an isomorphic copy of C .*

It is natural to ask if the groups constructed in proving Theorem 8.3 in [3] are in $H\mathcal{F}_\phi$.

Remark 8.4. It has been noted in [3], and it is also easy to see, that if Q is a group satisfying the statement of Theorem 8.3, then $Q \in H\mathcal{X}$, for any class \mathcal{X} , iff $Q \in \mathcal{X}$. Taking $\mathcal{X} = \mathcal{F}_\phi$, we get that if $Q \in H\mathcal{F}_\phi$, then Q is of type Φ . But if in the statement of Theorem 8.3, we take C to be the free abelian group of rank \aleph_0 , then Q cannot admit complete resolutions as C does not admit complete resolutions, and therefore Q cannot be of type Φ as groups of type Φ admit complete resolutions (see Remark 4.11). It is noteworthy that to reach this conclusion, we are having to choose a convenient countable group for C .

Another known concrete example of a group outside $H\mathcal{F}$ is the first Grigorchuk group (Theorem 4.11 of [17]). A major ingredient in the proof of Theorem 4.11 of [17] is the following result of Petrosyan [31].

Theorem 8.5. (Theorem 3.2 of [31]) *Take A to be a commutative ring, and let Γ be a discrete group with no A -torsion such that it has jump cohomology of height k over A , which means that for any subgroup $\Gamma_1 \leq \Gamma$, $cd_A(\Gamma_1) < \infty$ implies $cd_A(\Gamma_1) \leq k$. If $\Gamma \in H\mathcal{F}$, then $cd_A(\Gamma) \leq k$. So, an $H\mathcal{F}$ -group Γ can have jump cohomology of height k over \mathbb{Q} if and only if $cd_{\mathbb{Q}}(\Gamma) \leq k$.*

It is easy to see that the statement of Theorem 8.5 holds with $H\mathcal{F}$ replaced by $H\mathcal{F}_{\phi,A}$, for any commutative ring A of finite global dimension.

Corollary 8.6. *Let A be a commutative ring and let Γ be a discrete group with no A -torsion. Assume that there exists a non-negative integer k such that for any subgroup $\Gamma_1 \leq \Gamma$, $cd_A(\Gamma_1) < \infty$ implies $cd_A(\Gamma_1) \leq k$. Now, if $\Gamma \in H\mathcal{F}_{\phi,A}$, then $cd_A(\Gamma) \leq k$.*

Proof. Theorem 8.5 is proved in [31] by first proving it for the base case, i.e. when Γ is finite, and then proving it by transfinite induction on the level of Γ in $H\mathcal{F}$. We reproduce that proof for our case.

For our base case of type Φ groups, note that if Γ is of type Φ , then $cd_A(G) = 0$ for all finite $G \leq \Gamma$ since G needs to be A -torsion-free as per the hypothesis of Theorem 8.5. Thus, $cd_A(\Gamma) < \infty$ by definition of type Φ groups.

Now, we make the following induction hypothesis: for some fixed ordinal α , $cd_A(\Gamma') < k$ for any $H_{<\alpha}\mathcal{F}_{\phi,A}$ -subgroup Γ' of Γ (note that from the hypothesis of Corollary 8.6, it follows that $cd_A(\Gamma') < \infty$ iff $cd_A(\Gamma') < k$). Let Γ'' be an $H_{\alpha}\mathcal{F}_{\phi,A}$ -subgroup of Γ . Then, Γ'' acts on a finite dimensional contractible CW-complex X with stabilizers in $H_{<\alpha}\mathcal{F}_{\phi,A}$. By our induction hypothesis, all these stabilizers have cohomological dimension at most k over A . If the dimension of X is n , then using Lemma 3.3 of [31], we get that $cd_A(\Gamma'') \leq k + n$. Again from the hypothesis of Corollary 8.6 as noted in parentheses above, it now follows that $cd_A(\Gamma'') \leq k$.

We have thus proved that any $H\mathcal{F}_{\phi,A}$ -subgroup of Γ has cohomological dimension at most k over A . So, if $\Gamma \in H\mathcal{F}_{\phi,A}$, then $cd_A(\Gamma) \leq k$. □

Corollary 8.7. *The first Grigorchuk group is not in $H\mathcal{F}_{\phi,\mathbb{Q}}$.*

Proof. It is shown in Theorem 4.11 of [17] that the first Grigorchuk group has jump rational cohomology of height 1 and has infinite cohomological dimension over the rationals, so by Corollary 8.6, it is not in $H\mathcal{F}_{\phi, \mathbb{Q}}$. \square

Acknowledgements

This paper was written while the author was a postdoctoral fellow at the Max Planck Institute for Mathematics, Bonn. The author is thankful to MPI-B for their support.

Funding

The research presented in this article was undertaken while the author was a graduate student at the University of Manchester and they were supported by a research scholarship from the University of Manchester throughout this project.

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