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and a related conjecture**

by

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BENSON'S COFIBRANTS, GORENSTEIN PROJECTIVES AND A RELATED CONJECTURE

RUDRADIP BISWAS

ABSTRACT. In this short article, we will be principally investigating two classes of modules over any given group ring - the class of Gorenstein projectives and the class of Benson's cofibrants. We begin by studying various properties of these two classes and studying some of these properties comparatively against each other. There is a conjecture made by Fotini Dembegioti and Olympia Talelli that these two classes should coincide over the integral group ring for any group. We make this conjecture over group rings over commutative rings of finite global dimension and prove it for some classes of groups while also proving other related results involving the two classes of modules mentioned.

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As mentioned in the abstract, our dealings in this article will focus around two classes of modules, namely the class of Gorenstein projectives and the class of Benson's cofibrants which are conjectured to be coinciding all the time (see Conjecture 1.1).

Most of the arguments presented in this article are straightforward and easy to follow. There is a lot of background from the theory of cohomological invariants of groups, etc. without which one can struggle to appreciate the results, and this is why we have taken great care to expand on the background in the first section.

We start by providing a number of useful definitions and known results in **Section 1**. Then, in **Section 2**, we show that for any group G and any commutative ring R , both the classes of Gorenstein projective RG -modules and the class of RG -modules M such that $M \otimes_R B(G, R)$ (for the rest of this paper, we will be writing just \otimes to mean \otimes_R , with R being clear from the immediate

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context) is projective are good classes in the sense of Definition 3.5 of [5] (= Definition 2.1 of this paper) - the RG -module $B(G, R)$, which is given by all $G \rightarrow R$ functions that are only allowed to take finitely many values, plays a big role in the definition of Benson's cofibrants, we define this module in Definition 1.15. Then, in Section 3, we investigate the relations between the classes of modules generated by those two classes and show how, for groups satisfying certain properties or belonging to certain classes, we can derive information about the coincidence of these classes from the coincidence of the classes generated by them. When we say "generation" here, we mean generation in the sense introduced in [5]. And then, after investigating some less important or less relevant questions on Gorenstein projectivity in Section 4, we prove some general results on the coincidence of Gorenstein projectives and Benson's cofibrants for some classes of groups in Section 5.

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1. PRELIMINARIES AND DEFINITIONS

We begin by stating the main conjecture by Dembogiotti and Talelli that we will be dealing with in this article.

Conjecture 1.1. *(made over \mathbb{Z} in [10]) For any group G and any commutative ring R of finite global dimension, the class of Gorenstein projective RG -modules coincides with the class of Benson's cofibrant RG -modules, i.e. the class $\{M \in \text{Mod}(RG) : M \otimes_R B(G, R) \text{ is projective}\}$ (see Definition 1.15).*

All of our tensor products in this article will be over R .

We now look into the two classes of modules mentioned in the statement of Conjecture 1.1 and study some of their important properties.

To define Gorenstein projectives, we need to provide a definition of complete resolutions because Gorenstein projectives are defined to be precisely those modules that occur as kernels in complete resolutions.

Definition 1.2. *Let A be a ring and let M be an A -module. A complete resolution admitted by M is an infinite exact sequence of A -projective modules, (F_*, d_*) : $\dots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \rightarrow \dots$ such that*

a) *There exists a projective resolution of M , $(P_*, \delta_*) \rightarrow M$, such that for some $n \geq 0$, $(P_i, \delta_i)_{i \geq n} = (F_i, d_i)_{i \geq n}$. The smallest such n is called the coincidence index of the complete resolution with respect to the given projective resolution.*

b) *$\text{Hom}_A(F_*, P)$ is acyclic for any A -projective module P .*

An A -module is called Gorenstein projective iff it occurs as a kernel in a complete resolution.

When A is a group ring RG , where G is a group and R is a commutative ring, a group G is said to admit a complete resolution over R if the trivial RG -module R admits a complete resolution.

First, note that if we are working over a group algebra over a commutative ring R of finite global dimension, then Gorenstein projectives over the group algebra are projective over R . This follows straightforwardly from the definition of complete resolutions. We record this in the following lemma. We will be making use of this fact in Section 3.

Lemma 1.3. *Let G be a group and let R be a commutative ring of finite global dimension and let M be a kernel in an exact sequence of projective RG -modules that extends to infinity in both directions. Then, M is R -projective.*

Proof. Let the global dimension of R be r . We have M as a kernel in a doubly infinite acyclic complex of projectives, (F_*, d_*) . Let $M = \text{Ker}(d_t)$. Looking at the exact sequence $0 \rightarrow M \hookrightarrow F_t \rightarrow F_{t-1} \rightarrow \dots \rightarrow F_{t-s} \twoheadrightarrow \text{Im}(d_{t-s}) \rightarrow 0$ where $s \geq r-1$, we see that M is the $(s+1)$ -th syzygy of $\text{Im}(d_{t-s})$ and since r is the global dimension of R , this implies that M is R -projective. \square

Theorem 1.4. *(Thm 2.5, Prop. 2.27 of [15]) The class of Gorenstein projective R -modules, for any ring R , contains the class of projectives and is closed under arbitrary direct sums and direct summands. Additionally, Gorenstein projectives are either projective or of infinite projective dimension.*

The following result provides a nice class of examples of Gorenstein projectives.

Lemma 1.5. *(Lemma 2.21 of [1]) Let G be a group and R a commutative ring of finite global dimension. Then, any RG -permutation module with finite stabilisers is Gorenstein projective.*

It is important to note that, when working over group algebras over rings of finite global dimension, whenever we have the trivial module admitting complete resolutions, all other modules admit complete resolutions. This result can be stated in terms of the Gorenstein projective dimension, and since we will be dealing a lot with the Gorenstein projective dimension in the coming sections, we provide here its definition and a very useful property.

Definition 1.6. *Let R be a ring and let M be an R -module. The Gorenstein projective dimension of M as an R -module, denoted $\text{Gpd}_R(M)$, is defined as $\min\{n \in \mathbb{N} : \exists \text{ an exact sequence } 0 \rightarrow G_n \rightarrow \dots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0 \text{ where each } G_i \text{ is a Gorenstein projective } R\text{-module}\}$. If M does*

not admit a finite length resolution by Gorenstein projective R -modules, we say $Gpd_R(M)$ is not finite.

For any group G , the Gorenstein cohomological dimension of G with respect to R , denoted $Gcd_R(G)$, is defined as $Gpd_{RG}(R)$.

Any given group G achieves its highest Gorenstein cohomological dimension over the ring of integers among all rings. This was proved in [14].

Proposition 1.7. (Proposition 2.1 of [14]) *Let G be a group. Then, for any commutative ring R , $Gcd_R(G) \leq Gcd_{\mathbb{Z}}(G)$.*

Theorem 1.8. (Theorem 2.20 of [15]) *Let R be a ring and let M be an R -module such that $Gpd_R(M) < \infty$. The following are equivalent.*

- a) $Gpd_R(M) \leq n$.
- b) $\text{Ext}_R^{>n}(M, N) = 0$ for any R -module N satisfying $\text{proj. dim}_R(N) < \infty$.
- c) $\text{Ext}_R^{>n}(M, P) = 0$ for any projective P .
- d) In any exact sequence $0 \rightarrow K \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow M \rightarrow 0$, where each L_i is Gorenstein projective, K is Gorenstein projective.

It follows from Theorem 2.24 of [15] that we cannot have a situation where there is a short exact sequence with one module of infinite Gorenstein projective dimension and every other module of finite Gorenstein projective dimension. Using the Ext-formulation of Gorenstein projective dimension ((a)-(c) equivalence in Theorem 1.8), we get the following result.

Lemma 1.9. *Let R be a ring. Take a short exact sequence of R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. Then $Gpd_R(C) \leq \max\{Gpd_R(A) + 1, Gpd_R(B)\}$.*

Proof. If $Gpd_R(A)$ or $Gpd_R(B)$ is not finite, we have nothing to prove, so we can assume they are both finite. So, by Theorem 2.24 of [15], it follows that $Gpd_R(C)$ is finite.

The proof now is done in exactly the same way the very well-known analogous result is proved for projective dimension.

Let $Gpd_R(A) = m$ and $Gpd_R(B) = n$.

First, we deal with the case where $m+1 \geq n$. For any R -projective P , look at the long exact Ext-sequence: $\dots \rightarrow \text{Ext}_R^{m+1}(A, P) \rightarrow \text{Ext}_R^{m+2}(C, P) \rightarrow \text{Ext}_R^{m+2}(B, P) \rightarrow \dots$; here, $\text{Ext}_R^{m+1}(A, P) = 0$ by Theorem 1.8 as $Gpd_R(A) = m$ and $\text{Ext}_R^{m+2}(B, P) = 0$ again by Theorem 1.8 as $Gpd_R(B) = n$ and $m+2 > m+1 \geq n$. Therefore, $\text{Ext}_R^{m+2}(C, P) = 0$, and by Theorem 1.8, we have $Gpd_R(C) \leq m+1$.

Now, we deal with the case where $n > m+1$. Again, for any R -projective P , look at the long exact Ext-sequence: $\dots \rightarrow \text{Ext}_R^n(A, P) \rightarrow \text{Ext}_R^{n+1}(C, P) \rightarrow \text{Ext}_R^{n+1}(B, P) \rightarrow \dots$. Here, by Theorem 1.8 and since $n > 1 + Gpd_R(A)$, $\text{Ext}_R^n(A, P) = 0$, and again by Theorem 1.8 and since $Gpd_R(B) = n$, $\text{Ext}_R^{n+1}(B, P) = 0$. Therefore, $\text{Ext}_R^{n+1}(C, P) = 0$, and by Theorem 1.8, $Gpd_R(C) \leq n$.

Although the two cases dealt with above can be summed up in one argument, we dealt with them separately for clarity.

□

Remark 1.10. We will be making use of Lemma 1.9 later in Section 2 to see why the class of Gorenstein projectives is a “good” class in the sense of Definition 3.5 of [5]. However, it is noteworthy that instead of using Lemma 1.9, we can just use Theorem 2.24 of [15] in its place in Section 3.

It is easy to see that a module admits a complete resolution iff it has finite Gorenstein projective dimension - this is a very useful result. In our next result, we see, as touched upon before, that the trivial module admitting complete resolutions and all modules admitting complete resolutions are equivalent.

Theorem 1.11. *(Theorem 1.7 of [14]) Let G be a group and let R be a ring of finite global dimension. Then, $Gpd_{RG}(R) < \infty$ iff $Gpd_{RG}(M) < \infty$ for all RG -modules M .*

On the topic of groups admitting complete resolutions, it is to be noted that there are infinitely many groups that do not admit complete resolutions over any commutative ring of finite global dimension - free abelian groups of infinite rank, for example. Before looking at some interesting examples of infinite groups that admit complete resolutions, we make the following definition of two large classes of groups.

Definition 1.12. *(see [18],[24]) Let \mathcal{X} be a class of groups. We define $H_0\mathcal{X} := \mathcal{X}$, and for any successor ordinal α , a group G is said to be in $H_\alpha\mathcal{X}$ iff there exists a finite dimensional contractible CW-complex on which G acts by permuting the cells with cell stabilizers in $H_{\alpha-1}\mathcal{X}$. If α is a limit ordinal, then we define $H_\alpha\mathcal{X}$ as $\bigcup_{\beta < \alpha} H_\beta\mathcal{X}$. G is said to be in $H\mathcal{X}$ iff G is in $H_\alpha\mathcal{X}$ for some ordinal α (note that α may be chosen to be a successor ordinal here). We also use the notation $H_{<\alpha}\mathcal{X} := \bigcup_{\beta < \alpha} H_\beta\mathcal{X}$, for any ordinal α .*

A group G is said to be of type Φ over a ring R if for any RG -module M , $\text{proj. dim}_{RG}(M) < \infty$ iff $\text{proj. dim}_{RH}(M) < \infty$ for all finite subgroups H of G .

Notation 1.13. *Throughout this paper, \mathcal{F} will denote the class of all finite groups.*

Taking $\mathcal{X} = \mathcal{F}$, the class of all finite groups, in the above definition, we get many groups in $H\mathcal{F}$ that do not admit complete resolutions. For example, the free abelian group of rank t with $\aleph_0 \leq t < \aleph_\omega$ where ω is the first infinite ordinal, does not admit complete resolutions but is in $H_2\mathcal{F} \setminus H_1\mathcal{F}$ (see Theorem 7.10 of [11]). In fact, if t here is at least 1 but strictly smaller than \aleph_0 , then the free abelian group of rank t is in $H_1\mathcal{F} \setminus H_0\mathcal{F}$. The class $H_1\mathcal{F}$ is an interesting class here because all of its groups admit complete resolutions. This was proved in [7], but as we noted in [6], we can have a much shorter proof using the language of Gorenstein projectives:

Theorem 1.14. *(different proof in [7], also noted in Proposition 2.4 of [6]) If $G \in H_1\mathcal{F}$, then G admits complete resolutions over any commutative ring. And, for any commutative ring R of finite global dimension, all RG -modules admit complete resolutions.*

Proof. From the definition of $H_1\mathcal{F}$, it follows that there exists a finite dimensional contractible CW-complex X , say of dimension n , on which G acts by permuting its cells with finite cell stabilisers. We look at the augmented cellular complex of this action: $0 \rightarrow A_n \rightarrow A_{n-1} \rightarrow \dots \rightarrow A_1 \rightarrow A_0 \rightarrow \mathbb{Z} \rightarrow 0$ where each A_i is a direct sum of permutation $\mathbb{Z}G$ -modules with finite stabilisers. Permutation modules with finite stabilisers are Gorenstein projective (see Lemma 1.5) and as Gorenstein projectives are closed under direct sums (see Lemma 1.4), we conclude that each A_i is Gorenstein projective. Therefore, $Gpd_{\mathbb{Z}G}(\mathbb{Z}) \leq n$, and it follows that G admits complete resolutions over the ring of integers and by Proposition 1.7, it follows that G admits complete resolutions over any ring. The second part follows from Theorem 1.11 over rings of finite global dimension. \square

If G is of type Φ over a ring R of finite global dimension, then it also admits complete resolutions. This was shown in [20], but we can provide a very short proof using Benson’s cofibrants and a result of Cornick and Kropholler. To show this short proof, we need to make use of an RG -module which we usually denote by $B(G, R)$, which is also crucial in defining the class of modules that we have been calling “Benson’s cofibrants”. We define these modules below and state a couple of important and useful results involving $B(G, R)$.

Definition 1.15. *For any group G and any ring R , the module $B(G, R)$ is defined to be the RG -module of all $G \rightarrow R$ functions that are only allowed to take finitely many values. The module structure is given by $g.f(h) = f(g^{-1}h)$ for all $g, h \in G$ and $f \in B(G, R)$. When $R = \mathbb{Z}$, $B(G, \mathbb{Z})$ is the module of all bounded functions from G to \mathbb{Z} .*

We define an RG -module M to be Benson’s cofibrant if $M \otimes_R B(G, R)$ is projective as an RG -module. For $R = \mathbb{Z}$, this definition was made by Benson in [3]. As mentioned earlier, throughout this article, we will drop the subscript “ R ” while writing this tensor product.

For the rest of this article, whenever we say “cofibrant”, we mean “Benson’s cofibrant” as in Definition 1.15.

Lemma 1.16. *(Lemma 3.4 of [3]) For any group G and any ring R , $B(G, R)$ is R -free and RH -free for every finite subgroup H of G . Also, $B(G, R) = B(G, \mathbb{Z}) \otimes_{\mathbb{Z}} R$.*

Using the word “cofibrant” without a model theoretic context might seem strange, so the following is a natural question to ask.

Question 1.17. *What is the motivation behind using the word “cofibrant” in Definition 1.15?*

Answer 1.18. In [4], Benson put a closed model category structure on the module category of RG -modules, for any group G and any commutative Noetherian ring R , and defined cofibrations in the closed model category with the class of modules that we are calling “Benson’s cofibrants” as the cofibrant objects. He additionally showed, in Theorem 10.10 of [4], that the homotopy category of the closed model category, denoted $\text{Ho. Mod}(RG)$, is equivalent to the “stable module category” where the objects are the RG -modules and the arrows between modules M and N are given by $\widehat{\text{Ext}}_{RG}^0(M, N)$ (See Remark 1.19 below for more on $\widehat{\text{Ext}}_{RG}^*(M, N)$.)

For a group of type Φ over R , now making the assumption that R has finite global dimension in addition to being Noetherian, stable module categories for RG -modules were studied by Mazza and Symonds in [20], and their definition (Definition 3.2 of [20]) of the stable module category exactly coincided with Benson’s definition above (to note that the definitions for Hom-sets coincide, compare Definition 3.2 of [20] with the definition of $\widehat{\text{Ext}}_{RG}^0$ provided below in Remark 1.19). The hypotheses that R has finite global dimension and that G is of type Φ over R become important in [20] in a key result of the paper (Theorem 3.10 of [20]) where it is shown that the stable module category is equivalent, as a triangulated category, to several other known and well-behaved triangulated categories.

Because of the similarity in definition of the stable module category highlighted above, we can conclude that if one repeated the treatment in [4] of putting a closed model category structure on the module category of RG -modules, with G a group of type Φ over R where R is commutative Noetherian and of finite global dimension, we will get the same conclusion regarding the homotopy category being equivalent to the Mazza-Symonds stable module category and the class of “Benson’s cofibrants” being the cofibrant objects.

Remark 1.19. Following the treatment in Section 3 of [3], we denote by $\underline{\text{Hom}}_{RG}(M, N)$ the quotient of $\text{Hom}_{RG}(M, N)$ by the additive subgroup consisting of homomorphisms $M \rightarrow N$ that factor through an RG -projective. As noted in Section 3 of [3], there is a natural homomorphism $\underline{\text{Hom}}_{RG}(M, N) \rightarrow \underline{\text{Hom}}_{RG}(\Omega(M), \Omega(N))$. Now, for any integer r , $\widehat{\text{Ext}}_{RG}^r(M, N)$ is defined as $\varinjlim_i \underline{\text{Hom}}_{RG}(\Omega^{i+r}(M), \Omega^i(N))$.

Note that, $\widehat{\text{Ext}}_{RG}^r(M, N)$ can also be defined using satellite functors as shown in [21]. It is shown in Section 4 of [21] that there is a natural isomorphism between the complete cohomology functors defined using satellite functors (as done in [21]) and those defined as per the definition above.

Coming back to the question of complete resolutions, we show with the aid of the following result why groups of type Φ admit complete resolutions.

Theorem 1.20. (Theorem 3.5 of [7]) *For any group G and any commutative ring R , if M is an RG -module such that $M \otimes B(G, R)$ has finite projective dimension as an RG -module, then M admits complete resolutions.*

Corollary 1.21. *If G is a group of type Φ over R , then the trivial module admits complete resolutions. If, in addition, R is of finite global dimension, then all RG -modules admit complete resolutions.*

Proof. From Lemma 1.16 and the definition of type Φ , it follows that $B(G, R)$ is of finite projective dimension and then by Theorem 1.20, it follows that the trivial module admits complete resolutions. If R has finite global dimension, then from Theorem 1.11, it follows that all RG -modules admit complete resolutions. \square

We do not know of any groups that admit complete resolutions over a ring of finite global dimension but are not of type Φ over that ring or are not in $H_1\mathcal{F}$. The following is conjectured regarding the classes $H_1\mathcal{F}$ and groups of type Φ .

Conjecture 1.22. *(Conjecture A in [24], Conjecture 43.1 in [9]) For any group G , the following are equivalent.*

- a) G is of type Φ over \mathbb{Z} .
- b) G admits a finite dimensional model for its classifying space of proper actions.
- c) $G \in H_1\mathcal{F}$.
- d) G admits complete resolutions over \mathbb{Z} .

(c) \Rightarrow (a) is very easy to check and it has been proved in [20]. We checked (a) \Rightarrow (d) and (c) \Rightarrow (d) although it must be noted that they were already known in the literature.

We end this section with some more examples of groups in these classes. We have already mentioned some free abelian examples of $H_1\mathcal{F}$ -groups. Some other examples of groups that are type Φ (over the ring of integers) are groups of finite virtual cohomological dimension like discrete subgroups of Lie groups with finitely many components for example, groups acting on trees with finite stabilisers like \mathbb{Q}/\mathbb{Z} . See Section 3 in [23] for a reference.

2. BOTH CLASSES IN CONJECTURE 1.1 ARE GOOD CLASSES

For the rest of this article for a group G and a commutative ring R , we shall denote by $\text{GProj}(RG)$ the class of Gorenstein projective RG -modules and by $\text{CoF}(RG)$ the class of RG -modules M such that $M \otimes B(G, R)$ is projective. Our first result here is that $\text{GProj}(RG)$, for all groups G and all rings R , is a “good” class. We recall the definition of a good class first.

Definition 2.1. *(see Definition 3.5 of [5]) Let \mathcal{T} be a class of R -modules. An R -module M is generated in zero steps from \mathcal{T} iff it is in \mathcal{T} and in n steps iff there is an exact sequence $0 \rightarrow M_2 \rightarrow M_1 \rightarrow M \rightarrow 0$, where M_i is generated from \mathcal{T} in a_i steps, and $a_1 + a_2 \leq n - 1$. The class of all R -modules generated in finitely many steps from \mathcal{T} is denoted $\langle \mathcal{T} \rangle$.*

The \mathcal{T} -dimension of an R -module M , denoted $\mathcal{T}\text{-dim}(M)$, is defined to be the smallest integer n such that there exists an exact sequence $0 \rightarrow T_n \rightarrow \dots \rightarrow T_1 \rightarrow T_0 \rightarrow M \rightarrow 0$, where each $T_i \in \mathcal{T}$, for some n . The class of all R -modules with finite \mathcal{T} -dimension is denoted $[\mathcal{T}]$.

A class of R -modules is called good iff $[\mathcal{T}] = \langle \mathcal{T} \rangle$.

We proved the following characterisation result for good classes in [5].

Proposition 2.2. (part of Proposition 7.2 of [5]) *Let \mathcal{T} be a class of R -modules. Then, the following two statements are equivalent.*

- a) *For any short exact sequence of R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, if $A, B \in [\mathcal{T}]$, then $C \in [\mathcal{T}]$.*
- b) *For any exact sequence of R -modules $0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C \rightarrow 0$, for any $n > 1$, if each $C_i \in [\mathcal{T}]$, then $C \in [\mathcal{T}]$. Note that this is equivalent to saying $[[\mathcal{T}]] = [\mathcal{T}]$.*
- c) *\mathcal{T} is a good class.*

Proof. (a) \Rightarrow (b): We shall proceed by induction on n . If $n = 2$, (b) holds true, by (a). Let the statement of (b) hold true for $n = k$, this is our induction hypothesis. Now let $0 \rightarrow C_{k+1} \rightarrow C_k \rightarrow \dots \rightarrow C_1 \rightarrow C \rightarrow 0$ be an exact sequence where each C_i is in $[\mathcal{T}]$. We split this into two exact sequences.

$$(S0) \quad 0 \rightarrow C_{k+1} \rightarrow C_k \rightarrow \text{Im}(C_k \rightarrow C_{k-1}) \rightarrow 0.$$

$$(S1) \quad 0 \rightarrow \text{Im}(C_k \rightarrow C_{k-1}) \hookrightarrow C_{k-1} \rightarrow \dots \rightarrow C_1 \rightarrow C \rightarrow 0.$$

From (a), it follows that in (S0), $\text{Im}(C_k \rightarrow C_{k-1}) \in [\mathcal{T}]$. Therefore, in (S1), every module other than C is in $[\mathcal{T}]$. So, by our induction hypothesis, $C \in [\mathcal{T}]$.

(b) \Rightarrow (c): First note that we always have $[\mathcal{T}] \subseteq \langle \mathcal{T} \rangle$, i.e. for any R -module M , if the \mathcal{T} -dim(M) $< \infty$, then $M \in \langle \mathcal{T} \rangle$. This follows from an easy induction - if \mathcal{T} -dim(M) = 0, then $M \in \mathcal{T} \subseteq \langle \mathcal{T} \rangle$. Assume as an induction hypothesis that for all M satisfying \mathcal{T} -dim(M) $\leq n$, $M \in \langle \mathcal{T} \rangle$. Now, let \mathcal{T} -dim(M) = $n + 1$; then there exists an exact sequence $0 \rightarrow T_{n+1} \rightarrow T_n \rightarrow \dots \rightarrow T_0 \rightarrow M \rightarrow 0$ where each $T_i \in \mathcal{T}$. It follows that \mathcal{T} -dim(Ker($T_0 \rightarrow M$)) $\leq n$ and therefore Ker($T_0 \rightarrow M$) $\in \langle \mathcal{T} \rangle$ by our induction hypothesis, and from the exact sequence $0 \rightarrow \text{Ker}(T_0 \rightarrow M) \rightarrow T_0 \rightarrow M \rightarrow 0$, since all the modules except M are in $\langle \mathcal{T} \rangle$, we get from Definition 2.1 that $M \in \langle \mathcal{T} \rangle$.

Now, we need to show $\langle \mathcal{T} \rangle \subseteq [\mathcal{T}]$ assuming (b) to be true. For any $M \in \langle \mathcal{T} \rangle$, denote by $\alpha_{\mathcal{T}}(M)$ the least number of steps required to generate M from \mathcal{T} . We make the induction hypothesis that for all M such that $\alpha_{\mathcal{T}}(M) \leq n$, $M \in [\mathcal{T}]$. The base case is obvious as if $\alpha_{\mathcal{T}}(M) = 0$, then $M \in \mathcal{T} \subseteq [\mathcal{T}]$. Now assume $\alpha_{\mathcal{T}}(M) = n + 1$, then there is an exact sequence $0 \rightarrow M_2 \rightarrow M_1 \rightarrow M \rightarrow 0$ where $\alpha_{\mathcal{T}}(M_i) \leq n$, for $i = 1, 2$. By the induction hypothesis, $M_1, M_2 \in [\mathcal{T}]$ and by (b), we have $M \in [\mathcal{T}]$.

(c) \Rightarrow (a): Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence where $A, B \in [\mathcal{T}]$. From (c), it follows that $A, B \in \langle \mathcal{T} \rangle$, and therefore $C \in \langle \mathcal{T} \rangle = [\mathcal{T}]$ by Definition 2.1.

□

Lemma 2.3. *Let G be a group and let R be a commutative ring. Then, $\text{GProj}(RG)$ is a good class.*

Proof. We saw in Lemma 1.9 that if we have a short exact sequence of RG -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ where $A, B \in [\text{GProj}(RG)]$, i.e. $Gpd_{RG}(A), Gpd_{RG}(B) < \infty$, then $Gpd_{RG}(C) < \infty$. Now it follows from Proposition 2.2 that $\text{GProj}(RG)$ is a good class. \square

Before we show that $\text{CoF}(RG)$ is a good class for all G and R , note that since $B(G, R)$ is R -free, if $P_* \rightarrow M$ is an RG -projective resolution of an RG -module M , then $P_* \otimes B(G, R) \rightarrow M \otimes B(G, R)$ is an RG -projective resolution of $M \otimes B(G, R)$.

Lemma 2.4. *For any group G and any commutative ring R , $[\text{CoF}(RG)]$ and the class of all RG -modules M satisfying $\text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty$ coincide.*

Proof. Let $M \in [\text{CoF}(RG)]$. Then, there exists an exact sequence

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$$

where each C_i is a cofibrant module, i.e. $C_i \otimes B(G, R)$ is projective for all i . Since $B(G, R)$ is R -free, we can tensor the exact sequence by $B(G, R)$ to get

$$0 \rightarrow C_n \otimes B(G, R) \rightarrow \dots \rightarrow C_1 \otimes B(G, R) \rightarrow C_0 \otimes B(G, R) \rightarrow M \otimes B(G, R) \rightarrow 0$$

Now, as each term in the exact sequence, other than $M \otimes B(G, R)$, is projective, we can say that $\text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty$. Thus, $[\text{CoF}(RG)]$ is a subclass of the class of all RG -modules M satisfying $\text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty$.

Now let M be an RG -module satisfying $\text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty$. Let $(P_i, d_i)_{i \geq 0}$ be a projective resolution of M , and, for any positive integer r , let $K_r(M)$ denote the r -th kernel in this resolution. Let $\text{proj. dim}_{RG}(M \otimes B(G, R)) = t$. Then, $K_t(M) \otimes B(G, R)$ is projective. By definition of $K_t(M)$, we have the following exact sequence

$$0 \rightarrow K_t(M) \rightarrow P_{t-1} \rightarrow P_{t-2} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

$P_i \otimes B(G, R)$ is projective for all i (as $B(G, R)$ is R -free), so each P_i is cofibrant. And, $K_t(M) \otimes B(G, R)$ is projective as well, so $K_t(M)$ is cofibrant. Thus, M is in $[\text{CoF}(RG)]$. Therefore, the class of all RG -modules M satisfying $\text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty$ is a subclass of $[\text{CoF}(RG)]$. \square

Lemma 2.5. *For any group G and any commutative ring R , $\text{CoF}(RG)$ is a good class. In particular, we claim $[[\text{CoF}(RG)]] = [\text{CoF}(RG)]$.*

Proof. It is obvious that $[\text{CoF}(RG)]$ is a subclass of $[[\text{CoF}(RG)]]$.

Now let M be a module in $[[\text{CoF}(RG)]]$. There exists an exact sequence

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$$

where each C_i is in $[\text{CoF}(RG)]$. We can tensor the above exact sequence by $B(G, R)$, which is R -free, to get the following exact sequence

$$0 \rightarrow C_n \otimes B(G, R) \rightarrow C_{n-1} \otimes B(G, R) \rightarrow \dots \rightarrow C_0 \otimes B(G, R) \rightarrow M \otimes B(G, R) \rightarrow 0.$$

By Lemma 2.4, $\text{proj. dim}_{RG}(C_i \otimes B(G, R)) < \infty$ for $i = 0, 1, \dots, n$. Therefore, $\text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty$, and by Lemma 2.4 again, M is in $[\text{CoF}(RG)]$. Thus, the classes $[[\text{CoF}(RG)]]$ and $[\text{CoF}(RG)]$ coincide. Therefore, $\text{CoF}(RG)$ is a good class by Proposition 2.2. \square

3. RELATIONS BETWEEN THE CLASSES GENERATED BY GORENSTEIN PROJECTIVES AND THE COFIBRANTS

We start this section with a technical result.

Proposition 3.1. *Let G be a group and R be a commutative ring. Let \mathcal{T} be a class of RG -modules satisfying the following conditions:*

- a) \mathcal{T} is closed under tensoring with R -free modules.
- b) For any RG -module M , there exists a surjective map of RG -modules $\phi : T_M \rightarrow M$ for some $T_M \in \mathcal{T}$.
- c) For any RG -module M , if $\mathcal{T}\text{-dim}(M) \leq n$, then in any exact sequence $0 \rightarrow K_n \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow \dots \rightarrow T_1 \rightarrow T_0 \rightarrow M \rightarrow 0$ where each $T_i \in \mathcal{T}$, $K_n \in \mathcal{T}$.
- d) For any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, where $B \in \mathcal{T}$ and $\mathcal{T}\text{-dim}(A) = k > 0$, $\mathcal{T}\text{-dim}(C) = k + 1$.

For any R -free RG -module F , let $\mathcal{X}_{F, \mathcal{T}} := \{M \in \text{Mod}(RG) : M \otimes F \in \mathcal{T}\}$, and let \mathcal{S} denote the class of all RG -modules that occur as kernels of $(-\infty, \infty)$ -exact sequences of modules in \mathcal{T} (this means the exact sequences are extending to infinity in both directions). Then, $\mathcal{S} \cap [\mathcal{X}_{F, \mathcal{T}}] = \mathcal{S} \cap \mathcal{X}_{F, \mathcal{T}}$ if $F\mathcal{T}D(RG) < \infty$. Here, $F\mathcal{T}D(RG) := \sup\{\mathcal{T}\text{-dim}(M) : M \in \text{Mod}(RG) \text{ satisfying } \mathcal{T}\text{-dim}(M) < \infty\}$.

Proof. All tensor products are being taken over R .

First we prove that $[\mathcal{X}_{F, \mathcal{T}}] = \{M \in \text{Mod}(RG) : \mathcal{T}\text{-dim}(M \otimes F) < \infty\}$. To show this, we start with an RG -module M satisfying $\mathcal{T}\text{-dim}(M \otimes F) = r < \infty$. We know from condition (b) of our hypothesis, that there exists a surjective RG -linear map $\phi_0 : T_0 \rightarrow M$ for some $T_0 \in \mathcal{T}$. Similarly, there exists an RG -surjective map $\phi_1 : T_1 \rightarrow \text{Ker}(\phi_0)$. Going on like this, we get an exact sequence $0 \rightarrow \text{Ker}(\phi_{r-1}) \hookrightarrow T_{r-1} \rightarrow T_{r-2} \rightarrow \dots \rightarrow T_1 \rightarrow T_0 \rightarrow M \rightarrow 0$, where each $T_i \in \mathcal{T}$. When we tensor this exact sequence by F which is R -free, we get an exact sequence $0 \rightarrow \text{Ker}(\phi_{r-1}) \otimes F \rightarrow T_{r-1} \otimes F \rightarrow \dots \rightarrow T_0 \otimes F \rightarrow M \otimes F \rightarrow 0$, where each $T_i \otimes F \in \mathcal{T}$ as \mathcal{T} is closed under tensoring with R -free modules by condition (a) of our hypothesis. Now, as $\mathcal{T}\text{-dim}(M \otimes F) = r$, by condition (c) of our hypothesis, $\text{Ker}(\phi_{r-1}) \otimes F \in \mathcal{T}$, i.e. $\text{Ker}(\phi_{r-1}) \in \mathcal{X}_{F, \mathcal{T}}$. Thus, $M \in [\mathcal{X}_{F, \mathcal{T}}]$. Now if we take $M \in [\mathcal{X}_{F, \mathcal{T}}]$, then we have an exact sequence $0 \rightarrow X_n \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ for some n and some X_0, \dots, X_n where each $X_i \in \mathcal{X}_{F, \mathcal{T}}$. Tensoring this sequence by the R -free F , we get a finite length resolution of $M \otimes F$ by modules in \mathcal{T} as each $X_i \otimes F \in \mathcal{T}$. Thus, $\mathcal{T}\text{-dim}(M \otimes F) < \infty$.

We are now in a position to complete our proof of the proposition. Note that since $\mathcal{X}_{F, \mathcal{T}}$ is a subclass of $[\mathcal{X}_{F, \mathcal{T}}]$, we have $\mathcal{S} \cap \mathcal{X}_{F, \mathcal{T}}$ to be a subclass of $\mathcal{S} \cap [\mathcal{X}_{F, \mathcal{T}}]$. To prove the

other direction, we start with an RG -module $M_0 \in \mathcal{S} \cap [\mathcal{X}_{F, \mathcal{T}}]$. From our first paragraph, it follows that $\mathcal{T}\text{-dim}(M_0 \otimes F) < \infty$. We need to show $M_0 \otimes F \in \mathcal{T}$, so we start with the assumption that it is not the case and that $\mathcal{T}\text{-dim}(M_0 \otimes F) = r > 0$. By definition of \mathcal{S} , there exists an exact sequence $\dots \rightarrow T_1 \rightarrow T_0 \rightarrow T_{-1} \rightarrow T_{-2} \rightarrow \dots$ where each $T_i \in \mathcal{T}$ and $M_0 = \text{Ker}(T_0 \rightarrow T_{-1})$. For $i \neq 0$, let $M_i := \text{Ker}(T_i \rightarrow T_{i-1})$. We can tensor the short exact sequence $0 \rightarrow M_0 \hookrightarrow T_0 \twoheadrightarrow M_{-1} \rightarrow 0$ by F , which is R -free, to get the short exact sequence $0 \rightarrow M_0 \otimes F \hookrightarrow T_0 \otimes F \twoheadrightarrow M_{-1} \otimes F \rightarrow 0$ where $T_0 \otimes F \in \mathcal{T}$ since \mathcal{T} is closed under tensoring with R -free modules. By (d), we can say that $\mathcal{T}\text{-dim}(M_{-1} \otimes F) = r + 1$. Similarly, we get that $\mathcal{T}\text{-dim}(M_{-F\mathcal{T}D(RG)} \otimes F) = r + F\mathcal{T}D(RG) > F\mathcal{T}D(RG)$ which is not possible. So, $M_0 \otimes F \in \mathcal{T}$, i.e. $M_0 \in \mathcal{X}_{F, \mathcal{T}}$. Thus, $\mathcal{S} \cap [\mathcal{X}_{F, \mathcal{T}}]$ is a subclass of $\mathcal{S} \cap \mathcal{X}_{F, \mathcal{T}}$, and we are done. \square

The proposition above is a general result. We can use it to show that when $\text{fin.dim}(RG) < \infty$, the class generated by Benson's cofibrants coincides with the class generated by Gorenstein projectives if and only if the class of Benson's cofibrants coincides with the class of Gorenstein projectives. To see why that is true, we state this following result, which we have already mentioned in the first section (see Theorem 1.20), from the literature using the language of generation.

Theorem 3.2. *For any group G and any commutative ring R ,*

$$[\text{CoF}(RG)] = \langle \text{CoF}(RG) \rangle \subseteq \langle \text{GProj}(RG) \rangle = [\text{GProj}(RG)].$$

Remark 3.3. Note that in [7] although the statement of the result was exactly what we have stated above, the proof showed that cofibrants admit complete resolutions of index 0, i.e. they are Gorenstein projective. So we know $\text{CoF}(RG) \subseteq \text{GProj}(RG)$ for all G and R and the challenge resides in proving the other direction.

We can actually determine whether the class of modules generated by $\text{CoF}(RG)$ contains every module just by looking at the projective dimension of $B(G, R)$ as long as the ring R is of finite global dimension. It follows from Theorem 3.2 that if the class of modules generated by the cofibrants includes everything, then the same is true of the class of modules generated by the Gorenstein projectives. The reverse can be shown to be true if we assume a related conjecture to be true. We first state this conjecture below.

Conjecture 3.4. *For any group G and any commutative ring R of finite global dimension, $\text{Gcd}_R(G) = \text{proj. dim}_{RG}(B(G, R))$.*

In Remark 3.7 below, we deal with some progress that has been achieved on Conjecture 3.4.

Proposition 3.5. *For any group G and any commutative ring R which has finite global dimension, the following implications hold with the following statements: (a) \iff (b) \implies (c), and Conjecture 3.4 \implies ((c) \implies (b)).*

a) $\langle \text{CoF}(RG) \rangle$ contains all RG -modules.

- b) $\text{proj. dim}_{RG} B(G, R) < \infty$.
 c) $\langle \text{GProj}(RG) \rangle$ contains all RG -modules.

Proof. (a) \implies (b): If $\langle \text{CoF}(RG) \rangle$ contains all RG -modules, then it contains the trivial module, and since $\text{CoF}(RG)$ is a good class, we have $\langle \text{CoF}(RG) \rangle = [\text{CoF}(RG)] = \{M \in \text{Mod-}RG : \text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty\}$ (by Lemma 2.4 and Lemma 2.5), and so $R \otimes B(G, R) = B(G, R)$ must have finite projective dimension.

(b) \implies (a) : Let t be the global dimension of R . Let M be an RG -module. Let $(P_i, d_i)_{i \geq 0}$ be a projective resolution admitted by M (denote by $K_r(M)$ the r -th kernel in this projective resolution, for any positive integer r). As $B(G, R)$ is R -free, $(P_i \otimes B(G, R), d_i \otimes id)_{i \geq 0}$ is a projective resolution admitted by $M \otimes B(G, R)$ and $K_r(M) \otimes B(G, R)$ the r -th kernel in this projective resolution, for any positive integer r . $K_t(M)$ is R -projective for any RG -module M as t is the global dimension of R . As $\text{proj. dim}_{RG}(B(G, R)) < \infty$, we have $\text{proj. dim}_{RG}(K_t(M) \otimes B(G, R)) < \infty$ and since $(K_t(M) \otimes B(G, R))$ is the t -th kernel in the projective resolution $P_* \otimes B(G, R) \rightarrow M \otimes B(G, R)$, we have $\text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty$. So, M is in $[\text{CoF}(RG)]$, by Lemma 2.4. Thus, $[\text{CoF}(RG)]$ contains all RG -modules, and, as $\text{CoF}(RG)$ is a good class (by Lemma 2.5), $\langle \text{CoF}(RG) \rangle$ contains all RG -modules.

(a) \implies (c): obvious from Theorem 3.2.

Conjecture 3.4 \implies ((c) \implies (b)): note that if $\langle \text{GProj}(RG) \rangle$ is the class of all RG -modules, then $Gpd_{RG}(R) = Gcd_R(G) < \infty$, and therefore by Conjecture 3.4, $\text{proj. dim}_{RG}(B(G, R)) < \infty$. □

Remark 3.6. In the proof of Proposition 3.5, only the part (b) \implies (a) makes use of the fact that R is of finite global dimension.

Remark 3.7. In [6], we proved the following result regarding Conjecture 3.4: for any group G and any commutative ring R of finite global dimension, $\text{proj. dim}_{RG}(B(G, R)) = Gcd_R(G)$ if $\text{proj. dim}_{RG}(B(G, R))$ is finite. To do this, we need to invoke an invariant called the generalized cohomological dimension of a group that was first defined over the integers in [16], denoted $\underline{cd}_R(G)$. $\underline{cd}_R(G)$ is defined to be the supremum over all integers n such that there is an R -free RG -module M and an RG -free module F such that $\text{Ext}_{RG}^n(M, F) \neq 0$.

First note that, when $\text{proj. dim}_{RG}(B(G, R)) = n < \infty$, $\Omega^n(R)$ is Gorenstein projective by Remark 3.3 and thus $Gcd_R(G) \leq n$. We also have, under the same hypothesis, $\text{proj. dim}_{RG}(B(G, R)) \leq \underline{cd}_R(G)$ (see Lemma 3.8 below). And finally it follows from the proof of Theorem 2.5 of [2] that if $Gcd_R(G) < \infty$, then $Gcd_R(G) = \underline{cd}_R(G)$. It is noteworthy that here we do not need R to be Noetherian because as we see in the proof of Theorem 2.5 of [2], showing $Gcd_R(G) = \underline{cd}_R(G)$ when it is already known that $Gcd_R(G)$ is finite does not require using $\text{silp}(RG) = \text{spli}(RG)$ which is the only place where R is required to be Noetherian. Recall that for any ring A , $\text{silp}(A)$ and

$\text{spli}(A)$ denote, respectively, the supremum over the injective dimensions of A -projectives and the supremum over the projective dimensions of A -injectives.

Lemma 3.8. (Lemma 1.19 of [6]) *Let G be a group and let R be a commutative ring. If $\text{proj. dim}_{RG}(B(G, R))$ is finite, then $\underline{\text{cd}}_R(G) \geq \text{proj. dim}_{RG}(B(G, R))$.*

Proof. We can assume that $\underline{\text{cd}}_R(G)$ is finite. Recall that for any commutative ring R and any group G , $\underline{\text{cd}}_R(G) := \sup\{i \in \mathbb{Z} : \text{Ext}_{RG}^i(M, F) \neq 0 \text{ for some } R\text{-free } M \text{ and some } RG\text{-free } F\}$. Let us start with the assumption that $\text{proj. dim}_{RG}(B(G, R)) = k > \underline{\text{cd}}_R(G)$. There must exist some RG -module M such that $\text{Ext}_{RG}^k(B(G, R), M) \neq 0$ because otherwise $\text{proj. dim}_{RG}(B(G, R)) \leq k - 1$. Now take F_M , the RG -free module on M , and form the short exact sequence $0 \rightarrow \Omega(M) \rightarrow F_M \rightarrow M \rightarrow 0$. We now look at the associated long exact Ext-sequence associated to this short exact sequence and get $\dots \rightarrow \text{Ext}_{RG}^k(B(G, R), \Omega(M)) \rightarrow \text{Ext}_{RG}^k(B(G, R), F_M) \rightarrow \text{Ext}_{RG}^k(B(G, R), M) \rightarrow \text{Ext}_{RG}^{k+1}(B(G, R), \Omega(M)) \rightarrow \dots$. Here, $\text{Ext}_{RG}^k(B(G, R), F_M) = 0$ because $k > \underline{\text{cd}}_R(G)$ and $B(G, R)$ is R -free and F_M is RG -free. Also, $\text{Ext}_{RG}^{k+1}(B(G, R), \Omega(M)) = 0$ because $\text{proj. dim}_{RG}(B(G, R)) = k$. So, $\text{Ext}_{RG}^k(B(G, R), M) = 0$ which gives us a contradiction. \square

The following now follows directly from Proposition 3.5 and the more technical Proposition 3.1.

Theorem 3.9. *Let G be a group and let R be a commutative ring of finite global dimension. If $\text{proj. dim}_{RG}(B(G, R)) < \infty$, then $\text{CoF}(RG) = \text{GProj}(RG)$.*

Proof. From Proposition 3.5, it follows that $\langle \text{CoF}(RG) \rangle$ and $\langle \text{GProj}(RG) \rangle$ coincide with the class of all modules. Therefore, $\text{Gcd}_R(G) < \infty$. Now, we know from Theorem C of [8] that $\text{fin. dim}(RG) \leq \text{silp}(RG)$, where $\text{silp}(RG)$, following the notation mentioned in Remark 3.7, is the supremum over the injective dimensions of all projective RG -modules (note that although the statement of Theorem C in [8] has the hypothesis that G is an $H\mathcal{F}$ -group, for the proof of this inequality that assumption was not used). Also, from Corollary 1.6 of [14], we know that $\text{silp}(RG) \leq \text{Gcd}_R(G) + \text{global dimension of } R$. (We are dealing with weak Gorenstein projective modules here which are formally defined at the beginning of Section 4, see Remark 3.10 below for a slightly general remark) Thus, we have in this case that $\text{fin. dim}(RG) < \infty$, and therefore by Proposition 3.1, $\text{CoF}(RG) = \text{GProj}(RG)$. \square

Remark 3.10. It can be noted that, in the setup used by Theorem 3.9, if $\text{silp}(RG)$ is finite, then all weak Gorenstein projective RG -modules, i.e. modules occurring as kernels in infinite acyclic complexes of projectives satisfying condition (a) of Definition 1.2 but not necessarily condition (b) of Definition 1.2, are necessarily Gorenstein projective.

Although Theorem 3.9 implies that groups of type Φ over any ring of finite global dimension satisfy Conjecture 1.1 - this is because it follows from Lemma 1.16 and the definition of type Φ groups that if G is of type Φ over R then $B(G, R)$ is of finite projective dimension - we provide a simpler proof below.

Theorem 3.11. *Let G be a group of type Φ over a commutative ring R of finite global dimension. Then, $\text{CoF}(RG) = \text{GProj}(RG)$.*

Proof. We only need to show $\text{GProj}(RG) \subseteq \text{CoF}(RG)$ in light of Remark 3.3. Now, if $M \in \text{GProj}(RG)$, then $M \otimes B(G, R)$, since $B(G, R)$ is R -free, occurs as a kernel in a doubly infinite acyclic complex of projectives. Over type Φ groups, doubly infinite acyclic complexes of projectives are totally acyclic (note that, for any ring A , an acyclic complex of projective A -modules, C_* , is called *totally acyclic* iff $\text{Hom}_A(C_*, P)$ is acyclic for any A -projective P) - this result is proved in Lemma 3.14 of [20], see Remark 3.12 below. So, $M \otimes B(G, R)$ is Gorenstein projective. Now, M is R -projective by Lemma 1.3 and over any finite subgroup H of G , $B(G, R)$ is RH -free by Lemma 1.16, so $M \otimes B(G, R)$ is RH -projective. As G is of type Φ , this means $M \otimes B(G, R)$ has finite projective dimension as an RG -module. Now, since $M \otimes B(G, R)$ is Gorenstein projective, it follows from Theorem 1.4 that $M \otimes B(G, R)$ is projective. Therefore, $M \in \text{CoF}(RG)$. \square

Remark 3.12. In Lemma 3.14 of [20], the base ring R is Noetherian but the Noetherian assumption is only used (this is stated explicitly in Remark 3.15 of [20]) to get $\text{silp}(RG) < \infty$ where R is a commutative Noetherian ring of finite global dimension and G is of type Φ over R . Actually, we do not need the Noetherian assumption here because, for just commutative R of finite global dimension, we already have $B(G, R)$ to be of finite projective dimension as an RG -module and from Remark 3.7, we get $\text{Gcd}_R(G) < \infty$ and, then using Corollary 1.6 of [14], that $\text{silp}(RG) < \infty$.

Remark 3.13. As we can see, it is actually quite easy to prove that groups of type Φ satisfy Conjecture 1.1. We have provided two different proofs already and in Section 5, we get a slightly stronger result as a corollary from a technical result for type Φ groups again. Since for any ring R of finite global dimension, $B(G, R)$ being of finite projective dimension as an RG -module implies that G must admit complete resolutions over R , and since Conjecture 1.22 when stated over R claims that a group admitting complete resolutions and being of type Φ are equivalent conditions, it is plausible that $B(G, R)$ being of finite projective dimension and G being of type Φ are equivalent conditions as well, but if it is not, then Theorem 3.9 proves Conjecture 1.1 for a larger class of groups than the class of groups of type Φ .

We now look at some conjectured properties of Gorenstein projectivity and how those conjectures relate to Conjecture 1.1.

4. TWO OTHER QUESTIONS ON GORENSTEIN PROJECTIVITY

In this section, we introduce a class of modules called the *weak Gorenstein projectives*. Recall that Gorenstein projectives are defined as modules that arise as kernels in complete resolutions and if $(F_i, d_i)_{i \in \mathbb{Z}}$ is a complete resolution of projective RG -modules, we require by definition that $\text{Hom}_{RG}(F_*, P)$ be acyclic for all projective RG -modules P . We define F_* to be a *weak complete resolution* if it satisfies the same definition as complete resolutions except we do not put the

condition that $\text{Hom}_{RG}(F_*, P)$ is acyclic for all projective P . *Weak Gorenstein projectives* are defined as modules that occur as kernels in weak complete resolutions. We shall be denoting the class of all weak Gorenstein projective RG -modules by $\text{WGProj}(RG)$. Weak Gorenstein projectives share the following crucial property with Gorenstein projectives.

Lemma 4.1. *Let R be a ring of finite global dimension and let G be a group. Then, weak Gorenstein projectives over RG are R -projective. Also, $\text{WGProj}(RG)$ is closed under direct sums.*

Proof. The proof of the first part follows from Lemma 1.3. The proof for weak Gorenstein projectives being closed under direct sums is obvious as a direct sum of weak complete resolutions is still a weak complete resolution. \square

The following conjecture was made in [10] by Dembegioti and Talelli over the ring of integers, we make it here over rings of finite global dimension.

Conjecture 4.2. *For any group G and a commutative ring R of finite global dimension, an RG -module M admits a complete resolution iff it admits a weak complete resolution.*

We wish to focus on the following related conjecture regarding the class of weak Gorenstein projectives.

Conjecture 4.3. *For any group G and a commutative ring R of finite global dimension, the class of weak Gorenstein projective RG -modules coincides with the class of Gorenstein projective RG -modules.*

Before going further in this section, we wish to state another question on Gorenstein projectivity which will be important in our study of groups in the next section where we will deal with a situation where a slight variation of Conjecture 1.1 is satisfied at the level of finitely generated subgroups.

Conjecture 4.4. *For any group G and any commutative ring R of finite global dimension, if M is Gorenstein projective as an RG -module then it is also Gorenstein projective as an RH -module for any subgroup H of G , i.e. Gorenstein projectivity is closed upon restriction to subgroups.*

The following is an immediate observation.

Lemma 4.5. *If Conjecture 4.3 is satisfied for a subgroup-closed class of groups \mathcal{X} and a ring R of finite global dimension, then \mathcal{X} -groups satisfy Conjecture 4.4 over R .*

Proof. Take $G \in \mathcal{X}$. Let M be a Gorenstein projective RG -module that occurs as a kernel in the complete resolution $(F_i, d_i)_{i \in \mathbb{Z}}$ of RG -projective modules. Upon restriction to a subgroup H of G , M occurs as a kernel in the same weak complete resolution upon restriction to H (it is still a weak complete resolution because projectivity is closed under restriction to subgroups). This means,

upon restriction to H , M is weak Gorenstein projective, but since Conjecture 4.3 is satisfied for H and R (this is because \mathcal{X} is subgroup-closed), we have that M is Gorenstein projective upon restriction to H . \square

If it is known for some group G and some ring R that the class of weak Gorenstein projective RG -modules coincides with the class of Benson's cofibrants, then Conjecture 4.2, in fact a slightly stronger version of it, follows.

Lemma 4.6. *If $\text{WGProj}(RG) = \text{CoF}(RG)$, then, over RG , every weak complete resolution is a complete resolution.*

Proof. This follows from Corollary D of [10] where they have dealt with the exactly similar situation in different language over \mathbb{Z} . The same proof works for any ring R . \square

We now note the following straightforward result involving $B(G, R)$ that helps us to show why the statement of Conjecture 1.1 over a ring R implies Conjecture 4.4.

Lemma 4.7. *For any group G and any ring R , $B(H, R)$ is a summand of $\text{Res}_H^G B(G, R)$, where H is any subgroup of G .*

Proof. This result has been proved in [19] for $R = \mathbb{Z}$. It follows over any ring R because $B(G, R) = B(G, \mathbb{Z}) \otimes_{\mathbb{Z}} R$ for any group G . \square

Lemma 4.8. *Let G be a group and R a commutative ring. Then, if M is cofibrant as an RG -module, M is cofibrant as an RH -module for any subgroup $H \leq G$ as well.*

Proof. Let M be cofibrant as an RG -module. So, $M \otimes B(G, R)$ is projective as an RG -module. This implies that $\text{Res}_H^G(M \otimes B(G, R))$ is projective as an RH -module for any subgroup H of G . By Lemma 4.7, $B(H, R)$ is a direct summand of $\text{Res}_H^G B(G, R)$, and so $\text{Res}_H^G M \otimes B(H, R)$ is a summand of $\text{Res}_H^G(M \otimes B(G, R))$ which is projective. So, $\text{Res}_H^G M$ is cofibrant as an RH -module. \square

Corollary 4.9. *Let R be a commutative ring. If a group G satisfies Conjecture 1.1, then Conjecture 4.4 holds for G as well.*

Proof. When Conjecture 1.1 is satisfied for G , the class of Gorenstein projective RG -modules coincides with the class of RG -modules M such that $M \otimes B(G, R)$ is projective. Thus, by Lemma 4.8 and the fact that cofibrant RH -modules are Gorenstein projective for any subgroup $H \leq G$, our result follows. \square

5. GROUPS SATISFYING A VARIATION OF CONJECTURE 1.1 LOCALLY

We start this section by recalling the definition of $LH\mathcal{X}$ -groups, where \mathcal{X} is a class of groups.

Definition 5.1. We define a group G to be in $LH\mathcal{X}$ iff all finitely generated subgroups of G are in $H\mathcal{X}$. Similarly, for any class of groups \mathcal{X} , $L\mathcal{X}$ or the class of groups that are locally in \mathcal{X} , is defined to be the class of groups all of whose finitely generated subgroups are in \mathcal{X} .

It is important to note that it follows from Theorem 1.1 of [17] that if α is a countable ordinal, then $LH_\alpha\mathcal{F} \subset LH\mathcal{F}$. This implies that the class $LH\mathcal{F}$ contains groups that are not contained in $H_n\mathcal{F}$ for any integer n .

Before going forward, we need to prove a technical result which is a close variant of Theorem B of [10]. Before we state and prove this result, we need to state two lemmas that will be crucial in its proof and in the proofs of some other results in this section.

Lemma 5.2. (Lemma 5.6 of [3]) Let R be a commutative ring and let $G = \bigcup_{\alpha < \gamma} G_\alpha$ where $(G_\alpha)_{\alpha < \gamma}$ is an ascending chain of subgroups of G for some ordinal γ . Then for any RG -module M that is projective over each G_α , $\text{proj. dim}_{RG}(M) \leq 1$.

Lemma 5.3. (Lemma 2.5 of [17]) Every countable group admits an action on a tree with finitely generated vertex and edge stabilisers.

Proposition 5.4. (done over \mathbb{Z} and \mathcal{F} in Theorem B of [10], almost the same proof works here) Let G be a group and let R be a commutative ring of finite global dimension. Let \mathcal{X} be a class of groups and let $B_{G,\mathcal{X}}$ be an R -free RG -module that restricts to a projective module on every subgroup of G that is in \mathcal{X} . Now if M is a weak Gorenstein projective RG -module, then $M \otimes B_{G,\mathcal{X}}$ is projective over every subgroup of G that is in $LH\mathcal{X}$.

Proof. We first show the result is true over all $H\mathcal{X}$ -subgroups of G , i.e. over all $H_\alpha\mathcal{X}$ -subgroups of G where α is some ordinal. We shall proceed by transfinite induction on α . When $\alpha = 0$, then $B_{G,\mathcal{X}}$ is projective over all $H_0\mathcal{X}$ subgroups of G by definition and since M is R -projective by Lemma 4.1, we have that $M \otimes B_{G,\mathcal{X}}$ is projective over all $H_0\mathcal{X}$ subgroups. Now assume that the result is true over $H_\beta\mathcal{X}$ -subgroups of G for all $\beta < \alpha$ - this is our induction hypothesis. Now if H is a subgroup of G in $H_\alpha\mathcal{X}$, then H acts on, say, an n -dimensional contractible cell complex with $H_{<\alpha}\mathcal{X}$ -stabilisers and the trivial RH -module admits a length n resolution by direct sums of permutation modules with stabilisers in $H_\beta\mathcal{X}$ for some $\beta < \alpha$, i.e. modules that are direct sums of modules of the form $\text{Ind}_F^H(\text{trivial module})$, for some $F \in H_{<\alpha}\mathcal{X}$. For any $M \in \text{WGProj}(RG)$, we can tensor this resolution by $M \otimes B_{G,\mathcal{X}}$ and since by the induction hypothesis, $M \otimes B_{G,\mathcal{X}}$ is projective over all $H_{<\alpha}\mathcal{X}$ -subgroups of G , what we get after tensoring is a finite length resolution of $M \otimes B_{G,\mathcal{X}}$ by modules that are projective over RH . (This is because, for any $F \in H_{<\alpha}\mathcal{X}$, when we tensor $\text{Ind}_F^H(\text{trivial module})$ with $M \otimes B_{G,\mathcal{X}}$ as an RH -module, we get $\text{Ind}_F^H \text{Res}_F^H(M \otimes B_{G,\mathcal{X}})$ and it follows from the induction hypothesis that $\text{Ind}_F^H \text{Res}_F^H(M \otimes B_{G,\mathcal{X}})$ is projective over RH .) So, $\text{proj. dim}_{RH}(M \otimes B_{G,\mathcal{X}}) \leq n$. Now note that this is true for all weak Gorenstein projectives M . Now since $B_{G,\mathcal{X}}$ is R -free, if M is a kernel in a weak complete resolution $(F_i, d_i)_{i \in \mathbb{Z}}$ of RH -modules

(note that weak Gorenstein projectivity is closed under restriction to subgroups) then $M \otimes B_{G,\mathcal{X}}$ occurs as a kernel in the weak complete resolution $(F_i \otimes B_{G,\mathcal{X}}, d_i \otimes id)_{i \in \mathbb{Z}}$ and all the kernels in this weak complete resolution have finite projective dimension over RH , bounded by n , as we have just showed, so by Proposition 3.1, $M \otimes B_{G,\mathcal{X}}$ is projective over RH .

Now we show the statement of the proposition is true for $LH\mathcal{X}$ -subgroups of G . Let H be an $LH\mathcal{X}$ subgroup of G . Then, if H is countable, then by Lemma 5.3, H is in $H\mathcal{F}$. Let H be an uncountable $LH\mathcal{X}$ -subgroup of G . Assume as induction hypothesis that the statement of the proposition holds true for all $LH\mathcal{X}$ -subgroups of G of cardinality strictly smaller than the cardinality of H . Now as H is uncountable, it can be written as $\bigcup_{\gamma < \alpha} H_\gamma$ where $(H_\gamma)_{\gamma < \alpha}$ is a strictly ascending chain of subgroups of H and α is some ordinal, where each H_γ is strictly smaller than H in cardinality. By our induction hypothesis, $M \otimes B_{G,\mathcal{X}}$ is projective over RH_γ for each $\gamma < \alpha$, therefore by Lemma 5.2, $\text{proj. dim}_{RH}(M \otimes B_{G,\mathcal{X}}) \leq 1$. Again, if M , as a weak Gorenstein projective RH -module, occurs as the r -th kernel in a complete resolution $(F_i, d_i)_{i \in \mathbb{Z}}$ and if the $(r-1)$ -th kernel, which is also a weak Gorenstein projective RH -module, is denoted by $K_{-1}(M)$, then we have a short exact sequence $0 \rightarrow M \otimes B_{G,\mathcal{X}} \hookrightarrow F_{r-1} \otimes B_{G,\mathcal{X}} \rightarrow K_{-1}(M) \otimes B_{G,\mathcal{X}} \rightarrow 0$. So, if $\text{proj. dim}_{RH}(M \otimes B_{G,\mathcal{X}}) = 1$, then $\text{proj. dim}_{RH}(K_{-1}(M) \otimes B_{G,\mathcal{X}}) = 2$ which is not possible because since $K_{-1}(M)$ is weak Gorenstein projective, the above claim for weak Gorenstein projectives shows that $\text{proj. dim}_{RH}(K_{-1}(M) \otimes B_{G,\mathcal{X}}) \leq 1$. Thus, $\text{proj. dim}_{RH}(M \otimes B_{G,\mathcal{X}}) = 0$.

□

Corollary 5.5. *Let G be a group in $LH\mathcal{F}$ or of type Φ over a commutative ring R of finite global dimension. Then, the class of weak Gorenstein projective RG -modules, the class of Gorenstein projective RG -modules and the class of Benson's cofibrant RG -modules all coincide.*

Proof. For the case where G is in $LH\mathcal{F}$, it follows directly from Proposition 5.4, with $\mathcal{X} = \mathcal{F}$ and $B_{G,\mathcal{X}} = B(G, R)$, that weak Gorenstein projective RG -modules are Benson's cofibrant. Exactly this was handled and proved in Theorem B and subsequently in Corollary C of [10] by Dembegioti and Talelli.

Now, let us assume that G is of type Φ over a commutative ring R of finite global dimension. Let M be a weak Gorenstein projective RG -module which occurs as a kernel of a weak complete resolution $(F_i, d_i)_{i \in \mathbb{Z}}$ of projective RG -modules. Then $M \otimes B(G, R)$ is projective as an RF -module for all finite subgroups F of G (this is because M is R -projective and $B(G, R)$ is RF -free for all finite $F \leq G$). Since G is of type Φ , this implies that $\text{proj. dim}_{RG}(M \otimes B(G, R)) < \infty$. But $\text{fin. dim}(RG) < \infty$ as G is of type Φ (see Lemma 2.4 of [20]) and M can be any kernel in F_* ; so by Proposition 3.1, $M \otimes B(G, R)$ is projective as an RG -module. By Remark 3.3, it now follows that $M \in \text{GProj}(RG)$. Thus we have $\text{WGProj}(RG) \subseteq \text{CoF}(RG) \subseteq \text{GProj}(RG) \subseteq \text{WGProj}(RG)$. So, $\text{WGProj}(RG) = \text{CoF}(RG) = \text{GProj}(RG)$. This gives us a new proof of Theorem 3.11.

□

Although it is usually not much convenient, we can replace the class of finite subgroups in the definition of type Φ and have in its place subgroups belonging to some other arbitrary class.

Definition 5.6. *A group G is said to be of type $\Phi\text{-}\mathcal{X}$, for a class of groups \mathcal{X} , over a ring R if the following conditions are equivalent for all RG -modules M .*

- a) $\text{proj. dim}_{RG}(M) < \infty$.
- b) $\text{proj. dim}_{RH}(M) < \infty$ for all subgroups $H \leq G$ such that $H \in \mathcal{X}$.

Now, from Proposition 5.4, the following two corollaries follow.

Corollary 5.7. *Let G be a group and let R be a commutative ring of finite global dimension. Let \mathcal{X} be a class of groups, and let $B_{G,\mathcal{X}}$ be an R -free RG -module that restricts to a projective module on every \mathcal{X} -subgroup of G . Then, for any weak Gorenstein projective RG -module M , $M \otimes B_{G,\mathcal{X}}$ is projective over every subgroup of G that is of type $\Phi\text{-LH}\mathcal{X}$ over R .*

Proof. Let $M \in \text{WGProj}(RG)$. From Proposition 5.4, we get that $M \otimes B_{G,\mathcal{X}}$ is projective over $\text{LH}\mathcal{X}$ -subgroups of G , and then from Definition 5.6 it follows that $M \otimes B_{G,\mathcal{X}}$ has finite projective dimension over all $\Phi\text{-LH}\mathcal{X}$ -subgroups of G . Let us fix K to be a $\Phi\text{-LH}\mathcal{X}$ -subgroup of G . Note that as $\text{WGProj}(RG)$ is closed under restriction to subgroups, M , restricted as an RK -module, is in $\text{WGProj}(RK)$ and as $B_{G,\mathcal{X}}$ is R -free, $M \otimes B_{G,\mathcal{X}} \in \text{WGProj}(RK)$ (here, we are restricting both M and $B_{G,\mathcal{X}}$ as RK -modules, and we will be considering $B_{G,\mathcal{X}}$ as an RK -module for the rest of this proof).

We claim that $\sup\{\text{proj. dim}_{RK}(N \otimes B_{G,\mathcal{X}}) : N \in \text{WGProj}(RK)\}$ is finite. This is easy to see because otherwise for every integer $n > 0$, we will have an $N_n \in \text{WGProj}(RK)$ such that $\text{proj. dim}_{RK}(N_n \otimes B_{G,\mathcal{X}}) > n$, and then $\text{proj. dim}_{RK}(\bigoplus_n N_n \otimes B_{G,\mathcal{X}})$ will not be finite which will be a contradiction as $\text{WGProj}(RK)$ is closed under direct sums. Now, from Proposition 3.1, it follows that $M \otimes B_{G,\mathcal{X}}$ is actually projective over K . \square

Corollary 5.8. *Let G be a group of type $\Phi\text{-LH}\mathcal{F}$ over a commutative ring R of finite global dimension. Then, the class of weak Gorenstein projective RG -modules, the class of Gorenstein projective RG -modules and the class of Benson's cofibrant RG -modules all coincide.*

One of the main arguments in the proof of Proposition 5.4 is moving from the module $M \otimes B_{G,\mathcal{X}}$ being projective over all $H\mathcal{X}$ -subgroups to proving that it must be so over all subgroups that are locally $H\mathcal{X}$. The proof of Proposition 5.4 however allows us to prove the following.

Theorem 5.9. *Let G be a group and let R be a commutative ring. Let \mathcal{X} be a class of groups. Assume that for any weak Gorenstein projective RG -module M , $M \otimes B(G, R)$ is projective over every \mathcal{X} -subgroup of G . Then, $M \otimes B(G, R)$ is projective over every $\text{LH}\mathcal{X}$ -subgroup of G . Note that from this it follows that $M \otimes B(G, R)$ is projective over every $\text{L}\mathcal{X}$ -subgroup and over every $H\mathcal{X}$ -subgroup of G .*

Corollary 5.10. *Let R be a ring of finite global dimension. If a group G is in $LH\mathcal{F}_{\phi,R}$, where $\mathcal{F}_{\phi,R}$ is the class of all groups of type Φ over R , or locally Φ - $LH\mathcal{F}$ over R , then, the class of weak Gorenstein projectives, Benson's cofibrants and Gorenstein projectives all coincide.*

Proof. The proof is obvious from Theorem 5.9 and Corollary 5.7 which shows that the weak Gorenstein projectives form a subclass of Benson's cofibrants and from the logic in the proof of Corollary 5.5 it follows that we have $WGProj(RG) \subseteq CoF(RG) \subseteq GProj(RG) \subseteq WGProj(RG)$. \square

Remark 5.11. I do not know of any examples of groups that are locally Φ - $LH\mathcal{F}$ over, say, \mathbb{Z} , but are not of type Φ - $LH\mathcal{F}$ over the same. The following question becomes crucial in investigating locally Φ - $LH\mathcal{F}$ should imply Φ - $LH\mathcal{F}$: is a module over a group of finite projective dimension if it is of finite projective dimension over any finitely generated subgroup of the given group?

For any commutative ring R of finite global dimension, taking as base class $\mathcal{F}_{\phi,R}$, the class of all groups of type Φ over R , we get that Kropholler's class $H\mathcal{F}_{\phi,R}$ is much larger than $\mathcal{F}_{\phi,R}$ because $H\mathcal{F}_{\phi,R}$ contains $H\mathcal{F}$ as $\mathcal{F}_{\phi,R}$ contains \mathcal{F} and there are examples of free abelian groups of infinite rank that are in $H\mathcal{F}$ which do not admit complete resolutions as we have discussed before and therefore cannot be in $\mathcal{F}_{\phi,R}$. Free abelian groups of infinite rank are also locally type Φ but not type Φ themselves over any ring of finite global dimension. However, if $(a) \Leftrightarrow (c)$ holds in Conjecture 1.22, then $H\mathcal{F}_{\phi,\mathbb{Z}} = H(H_1\mathcal{F}) = H\mathcal{F}$, and $LH\mathcal{F}_{\phi,\mathbb{Z}} = LH\mathcal{F}$, and that would mean, for $R = \mathbb{Z}$, our result (Corollary 5.10) for $LH\mathcal{F}_{\phi,\mathbb{Z}}$ -groups does not give us any groups for which Conjecture 1.1 is not already known to be true.

We end this section with the following corollary which shows that, for a given commutative ring R of finite global dimension, Conjecture 4.4 is satisfied by $LH\mathcal{F}_{\phi,R}$ -groups and groups that are either fully or locally Φ - $LH\mathcal{F}$.

Corollary 5.12. *Fix a ring R of finite global dimension. Using the notations of Corollary 5.10, Gorenstein projectivity is closed under*

- a) *subgroups that are in $LH\mathcal{F}_{\phi,R}$.*
- b) *subgroups that are either locally or fully Φ - $LH\mathcal{F}$ (or even Φ - $LH\mathcal{F}_{\phi,R}$) over R .*

Proof. If M is a Gorenstein projective RG -module, then for any subgroup H of G , $Res_H^G(M)$ is a weak Gorenstein projective RH -module, and if H is a subgroup of G satisfying (a) or (b) of the hypothesis of Corollary 5.12, then $Res_H^G(M)$ is a Gorenstein projective RH -module by Corollary 5.10. \square

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