Macroscopic behavior of the P_1 (β) and P_2 (distorted β) phases of superfluid ^{3}He: Soundlike excitations

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We describe the macroscopic behavior of superfluid ^{3}He in the P_1 (β) phase in an anisotropic aerogel. It turns out that the P_1 phase shares many features with superfluid ^{3}He-A$_1$. It exists only in an external magnetic field and has only one spin orientation in its superfluid condensate. In the P_1 phase, the direction of the external magnetic field, the preferred direction in orbit space $\hat{\mathbf{m}}$ and the average preferred direction of the silica strands ζ_i, are all parallel. While the preferred direction in orbit space and the average preferred direction of the silica strands are even under time reversal, the preferred direction $\hat{\mathbf{w}}$ in spin space is odd under time reversal. As new macroscopic variables compared with superfluid ^{3}He we have for the superfluid P_1 phase in an aerogel the strain field associated with the aerogel network. As a result of these additional macroscopic variables we find new static and dynamic cross-coupling terms, which come as reversible (zero entropy production) as well as as irreversible contributions. As an outstanding feature of the P_1 phase we find that second sound and, to a lesser extent, fourth sound assume spin-wave character, a feature that should be testable experimentally. This result closely parallels that for the A_1 phase of bulk superfluid ^{3}He in spite of the fact that the $\hat{\mathbf{l}}$ vector of the A_1 phase, which is odd under time reversal, does not exist in the P_1 phase. We also discuss briefly the macroscopic behavior of the distorted β (P_2) phase, which shares several features with the distorted A phase of bulk superfluid ^{3}He, in particular with respect to its properties in spin space.

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I. INTRODUCTION

Quite recently two new superfluid phases of ^{3}He have been discovered in strongly anisotropic aerogels [1], which both exist only in a sufficiently strong external magnetic field. They have been found using a vibrating aerogel resonator via the splitting of the superfluid phase transition into two discrete transitions as a function of temperature in a strong magnetic field. The two new superfluid phases have been denoted as β (or P_1) phase and as distorted β (or P_2) phase as one is cooling from the isotropic liquid phase. Their existence has been anticipated on the basis of a Landau energy for superfluid ^{3}He incorporating high magnetic fields [2,3]. The discovery followed earlier vibrating-wire experiments on the same type of mullite samples at lower magnetic fields [4] by the same group of authors.

The recent discovery of the P_1 and P_2 phases followed about five years after the detection of the polar phase, anticipated for a long time [5,6], using NMR [7] in strongly anisotropic aerogels called Nafen [8–10]. In addition, a distorted A and a distorted B phase [11–13] due to the strong influence of the anisotropic aerogel have been found [7,14] further elucidated experimentally in detail using cw and pulsed NMR techniques [14,15]. More recently, the polar phase has also been found [16] for a second class of strongly anisotropic aerogels called mullite, the same system that has been used in detecting the superfluid P_1 and P_2 phases.

In parallel to the various experimental developments, the modeling of the various superfluid phases in an anisotropic aerogel has been advanced [17–19]. In addition it had been shown previously that the polar phase could be stable in anisotropic aerogels [20].

With the discovery of the polar phase and the distorted A and distorted B phase the situation for superfluid ^{3}He in an anisotropic aerogel closely resembled that of ^{3}He-A and B in the bulk. However, what was missing was the analog of bulk ^{3}He-A_1, which exists only in a (strong) magnetic field. It has only one spin projection, that is, only up-up or down-down pairs [5,6]. And this missing link is the P_1 phase, which has also only pairs with one spin projection [1] and exists only in an anisotropic aerogel and an external magnetic field.

In contrast with the A_1 phase, however, the orbit part of the order parameter in the P_1 phase—and also of the P_2 phase—is the same as for the polar phase. This in turn leads to different macroscopic properties when comparing the P_1 and P_2 phases with ^{3}He-A_1 and ^{3}He-A in high magnetic fields (sometimes called ^{3}He-A_2 phase). It is therefore the goal of this paper to derive the macroscopic dynamics of P_1 and P_2 and to

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elucidate the similarities and the differences to ^{3}He -A_1 and ^{3}He -A in high magnetic fields. To achieve this we draw on our recent paper on the macroscopic behavior of the polar phase in anisotropic aerogels [21], as well as on previous work on the macroscopic behavior of the A_1 phase [22–25] and the A phase in high magnetic fields in the bulk [26,27].

The paper is organized as follows: In Sec. II we present the relevant variables due to hydrodynamic conservation laws, including the elastic and magnetic ones (II A), due to broken gauge (II B) and rotational symmetries, in particular those resulting from the structures of the P_1 phase (II C) and the P_2 phase (II D), and finally due to the slowly relaxing rotations (II E). The macroscopic orbital dynamics of the P_1 phase is discussed in detail in Sec. III, also including spin-orbit coupling (III G) and sound spectra (III E and III F). In Sec. IV we analyze selected aspects of the macroscopic behavior of the P_2 phase, including sound spectra (IV D) and spin-orbit coupling (IV E). In Sec. V we give a summary.

II. THE RELEVANT MACROSCOPIC VARIABLES FOR THE P_1 AND P_2 PHASES

A. Hydrodynamic variables

In this paper we use linearized hydrodynamics [28] to describe the macroscopic behavior of superfluid ^{3}He in the P_1 and P_2 phases. We derive the balance equations describing the behavior of the system in the low-frequency, long-wavelength limit. Low frequencies in this context mean small compared with all collisional frequencies while wavelengths are considered to be long if they are large compared with all microscopic lengths. Naturally these conditions for the purely hydrodynamic regime impose rather severe constraints on the frequencies and wave vectors for which this approach is strictly valid. Nevertheless, the hydrodynamic description and its generalization to include variables that relax on a long but finite timescale have turned out to be rather useful [29].

The conserved quantities in superfluid ^{3}He are ρ (mass density), ε (energy density), and g_i (momentum density), just as in any normal fluid. The entropy density σ is conserved only for reversible processes and contains the entropy production of irreversible processes as a source term. They act as hydrodynamic variables and are related to the energy density by

$$d\varepsilon_h = T d\sigma + \mu d\rho + \mathbf{v^n} \cdot d\mathbf{g}, \tag{1}$$

with μ being the chemical potential, T the temperature, and v_i^n the normal velocity.

Since the P_1 and P_2 phases only exist in the presence of an anisotropic aerogel, we have to take into account its elastic strain ϵ_{ij} as a hydrodynamic variable. The aerogel breaks translational symmetry, which gives rise to a translation vector as a symmetry variable. To exclude homogeneous translations and homogeneous rotations, the strain symmetric and is, to linear order, given by $\epsilon_{ij} = (1/2)(\nabla_i u_j + \nabla_j u_i)$. Elastic strains enter the energy density by

$$d\varepsilon_e = \Psi_{ij} d\epsilon_{ij}, \tag{2}$$

thereby defining the elastic stresses Ψ_{ij}.

The anisotropy of the elastic network is described by a unit vector ζ_i. Its role as a variable is discussed in Sec. II E.

Since the ^{3}He atoms have spin $1/2$ each, they give rise to a magnetization density M_ν. The frame to describe the orientation of spins is *a priori* not the same as that of, e.g., the flow variables. Therefore, it is customary to use in "spin space" a different Cartesian frame indicated by Greek indices. The magnetization enters the energy density by

$$d\varepsilon_M = (h_\nu - H_\nu) dM_\nu, \tag{3}$$

thereby defining the internal field h_ν. The external field is H_ν. In equilibrium $M_\nu^0 = H_\nu$.

B. Order parameter for superfluid ^{3}He

In superfluid ^{3}He the neutral ^{3}He atoms combine to form Cooper pairs similar to those found in superconductors which can be viewed as composite bosons. While the electrons in conventional superconductors are in a spin-singlet s-wave state, the ^{3}He atoms are in a spin-triplet p-wave state. This fact clearly distinguishes the two situations: The pair of electrons has no internal structure, but the pair of ^{3}He atoms is intrinsically anisotropic. Because of the spin-triplet and p-wave pairing the order parameter $T_{\nu j}$ has to be a complex 3×3 matrix whose expectation value can in general be written as [5,30]

$$\langle T_{\nu j}(|\mathbf{c}|, \mathbf{r})\rangle = \mathcal{F}(||\mathbf{c}||) A_{\nu j}(\mathbf{r}) e^{i\varphi(\mathbf{r})}, \tag{4}$$

where ν is an index in spin space, j is an index in orbital space, $\mathbf{r}$ is the position vector of the center of gravity, and $\mathbf{c}$ is the relative vector between the two ^{3}He atoms.

The macroscopic properties of the different ^{3}He phases are represented by the matrix $A_{\nu j}$. The normalization amplitude $\mathcal{F}$ describes the degree of ordering and is considered as a microscopic variable, which does not appear in the macroscopic dynamics of the system. Therefore, one can impose the restriction of normalization

$$A_{\nu j} A^*_{\nu j} = 1. \tag{5}$$

The overall phase factor is characteristic for superfluids and superconductors because it reflects the fact that gauge invariance is spontaneously broken and the phase variation $\delta\varphi$ has to be used in the hydrodynamic description. Therefore, we will use the superfluid velocity

$$v_i^s = \frac{\hbar}{2m_H} \nabla_i \varphi \tag{6}$$

according to the two-fluid model developed by Landau [28] and Khalatnikov [31] to describe the macroscopic behavior of the system (m_H is the bare mass of ^{3}He).

In equilibrium the superfluid velocity is zero and a finite superfluid velocity gives a contribution to the free-energy density

$$d\varepsilon_\lambda = \lambda_i^s dv_i^s = (\hbar/2m_H)\lambda_i^s d\nabla_i\varphi, \tag{7}$$

with λ_i^s being the conjugate quantity.

To make contact with a more microscopic description [30], we recall that the triplet pairing in the superfluid phases of ^{3}He can be described by the matrix of anomalous expectation values

$$\langle \hat{T}_{\nu j}(|\mathbf{c}|,\mathbf{r})\rangle = \mathrm{Tr}\{\hat{\rho}\hat{T}_{\nu j}(|\mathbf{c}|,\mathbf{r})\}, \tag{8}$$

where the order parameter $\hat{T}_{\nu j}$ is defined by

$$\hat{T}_{\nu j}(|\mathbf{c}|,\mathbf{r})\rangle = \frac{3}{4\pi}\int d\Omega_c \hat{\psi}_\alpha\left(\mathbf{r}-\frac{1}{2}\mathbf{c}\right)(\sigma_\nu\sigma_2)_{\alpha\beta} \times\frac{c_j}{|\mathbf{c}|}\hat{\psi}_\beta\left(\mathbf{r}+\frac{1}{2}\mathbf{c}\right), \tag{9}$$

which includes an integration over the solid angle of the relative coordinate. $\hat{\psi}_\alpha$ is the fermion operator annihilating a ^{3}He particle with spin α, and σ_ν are the Pauli matrices.

The restricted ensemble $\hat{\rho}$ in Eq. (8) is

$$\hat{\rho} = Z^{-1}\exp(-\beta_0\hat{H}_r), \tag{10}$$

with Z being the proper normalization and

$$\hat{H}_r = \hat{H} - \mu m_H\hat{N} - \mathbf{v}^\mathbf{n}\cdot\hat{\mathbf{P}} - \gamma\mathbf{h}\cdot\hat{\mathbf{S}} - \eta\int d^3r[\lambda^*_{\nu j}(\mathbf{r})\hat{A}_{\nu j}(\mathbf{r}) + \lambda_{\nu j}(\mathbf{r})\hat{A}^\dagger_{\nu j}(\mathbf{r})], \tag{11}$$

where $\hat{H}$ is the Hamiltonian, $\hat{N}$ is the number operator, $\hat{\mathbf{P}}$ is the total momentum operator, $\hat{\mathbf{S}}$ is the spin operator, and

$$\hat{A}_{\nu j}(\mathbf{r}) = \int_0^\infty c^2F^*(c)\hat{T}_{\nu j}(|\mathbf{c}|,\mathbf{r})dc \tag{12}$$

with the normalization $\int_0^\infty c^2F^*(c)F(c)dc = 1$. The quantities μ (chemical potential), $\mathbf{v}^n$, $\mathbf{h}$, β_0 (Boltzmann factor), and $\eta\lambda_{ij}$ act as Lagrange parameters with m_H being the bare ^{3}He mass and γ the gyromagnetic ratio [30].

C. Structural variables for the P_1 phase

In the P_1 phase, the matrix $A_{\nu j}$ defined in Eq. (4) can be written in the non-normalized form

$$A_{\nu j}(\mathbf{r}) = \Delta_1\hat{v}_\nu(\mathbf{r})\hat{m}_j(\mathbf{r}), \tag{13}$$

with the gap function Δ_1. The unit vector $\hat{m}_i$ is the preferred direction in orbital space breaking rotational symmetry. The orbital part of the order parameter in the P_1 phase is identical to the orbit part of the polar phase.

Since there is only $m_s = +1$ (or $m_s = -1$) pairing in the P_1 phase, one can write

$$\hat{v}_\nu(\mathbf{r}) = \frac{1}{\sqrt{2}}(\hat{d}_\nu(\mathbf{r}) + i\hat{e}_\nu(\mathbf{r})), \tag{14}$$

with $\mathbf{d}$ and $\mathbf{e}$ being mutually orthogonal unit vectors in spin space, which are perpendicular to the external magnetic field in equilibrium. They allow the introduction of the real unit vector as the preferred direction in spin space (indicating spontaneously broken rotational symmetry):

$$\hat{\mathbf{w}} = i(\hat{\mathbf{v}}\times\hat{\mathbf{v}}^*), \tag{15}$$

which is odd under time reversal and is an axial vector. Thus, $\hat{\mathbf{w}}$ can be introduced for the P_1 phase in the same spirit as in the A_1 phase with

$$d\varepsilon_w = \Phi^w_{\nu i}d\nabla_i w_\nu \tag{16}$$

defining the conjugate quantity $\Phi^w_{\nu i}$.

As m_i follows from Eq. (13) via the contraction $A_{\nu i}A_{\nu j}\sim m_im_j$, a form similar to the nematic order parameter, it is obvious that m_i is not really a vector but a director, meaning a substitution of m_i with $-m_i$ must not change the hydrodynamics. It can be viewed as a direction that cannot distinguish head from tail. Breaking spontaneously rotational symmetry in orbital space, rotations give rise to two hydrodynamic variables in orbital space that enter the energy density

$$d\varepsilon_p = \Phi^m_{ij}d\nabla_j m_i \tag{17}$$

defining the conjugate quantity Φ^m_{ij}.

Both preferred directions are even under space inversion, while $\hat{\mathbf{m}}$ is also even under time reversal. This is in contrast with the A phase, where the preferred direction in orbital space is odd under time reversal. Since m_i and w_ν are unit vectors, there is $m_i\nabla_j m_i = 0 = w_\nu\nabla_\mu w_\nu$.

The small dipole interaction leads to a coupling of the preferred directions in orbital and spin space such that $w_i \parallel m_i$. As a consequence also the preferred direction of the anisotropic network ζ_i and the external magnetic field H_ν are parallel to that direction in equilibrium, rendering the P_1 phase uniaxial. In the following we disregard the dipole coupling and comment on its effect briefly in Sec. III G.

Returning to the operator representation we have for the variables in orbit space the following Hermitian operators [30]:

$$\delta\hat{m}_i = \epsilon_{ijk}\delta\hat{\Theta}_j m^0_k, \tag{18}$$

$$\delta\hat{\Theta}_i(\mathbf{r}) = \frac{1}{2}\epsilon_{ijk}\left[A^{*0}_{\nu j}\hat{A}_{\nu k}(\mathbf{r}) + A^0_{\nu j}\hat{A}^+_{\nu k}(\mathbf{r})\right], \tag{19}$$

$$\delta\hat{\varphi}(\mathbf{r}) = \frac{1}{2i}\left[A^{*0}_{\nu j}\hat{A}_{\nu j}(\mathbf{r}) - A^0_{\nu j}\hat{A}^+_{\nu j}(\mathbf{r})\right]. \tag{20}$$

They satisfy a number of commutation relations with the magnetization $\gamma\hat{S}_\nu = \hat{M}_\nu$, the angular momentum $\hat{L}_i$, and the particle number $\hat{N}$:

$$\langle[\delta\hat{m}_i,\hat{M}_\nu]\rangle = 0, \tag{21}$$

$$\langle[\delta\hat{\varphi},\hat{M}_\nu]\rangle = i\gamma\frac{H_\nu}{|\mathbf{H}|}, \tag{22}$$

$$\langle[\delta\hat{m}_i,\hat{L}_j]\rangle = -i\hbar\epsilon_{ijk}m^0_k, \tag{23}$$

$$\langle[\delta\hat{\varphi},\hat{L}_j]\rangle = 0, \tag{24}$$

$$\langle[\delta\hat{m}_i,\hat{N}]\rangle = 0, \tag{25}$$

$$\langle[\delta\hat{\varphi},\hat{N}]\rangle = -2i, \tag{26}$$

from which we make a number of important observations. From Eqs. (21), (23), and (25) we see that $\hat{m}_i$ commutes with rotations in spin space and is invariant under gauge transformations, while it describes rotations in orbit space. This is the behavior one expects for a director-like quantity. We see from Eq. (26) that $\delta\hat{\varphi}$ breaks gauge invariance as expected for a superfluid. In addition it is invariant under rotations in real space. The most important result, however, follows from Eq. (22): $\delta\hat{\varphi}$ is conjugate to the magnetization in the direction of the external field, necessary for the P_1 phase to exist. The latter effect is reminiscent of a similar behavior known from the A_1 phase of bulk superfluid ^{3}He. In the following we show that this coupling leads to important consequences for the velocities of second and fourth sound: These hydrodynamic excitations acquire spin-wave character, a feature unknown from the polar phase without external magnetic field.

D. Structural variables of the P_2 phase

In the P_2 phase (sometimes also called the distorted β phase [1]) the orbit part of the matrix $A_{\nu j}$ is unchanged, while the spin part is changing when compared with the P_1 phase:

$$A_{\nu j} = m_j \frac{1}{\sqrt{2(\Delta_1^2 + \Delta_2^2)}}[(\Delta_1 + \Delta_2)\delta_{\nu y} + i(\Delta_1 - \Delta_2)\delta_{\nu x}], \tag{27}$$

where we have now two gap parameters. It is easily checked that the expression given in Eq. (27) is equivalent to Eq. (2) of Ref. [1].

In the limit $\Delta_1 = 0$ (or $\Delta_2 = 0$), we obtain the result for the P_1 phase for which one has only up-up or down-down pairs in the spin projection. In particular, there is the preferred direction $\hat{\mathbf{w}}^0$ parallel to the magnetic field. For $\Delta_1 \to \Delta_2$ (or magnetic field $H \to 0$) we obtain the corresponding result for the polar phase, i.e., a preferred direction $\hat{\mathbf{d}}^0$, which is perpendicular to $\hat{\mathbf{w}}^0$ (and to the field). As a result there are two preferred spin space directions in the P_2 phase. To be definite we have chosen a fixed coordinate system with $\hat{\mathbf{w}}^0 \parallel \hat{e}_z$ and $\hat{\mathbf{d}}^0 \parallel \hat{e}_y$.

There is only one rotation of the $\hat{\mathbf{w}}^0/\hat{\mathbf{d}}^0$ system that breaks rotational symmetry spontaneously, δd_α, which is orthogonal to both $\hat{\mathbf{w}}^0$ and $\hat{\mathbf{d}}^0$. It gives rise to a hydrodynamic variable $\delta n \equiv -\epsilon_{\alpha\beta\gamma}\delta d_\alpha \hat{d}^0_\beta \hat{w}^0_\gamma$ that enters the free energy:

$$d\varepsilon_n = \psi_i d\nabla_i n, \tag{28}$$

defining the conjugate ψ_i. Since n is a scalar, this degree of freedom also enters the orbit dynamics of the P_2 phase. It transforms even under parity and odd under time reversal.

The orbit part of the order parameter is the same for the P_1 and P_2 phases [1–3] (as well as for the polar phase [5,6,29,32]). They all have a line node, since the energy gap is zero in the plane normal to the preferred direction m_j. Breaking spontaneously the rotational symmetry in orbit space, rotations of m_i give rise to two hydrodynamic variables, as in the P_1 phase, Eq. (17):

$$d\varepsilon_A = \Phi^m_{ij} d\nabla_j m_i, \tag{29}$$

defining the conjugate quantity Φ^m_{ij}. Thus we conclude that the macroscopic variables in orbit space are the same in the P_1 and P_2 phase, except for n that does not exist in the P_1 phase.

Spin-orbit coupling effects are briefly discussed in Sec. IV E.

The commutation relations of the P_1 phase, Eqs. (21)–(26), can be taken over to the P_2 phase, with the exception of Eq. (22), which is replaced by

$$\langle[\delta\hat{\varphi}, \hat{M}]\rangle = -i\gamma \frac{2\alpha\beta}{\alpha^2 + \beta^2} \equiv -i\gamma\beta_1, \tag{30}$$

involving the longitudinal magnetization $\hat{M} = \hat{M}_\nu \hat{w}_\nu$ and the phase variations, with $\alpha = |\Delta_1 - \Delta_2|$ and $\beta = \Delta_1 + \Delta_2$. For $H \to 0$, we obtain $\alpha = 0$ and thus the result for the polar phase, while for Δ_1 (or $\Delta_2 = 0$) the result for the P_1 phase is regained.

E. Slowly relaxing rotations relative to the elastic network

As an additional input from experimental results it is known that, in the polar phase [7,14] as well as in the P_1 and P_2 phases [1], the preferred direction in orbit space, m_i, is parallel to ζ_i, the averaged strand direction, which is the preferred anisotropy direction of the aerogel. This condition is not due to a broken symmetry, but results from molecular interaction forces. Therefore, deviations from $m_i^0 \parallel \zeta_i^0$, e.g., $\delta m_i - \delta\zeta_i$ are relaxing variables, which, however, interact with the hydrodynamic variables on timescales shorter than the relaxation times. We take into account these slowly relaxing relative rotations as has been done, e.g., for nematic liquid crystal elastomers by de Gennes [33].

This variable can be written as

$$\Omega_i = \delta(\mathbf{m} \times \boldsymbol{\zeta})_i = \epsilon_{ijk} m_j^0 (\delta\zeta_k - \delta m_k) \tag{31}$$

because of $m_i^0 \parallel \zeta_i^0$. Another possible definition is

$$\Omega_i = \delta m_i - \tfrac{1}{2}\zeta_j^0(\nabla_i u_j - \nabla_j u_i) \tag{32}$$

involving rotations expressed by antisymmetric gradients of the elastic translation vector.

The variables Ω_i include the orientational dynamics of the preferred direction of the strands and are independent of the elastic deformations that only contain symmetric gradients of u_i.

The relative rotations lead in the energy density to

$$d\varepsilon_r = W_i d\Omega_i, \tag{33}$$

defining the conjugates W_i.

III. THE (ORBITAL) DYNAMICS OF THE P_1 PHASE

In this section we present the derivation of the macroscopic dynamics of the P_1 phase mainly in orbit space. Spin space dynamics will be discussed in connection with spin-orbit coupling in Sec. III G. The procedure follows closely earlier macroscopic descriptions of the superfluid ^{3}He phases in the bulk, i.e., the A phase [26,30,34–36], the B phase [30,37], as well as the phases in high magnetic fields, i.e., the A_1 phase [22–24] and the A phase in strong magnetic fields [26,27]. Quite recently such a program has been carried out [21] for the polar phase and for the distorted A and B phases in anisotropic aerogels.

A. Statics and thermodynamics

To obtain the static properties of our system we formulate the local first law of thermodynamics relating changes in the entropy density σ to changes in the hydrodynamic and macroscopic variables discussed above. According to the discussions in Sec. II, Eqs. (1)–(3), (7), (17), and (33), we get the Gibbs relation for the variables acting in orbit space [38,39]:

$$\begin{aligned} d\varepsilon = {} & T d\sigma + \mu d\rho + v_i^n dg_i + \lambda_i^s dv_i^s + (h - H)dM \\ & + \Phi^m_{ij} d(\nabla_j m_i) + \Psi_{ij} d\epsilon_{ij} + W_i d\Omega_i, \end{aligned} \tag{34}$$

with $M = M_\nu \hat{w}_\nu$. Of the magnetic degrees of freedom we have kept only the longitudinal one, since it is a scalar variable and therefore effective also in orbit space. For the spin space part, cf. Sec. III G. The thermodynamic conjugates are defined as variational or partial derivatives of the energy density with respect to the appropriate variables [38].

Let us list the symmetry properties used: Scalar quantities are ε, σ, ρ, and M, while g_i and v_i^s are polar vectors, Ω_i is an axial vector, and ε_{ij} and $\nabla_j m_i$ are tensors of second rank. Even under time reversal are ε, σ, M, Ω_i, ε_{ij}, and $\nabla_j m_i$, while g_i and v_i^s are odd under time reversal. Odd under parity are v_i^s, g_i, and $\nabla_j m_i$, while ε, σ, ρ, ε_{ij}, and Ω_i are even under parity. The behavior of the thermodynamic conjugates defined via Eq. (34) under time reversal and parity can then be read off immediately.

To determine the thermodynamic conjugate variables we need an expression for the local energy density. This energy density must be invariant under time reversal as well as under parity and it must be invariant under rigid rotations and rigid translations, and covariant under Galilei transformations. In addition to that, this energy density must have a minimum, because there exists an equilibrium state for the gel. Therefore, the expression for the energy density needs to be convex. Taking into account these symmetry arguments we write down an expansion for the generalized energy density up to second order in the variables that describe deviations out of that equilibrium:

$$
\begin{aligned}
\varepsilon = {} & \frac{1}{2}\rho_0\left(\frac{\rho^s}{\rho^n}\right)_{ij} v_i^s v_j^s + \frac{1}{2}\left(\frac{1}{\rho^n}\right)_{ij} g_i g_j - \left(\frac{\rho^s}{\rho^n}\right)_{ij} v_i^s g_j \\
& + \frac{1}{2}\bar{\mu}_{ijkl}\epsilon_{ij}\epsilon_{kl} + \frac{1}{2}\chi^{-1}(\delta M)^2 + c_{M\sigma}(\delta M)(\delta\sigma) \\
& + c_{M\rho}(\delta M)(\delta\rho) + \frac{1}{2}K_{ijkl}(\nabla_j m_i)(\nabla_l m_k) + \frac{1}{2}D_1\Omega_i\Omega_i \\
& + D_2\left(m_j^0\delta_{ik}^{\perp} + m_k^0\delta_{ij}^{\perp}\right)\Omega_i\epsilon_{jk} + \sigma_{ijk}^{\sigma}(\nabla_j m_i)(\nabla_k\sigma) \\
& + \sigma_{ijk}^{\rho}(\nabla_j m_i)(\nabla_k\rho) + \sigma_{ijk}^{M}(\nabla_j m_i)(\nabla_k M) \\
& + \epsilon_{ij}\left(\chi_{ij}^{\sigma}\delta\sigma + \chi_{ij}^{\rho}\delta\rho + \chi_{ij}^{M}\delta M\right) \\
& + \frac{1}{2}c_{\rho\rho}(\delta\rho)^2 + \frac{1}{2}c_{\sigma\sigma}(\delta\sigma)^2 + c_{\rho\sigma}(\delta\rho)(\delta\sigma),
\end{aligned}
\tag{35}
$$

with $\delta_{ij}^{\perp} = \delta_{ij} - m_i^0 m_j^0$. A "$\delta$" denotes deviations from the (constant) equilibrium value of the appropriate variable, e.g., $\delta\rho \equiv \rho - \rho_0$ with $\rho_0\delta_{ij} = \rho_{ij}^n + \rho_{ij}^s$.

Equation (35) contains the generalized energy density of the polar phase as a special case. In addition, we have the terms related to the longitudinal magnetization, M. Therefore we will focus in the following on the contributions involving M and refer for a discussion of all other terms in Eq. (35) to Ref. [21].

For the couplings between M and the strain field as well as for the coupling between the gradient of M and the gradient of the **m** vector, we have for the appropriate tensors the following form, respectively:

$$
\chi_{ij}^{M} = \chi_{||}^{M} m_i^0 m_j^0 + \chi_{\perp}^{M}\delta_{ij}^{\perp}, \tag{36}
$$

$$
\sigma_{ijk}^{M} = \sigma^{M}\left(\delta_{ik}^{\perp} m_j^0 + \delta_{ij}^{\perp} m_k^0\right). \tag{37}
$$

In the following we give the expressions for the conjugated variables in terms of the hydrodynamic and macroscopic variables. They are defined as the partial derivative of the energy density with respect to the appropriate variable, while all the other variables are kept constant, denoted by dots at the brackets in the following. We obtain for the contributions in Eq. (35) related to M,

$$
\Phi_{ij}^{m} = \left(\frac{\partial\varepsilon}{\partial(\nabla_j m_i)}\right)_{\dots} = \dots + \sigma_{ijk}^{M}\nabla_k M, \tag{38}
$$

$$
\Psi_{ij} = \left(\frac{\partial\varepsilon}{\partial\epsilon_{ij}}\right)_{\dots} = \dots + \chi_{ij}^{M}\delta M, \tag{39}
$$

$$
\begin{aligned}
\delta h = \left(\frac{\partial\varepsilon}{\partial\delta M}\right)_{\dots} = {} & \chi^{-1}\delta M + c_{M\sigma}\delta\sigma + c_{M\rho}\delta\rho \\
& - \sigma_{ijk}^{M}\nabla_j\nabla_k m_i + \chi_{ij}^{M}\epsilon_{ij},
\end{aligned}
\tag{40}
$$

$$
\delta T = \left(\frac{\partial\varepsilon}{\partial\delta\sigma}\right)_{\dots} = \dots + c_{M\sigma}\delta M, \tag{41}
$$

$$
\delta\mu = \left(\frac{\partial\varepsilon}{\partial\delta\rho}\right)_{\dots} = \dots + c_{M\rho}\delta M, \tag{42}
$$

with, e.g., $\delta T = T - T_0$ and $\delta h = h - H$. The ellipses denote those contributions that already exist in the polar phase, Eqs. (26)–(32) in Ref. [21].

B. Dynamic equations

To determine the dynamics of the variables we take into account that the first class of our set of variables, the conserved quantities, obey a local conservation law, while the dynamics of the other two classes of variables can be described by a simple balance equation, where the counter term to the temporal change of the quantity is called a quasicurrent. For the set of dynamical equations we get

$$
\dot{\rho} + \nabla_i g_i = 0, \tag{43}
$$

$$
\dot{\sigma} + \nabla_i j_i^{\sigma} = \frac{2R}{T}, \tag{44}
$$

$$
\dot{g}_i + \nabla_j(\delta_{ij}p + \sigma_{ij}) = 0, \tag{45}
$$

$$
\dot{m}_i + \epsilon_{ijk}m_j^0\omega_k + X_i^m = 0, \tag{46}
$$

$$
\dot{v}_i^s + \nabla_i I_\varphi = 0, \tag{47}
$$

$$
\dot{\epsilon}_{ij} + Y_{ij} = 0, \tag{48}
$$

$$
\dot{\Omega}_i + Z_i = 0, \tag{49}
$$

$$
\dot{M} + \nabla_i j_i^{M} = 0, \tag{50}
$$

with $\omega_i = (1/2)\epsilon_{ijk}\nabla_j v_k^n$ being the vorticity. The entropy production R/T, with R being the dissipation function, acts as a source term in Eq. (44). The pressure p in Eq. (45) is given by $-\partial E/\partial V$, with E being the total energy, cf. Ref. [38], and reads for our system

$$
p = -\varepsilon + \mu\rho + T\sigma + Mh + v_i^n g_i. \tag{51}
$$

The time derivatives in Eqs. (43)–(49) have the following behavior under time reversal: $\dot{\rho}$, $\dot{\sigma}$, $\dot{M}$, $\dot{m}_i$, $\dot{\varepsilon}_{ij}$, and $\dot{\Omega}_i$ are odd, while $\dot{g}_i$ and $\dot{v}_i^s$ are even under time reversal.

Now we can decompose all currents and quasicurrents listed in Eqs. (43)–(49) into reversible and irreversible contributions. To describe reversible dynamics, the reversible part of a currents (superscript R) must have the same behavior under time reversal as the time derivative of the appropriate variable in Eqs. (43)–(49). In contrast, the dissipative parts of the currents (superscript D) have the opposite sign under time reversal as the time derivatives of the variables. According to the second law of thermodynamics, the entropy production has to vanish identically for reversible processes, $R \equiv 0$, and the entropy obeys a conservation law, Eq. (44). For irreversible processes $R > 0$ is required. With the help of the full set of dynamic equations the Gibbs relation, Eq. (34) leads to an expression for the entropy production R/T, bilinear in the currents and thermodynamic conjugates. This can be used to impose the restrictions on the reversible and irreversible parts of the currents, separately, in particular on the form of cross-coupling terms.

Since we restrict ourselves to a linear description, we do not have to worry about which velocity should be chosen for the transport derivatives [40].

C. Reversible dynamics

Implementing the condition $R = 0$ and the required behavior under time reversal and parity, we obtain the following expressions for the reversible contributions to the currents:

$$g_i^R = \rho_0 v_i^n + \lambda_i^s, \tag{52}$$

$$j_i^{\sigma R} = T_0 \sigma_0 v_i^n, \tag{53}$$

$$\sigma_{ij}^R = -\Psi_{ij} - \lambda_{kji} \nabla_l \Phi_{kl}^m + \xi_{kji}^R W_k, \tag{54}$$

$$Y_{ij}^R = -A_{ij}, \tag{55}$$

$$X_i^{mR} = -\lambda_{ijk} A_{jk}, \tag{56}$$

$$I_\varphi^R = \mu + \tilde{\gamma}\delta h, \tag{57}$$

$$Z_i^R = -\xi_{ijk}^R A_{jk}, \tag{58}$$

$$j_i^{MR} = M_0 v_i^n + \tilde{\gamma}\lambda_i^s, \tag{59}$$

where $A_{ij} = (1/2)(\nabla_i v_j^n + \nabla_j v_i^n)$ and $\tilde{\gamma} = (\hbar/2m_H)\gamma$.

Inspecting Eqs. (54)–(58), we see that σ_{ij}^R and I_φ^R are even under time reversal, while Y_{ij}^R, X_i^{mR}, and Z_i^R are odd under time reversal—just as expected from the general analysis discussed above for the time derivative of the variables and their time-reversal properties.

The density current g_i is at the same time the momentum density and therefore cannot have dissipative contributions. It is part of the kinetic energy and has been given in Eqs. (35). The result for I_φ^R, Eq. (57), follows from the fact that φ is the canonical conjugate to the particle number [28,31] as well as to the component of the magnetization parallel to the external magnetic field, M [compare Eqs. (26) and (22)].

The material tensors λ_{ijk} and ξ_{ijk}^R describe the coupling of the stress tensor with the quasicurrents X_i^{mR} and Z_i^R and have the form

$$\alpha_{ijk} = \alpha\left(m_k^0 \delta_{ij}^\perp + m_j^0 \delta_{ik}^\perp\right). \tag{60}$$

These tensors have to be symmetric in the last two indices, the first index has to be transverse to m_i, and they must contain an odd number of m_i factors because m_i is a director.

We also point out that—except for the terms related to superfluidity and to M—the reversible currents for the P_1 phase are isomorphic to those given in uniaxial nematic elastomers [41].

D. Irreversible dynamics and entropy production

We can use the dissipation function R as a Lyapunov functional to derive the irreversible currents and quasicurrents. This automatically includes the famous reciprocity rules for dissipative cross-couplings [39]. One can expand the function R (R/T is the amount of entropy produced within a unit volume per unit time) into the thermodynamic forces using the same symmetry arguments as in the case of the energy density. We obtain

$$\begin{aligned} R = {} & \tfrac{1}{2}\kappa_{ij}(\nabla_i T)(\nabla_j T) + \tfrac{1}{2}\nu_{ijkl} A_{ij} A_{kl} \\ & + \tfrac{1}{2}\mu_{ij}(\nabla_i h)(\nabla_j h) + \kappa'_{ij}(\nabla_i h)(\nabla_j T) \\ & + \tfrac{1}{2}\xi_{ij}(\nabla_k \Psi_{ik})(\nabla_l \Psi_{jl}) + \xi_{ij}^T(\nabla_i T)(\nabla_k \Psi_{jk}) \\ & + \zeta\left(\nabla_i \lambda_i^s\right)\left(\nabla_j \lambda_j^s\right) + \zeta_{ij}^n A_{ij}\left(\nabla_k \lambda_k^s\right) \\ & + \tfrac{1}{2} b \delta_{ij}^\perp \left(\nabla_l \Phi_{il}^m\right)\left(\nabla_m \Phi_{jm}^m\right) + \xi^m \delta_{ij}^\perp W_i \left(\nabla_l \Phi_{jl}^m\right) \\ & + \tfrac{1}{2}\tau \delta_{ij}^\perp W_i W_j + \xi_{ijk}^D (\nabla_i W_k)(\nabla_l \Psi_{jl}). \end{aligned} \tag{61}$$

Since there is also considerable overlap between the dissipation functions of the polar phase and the P_1 phase, respectively, we focus on the following on the dissipative contributions involving M and refer for all other contributions to Ref. [21]. For the two additions involving $\nabla_i h$, the second-rank tensors μ_{ij} and κ'_{ij} take the form

$$\alpha_{ij} = \alpha_\parallel m_i^0 m_j^0 + \alpha_\perp \delta_{ij}^\perp. \tag{62}$$

To obtain the dissipative parts of the currents and quasicurrents we take the partial derivatives with respect to the appropriate thermodynamic force. We find for the additional contributions due to the longitudinal magnetization, M

$$j_i^{\sigma D} = -\left(\frac{\partial R}{\partial(\nabla_i T)}\right)_{\dots} = -\kappa'_{ij}\nabla_j h, \tag{63}$$

$$j_i^{MD} = -\left(\frac{\partial R}{\partial(\nabla_i h)}\right)_{\dots} = -\mu_{ij}\nabla_j h - \kappa'_{ij}\nabla_j T. \tag{64}$$

In this section we have so far taken into account (as macroscopic variables with a finite, but sufficiently long relaxation time) the relative rotations between the preferred direction for the aerogel, ζ_i, and the preferred direction in orbit space, m_i. Should this relaxation time be very short, the relative rotations are no longer macroscopic variables, meaning $\dot{\Omega}_i = 0$ macroscopically, and therefore $W_i = 0$. In this case the cross-coupling terms between the relative rotations and the other macroscopic variables no longer exist. This affects in the reversible dynamics, Eq. (54), the coupling to the stress tensor ($\xi_{kji}^R = 0$), and in the dissipative dynamics, Eq. (61), the coupling terms to strains ($\xi^D_{ijk} = 0$) and to gradients of the vector m_i, ($\xi^m = 0$). In the statics, $W_i = 0$ leads to an irrelevant renormalization of the elastic tensor μ_{ijkl} in Eq. (35).

E. On the dispersion relation for fourth sound

Fourth sound is obtained when normal flow of the liquid is suppressed and one has $\mathbf{v}^n \equiv 0$. In this case one has as macroscopic variables σ, $\delta\varphi$, ρ, δm_i, and the longitudinal magnetization M. To lowest order in k ($\omega \sim k$) the director variables δm_i are decoupled from the rest. In addition, δm_i does not lead to a propagating mode, just like the director in nematic liquid crystals. We then obtain for the dispersion relation of fourth sound

$$\omega_4^2 = \hat{\rho}_s^2 N + \frac{i}{2}\Big(\zeta\hat{\rho}_s^2 + \frac{P}{N}\Big), \tag{65}$$

$$\hat{\rho}_s^2 = \rho^s_{\parallel} k_{\parallel}^2 + \rho^s_{\perp} k_{\perp}^2, \tag{66}$$

$$N = 2\tilde{\gamma} c_{M\rho} + \tilde{\gamma}^2\chi^{-1} + c_{\rho\rho}, \tag{67}$$

$$\begin{aligned} P = {}& \hat{\mu}^2\big(\tilde{\gamma}^2\chi^{-2} + 2\tilde{\gamma}\chi^{-1}c_{M\rho} + c_{M\rho}^2\big) \\ &+ \hat{\kappa}^2\big(\tilde{\gamma}^2 c_{M\sigma}^2 + 2\tilde{\gamma} c_{M\sigma}c_{\rho\sigma} + c_{\rho\sigma}^2\big) \\ &+ 2\hat{\kappa}'^2(\tilde{\gamma}^2\chi^{-1}c_{M\sigma} + \tilde{\gamma} c_{\rho\sigma}\chi^{-1} \\ &+ \tilde{\gamma} c_{M\sigma}c_{M\rho} + c_{M\rho}c_{\rho\sigma}), \end{aligned} \tag{68}$$

with $\hat{\pi}^2 = \pi_{\parallel} k_{\parallel}^2 + \pi_{\perp} k_{\perp}^2$ for $\pi \in \{\kappa, \kappa', \mu\}$ and $\tilde{\gamma} = \frac{\hbar}{2m}\gamma$.

The influence of the strain field ε_{ij} on fourth sound has been neglected so far. Since the aerogel network cannot move, we assume $\dot{\varepsilon}_{ij} = 0$ and there is no transverse elastic propagating mode, nor an elastic contribution to first sound velocity. However, there are elastic strains coupled statically to the scalar variables, expressed by $\approx\chi^{\xi}_{ij}$ for $\xi = \rho$, σ, or M in Eq. (35). As a result, all static cross-couplings among these variables, $c_{\rho\rho}$, $c_{\rho\sigma}$, $c_{M\rho}$, $c_{\sigma\sigma}$, $c_{M\sigma}$, and $c_{MM} \equiv \chi^{-1}$ are renormalized, e.g., $c_{M\sigma}$ is replaced by

$$c_{M\sigma} \to c_{M\sigma} + \sum_{ab} \chi^M_a \chi^\sigma_b (\bar{\mu}^{-1})_{ab}, \tag{69}$$

with $\{a, b\} \in \{\perp, \parallel\}$, and $\bar{\mu}_{ijkl}$ the elastic tensor. The replacements of Eq. (69) apply to all propagating modes in the P_1 and P_2 described below.

Inspection of the velocity of fourth sound Eq. (65) shows that in the P_1 phase acquires a spin-wave contribution—just as for the A_1 phase in bulk superfluid ^{3}He [23]. We note that ω_4 contains the velocity of fourth sound for the polar phase as a special case in the limit of zero magnetization (formally $\gamma \to 0$). As it is easily checked, all dissipative channels except for those associated with the director variables δm_i contribute to the damping of fourth sound. This is common for hydrodynamic soundlike excitations and already well known for simple fluids [28] and superfluid ^{4}He [31].

We also notice the close similarity between the structure of the velocity of fourth sound in the A_1 phase and P_1 phase. In lowest order in k the $\hat{\mathbf{l}}$-vector in the A_1 phase as well as the $\hat{\mathbf{m}}$ vector in the P_1 phase are decoupled from the other variables, respectively.

On the other hand, there is a qualitative difference in the spectra between ^{3}He-A_1 and the P_1 phase. In ^{3}He-A_1 the $\hat{\mathbf{l}}$ vector is odd under time reversal allowing for orbit waves. In P_1 orbit waves cannot exist and are replaced by purely diffusive processes, since $\hat{\mathbf{m}}$ is even under time reversal and the stiffness coefficient for orbit waves (β in ^{3}He-A_1 and A) cannot exist.

F. On the dispersion relation for first and second sound

First and second sound are obtained when normal flow of the liquid is present and one has as macroscopic variables σ, v^s_i, ρ, the longitudinal magnetization M and $\nabla_i g_i$ to lowest order in k ($\omega \sim k$). In this case the director variables δm_i and the vorticity $\epsilon_{ijk}\nabla_j v^n_k$ are decoupled and one obtains the velocities of first and second sound. Neglecting the static couplings $c_{\rho\sigma}$ (between $\delta\rho$ and $\delta\sigma$) and $c_{M\rho}$ (between $\delta\rho$ and δM), rather simple expressions for the velocities are found for second sound:

$$\omega_2^2 = \hat{\rho}_D^2 \frac{D}{\rho_0} \equiv C_{20}^2 k^2, \tag{70}$$

where

$$D = \sigma_0^2 c_{\sigma\sigma} + 2\sigma_0\gamma_1 c_{M\sigma} + \gamma_1^2\chi^{-1}, \tag{71}$$

$$\hat{\rho}_D^2 = \frac{\rho^s_{\parallel}}{\rho^n_{\parallel}} k_{\parallel}^2 + \frac{\rho^s_{\perp}}{\rho^n_{\perp}} k_{\perp}^2, \tag{72}$$

with $\gamma_1 = \rho_0\tilde{\gamma} - M_0$.

For first sound we find

$$\begin{aligned} \omega_1^2 = {}& \Big(\rho_0 c_{\rho\rho} - \frac{1}{\rho_0}\big[M_0^2\chi^{-1} - 2M_0\sigma_0 c_{M\sigma} + c_{\sigma\sigma}\sigma_0^2\big]\Big)k^2 \\ \equiv {}& C_{10}^2 k^2, \end{aligned} \tag{73}$$

with $M_0 = \chi H$. We note that, in this approximation, the velocity of first sound remains isotropic.

We emphasize that second sound acquires mainly spin-wave character, quite differently from the usual second sound in the polar phase or the superfluid phase of ^{4}He. Most likely the spin contributions to second sound dominate—just as is the case for superfluid ^{3}He-A_1—and we obtain from Eq. (70) the simplified expression

$$\omega_2^2 \approx \hat{\rho}_D^2\gamma_1^2/\rho_0\chi. \tag{74}$$

Without the approximations mentioned above to get first and second sound we obtain

$$\omega_1^2 + \omega_2^2 = C_{10}^2 k^2 + C_{20}^2 k^2, \tag{75}$$

$$\omega_1^2\omega_2^2 = C_{10}^2 C_{20}^2 k^4\Big(1 - \frac{\rho_0\Gamma}{D\Sigma}\Big), \tag{76}$$

where

$$\Gamma = (\gamma_1\varepsilon_2 - \sigma_0\varepsilon_1)^2, \tag{77}$$

$$\Sigma = c_{\rho\rho}\rho_0 + (c_{M\rho} + \varepsilon_2)M_0 + (c_{\sigma\rho} + \varepsilon_1)\sigma_0, \tag{78}$$

with

$$\rho_0\varepsilon_1 = c_{\sigma\sigma}\sigma_0 + c_{M\sigma}M_0 + c_{\sigma\rho}\rho_0, \tag{79}$$

$$\rho_0\varepsilon_2 = \chi^{-1}M_0 + c_{M\sigma}\sigma_0 + c_{M\rho}\rho_0. \tag{80}$$

We note that there are, up to now, no experiments on first and second sound—as well as on fourth sound—for the superfluid P_1 phase in uniaxial aerogels. Therefore, the results presented inSecs. III E and III F are predictions which should be tested experimentally.

Reviewing the experimental literature we see that, for bulk superfluid ^{3}He -A$_1$, second sound has been demonstrated to be predominantly a spin wave using NMR as well as an acoustic cell [42], following the suggestions of Liu [23]. Given this background, we conjecture that, for superfluid ^{3}He in anisotropic aerogels, second sound should also be the prime candidate to check the predictions made here experimentally.

G. The spin space dynamics and its coupling to orbit space

In the P_1 phase there are two different types of variables in spin space—the magnetization δM_ν and the rotations of the preferred axis $\hat{w}_\nu^0$, Eq. (16), δw_ν. The longitudinal magnetization is a scalar quantity that enters orbit space and its influence on the propagating soundlike excitations has been discussed in the previous two subsections. The transverse magnetization gives rise to Larmor precession about the magnetic field, which is strongly damped due to the strong field present. The symmetry variables δw_ν are due to a spontaneously broken rotational invariance in spin space and do not lead to a propagating mode, similar to the case of director rotations in nematic liquid crystals and to the rotations δm_i, Eq. (17), of the preferred direction in orbit space, m_i^0.

If spin-orbit coupling due to the tiny magnetic dipole-dipole interaction is taken into account, rotational symmetry breaking in spin and orbit space is no longer independent, but restricted by $\hat{w}_i^0 \parallel \hat{m}_i^0$, rendering the distinction between Greek and Latin indices obsolete. The spin-orbit coupling energy $\varepsilon_{so} \sim (\delta m_i - \delta w_i)^2$ gives rise to the coupling $\delta m_i = \delta w_i$ and to two coupled, nonpropagating, spin-orbit excitations with $\omega \sim k^2$.

IV. THE ORBITAL DYNAMICS OF THE P_2 PHASE

A. Statics and thermodynamics

The relevant variables to describe the orbital dynamics of the P_2 phase have been discussed in Sec. II. Thus we get for the Gibbs relation in the P_2 phase:

$$\begin{aligned} d\varepsilon = {} & T d\sigma + \mu d\rho + v_i^n dg_i + \lambda_i^s dv_i^s + (h - H)dM \\ & + \psi_i d\nabla_i n + \Phi_{ij}^m d(\nabla_j m_i) + \Psi_{ij} d\epsilon_{ij} + W_i d\Omega_i. \end{aligned} \quad (81)$$

Equation (81) is thus very similar to the Gibbs relation for the P_1 phase, Eq. (34), except for the additional variable n, which is associated with the additional broken symmetry in the P_2 phase in spin space but also enters the orbital dynamics, similarly to M, the longitudinal component of the magnetization.

Next we discuss the equations of state. We focus on the equations of state which change when going from the P_1 to the P_2 phase. To obtain explicit expressions for the thermodynamic conjugates in terms of the hydrodynamic and macroscopic variables we expand g_i, λ_i^s, and ψ_i in terms of v_i^n, v_i^s, and $\nabla_i n$ and obtain the result

$$g_i = \rho_{ij}^n v_j^n + \rho_{ij}^s v_j^s + \rho_{ij}^w \nabla_j n, \quad (82)$$

$$\lambda_i^s = \rho_{ij}^s \big(v_j^s - v_j^n\big) + \big(\rho_{ij}^w - \beta_1 \mu_{ij}\big)\big(\nabla_j n - \beta_1 v_j^s\big), \quad (83)$$

$$\psi_i = \rho_{ij}^w \big(v_j^s - v_j^n\big) + \mu_{ij}\big(\nabla_j n - \beta_1 v_j^s\big). \quad (84)$$

To arrive at Eqs. (82)–(84) we have used general symmetry arguments and that the Legendre transformed energy $d\tilde{\varepsilon} \equiv d\varepsilon - d(v_i^n g_i)$ [Eq. (81)] is a complete differential form. In addition we take into account the commutation relations discussed in section II D and the fact that g_i, the density of linear momentum, also serves as the current for ρ:

$$g_i = \rho_0 v_i^n + \lambda_i^s + \beta_1 \psi_i, \quad (85)$$

with $\rho_0 \delta_{ij} = \rho_{ij}^n + \rho_{ij}^s + \beta_1 \rho_{ij}^w$.

We note that $\frac{1}{2}\mu_{ij}(\nabla_i n)(\nabla_j n)$ is the gradient energy for deformations of the additional variable δn, due to the spontaneously broken continuous symmetry in spin space, with μ_{ij} of the standard uniaxial form.

For the static behavior of the P_2 phase it is important to emphasize that it contains the statics of the P_1 phase as a special case, when the additional variable is discarded. We also mention in passing that there is, for the static behavior, considerable overlap between the P_2 phase and the A phase in high magnetic fields [24,27]: when discarding all terms with the $\hat{\mathbf{l}}$ vector in Refs. [24,27], the results presented here are recovered.

B. Dynamic equations and reversible dynamics

The dynamic Eqs. (43)–(50) of the P_1 phase can be taken over for the P_2 phase. In addition, however, we have a dynamic equation for δn, the additional variable associated with the additional broken continuous rotational symmetry as discussed above:

$$\dot{n} + Y = 0. \quad (86)$$

For the reversible currents up to linear order in the thermodynamic forces we find for the changes compared with the P_1 phase, where ... refers to the contributions already present in the P_1 phase, Eqs. (52)–(59),

$$g_i^R = \cdots + \beta_1 \psi_i, \quad (87)$$

$$j_i^{\sigma R} = \cdots + \beta_2 \psi_i, \quad (88)$$

$$Y^R = \beta_1 \mu + \beta_2 T + \tilde{\gamma}\delta h, \quad (89)$$

$$j_i^{MR} = \cdots + \tilde{\gamma}\psi_i + \tilde{\gamma}\beta_1 \lambda_i^s, \quad (90)$$

$$I_\varphi^R = \cdots + \tilde{\gamma}\beta_1 \delta h, \quad (91)$$

where β_1, Eq. (30), describes couplings between spin order and density, and $\beta_2 \equiv (T_0\sigma_0/\rho_0)\beta_1 + \tilde{\gamma} H$ between spin order and entropy density. They can be calculated from an equal-time commutator [27].

C. Irreversible dynamics and entropy production

Compared with the dissipation function for the P_1 phase we have additional contributions associated with n:

$$R_{P_2} = \cdots + \tfrac{1}{2}\nu(\nabla_j\psi_j)(\nabla_i\psi_i) + \eta_7(\nabla_i\psi_i)\big(\nabla_l\lambda_l^s\big) + \eta_{ij}^{(8)}(\nabla_k\psi_k)A_{ij}, \tag{92}$$

where … denotes all the terms already arising for the P_1 phase, which were given in Sec. III. The second-rank tensors $\eta_{ij}^{(8)}$ are of the standard uniaxial form [43]. From Eq. (92) we see that there are cross-coupling terms of n to the conjugate of the superfluid velocity as well as to symmetrized velocity gradients A_{ij}.

To obtain the dissipative parts of the additional currents and quasicurrents due to n when compared with P_1 we take the partial derivatives with respect to the appropriate thermodynamic forces

$$Y^D = -\nu\nabla_k\psi_k - \eta_7\nabla_k\lambda_k^s - \eta_{ij}^{(8)}A_{ij}, \tag{93}$$

$$I_\varphi{}^D = \cdots - \eta_7\nabla_k\psi_k, \tag{94}$$

$$\sigma_{ij}^D = \cdots - \eta_{ij}^{(8)}\nabla_k\psi_k. \tag{95}$$

D. Propagating modes in the P_2 phase: Selected aspects

In the present section we discuss briefly selected aspects of propagating modes in the P_2 phase. We focus exclusively of the velocities of these propagating modes. It turns out that, for the P_2 phase, the hydrodynamic variable δn associated with the spontaneously broken symmetry in spin space couples in an intricate manner to fourth sound ($\mathbf{v^n} \equiv 0$) on the one hand and to second sound ($\mathbf{v^n} \neq 0$) on the other.

We will give here explicitly the expressions for the coupled excitations of fourth sound and spin waves and discuss only qualitatively the coupling of spin waves to second sound.

To address this task it is useful to recognize the close similarity of the problem at hand to that of the superfluid ^{3}He-A phase in high magnetic fields, which has been studied in quite some detail in Refs. [27] and [24].

First we analyze the coupled excitations of fourth sound and spin waves. As macroscopic variables entering the picture to lowest order in the wave vector k we have the variations of the superfluid velocity v_i^s, the entropy density $\delta\sigma$, the density $\delta\rho$, the longitudinal component of the magnetization δM, and the broken-symmetry variable δn.

We obtain a quintic equation with one root $\omega \equiv 0$ and the equations for the sum and the product of fourth sound $\omega_4^2 = c_4^2k^2$ and spin waves $\omega_s^2 = c_s^2k^2$ read

$$\omega_4^2 + \omega_s^2 = \big[a\big(\rho_0k^2 - \hat{\rho}_n^2\big) + b\hat{\mu}^2\big], \tag{96}$$

$$\omega_4^2\omega_s^2 = (ab - c^2)\big[\hat{\mu}^2\big(\rho_0k^2 - \hat{\rho}_n^2\big) - \big(\hat{\rho}_w^2\big)^2\big], \tag{97}$$

where

$$a = c_{\rho\rho} + 2\beta_1\tilde{\gamma}c_{M\rho} + \beta_1^2\tilde{\gamma}^2\chi^{-1}, \tag{98}$$

$$b = \beta_2^2c_{\sigma\sigma} - 2\big(1-\beta_1^2\big)\beta_2\tilde{\gamma}c_{M\sigma} + \big(1-\beta_1^2\big)\tilde{\gamma}^2\chi^{-1}, \tag{99}$$

$$c = \beta_2(c_{\rho\sigma} + \beta_1\tilde{\gamma}c_{M\sigma}) + \big(1-\beta_1^2\big)(\tilde{\gamma}c_{M\rho} + \beta_1\tilde{\gamma}^2\chi^{-1}), \tag{100}$$

with $\hat{\pi}_q^2 = \pi_\parallel^qk_\parallel^2 + \pi_\perp^qk_\perp^2$ for $\pi \in \{\rho_n, \rho_w, \mu\}$ and $\parallel$ and $\perp$ referring to the preferred direction in orbit space, $\hat{m}_i^0$, which is also the elastic anisotropy axis. The static susceptibilities $c_{\rho\rho}$, $c_{\rho\sigma}$, $c_{M\sigma}$, and $c_{M\rho}$ have already been introduced for the P_1 phase in Eq. (35). Equations (96)–(100) show that $\hat{\mu}^2$, $\hat{\rho}_w^2$, and $\beta_1 \neq 1$, which are characteristic for the P_2 phase when compared with the P_1 phase, are instrumental for the existence of spin waves and their coupling to fourth sound even to lowest order in the wave vector k. The fundamental difference is that in the P_1 phase the hydrodynamic variable δn does not exist.

In closing this section we mention briefly that, for $v^n \neq 0$, one gets a mixture of second sound, first sound, and spin waves. And even neglecting cross-coupling terms between $\delta\rho$ and the other macroscopic variables, the resulting expressions for the velocities of second sound, c_2^2, and for the spin waves, c_s^2, are rather lengthy, and we therefore refrain from giving them explicitly. We note, however, that their structure is rather similar to that found for ^{3}He-A in high magnetic fields in Ref. [24].

E. Spin space dynamics and spin-orbit coupling

In spin space there are two orthogonal preferred directions, $\hat{d}_i^0$ and $\hat{w}_i^0$. Since $\hat{w}_i^0$ is parallel to the external field, only the rotation δn about the third direction $\hat{\mathbf{w}}^0 \times \hat{\mathbf{d}}^0$ is a hydrodynamic variable. It is a scalar quantity, independent of the relative orientation of spin and orbit space and therefore enters orbit space dynamics, as discussed in Secs. IV A–IV D. The other variables in spin space is the transverse magnetization, giving rise to a strongly damped Larmor precession.

The small spin-orbit coupling leads to $\hat{\mathbf{m}}^0$, $\hat{\mathbf{w}}^0$, $\mathbf{H}_0$ being all parallel and $\mathbf{d}^0$ perpendicular to them (we always consider the case $\boldsymbol{\zeta}^0 \parallel \hat{\mathbf{m}}^0$). As a result, the P_2 phase becomes biaxial. The main consequence of biaxiality is a more complicated structure of the material tensors, e.g., for symmetric second-rank tensors $\mu_{ij} = \mu_\parallel m_i^0m_j^0 + \mu_3d_i^0d_j^0 + \mu_\perp(\delta_{ij} - m_i^0m_j^0 - d_i^0d_j^0)$. The spin-orbit coupling energy $\varepsilon_{so} \sim (\delta m_i - \delta w_i)^2$ does not lead to a coupling to δn_i and there are no contributions to the propagating modes in lowest order, similar to the P_1 phase.

V. SUMMARY AND PERSPECTIVE

In this paper we have studied the macroscopic behavior of the P_1 phase and the P_2 phase of superfluid ^{3}He observed experimentally in anisotropic aerogels. Both phases exist only in an external magnetic field, an aspect which is similar to the A_1 phase and the A phase in high magnetic fields in bulk superfluid ^{3}He. It turns out that the order parameter in real space is the same for the P_1 and the P_2 phase as for the polar phase without an external field studied before.

In the P_1 phase one has only one type of pairs in spin space, either up-up or down-down pairs for the spin projection, just as for the A_1 phase. As an additional hydrodynamic variable contributing to the orbital dynamics, one has in this case the longitudinal component of the magnetization, which is in equilibrium parallel to the external magnetic field and also parallel to the preferred direction in real space characterized by a director and also to the average preferred direction of the silica strands of the mullite aerogel. For the P_1 phase we have presented the full macroscopic equations and also discussed in some detail the overlap with the dynamics of the polar phase. In addition, we have discussed the velocities of fourth sound for vanishing normal velocity as well as of second and first sound. It turns out that second sound acquires predominantly spin-wave character. This result is closely in parallel to that predicted [23] and observed [42] for the A_1 phase of bulk superfluid ^{3}He. Thus there is a clear-cut prediction for the P_1 phase, which could be tested experimentally using NMR as well as sound measurements.

In the P_2 phase the order parameter in spin space differs from that of the P_1 phase, where one has only one spin projection, as well as from that of the polar phase, in which one has an equal amount of "up" and "down" pairs in the spin projection. This fact leads to two preferred directions in spin space, one which is parallel to the preferred direction in the P_1 phase, $\hat{\mathbf{w}}^0$, and the other one, which is parallel to the preferred direction in the polar phases, $\hat{\mathbf{d}}^0$. As a consequence one can construct a variable δn associated with a spontaneously broken continuous rotational symmetry in spin space along the direction, which is orthogonal to $\hat{\mathbf{w}}^0$ as well as to $\hat{\mathbf{d}}^0$. The additional variable n is even under parity, odd under time reversal, and can couple to the other macroscopic variables familiar from the P_1 phase in orbit space. As a result one finds for the propagating excitations even to lowest order in the wave vector a coupling between spin waves on the one hand and fourth sound (for $v^n \equiv 0$) or first and second sound (for $v^n \neq 0$) on the other. In closing we point out that there are so far no experimental results available on the hydrodynamic propagating modes in both, the P_1 and P_2 phases.

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