

Peculiar thermal states in the first-order thermodynamics of gravity

Valerio Faraoni,^{1,*} Andrea Giusti,^{2,†} Sonia Jose,^{1,‡} and Serena Giardino^{3,§}

¹*Department of Physics & Astronomy, Bishop's University,
2600 College Street, Sherbrooke, Québec, Canada J1M 1Z7*

²*Institute for Theoretical Physics, ETH Zurich,
Wolfgang-Pauli-Strasse 27, 8093 Zurich, Switzerland*

³*Max Planck Institute for Gravitational Physics (Albert Einstein Institute),
Callinstraße 38, 30167 Hannover, Germany*

In the context of the recently proposed first-order thermodynamics of scalar-tensor gravity, we discuss the possibility of zero-temperature states of equilibrium other than Einstein gravity, including pathological Brans-Dicke theory, Palatini $f(R)$ gravity, and cuscuton gravity, all with non-dynamical scalar fields. The formalism is extended to Nordström gravity, which contains only a scalar degree of freedom and has negative temperature relative to general relativity.

I. INTRODUCTION

The idea that gravity may not be fundamental but, rather, emergent has been the subject of many works. A significant step in this line of research was the derivation of the Einstein equations with purely thermodynamical considerations [1], followed by the realization that general relativity (GR) could be the equilibrium state of $f(R)$ gravity [2] (see also [3]). This second idea could, in principle, apply to a vast landscape of theories of gravity: GR could be the “zero-temperature” equilibrium state and modified theories of gravity would then be excited states with positive “temperature”. This idea makes sense when one considers that theories of gravity alternative to GR usually contain extra degrees of freedom and that, if the latter are excited together with the two spin two massless degrees of freedom of Einstein theory, one would then have an “excited state” with respect to GR.

The problem is that, in spite of many years of research on the thermodynamics of spacetime initiated by Refs. [1, 2], the “temperature of gravity” and the equations describing the approach to the GR equilibrium state have not been found. Recently [4, 5], we proposed an effective scalar-tensor thermodynamics completely different from the thermodynamics of spacetime. We identified a “temperature of gravity” with GR as the zero-temperature state and we provided an equation describing the approach to equilibrium (or, possibly, departures from it). This approach was originally formulated for “first-generation” scalar-tensor gravity [6–9] and $f(R)$ gravity (which is a subclass of scalar-tensor theories [10–12]) and then extended to “viable” Horndeski gravity [13]. The key idea consists of writing the field equations of modified gravity as effective Einstein equations of the form¹ $G_{ab} = 8\pi T_{ab}^{(\text{eff})}$ by grouping all terms other than

the Einstein tensor G_{ab} (which contain the gravitational scalar field ϕ of the theory and its first and second derivatives) to the right-hand side, where they form an effective stress-energy tensor $T_{ab}^{(\text{eff})}$. It is a fact that, in “old” scalar-tensor and in “viable” Horndeski gravity, when the gradient $\nabla^c \phi$ is timelike this effective stress-energy tensor has the form of an imperfect fluid stress-energy tensor

$$T_{ab} = \rho u_a u_b + 2 u_{(a} q_{b)} + P h_{ab} + \pi_{ab} \quad (1.1)$$

where ρ , P , q_a , and π_{ab} are the energy density, (total) isotropic pressure, heat flux density, and trace-free part of the anisotropic stress tensor, respectively, $h_{ab} \equiv g_{ab} + u_a u_b$, and

$$u^a = \pm \frac{\nabla^a \phi}{\sqrt{-\nabla^c \phi \nabla_c \phi}} \quad (1.2)$$

is the four-velocity of the effective fluid (the sign of the right-hand side of Eq. (1.2) must be chosen so that u^a is future-pointing).

Detailed expressions of the kinematic quantities, heat flux density, pressure, and stresses of the effective ϕ -fluid are given in [15, 16] for first generation scalar-tensor gravity and in [13, 17] for Horndeski gravity. The next step is to take this dissipative effective fluid seriously and apply Eckart’s first-order thermodynamics to it [18]. In fact, much less is needed: one only needs [4, 5, 13] the three constitutive relations postulated in Eckart’s theory to obtain the product of the “thermal conductivity” \mathcal{K} and the “temperature of gravity” \mathcal{T} (which follows from Eckart’s generalized Fourier law [18]), and the coefficients of bulk and shear viscosity.

Focusing on “old” scalar-tensor gravity for simplicity, one obtains [4, 5]

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^c \phi \nabla_c \phi}}{8\pi\phi} \quad (1.3)$$

constant G and the speed of light c are unity and the metric signature is $(-+++)$.

* vfaraoni@ubishops.ca

† agiusti@phys.ethz.ch

‡ sjose21@ubishops.ca

§ serena.giardino@aei.mpg.de

¹ We follow the notation of Ref. [14], using units in which Newton’s

and the thermal evolution of the system is described by

$$\frac{d(\mathcal{KT})}{d\tau} = 8\pi(\mathcal{KT})^2 - \Theta\mathcal{KT} + \frac{\square\phi}{8\pi\phi}, \quad (1.4)$$

where τ is the proper time along the flow lines of the effective fluid, $\Theta = \nabla_c u^c$ is its expansion scalar, and $\square \equiv g^{ab}\nabla_a\nabla_b$ denotes the Laplace–Beltrami operator.

This new formalism was applied to generic Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology and to specific FLRW exact solutions of scalar-tensor gravity in [19], while the existence of special metastable states was highlighted in [20].

Here we want to explore the possible existence of other peculiar states of gravity, in addition to the zero-temperature GR state: such states would correspond to theories of gravity that are special in some physical sense. In particular, we want to probe the possibility of states of equilibrium other than GR, corresponding to $\mathcal{KT} = \text{constant}$. We apply the formalism to entire classes of gravitational theories rather than to specific solutions of one such class (which was done in [19] and [20]). The aim of this study is to better understand the regime of validity of the thermodynamical formalism, by testing it on theories that – while not always physically viable – are useful for our purposes since they allow to clarify the existence of other equilibrium states.

We begin with a simple consideration: one expects that theories of gravity containing non-dynamical fields in addition to the two spin two massless modes of GR will have either zero \mathcal{KT} or that the latter will be completely arbitrary if these extra non-dynamical fields are. Indeed, we show that this is the case for $\omega = -3/2$ Brans–Dicke theory, for Palatini $f(R)$ gravity, and for cuscuton gravity (a special Horndeski theory that is also a special case of Hořava–Lifschitz gravity). All these cases correspond to a non-dynamical scalar field ϕ . Since they are all contained in the subclass of “viable” Horndeski theories previously studied, the first-order thermodynamical formalism of [4, 5, 13] can be applied without changes.

Next, one wonders what a theory with *less* degrees of freedom than GR would look like from the point of view of the thermodynamics of modified gravity. In particular, one expects that, if one can define a concept of temperature as done in scalar-tensor and Horndeski gravity, this temperature should be *negative*, corresponding to the ex-

citation of less degrees of freedom than GR. To this end, we study Nordström’s theory of gravity [21], in which the metric is forced to be conformally flat and only a scalar field degree of freedom (but not the two spin two modes of GR) is excited. This theory was considered a serious candidate for the description of gravity only for a very brief period of time and contradicts the classical Solar System tests of GR, therefore it has only historical importance [22, 23]. However, it is still useful as a toy model when studying fundamental questions such as the validity of the strong equivalence principle in modified gravity [24, 25], the emergence of gravity and Lorentzian spacetime [26], theoretical frameworks for testing gravity [27] or, in our case, the thermodynamics of gravity in a landscape of theories. Describing Nordström gravity as an effective dissipative fluid proceeds in a manner very similar to scalar-tensor gravity since the effective scalar field (*i.e.*, the conformal factor of the conformally flat metric of this theory), is truly dynamical. As expected, the resulting temperature turns out to be negative. We also provide kinematic quantities, the decomposition of the effective stress-energy tensor, and the dissipative quantities (bulk and shear viscosity and heat current density) for this effective fluid.

The plan of this paper is as follows: the next section discusses two theories with non-dynamical scalar field, $\omega = -3/2$ Brans–Dicke theory and Palatini $f(R)$ gravity. Sec. III studies cuscuton gravity from the lens of first-order thermodynamics of scalar-tensor gravity (again, the cuscuton is non-dynamical), while Sec. IV discusses Nordström gravity. Section V contains the conclusions and interprets the results in the general framework of the effective thermodynamics of gravity.

II. $\omega = -3/2$ BRANS-DICKE GRAVITY AND PALATINI $f(R)$ GRAVITY

Let us consider the non-dynamical case of Brans–Dicke gravity [6]. It is well-known that $\omega = -3/2$ Brans–Dicke gravity and Palatini $f(R)$ gravity contain a scalar field that is not dynamical.

The (Jordan frame) field equations of scalar-tensor gravity are [6–9]

$$G_{ab} = \frac{8\pi}{\phi} T_{ab}^{(m)} + \frac{\omega}{\phi^2} \left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi \right) + \frac{1}{\phi} (\nabla_a\nabla_b\phi - g_{ab}\square\phi) - \frac{V}{2\phi} g_{ab}, \quad (2.1)$$

$$(2\omega + 3)\square\phi = 8\pi T^{(m)} + \phi V' - 2V - \omega'\nabla^c\phi\nabla_c\phi, \quad (2.2)$$

where a prime denotes differentiation with respect to ϕ . Note that in Refs. [4, 5, 16, 19] the trace of the stress-

energy tensor of matter $T^{(m)}$ in Eq. (2.2) appears erroneously divided by ϕ . This typographical error, however,

does not affect any of the results of [4, 5, 16, 19] since that equation is never employed in computations.

Setting $\omega = -3/2$, we lose the wave equation (2.2) for the scalar field ϕ , which reduces to the algebraic identity

$$8\pi T^{(m)} = 2V - \phi V', \quad (2.3)$$

making it clear that the scalar ϕ is not dynamical in this theory. The other field equation (2.1) becomes

$$R_{ab} - \frac{R}{2} g_{ab} = \frac{8\pi}{\phi} T_{ab}^{(m)} - \frac{3}{2\phi^2} \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) + \frac{1}{\phi} (\nabla_a \nabla_b \phi - g_{ab} \square \phi) - \frac{V}{2\phi} g_{ab} \quad (2.4)$$

which, upon contraction, gives

$$R = -\frac{8\pi T^{(m)}}{\phi} - \frac{3}{2\phi^2} \nabla^c \phi \nabla_c \phi + \frac{3}{\phi} \square \phi + \frac{2V}{\phi}. \quad (2.5)$$

Substituting Eq. (2.3) gives

$$R = V' - \frac{3}{2\phi^2} \nabla^c \phi \nabla_c \phi + \frac{3\square\phi}{\phi}. \quad (2.6)$$

Let us differentiate the identity (2.3), which yields

$$(V' - \phi V'') \nabla_c \phi = 8\pi \nabla_c T^{(m)}. \quad (2.7)$$

From Eq. (2.3) it is easy to see that in the absence of a potential, or when the latter is a pure mass term $V = m^2 \phi^2/2$, it must be

$$T^{(m)} = 0, \quad (2.8)$$

i.e., we can only have vacuum or conformally invariant matter, or else the trace $T^{(m)}$ is constant.

If the gradient $\nabla^c \phi$ is timelike and $V(\phi) \neq m^2 \phi^2/2$ (with $m^2 \geq 0$), then one can rewrite $\nabla^c \phi$ in terms of $\nabla^c T^{(m)}$ by taking advantage of Eq. (2.7). Therefore, the effective temperature of $\omega = -3/2$ Brans-Dicke theory with non-dynamical scalar is given by

$$\mathcal{KT} = \frac{\sqrt{|\nabla^c \phi \nabla_c \phi|}}{8\pi\phi} = \frac{\sqrt{|\nabla^c T^{(m)} \nabla_c T^{(m)}|}}{\phi |V' - \phi V''|}. \quad (2.9)$$

If, instead, $V(\phi) = m^2 \phi^2/2$, then $T^{(m)} = 0$ and there is no relation between $\nabla_c \phi$ and $\nabla_c T^{(m)}$.

For general forms of matter, in both cases the temperature is almost completely arbitrary. This is not too surprising because the scalar field ϕ is non-dynamical and, essentially, arbitrary. The temperature \mathcal{KT} relative to GR is defined using this non-dynamical scalar field and is, therefore, ill-defined as a consequence of the arbitrariness of ϕ .

The situation changes *in vacuo*, possibly in the presence a cosmological constant. In this case

$$T_{ab}^{(m)} = -\Lambda g_{ab}, \quad (2.10)$$

hence one has that $T^{(m)} = -4\Lambda$ is constant, which implies $\nabla_c T^{(m)} = 0$ and $\mathcal{KT} = 0$.

A. Palatini $f(R)$ gravity

It is well-known that Palatini $f(R)$ gravity is equivalent to $\omega = -3/2$ Brans-Dicke theory with a complicated potential [10–12] and that, *in vacuo*, it reduces to general relativity with (possibly) a cosmological constant. Therefore, vacuum Palatini $f(R)$ gravity has effective “temperature of gravity” given by $\mathcal{KT} = 0$. In any case, the scalar field is non-dynamical and, in the presence of matter, the theory runs into all sorts of problems, including unacceptably strong couplings to the Standard Model, impossibility to build polytropic stars (which should always be possible in any reasonable theory of gravity, as it is possible in Newtonian gravity), ill-posed Cauchy problem, *etc.* [10–12].

III. CUSCUTON AND THE THERMODYNAMICS OF HORNDESKI GRAVITY

Cuscuton gravity [28–32] is interesting from various points of view: it is a special case of Hořava-Lifschitz theory, is a model of a Lorentz-violating theory, it can implement the idea of limiting curvature without cosmological instabilities [29, 33] and cosmological singularities [34–36] (this is not true in more general Horndeski theories [37–40]), and has been obtained as the ultraviolet limit of an anti-Dirac-Born-Infeld theory [41]. Other phenomenological properties are studied in [42–44].

The cuscuton is realized by a scalar field that does not propagate new degrees of freedom with respect to GR (at least in the unitary gauge [45], but this property is believed to hold in any gauge [46]). This scalar (cuscuton field) satisfies a first-order equation of motion, *i.e.*, a constraint and the perturbed scalar action does not contain a kinetic term for this field, at all orders [45]. Denoting the cuscuton field with ϕ , its potential with $V(\phi)$, and using

$$X \equiv -\frac{1}{2} \nabla^c \phi \nabla_c \phi, \quad f_{,\phi} \equiv \frac{\partial f}{\partial \phi}, \quad f_{,X} \equiv \frac{\partial f}{\partial X}, \quad (3.1)$$

for any $f = f(\phi, X)$, the cuscuton Lagrangian density is [36]

$$\mathcal{P}(\phi, X) = \pm \mu^2 \sqrt{2X} - V(\phi), \quad (3.2)$$

where μ is a mass scale. The total action is

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi} + \mathcal{P} \right) + S_{\text{matter}}. \quad (3.3)$$

The cuscuton satisfies the equation of motion

$$g^{ab} \nabla_a (\mathcal{P}_{,X} \nabla_b \phi) + \mathcal{P}_{,\phi} = 0, \quad (3.4)$$

or

$$\pm \mu^2 \nabla^b \left(\frac{\nabla_b \phi}{\sqrt{2X}} \right) = V_{,\phi}, \quad (3.5)$$

which reduces to a first-order constraint (see, *e.g.*, [36]). The field equations for g_{ab} can be written in the form of effective Einstein equations with the effective stress-energy tensor [36]

$$\begin{aligned} T_{ab}^{(\phi)} &= \mathcal{P}g_{ab} + \mathcal{P}_{,X}\nabla_a\phi\nabla_b\phi \\ &= \left[\pm\mu^2\sqrt{2X} - V\right]g_{ab} \pm \mu^2\frac{\nabla_a\phi\nabla_b\phi}{\sqrt{2X}} \end{aligned} \quad (3.6)$$

on the right-hand side as the effective source. $T_{ab}^{(\phi)}$ has the form a perfect fluid stress-energy tensor

$$T_{ab} = (P + \rho)u_a u_b + P g_{ab} \quad (3.7)$$

where the energy density, pressure, and 4-velocity are

$$\rho^{(\phi)}(\phi, X) = 2X\mathcal{P}_{,X} - P^{(\phi)} = V(\phi), \quad (3.8)$$

$$P^{(\phi)}(\phi, X) = \mathcal{P}(\phi, X) = \pm\mu^2\sqrt{2X} - V(\phi), \quad (3.9)$$

$$u^a = \pm\frac{\nabla^a\phi}{\sqrt{2X}}, \quad (3.10)$$

respectively. The fact that the Lagrangian $\mathcal{P}(\phi, X)$ coincides with the pressure is a trademark of a perfect fluid, for which a Lagrangian description is known [47–49]. The \pm sign in Eq. (3.10) ensures that u^c can be chosen so that it is future-pointing.

The speed of sound in the cuscuton fluid, given by

$$c_s^2 = \frac{P_{,X}^{(\phi)}}{\rho_{,X}^{(\phi)}} = \frac{\mathcal{P}_{,X}}{\mathcal{P}_{,X} + 2X\mathcal{P}_{,XX}}, \quad (3.11)$$

diverges because the denominator vanishes, a rigidity property typical of the incompressible cuscuton fluid [28]. In the unitary gauge, where $\phi = \phi(t)$, it is obvious that $\nabla^c\phi$ is timelike.

Since there is no dissipation, the cuscuton field corresponds to a state of equilibrium: one can argue that no dissipation occurs in this fluid because it is already in a state of equilibrium. This is not really surprising. Since no propagating degree of freedom is excited in addition to the two massless spin two modes of GR, the cuscuton theory cannot be an “excited state” of GR.

From a more general point of view, the cuscuton is a special case of the “viable” class of Horndeski gravities corresponding to the choice of functions

$$G_4(\phi, X) = \frac{1}{16\pi}, \quad (3.12)$$

$$G_2(\phi, X) = \mathcal{P}(\phi, X) = \pm\mu^2\sqrt{2X} - V(\phi), \quad (3.13)$$

$$G_3(\phi, X) = G_5(\phi, X) = 0, \quad (3.14)$$

in the (general) Horndeski Lagrangian

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \quad (3.15)$$

where

$$\mathcal{L}_2 = G_2, \quad \mathcal{L}_3 = -G_3\Box\phi, \quad (3.16)$$

$$\mathcal{L}_4 = G_4 R + G_{4X} [(\Box\phi)^2 - (\nabla_a\nabla_b\phi)^2],$$

$$\begin{aligned} \mathcal{L}_5 &= G_5 G_{ab}\nabla^a\nabla^b\phi - \frac{G_{5X}}{6} [(\Box\phi)^3 - 3\Box\phi(\nabla_a\nabla_b\phi)^2 \\ &\quad + 2(\nabla_a\nabla_b\phi)^3]. \end{aligned} \quad (3.17)$$

If one thinks of starting from a viable Horndeski theory, in which the scalar field is equivalent to a dissipative fluid to which we assign [13]

$$\mathcal{KT} = \sqrt{2X} \frac{(G_{4,\phi} - XG_{3,X})}{G_4}, \quad (3.18)$$

then taking the limit in which $G_3 \rightarrow 0$, $G_4 \rightarrow \text{const.}$, and G_2 is as in Eq. (3.13), one obtains the cuscuton theory without dissipation. In this limit, Eq. (3.18) yields $\mathcal{KT} \rightarrow 0$.

By contrast, consider extended cuscuton theories [50–52], in some of which (Galileon generalizations of the cuscuton) spherical waves of the scalar field are free from caustic singularities [50]: they generally contain a dynamical scalar field. For example, consider the theory described by [36]

$$G_4(\phi, X) = \frac{1}{16\pi}, \quad (3.19)$$

$$G_2(\phi, X) = \pm\mu^2\sqrt{2X} - V(\phi), \quad (3.20)$$

$$G_3(\phi, X) = -a_3 \ln\left(\frac{X}{\Lambda^4}\right), \quad (3.21)$$

$$G_5(\phi, X) = 0, \quad (3.22)$$

in the standard Horndeski notation. For this theory, Eq. (3.18) gives a non-zero effective temperature,

$$\mathcal{KT} = 16\pi a_3 \sqrt{-\nabla^c\phi\nabla_c\phi}; \quad (3.23)$$

in fact, in spite of its name, this model has three degrees of freedom unlike the original cuscuton theory, which means that the scalar degree of freedom is excited and propagates [51]. Taking the limit $a_3 \rightarrow 0$ recovers the usual cuscuton, sending \mathcal{KT} to zero.

IV. NORDSTRÖM GRAVITY

In Nordström’s scalar theory of gravity [21], the space-time metric \tilde{g}_{ab} is conformally flat,

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \quad (4.1)$$

where g_{ab} is the Minkowski metric and the conformal factor Ω satisfies

$$\square\Omega = 0. \quad (4.2)$$

Under a generic conformal map (4.1), geometric quantities transform according to the well-known laws [14]

$$\tilde{\Gamma}_{bc}^a = \Gamma_{bc}^a + \frac{1}{\Omega} \left(\delta_b^a \nabla_c \Omega + \delta_c^a \nabla_b \Omega - g_{bc} \nabla^a \Omega \right), \quad (4.3)$$

$$\begin{aligned} \tilde{R}_{ab} &= R_{ab} - 2\nabla_a \nabla_b \ln \Omega - g_{ab} g^{ef} \nabla_e \nabla_f \ln \Omega \\ &\quad + 2\nabla_a \ln \Omega \nabla_b \ln \Omega - 2g_{ab} g^{ef} \nabla_e \ln \Omega \nabla_f \ln \Omega, \end{aligned} \quad (4.4)$$

$$\tilde{R} = \frac{1}{\Omega^2} \left(R - \frac{6\square\Omega}{\Omega} \right). \quad (4.5)$$

In our case $\square\Omega = 0$ and g_{ab} is the Minkowski metric, thus $R_{ab} = 0$ and $R = 0$. This implies that $\tilde{R} = 0$ and the Einstein tensor transforms as

$$\begin{aligned} \tilde{G}_{ab} &= \tilde{R}_{ab} - \frac{\tilde{R}}{2} \tilde{g}_{ab} \\ &= -\frac{2\nabla_a \nabla_b \Omega}{\Omega} + \frac{4\nabla_a \Omega \nabla_b \Omega}{\Omega^2} - g_{ab} \frac{\nabla^c \Omega \nabla_c \Omega}{\Omega^2}, \end{aligned} \quad (4.6)$$

Inverting Eq. (4.3) one has that

$$\Gamma_{bc}^a = \tilde{\Gamma}_{bc}^a - \frac{1}{\Omega} \left(\delta_b^a \tilde{\nabla}_c \Omega + \delta_c^a \tilde{\nabla}_b \Omega - \tilde{g}_{bc} \tilde{\nabla}^a \Omega \right), \quad (4.7)$$

where one has to recall that $\nabla_a \Omega = \partial_a \Omega = \tilde{\nabla}_a \Omega$ since $\Omega = \Omega(x)$ is a scalar function. Therefore, it is easy to see that

$$\begin{aligned} \nabla_a \nabla_b \Omega &= \tilde{\nabla}_a \tilde{\nabla}_b \Omega \\ &\quad + \frac{1}{\Omega} \left(\delta_a^c \tilde{\nabla}_b \Omega + \delta_b^c \tilde{\nabla}_a \Omega - \tilde{g}_{ab} \tilde{\nabla}^c \Omega \right) \tilde{\nabla}_c \Omega, \end{aligned} \quad (4.8)$$

then taking advantage of $g^{ab} = \Omega^2 \tilde{g}^{ab}$, easily derived from Eq. (4.1), one has that

$$\square\Omega = \Omega^2 \tilde{\square}\Omega - 2\Omega \tilde{g}^{ef} \tilde{\nabla}_e \Omega \tilde{\nabla}_f \Omega, \quad (4.9)$$

which reduces to

$$\tilde{\square}\Omega = \frac{2}{\Omega} \tilde{g}^{ef} \tilde{\nabla}_e \Omega \tilde{\nabla}_f \Omega \quad (4.10)$$

if one recalls the condition in Eq. (4.2). Additionally, one can use Eq. (4.8) to rewrite Eq. (4.6) as

$$\tilde{G}_{ab} = -\frac{2\tilde{\nabla}_a \tilde{\nabla}_b \Omega}{\Omega} - \tilde{g}_{ab} \frac{\tilde{\nabla}^c \Omega \tilde{\nabla}_c \Omega}{\Omega^2}. \quad (4.11)$$

We can now use this Einstein tensor for the conformally flat solutions of Nordström theory to write the

vacuum Nordström quantity \tilde{G}_{ab} in the form of effective Einstein equations

$$\tilde{G}_{ab} = 8\pi \tilde{T}_{ab}^{(\Omega)}, \quad (4.12)$$

where

$$8\pi \tilde{T}_{ab}^{(\Omega)} = -\frac{2\tilde{\nabla}_a \tilde{\nabla}_b \Omega}{\Omega} + \tilde{g}_{ab} \frac{\tilde{\nabla}^c \Omega \tilde{\nabla}_c \Omega}{\Omega^2}. \quad (4.13)$$

This tensor is traceless, $\tilde{T}^{(\Omega)} = 0$, as a result of Eq. (4.10).

Assuming the gradient $\nabla^c \Omega$ to be timelike and following the usual procedure to associate an effective fluid with a scalar field [5, 15–17], we introduce the effective fluid four-velocity

$$\tilde{u}_a \equiv \pm \frac{\tilde{\nabla}_a \Omega}{\sqrt{-\tilde{g}^{cd} \tilde{\nabla}_c \Omega \tilde{\nabla}_d \Omega}}, \quad (4.14)$$

where the sign of the right-hand side is chosen so that $\nabla^c \Omega$ is future-oriented, and the Nordström metric undergoes the 3 + 1 splitting

$$\tilde{g}_{ab} = -\tilde{u}_a \tilde{u}_b + \tilde{h}_{ab}, \quad (4.15)$$

where \tilde{h}_a^b is the projection operator on the 3-space of the observers comoving with the fluid, who have four-velocities \tilde{u}^c .

The effective stress-energy tensor $\tilde{T}_{ab}^{(\Omega)}$ has the structure (1.1) of an imperfect fluid. It is straightforward to compute the heat current density

$$\tilde{q}_a^{(\Omega)} = -\tilde{T}_{cd}^{(\Omega)} \tilde{u}^c \tilde{h}_a^d = \frac{\sqrt{2\tilde{X}}}{4\pi\Omega} \dot{\tilde{u}}_a, \quad (4.16)$$

where

$$\tilde{X} \equiv -\frac{1}{2} \tilde{g}^{ef} \tilde{\nabla}_e \Omega \tilde{\nabla}_f \Omega. \quad (4.17)$$

Eckart's generalized Fourier law (which is one of the three constitutive relations of Eckart's theory) [18]

$$q^a = -\mathcal{K} (h^{ab} \nabla_b \mathcal{T} + \mathcal{T} \dot{u}^a) \quad (4.18)$$

then yields

$$\mathcal{K}\mathcal{T} = -\frac{\sqrt{2\tilde{X}}}{4\pi\Omega} = -\frac{\sqrt{-\tilde{g}^{ef} \tilde{\nabla}_e \Omega \tilde{\nabla}_f \Omega}}{4\pi\Omega}, \quad (4.19)$$

which is negative. This result is interpreted by saying that, defining the effective temperature of gravity in the same manner as done in scalar-tensor gravity (where \mathcal{T} is a notion of temperature relative to GR), Nordström's scalar gravity is de-excited with respect to GR since it contains only one scalar degree of freedom, as opposed to the two massless spin two degrees of freedom of Einstein theory.

In order to draw an explicit parallel with scalar-tensor gravity it is useful to remember that, if g_{ab} is an electrovacuum solution of the Einstein equations, the conformally transformed metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ is a solution of $\omega = -3/2$ Brans-Dicke theory with Brans-Dicke field [53]

$$\phi = \frac{1}{\Omega^2}. \quad (4.20)$$

Here however, contrary to $\omega = -3/2$ Brans-Dicke gravity considered in Sec. II, the scalar field ϕ is not arbitrary but must satisfy Eq. (4.2), equivalent to

$$\square\phi = \frac{3}{2\phi} \nabla^c \phi \nabla_c \phi. \quad (4.21)$$

The temperature (4.19) found for Nordström gravity is then written as

$$\mathcal{KT} = -\frac{\sqrt{-\tilde{g}^{ab} \tilde{\nabla}_a \phi \tilde{\nabla}_b \phi}}{8\pi\phi}, \quad (4.22)$$

which is exactly the negative of what was found in scalar-tensor gravity in Refs. [4, 5]. Solutions with constant ϕ ,

or constant Ω , correspond to the Minkowski metric and the absence of (scalar) gravity.

Next, one can consider the stability of Nordström gravity seen as a (peculiar) thermal state of gravity. The Nordström field equation (4.10) can be rewritten in the form of an effective Klein-Gordon equation

$$\tilde{\square}\Omega - m_{\text{eff}}^2 \Omega = 0, \quad (4.23)$$

where

$$m_{\text{eff}}^2 \equiv \frac{2}{\Omega^2} \tilde{g}^{cd} \tilde{\nabla}_c \Omega \tilde{\nabla}_d \Omega \quad (4.24)$$

must be non-negative for stability (see [54]). Since $\tilde{\nabla}^c \Omega \tilde{\nabla}_c \Omega < 0$, we have what is called an effective thermal instability of Nordström gravity in [54].

To complete the first-order thermodynamical description of Nordström gravity, we compute the other effective fluid quantities. The kinematic quantities acceleration, expansion, and shear derived from the four-velocity and its gradient coincide with those already derived in scalar-tensor gravity [15, 16], with the provision that the Brans-Dicke-like field ϕ must be replaced with Ω and that Eq. (4.10) be substituted into the equations of [16] (in fact, the kinematic quantities do not depend on the field equations of the theory). Using the definition of \tilde{X} , the result consists of the fluid four-acceleration

$$\dot{u}_a \equiv \tilde{u}^c \tilde{\nabla}_c \tilde{u}_a = \frac{\tilde{\nabla}^b \Omega}{(2\tilde{X})^2} \left[2\tilde{X} \tilde{\nabla}_a \tilde{\nabla}_b \Omega + \tilde{\nabla}^d \Omega \tilde{\nabla}_b \tilde{\nabla}_d \Omega \tilde{\nabla}_a \Omega \right], \quad (4.25)$$

the expansion

$$\tilde{\Theta} = \tilde{\nabla}_c \tilde{u}^c = -\frac{2\sqrt{2\tilde{X}}}{\Omega} + \frac{\tilde{\nabla}^a \Omega \tilde{\nabla}^b \Omega \tilde{\nabla}_a \tilde{\nabla}_b \Omega}{(2\tilde{X})^{3/2}}, \quad (4.26)$$

and the shear tensor

$$\begin{aligned} \tilde{\sigma}_{ab} &= \frac{1}{(2\tilde{X})^{3/2}} \left[2\tilde{X} \tilde{\nabla}_a \tilde{\nabla}_b \Omega + \frac{4\tilde{X}}{3\Omega} \left(\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + 2\tilde{X} \tilde{g}_{ab} \right) \right. \\ &\quad \left. - \frac{1}{3} \left(\tilde{g}_{ab} - \frac{\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega}{\tilde{X}} \right) \tilde{\nabla}^c \Omega \tilde{\nabla}^d \Omega \tilde{\nabla}_c \tilde{\nabla}_d \Omega + 2\tilde{\nabla}^c \Omega \tilde{\nabla}_{(a} \Omega \tilde{\nabla}_{b)} \tilde{\nabla}_c \Omega \right] \\ &= \frac{1}{\sqrt{2\tilde{X}}} \left[\tilde{\nabla}_a \tilde{\nabla}_b \Omega + \frac{\tilde{\nabla}^e \Omega \tilde{\nabla}_{(a} \Omega \tilde{\nabla}_{b)} \tilde{\nabla}_e \Omega}{\tilde{X}} + \frac{\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega \tilde{\nabla}^c \Omega \tilde{\nabla}^d \Omega \tilde{\nabla}_c \tilde{\nabla}_d \Omega}{4\tilde{X}^2} + h_{ab} \left(\frac{4\tilde{X}}{3\Omega} - \frac{\tilde{\nabla}^c \Omega \tilde{\nabla}^d \Omega \tilde{\nabla}_c \tilde{\nabla}_d \Omega}{6\tilde{X}} \right) \right] \end{aligned} \quad (4.27)$$

while, of course, the effective fluid is irrotational because it is derived from a scalar, $\tilde{\omega}_{ab} = 0$.

The effective fluid quantities derived from the effective

stress-energy tensor (4.13) include the energy density

$$\tilde{\rho}^{(\Omega)} = \tilde{T}_{ab}^{(\Omega)} \tilde{u}^a \tilde{u}^b = \frac{1}{4\pi} \left(-\frac{\tilde{\nabla}^a \Omega \tilde{\nabla}^b \Omega \tilde{\nabla}_a \tilde{\nabla}_b \Omega}{2\tilde{X}\Omega} + \frac{\tilde{X}}{\Omega^2} \right), \quad (4.28)$$

the spatial stress tensor

$$8\pi \tilde{\Pi}_{ab}^{(\Omega)} = 8\pi \tilde{T}_{cd}^{(\Omega)} \tilde{h}_c^a \tilde{h}_b^d = -\frac{2}{\Omega} \left[\tilde{\nabla}_a \tilde{\nabla}_b \Omega + \frac{\tilde{\nabla}^e \Omega \tilde{\nabla}_{(a} \Omega \tilde{\nabla}_{b)} \tilde{\nabla}_e \Omega}{\tilde{X}} + \frac{\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega \tilde{\nabla}^c \Omega \tilde{\nabla}^d \Omega \tilde{\nabla}_c \tilde{\nabla}_d \Omega}{4\tilde{X}^2} + \frac{\tilde{X}}{\Omega} h_{ab} \right], \quad (4.29)$$

the isotropic pressure

$$8\pi \tilde{P}^{(\Omega)} = \frac{8\pi}{3} \tilde{h}^{ab} \tilde{\Pi}_{ab}^{(\Omega)} = \frac{2\tilde{X}}{3\Omega^2} - \frac{\tilde{\nabla}^a \Omega \tilde{\nabla}^b \Omega \tilde{\nabla}_a \tilde{\nabla}_b \Omega}{3\tilde{X}\Omega}, \quad (4.30)$$

and the anisotropic stresses

$$\begin{aligned} 8\pi \tilde{\pi}_{ab}^{(\Omega)} &= 8\pi (\tilde{\Pi}_{ab}^{(\Omega)} - \tilde{P}^{(\Omega)} \tilde{h}_{ab}) \\ &= -\frac{2}{\Omega} \left[\tilde{\nabla}_a \tilde{\nabla}_b \Omega + \frac{\tilde{\nabla}^e \Omega \tilde{\nabla}_{(a} \Omega \tilde{\nabla}_{b)} \tilde{\nabla}_e \Omega}{\tilde{X}} + \frac{\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega \tilde{\nabla}^c \Omega \tilde{\nabla}^d \Omega \tilde{\nabla}_c \tilde{\nabla}_d \Omega}{4\tilde{X}^2} + h_{ab} \left(\frac{4\tilde{X}}{3\Omega} - \frac{\tilde{\nabla}^c \Omega \tilde{\nabla}^d \Omega \tilde{\nabla}_c \tilde{\nabla}_d \Omega}{6\tilde{X}} \right) \right]. \end{aligned} \quad (4.31)$$

The shear tensor is therefore proportional to the contribution of the anisotropic stresses. Indeed, it is easy to see that

$$\tilde{\pi}_{ab}^{(\Omega)} = -\frac{\sqrt{2\tilde{X}}}{4\pi\Omega} \tilde{\sigma}_{ab}. \quad (4.32)$$

If we then recall the third constitutive relation of Eckart's first-order thermodynamics [18], *i.e.*, $\pi_{ab} = -2\eta\sigma_{ab}$, one can conclude that for Nordström gravity the shear viscosity reads

$$\eta = \frac{\sqrt{2\tilde{X}}}{8\pi\Omega} = -\frac{\mathcal{KT}}{2}, \quad (4.33)$$

in analogy with scalar-tensor gravity.

To compute the bulk viscosity coefficient, the expression (4.26) of the expansion scalar yields

$$\frac{\tilde{\nabla}^a \Omega \tilde{\nabla}^b \Omega \tilde{\nabla}_a \tilde{\nabla}_b \Omega}{2\tilde{X}} = \sqrt{2\tilde{X}} \tilde{\Theta} + \frac{4\tilde{X}}{\Omega} \quad (4.34)$$

which, substituted in the effective pressure (4.30), highlights the two distinct contributions (non-viscous and viscous, respectively) to the total isotropic pressure

$$\tilde{P}^{(\Omega)} = \tilde{P}_{\text{non-viscous}}^{(\Omega)} + \tilde{P}_{\text{viscous}}^{(\Omega)} = -\frac{\tilde{X}}{4\pi\Omega^2} - \frac{\sqrt{2\tilde{X}}}{12\pi\Omega} \tilde{\Theta}. \quad (4.35)$$

Remembering Eckart's constitutive relation [18]

$$P_{\text{viscous}} = -\zeta \Theta, \quad (4.36)$$

the bulk viscosity coefficient for Nordström gravity is

$$\zeta = \frac{\sqrt{2\tilde{X}}}{12\pi\Omega} = \frac{2}{3} \eta = -\frac{\mathcal{KT}}{3}. \quad (4.37)$$

V. CONCLUSIONS

In the context of the first-order thermodynamics of gravity developed in previous works, the question regarding the existence of equilibrium states other than GR remained unanswered. Here, we have studied peculiar classes of gravitational theories that are interesting from this point of view. While not always physically viable, these theories help to test the boundaries of the new thermodynamical formalism and to better grasp the meaning of the zero-temperature equilibrium states.

In $\omega = -3/2$ Brans-Dicke theory and in Palatini $f(R)$ gravity with matter, the scalar field ϕ can be assigned completely arbitrarily (apart from the requirement of being positive to keep the effective gravitational coupling $G_{\text{eff}} \sim 1/\phi$ positive) and does not have to satisfy even a first-order constraint. Correspondingly, the temperature of gravity given by Eq. (1.3) is also arbitrary. *A posteriori*, this fact makes sense because a well-defined temperature, even if vanishing, cannot be meaningfully derived from a completely arbitrary ϕ . In vacuum, this theory is equivalent to GR with, possibly, a cosmological constant and thus \mathcal{KT} vanishes.

The cuscuto field discussed in Sec. III is not a propagating degree of freedom but must satisfy a first-order constraint [28–32, 36]. As a consequence, it is not arbitrary and the corresponding temperature of cuscuto gravity turns out to be zero, which makes sense in the physical interpretation of excited or “hot” states as states endowed with new propagating degrees of freedom compared to GR. Since the cuscuto is somehow specified by the first-order constraint but is not dynamical, it corresponds to zero temperature and first-order thermodynamics does not distinguish between GR and cuscuto gravity.

Finally, we have discussed Nordström gravity which, with its reduced freedom compared to GR, corresponds

to much less excitation (one scalar mode versus two spin two modes). It is more difficult to compare Nordström gravity with GR because, contrary to scalar-tensor or Horndeski gravity which have a GR limit, Nordström gravity – strictly speaking – does not. Indeed, even conformally flat GR solutions $\tilde{g}_{ab} = \Omega^2 \eta_{ab}$ are not automatically solutions of Nordström gravity because the extra condition required $\square\Omega = 0$ is quite restrictive. However, the effective stress-energy tensor of the Nordström scalar Ω is quite similar to (a part of) the effective stress-energy tensor of the Brans-Dicke-like (or Horndeski) scalar ϕ . Since the scalar degree of freedom of Nordström gravity, which is the conformal factor Ω of the conformally flat Nordström metric, is not arbitrary but must obey the dynamical second order equation $\square\Omega = 0$, its effective temperature is well-defined and turns out to be negative, as naively expected. In this theory there are both bulk and shear viscosity and we have calculated the viscosity coefficients according to Eckart’s constitutive relations.

Overall, the theories analysed here are all quite peculiar and, in some cases, even pathological. This confirms the expected conclusion that, in the thermodynamical formalism, general relativity does retain a special status as an equilibrium state, just as it does in the landscape of gravity theories.

Other theories of gravity could be examined from the point of view of the effective thermodynamics, provided that their field equations can be written as effective Einstein equations with effective dissipative fluids and that the Eckart constitutive relations deliver a meaningful effective temperature. It is easy to include in the list Rastall theory, which has seen a recent resurgence of in-

terest: it is shown in [55] that this theory is just GR with a cosmological constant, so we have trivially $\mathcal{KT} = 0$. Similarly, Eddington-inspired Born-Infeld gravity is very similar to Palatini $f(R)$ gravity and *in vacuo* it is equivalent to GR plus a cosmological constant (see the discussion of [56]), so one expects a similar conclusion. Likewise, unimodular gravity [57–59] is equivalent to GR with Λ [60, 61], yielding $\mathcal{KT} = 0$. Many, more complicated, theories of gravity have been proposed in the literature (*e.g.*, [62–67]) and it will require a lengthy and detailed analysis to assess whether it is possible to formulate a first-order thermodynamical description à la Eckart for them. They will be explored in future work.

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