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ABSTRACT

We put forward a definition for spectral triples and algebraic backgrounds based on Jordan coordinate algebras. We also propose natural and gauge-invariant bosonic configuration spaces of fluctuated Dirac operators and compute them for general, almost-associative, Jordan, coordinate algebras. We emphasize that the theory so obtained is not equivalent with usual associative noncommutative geometry, even when the coordinate algebra is the self-adjoint part of a C^*-algebra. In particular, in the Jordan case, the gauge fields are always unimodular, thus curing a long-standing problem in noncommutative geometry.

I. INTRODUCTION

One of the major achievements of the 20th century fundamental physics was the discovery that elementary particles are subject to internal symmetries, i.e., symmetries that are not associated with the four dimensions of spacetime. These symmetries are described by gauge theory, which successfully incorporates non-gravitational fundamental forces among the elementary particles, with mass generation accounted for by the presence of scalar fields (also known as Higgs fields). While this description is highly successful from a phenomenological perspective, the situation remains somewhat unsatisfactory, as already at the classical level gravitational and non-gravitational forces are described by different kinds of theories. A common theme has been the development of a quantum theory that unifies gravity with the other fundamental forces associated with internal gauge symmetries.

In the 1990s, noncommutative geometry (NCG) emerged as an intriguing unifying framework. (See Ref. 50 for a historical survey.) NCG provides an approach very similar to Kaluza–Klein theory, but in which the extra-dimensions are noncommutative, which means they are described by a (finite-dimensional) noncommutative C^*-algebra. In short, noncommutative geometry allows one to replace the usual internal Riemannian space of a Kaluza–Klein theory with a more general kind of geometry that avoids all of the stability issues that would otherwise arise through compactification. Within the NCG framework, all of the fundamental forces are unified, with the internal forces corresponding to "gravity" in the internal geometry, with the Dirac operator taken to be the dynamical variable. Moreover, Higgs fields do not have to be put in by hand and have a natural interpretation as the finite components of the Dirac operator. It must be added that NCG is not only a reformulation but is also more restrictive than gauge theory and predictive.[1–2]

Despite the many successful features of the NCG approach to unification, it has suffered from some technical problems (see the introduction of Ref. 3 for a list). By now, most of these problems have admitted at least partial solutions. For instance, only a Euclidean version of NCG was available at first, but some aspects have now been extended to general signature. However, the only action principle that is known to work for those extensions does not include gravity. (Some progress on this front may be close; see Ref. 51.) Hence, the *signature problem* may be said to be partially resolved. On the other hand, the so-called *unimodularity problem* remains a puzzle within the NCG framework. In brief, this problem is that one has to remove a $U(1)$-factor by hand through an *ad hoc* "unimodularity condition" in order to recover the correct gauge group for the standard model in NCG.[4] As was argued in Ref. 6, the unimodularity problem is automatically resolved by replacing associative noncommutative coordinate algebras by Jordan algebras, hence moving from noncommutative to nonassociative geometry of Jordan type.[5] One of the key motivations for the present work is to back this claim with a more formal proof.

This paper focuses on the description of gauge theories as nonassociative geometries of Jordan type. Several works have already been devoted to non-associative geometry and, in particular, Jordan geometry (Refs. 6–8), with one of the original motivations being a resolution to the so-called "fermion doubling" problem, which arises in the associative NCG construction of the standard model of particle physics,[9] but which is neatly circumvented in the Jordan setting[7] (see also Barrett's related solution[10] and that proposed by Connes[11]). Further motivation arises from the geometric construction of E_6 grand unified theories, where a related approach based on the exceptional Jordan algebra has emerged (Refs. 12 and 13; see also Refs. 14 and 15) and drawn legitimate attention. (See also Ref. 52 for an unrelated but interesting approach to finding standard model representation spaces using nonassociative algebras.) However, despite this attention, the detailed framework of NCG, including the axioms of spectral triples, the definition of the fields, and so on, has not yet been completely formalized in the Jordan setting. The main goal of this paper is to propose a complete translation from the NCG framework. In doing so, we will observe that the Jordan formalism admits a more natural representation of the symmetries of the algebra inside the triple. Moreover, we will show in more explicit detail that the unimodularity problem disappears. Our primary focus will be on special Jordan coordinate algebras although we will also briefly discuss the more general case at the end of this paper. It is important to observe that Jordan geometry is not equivalent to associative NCG even in the special (i.e., non-exceptional) case as the resolution of the unimodularity and fermion doubling problems shows.

Let us summarize more precisely the content of this paper. In Sec. II, we briefly recall the axioms of (associative) spectral triples and how particle models, such as the standard model, can be defined using them. In particular, we recall the existence of a map, called Υ, which sends the unitary elements of the algebra to the symmetries of the model. Hence, a factor $M_3(\mathbb{C})$ in the algebra generates a $U(3)$ group of symmetries, while we would need $SU(3)$ for the standard model, thus creating the unimodularity problem. In Sec. III, we recall the notion of the algebraic background, which one of the authors has proposed as an analog of the background manifold for a noncommutative Kaluza–Klein theory. Algebraic backgrounds permit one to define a configuration space in which the Dirac operator (i.e., the dynamic variable capturing all Bosonic degrees of freedom) lives. In Sec. IV, we recall the definition of this space and explain how fluctuated Dirac operators form a subspace, which is invariant under non-gravitational symmetries. Hence, the space of fluctuated Dirac operators can be used as the bosonic configuration space for a particle model in which gravity is ignored. This will end the reminders about associative noncommutative geometry.

In order to deal with Jordan geometry, we first recall a few facts on Jordan algebras, Jordan–Banach algebras, and their derivations in Sec. V. In particular, we recall the notions of associative and multiplicative representations. The first kind of representation only exists for special algebras, while the latter is a generalization, which is available for all Jordan algebras. In Sec. VI, we propose a definition of Jordan spectral triples and Jordan algebraic backgrounds in the non-exceptional case. In particular, we observe that a real special Jordan triple or background is always equipped not only with an associative representation π and an opposite action π^o (i.e., a bi-representation) but, thanks to the order 0 condition, also with a multiplicative representation $S = \frac{1}{2}(\pi + \pi^o)$, which turns out to play a key role. It should also be noted that the module of Jordan 1-forms (replacing noncommutative 1-forms) is both a Lie module and a Jordan module.

Section VII is devoted to fluctuations and symmetries, a key part of the present paper. In particular, we show that, at least for *almost-associative algebras*, i.e., algebras $A = \mathcal{C}(M, A_F)$ of continuous functions with values in a finite-dimensional Jordan algebra A_F,[16] the Lie algebra of inner derivations is isomorphic through the derivative of Υ to $[S(A), S(A)]$. This means that inner derivations and, by extension, those inner automorphisms obtained through exponentiation of the latter are directly represented on the Hilbert space of the Jordan triple, while in the associative case, one had to use unitary elements. This is the reason why Jordan triples naturally implement unimodularity: the elements of $[S(A), S(A)]$ are traceless. In Sec. VIII, we give an interpretation of the order 1 condition, which is peculiar to the Jordan setting and allows us to define a generalized form of fluctuations. Section IX is a specialization to the case of "almost-associative" Jordan algebras. We give a motivated definition for the module of Jordan 1-forms in this case, and most importantly, we compute the space of fluctuated Dirac operators of an almost-associative triple and put forward a condition (weaker that the order 1 condition) under which it is automorphism invariant. Sections X and XI are applications to the B–L extension of the standard model studied in Ref. 7 and the Pati–Salam model, respectively. In Sec. XII, we discuss the extension of the results of this paper to the general setting (i.e., including exceptional Jordan coordinate algebras) by directly using a multiplicative representation ρ, without assuming that it derives from an associative one. This generalization applies, in particular, to the exceptional case where no associative representation exists. We show that inner derivations are still directly implemented on the Hilbert space, and we discuss the representation of 1-forms and the fluctuation of Dirac operators in this more general setting.

In this whole paper, the "standard model" we discuss includes three generations of right-handed neutrinos and the see–saw mechanism. We will use the symbol $\dagger$ to denote the involution in an $*$-algebra. In particular, if T is an operator, we will write $T^\dagger$ for its adjoint. The symbol $*$ will be used for complex conjugates. We will generally use the symbol A for a Jordan algebra and $\mathcal{A}$ for an associative one.

II. ASSOCIATIVE SPECTRAL TRIPLES AND THE STANDARD MODEL

In this section, we recall some basics on spectral triples in the usual setting involving C^*-algebras, as well as the less familiar notion of algebraic backgrounds and what they are good for. We will talk about *associative* spectral triples in order to distinguish them from the Jordan spectral triples to be defined below. A real, even, associative, spectral triple[17] is a multiplet $\mathcal{T} = (\mathcal{A}, H, \pi, D, \chi, J)$, where $\mathcal{A}$ is a unital $*$-algebra;

H is a Hilbert space; π is an $\star$-representation of $\mathcal{A}$ on H, which we suppose to be faithful; D is a formally self-adjoint operator on H; χ is a bounded self-adjoint operator; and J is an anti-unitary operator such that the following holds.

- The norm closure of $\pi(\mathcal{A})$ is a C^*-algebra.
- For each $a \in \mathcal{A}$, $\pi(a)$ is in the domain of the derivation $[D, .]$.
- χ commutes with $\pi(\mathcal{A})$ and anticommutes with D, while J commutes with D.
- $\chi^2 = 1$.
- For each $a, b \in \mathcal{A}$, $[\pi(a), \pi(b)^o] = 0$ (C_0).
- $\forall a, b \in \mathcal{A}$, $[\pi(a)^o, [D, \pi(b)]] = 0$ (C_1).
- $J\chi = \epsilon''\chi J$, $J^2 = \epsilon$, where $\epsilon, \epsilon'' = \pm 1$.

Here, we have used the very handy notation $T^o := JT^\dagger J^{-1}$. Note that since we are dealing with the even case, we can, without loss of generality, choose the real structure such that $JD = DJ$.[20]

The idea behind these axioms is that the elements of $\mathcal{A}$ represent virtual differentiable functions on a noncommutative manifold. We will often refer to the "manifold case," which is the paradigmatic example where $\mathcal{A}$ is really the algebra of differentiable functions and D is the canonical Dirac operator on a spin manifold. (For the precise role of each object, see Ref. 35 or Ref. 53.) In order to state a reconstruction theorem,[18] additional conditions having to do with the topology of the manifold under consideration would be required, but we will not need them in this paper. Note that the order 1 condition (C_1) plays an essential role in the manifold case since it ensures that D is a first-order differential operator. One gains access to more general geometries, however, by dropping C_1. As many of these more general geometries are physically interesting, including the B–L extension given in Refs. 19 and 20 or the Pati–Salam model,[24] our discussion will focus primarily on the more general setting. We also need to recall the definition of *noncommutative 1-forms* since they will soon turn out to play a key role.[21] A noncommutative 1-form ω is a finite sum

$$\omega = \sum_i \pi(a_i)[D, \pi(b_i)] \tag{1}$$

with $a_i, b_i \in \mathcal{A}$. The space of all such 1-forms is an associative $\mathcal{A}$-bimodule, denoted as $\Omega^1_D\mathcal{A}$. In the manifold case, a noncommutative 1-form is a field with values in the space generated by gamma-matrices. It can be understood as a vector field seen as an operator on spinors.

Let us now briefly explain how particle models are defined using spectral triples. There are three fundamental insights:

a. The Dirac operator is a replacement for the manifold metric.
b. The symmetry group of a gauge theory coupled to gravity, which is the semi-direct product of the diffeomorphism group with the gauge symmetry group, can be interpreted (roughly) as the automorphism group of the $\star$-algebra $\mathcal{A} = \mathcal{C}^\infty(M) \otimes \mathcal{A}_F$, where $\mathcal{A}_F$ is a finite-dimensional algebra.
c. The bosonic part of the standard model coupled to gravity can, thus, be rewritten as a unified theory with the Dirac operator as the variable, defined on an almost-commutative manifold, i.e., the tensor product of the canonical triple over a manifold with a finite-dimensional noncommutative triple. As a bonus, we shall recover the Higgs fields as the finite component of the Dirac operator.

Let us now look at the technical implementation of these beautiful ideas. A first difficulty arises with the second one. The gauge group of the standard model is $U(1) \times SU(2) \times SU(3)/(\mathbb{Z}_2 \times \mathbb{Z}_3)$. Now, $SU(2) \times SU(3)/(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is the automorphism group of the algebra $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$, but we cannot make $U(1)$ appear in this way. This problem can be solved by considering unitary elements of the algebra instead of automorphisms (this will be justified below). Then, we can consider the algebra $\mathcal{A}_{SM} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, which has unitary group $U(1) \times SU(2) \times U(3)$. We see that we still need to reduce $U(3)$ to $SU(3)$, which is dealt with by the *unimodularity condition* to be explained below.

Remark. Let us observe that the algebra $\mathcal{A}_{SM}$ deviates a little bit from the C^*-paradigm since it is not a complex but a real algebra. There is a theory of real C^*-algebras,[22] but it has some subtleties, in particular concerning the Gelfand–Naimark duality, which is at the root of the noncommutative geometry program. This difficulty, which is seldom emphasized, might be an additional clue hinting toward Jordan algebras, which are better behaved than real C^*-algebras in many respects.

Let us now turn to the gauge fields. Gauge fields appear when we "fluctuate" the manifold Dirac operator D_M with an inner automorphism u of the algebra, i.e., we replace D_M with $\pi(u)D_M\pi(u)^{-1}$, where

$$u : x \mapsto u(x) \tag{2}$$

is a smooth map from M to the unitary group of the finite algebra. Rewriting $\pi(u)D_M\pi(u)^{-1}$ as $D_M + \pi(u)[D_M, \pi(u)^{-1}]$, we see the term $\pi(u)[D_M, \pi(u)^{-1}]$ appear, which is a "pure gauge" field with values in the Lie algebra of the gauge group. The idea with the replacement $D_M \mapsto \pi(u)D_M\pi(u)^{-1}$ is that $\pi(u)$ acts as an automorphism of the spectral triple and that $\pi(u)D_M\pi(u)^{-1}$ is just as good a Dirac operator as D_M is. The problem is that we have not yet specified what a spectral triple automorphism is exactly and what *a* Dirac operator is as opposed to *the* Dirac operator we start with. Clearly, we need a structure in which the Dirac operator is allowed to vary, which is not the case with a spectral triple. Let us use the following definitions.

Definition 1. A pre-spectral triple $B = (\mathcal{A}, H, \pi, \chi, J)$ is the same thing as a spectral triple with the Dirac operator removed. A pre-spectral triple automorphism is a unitary operator U such that the following holds.

1. $U\pi(\mathcal{A})U^{-1} = \pi(\mathcal{A})$.
2. *U commutes with J and χ.*

A spectral triple automorphism is a pre-spectral triple automorphism, which also commutes with the Dirac operator.

Note that pre-spectral triples are called *fermion spaces* in Ref. 23.

Let $\mathrm{Aut}(\mathcal{A})$ be the group of $*$-automorphisms of $\mathcal{A}$ and $\mathrm{Aut}(B)$ be the group of pre-spectral triple automorphisms of B. There is a natural homomorphism $\mathrm{Aut}(B) \to \mathrm{Aut}(\mathcal{A})$, which sends U to the restriction of Ad_U on $\pi(\mathcal{A})$ and, thus, defines an automorphism of $\mathcal{A}$ since π is faithful. What we would need in order to fully implement insight **b** above is a section of this homomorphism. This does not exist, in general; however, we can always lift the unitary elements of $\mathcal{A}$ thanks to the following map: for any $a \in \mathcal{A}$, define

$$\Upsilon(a) = \pi(a)J\pi(a)J^{-1}. \tag{3}$$

Thanks to C_0, the map Υ satisfies for all $a, b \in \mathcal{A}$,

1. $[J, \Upsilon(a)] = 0$,
2. $\Upsilon(ab) = \Upsilon(a)\Upsilon(b)$, and
3. $\Upsilon(a^\dagger) = \Upsilon(a)^\dagger$.

In particular, Υ defines a group homomorphism from $U(\mathcal{A})$, the unitary group of $\mathcal{A}$, to $U(H)$, the group of unitary operators on H. Moreover, we immediately see using C_0 that for $u \in U(\mathcal{A})$, $\Upsilon(u)$ is a pre-spectral triple automorphism such that $\mathrm{Ad}_{\Upsilon(u)} = \mathrm{Ad}_u$. We, thus, have the following commutative diagram:

$$\begin{array}{ccc} U(\mathcal{A}) & \xrightarrow{\Upsilon} & \mathrm{Aut}(B) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}} \\ U(\mathcal{A}) & \xrightarrow[\mathrm{Ad}]{} & \mathrm{Aut}_{\mathrm{Inn}}(\mathcal{A}) \end{array}, \tag{4}$$

where $\mathrm{Aut}_{\mathrm{Inn}}(\mathcal{A})$ is the group of inner $*$-automorphisms of $\mathcal{A}$.

Remark. As explained by one of the authors in Ref. 8, a beautiful and compact way of understanding pre-spectral triples and their automorphisms is in terms of "Eilenberg" algebras. That is, a pre-spectral triple can be thought of as a graded, involutive algebra B[24] with vector space $\mathcal{A} \bigoplus H$. The algebra product between elements in $\mathcal{A}$ is provided by the usual product on $\mathcal{A}$, while the product between elements in $\mathcal{A}$ and H is provided by the representation of $\mathcal{A}$ on H, and the product between elements in H is always equal to zero. The involution on elements in $\mathcal{A}$ is simply inherited from $\mathcal{A}$ itself, while the involution on elements in H is provided by J. The operator χ provides a grading on H. As indicated in the above diagram, the unitary elements $\Upsilon(u)$ for $u \in \mathcal{A}$, then, satisfy the defining properties of automorphisms on the "Eilenberg" algebra B, and, in particular,

$$\Upsilon(u)(\pi(a)h) = \pi(Ad_u a)\Upsilon(u)(h) \tag{5}$$

for all $a \in \mathcal{A}$ and $h \in H$. The observation that Upsilon should be viewed as an automorphism on B is key to generalizing to the non-special Jordan setting and to the non-associative setting more generally.[6]

Let us see how Υ works in the example of the standard model. Since a unitary element of $\mathcal{C}(M, \mathcal{A}_{\mathrm{SM}})$ is a function with values in $U(\mathcal{A}_{\mathrm{SM}})$, we only need to examine the finite part of the model. The finite Hilbert space is, then, given by

$$H_F = \mathbb{C}^2_{\mathrm{weak}} \otimes \mathbb{C}^4_{\mathrm{colour}} \otimes \mathbb{C}^4_\chi \otimes \mathbb{C}^N, \tag{6}$$

where $\mathbb{C}^2_{\mathrm{weak}}$ is the weak isospin space; the *color space* $\mathbb{C}^4_{\mathrm{color}}$ is $\mathbb{C}^4 = \mathbb{C}_\ell \bigoplus \mathbb{C}^3_{rgb}$, where "leptonicity" is seen as a fourth color; and the *chirality space* $\mathbb{C}_\chi$ is generated by the letters $R, L, \bar{R}, \bar{L}$. The last factor $\mathbb{C}^N$ is the generation space, where N is usually equals to 3. The finite representation $\pi_F : \mathcal{A}_{\mathrm{SM}} \to \mathrm{End}(H_F)$ is

$$\pi_F(\lambda, q, m) = \left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix} \otimes 1_4, q \otimes 1_4, \lambda 1_2 \bigoplus 1_2 \otimes m, \lambda 1_2 \bigoplus 1_2 \otimes m\right] \otimes 1_N, \tag{7}$$

where $[A, B, C, D]$ denotes a matrix that is block-diagonal with respect to the chirality space. The real structure J_F acts trivially on $\mathbb{C}^2_{\mathrm{weak}}$ and $\mathbb{C}^4_{\mathrm{colour}}$, and it acts as $(R, L, \bar{R}, \bar{L}) \mapsto (\bar{R}, \bar{L}, R, L)$ on the basis of $\mathbb{C}^4_\chi$. Hence, $J_F[A, B, C, D]J_F^{-1} = [C^*, D^*, A^*, B^*]$. Now, if $u = (e^{i\theta}, q, m)$ is a unitary element of $\mathcal{A}_{\mathrm{SM}}$, with $q \in SU(2)$ and $m \in U(3)$, we have

$$\Upsilon(u) = \left[\begin{pmatrix}1 & 0\\ 0 & e^{-2i\theta}\end{pmatrix} \bigoplus \begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix} \otimes m^*, qe^{-i\theta} \bigoplus q \otimes m^*, c.c., c.c.\right] \otimes 1_N, \tag{8}$$

where c.c. means that the last two blocks are the complex conjugate of the first two. To get rid of the extra $U(1)$, we impose the *unimodularity condition* $\det(\pi_F(u)) = 1$. This is equivalent (up to a finite Abelian group, which we neglect here) to ask m to be of the form $m = e^{-i\theta/3}g$, with $g \in SU(3)$, and (8) becomes

$$\Upsilon(u) = \left[\begin{pmatrix}1 & 0\\ 0 & e^{-2i\theta}\end{pmatrix} \bigoplus \begin{pmatrix}e^{4i\theta/3} & 0\\ 0 & e^{-2i\theta/3}\end{pmatrix} \otimes g^*, qe^{-i\theta} \bigoplus qe^{i\theta/3} \otimes g^*, c.c., c.c.\right] \otimes 1_N, \tag{9}$$

where g^* is a generic element of $SU(3)$. We see that this yields the correct representation of the gauge group, which is remarkable. Nonetheless, we would prefer to obtain the unimodularity condition in a natural way rather than by simply imposing it by hand. This would be possible if the finite algebra $\mathcal{A}_F$ was complex and π_F a complex representation. Indeed, we would have $\Upsilon(e^{i\theta}u) = e^{i\theta}\pi(u)Je^{i\theta}\pi(u)J^{-1} = \Upsilon(u)$ so that Υ would yield a representation of $U(\mathcal{A}_F)/U(1)$, which can be identified up to a finite center with $SU(\mathcal{A}_F)$. However, even ignoring the problem of the algebra $\mathbb{H}$, we could not have $\mathbb{C}$ act differently on up and down right-handed leptons in (7), and thus, we would not end up with the correct hypercharges. It, thus, seems that as far as this problem is concerned, we have come to a dead end.[25]

III. ALGEBRAIC BACKGROUNDS

Pre-spectral triples have been in use implicitly in noncommutative geometry since the first particle models were defined, but have been investigated explicitly only recently.[3] A number of problems have been found. To understand them, let us consider the case of pure gravity on a manifold. Since we have a Hilbert space of spinors, we will consider tetradic gravity on a spin manifold. Pre-spectral triple automorphisms should correspond to the symmetries of this theory, i.e., to diffeomorphisms coupled with lifts of the spin group (sometimes called local Lorentz transformations in physics). However, they do not: as soon as $\dim(M) \geq 6$, there are extra local automorphisms.[3] It means that we are missing a background structure, left invariant by the true automorphisms. We can understand what it is by considering the algebraic definition of the spin group. Let g be a metric (of possibly indefinite signature) on $\mathbb{R}^n$, with n being even. Let us embed $\mathbb{R}^n$ in the complex Clifford algebra $\mathbb{C}l(\mathbb{R}^n, g)$. The latter is equipped with a natural bilinear form $\tilde{g}$ extending g, a real structure J, and a chirality χ (see Ref. 30 for details). Moreover, the spin group $\mathrm{Spin}(g)$ precisely contains the elements u of $\mathbb{C}l(\mathbb{R}^n, g)$ such that the following holds.

1. u is unitary.
2. u commutes with J and χ.
3. Ad_u leaves $\mathbb{R}^n$ invariant.

If we translate these conditions in the case of the canonical spectral triple over a spin manifold, we obtain a pre-spectral triple automorphism with an additional property: it leaves the bimodule of 1-forms $\Omega^1_D\mathcal{A}$ invariant. Despite the notation and Eq. (1), the latter does not really depend on D, but rather on the differential and spin structures. [On a Riemannian manifold, it is possible to define spin structures independently of the metric.[54] An even more straightforward solution is available (in any signature) if the manifold is parallelizable.[4]] This motivates us to define a new structure, falling in between a pre-spectral triple and a spectral triple, which we call an *algebraic background*.

Definition 2. An algebraic background (or background for short) is a pre-spectral triple equipped with an $\mathcal{A}$-bimodule Ω^1, which is odd[26] *(i.e., it anticommutes with χ). A background automorphism is a pre-spectral triple automorphism U such that $U\Omega^1U^{-1} = \Omega^1$.*

With this definition, automorphisms of the canonical background over a manifold bijectively correspond to products of diffeomorphisms with local Lorentz transformations, as expected. Moreover, in the case of the standard model background, we find in addition to the latter the gauge transformations and gauged B–L symmetries[27] (Ref. 28, Sec. 4.2). There are many more pre-spectral triple automorphisms in this case, notably flavor changing symmetries. Note that $\Upsilon(u)$ is a background automorphism, provided that

$$\pi(u)^o\Omega^1\pi(u^{-1})^o = \Omega^1, \tag{10}$$

which is a weaker condition than C_1, so we call it *weak* C_1. An example of a model in which C_1 fails while *weak* C_1 continues to hold can be found in Ref. 29, in which a B–L extension of the standard model was obtained by enlarging the standard model coordinate algebra. We will discuss the relationship between C_1 and symmetries in more detail in Sec. VII.

Let us summarize this section so far with the following analogy: just as General Relativity (GR) is not defined on a Riemannian manifold, but on the configuration space of all metrics compatible with the differential structure of the background manifold, a generalized gravity model in NCG cannot be defined on a spectral triple, but instead on the configuration space of all Dirac operators compatible with a more primitive structure not including the Dirac operator. There are two options (as of yet): pre-spectral triples and algebraic backgrounds. We prefer the latter since it gives better results as far as automorphisms are concerned. Hence, we will view algebraic backgrounds as the noncommutative equivalent of the background manifold with its differential (and spin) structure. It should be noted that no reconstruction theorem in the spirit of Ref. 18 has been proven to date for this structure. (However, in the commutative and finite case, an algebraic background encodes a finite graph,[55] while a pre-spectral triple only encodes a set of points, and it is known that giving a differential structure to a finite set is equivalent to turning it into a graph.[56])

IV. CONFIGURATION SPACE AND FLUCTUATIONS

Algebraic backgrounds, or, for that matter, pre-spectral triples, permit one to define a configuration space for a generalized gravity theory in a natural way.

Definition 3. The configuration space of an algebraic background is the space of all operators D satisfying the conditions in the definition of a spectral triple and such that $\Omega^1_D\mathcal{A} \subset \Omega^1$. If D is such that $\Omega^1_D\mathcal{A} = \Omega^1$, we say that D is regular. If there exists a regular D on a background, we say that the background is regular.

The configuration space of a pre-spectral triple can be similarly defined, except that the condition $\Omega^1_D\mathcal{A} \subset \Omega^1$ is replaced with C_1. It is worthy to note that the two definitions give the same result in the case of a manifold (compare Ref. 30, Chap. 11 and Ref. 3).

The configuration space of a background or pre-sprectral triple is immediately checked to be invariant under the relevant automorphisms. In particular, if D is a Dirac operator and u is a unitary element of the algebra, then $\Upsilon(u)D\Upsilon(u)^{-1}$ is a Dirac operator. Let us write it in the form $D + F_u$, where

$$F_u := \Upsilon(u)D\Upsilon(u)^{-1} - D \tag{11}$$

is called an *inner fluctuation* of D since it is associated with an inner automorphism of the algebra, while a Dirac operator such as $D + F_u$ is called a *fluctuated Dirac*. It is from such fluctuations that non-gravitational bosonic fields arise in noncommutative geometry. Physically, fluctuations such as (11) only yield pure gauge fields, so we need a larger space. The standard choice is $D + \mathcal{F}_D$, where

$$\mathcal{F}_D = \{\omega + \omega^o | \omega \in \Omega^1, \omega^\dagger = \omega\}. \tag{12}$$

An element of $\mathcal{F}_D$ is a called a (general) *fluctuation*. If we neglect gravity, it is enough to define the physical theory on a subspace of the configuration space containing all fluctuated Dirac operators $D + F$, where $F \in \mathcal{F}_D$, with fixed D (corresponding, for instance, to the Minkowski metric). We can justify the choice given in Eq. (12) by observing that fluctuations given in (11) are, indeed, of the form $\omega + \omega^o$ with $\omega = \pi(u)[D, \pi(u)^{-1}]$ and that $D + \mathcal{F}_D$ is an affine space invariant under the relevant symmetries, namely, the background automorphisms fixing D. In addition, observe that if $u = \exp(a)$ with $a^\dagger = -a$, then $\mathcal{F}_D$ contains $[D, \pi(a)] + [D, \pi(a)]^o$, which is the infinitesimal version of (11). Note that ad_a is an inner derivation of $\mathcal{A}$, which commutes with the $*$-operation, and that such derivations exponentiate to $*$-automorphisms. There is, thus, the following commutative diagram:

$$\begin{tikzcd} Aut_{\mathrm{Inn}}(\mathcal{A}) & U(\mathcal{A}) \arrow[l, "\mathrm{Ad}"'] \arrow[r, "\Upsilon"] & \mathcal{U}(\mathcal{A}) \\ Der_{\mathrm{Inn}}(\mathcal{A}) \arrow[u, "\exp"] & \mathrm{Skew}(\mathcal{A}) \arrow[l, "\mathrm{ad}"'] \arrow[r, "d_e\Upsilon"] \arrow[u, "\exp"] & \mathcal{G}(\mathcal{A}) \arrow[u, "\exp"] \end{tikzcd}, \tag{13}$$

where have used the following notations: $\mathrm{Skew}(\mathcal{A})$ is the space of anti-self-adjoint elements of $\mathcal{A}$, $\mathcal{U}(\mathcal{A})$ is $\Upsilon(U(\mathcal{A}))$ [and is a subgroup of $\mathrm{Aut}(S)$, which we can call the subgroup of gauge transformations], $d_e\Upsilon(a) = a - a^o$ is the differential of Υ at the identity, and $\mathcal{G}(\mathcal{A}) := d_e\Upsilon(\mathrm{Skew}(\mathcal{A}))$ is the space of infinitesimal gauge transformations. Note that both $\mathcal{U}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A})$ act on $\mathcal{F}_D$ thanks to the bimodule structure of Ω^1_D. However, since the arrows on the left cannot be inversed (because of central elements), we cannot make $\mathrm{Aut}_{\mathrm{Inn}}(\mathcal{A})$ or $\mathrm{Der}_{\mathrm{Inn}}(\mathcal{A})$ directly act on fluctuations.

V. PRELIMINARIES ON JORDAN ALGEBRAS

A. Generalities

A (real) Jordan algebra[36] is a real vector space A equipped with a bilinear, commutative product $\circ$, satisfying the Jordan identity,[31]

$$\forall a, b \in A,\ (a^2 \circ b) \circ a = a^2 \circ (b \circ a). \tag{14}$$

Note that while Jordan algebras are, in general, not associative, every Jordan algebra is power associative, i.e., a^n has an unambiguous meaning for all $n \in \mathbb{N}$. All the Jordan algebras considered in this paper will be unital.

Let $\mathcal{A}$ be an associative algebra equipped with the product

$$a \circ b = \frac{1}{2}(ab + ba), \tag{15}$$

and let A be a real subspace of $\mathcal{A}$ stable under $\circ$ (an important example is when $\mathcal{A}$ is an $*$-algebra and A is the space of self-adjoint elements). Then, $(A, \circ)$ is a Jordan algebra. A Jordan algebra isomorphic to one of this kind is called *special*; otherwise, it is called *exceptional*. We will be primarily interested in the special case in this paper.

Let us now introduce some notations and examples. For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $n \in \mathbb{N}^*$, we denote by $H_n(\mathbb{K})$ the Jordan algebra of self-adjoint elements of $M_n(\mathbb{K})$. We similarly define $H_3(\mathbb{O})$, where $\mathbb{O}$ is the non-associative algebra of octonions. It turns out that $H_3(\mathbb{O})$ is a Jordan algebra (called the Albert algebra), which is exceptional. Let us also introduce the *spin factor* $\mathrm{JSpin}(n)$, which is the sub-Jordan algebra generated by $\mathbb{R}^n$ in the Clifford algebra $\mathrm{Cl}(\mathbb{R}^n, g)$, where g is the canonical scalar product, equipped with (15). It is, thus, generated by vectors of $\mathbb{R}^n$ with the product $u \circ v = g(u, v)$.

Let us now introduce *Jordan–Banach* algebras (or JB-algebras).

Definition 4. A JB algebra is a normed Jordan algebra A, which is complete in the norm and satisfies, for all $a, b \in A$, the following:

1. $\|a \circ b\| \le \|a\|\|b\|$,
2. $\|a^2\| = \|a\|^2$, *and*
3. $\|a^2\| \le \|a^2 + b^2\|$.

The main example of a JB algebra is the self-adjoint part of a C^*-algebra equipped with the symmetrized product given in Eq. (15). Just as with C^*-algebras, JB algebras admit a continuous functional calculus. Moreover, just as commutative C^*-algebras correspond to algebras of complex functions on compact Hausdorff spaces by the Gelfand transform, the same is true for associative JB algebras. More precisely, a JB algebra is associative iff it is isomorphic to the algebra of real functions on a compact Hausdorff space (see Ref. 6 and references therein for a more in depth discussion of the relationship between JB algebras and C^*-algebras). Finite-dimensional JB algebras can be completely classified.

Theorem 1. *Every finite-dimensional JB-algebra is a direct sum of ones on this list:*

1. $H_n(\mathbb{K})$ *for* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$,
2. $\mathrm{JSpin}(n)$, *and*
3. $H_3(\mathbb{O})$.

Remark. Note that the same theorem holds for the finite-dimensional formally real Jordan algebra, i.e., Jordan algebras such that $a_1^2 + \cdots + a_k^2 = 0 \Rightarrow a_1 = \cdots = a_k = 0$ for any sum of squares.

The following definitions can be found[32] in Ref. 33, Chap. 2.

Definition 5. Let A be a Jordan algebra and $\mathcal{A}$ be an associative algebra. A linear map $\sigma : A \to \mathcal{A}$ is called

1. *an associative specialization of A in $\mathcal{A}$ if for all $a, b \in A$,*

$$\sigma(a \circ b) = \sigma(a) \circ \sigma(b), \tag{16}$$

2. *a multiplicative specialization of A in $\mathcal{A}$ if for all $a, b \in A$,*

$$\begin{gathered} [\sigma(a), \sigma(a^2)] = 0, \\ 2\sigma(a)\sigma(b)\sigma(a) + \sigma(a^2 \circ b) = 2\sigma(a \circ b)\sigma(a) + \sigma(a^2)\sigma(b). \end{gathered} \tag{17}$$

Theorem 2. *Let σ_1 and σ_2 be two associative specializations of A in $\mathcal{A}$ such that $[\sigma_1(A), \sigma_2(A)] = 0$. Then, $\rho = \frac{1}{2}(\sigma_1 + \sigma_2)$ is a multiplicative specialization of A in $\mathcal{A}$.*

An immediate corollary is that $\frac{1}{2}$ times an associative specialization is automatically a multiplicative specialization so that the latter are more or less a generalization of the former. A specialization in $\mathrm{End}(H)$ will be called a *representation*. The reason for such a terminology is the relationship between multiplicative representations and Jordan modules. There is a general theory for modules over non-associative algebras.[34] In the Jordan case, this boils down to the following definition.

Definition 6. Let A be a Jordan algebra, H be a vector space, and $S : A \to \mathrm{End}(H)$ be a linear map. Let $\circ$ be the bilinear product on $A \oplus H$ extending $\circ$ on A and such that $a \circ h = h \circ a = S_a h$, $h \circ h' = 0$, for all $a \in A$, $h, h' \in H$. We say that (H, S) is a Jordan module if $(A \oplus H, \circ)$ is a Jordan algebra.

Theorem 3. *With the notations of Definition 6, (H, S) is a Jordan module iff S is a multiplicative representation.*

An equivalent requirement is[41]

$$
\begin{aligned}
&[S_a, S_{b\circ c}] + [S_c, S_{a\circ b}] + [S_b, S_{c\circ a}] = 0,\\
&S_a S_b S_c + S_c S_b S_a + S_{(ac)b} = S_a S_{bc} + S_b S_{ac} + S_c S_{ab},
\end{aligned}
\tag{18}
$$

which is obtained by linearization of (17). Summing all the cyclic permutations of the second equation in (18) and using Jacobi's identity also entail

$$
[[S_x, S_y], S_z] = S_{[y,z,x]}, \tag{19}
$$

where *the associator* is defined to be $[x,y,z] := (xy)z - x(yz)$. We will make extensive use of (19) in what follows. Note also the following natural properties.

Proposition 1. Let A be a Jordan algebra and A' be a subalgebra. Then, A is an A'-module for the action $L_a b = a\circ b$.

A Jordan algebra is, thus, a Jordan module over itself.

Proposition 2. Let A, A' be Jordan algebras and (H,S) be a A'-module. Let $\phi : A \to A'$ be a homomorphism. Then, $(H, S\circ\phi)$ is an A-module.

It follows from this that if σ is an associative representation of A on H, then $B(H)$ is an A-module for the action $S_a T = \sigma(a)\circ T$ with $T \in B(H)$. The reason is that $[B(H),\circ]$ is a Jordan algebra, hence a Jordan module over itself.

Let (H,S) and (H',S') be two modules over A and $f : H \to H'$ be a linear map. Then, f is defined to be a *module homomorphism*, or module map, iff $f(S_a h) = S'_a f(h)$ for all $(a,h) \in A\times H$. This is equivalent to requiring $\mathrm{Id}\oplus f$ to be a homomorphism from $A\oplus H$ to $A\oplus H'$ seen as Jordan algebras. The image and kernel of module maps are submodules; the quotient of a module by a submodule is a module: all of these results extend without change from the associative to the Jordan case. However, care must be taken with the tensor product in the non-associative setting. For instance, let M be a vector space. We want to define the *free A-module generated by M* to be an A-module $\langle M\rangle_A$ with a linear inclusion map $\iota : M \to \langle M\rangle_A$ satisfying the following universal property: for all A-modules H and linear maps $f : M \to H$, there exists a module map $\tilde f : \langle M\rangle_A \to H$ extending f, i.e., such that the following diagram commutes:

$$
\begin{tikzcd}
M \arrow[d, "\iota"] \arrow[dr, "f"] & \\
\langle M\rangle_A \arrow[r, "\tilde f"] & H
\end{tikzcd}
\quad . \tag{20}
$$

For an associative algebra $\mathcal{A}$, such an object exists and can be taken to be $\mathcal{A}\otimes_{\mathbb{R}} M$ with inclusion map $m \mapsto 1\otimes m$. The extension of f is defined by $\tilde f(a\otimes m) = af(m)$. However, when $\mathcal{A}$ is not associative, this fails to be a module map. We do not know if there exists an object with the universal property (20) in all generality in the Jordan category; however, we provide below a solution in the special case, which will be useful in Sec. VI when defining universal 1-forms. Let A be special Jordan algebra embedded in an associative algebra $\mathcal{A}$. If there is no natural choice for $\mathcal{A}$ in the context at hand, one can always consider the universal associative envelop of A.[33] A *special Jordan module* for A will be an A-module (H,S), which is also an $\mathcal{A}$-bimodule with $S_a h = \frac{1}{2}(a\cdot h + h\cdot a)$ for all $(a,h) \in A\times H$ (see also Ref. 33, p. 100). Now, for any vector space M, let $\langle M\rangle_{\mathcal{A}} := \mathcal{A}\otimes\mathcal{A}^{opp}\otimes M$, where the superscript *opp* denotes the opposite algebra. We have the inclusion map $m \mapsto 1\otimes 1\otimes m$, and it is easy to see that $\langle M\rangle_{\mathcal{A}}$ is canonically an $\mathcal{A}$-bimodule with the following universal property: for any linear map $f : M \to H$, with H being an $\mathcal{A}$-bimodule, there exists a bimodule map $\tilde f$ such that

$$
\begin{tikzcd}
M \arrow[d, "\iota"] \arrow[dr, "f"] & \\
\langle M\rangle_{\mathcal{A}} \arrow[r, "\tilde f"] & H
\end{tikzcd}
\tag{21}
$$

commute. Being an $\mathcal{A}$-bimodule, $\langle M\rangle_{\mathcal{A}}$ is also an A-module containing M. We define $\langle M\rangle_A$ to be the sub-A-module of $\langle M\rangle_{\mathcal{A}}$ generated by M. We call it the *special free A-module generated by M*. Since $\tilde f$ satisfies, for all $a\in A$ and $m\in M$,

$$
\begin{aligned}
\tilde f(S_a m) &= \frac{1}{2}\tilde f(a\cdot m + m\cdot a)\\
&= \frac{1}{2}(\tilde f(a)m + m\tilde f(a))\\
&= S_a\tilde f(m),
\end{aligned}
\tag{22}
$$

it is an A-module map. Hence, $\langle M\rangle_A$ satisfies the universal property (20) for all special Jordan module H and linear map f. Its universal property, then, makes it unique up to isomorphism, as is usual.

B. Derivations and automorphisms of JB algebras

We start by recalling some facts about order derivations. The following definition was introduced in Ref. 35 (see also Ref. 31).

Definition 7. A bounded linear operator δ on a JB-algebra A is called an order derivation if for all $t \in \mathbb{R}$, $e^{t\delta}(A^+) \subset A^+$, where A^+ is the subset of positive elements of A.

Note that in that case, $e^{t\delta}$ will be an *order automorphism*, i.e., a bijective linear map ϕ such that ϕ and ϕ^{-1} preserve order. From Proposition 1, every Jordan algebra acts as a module over itself. For every x in a Jordan algebra A, we write L_x for the Jordan multiplication by x, i.e., $L_x y = x \circ y$ for $x, y \in A$. In this case, L_x is an order derivation.

Definition 8. An order derivation on a unital JB-algebra A is called self-adjoint iff it is of the form L_a for some $a \in A$, and it is called skew if $\delta(1) = 0$.

Proposition 3. The set $\mathrm{Der}(A)$ of order derivations of a JB-algebra A is norm closed and closed under the Lie bracket. Moreover, it is the direct sum $\mathrm{Der}(A)_{sa} \oplus \mathrm{Der}(A)_{skew}$ of the real subspaces of self-adjoint and skew order derivations.

Proof. See Ref. 31, Propositions 1.59 and 1.60. □

Using the fact that unital order automorphisms are Jordan automorphisms (see Ref. 31, Theorem 2.80), it is possible to give the following characterization of skew derivations.

Proposition 4. Let A be JB-algebra and δ be an order derivation on A. The following are equivalent.

1. *δ is skew.*
2. *$\forall t \in \mathbb{R}$, $e^{t\delta}$ is a Jordan automorphism.*
3. *δ is a Jordan derivation.*

Here, a Jordan automorphism on a Jordan algebra A is an invertible linear map $\alpha : A \to A$, which respects the product on A,

$$\alpha(ab) = \alpha(a)\alpha(b), \tag{23}$$

and a Jordan derivation is a linear map $\delta : A \to A$, which satisfies the Leibniz rule,

$$\delta(ab) = \delta(a)b + a\delta(b). \tag{24}$$

Definition 9. An inner derivation of A is a Jordan derivation of the form

$$\delta = \sum_i [L_{a_i}, L_{b_i}].$$

The Lie algebra of inner derivation will be denoted by $\mathrm{Der}_{Inn}(A)$.

The above definition for inner derivations is standard and generalizes to all Jordan algebras. Similarly, we have the following definition.

Definition 10. The group $\mathrm{Aut}_{\mathrm{Inn}}(A)$ of inner automorphisms of A is by definition the subgroup of order automorphisms of A generated by exponentials of inner derivations.

The next definition can be found in Ref. 36.

Definition 11. The Lie multiplication algebra $M(A)$ is the sub-Lie algebra of $\mathrm{End}(A)$ generated by the operators L_a, $a \in A$.

The Lie multiplication is equal to the direct sum $\mathrm{Der}(A)_{sa} \oplus \mathrm{Der}_{Inn}(A)$. If A is finite-dimensional (or, more generally, if it is a JBW algebra, see Ref. 31 for the definition), then $M(A) = \mathrm{Der}(A)$.

We will now apply the previous concepts to the case where A has an associative representation in a Hilbert space.

Definition 12. Let $\pi : A \to B(H)$ be an associative representation of the Jordan algebra A. Then, we define $\mathrm{Lie}_\pi(A)$ to be the Lie algebra generated by $\pi(A)$ in $B(H)$.

To prove the next proposition, we need to recall that for $x, y \in \pi(A)$,

$$[L_x, L_y] = \frac{1}{4} ad_{[x,y]}. \tag{25}$$

Proposition 5. $\mathrm{Lie}_\pi(A) = \pi(A) \oplus [\pi(A), \pi(A)]$. Moreover, $[\pi(A), [\pi(A), \pi(A)]] \subset \pi(A)$ and $[\pi(A), \pi(A)]$ is a Lie subalgebra.

Proof. Let $x, y, z, t \in \pi(A)$. Then, $[x, [y, z]] = -4[S_y, S_z]x$ by (25). Hence, $[x, [y, z]] = -4y \circ (z \circ x) + 4z \circ (y \circ x) \in \pi(A)$ since $\pi(A)$ is a Jordan algebra. Moreover, $[[x, y], [z, t]] = [[[x, y], z], t] + [z, [[x, y], t]]$ by Jacobi's identity. Now, $[[x, y], z]$ and $[[x, y], t]$ are both in $\pi(A)$ by the above; hence, $[[x, y], [z, t]] \in [\pi(A), \pi(A)]$. This proves the result by linearity. □

For all $a, g, x \in B(H)$, let us define $ad^*_a(x)$ and $Ad^*_g(x)$, respectively, by

$$ad^*_a(x) = ax + xa^\dagger, \;\; Ad^*_g(x) = gxg^\dagger. \tag{26}$$

Note that Ad^*_g is the natural action of g on $B(H)$ if elements of $B(H)$ are interpreted as sesquilinear forms through $x \mapsto \langle ., x.\rangle$, rather than operators on H.

Proposition 6. ad^*_a defines a Lie homomorphism from $\mathrm{Lie}_\pi(A)$ *onto* $M(\pi(A)) \subset \mathrm{Der}(\pi(A))$.

Proof. Note that if $a = a^\dagger$, $ad^*_a = 2L_a$, and if $a = -a^\dagger$, $ad^*_a = ad_a$ so that ad^*_a is always an order derivation of $\pi(A)$. A direct computation shows that $ad^*_{[a,b]}x = [a, b]x + x[a, b]^\dagger = [ad^*_a, ad^*_b]x$. □

Note that by definition, the image of $\pi(A)$ is $\mathrm{Der}(\pi(A))_{sa}$ and the image of $[\pi(A), \pi(A)]$ is $\mathrm{Der}_{inn}(\pi(A))$. In the cases of interest, π will be faithful so that ad^* can be thought of as a surjective morphism from $\mathrm{Lie}_\pi(A)$ to $M(A)$, sending an operator on H to an operator on A. In the case that interests us the most, it is 1–1.

Proposition 7. Let A be finite-dimensional or of the form $\mathcal{C}(M, A_F)$, where A_F is finite-dimensional and M is a Hausdorff space. Then, $\mathrm{ad}^* : \mathrm{Lie}_\pi(A) \to M(\pi(A))$ *is an isomorphism. In particular,* $\mathrm{ad} : [\pi(A), \pi(A)] \to \mathrm{Der}_{\mathrm{Inn}}(\pi(A))$ *is an isomorphism, whose inverse will be denoted by a.*

Proof. First, let us observe that an element $z \in \ker \mathrm{ad}^*$ is skew since $ad^*_z(1) = z + z^\dagger = 0$. Now, if A is finite-dimensional and $z = \sum_i [x_i, y_i] \in \ker(\mathrm{ad})$, then z will commute with the C^*-algebra generated by $\pi(A)$, and hence, $\theta(z)$ will be a multiple of the identity for every irreducible representation θ of A. Since $\mathrm{Tr}(\theta(z)) = 0$, we have $\theta(z) = 0$ for all θ irreducible; hence, $z = 0$.

In the second case, if $g \in \ker \mathrm{ad}^*$, then g is skew for the same reason as above. Hence, $g \in \mathcal{C}(M, [\pi(A_F), \pi(A_F)])$ such that for every $f \in A$ and every $x \in M$, $[g(x), f(x)] = 0$. In particular, if f is constant, we see that $g(x)$ commutes with $\pi(A_F)$ so that the above argument yields $g(x) = 0$. □

Remark. In the general infinite-dimensional case, we can say the following: if $[x, y] \in \ker \mathrm{ad}$, then $[[x, y], x] = 0$, which implies that $[x, y]$ is quasi-nilpotent (Kleinecke–Shirokov theorem); hence, $[x, y] = 0$ since $i[x, y]$ is self-adjoint. We do not know, however, if this result extends to sums of commutators.

Now, let us define the group $\mathcal{L}\mathrm{ie}_\pi(A)$, which is generated by $\exp(\mathrm{Lie}_\pi(A))$, and its subgroup $U(A)$, which is also a subgroup of $U(H)$ and is generated by $\exp([\pi(A), \pi(A)])$. Then, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Aut}_{order}(\pi(A)) & \mathcal{L}\mathrm{ie}_\pi(A) \arrow[l, "\mathrm{Ad}^*"'] \\ M(\pi(A)) \arrow[u, "\exp"] & \mathrm{Lie}_\pi(A) \arrow[l, "\mathrm{ad}^*"'] \arrow[u, "\exp"'] \end{tikzcd} . \tag{27}$$

Over the subalgebra $[\pi(A), \pi(A)]$, (27) reduces to

$$\begin{tikzcd} \mathrm{Aut}_{\mathrm{Inn}}(\pi(A)) & U(A) \arrow[l, "\mathrm{Ad}"'] \\ \mathrm{Der}_{\mathrm{Inn}}(\pi(A)) \arrow[u, "\exp"] & {[\pi(A), \pi(A)]} \arrow[l, "\mathrm{ad}"'] \arrow[u, "\exp"'] \end{tikzcd} . \tag{28}$$

When ad is 1–1, since exp is also near the identity, we can infer that Ad will be invertible near 1. In this case, we obtain the following commutative diagram, where the wriggling arrow is defined near the identity:

$$\begin{tikzcd} \mathrm{Aut}_{\mathrm{Inn}}(\pi(A)) \arrow[r, squiggly] & U(A) \\ \mathrm{Der}_{\mathrm{Inn}}(\pi(A)) \arrow[u, "\exp"] \arrow[r, "a"] & {[\pi(A), \pi(A)]} \arrow[u, "\exp"'] \end{tikzcd} . \tag{29}$$

VI. JORDAN BACKGROUND AND TRIPLES

After these preliminaries, we come to the main course. We begin in this section with special Jordan triples, leaving the discussion of the generalization to the exceptional case for Sec. XII.

Definition 13. A (real, even) special Jordan triple is gadget $\mathcal{T} = (A, H, \pi, D, \chi, J)$ *such that the following holds.*

1. A *is a special Jordan algebra.*
2. H *is a Hilbert space.*
3. π *is a faithful associative representation of* A *to* $\mathrm{End}(H)$.
4. $\overline{\pi(A)}$ *is a JB algebra.*
5. D, J, *and* χ *are the usual things, with the usual conditions (in particular,* C_0, *but not* C_1 *yet).*
6. *For all* $a \in A$, $[D, \pi(a)]$ *is well-defined and bounded.*

Remark. We need to be careful here as A (whose elements play the role of differentiable functions) is not a JB-algebra as in Sec. V. However, it can readily be checked that the main result of this section, namely, Proposition 7, still holds with an algebra of the form $\mathcal{C}^\infty(M, A_F)$ with an unchanged proof.

We now turn to 1-forms. There is a standard way to construct a graded algebra of differential forms $\Omega_d A$ from an algebra A, which continues to make sense in the Jordan setting.[41] In this paper, we will not consider forms of degree higher than one. We obviously have $\Omega_d^0 A = A$. We want the space $\Omega_d^1 A$ of one-forms to be generated, as a Jordan A-module, by symbols $d[a]$, $a \in A$, with relations

$$d[a \circ b] = d[a] \circ b + a \circ d[b], \qquad \forall a, b \in A, \tag{30}$$

$$d[\alpha a + \beta b] = \alpha d[a] + \beta d[b], \qquad \forall a, b \in A, \quad \alpha, \beta \in \mathbb{R}, \tag{31}$$

where the product "$\circ$" is symmetric and satisfies the Jordan identity. That is, we want $d : \Omega_d^0 A \to \Omega_d^1 A$ to be a derivation. A generic element $\omega \in \Omega_d^1 A$ is a finite sum of the form

$$\omega = \sum a_1 \circ (a_2 \circ (\cdots (a_{n-1} \circ d[a_n]))) \tag{32}$$

with $a_i \in \Omega_d^0 A$. Here is an explicit construction of $\Omega_d^1 A$. We first consider the vector space $dA := A/\mathbb{R}.1_A$ and call d the quotient map. We now make use of the fact that A is a special Jordan algebra embedded in $\mathrm{End}(H)$ and consider the special free module $\langle dA \rangle_A$ defined in Sec. V. We let R be the sub-module of $\langle dA \rangle_A$ generated by $d(a \circ b) - d(a) \circ b - a \circ d(b)$ for all $a, b \in A$, and finally, we set $\Omega_d^1 A := \langle dA \rangle_A / R$. By construction, it is an A-module and d has been extended to a derivation of A into it. Moreover, it satisfies the following universal property.

Theorem 4. *Let* $\mathcal{M}$ *be any special* A*-module and* $\delta : A \to \mathcal{M}$ *be a derivation. Then, there exists an* A*-module map* $p : \Omega_d^1 A \to \mathcal{M}$ *such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[d, "d"'] \arrow[dr, "\delta"] & \\ \Omega_d^1 A \arrow[r, "p"] & \mathcal{M} \end{tikzcd}. \tag{33}$$

Proof. For all $a \in A$, we set $p(da) := \delta(a)$. This is well-defined because d annihilates constants. Since δ and d are linear, p also is and we, thus, have a well-defined linear map $p : dA \to \mathcal{M}$. By the universal property (20) of $\langle dA \rangle_A$, we obtain a module map, which we still call p from $\langle dA \rangle_A$ to $\mathcal{M}$. Since δ is a derivation, R is in the kernel of p, which, thus, goes to the quotient, ending the proof. □

For this reason, elements of $\Omega_d^1 A$ will be called *(special) universal 1-forms*. Note that a module of universal forms in the above sense has been constructed in Ref. 41 for the Albert algebra (using a different approach) so that this work is complementary to ours in this respect. Now, we can make use of the previous theorem to define two natural representations of 1-forms on H. First, we observe that since π is an associative representation, the map

$$\delta(a) := [D, \pi(a)] \tag{34}$$

is a derivation of A into $\mathrm{End}(H)$ so that by the universal property of $\Omega_d^1 A$, there exists a well-defined module map p such that

$$p(da) = [D, \pi(a)], \quad a \in A. \tag{35}$$

We will write $\pi(da)$ instead of $p(da)$ since no confusion should arise. We let $\Omega^1_D A$ be the image of this map π. In other words, $\Omega^1_D A$ is given by the Jordan A-submodule of $\mathrm{End}(H)$ generated by $\{[D,\pi(a)], a\in A\}$. Following the definition of an associative representation given in Eq. (16), a general element in $\Omega^1_D A$ is, then, given by

$$\pi(a_1\circ(a_2\circ(\cdots(a_{n-1}\circ d[a_n])))) = \sum \pi(a_1)\circ(\pi(a_2)\circ(\cdots(\pi(a_{n-1})\circ[D,\pi(a_n)]))). \tag{36}$$

For the case in which $\Omega^0 A = A$ is a special Jordan algebra, we will continue to refer to the corresponding differential, graded, Jordan algebras $\Omega = \Omega^0 A \oplus \Omega^1 A \oplus \cdots$ as "special." Note that the above definition collapses to the standard and familiar form when dealing with associative, Jordan algebras (e.g., in the canonical setting of a Riemannian manifold). Note, furthermore, that because $\Omega^1_D A$ acts as a Jordan module over A, it will also act naturally as a Lie module over $\mathrm{Der}_{Inn}(A)$ for the exact same reasons that H acts as a Lie module over $\mathrm{Der}_{Inn}(A)$.

Furthermore, there is another perfectly natural representation of universal 1-forms on H based on changing the embedding of A in $\mathrm{End}(H)$ from π to π^o. This will yield a module map p such that $p(da) = [D,\pi^o(a)]$. We will also write it π^o instead of p, but this time we need to be careful since $\pi^o(da) = [D^o,\pi(a)^o] = -[D,\pi(a)]^o = -\pi(da)^o$ so that π^o is not the composition of π with o on 1-forms.

Definition 14. A (real, even) special Jordan background is a gadget $\mathcal{B} = (A, H, \pi, \Omega^1, \chi, J)$ such that the following holds.

1. *A is a special Jordan algebra.*
2. *H is a Hilbert space.*
3. *π is a faithful associative representation of A.*
4. *$\overline{\pi(A)}$ is a JB algebra.*
5. *J and χ are the usual things, with the usual conditions (in particular, C_0).*
6. *Ω^1 is a $\pi(A)$-module for the Jordan multiplication, whose elements anticommute with χ.*

Just as in Sec. II, a Dirac operator for $\mathcal{B}$ is an operator D such that (A,H,π,D,χ,J) is a Jordan triple and $\Omega^1_D A\subset \Omega^1$. The configuration space and automorphisms are also defined exactly as in the associative case.

Condition C_0 has a very important interpretation in the Jordan setting.

Proposition 8. The "symmetrized" action $S = \frac{1}{2}(\pi + \pi^\circ)$ is a multiplicative representation of A on H.

Proof. We already know that π is an associative representation. Let us prove that π^o is also an associative representation: for all $a, b\in A$, we have $\pi^o(a\circ b) = \pi(a\circ b)^o = (\pi(a)\circ\pi(b))^o = \pi^o(b)\circ\pi^o(a) = \pi^o(a)\circ\pi^o(b)$ since $\circ$ is commutative. Now, π and π^o satisfy the hypotheses of Theorem 2 by C_0, and the proposition follows. □

Condition C_1 is generalized without change; it reads $[\pi(a),\pi(\omega)^o] = 0$ or equivalently $[\pi(a)^o,\pi(\omega)] = 0$ for all $a\in A$ and $\omega\in\Omega^1$. Unless specified otherwise, we will generally prefer to use the infinitesimal version of condition weak C_1,

$$[[\pi(A),\pi(A)]^o,\Omega^1]\subset\Omega^1 \qquad (\text{weak } C_1).$$

We will see in Proposition 10 that weak C_1 keeps the same meaning as in the associative case.

We will have more to say about the manifestations of the order 0 and 1 conditions in the Jordan setting in Sec. VIII, but for now, let us take a closer look at the manifold case. We suppose M to be parallelizable, and we define the canonical Jordan background over M to be the same as in the associative setting, which makes sense since the algebra $A = \mathcal{C}^\infty(M)$ is associative. Let us quickly recall the construction. We pick a moving frame (e_a). It defines a metric g_0 and a spin structure at the same time. In particular, it defines the Clifford mapping γ. Note that g_0 is not a background structure since it is not fixed by the automorphisms. The definitions of J_M, χ_M, and Ω^1_M are the same as usual.[3] In particular,

$$\Omega^1_M = \{i\gamma(\alpha)|\alpha \text{ is a real smooth } 1-\text{form on } M\}, \tag{37}$$

and D_0, the canonical Dirac operator associated with e_0, is a regular Dirac operator. Note that since J_M anticommutes with i and with Clifford elements of odd degree, it commutes with the elements of Ω^1_M, and since those are anti-self-adjoint, they are self-opposite, a fact which will be useful in Sec. IX.

VII. LIFTED INNER AUTOMORPHISMS AND MINIMAL FLUCTUATIONS

In this section, we seek to define a fluctuation space $\mathcal{F}_D$ for Jordan spectral triples. In order to do this, we first guess the correct algebraic structure of this fluctuation space, basing ourselves on three first principles. As we have recalled in Sec. II, in the associative case, the fluctuation space is a certain affine space containing pure gauge fluctuations (11) and is stable under automorphisms. Pure gauge fluctuations are associated with inner automorphisms of the algebra though not in a 1–1 way: this is the reason why the left arrows in diagram (13) go in the wrong direction. In the Jordan case, we can actually do better thanks to diagram (29) and obtain the following:

$$\begin{tikzcd}
\mathrm{Aut}_{\mathrm{Inn}}(\pi(A)) \arrow[r, squiggly] & U(A) \arrow[r, "\Upsilon"] & \mathcal{U}(A) \\
\mathrm{Der}_{\mathrm{Inn}}(\pi(A)) \arrow[u, "\exp"] \arrow[r, "a"] & {[\pi(A),\pi(A)]} \arrow[u, "\exp"] \arrow[r, "d_e\Upsilon"] & \mathcal{G}(A) \arrow[u, "\exp"]
\end{tikzcd} . \tag{38}$$

The notations are the same as in Sec. II, namely, $\mathcal{U}(A) := \Upsilon(U(A))$ and $\mathcal{G}(A) := d_e\Upsilon([\pi(A),\pi(A)])$. For the sake of clarity, let us follow the bottom line in detail. We start with $\delta \in \mathrm{Der}_{\mathrm{Inn}}(\pi(A))$. It is of the form $\delta = \mathrm{ad}(a_\delta)$, where

$$a_\delta = \frac{1}{4}\sum_i [\pi(a_i),\pi(b_i)] \tag{39}$$

is a uniquely defined element of $[\pi(A),\pi(A)]$ thanks to Proposition 7. Then, $d_e\Upsilon(a_\delta) = a_\delta + Ja_\delta J^{-1} = a_\delta - a_\delta^o$. Now, observe that $a_\delta - a_\delta^o = \sum_i [S(a_i),S(b_i)]$ by C_0. Hence, we obtain the following.

Proposition 9. One has $\mathcal{G}(A) = [S(A),S(A)]$.

Proposition 10. Under weak C_1, $\mathcal{U}(A)$ is a subgroup of the automorphism group of $\mathcal{B}$.

Proof. Let $\Upsilon(u) \in \mathcal{U}(A)$. It is clear that it is unitary and commutes with χ and J. Moreover, $\mathrm{Ad}_{\Upsilon(u)}(\pi(A)) = \mathrm{Ad}(u)(\pi(A)) = \pi(A)$ by C_0. Finally, let us write $u = \exp(x_k)\ldots\exp(x_1)$, $x_1,\ldots,x_k \in [\pi(A),\pi(A)]$, and let $\omega \in \Omega^1$. Then, $Ad_{\Upsilon(u)}(\omega) = Ad_{(u^{-1})^o}(Ad_u(\omega))$ by C_0. Now,

$$\begin{aligned} Ad_u(\omega) &= Ad_{\exp(x_k)}(\cdots(Ad_{\exp(x_1)}(\omega)\cdots) \\ &= \exp(ad_{x_k}(\cdots(\exp(ad_{x_1}(\omega)\cdots). \end{aligned} \tag{40}$$

Since Ω^1 is stable under the adjoint action of $[\pi(A),\pi(A)] \subset \mathrm{Lie}_\pi(A)$ by definition, we have $\mathrm{Ad}_u(\omega) \in \Omega^1$. By weak C_1, Ω^1 is also a $[\pi(A),\pi(A)]^o$-module, and we can repeat the same proof to show that $Ad_{(u^{-1})^o}(Ad_u(\omega)) \in \Omega^1$. □

As can be seen from Proposition 10, diagram (38) can be completed as the commutative cube,

$$\begin{tikzcd}
 & \mathrm{Aut}_{\mathrm{Inn}}(\pi(A)) & & \mathcal{U}(A) \arrow[ll, "\mathrm{Ad}"'] \\
\mathrm{Aut}_{\mathrm{Inn}}(\pi(A)) \arrow[ur, "\mathrm{Id}"] \arrow[rr, squiggly] & & U(A) \arrow[ur, "\Upsilon"'] & \\
 & \mathrm{Der}_{\mathrm{Inn}}(\pi(A)) \arrow[uu, dashed, "\exp"] & & {[S(A),S(A)]} \arrow[ll, dashed, "\mathrm{ad}"'] \arrow[uu, "\exp"'] \\
\mathrm{Der}_{\mathrm{Inn}}(\pi(A)) \arrow[uu, "\exp"] \arrow[ur, dashed, "\mathrm{Id}"] \arrow[rr, "a"] & & {[\pi(A),\pi(A)]} \arrow[uu, "\exp"] \arrow[ur, "d_e\Upsilon"'] &
\end{tikzcd} . \tag{41}$$

For instance, the path followed by the inner derivation $[L_{\pi(a)},L_{\pi(b)}]$ is

$$[L_{\pi(a)},L_{\pi(b)}] \mapsto \frac{1}{4}[\pi(a),\pi(b)] \mapsto \frac{1}{4}[\pi(a),\pi(b)] - \frac{1}{4}[\pi(a),\pi(b)]^o \mapsto ad_{\frac{1}{4}[\pi(a),\pi(b)]-\frac{1}{4}[\pi(a),\pi(b)]^o} = [L_{\pi(a)},L_{\pi(b)}]. \tag{42}$$

Note that Ad and ad could also be replaced with Ad^* and ad^*.

Remark. When Proposition 7 holds, all the arrows in the bottom face of the cube (41) are isomorphisms, in particular $d_e\Upsilon$. It means that there cannot be non-trivial self-opposite elements in $[\pi(A),\pi(A)]$. Note that the situation for associative backgrounds is much more involved (see Ref. 27).

Now, let us come to the fluctuation space $\mathcal{F}_D$. We want to guess what this space is basing ourselves on the following postulates.

1. It is a real vector space.

2. It contains the pure gauge fluctuations $UDU^{-1} - D$ for all $U \in \mathcal{U}(A)$.
3. For all $F \in \mathcal{F}_D$, $\mathcal{F}_{D+F} \subset \mathcal{F}_D$.

The third postulate says that "a fluctuation of a fluctuated Dirac is a fluctuation of the original Dirac." This property holds in the associative case and seems desirable to maintain. However, we could think of requiring, instead, a minimality property such as the following one.

$3'$. $\mathcal{F}_D$ is generated as a vector space by 2.

Later on, we will see that system $1, 2, 3'$ is actually stronger than $1, 2, 3$. However, first, we prove the following result.

Proposition 11. If $\mathcal{F}_D$ satisfies either $1, 2, 3$ or $1, 2, 3'$, then it satisfies the following:

4. *for all $F \in \mathcal{F}_D$ and $U \in \mathcal{U}(A)$, $Ad_U(F) \in \mathcal{F}_D$.*

Proof. By 2, $\mathrm{Ad}_{UV}(D) - D$ and $\mathrm{Ad}_U(D) - D$ are both in $\mathcal{F}_D$; hence, their difference $Ad_U(Ad_V(D) - D) \in \mathcal{F}_D$ by 1. By linearity, this shows that $1, 2, 3'$ implies 4.

Now, let $F \in \mathcal{F}_D$. Then, $Ad_U(D + F) - (D + F) \in \mathcal{F}_{D+F}$ by 2. If 3 holds, it is also in $\mathcal{F}_D$. Thus, $(Ad_U(D) - D) + Ad_U(F) - F \in \mathcal{F}_D$. However, $Ad_U(D) - D \in \mathcal{F}_D$ by 2 and $F \in \mathcal{F}_D$ by hypothesis. Hence, $Ad_U(F) \in \mathcal{F}_D$ by 1. □

Proposition 12. $1, 2, 3' \Rightarrow 3$.

Proof. Let $F \in \mathcal{F}_D$. By $3'$, a general element of $\mathcal{F}_{D+F}$ is a linear combination of $Ad_{U_i}(D + F) - (D + F) = (Ad_{U_i}(D) - D) + Ad_{U_i}(F) - F$. The first summand is in $\mathcal{F}_D$ by 2, the third one by hypothesis, and the middle one by the previous proposition. Hence, by 1, $\mathcal{F}_{D+F} \subset \mathcal{F}_D$. □

In this section, we will investigate a minimal fluctuation space consistent with the stronger set of axioms $1, 2, 3'$. In Sec. VIII, we will consider more general fluctuations consistent with $1, 2, 3$ for the case in which C_1 holds. Proposition 11 tells us that $\mathcal{F}_D$ is an invariant space for the adjoint representation of $U(A)$ on $B(H)$. If $\mathcal{F}_D$ is closed, we obtain by differentiation that $\mathcal{F}_D$ is a module over the Lie algebra $\mathcal{G}(A) = [S(A), S(A)]$ for the action ad. Moreover, it contains $ad_h(D)$ for all $h \in [S(A), S(A)]$. All of this suggests the following possible definition.

Definition 15. The minimal fluctuation space $\mathcal{F}_D$ is the Lie module over $\mathcal{G}(A)$ generated by elements of the form $[\delta, D]$ for $\delta \in \mathcal{G}(A)$.

Remark. Since π is faithful, the fluctuation space $\mathcal{F}_D$ is also a Lie module over inner derivations of A.

Conversely, it is easy to see that with Definition 15, $1, 2, 3$ hold. First, property 1 holds by definition. Let us prove 2. If $U = \exp(h)$ with $h \in \mathcal{G}(A)$, then $UDU^{-1} - D = \sum_{k=1}^{\infty} \frac{1}{k!} ad_h^k(D) \in \mathcal{F}_D$. Suppose that we have proved the result for any U, which is a product of $n - 1$ exponentials, and let $V = \exp(h)U$. Then,

$$\begin{aligned} VDV^{-1} - D &= Ad_{\exp(h)}(UDU^{-1} - D) + Ad_{\exp(h)}(D) - D \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} ad_h^k(UDU^{-1} - D) + UDU^{-1} - D + Ad_{\exp(h)}(D) - D \in \mathcal{F}_D. \end{aligned} \tag{43}$$

Property 2 follows by induction. Now, let us prove 3. We let $F \in \mathcal{F}_D$. Then, $ad_h(D + F) = ad_h(D) + ad_h(F) \in \mathcal{F}_D$. It follows that the submodule generated by $D + F$ is a subset of $\mathcal{F}_D$.

Now that we have a well-motivated definition for the minimal fluctuation space, let us take a closer look at the form they take. It will be a linear combination of terms $\delta_k, \ldots, \delta_1 \cdot D$ for $k \geq 1$, where $\cdot$ means the adjoint action and $\delta_j = T_j - T_j^o$ with $T_j \in [\pi(A), \pi(A)]$.

Proposition 13. If weak C_1 holds, $D + \mathcal{F}_D$ is a subspace of the configuration space.

Proof. We must prove that each $D + F$ with $F \in \mathcal{F}_D$ is a Dirac operator. All the required properties are obviously satisfied except $\{[D + F, \pi(a)], a \in A\} \subset \Omega^1$. To see that it also holds under weak C_1, it suffices to show that for all $a \in A$, $[F, \pi(a)] \in \Omega^1$. We can suppose without loss of generality that F is of the form $F = \delta_k, \ldots, \delta_1 \cdot D$, with $\delta_j = T_j - T_j^o$ for $k \geq 1$, and $T_j \in [\pi(A), \pi(A)]$, as above. By linearity, we only need to consider the case $F = R_k, \ldots, R_1 \cdot D$ with $R_j \in [\pi(A), \pi(A)]$ or $R_j \in [\pi(A), \pi(A)]^o$, with $j = 1, \ldots, k$. If $k = 1$, then $[\pi(a), F] = \pi(a) \cdot R_1 \cdot D = (\pi(a) \cdot R_1) \cdot D + R_1 \cdot (\pi(a) \cdot D)$ by Jacobi's identity. Now, $\pi(a) \cdot R_1$ is an element of $\pi(A)$ if $R_1 \in [\pi(A), \pi(A)]$ since ad_{R_1} is a derivation of $\pi(A)$, and it vanishes by C_0 if $R_1 \in [\pi(A), \pi(A)]^o$. On the other hand, $\pi(a) \cdot D \in \Omega^1$ so that $R_1 \cdot (\pi(a) \cdot D)$ also belongs to Ω^1 using the fact that Ω^1 is a both a $[\pi(A), \pi(A)]$-module and a $[\pi(A), \pi(A)]^o$-module. Now, suppose that the property is proved for some k. By Jacobi, again, we have $a \cdot R_{k+1}, \ldots, R_1 \cdot D = \sum_{j=1}^{k+1} R_{k+1}, \ldots, (a \cdot R_j), \ldots, R_1 \cdot D$. If $R_j \in [\pi(A), \pi(A)]^o$, the summand vanishes by C_0. If $R_j \in [\pi(A), \pi(A)]$, then $a \cdot R_j \in \pi(A)$ and $(a \cdot R_j), \ldots \cdot R_1 \cdot D \in \Omega^1$ by induction. Then, the summand belongs to Ω^1 since Ω^1 is a $[\pi(A), \pi(A)]$-module and a $[\pi(A), \pi(A)]^o$-module. □

Remark. It is remarkable that Proposition 13 holds under weak C_1 alone in the Jordan case, whereas an additional condition called weak C_1' is needed in the associative case.[29]

We have seen that Definition 16 entails properties 1, 2, 3, and from Proposition 11, it follows that $\mathcal{F}_D$ is always automorphism invariant. Now, the same will be true for $D + \mathcal{F}_D$ since $U(D+F)U^{-1} = D + (UDU^{-1} - D) + UFU^{-1}$. Thus, under weak C_1, $D + \mathcal{F}_D$ is an automorphism invariant subspace of the configuration space. This means that a particle model can consistently be defined on this space.

The fluctuation space given by Definition 16 is the smallest one, which respects the principles we have set forth, among them gauge-invariance. Let us observe that our approach here differs somewhat from what is usually done in the associative setting. The definition of the fluctuation space given by Connes is not only guided by the physically well-motivated principle of gauge-invariance but also by a deep generalization, namely, Morita self-equivalence of spectral triples. Furthermore, Connes' fluctuation space is usually defined in the presence of C_1, with the story becoming somewhat more complicated in the absence of C_1.[37] As we are dealing with special Jordan triples satisfying weak C_1 and it is not clear what the analog of Morita self-equivalence might be in the Jordan setting, we opt for gauge-invariance and minimality. An obvious question is what this same approach would yield in the associative setting. In Sec. VIII, we consider the construction of more general fluctuations for special Jordan triples that satisfy C_1.

VIII. ORDER CONDITIONS AND GENERAL FLUCTUATIONS

In Secs. VI and VII, we limited our discussion of the order conditions C_0 and C_1. In general, we will not restrict attention to geometries satisfying C_1; however, it is important to understand the implications that these conditions have as many physically relevant and interesting geometries will satisfy both conditions. In this section, we take a closer look at special Jordan representations in the presence of both C_0 and C_1, focusing, in particular, on their symmetries.

We begin with C_0. Consider a (real, even) special, Jordan, pre-spectral triple $B = (A, H, \pi, \chi, J)$. The representations π and π^0 both individually satisfy the properties of an associative specialization. We have already seen (Proposition 8) that it is possible to form a new "symmetrized" action $S = \frac{1}{2}(\pi + \pi^\circ)$, which satisfies all of the properties of a multiplicative representation. Following Definition 6, we, therefore, see that (H, S) is a Jordan module, or equivalently, the pre-spectral triple can be viewed as a Jordan algebra $B = A \bigoplus H$, where the bilinear product extends $\circ$ on A such that $a \circ h = h \circ a = S_a h$, $h \circ h' = 0$ for all $a \in A$, $h, h' \in H$. Note that both J and χ will commute with the "symmetrized" representation $S = \frac{1}{2}(\pi + \pi^\circ)$, a fact which has deep implications for the construction of physical theories with Majorana fermions. In particular, note that only representations that commute with J are compatible with the Majorana condition $JH = H$. This is a key motivation that underlies the construction of the standard model as a Jordan geometry.[7]

The meaning of C_0 is clear. Following Theorem 2, its imposition ensures that the symmetrized action of A on H satisfies all of the properties of a Jordan action. The upshot is that for a "symmetric" representation satisfying C_0, $B = A \bigoplus H$ will be a Jordan algebra, and following Definition 9, its inner derivations will be of the form

$$\delta = \sum [L_a, L_b], \tag{44}$$

where $a, b \in B$ and L_a is the Jordan multiplication by a when acting on A and is equal to S_a when acting on H. Note, however, that when either a or b is drawn from H, we have $\delta_{a,b} h = 0$ for all $h \in H$. Following Definition 10, the inner automorphisms of B acting on H will, therefore, be of the form $\alpha = e^\delta \in \mathcal{U}(A)$, where δ is constructed from elements from A and not from $B = A + H$, more generally, i.e., $\in \delta \in \mathcal{G}(A)$.[38] Inner automorphisms of B acting on H will, therefore, automatically commute with both J and χ (because the symmetric action on H commutes with both J and χ). In other words, C_0 ensures that S is multiplicative and as such that the symmetries of A can be "lifted" in a consistent way to the symmetrized representation of A on H.

Next, let us consider the meaning of C_1. We restrict attention to expressions of degree one and lower, meaning that we will not consider the representation of products of forms on H. The representation of higher order forms is an involved discussion even in the associative setting and deserves a paper in its own right. As a representation of higher order forms will not be necessary for deriving any of the results in this paper, we will return to the discussion in a follow-up paper where we will show that Jordan geometries are somewhat better behaved than associative geometries at higher order (see, however, Refs. 8 and 39 for a more involved discussion of Junk forms and the second order condition in the associative setting).

We begin by equipping the pre-spectral triple B with a Dirac operator D to form a special Jordan triple $\mathcal{T} = (A, H, \pi, D, \chi, J)$, which satisfies all of the usual properties. We observe that we have a map $\pi : A \bigoplus \Omega^1_d A \to \mathrm{End}(H)$ such that π is an associative representation on A and a module map on $\Omega^1_d A$. This means that π is an associative representation of the split null extension $A \bigoplus \Omega^1_d A$ *up to degree 1*. Similarly, π^o is an associative representation of $A \bigoplus \Omega^1_d A$ up to degree 1, and thanks to C_0 and C_1, these two "representations" commute. Thus, by Theorem 2, $S = \frac{1}{2}(\pi + \pi^\circ)$, will be a multiplicative representation up to degree 1. Since one might feel uncomfortable using this theorem "up to degree one," we provide a formal proof below.

Proposition 14. The "symmetrized" action of forms $S = \frac{1}{2}(\pi + \pi^\circ)$ defined on $A \bigoplus \Omega^1_d A$ satisfies the properties of a multiplicative specialization for all expressions up to degree 1.

Proof. We need to show that the linearization of Eq. (17) given in Eq. (18) holds for the case in which a single element $a, b, c \in \Omega A$ is of degree 1 and the remaining elements are of degree 0. We begin with the first equation:

$$\begin{aligned}[S_a, S_{b\circ\omega}] + [S_b, S_{a\circ\omega}] + [S_\omega, S_{b\circ a}] &= -\frac{1}{4}[\pi(a), \pi(b\circ\omega)^\circ] + \frac{1}{4}[\pi(a)^\circ, \pi(b\circ\omega)] \\ &\quad - \frac{1}{4}[\pi(b), \pi(a\circ\omega)^\circ] + \frac{1}{4}[\pi(b)^\circ, \pi(a\circ\omega)] \\ &\quad + \frac{1}{4}[\pi(\omega), \pi(a\circ b)^\circ] - \frac{1}{4}[\pi(\omega)^\circ, \pi(a\circ b)] \\ &= 0 \end{aligned} \tag{45}$$

for $a, b \in A$, $\omega \in \Omega^1 A$, where the first equality holds because as π and π° are associative specializations, $\frac{1}{2}\pi$ and $\frac{1}{2}\pi^\circ$ satisfy the equation separately. The second equality holds due to C_1. For the second equation, we find

$$\begin{aligned} S_a S_b S_\omega + S_\omega S_b S_a + S_{(a\omega)b} - S_a S_{b\omega} - S_b S_{a\omega} - S_\omega S_{ab} &= \frac{1}{8}[\pi(a), \pi(b)^\circ]\pi(\omega) + \frac{1}{8}[\pi(\omega), \pi(b)^\circ]\pi(a) \\ &\quad + \frac{1}{8}[\pi(b), \pi(\omega)^\circ]\pi(a)^\circ + \frac{1}{8}[\pi(b), \pi(a)^\circ]\pi(\omega)^\circ \\ &\quad + \frac{1}{8}[\pi(\omega)^\circ, \pi(a)\pi(b)] + \frac{1}{8}[\pi(\omega)\pi(b), \pi(a)^\circ] \\ &\quad + \frac{1}{8}[\pi(a)^\circ\pi(b)^\circ, \pi(\omega)] + \frac{1}{8}[\pi(a), \pi(\omega)^\circ\pi(b)^\circ] \\ &= 0 \end{aligned} \tag{46}$$

for $a, b \in A$ and $\omega \in \Omega^1 A$, where the first equality holds because $\frac{1}{2}\pi$ and $\frac{1}{2}\pi^\circ$ satisfy the equation separately and the second equality holds due to C_1. Similarly,

$$\begin{aligned} S_a S_\omega S_c + S_c S_\omega S_a + S_{(ac)\omega} - S_a S_{\omega c} - S_\omega S_{ac} - S_c S_{a\omega} &= \frac{1}{8}[\pi(\omega)^\circ, \pi(a)]\pi(c) + \frac{1}{8}[\pi(\omega)^\circ, \pi(c)]\pi(a) \\ &\quad + \frac{1}{8}[\pi(a)^\circ, \pi(\omega)]\pi(c)^\circ + \frac{1}{8}[\pi(c)^\circ, \pi(\omega)]\pi(a)^\circ \\ &\quad + \frac{1}{8}[\pi(a), \pi(c)^\circ\pi(\omega)^\circ] + \frac{1}{8}[\pi(c), \pi(a)^\circ\pi(\omega)^\circ] \\ &\quad + \frac{1}{8}[\pi(a)\pi(\omega), \pi(c)^\circ] + \frac{1}{8}[\pi(c)\pi(\omega), \pi(a)^\circ] \\ &= 0 \end{aligned} \tag{47}$$

for $a, c \in A$, $\omega \in \Omega^1 A$. □

For special Jordan triples that satisfy C_1, the symmetrized action of A on H, therefore, satisfies the properties of a multiplicative specialization for all expressions of degree 1 and lower. In effect, just as C_0 extends the Jordan product $\circ$ on A to all of $B = A \oplus H$ as a Jordan product, the first order condition C_1 extends the product further as a Jordan product to $\mathcal{T} = A \oplus \Omega^1_D A \oplus H$, so long as one restricts attention to the expression of degree 1 or lower. The upshot is that for special Jordan triples satisfying C_1, we are able to extend the "degree zero" inner derivations on $B = A \oplus H$ to include "degree one" elements $\delta : A \oplus H \to \Omega^1_d \oplus H$ of the form

$$\sum [L_a, L_\omega] \tag{48}$$

for $a \in A$, $\omega \in \Omega^1_d A$.[40] Following the same route as in (42) yields the operator

$$F = \sum [S_a, S_\omega]. \tag{49}$$

Let it be clear that a rigorous justification of this step would require to define the full algebra of forms, the extension of A by this algebra (in the spirit of Ref. 8), and the upgrading of (41) to this new setting, which is beyond the scope of this paper. Here, we will be happy to observe that things are working at first order and take it as a motivation to consider operators of the form (49). It is easy to show that they satisfy the Leibniz rule, and in particular, following Eq. (19),

$$[[S_a, S_\omega], S_b] = S_{[\omega,b,a]} \tag{50}$$

for all $a, b \in A$ and $\omega \in \Omega^1_d A$. We denote the space of degree one derivation elements of the form given in (49) by $\mathcal{F}_\omega$. Note that unlike at degree zero, these derivations are self-adjoint, commute with J, and anti-commute with χ.

We now turn to the discussion of fluctuated Dirac operators in the presence of C_1.

Definition 16. Given a special Jordan triple $\mathcal{T} = (A, H, \pi, D, \chi, J)$, we define the general fluctuation space to be given by $\mathcal{F}_\omega$.

In addition to having Hermitian elements that commute with J, anti-commute with χ, and map zero forms to one forms through commutation, the fluctuation space $\mathcal{F}_\omega$ satisfies the three postulates that we set out in Sec. VII. In particular, $\mathcal{F}_\omega$ is a real vector space that contains $\mathcal{F}_\mathcal{D}$, and for all $F \in \mathcal{F}_\omega$, $\mathcal{F}_{D+F} \subset \mathcal{F}_D$.

Proof. $\mathcal{F}_\omega$ is a real vector space by definition, so we focus on the other two postulates. For the second postulate, we have only to show that elements of the form $\delta_k, \ldots, \delta_1 \cdot D$, for $k \geq 1$, are in $\mathcal{F}_\omega$. To begin with, for $k = 1$, we have

$$[D, [S_a, S_b]] = [S_{d[a]}, S_b] + [S_a, S_{d[b]}] \in \mathcal{F}_\omega \tag{51}$$

for $a, b \in A$, which follows directly from Eq. (35) and the definition of the symmetrized action. Moreover, for all $F \in \mathcal{F}_\omega$, one has $[[S_a, S_b], F] \in \mathcal{F}_\omega$ by Jacobi's identity and Eq. (50). It, then, follows that $\mathcal{F}_\omega$ is a Lie module over $\mathcal{G}(A)$, and as a result, all elements of the form $\delta_k, \ldots, \delta_1 \cdot D$ are in $\mathcal{F}_\omega$, proving the second postulate. Similarly, postulate 3 follows directly from Eq. (50). □

Remark. Finally, before closing this section, we make a brief comparison between $\mathcal{F}_\omega$ and the general associative fluctuations given in Eq. (12). The analog of Eq. (44) in the associative setting is given by

$$\delta = \sum L_a - R_a, \tag{52}$$

where $a \in B$ and where for $a \in \mathcal{A}$, the "left action" is given by $L_a a' = aa'$ and $L_a h = \pi(a)h$, while the "right action" is given by $R_a a' = a'a$ and $R_a h = \pi^\circ(a)h$ for $a' \in \mathcal{A}$ and $h \in H$. For associative geometries satisfying C_1, the "degree zero" inner derivations on $B = A \oplus H$ can, then, be extended to include "degree one" elements $\delta : A \oplus H \to \Omega^1_d \oplus H$ of the form

$$\sum L_\omega - R_\omega \tag{53}$$

for $\omega \in \Omega^1_d A$. The analog of Eq. (49) is, then, given by

$$\sum \omega + \omega^\circ. \tag{54}$$

Restricting to Hermitian elements of this form, we obtain Connes' fluctuations [Eq. (12)]. Note that in the Jordan setting, we obtain Hermiticity for free, and it is not put in by hand.

Furthermore, a comparison is able to be made between the associative and Jordan settings by expressing Eq. (49) more explicitly in terms of the associative representations π and π°,

$$\begin{aligned}[S_a, S_\omega] &= [\pi(a) + \pi(a)^\circ, \pi(\omega) - \pi(\omega)^\circ] \\ &= [\pi(a), \pi(\omega)] + J[\pi(a), \pi(\omega)]J^{-1}.\end{aligned} \tag{55}$$

We see that the general fluctuation space of a Jordan geometry is slightly more restrictive than the corresponding fluctuation space for an associative geometry since $[\pi(a), \pi(\omega)]$ is a traceless 1-form. Hence, unimodularity is an automatic feature of the general fluctuation space. A curious question is what phenomenological restrictions on the scalar sector of particle theories these additional restrictions will bring.

IX. JORDAN 1-FORMS AND FLUCTUATIONS FOR ALMOST-ASSOCIATIVE SPECIAL JORDAN TRIPLES

The tensor product of two Jordan algebras with product $(a \otimes b) \circ (c \otimes d) = a \circ b \otimes c \circ d$ is generally not a Jordan algebra. However, this works if at least one of the algebras is associative. Here, we will consider algebras of the form

$$A = \mathcal{C}^\infty(M) \otimes A_F = \mathcal{C}(M, A_F), \tag{56}$$

where M is a manifold and A_F is a finite-dimensional Jordan algebra. We call these algebras *almost-associative* by analogy with the almost-commutative ones.

Let $\mathcal{B}_M$ be the canonical Jordan background over M and $\mathcal{B}_F$ be a finite Jordan background. The definition of the almost-associative Jordan background $\mathcal{B}_M \hat{\otimes} \mathcal{B}_F$ is the same as in the almost-commutative case. The algebra, real structure, and chirality operators are graded tensor products and follow the same rules as given in Refs. 20 and 41. In particular, one has $(T_1 \hat{\otimes} T_2)^o = (-1)^{|T_1||T_2|} T_1^o \hat{\otimes} T_2^o$, where $|T_{1,2}|$ is the grading of the corresponding operator, defined by its commutation property with $\chi_{1,2}$. All the necessary checks are exactly the same for the almost-associative and almost-commutative cases, except for the module of 1-forms, which is given by

$$\Omega^1_{M\times F} = \Omega^1_M \otimes \pi(A_F) \bigoplus \mathcal{C}^\infty(M) \otimes \Omega^1_F. \tag{57}$$

It is immediate to check that Eq. (57) defines an odd Jordan $\pi(A)$-module. Moreover, let D be a *product Dirac,*

$$D = D_M \hat{\otimes} 1 + 1 \hat{\otimes} D_F, \tag{58}$$

where D_M is in the configuration space of $\mathcal{B}_M$ and D_F is in the configuration space of $\mathcal{B}_F$. Then, $\Omega^1_D A \subset \Omega^1_{M\times F}$, with equality whenever D_F is regular.

Let us turn to fluctuations. We know that $\mathcal{F}_D$ is the Lie module generated by the orbit of D under the action of $[S(A), S(A)]$. Let $A = \mathcal{C}^\infty(M, A_F)$, and D be a product Dirac as in (58). We will need the following lemma. We recall that a Lie algebra $\mathcal{G}$ is called *perfect* if $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$. Semisimple Lie algebras are perfect.

Lemma 1. Consider a finite Jordan triple over A_F and its fluctuation space $\mathcal{F}_{D_F}$. Suppose that $[S(A_F), S(A_F)]$ is a perfect Lie algebra, and let $\mathcal{F}'_{D_F}$ be the derived fluctuation space $\mathcal{F}'_{D_F} := [S(A_F), S(A_F)] \cdot \mathcal{F}_{D_F}$. Then, $\mathcal{F}'_{D_F} = \mathcal{F}_{D_F}$.

Proof. Since the inclusion $\subset$ is obvious, we only need to prove the converse. Every finite fluctuation is a sum of terms such as $T_k \cdots T_1 \cdot D_F$ with $T_i \in [S(A_F), S(A_F)]$. We only need to prove that a fluctuation of the form $T \cdot D_F$ with $T \in [S(A_F), S(A_F)]$ can be written as a sum of terms $T_k \cdots T_1 \cdot D_F$ with $k \geq 2$. Since $[S(A_F), S(A_F)]$ is perfect, we can write $T = \sum_i [\alpha_i, \beta_i]$ with $\alpha_i, \beta_i \in [S(A_F), S(A_F)]$. Now, we have

$$\begin{aligned} T \cdot D_F &= \sum_i [\alpha_i, \beta_i] \cdot D_F \\ &= \sum_i \alpha_i \cdot \beta_i \cdot D_F - \beta_i \cdot \alpha_i \cdot D_F \text{ by Jacobi's identity} \\ &\in \mathcal{F}'_{D_F}. \end{aligned} \tag{59}$$

□

Theorem 5. *Let $D = D_M \hat{\otimes} 1 + 1 \hat{\otimes} D_F$ be the product Dirac operator of an almost-associative Jordan triple. Then,*

$$\mathcal{F}_D \subset \Omega^1_M \otimes [S(A_F), S(A_F)] \bigoplus \mathcal{C}^\infty(M, \mathcal{F}_{D_F}) \tag{60}$$

with equality if $[S(A_F), S(A_F)]$ is a perfect Lie algebra.

Proof. Let us call M the RHS of (60). To prove that $\mathcal{F}_D \subset M$, it suffices to prove that M is a $[S(A), S(A)]$-module, which contains $[S(A), S(A)] \cdot D$. Let $f \otimes \alpha \in [S(A), S(A)] = \mathcal{C}^\infty(M, [S(A_F), S(A_F)])$. Then,

$$(f \otimes \alpha) \cdot D = f \cdot D_M \otimes \alpha + f \otimes \alpha \cdot D_F, \tag{61}$$

which belongs to the RHS of (60). Now, let $\omega \in \Omega^1_M$, $\beta \in [S(A_F), S(A_F)]$, $g \in \mathcal{C}^\infty(M)$, and $\phi \in \mathcal{F}_{D_F}$. Then, $f \otimes \alpha \cdot (\omega \otimes \beta + g \otimes \phi) = f\omega \otimes [\alpha, \beta] + fg \otimes \alpha \cdot \phi$, which also belongs in the right space since $[S(A_F), S(A_F)]$ is a Lie algebra and $\mathcal{F}_{D_F}$ is an $[S(A_F), S(A_F)]$-module.

Let us now prove the converse inclusion when $[S(A_F), S(A_F)]$ is a perfect Lie algebra. First, using $(1 \otimes \alpha) \cdot D = 1 \otimes (\alpha \cdot D_F)$, we see that $1 \otimes \mathcal{F}_{D_F} \subset \mathcal{F}_D$. Acting with $f \otimes \beta \in [S(A), S(A)]$ on $1 \otimes \phi \in 1 \otimes \mathcal{F}_{D_F}$, we obtain $f \otimes \beta \cdot \phi \in \mathcal{F}_D$, and we conclude that $\mathcal{C}^\infty(M, \mathcal{F}'_{D_F}) \subset \mathcal{F}_D$. Using the lemma, this shows that $\mathcal{C}^\infty(M, \mathcal{F}_{D_F}) \subset \mathcal{F}_D$. There just remains to prove that $\Omega^1_M \otimes [S(A_F), S(A_F)] \subset \mathcal{F}_D$. From (61), we see that $f \cdot D_M \otimes \alpha$ is the difference of two elements of $\mathcal{F}_D$ and it is, thus, also in $\mathcal{F}_D$. Hence, any tensor of the form $\omega \otimes \alpha$ with ω exact and $\alpha \in [S(A_F), S(A_F)]$ belongs to $\mathcal{F}_D$. Now, if we act on such a tensor with $g \otimes \beta \in \mathcal{C}^\infty(M) \otimes [S(A_F), S(A_F)]$, we obtain $(g \otimes \beta) \cdot (\omega \otimes \alpha) = g\omega \otimes [\beta, \alpha]$. Using sums of terms such as this and the fact that $[S(A_F), S(A_F)]$ is perfect, we see that $\Omega^1_M \otimes [S(A_F), S(A_F)] \subset \mathcal{F}_D$, and the theorem is proved. □

Let us apply Theorem 5 to find the nature of gauge fields in the case where $A_F = \bigoplus_{i=1}^k J_i$ with $J_i = H_{n_i}(\mathbb{K})$ or $\mathrm{JSpin}(n_i)$ (see Theorem 1). Using the cube (41) and $\ker \pi = 0$, we obtain that

$$[S(A_F), S(A_F)] = \bigoplus_{i=1}^k \mathrm{Der}_{\mathrm{Inn}}(J_i). \tag{62}$$

If $J_i = H_{n_i}(\mathbb{R}), H_{n_i}(\mathbb{C}), H_{n_i}(\mathbb{H})$ or $\mathrm{JSpin}(\mathrm{n_i})$, then $\mathrm{Der}_{\mathrm{Inn}}(J_i) = so(n_i), su(n_i), sp(n_i)$ or $\mathrm{spin}(n_i) = so(n_i)$, respectively. This directly gives the nature of the gauge fields from the algebra A_F. In particular, we see that unimodularity is a natural feature of the Jordan setting.

We now turn our attention to the general fluctuation space.

Theorem 6. *Let $\mathcal{F}^{M\times F}_\omega$ be the general fluctuation space of the almost-associative background $\mathcal{B}_M \hat{\otimes} \mathcal{B}_F$. Then,*

$$\mathcal{F}^{M\times F}_\omega = \Omega^1_M \otimes [S(A_F), S(A_F)] \bigoplus \mathcal{C}^\infty(M, \mathcal{F}^F_\omega), \tag{63}$$

where $\mathcal{F}^F_\omega$ is the general fluctuation space of $\mathcal{B}_F$.

Proof. With the same notations as above, we let $a = f \otimes a_F \in A$ and $\omega = \omega_M \otimes b_F + g \otimes \omega_F \in \Omega^1_{M\times F}$. Then, using $f = f^o$ and $\omega^o_M = -\omega_M$ and suppressing π for simplicity, we have

$$\begin{aligned}[S_a, S_\omega] &= \frac{1}{4}[a + a^o, \omega - \omega^o] \\ &= \frac{1}{4}[f \otimes (a_F + a^o_F), \omega_M \otimes b_F + g \otimes \omega_F - (-\omega_M \otimes b^o_F + g \otimes \omega^o_F] \\ &= \frac{1}{4} f\omega_M \otimes [a_F + a^o_F, b_F + b^o_F] + fg \otimes [a_F + a^o_F, \omega_F - \omega^o_F] \\ &= f\omega_M \otimes [S_{a_F}, S_{a_F}] + fg \otimes [S_{a_F}, S_{\omega_F}]. \end{aligned} \tag{64}$$

The result follows. □

Note, in particular, that both the minimal and general gauge fluctuations will be the same. Differences may arise in the Higgs sector of a model, however.

X. BOYLE–FARNSWORTH MODEL

A. Definition of the model

Let us consider the model with finite algebra $A_F = \mathrm{JSpin}(2) \oplus H_2(\mathbb{C}) \oplus H_3(\mathbb{C}) \oplus \mathbb{R}$, which has been proposed by Boyle and Farnsworth.[7] We will identify the elements of JSpin(2) with matrices of the form $\begin{pmatrix} x & z^* \\ z & x \end{pmatrix}$ with $x \in \mathbb{R}$ and $z \in \mathbb{C}$. The Hilbert space H_F is the same as for the Standard Model (SM); cf. (6). The associative representation is

$$\pi(\lambda, h, m, r) = [\lambda \otimes 1_4, h \otimes 1_4, r \bigoplus 1_2 \otimes m, r \bigoplus 1_2 \otimes m], \tag{65}$$

where $[A, B, C, D]$ denotes a matrix, which is block-diagonal with respect to the chirality space. More generally, we will write a_{XY} with $X, Y = R, L, \bar{R}, \bar{L}$ for a matrix decomposed into chiral blocks.

B. Gauge fields

From Sec. IX, we know that gauge fields will take values in some representation of $u(1) \oplus su(2) \oplus su(3)$ [since $u(1) \simeq so(2)$]. To work out the precise representations, we first express the symmetrized action $S(A_F)$, which contains general elements that are sums of the form

$$\begin{aligned} 2S(\lambda) &= [\lambda \otimes 1_4, 0, \lambda \otimes 1_4, 0], \\ 2S(h) &= [0, h \otimes 1_4, 0, h \otimes 1_4], \\ 2S(m) &= [0 \bigoplus 1_2 \otimes m, 0 \bigoplus 1_2 \otimes m, 0 \bigoplus 1_2 \otimes m, 0 \bigoplus 1_2 \otimes m], \\ 2S(r) &= [r \bigoplus 0, r \bigoplus 0, r \bigoplus 0, r \bigoplus 0]. \end{aligned} \tag{66}$$

Because elements of different kinds commute with one another and $S(r)$ commutes with everything, we, therefore, find that $[S(A_F), S(A_F)]$ is generated as a vector space by elements of the form

$$\begin{aligned} T(\lambda') &:= [\lambda' \otimes 1_4, 0, \lambda' \otimes 1_4, 0], \\ T(h') &:= [0, h' \otimes 1_4, 0, h' \otimes 1_4], \\ T(m') &:= [0 \bigoplus 1_2 \otimes m', 0 \bigoplus 1_2 \otimes m', 0 \bigoplus 1_2 \otimes m', 0 \bigoplus 1_2 \otimes m'], \end{aligned} \tag{67}$$

where edit $\lambda' \in \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = u(1)$, $h' \in [H_2(\mathbb{C}), H_2(\mathbb{C})] = su(2)$, and $m' \in [H_3(\mathbb{C}), H_3(\mathbb{C})] = su(3)$.

As noted by the authors in Ref. 7, even though the Lie algebra is the correct one, the $u(1)$ charges correspond to a linear combination of hypercharge and B–L. In order to obtain the correct hyper-charges, the gauge symmetries would need to be extended to include the anomaly-free outer automorphisms of the representation. In this case, one obtains the correct hypercharges, but the model is extended by an additional gauged B–L symmetry. We will not consider such an extension in this paper.

C. Higgs sector

We recall that for the noncommutative standard model, the Dirac operator takes the form

$$D_F = \begin{pmatrix} 0 & Y^\dagger & M^\dagger & 0 \\ Y & 0 & 0 & 0 \\ M & 0 & 0 & Y^T \\ 0 & 0 & Y^* & 0 \end{pmatrix}, \tag{68}$$

where $Y = Y_\ell \oplus Y_q$, $Y_\ell = \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}$, $Y_q = \begin{pmatrix} Y_u & 0 \\ 0 & Y_d \end{pmatrix}$, and $M = \begin{pmatrix} m_\nu & 0 \\ 0 & 0 \end{pmatrix} \oplus 0$.

We will take this D_F as our starting point. To check that C_1 (respectively, weak C_1) holds, we need only consider commutators of the form $[(a')^o, [D, a]]$ with $a, a' \in \pi(A_F)$ (respectively, $a' \in [\pi(A_F), \pi(A_F)]$). Consider first the case where $a' \in \pi(A_F)$. Using the same notations as in (65) for a (and the same with primes for a'), we easily find that $[(a')^o, [D, a]]$ is a self-adjoint matrix with all blocks vanishing except the $(R, \bar{R})$ and $(\bar{R}, R)$ ones, the latter being given by $(\lambda' - r')M(\lambda - r)$. Thus, C_1 is not satisfied unless $M = 0$, in which case both blocks vanish. On the other hand, if $a' \in [\pi(A_F), \pi(A_F)]$, then $r' = 0$ and $\lambda' = t\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $t \in \mathbb{R}$. Such an element is easily seen to be in $\Omega^1_{D_F}$. In other words, D_F is seen to satisfy weak C_1, while C_1 is only satisfied for $M = 0$. We, therefore, see that stronger restrictions arise on the Higgs sector under C_1 than occurring in the associative NCG SM.

Satisfied that this Dirac operator is compatible with weak C_1, let us next determine the minimal finite fluctuations. These are obtained by taking iterated commutators of elements (67) with D_F. As the matrix D_F commutes with $T(m')$, however, we only have to focus on iterated commutators of D_F with elements of the form $T(h')$ and $T(\lambda')$. Beginning with $T(h')$ yields

$$\Phi(q) := \begin{pmatrix} 0 & Y(q)^\dagger & 0 & 0 \\ Y(q) & 0 & 0 & 0 \\ 0 & 0 & 0 & Y(q)^T \\ 0 & 0 & Y(q)^* & 0 \end{pmatrix}, \tag{69}$$

where $Y(q) = qY$, $q \in su(2)$ (a pure quaternion). Commuting with $T(\lambda')$ boils down to multiplying q with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, which is another quaternion, which gives nothing new. The Higgs sector is, thus, the same as in the standard model when we consider the minimal fluctuation space.

Let us next look at the general fluctuation space. We begin by enforcing C_1, which sets $M = 0$. We, then, need to determine the form of the Jordan module of finite 1-forms. Using hermiticity, we only need to consider the (L, R)-block. Now, if $a = [a_R, a_L, a_{\bar{L}}, a_{\bar{L}}]$ and $b = [b_R, b_L, \ldots, \ldots]$, then the (L, R)-block of $a \circ [D, b]$ is

$$(a \circ [D_F, b])_{L,R} = a_L Y b_R - a_L b_L Y + Y b_R a_R - b_L Y a_R. \tag{70}$$

It is, then, easy to see that a general 1-form ω will have a block $\omega_{LR} = \sum_{i,j} a_i Y b_j$, where a_i is in the associative $\mathbb{R}$-algebra generated by $\mathrm{JSpin}(2)$ and b_j is in the associative $\mathbb{R}$-algebra generated by $H_2(\mathbb{C})$, that is, $M_2(\mathbb{C})$. Now, let us introduce the notation $\tilde{Y} := Y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Using $Y\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} Y + \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} \tilde{Y}$, we can rewrite ω_{LR} in the form $\sum x_i Y + \tilde{x}_i \tilde{Y}$, where $x_i, \tilde{x}_i$ belong to the algebra generated by $\mathrm{JSpin}(2)$ and diagonal matrices, which is $M_2(\mathbb{C})$. Thus, Ω^1_F is the set of matrices of the form

$$\omega_g = \begin{pmatrix} 0 & Z^\dagger g^\dagger & 0 & 0 \\ gZ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{71}$$

where $g \in M_2(\mathbb{C})$ and Z is either equal to Y or to $\tilde{Y}$. From (55), the general fluctuations are, thus, of the form $[\pi(a), \omega_g] + J_F[\pi(a), \omega_g]J_F^{-1}$. Using the fact that $M_2(\mathbb{C}) = \mathbb{H} \oplus i\mathbb{H}$, this yields four $SU(2)$-doublets $\Phi(q), \Phi(iq'), \tilde{\Phi}(p), \tilde{\Phi}(ip')$, where q, q', p, p' are quaternions, Φ is defined by (69), and $\tilde{\Phi}$ is the same as Φ with $\tilde{Y}$ replacing Y. These fields will be independent iff the up and down components of Y are. This example provides a proof that the general fluctuation space and the minimal one are, in general, different.

XI. THE PATI–SALAM MODEL

A. Definition of the model

The finite Hilbert space H_F is still the SM one, and D_F still has the form (68). We take $A_F = A_{\mathrm{PS}} := H_2(\mathbb{C}) \bigoplus H_2(\mathbb{C}) \bigoplus H_4(\mathbb{C})$ represented as

$$\pi(p,q,m) = [p \otimes 1_4, q \otimes 1_4, 1_2 \otimes m, 1_2 \otimes m] \otimes 1_N. \tag{72}$$

We have

$$\pi(p,q,m)^o = [1_2 \otimes m^*, 1_2 \otimes m^*, p^* \otimes 1_4, q^* \otimes 1_4] \otimes 1_N. \tag{73}$$

Hence,

$$2S(p,q,m) = [p \otimes 1_4 + 1_2 \otimes m^*, q \otimes 1_4 + 1_2 \otimes m^*, c.c., c.c.] \otimes 1_N. \tag{74}$$

B. Gauge fields

Taking the commutator of two elements such as (74), we obtain that

$$[S(A_F), S(A_F)] = su(2)_R \bigoplus su(2)_L \bigoplus su(4), \tag{75}$$

represented as

$$T(g_R, g_L, g) = [g_R \otimes 1_4 + 1_2 \otimes g, g_L \otimes 1_4 + 1_2 \otimes g, c.c, c.c] \otimes 1_N. \tag{76}$$

C. Higgs sector

The finite fluctuations are iterated commutators of elements of the form (76) with D_F. More precisely, since the summands of (75) commute among one another, a finite fluctuation has the form $T_1, \ldots, T_i \cdots S_1, \ldots, S_j \cdot R_1, \ldots, R_k \cdot D_F$, with $T_1, \ldots, T_i$ in $su(4)$, $S_1, \ldots, S_j \in su(2)_L$, and $R_1 \cdots R_k \in su(2)_R$ (with the now usual convention that R_k operates first and T_1 last). The action of R_1 to R_k on the L–R sector of (68) replaces Y with $Yg_R^1 \ldots g_R^k \otimes 1_4$ (up to an irrelevant sign). Then, we act with $S_1, \ldots, S_j$ and get $g_L^j \ldots g_L^1 \otimes 1_4 Y g_R^1 \ldots g_R^k \otimes 1_4$. Finally, we act with $T_1, \ldots, T_i$, which in the (L,R)-sector amounts to taking the commutators with $g_1, \ldots, g_i$. Hence, the L–R sector of a finite fluctuation contains a linear combination of elements of the form

$$1_2 \otimes g_1, \ldots g_i \cdot (p \otimes 1_4) Y (q \otimes 1_4), \tag{77}$$

where p and q are products of elements of $su(2)$. Such products are just generic elements of $M_2(\mathbb{C})$. Now, let us write $Y_{\alpha,i}$ for the elements of the tensor Y, where α runs through uu, ud, du, dd and i runs through $\{\ell, r, g, b\}^2$. Hence, for each i, $Y_{.,i}$ is a 2×2 matrix acting on $\mathbb{C}^2_{\mathrm{weak}}$. If there are N generations, $Y_{\alpha,i}$ will be a $N \times N$-matrix. For each i, if $Y_{.,i} \neq 0$, the products $pY_{.,i}q$ generate $M_2(\mathbb{C})$. Moreover, $M_4(\mathbb{C})$ decomposes as $\mathbb{C} \bigoplus su(4) \bigoplus isu(4)$ under the adjoint action of $su(4)$. Hence, (77) will be a completely generic traceless element of $M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$ unless $Y_{\alpha,.}$ belongs to one of the submodules $\mathbb{C}, su(4), isu(4), \mathbb{C} \bigoplus su(4), \ldots$ for every α. Given that $pY_{.,i}q$ generate $M_2(\mathbb{C})$ unless $Y_{.,i}$ vanishes, this is impossible for $Y \neq 0$. In conclusion, a Higgs field takes values in general traceless elements $M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$. The action of $su(2)_R$ is by multiplication on the right of $M_2(\mathbb{C})$, the action of $su(2)_L$ is by multiplication on the left of $M_2(\mathbb{C})$, and the action of $su(4)$ is the adjoint action on $M_4(\mathbb{C})$. This coincides with the results in Ref. 24.

XII. BEYOND THE SPECIAL CASE

Our discussion has focused on special Jordan coordinate algebras with associative specializations. We have looked specifically at real, even, special Jordan spectral triples (i.e., special Jordan triples equipped with real structure and grading operators) as these are the geometries that are of most interest when constructing physical theories. In this section, we discuss the generalization to the exceptional setting, in which it is no longer possible to construct associative specializations. Our goal is not to give a complete and rigorous account, but rather to outline the most natural path forward that we see for constructing exceptional Jordan geometries. For this purpose, we will focus on geometries without the real structure, which will allow us to highlight the primary distinctions between the special and exceptional cases without becoming tied down by interesting but tangential discussions concerning the real structure and the (weak) order conditions.

Consider a special, Jordan, spectral triple without real structure $\mathcal{T} = (A, H, \pi, D, \chi)$. A key feature of our construction is the extension of the associative specialization π of A on H to all expressions of degree 1 or less such that

$$\pi(a \circ b) = \pi(a) \circ \pi(b), \tag{78}$$

$$\pi(a \circ \omega) = \pi(a) \circ \pi(\omega) \tag{79}$$

for all $a, b \in A$ and $\omega \in \Omega^1 A$. Associative specializations readily allow for the representation of general 1-forms once the representation of exact one forms is defined. Note, however, that all other features of our construction actually derived from the properties of multiplicative specializations. Stated explicitly, an immediate consequence of Theorem 2 is that $\rho = \frac{1}{2}\pi$ satisfies all of the properties of a multiplicative specialization for expressions of degree 1 or less. The relationship between multiplicative specializations and Jordan modules, then, ensures that (H, ρ) satisfies all the properties of a Jordan module over $A \oplus \Omega^1 A$ or, equivalently, that $\mathcal{T}$ can be viewed as a Jordan algebra [i.e., Eqs. (18) and (19) are satisfied up to degree 1]. It is the Jordan module structure that ensures the form of the inner derivations of $\mathcal{T}$ restricted to H, the form that inner fluctuations of D take, and the Lie module structure of $\Omega^1_D A$.

The question of this section is how to extend the construction to accommodate exceptional Jordan coordinate algebras. Exceptional Jordan algebras do not have associative specializations, but they do have multiplicative specializations, and as such, it is natural to consider Jordan spectral triples of the form $\mathcal{T} = (A, H, \rho, D, \chi)$, where we have the following.

1. A is a Jordan algebra with a JB algebra completion $\overline{A}$.
2. H is a Hilbert space.
3. ρ is a faithful multiplicative representation of A.
4. D and χ satisfy all the usual properties.
5. For all $a \in A$, $[D, \rho(a)]$ is well-defined and bounded.

Let us unpack this construction, beginning with the coordinate algebra and its representation on the Hilbert space (A, H, ρ). It should be stressed that in the general Jordan setting, we have, in general, that

$$\rho_{a\circ b} \neq \rho_a \circ \rho_b \tag{80}$$

for $a, b \in A$. Equation (16) only holds for the special case in which one has an associative specialization, which is the key assumption that we are dropping in this section. In other words, the algebra generated by elements of the form ρ_a for $a \in A$, equipped with the symmetrized product $\circ$, will be a special Jordan algebra that will not, in general, be isomorphic to A. Nonetheless, the properties of ρ ensure that $B = A \oplus H$ is a Jordan algebra, and as such, the forms of its inner derivations are known. In particular, following Eq. (44), when restricted to H, the inner derivations of B are given by $\delta_{ab} = [\rho_a, \rho_b]$ for $a, b \in A$. Furthermore, following Eq. (19), we have

$$[\delta_{ab}, \rho_c] = \rho_{[b,c,a]} \tag{81}$$

for $c \in A$. The "lift" of the inner derivations of A, therefore, remain exactly as in the special Jordan setting with no alteration. At order zero, our construction, therefore, ports readily to the exceptional setting. The same observation applies to the minimal space $\mathcal{F}_D$ defined in Sec. VII: it continues to make sense as the Lie module over $\mathcal{G}(A)$ generated by elements of the form $[\delta, D]$ for $\delta \in \mathcal{G}(A)$. Hence, a spectral triple over a non-special Jordan algebra together with its minimal fluctuation space could, in principle, be constructed.

In order to define an algebraic background and/or a general fluctuation space, one would need a general theory for the representation of Jordan 1-forms. Let us say a few words about why such a construction is difficult. Assuming that one is able to obtain a module of universal 1-forms with the universal property (33),[42] it would, then, be natural to consider the action of exact one forms on H following Eq. (35),

$$\rho_{d[a]} = [D, \rho_a], \tag{82}$$

where $a \in A$. Unfortunately, the degree 1 extension of a multiplicative representation will have, in general, $\rho_{a\circ\omega} \neq \rho_a \circ \rho_\omega$ for $a \in A$, $\omega \in \Omega^1_d A$. It is, therefore, not possible to use our approach for special Jordan algebras to define the representation of general 1-forms or to prove, in general, that the map $a \mapsto [D, \rho(a)]$ is a derivation into $\rho(A)$. This is the primary obstruction to developing a completely general framework for Jordan coordinate algebras. Fortunately, while a general theory of Jordan 1-forms seems to be difficult to reach, an approach tailored to the particular properties of the Albert algebra appears easier to obtain. Work in this direction is currently in progress and is planned for an upcoming paper.

XIII. CONCLUSION

In this paper, we have proposed a definition for spectral triples and backgrounds based on Jordan coordinate algebras. Our treatment primarily focuses on special Jordan algebras with associative representations. We have also outlined what we view to be the most natural path toward a generalization from the special Jordan case to the exceptional setting. The key elements outlined in the extension of the spectral triple formalism to nonassociative geometries of Jordan type are as follows:

1. The definition of coordinate algebra representations and the generalization of order conditions.
2. The definition of internal symmetries and their lifts to the representation space.
3. The construction of differential forms and their representations.
4. Inner fluctuations of the Dirac operator.

Regarding the fluctuation space, we have shown that under the assumption of weak C_1, the minimal space of fluctuated Dirac operators provides a subspace of the configuration space, which is automorphism invariant and, thus, can be used to define a consistent particle model.

We have also explained the relationship of the general fluctuation space to the fluctuation space given by Connes in the associative setting. An insight we have used is the construction of fluctuations as general derivation elements of degree 1. This was also seen when constructing nonassociative geometries of alternative type[43] [see Eq. (3.23) in that paper]. This observation should allow one to extend the spectral formalism to general non-associative geometries, of which Jordan geometries are just one example.

We highlight a number of points in which Jordan geometries appear to do a better job of describing gauge theories than in the associative noncommutative setting.

1. As outlined in Ref. 7, the Jordan setting appears to provide a more natural setting for describing Majorana fermions (and by extension provides a natural solution to the fermion doubling problem).
2. The infinitesimal automorphisms of the coordinate algebra (inner derivations) are faithfully represented as operators on the Hilbert space.
3. Because the infinitesimal automorphisms of the coordinate algebra are represented by commutators, unimodularity is naturally implemented.
4. Jordan–Banach algebras are more natural candidates for coordinatizing geometric spaces than real C^* algebras.

Despite these benefits, there does appear to be one (potential) downside. In particular, the representation of $U(1)$ factors appears to be far more restrictive in the Jordan setting. This, however, might also be seen as a benefit from the perspective of constructing realistic gauge theories as it limits the allowable models that can be built (which is one of the key reasons for the interest in spectral geometry in the first place).

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

F. Besnard: Conceptualization (equal); Investigation (equal); Writing – original draft (equal). **S. Farnsworth**: Conceptualization (equal); Investigation (equal); Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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