

# RELATIVE $K$ -THEORY VIA 0-CYCLES IN FINITE CHARACTERISTIC

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*Abstract.* Let  $R$  be a regular semi-local ring, essentially of finite type over an infinite perfect field of characteristic  $p > 0$ . We show that the known cycle class map from the Chow group of 0-cycles with modulus to the relative  $K$ -theory induces a pro-isomorphism between the additive higher Chow groups of relative 0-cycles and the relative  $K$ -theory of truncated polynomial rings over  $R$ . This settles the problem of completely describing the relative  $K$ -theory of such rings via the cycle class map.

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## 1. INTRODUCTION

Ever since the invention of additive Chow groups and higher Chow groups with modulus, it has been an open question whether these groups together would give rise to a motivic cohomology which could describe the algebraic  $K$ -theory of non-reduced schemes. Existence of such a motivic cohomology was conjectured in the seminal work of Bloch and Esnault [5].

There are usually two ways to solve this question; either construct a direct cycle class map from the Chow groups with modulus to relative  $K$ -theory, or, construct an Atiyah-Hirzebruch type spectral sequence. For smooth schemes, both approaches have been shown to be successful in describing the algebraic  $K$ -theory in terms of algebraic cycles. However, this question remains unsolved for singular schemes.

In [30], Levine constructed cycle class maps with rational coefficients from Bloch's higher Chow groups [4] to the algebraic  $K$ -groups of a smooth scheme over a field. He showed that these maps induce isomorphisms between the higher Chow groups and the Adams graded pieces of the algebraic  $K$ -groups of the scheme.

Motivated by Levine's work, the authors constructed in [10] an explicit cycle class map (with integral coefficients) from the higher Chow groups of 0-cycles with modulus to the relative  $K$ -theory in the setting of pro-abelian groups. The main result of [10] was that this cycle class map induces a pro-isomorphism between the additive higher Chow

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groups of relative 0-cycles and relative  $K$ -theory of truncated polynomial rings over a regular semi-local ring, essentially of finite type over a characteristic zero field. The goal of this manuscript is to extend this result to positive characteristic.

To state our main result, recall from [2] and [27] that for a smooth scheme  $X$  of dimension  $d$  which is essentially of finite type over a field  $k$  and an effective Cartier divisor  $D \subset X$ , the higher Chow groups of codimension  $q$ -cycles with modulus are denoted by  $\mathrm{CH}^q(X|D; n)$ . Let  $K(X, D)$  denote the relative  $K$ -theory spectrum. In order to study the relative algebraic  $K$ -theory in terms of 0-cycles with modulus, it was shown in [10] that there exists a cycle class map

$$(1.1) \quad \mathrm{cyc}_{X|D}: \{\mathrm{CH}^{n+d}(X|mD; n)\}_{m \geq 1} \rightarrow \{K_n(X, mD)\}_{m \geq 1}$$

in the setting of pro-abelian groups. This cycle class map coincides with that of Levine when  $D = \emptyset$ .

Recall now that for an equi-dimensional scheme  $X$  over  $k$ , the Chow group with modulus  $\mathrm{CH}^q(X \times \mathbb{A}_k^1 | X \times (m+1)\{0\}, n)$  is the same thing as the additive higher Chow group of codimension  $q$ -cycles  $\mathrm{TCH}^q(X, n+1; m)$  (see [24]). Applying (1.1) to  $X = \mathrm{Spec}(k)$  and using the natural connecting isomorphism  $\partial: K_{n+1}(k[x]/(x^m), (x)) \xrightarrow{\cong} K_n(\mathbb{A}_k^1, m\{0\})$ , we see that (1.1) is the same thing as the map

$$(1.2) \quad \mathrm{cyc}_k: \{\mathrm{TCH}^{n+1}(k, n+1; m)\}_{m \geq 1} \rightarrow \{K_{n+1}(k[x]/(x^m), (x))\}_{m \geq 1}.$$

The main property of  $\mathrm{cyc}_k$  is that its definition is very explicit on the set of generators  $\mathrm{Tz}^{n+1}(k, n+1; m)$  (see § 2.5). This property of  $\mathrm{cyc}_k$  often turns out to be very useful in the study of  $K$ -theory via algebraic cycles. If  $R$  is, more generally, a regular semi-local ring containing  $k$ , the map  $\mathrm{cyc}_k$  directly extends to an explicit cycle class map

$$(1.3) \quad \mathrm{cyc}_R: \mathrm{Tz}_{\mathrm{sfs}}^{n+1}(R, n+1; m) \rightarrow K_{n+1}(R[x]/(x^{m+1}), (x)),$$

where  $\mathrm{Tz}_{\mathrm{sfs}}^{n+1}(R, n+1; m) \subset \mathrm{Tz}^{n+1}(R, n+1; m)$  is a subgroup of ‘sfs’ cycles (see § 6.2 for the definition of sfs cycles and § 6.3 for the definition of  $\mathrm{cyc}_R$ ).

The initial motivation behind the discovery of additive higher Chow groups by Bloch and Esnault [5] was to know if a cycle class map such as  $\mathrm{cyc}_R$  passes through rational equivalence and if the resulting map is an isomorphism. The main objective of this paper is to provide the following partial answer to the question of Bloch and Esnault. In fact, we construct a new cycle class map in the pro-setting and show that it is an isomorphism when  $k$  is any perfect field. We subsequently show that this new cycle class map coincides with  $\mathrm{cyc}_R$  of (1.3) when  $k$  is furthermore infinite.

**Theorem 1.1.** *Let  $R$  be a regular semi-local ring which is essentially of finite type over a field  $k$  such that  $\mathrm{char}(k) > 0$ . Let  $n \geq 1$  be an integer. Then there exists a cycle class map*

$$\mathrm{cyc}'_R: \{\mathrm{TCH}^n(R, n; m)\}_{m \geq 1} \rightarrow \{K_n(R[x]/(x^m), (x))\}_{m \geq 1}.$$

*This map satisfies the following.*

- (1)  $\mathrm{cyc}'_R$  is natural in  $R$ .
- (2)  $\mathrm{cyc}'_R$  is injective.
- (3)  $\mathrm{cyc}'_R$  is an isomorphism if  $k$  is perfect.
- (4) The composite map

$$\{\mathrm{Tz}_{\mathrm{sfs}}^n(R, n; m)\}_{m \geq 1} \rightarrow \{\mathrm{TCH}^n(R, n; m)\}_{m \geq 1} \xrightarrow{\mathrm{cyc}'_R} \{K_n(R[x]/(x^m), (x))\}_{m \geq 1}$$

*coincides with  $\mathrm{cyc}_R$  if  $R = k$  or  $k$  is infinite.*

Using [10] in characteristic zero and Theorem 6.2 otherwise, we obtain the following.

**Corollary 1.2.** *Let  $R$  be a regular semi-local ring which is essentially of finite type over an infinite perfect field. Then  $cyc_R$  induces an isomorphism*

$$cyc_R: \{\mathrm{TCH}^n(R, n; m)\}_{m \geq 1} \rightarrow \{K_n(R[x]/(x^m), (x))\}_{m \geq 1}.$$

We remark that the only hurdle in extending the above corollary to finite base fields is the lack of sfs-moving lemma. The proof of this moving lemma given in [28] breaks down if the base field is finite. However, we expect that the new Bertini theorems of [9] may be enough to prove the sfs-moving lemma over finite base fields.

For a semi-local ring  $R$ , let  $K_*^M(R)$  denote the Milnor  $K$ -theory of  $R$ . When  $R$  has a finite residue field,  $K_*^M(R)$  is taken to be the one defined by Gabber (unpublished) and Kerz [22]. Let  $K_n^M(R[x]/(x^m), (x))$  denote the kernel of the canonical restriction map  $K_n^M(R[x]/(x^m)) \rightarrow K_n^M(R)$ . Unless  $R$  is local, there may not exist a natural map from the Milnor  $K$ -theory (à la Gabber-Kerz) to the Quillen  $K$ -theory of  $R[x]/(x^m)$ . Nonetheless, we show in this manuscript (see Corollary 5.4) that there is indeed a natural map of pro-abelian groups

$$\psi_R: \{K_n^M(R[x]/(x^m))\}_{m \geq 1} \rightarrow \{K_n(R[x]/(x^m))\}_{m \geq 1}.$$

This induces a natural map between the pro-relative  $K$ -groups as well. Furthermore, this restricts to the known canonical map when we replace  $R$  by any of its localizations. The main step in the proof of Theorem 1.1 (except its last part) is the following extension of [10, Theorem 1.3(1)] to positive characteristic.

**Theorem 1.3.** *Let  $R$  be a regular semi-local ring which is essentially of finite type over a field of characteristic  $p > 0$ . Let  $n \geq 1$  be an integer. Then there exists a cycle class map*

$$cyc_R^M: \{\mathrm{TCH}^n(R, n; m)\}_{m \geq 1} \rightarrow \{K_n^M(R[x]/(x^m), (x))\}_{m \geq 1}$$

*which is natural in  $R$  and is an isomorphism.*

The cycle class map  $cyc_R^M$  in Theorem 1.1 is, by definition, the composition  $\psi_R \circ cyc_R^M$ . We remark that by a result of Morrow [32] (which implicitly uses Theorem 4.4 of this paper, see the proof of Proposition 5.8), one knows that the canonical map from the relative Milnor  $K$ -theory to the relative Quillen  $K$ -theory is a pro-isomorphism when  $R$  is local. However, there are two points to be noted regarding Theorem 1.3. First, the pro-isomorphism between the relative Milnor  $K$ -theory and the Quillen  $K$ -theory for semi-local rings is not a straightforward deduction from the local case. Second, and more important, Theorem 1.3, along with Theorem 1.1(4), asserts that the cycle class map  $cyc_R$  (whose study is our main interest) from the additive higher Chow groups to the relative Quillen  $K$ -theory in (1.3) factors through the relative Milnor  $K$ -theory if the base field is infinite and perfect. This plays a very important role in understanding the cycle class map and in our proof. A similar result in characteristic zero was proven in [10].

A fundamental fact in Voevodsky's theory of motivic cohomology is that if  $R$  is an equi-characteristic regular semi-local ring, then its motivic cohomology in the equal bi-degree (the Milnor range) is isomorphic to the Milnor  $K$ -theory of  $R$  (see [7], [33] and [38]). Theorem 1.3 (3) says that this isomorphism also holds for truncated polynomial algebras over such rings. This provides a key evidence that if one could extend Voevodsky's theory of motives to so-called fat points (infinitesimal extensions of spectra of fields), then the underlying motivic cohomology groups must coincide with the additive higher Chow groups, at least in the setting of pro-abelian groups (see [25]).

**1.1. A brief outline of the proofs.** Our strategy for proving Theorem 1.1 (except part (4)) is to define the cycle class map  $cyc_R$  as the composition of  $\psi_R$  with a cycle class map  $cyc_R^M$ , to the relative Milnor  $K$ -theory of the truncated polynomial ring over the underlying regular semi-local ring  $R$ . The two known results that are used to achieve this are the ‘Chow-Witt isomorphism theorems’ of Rülling [35] and Krishna-Park [29] and the ‘Milnor-Witt isomorphism theorem’ of Rülling-Saito [36]. But these two results are not quite enough.

Going beyond, a fundamental result of independent interest that we need to prove is that there is a pro-isomorphism between the Milnor  $K$ -theories of Rülling-Saito [36], Kato-Saito [20] and Gabber-Kerz [22] (see Theorem 4.4). This allows us to define a cycle class map from the additive higher Chow groups to the relative Milnor  $K$ -theory of the truncated polynomial ring (see § 5.3). This map is easily seen to be an isomorphism by its construction.

The next step is to show the existence of a canonical map from the relative Milnor  $K$ -theory to relative Quillen  $K$ -theory of truncated polynomial rings over a regular semi-local ring (see Corollary 5.4). This requires care if the base field is finite and the ring is not local. The third step is show that the above described map is a pro-isomorphism. This is done in Proposition 5.8 using Theorem 4.4 and a result of Morrow [32, Corollary 5.5].

The final step to explicitly describe this composition when  $R$  is a field or is essentially of finite type over an infinite field. Under these assumptions, we define an explicit cycle class map to the relative Quillen  $K$ -theory of truncated polynomials in § 2.5 and § 6. The proof of Theorem 1.1 is then completed in Lemma 7.4.

In all the above steps, we have to pay special care if the underlying regular semi-local ring is not local. This is because the Gabber-Kerz Milnor  $K$ -theory is not very well behaved for such rings. This forces us to work with their sheafified versions. But this brings in another technical problem. Namely, a local isomorphism between two pro-sheaves does not necessarily imply an isomorphism between them. This is in contrast to the case of sheaves. To take care of this problem, we always have to prove a stronger assertion than merely an isomorphism whenever we work with pro-abelian groups of  $K$ -theories and de Rham-Witt forms on a local ring (see Lemma 2.1 for a precise version we need to prove).

## 2. THE CYCLE CLASS MAP

In this section, we shall recall our main object of study, the cycle class map for the additive higher Chow groups of 0-cycles, from [10]. Before this, we shall fix our notations, recall the basic definitions and prove some intermediate results to be used in the proofs of the main theorems.

**2.1. Notations.** Given a field  $k$ , we let  $\mathbf{Sch}_k$  denote the category of separated finite type schemes over  $k$  and let  $\mathbf{Sm}_k$  denote the full subcategory of non-singular schemes over  $k$ . For  $X, Y \in \mathbf{Sch}_k$ , we shall denote the product  $X \times_k Y$  simply by  $X \times Y$ . For any point  $x \in X$ , we shall let  $k(x)$  denote the residue field of  $x$ . For a reduced scheme  $X \in \mathbf{Sch}_k$ , we shall let  $X^N$  denote the normalization of  $X$ . Given a closed immersion  $D \subset X$  in  $\mathbf{Sch}_k$  defined by a sheaf of ideals  $\mathcal{I}_D \subset \mathcal{O}_X$ , we shall let  $mD \subset X$  denote the closed subscheme of  $X$  defined by the sheaf of ideals  $\mathcal{I}_D^m$  for  $m \geq 1$ . In this article, we shall always consider the Zariski topology on a Noetherian scheme whenever we talk about sheaves.

We shall let  $\bar{\square}$  denote the projective space  $\mathbb{P}_k^1 = \text{Proj}(k[Y_0, Y_1])$  and let  $\square = \bar{\square} \setminus \{1\}$ . We shall let  $\mathbb{A}_k^n = \text{Spec}(k[y_1, \dots, y_n])$  be the open subset of  $\bar{\square}^n$ , where  $(y_1, \dots, y_n)$  denotes the coordinate system of  $\bar{\square}^n$  with  $y_j = Y_1^j / Y_0^j$ . Given a rational map  $f: X \rightarrow \bar{\square}^n$  in  $\mathbf{Sch}_k$  and a point  $x \in X$  lying in the domain of definition of  $f$ , we shall let  $f_i(x) = (y_i \circ f)(x)$ ,

where  $y_i: \bar{\square}^n \rightarrow \bar{\square}$  is the  $i$ -th projection. For any  $1 \leq i \leq n$  and  $t \in \bar{\square}(k)$ , we let  $F_{n,i}^t$  denote the closed subscheme of  $\bar{\square}^n$  given by  $\{y_i = t\}$ . We let  $F_n^t = \sum_{i=1}^n F_{n,i}^t$ .

All rings in this text will be commutative and Noetherian. For such a ring  $R$  and an integer  $m \geq 0$ , we shall let  $R_m = R[t]/(t^{m+1})$  denote the truncated polynomial algebra over  $R$ . If in proving a statement in this manuscript, we have to deal with a ring  $R$  and an ideal  $I \subset R$ , we shall use the notation  $\bar{a}$  for the residue class in  $R/I$  of an element  $a \in R$ . If there are several ideals, we shall indicate the quotient in which we consider the residue class  $\bar{a}$ . The tensor product  $M \otimes_{\mathbb{Z}} N$  will be denoted simply as  $M \otimes N$ . Tensor products over other bases will be explicitly indicated. For an abelian group  $M$  and an integer  $n \geq 1$ , we let  ${}_n M$  denote the  $n$ -torsion subgroup of  $M$ .

**2.2. The category of pro-objects.** Since we shall mostly work in the category of pro-abelian groups, we recall here the notion of pro-objects in a general category. By a pro-object in a category  $\mathcal{C}$ , we shall mean a sequence of objects  $\{A_m\}_{m \geq 0}$  together with a map  $\alpha_m^A: A_{m+1} \rightarrow A_m$  for each  $m \geq 0$ . We shall write this object often as  $\{A_m\}$ . We let  $\text{Pro}(\mathcal{C})$  denote the category of pro-objects in  $\mathcal{C}$  with the morphism set given by

$$(2.1) \quad \text{Hom}_{\text{pro}(\mathcal{C})}(\{A_m\}, \{B_m\}) = \lim_n \lim_m \text{Hom}_{\mathcal{C}}(A_m, B_n).$$

It follows that giving a morphism  $f: \{A_m\} \rightarrow \{B_m\}$  in  $\text{Pro}(\mathcal{C})$  is equivalent to finding a function  $\lambda: \mathbb{N} \rightarrow \mathbb{N}$ , a map  $f_n: A_{\lambda(n)} \rightarrow B_n$  for each  $n \geq 0$  such that for each  $n' \geq n$ , there exists  $l \geq \lambda(n), \lambda(n')$  so that the diagram

$$(2.2) \quad \begin{array}{ccc} A_l & \rightarrow & A_{\lambda(n')} \xrightarrow{f_{n'}} B_{n'} \\ & \searrow & \downarrow \\ & & A_{\lambda(n)} \xrightarrow{f_n} B_n \end{array}$$

is commutative, where the unmarked arrows are the structure maps of  $\{A_m\}$  and  $\{B_m\}$ . We shall say that  $f$  is strict if  $\lambda$  is the identity function and  $l = \lambda(n') = n'$ . If  $\mathcal{C}$  admits all sequential limits, we shall denote the limit of  $\{A_m\}$  by  $\varprojlim_m A_m \in \mathcal{C}$ . If  $\mathcal{C}$  is an abelian category, then so is  $\text{Pro}(\mathcal{C})$ . We refer the reader to [1, Appendix 4] for further details about  $\text{Pro}(\mathcal{C})$ .

Let  $\mathcal{C}$  be an abelian category and let  $\{A_m\}_m$  be a pro-object in  $\mathcal{C}$ . We shall say that  $\{A_m\}$  is bounded by an integer  $N \in \mathbb{N}$  if the structure map  $A_{m+N} \rightarrow A_m$  is zero for all  $m \geq 0$ . A pro-object  $\{A_m\}$  which is bounded by an integer is classically also called AR-zero (Artin-Rees zero), see [18, Exposé V, Definition 2.2.1]. We shall say that  $\{A_m\}$  is bounded by  $\infty$  if  $\{A_m\} = 0$  in  $\text{Pro}(\mathcal{C})$ .

Let  $X$  be a Noetherian scheme. By a sheaf (or pre-sheaf) of pro-abelian groups on  $X$ , we shall mean a pro-object in the abelian category of sheaves (or pre-sheaves) of abelian groups on  $X$ . We caution the reader that if  $\{\mathcal{F}_m\}$  is a sheaf of pro-abelian groups such that the pro-abelian group of stalks  $\{\mathcal{F}_{m,x}\}$  is zero for all  $x \in X$ , then we can not in general conclude that  $\{\mathcal{F}_m\}$  is zero. However, the following is still true and will be used repeatedly in this article.

**Lemma 2.1.** *Let  $\{\mathcal{F}_m\}$  be a sheaf of pro-abelian groups on a Noetherian scheme  $X$ . Suppose that for every integer  $m \geq 0$ , there is an integer  $N(m) \geq 0$  such that the structure map  $\mathcal{F}_{m+N(m),x} \rightarrow \mathcal{F}_{m,x}$  is zero for all  $x \in X$ . Then  $\{\mathcal{F}_m\} = 0$ . If there is an integer  $N \geq 0$  such that  $\{\mathcal{F}_{m,x}\}$  is bounded by  $N$  for all  $x \in X$ , then  $\{\mathcal{F}_m\}$  is bounded by  $N$ .*

If  $f: \{\mathcal{F}_m\} \rightarrow \{\mathcal{F}'_m\}$  is an isomorphism of sheaves of pro-abelian groups on  $X$ , then the morphism of pro-abelian groups  $H^i(f): \{H^i(X, \mathcal{F}_m)\} \rightarrow \{H^i(X, \mathcal{F}'_m)\}$  is an isomorphism for all  $i \geq 0$ .

*Proof.* It is elementary and is left to the reader.  $\square$

**2.3. The relative algebraic  $K$ -theory.** Given a closed immersion  $D \subset X$  of schemes, we let  $K(X, D)$  be the homotopy fiber of the restriction map between the Bass-Thomason-Trobaugh non-connective algebraic  $K$ -theory spectra  $K(X) \rightarrow K(D)$ . We shall let  $K_i(X)$  denote the homotopy groups of  $K(X)$  for  $i \in \mathbb{Z}$ . We similarly define  $K_i(X, D)$ . The canonical maps of spectra  $K(X, (m+1)D) \rightarrow K(X, mD)$  together give rise to a pro-spectrum  $\{K(X, mD)\}$  and hence a pro-abelian group  $\{K_i(X, mD)\}$  for every  $i \in \mathbb{Z}$ .

If  $X = \text{Spec}(R)$  is affine and  $D = V(I)$ , we shall often write  $K(X, mD)$  as  $K(R, I^m)$  and  $K(X)$  as  $K(R)$ . For a ring  $R$ , we shall let  $\tilde{K}(R_m)$  denote the reduced  $K$ -theory of  $R_m$ , defined as the homotopy fiber of the augmentation map  $K(R_m) \rightarrow K(R)$ .

We need to use a push-forward map between the relative  $K$ -groups in a special situation. We describe this below. Let  $R \rightarrow R'$  be a finite and flat extension of rings and let  $(m, n)$  be a pair of integers such that  $m \geq n \geq 0$ . Let  $f: \text{Spec}(R') \rightarrow \text{Spec}(R)$  denote the corresponding maps between the schemes. Since  $R'_m \cong R_m \otimes_R R'$ , it follows that  $R'_m \otimes_{R_m} R_n \cong R'_n$ , where  $R_m \twoheadrightarrow R_n$  and  $R'_m \twoheadrightarrow R'_n$  are the canonical surjections. In particular, the diagram of schemes

$$(2.3) \quad \begin{array}{ccc} \text{Spec}(R'_n) & \twoheadrightarrow & \text{Spec}(R_n) \\ \downarrow & & \downarrow \\ \text{Spec}(R'_m) & \twoheadrightarrow & \text{Spec}(R_m) \end{array}$$

is Cartesian, where the vertical arrows are the closed immersions induced by the surjections  $R_m \twoheadrightarrow R_n$  and  $R'_m \twoheadrightarrow R'_n$ . Since the horizontal arrows in this diagram are flat, it follows that  $\text{Spec}(R_n)$  and  $\text{Spec}(R'_m)$  are Tor-independent over  $\text{Spec}(R_m)$ . Since  $R'_m$  is finite and flat over  $R_m$ , it follows from [37, Proposition 3.18] that (2.3) induces a homotopy commutative diagram of spectra

$$(2.4) \quad \begin{array}{ccc} K(R'_m) & \xrightarrow{f_{m*}} & K(R_m) \\ \downarrow & & \downarrow \\ K(R'_n) & \xrightarrow{f_{n*}} & K(R_n), \end{array}$$

where the horizontal arrows are the push-forward and the vertical arrows are the pull-back maps between  $K$ -theory spectra. Considering the map induced between the vertical homotopy fibers, we get a push-forward map  $f_{(m,n)*}: K(R'_m, (x^{n+1})/(x^{m+1})) \rightarrow K(R_m, (x^{n+1})/(x^{m+1}))$  between the relative  $K$ -theory spectra. The special case of the pair  $(m, 0)$  yields the push-forward map  $f_{m*}: \tilde{K}(R'_m) \rightarrow \tilde{K}(R_m)$  between the reduced  $K$ -theory spectra.

**Lemma 2.2.** *Let  $R \rightarrow R'$  be as above and let  $m \geq n \geq 0$  be two integers. Then the diagram*

$$(2.5) \quad \begin{array}{ccc} \tilde{K}_i(R'_m) & \xrightarrow{f_{m*}} & \tilde{K}_i(R_m) \\ \downarrow & & \downarrow \\ \tilde{K}_i(R'_n) & \xrightarrow{f_{n*}} & \tilde{K}_i(R_n) \end{array}$$

is commutative for every  $i \in \mathbb{Z}$ , where the vertical arrows are the pull-back maps induced by the quotients  $R_m \twoheadrightarrow R_n$  and  $R'_m \twoheadrightarrow R'_n$ . In particular, there is a push-forward map between the pro-abelian groups  $f_*: \{\tilde{K}_i(R'_m)\}_m \rightarrow \{\tilde{K}_i(R_m)\}_m$  for every  $i \in \mathbb{Z}$ .

*Proof.* We fix  $i \in \mathbb{Z}$  and consider the diagram

$$(2.6) \quad \begin{array}{ccccc} \tilde{K}_i(R'_m) & \xrightarrow{f_{m*}} & \tilde{K}_i(R_m) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & K_i(R'_m) & \xrightarrow{f_{m*}} & K_i(R_m) & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \tilde{K}_i(R'_n) & \xrightarrow{f_{n*}} & \tilde{K}_i(R_n) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & K_i(R'_n) & \xrightarrow{f_{n*}} & K_i(R_n) & \end{array}$$

We have seen in (2.4) that the front face of (2.6) is commutative. The left and the right faces clearly commute. The top and the bottom faces commute by applying (2.4) corresponding to the pairs  $(m, 0)$  and  $(n, 0)$ , respectively. Since the map  $\tilde{K}_i(R_n) \rightarrow K_i(R_n)$  is injective, it follows that the back face commutes, as desired.  $\square$

Let us now assume that  $k \hookrightarrow k'$  is a finite extension of fields and let  $f: \text{Spec}(k') \rightarrow \text{Spec}(k)$  denote the induced morphism of schemes. In this case,  $k[t] \hookrightarrow k'[t]$  is clearly a finite and flat extension of polynomial rings. Let  $A = k[t]_{(t)}$  denote the localization of  $k[t]$  at the maximal ideal  $(t)k[t] \subset k[t]$ . Let  $A' = k'[t]_{(t)}$  denote the localization of  $k'[t]$  at  $(t)k'[t]$ . We claim that  $A \hookrightarrow A'$  is a finite and flat extension of discrete valuation rings.

Indeed, it is clear that there are ring extensions  $A \hookrightarrow S^{-1}k'[t] \hookrightarrow A'$  in which the first inclusion is finite and flat if we let  $S = k[t] \setminus (t)$ . For every integer  $m \geq 0$ , we have a sequence of isomorphisms of  $k$ -algebras:

$$(2.7) \quad \begin{aligned} S^{-1}k'[t] \otimes_A A / (t^{m+1}) &\cong (k'[t] \otimes_{k[t]} A) \otimes_A k_m \cong k'[t] \otimes_{k[t]} k_m \\ &\cong (k' \otimes_k k[t]) \otimes_{k[t]} k_m \cong k' \otimes_k k_m \cong k'_m. \end{aligned}$$

If we let  $\mathfrak{m} = (t)k[t] \subset A$  and  $m = 0$ , it follows that  $A \hookrightarrow S^{-1}k'[t]$  is a finite extension of regular semi-local integral domains of dimension one such that  $\mathfrak{m}S^{-1}k'[t]$  is a maximal ideal of  $S^{-1}k'[t]$ . This forces  $S^{-1}k'[t]$  to be a discrete valuation ring (since  $A$  is). Since  $A'$  is also a discrete valuation ring which is a localization of  $S^{-1}k'[t]$ , different from the fraction field of  $S^{-1}k'[t]$ , we must have  $S^{-1}k'[t] = A'$ . This proves the claim.

For any integer  $m \geq 0$ , we now get finite and flat ring extensions

$$k \hookrightarrow k', \quad k_m \hookrightarrow k'_m, \quad A \hookrightarrow A', \quad \text{and} \quad k(t) \hookrightarrow k'(t).$$

Each of these extensions induces a push-forward map on the  $K$ -theory spectra. We denote all these push-forward maps by the common notation  $f_*$ .

**Lemma 2.3.** *For each  $i \in \mathbb{Z}$ , there is a commutative diagram*

$$(2.8) \quad \begin{array}{ccccccc} \tilde{K}_i(k'_m) & \hookrightarrow & K_i(k'_m) & \leftarrow & K_i(A') & \hookrightarrow & K_i(k'(t)) \\ f_* \downarrow & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ \tilde{K}_i(k_m) & \hookrightarrow & K_i(k_m) & \leftarrow & K_i(A) & \hookrightarrow & K_i(k(t)), \end{array}$$

where the horizontal arrows in the middle and the right side squares are the natural maps on  $K$ -theory induced by the ring homomorphisms. The horizontal arrows in the left square are the canonical maps.

*Proof.* The left square commutes by the construction of  $f_*$  in (2.4). The horizontal arrows in this square are split injective via the augmentation maps. The middle square commutes by exactly the same argument as for the commutativity of (2.4) since  $A'$  is finite and flat over  $A$ , hence  $A'$  and  $k_m$  are Tor-independent over  $A$ . Furthermore,  $A' \otimes_A k_m \cong k'_m$  by (2.7). The square on the right side commutes by a similar reason once we know that  $k(t) \otimes_A A' \cong k'(t)$ . But this is obvious because  $k'(t)$  is the field of fractions of  $A'$  on the one hand and  $k(t) \otimes_A A'$  is a localization of the integral domain  $A'$  which is finite over  $k(t)$  on the other hand. It follows that  $k'(t) \subset k(t) \otimes_A A' \subset k'(t)$ . The horizontal arrows in the right side square are injective by the Gersten resolution of  $K$ -theory.  $\square$

**2.4. The additive higher Chow groups.** We recall the definition of the higher Chow groups with modulus and the additive higher Chow groups (see [2], [24] and [27]). Let  $k$  be a field and let  $X$  be an equidimensional scheme over  $k$ . Let  $D \subset X$  be an effective Cartier divisor.

For any integers  $n, q \geq 0$ , we let  $\underline{z}^q(X|D, n)$  denote the free abelian group on the set of integral closed subschemes of  $X \times \square^n$  of codimension  $q$  satisfying the following.

- (1)  $Z$  intersects  $X \times F$  properly for each face  $F \subset \square^n$ .
- (2) If  $\overline{Z}$  is the closure of  $Z$  in  $X \times \overline{\square}^n$  and  $\nu: \overline{Z}^N \rightarrow X \times \overline{\square}^n$  is the canonical map from the normalization of  $\overline{Z}$ , then the inequality (called the modulus condition)

$$\nu^*(D \times \overline{\square}^n) \leq \nu^*(X \times F_n^1)$$

holds in the group of Weil divisors on  $\overline{Z}^N$ .

An element of the group  $\underline{z}^q(X|D, n)$  will be called an admissible cycle. It is known that  $\{n \mapsto \underline{z}^q(X|D, n)\}$  is a cubical abelian group (see [23, § 1]). We denote this by  $\underline{z}^q(X|D, *)$ . We let  $z^q(X|D, *) = \frac{\underline{z}^q(X|D, *)}{\underline{z}_{\text{degn}}^q(X|D, *)}$ , where  $\underline{z}_{\text{degn}}^q(X|D, *)$  is the degenerate part of the cubical abelian group  $\underline{z}^q(X|D, *)$ .

The higher Chow groups with modulus of  $(X, D)$  are defined as  $\text{CH}^q(X|D, n) = H_n(z^q(X|D, *))$ . It is clear that there is a canonical map  $\text{CH}^q(X|(m+1)D, n) \rightarrow \text{CH}^q(X|mD, n)$  for every integer  $m \geq 1$ . In particular,  $\{\text{CH}^q(X|mD, n)\}_{m \geq 1}$  is a pro-abelian group.

For an equidimensional scheme  $X$  over  $k$  and integers  $m, n \geq 0, q \geq 1$ , the additive higher Chow group of  $X$  is defined by

$$(2.9) \quad \text{TCH}^q(X, n+1; m) := \text{CH}^q(X \times \mathbb{A}_k^1 | X \times (m+1)\{0\}, n).$$

As with the Chow groups with modulus, the datum  $(X, n, q)$  for  $n, q \geq 1$  gives rise to a pro-abelian group  $\{\text{TCH}^q(X, n; m)\}_{m \geq 0}$ .

**2.5. The cycle class map.** In this subsection, we recall our main object of study, the cycle class map for 0-cycles with modulus, which was constructed in [10]. Let  $X$  be a smooth quasi-projective scheme of dimension  $d \geq 1$  over a field  $k$  and let  $D \subset X$  be an effective Cartier divisor. We fix an integer  $n \geq 0$ .

Let  $z \in X \times \square^n$  be an admissible closed point and let  $f: z = \text{Spec}(k(z)) \rightarrow \square^n$  be the projection map. For  $1 \leq i \leq n$ , we let  $y_i: \text{Spec}(k(z)) \xrightarrow{f} \square^n \rightarrow \square$  be the projection to the  $i$ -th factor of  $\square^n$ . It follows from the face condition of  $z$  that  $y_i(z)$  does not meet  $0, \infty \in \square$  for any  $i$ . Hence, we get an element  $\underline{y}(z) = \{y_1(z), \dots, y_n(z)\} \in K_n^M(k(z))$ . Composing with the canonical map  $K_n^M(k(z)) \rightarrow K_n(k(z))$ , we get an element  $\underline{y}(z) \in K_n(k(z))$ .



Since  $\text{Spec}(k(z)) \rightarrow X$  is finite and  $z \notin D \times \square^n$ , it follows that there is a push-forward map  $K(k(z)) \cong K(z, \emptyset) \rightarrow K(X, D)$ . Letting  $\text{cyc}_{X|D}([z])$  be the image of  $\underline{y}(z) \in K_n(k(z))$  in  $K_n(X, D)$  and extending it linearly, we obtain a cycle class map

$$(2.10) \quad \text{cyc}_{X|D}: z^{d+n}(X|D, n) \rightarrow K_n(X, D).$$

The key observation in the construction of the cycle class map at the level of the Chow group of 0-cycles with modulus is that the composite map  $z^{d+n}(X|(n+1)D, n+1) \hookrightarrow z^{d+n}(X|D, n+1) \xrightarrow{\partial} z^{d+n}(X|D, n)$  is zero. This yields for every  $m \geq 1$ , the map

$$(2.11) \quad \text{cyc}_{X|mD}: \text{CH}^{d+n}(X|(n+1)mD, n) \rightarrow K_n(X, mD).$$

This coincides with the cycle class maps of Levine [30] and Totaro [38] when  $D = \emptyset$ .

It is immediate from the above construction that the maps  $\{\text{cyc}_{X|mD}\}_{m \geq 1}$  are compatible with respect to the change in  $m \geq 1$ . In particular, they together give rise to a map of pro-abelian groups  $\text{cyc}_{X|D}: \{\text{CH}^{d+n}(X|mD, n)\}_m \rightarrow \{K_n(X, mD)\}_m$ .

Applying (2.11) to the additive higher Chow groups of the field  $k$ , we see that for every  $m \geq 0$  and  $n \geq 1$ , there is a cycle class map  $\text{cyc}_{k|m}: \text{TCH}^n(k, n; n(m+1) - 1) \rightarrow K_{n-1}(\mathbb{A}_k^1, (m+1)\{0\})$ . The homotopy fiber sequence  $K(\mathbb{A}_k^1, (m+1)\{0\}) \rightarrow K(\mathbb{A}_k^1) \rightarrow K(\text{Spec}(k_m))$  and the homotopy invariance of  $K$ -theory for smooth schemes together show that the connecting homomorphism  $\partial: \Omega \tilde{K}(\text{Spec}(k_m)) \rightarrow K(\mathbb{A}_k^1, (m+1)\{0\})$  is a functorial weak equivalence. Hence, we get a cycle class map

$$(2.12) \quad \text{cyc}_{k|m}: \text{TCH}^n(k, n; n(m+1) - 1) \rightarrow \tilde{K}_n(k_m).$$

The compatibility of these maps for varying values of  $m \geq 0$  yields the cycle class map at the level of pro-abelian groups

$$(2.13) \quad \text{cyc}_k: \{\text{TCH}^n(k, n; m)\}_m \rightarrow \{\tilde{K}_n(k_m)\}_m$$

for which the associated function  $\lambda_n: \mathbb{N} \rightarrow \mathbb{N}$  (see § 2.2) is given by  $\lambda_n(m) = n(m+1) - 1$ .

About the above cycle class map, the following were shown in [10] when the base field has characteristic zero.

- (1) The map  $\text{cyc}_k$  of (2.13) extends to the additive Chow group of relative 0-cycles over a regular semi-local domain  $R$ , essentially of finite type over  $k$ .
- (2) The resulting map  $\text{cyc}_R$  is an isomorphism.

In this manuscript, we wish to study this problem when  $k$  has positive characteristic.

### 3. THE RELATIVE MILNOR $K$ -GROUPS

The relative Milnor  $K$ -groups were defined by Kato-Saito [20, § 1.3], Kerz [22] and Rülling-Saito [36, § 2.7]. The groups defined by Kato-Saito and Kerz agree when all residue fields of the underlying ring are infinite. However, they differ from the one defined by Rülling-Saito even if all residue fields are infinite. When the underlying ring has a finite residue field, all three are in general different from each other. We need to establish some isomorphisms between these  $K$ -groups in pro-setting in order to prove our main results. We shall prove these isomorphisms in the next two sections.

**3.1. Kato-Saito relative Milnor  $K$ -groups.** For a ring  $R$ , the Milnor  $K$ -group  $K_n^M(R)$  was defined by Kato [19] to be the  $n$ -th graded piece of the graded ring  $K_*^M(R)$ . The latter is the quotient of the tensor algebra  $T_*(R^\times)$  by the two-sided graded ideal generated by the homogeneous elements  $a_1 \otimes \cdots \otimes a_n$  such that  $n \geq 2$  and  $a_i + a_j = 1$  for some  $1 \leq i \neq j \leq n$ . The residue class  $a_1 \otimes \cdots \otimes a_n \in T_n(R^\times)$  in  $K_n^M(R)$  is denoted by the Milnor symbol  $\underline{a} = \{a_1, \dots, a_n\}$ . If  $a_i + a_j = 1$  for some  $1 \leq i \neq j \leq n$ , we shall usually say that  $\underline{a}$  is a Kato symbol (or Kato relation). If  $I \subset R$  is an ideal, the relative Milnor

$K$ -group  $K_n^M(R, I)$  was defined by Kato-Saito [20, § 1.3] as the kernel of the natural map  $K_n^M(R) \rightarrow K_n^M(R/I)$ . In order to give a simple description of  $K_*^M(R, I)$ , we need the following elementary step.

**Lemma 3.1.** *Let  $R$  be a semi-local ring containing a field of cardinality at least three. Let  $I \subset R$  be a proper ideal. Suppose  $a \in R$  is such that  $\bar{a} \in (R/I)^\times$ . We can then find an element  $b \in I$  such that  $a + b \in R^\times$ . If  $\bar{a}, 1 - \bar{a} \in (R/I)^\times$ , then we have  $a + b, 1 - (a + b) \in R^\times$ .*

*Proof.* Let  $\{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$  be the set of maximal ideals of  $R$  which contain  $I$  and let  $\{\mathfrak{n}_1, \dots, \mathfrak{n}_s\}$  be the set of remaining maximal ideals of  $R$ . Since  $R$  contains a field of cardinality at least three, we can find an element  $u \in R$  such that  $u, 1 - u \in R^\times$ . By the Chinese remainder theorem, we can find an element  $b \in I$  such that  $b \equiv u - a$  modulo  $\mathfrak{n}_i$  for  $1 \leq i \leq s$ .

If  $\bar{a} \in (R/I)^\times$ , then we must have that  $a \notin \mathfrak{m}_i$  for any  $i$ . It follows that  $a + b \notin \mathfrak{m}_i$  for all  $i$ . Since  $a + b$  is a unit modulo  $\mathfrak{n}_j$  for each  $j$ ,  $a + b$  can not belong to  $\mathfrak{n}_j$  either. It follows that  $a + b \in R^\times$ . Suppose now that  $\bar{a}, 1 - \bar{a} \in (R/I)^\times$ . Then  $1 - (a + b)$  can not be in any  $\mathfrak{m}_i$ . On the other hand, we have  $1 - (a + b) \equiv 1 - u$ , which is a unit modulo  $\mathfrak{n}_j$  for every  $j$  and hence  $1 - (a + b)$  can not be in any  $\mathfrak{n}_j$  either. It follows that  $a + b, 1 - (a + b) \in R^\times$ .  $\square$

The next lemma is due to Kato-Saito (see [20, Lemma 1.3.1]) when  $R$  is local. We shall need a version of this also for the relative  $K$ -theory of Kerz [21] (see Lemma 3.4).

**Lemma 3.2.** ([20, Lemma 1.3.1]) *Let  $R$  be a semi-local ring and  $I \subset R$  a proper ideal. Then  $K_n^M(R) \rightarrow K_n^M(R/I)$  is surjective. If  $R$  contains a field of cardinality at least three, then  $K_n^M(R, I)$  is generated by the Milnor symbols  $\{a_1, \dots, a_n\}$  such that  $a_i \in \text{Ker}(R^\times \rightarrow (R/I)^\times)$  for some  $1 \leq i \leq n$ .*

*Proof.* The first part of the lemma follows from Lemma 3.1. The reader can check from the proof of Lemma 3.1 that this part does not require  $R$  to contain a field (take  $u = 1$ ). To prove the second part, let  $N$  be the ideal of  $T_*(R)$  generated Kato relations (see above) and the Milnor symbols of the type given in the statement of the lemma. It is clear that the map  $T_*(R)/N \rightarrow K_*^M(R/I)$  is surjective. It suffices therefore to construct a map  $\eta_I: K_*^M(R/I) \rightarrow T_*(R)/N$  such that the composite  $T_*(R)/N \rightarrow K_*^M(R/I) \rightarrow T_*(R)/N$  is identity.

Given  $a'_1, \dots, a'_n \in (R/I)^\times$ , we can use the first part of the lemma to find  $a_1, \dots, a_n \in R^\times$  such that  $\bar{a}_i = a'_i$  for each  $i$ . We let  $\eta_I(a'_1 \otimes \dots \otimes a'_n) = \{a_1, \dots, a_n\}$  modulo  $N$ . To show that this does not depend on the choice of the lifts, we first let  $n = 2$  (note that the  $n = 1$  case is clear). We let  $a_i, b_i \in R^\times$  be such that  $a_i b_i^{-1} \equiv 1$  modulo  $I$  for  $i = 1, 2$ . We then have the identities  $\{a_1, a_2\} = \{a_1, a_2 b_2^{-1}\} + \{a_1, b_2\}$  and  $\{b_1, b_2\} = \{a_1^{-1} b_1, b_2\} + \{a_1, b_2\}$ . The  $n = 2$  case follows immediately from these two identities.

Suppose now that  $n \geq 3$  and we are given  $a_i, b_i \in R^\times$  such that  $a_i b_i^{-1} \equiv 1$  modulo  $I$  for  $1 \leq i \leq n$ . We then have the identities

$$\begin{aligned} \{a_1, \dots, a_n\} &= \{a_1, \dots, a_{n-1}, a_n b_n^{-1}\} + \{a_1, \dots, a_{n-2}\} \cdot \{a_{n-1}, b_n\} \text{ and} \\ \{b_1, \dots, b_n\} &= \{b_1, \dots, a_{n-1}^{-1} b_{n-1}, b_n\} + \{b_1, \dots, b_{n-2}\} \cdot \{a_{n-1}, b_n\}. \end{aligned}$$

Using the induction and the above two identities, we conclude the proof of well-definedness of  $\eta_I$ . It is easy to check that  $\eta_I$  is multi-linear and hence defines a ring homomorphism  $\eta_I: T_*(R/I) \rightarrow T_*(R)/N$ . Furthermore, it follows from Lemma 3.1 that  $\eta_I$  kills Kato relations. In particular, we get a ring homomorphism  $\eta_I: K_*^M(R/I) \rightarrow T_*(R)/N$ . It is clear that the composite map  $T_*(R)/N \rightarrow K_*^M(R/I) \rightarrow T_*(R)/N$  is identity. This finishes the proof.  $\square$

*Remark 3.3.* Lemmas 3.1 and 3.2 hold even if  $R$  does not contain a field as long as  $R/\mathfrak{m}$  contains at least three elements for every maximal ideal  $\mathfrak{m} \subset R$  not containing  $I$ . In

particular, they hold if  $R$  is local. It is however not clear that they hold for all semi-local rings. The problem lies in lifting Kato relations from  $R/I$  to  $R$ .

**3.2. Kerz's relative Milnor  $K$ -groups.** In [21], Kerz defined the Milnor  $K$ -group  $K_n^{MS}(R)$  for a ring  $R$  to be the  $n$ -th graded piece of the graded ring  $K_*^{MS}(R)$ . The latter is the quotient of the tensor algebra  $T_*(R^\times)$  by the two-sided graded ideal generated by the Steinberg symbols  $a \otimes (1 - a) \in T_2(R^\times)$ , where  $a, 1 - a \in R^\times$ . This is the direct extension of  $K$ -theory of fields defined by Milnor [31] to rings. If  $I \subset R$  is an ideal, we let  $K_n^{MS}(R, I)$  be the kernel of the natural map  $K_n^{MS}(R) \rightarrow K_n^{MS}(R/I)$ . A straightforward imitation of the proof of Lemma 3.2 shows the following.

**Lemma 3.4.** *Lemma 3.2 is valid for the map  $K_*^{MS}(R) \rightarrow K^{MS}(R/I)$  and the group  $K_*^{MS}(R, I)$ .*

It is evident from the above definitions and Lemmas 3.2 and 3.4 that there are natural surjections

$$(3.1) \quad K_*^{MS}(R) \twoheadrightarrow K_*^M(R) \text{ and } K_*^{MS}(R, I) \twoheadrightarrow K_*^M(R, I),$$

where the second surjectivity holds under the assumption that  $R$  is a semi-local ring containing a field of cardinality at least three. It follows from [21, Lemma 2.2] that the maps of (3.1) are isomorphisms if  $R$  is a semi-local ring with infinite residue fields. In fact, the reader can easily check that the proof of [21, Lemma 2.2] remains valid if  $R$  is a local ring whose residue field contains at least five elements (if  $a = 1 - b$  with  $b \notin R^\times$ , take  $s_1 = s_2 = 2 - b$  in Kerz's proof). We therefore get:

**Lemma 3.5.** *Let  $R$  be either a semi-local ring with infinite residue fields or a local ring with residue field having cardinality at least five. Let  $I \subset R$  be a proper ideal. Then the maps  $K_*^{MS}(R) \rightarrow K_*^M(R)$  and  $K_*^{MS}(R, I) \rightarrow K_*^M(R, I)$  are isomorphisms.*

**3.3. Gabber-Kerz improved relative Milnor  $K$ -groups.** When the residue fields of  $R$  are not all infinite, then the Milnor  $K$ -theories  $K_*^M(R)$  and  $K_*^{MS}(R)$  do not have good properties. For example, the Gersten conjecture does not hold for them even if  $R$  is a regular local ring containing a field. If  $R$  is a finite product of local rings containing a field, Gabber (unpublished) and Kerz [22] defined an improved version of Milnor  $K$ -theory, which is denoted as  $\widehat{K}_*^M(R)$ . This is a graded commutative ring and [22, Proposition 10 (3), Theorem 13] imply that there are natural maps of graded commutative rings

$$(3.2) \quad K_*^{MS}(R) \twoheadrightarrow K_*^M(R) \xrightarrow{\eta_R} \widehat{K}_*^M(R) \xrightarrow{\psi_R} K_*(R).$$

The first two arrows are isomorphisms if  $R$  is a field. Moreover, the Gersten resolution holds for  $\widehat{K}_*^M(R)$  if  $R$  is regular. Given an ideal  $I \subset R$ , one defines  $\widehat{K}_*^M(R, I)$  similar to  $K_*^M(R, I)$ . The product structures on the (improved) Milnor and Quillen  $K$ -theories yield natural graded homomorphisms of  $K_*^{MS}(R)$ -modules

$$(3.3) \quad K_*^{MS}(R, I) \twoheadrightarrow K_*^M(R, I) \xrightarrow{\eta_{R/I}} \widehat{K}_*^M(R, I) \xrightarrow{\psi_R} \widehat{K}_*(R, I),$$

where the latter is the kernel of the canonical map  $K_*(R) \rightarrow K_*(R/I)$ .

If  $k$  is a field and  $X$  is a scheme over  $k$ , let  $\mathcal{K}_{n,X}^M$  be the Zariski sheaf on  $X$  whose stalk at a point  $x$  is  $K_n^M(\mathcal{O}_{X,x})$  for  $n \geq 0$ . In [22], Kerz actually shows that there is a Zariski sheaf  $\widehat{\mathcal{K}}_{n,X}^M$  on  $X$  with a natural surjective map  $\mathcal{K}_{n,X}^M \rightarrow \widehat{\mathcal{K}}_{n,X}^M$  such that the stalk of  $\widehat{\mathcal{K}}_{n,X}^M$  at  $x$  is  $\widehat{K}_n^M(\mathcal{O}_{X,x})$ . If  $X = \text{Spec}(A)$ , we let  $\widehat{K}_n^M(A) = H^0(X, \widehat{\mathcal{K}}_{n,X}^M)$ . If  $I \subset A$  is an ideal, we let  $\widehat{K}_n^M(A, I)$  be the kernel of the canonical map  $\widehat{K}_n^M(A) \rightarrow \widehat{K}_n^M(A/I)$ .

**Definition 3.6.** For  $n \geq 0$ , we let  $\overline{K_n^M(A, I)}$  denote the image of the canonical map  $K_n^M(A, I) \rightarrow \widehat{K}_n^M(A, I)$ . We similarly define  $\overline{K_n^M(A)}$  to be the image of the canonical map  $K_n^M(A) \rightarrow \widehat{K}_n^M(A)$ .

*Remark 3.7.* Observe that if  $A$  is a local ring, then  $\overline{K_n^M(A)} = \widehat{K}_n^M(A)$  but it may happen that  $\overline{K_n^M(A, I)} \neq \widehat{K}_n^M(A, I)$  if  $I \neq 0$  is a proper ideal in  $A$ . Moreover, for a semi-local ring  $A$ , we may have that  $\overline{K_n^M(A)} \neq \widehat{K}_n^M(A)$ .  $\square$

If  $R$  is a regular semi-local ring containing  $k$  and if  $F$  is its total ring of quotients, then the Gersten complex [19]

$$(3.4) \quad 0 \rightarrow \widehat{K}_n^M(R) \rightarrow K_n^M(F) \rightarrow \bigoplus_{\text{ht}(\mathfrak{p})=1} K_{n-1}^M(k(\mathfrak{p})) \rightarrow \cdots \rightarrow \bigoplus_{\text{ht}(\mathfrak{p})=n} K_0^M(k(\mathfrak{p}))$$

coincides with the Gersten complex for higher Chow groups except at the first place. Bloch [4] showed that the latter is exact when  $R$  is local. In general, it follows from the above definition of  $\widehat{K}_n^M(R)$  that (3.4) is exact at the first two terms.

We refer to [10, § 3] for more details about other properties of the improved Milnor  $K$ -theory. In this paper, we shall always use the improved Milnor  $K$ -theory of rings, whenever it is defined. For fields, we shall use the notations of Milnor and improved Milnor  $K$ -groups interchangeably.

**3.4. Rulling-Saito relative Milnor  $K$ -groups.** Let  $R$  be a local domain with fraction field  $F$  and let  $I = (f)$  be a principal ideal, where  $f \in R$  is a non-zero divisor. Let  $R_f$  denote the localization  $R[f^{-1}]$  obtained by inverting the powers of  $f$ . Let  $F$  denote the ring of total quotients of  $R$  so that there are inclusions of rings  $R \hookrightarrow R_f \hookrightarrow F$ . We let  $\widehat{K}_1^M(R|I) = K_1^M(R, I)$  and for  $n \geq 2$ , we let  $\widehat{K}_n^M(R|I)$  denote the image of the canonical map of abelian groups

$$(3.5) \quad \widehat{K}_1^M(R|I) \otimes (R_f)^\times \otimes \cdots \otimes (R_f)^\times \rightarrow \widehat{K}_n^M(F),$$

induced by the product in the Milnor  $K$ -theory. These groups were defined by Rulling-Saito [36, § 2.7] as stalks of a sheaf (see [36, Definition 2.4 and Lemma 2.1]). We shall call  $\widehat{K}_*^M(R|I)$  the Rulling-Saito relative Milnor  $K$ -groups. The basic relation between the Kato-Saito and Rulling-Saito relative Milnor  $K$ -groups is given by the following.

**Lemma 3.8.** *Let  $R$  be a regular local domain and let  $I = (f)$  be a non-zero principal ideal. Then there is a commutative diagram of graded groups*

$$(3.6) \quad \begin{array}{ccccc} K_*^M(R, I) & \twoheadrightarrow & \overline{K_*^M(R, I)} & \hookrightarrow & \widehat{K}_*^M(R|I) \\ \downarrow & & \downarrow & & \downarrow \\ K_*^M(R) & \longrightarrow & \widehat{K}_*^M(R) & \hookrightarrow & \widehat{K}_*^M(R_f). \end{array}$$

*Proof.* Let  $F$  be the fraction field of  $R$ . We then note that the image of  $K_*^M(R, I)$  in  $K_*^M(F)$  under the composite map  $K_*^M(R) \twoheadrightarrow \widehat{K}_*^M(R) \hookrightarrow \widehat{K}_*^M(F)$  is  $\overline{K_*^M(R, I)}$ . We therefore only need to verify that the image of this composite map lies in the subgroup  $\widehat{K}_*^M(R|I) \subset \widehat{K}_*^M(R_f)$ .

To prove this claim, it suffices to show that the canonical map  $\zeta_R: K_n^M(R) \rightarrow K_n^M(F)$  sends  $K_n^M(R, I)$  into  $\widehat{K}_n^M(R|I) \subset K_n^M(F)$  for all  $n \geq 1$ . We can assume  $n \geq 2$  as the assertion is clear for  $n = 1$ .

Now, by Lemma 3.2, we need to show that if  $\underline{a} = \{a_1, \dots, a_n\} \in K_n^M(R)$  is such that  $a_i \in K_1^M(R, I)$  for some  $1 \leq i \leq n$ , then  $\zeta_R(\underline{a}) \in \widehat{K}_n^M(R|I)$ . In other words, we have to show that as an element of  $K_n^M(F)$ , the symbol  $\underline{a}$  actually lies in  $\widehat{K}_n^M(R|I)$ . However, this is immediate (see (3.5)) because the ring  $K_*^M(F)$  is anti-commutative [31, Lemma 1.1].  $\square$

**3.5. Connection between  $\widehat{K}_*^M(R, I)$  and  $\widehat{K}_*^M(R|I)$ .** The canonical map  $K_*^M(R, I) \rightarrow \widehat{K}_*^M(R|I)$  of Lemma 3.8 in general may not factor through  $\widehat{K}_*^M(R, I)$ . We shall show however that this is indeed the case in some situations if we replace  $K_*^M(R, I)$  and  $\widehat{K}_*^M(R|I)$  by the pro-abelian groups  $\{K_*^M(R, I^m)\}_m$  and  $\{\widehat{K}_*^M(R|I^m)\}_m$ , respectively. In this subsection, we construct a map in the opposite direction, which is slightly easier.

Let  $R$  be a semi-local ring with the maximal ideals  $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ . Let  $R[T]$  denote the polynomial ring over  $R$  and let  $A$  denote the localization of  $R[T]$  obtained by inverting all polynomials having invertible constant term. Then  $A$  is a semi-local ring of Krull dimension  $\dim(R) + 1$  and the maximal ideals  $\mathfrak{m}_i A + (T)$ , for  $1 \leq i \leq r$ . We now let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$  and let  $R(T)$  be the localization of  $A[T^{-1}]$  at maximal ideal  $\mathfrak{m}[T^{\pm 1}]$ . Then we have the inclusions  $R[T] \hookrightarrow A \hookrightarrow R(T)$ . The ring  $R(T)$  is local with infinite residue field  $\frac{R}{\mathfrak{m}}(T)$ . When  $R$  is a field, then  $A = R[T]_{(T)}$ . If  $R$  is an integral domain with fraction field  $F$ , then  $A$  is an integral domain with fraction field  $F(T)$ .

We now fix a local integral domain  $R$  containing a field. Let  $A$  be the local ring defined above. We fix the ideal  $I = (T) \subset A$ . Let  $F$  denote the fraction field of  $R$ . The key lemma to connect  $\widehat{K}_*^M(A|I)$  with  $\widehat{K}_*^M(A, I)$  is the following.

**Lemma 3.9.** *Assume that  $R$  is a regular local ring. Then the following inclusions hold for every pair of integers  $n \geq 0, m \geq 1$ .*

$$(3.7) \quad \widehat{K}_{n+1}^M(A|I) \subset (1+I)\widehat{K}_n^M(A) \subset \overline{K_{n+1}^M(A, I)} \subset \widehat{K}_{n+1}^M(A, I).$$

$$(3.8) \quad \widehat{K}_{n+1}^M(A|I^{m+1}) \subset (1+I^m)\widehat{K}_n^M(A) \subset \overline{K_{n+1}^M(A, I^m)} \subset \widehat{K}_{n+1}^M(A, I^m).$$

*Proof.* The map  $K_*^M(A) \rightarrow \widehat{K}_*^M(A)$  is surjective as  $A$  is local. This immediately implies the second inclusions in (3.7) and (3.8). So we only need to show the first set of inclusions. Since  $A/(T) = R$  is a regular local ring,  $I$  defines an effective Cartier divisor on  $\text{Spec}(A)$  which is regular (and hence simple normal crossing). The first inclusion in (3.7) now follows from [36, Proposition 2.8(2)]. To prove (3.8), we first observe that  $(A[T^{-1}])^\times = A^\times \cdot T^{\mathbb{Z}}$ . Since  $\{T, T\} = \{T, -1\}$  in  $K_2^M(F(T))$ , it follows that  $\widehat{K}_{n+1}^M(A|I^{m+1})$  is generated by the subgroup  $(1+I^{m+1})\widehat{K}_n^M(A)$  and the element of the form  $\{1+aT^{m+1}, T, u_1, \dots, u_{n-1}\} \in K_{n+1}^M(F(T))$ , with  $a \in A$  and  $u_i \in A^\times$ . It therefore suffices to show that for  $a \in A$ , we have  $\{1+aT^{m+1}, T\} \in (1+I^m) \cdot A^\times \subset K_2^M(F(T))$ . But this follows from [36, Lemma 2.7(2)]. This completes the proof of the lemma.  $\square$

Let  $R$  be a regular local ring containing a field. We let  $\widetilde{K}_n^M(R_m) := \text{Ker}(\widehat{K}_n^M(R_m) \rightarrow \widehat{K}_n^M(R))$ . To apply Lemma 3.9, we observe that the canonical injection  $R[T] \hookrightarrow A$  induces an isomorphism  $R_m \xrightarrow{\cong} A/I^{m+1}$  for all  $m \geq 0$ . Since the maps  $K_n^M(A/I^{m+1}) \rightarrow \widehat{K}_n^M(A/I^{m+1})$  and  $K_n^M(A) \rightarrow K_n^M(A/I^{m+1})$  are surjective, our assumption implies that

$$0 \rightarrow \widehat{K}_n^M(A, I^{m+1}) \rightarrow \widehat{K}_n^M(A) \rightarrow \widehat{K}_n^M(A/I^{m+1}) \rightarrow 0$$

is an exact sequence. Using this, we therefore see that the canonical restriction map  $\widehat{K}_n^M(A) \rightarrow \widehat{K}_n^M(R_m)$  induces a natural (in  $R$ ) isomorphism

$$(3.9) \quad \theta_{R_m}: \frac{\widehat{K}_n^M(A, I)}{\widehat{K}_n^M(A, I^{m+1})} \xrightarrow{\cong} \widetilde{K}_n^M(R_m).$$

Since the map  $R \rightarrow A/I$  is an isomorphism, the quotient maps  $K_n^M(A) \twoheadrightarrow K_n^M(A/I)$  and  $\widehat{K}_n^M(A) \twoheadrightarrow \widehat{K}_n^M(A/I)$  compatibly split via the augmentation. This implies that the induced map on the kernels  $K_n^M(A, I) \rightarrow \widehat{K}_n^M(A, I)$  is also surjective. In particular, the map

$$(3.10) \quad \frac{\overline{K_n^M(A, I)}}{K_n^M(A, I^{m+1})} \rightarrow \frac{\widehat{K}_n^M(A, I)}{\widehat{K}_n^M(A, I^{m+1})}$$

is surjective.

Recall from Lemma 3.8 that  $\overline{K_n^M(A, I^m)}$  is a subgroup of  $\widehat{K}_n^M(A|I^m)$  for every  $m \geq 1$ . The main result we wished to prove in this section is the following.

**Proposition 3.10.** *Let  $R$ ,  $A$  and  $F$  be as in Lemma 3.9 and let  $n \geq 0$  be an integer. Then the kernel (resp. cokernel) of the natural morphism of pro-abelian groups  $\{\overline{K_n^M(A, I^m)}\}_m \rightarrow \{\widehat{K}_n^M(A|I^m)\}_m$  is bounded by 0 (resp. 1). In particular, the inclusions  $\widehat{K}_n^M(A|I^m) \hookrightarrow \widehat{K}_n^M(F(T))$  induce a morphism of pro-abelian groups*

$$(3.11) \quad \left\{ \frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^m)} \right\}_m \rightarrow \left\{ \frac{\widehat{K}_n^M(A, I)}{\widehat{K}_n^M(A, I^m)} \right\}_m$$

with  $\lambda(m) = m + 1$  whose cokernel is bounded by zero.

*Proof.* For  $n = 0$ , all the groups are zero. For  $n \geq 1$ , the first assertion is a direct consequence of Lemmas 3.8 and 3.9. Since the map  $\overline{K_n^M(A, I)} \rightarrow \widehat{K}_n^M(A|I)$  is an isomorphism, also by Lemma 3.9, it follows that the kernel of the map

$$\left\{ \frac{\overline{K_n^M(A, I)}}{\overline{K_n^M(A, I^m)}} \right\}_m \rightarrow \left\{ \frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^m)} \right\}_m$$

is bounded by 1 and the cokernel is bounded by zero. Equivalently,  $\left\{ \frac{\widehat{K}_n^M(A|I^m)}{\overline{K_n^M(A, I^m)}} \right\}_m$  is bounded by 1. By combining this with (3.10), we see that the inclusions  $\widehat{K}_n^M(A|I^m) \hookrightarrow \widehat{K}_n^M(A|I)$  induce a morphism of pro-abelian groups such as in (3.11) with  $\lambda(m) = m + 1$ . It is clear that the map  $\frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^{m+1})} \rightarrow \frac{\widehat{K}_n^M(A, I)}{\widehat{K}_n^M(A, I^m)}$  is surjective for each  $m \geq 1$  since  $\widehat{K}_n^M(A|I) = \overline{K_n^M(A, I)} = \widehat{K}_n^M(A, I)$ . This finishes the proof.  $\square$

#### 4. THE DE RHAM-WITT COMPLEX AND $K$ -THEORY

Proposition 3.10 is not quite enough to prove our main results. We need the map (3.11) to be actually an isomorphism. We shall prove this stronger assertion in this section using the de Rham-Witt complex. We shall use this isomorphism in § 5.3 to obtain our cycle class map to the relative Milnor  $K$ -theory of truncated polynomial rings. We shall also use the de Rham-Witt complex to prove some more results on Milnor and Quillen  $K$ -groups of truncated polynomial rings in § 5.

We shall not recall the definition of the de Rham-Witt complex here. Instead, we refer the reader to [13] and [35, § 1] for its definition and basic properties. We only recall that for a regular semi-local ring  $R$  and integer  $m \geq 1$ , there are natural isomorphisms of abelian groups

$$(4.1) \quad \gamma_{R,m}: \mathbb{W}_m(R) \xrightarrow{\cong} \frac{(1 + TR[[T]])^\times}{(1 + T^{m+1}R[[T]])^\times} \xrightarrow{\cong} \widetilde{K}_1^M(R_m);$$

$$\gamma_{R,m}(\underline{a}) = \prod_{i=1}^m (1 - a_i T^i).$$

We shall often write  $\gamma_{R,m}(\underline{a}) = \gamma_{R,m}((a_1, \dots, a_m))$  as  $\gamma(\underline{a})$  if the context of its usage is clear. For any  $a \in R$ , we recall that  $[a] = (a, 0, \dots, 0) \in \mathbb{W}_m(R)$  denotes the Teichmüller lift of  $a$ . Note that  $\gamma_{R,m}$  is clearly natural in  $R$  and  $m \geq 1$ . The following lemma is a direct consequence of [29, Proposition 2.3]. We state it separately as we will need it a few times in our proofs.

**Lemma 4.1.** *Let  $R$  be a regular semi-local ring containing a field and let  $F$  be its total ring of quotients. Then the canonical map  $\mathbb{W}_m \Omega_R^n \rightarrow \mathbb{W}_m \Omega_F^n$  is injective for all  $m \geq 1$  and  $n \geq 0$ .*

**4.1. Generators of de Rham-Witt complex.** We give a generating set of the de Rham-Witt complex of semi-local rings in this subsection. After proving this result, we realized that we only need it when the underlying ring is a field (see the proofs of Lemmas 7.2 and 7.3) to prove the main results of this paper. In this special case, a presentation of de Rham-Witt complex is already known by Hyodo-Kato [16] and Rülling-Saito [36]. We however present this generalization here since it has independent interest. For instance, it is indispensable for the proof of [29, Corollary 6.4].

**Proposition 4.2.** *Let  $R$  be a regular semi-local ring containing a field  $k$  of characteristic  $p > 0$  and let  $m \geq 1, n \geq 0$  be two integers. Assume that either  $k$  is infinite or  $R$  is local. Then the map*

$$(4.2) \quad (\mathbb{W}_m(R) \otimes \bigwedge_{\mathbb{Z}}^n R^\times) \oplus (\mathbb{W}_m(R) \otimes \bigwedge_{\mathbb{Z}}^{n-1} R^\times) \rightarrow \mathbb{W}_m \Omega_R^n,$$

defined by

$$\begin{aligned} w \otimes (a_1 \wedge \cdots \wedge a_n) &\mapsto w \mathrm{dlog}[a_1] \cdots \mathrm{dlog}[a_n] \text{ and} \\ w \otimes (a_1 \wedge \cdots \wedge a_{n-1}) &\mapsto d w \mathrm{dlog}[a_1] \cdots \mathrm{dlog}[a_{n-1}], \end{aligned}$$

is surjective.

*Proof.* When  $R$  is a local ring, this result was proven (with a description of the kernel of this map) by Hyodo-Kato [16, Proposition 4.6] in the  $p$ -typical case and the reduction of the general case to the  $p$ -typical case was shown by Rülling-Saito [36, Proposition 4.4]. The new assertion is that the surjectivity part of the result of Hyodo-Kato holds for semi-local rings as well.

It suffices to prove the proposition for the  $p$ -typical de Rham-Witt complex. This reduction is routine (see the proof of [36, Proposition 4.4]) and requires no condition on  $R$ . We shall use the notations of  $p$ -typical de Rham-Witt complex  $W_m \Omega_R^n$ , where  $W_m(-) := \mathbb{W}_{\{1, p, \dots, p^{m-1}\}}(-)$ .

We let  $E_{R,m}^n$  denote the abelian group on the left hand side of (4.2). We need to show that the map  $\theta_{R,m}^n: E_{R,m}^n \rightarrow W_m \Omega_R^n$  is surjective. We shall fix  $n \geq 0$  and prove that  $\theta_{R,m}^n$  is a surjection by induction on  $m \geq 1$ .

If  $m = 1$ , then we know that  $W_m \Omega_R^n \cong \Omega_R^n$ . In this case, it follows from our assumption and [10, Lemma 7.4] that  $R$  is additively generated by its units. The latter immediately implies the desired surjectivity.

In the general case, we know by [17, I.3.15.2, p. 576] that there is an exact sequence

$$0 \rightarrow \mathrm{gr}_C^m W \Omega_R^n \rightarrow W_{m+1} \Omega_R^n \rightarrow W_m \Omega_R^n \rightarrow 0,$$

where  $\mathrm{gr}_C^\bullet W \Omega_R^n$  is the associated graded module for the canonical filtration on the de Rham-Witt complex of  $R$ . One knows that the canonical filtration of  $W_{m+1} \Omega_R^n$  coincides with its  $V$ -filtration (see [15, Lemma 3.2.4]). Equivalently, we have  $\mathrm{gr}_C^m W \Omega_R^n = \mathrm{gr}_V^m W \Omega_R^n := V^m W_1 \Omega_R^n + dV^m W_1 \Omega_R^{n-1}$ .

We now consider the commutative diagram with exact rows

$$(4.3) \quad \begin{array}{ccccccc} R \otimes (\bigwedge_{\mathbb{Z}}^n R^\times \oplus \bigwedge_{\mathbb{Z}}^{n-1} R^\times) & \xrightarrow{V_m \otimes \mathrm{id}} & E_{R,m+1}^n & \longrightarrow & E_{R,m}^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{gr}_V^m W \Omega_R^n & \longrightarrow & W_{m+1} \Omega_R^n & \longrightarrow & W_m \Omega_R^n \longrightarrow 0. \end{array}$$

The  $m = 1$  case of the proposition and the expression of  $\mathrm{gr}_V^m W \Omega_R^n$  given above show that the left vertical arrow in (4.3) is surjective. The right vertical arrow is surjective by induction. It follows that the middle vertical arrow is surjective, as desired.  $\square$

**4.2. Milnor  $K$ -theory and de Rham-Witt complex.** Let  $X$  be a Noetherian scheme. For an integer  $m \geq 0$ , we let  $X_m = X \times \text{Spec}(k_m)$ . We let  $\widehat{\mathcal{K}}_{n,m,X}^M$  be the Zariski sheaf on  $X$  whose stalk at a point  $x \in X$  is the Milnor  $K$ -group  $\widehat{K}_n^M(\mathcal{O}_{X,x}[t]/(t^{m+1}))$ . We let  $\widetilde{\mathcal{K}}_{n,m,X}^M$  be the kernel of the split surjection  $\widehat{\mathcal{K}}_{n,m,X}^M \rightarrow \widehat{\mathcal{K}}_{n,X}^M$ . We let  $\mathcal{K}_{n,m,X}$  be the Zariski sheaf on  $X$  whose stalk at a point  $x \in X$  is the Quillen  $K$ -group  $K_n(\mathcal{O}_{X,x}[t]/(t^{m+1}))$ . We define  $\widetilde{\mathcal{K}}_{n,m,X}$  just as we defined  $\widetilde{K}_{n,m,X}^M$ .

For a point  $x \in X$ , we shall denote the ring  $A(\mathcal{O}_{X,x})$  (which is obtained exactly as in § 3.5, where  $R$  is replaced by  $\mathcal{O}_{X,x}$ ) in short by  $A_x$ . Note that  $A_x$  is local. We let  $\widehat{\mathcal{K}}_{n,X|I^m}^M$  denote the Zariski sheaf on  $X$  whose stalk at a point  $x \in X$  is the group  $\widehat{K}_n^M(A_x|I^m)$ . We let  $\widehat{K}_{n,(X,I^m)}^M$  denote the Zariski sheaf on  $X$  whose stalk at a point  $x \in X$  is the group  $\widehat{K}_n^M(A_x, I^m)$ .

We now fix a regular semi-local ring  $R$  containing a field  $k$  of characteristic  $p > 0$  and let  $X = \text{Spec}(R)$ . We let  $A := A(R)$  be the semi-local ring defined in § 3.5 and let  $I = (T) \subset A$ . We shall continue to follow the notations of § 3. We shall use the following result of Rülling-Saito which gives an explicit relation between the de Rham-Witt complex and the Rülling-Saito relative Milnor  $K$ -theory.

**Theorem 4.3.** ([36, Theorem 4.8]) *Let  $n \geq 0$  and  $m \geq 1$  be two integers. Then the map of sheaves*

$$(4.4) \quad \lambda_X: \mathbb{W}_m \Omega_X^n \rightarrow \frac{\widehat{\mathcal{K}}_{n+1,X|I}^M}{\widehat{\mathcal{K}}_{n+1,X|I^m}^M},$$

which on local sections is given by

$$w \text{dlog}[a_1] \cdots \text{dlog}[a_n] \mapsto \{\gamma(w), a_1, \dots, a_n\}$$

and

$$dwd \text{dlog}[a_1] \cdots \text{dlog}[a_{n-1}] \mapsto (-1)^n \{\gamma(w), a_1, \dots, a_{n-1}, T\},$$

is an isomorphism.

By combining Proposition 3.10, Lemma 4.1 and Theorem 4.3, we now prove the following key result. This will play a crucial role in the proofs of the main theorems. This result is also used in [32, § 6] without a proof when  $R$  is local.

**Theorem 4.4.** *Let  $X$  be a regular scheme over a field  $k$  of characteristic  $p > 0$  and let  $n \geq 0$  be an integer. Then the canonical maps of sheaves of pro-abelian groups*

$$\left\{ \frac{\widehat{\mathcal{K}}_{n,X|I}^M}{\widehat{\mathcal{K}}_{n,X|I^m}^M} \right\}_m \rightarrow \left\{ \frac{\widehat{\mathcal{K}}_{n,(X,I)}^M}{\widehat{\mathcal{K}}_{n,(X,I^m)}^M} \right\}_m \rightarrow \{\widetilde{\mathcal{K}}_{n,m,X}^M\}_m$$

are isomorphisms.

*Proof.* The theorem is obvious for  $n = 1$  from the definitions of various groups. We thus assume that  $n \geq 2$ . Note here that the first map is induced by (3.7) and (3.8) and second by (3.9). Indeed, note that all the sheaves  $\widehat{\mathcal{K}}_{n,X|I}^M, \widehat{\mathcal{K}}_{n,X|I^m}^M, \widehat{\mathcal{K}}_{n,(X,I)}^M$  and  $\widehat{\mathcal{K}}_{n,(X,I^m)}^M$  are sub-sheaves of the constant sheaf  $K_n(F(T))$ , where  $F$  is the total ring of quotients of  $R$ . It then follows by (3.7) and (3.8) that  $\widehat{\mathcal{K}}_{n,X|I}^M = \widehat{\mathcal{K}}_{n,(X,I)}^M$  and  $\widehat{\mathcal{K}}_{n,X|I^m}^M \subset \widehat{\mathcal{K}}_{n,(X,I^m-n)}^M$ . A similar argument yields the second map using (3.9) so that it is a level-wise isomorphism. We only need to show that the first map is an isomorphism too. By Lemma 2.1 and Proposition 3.10, we only need to show that if  $R$  is local, then the kernel of the level-wise surjective map

$$(4.5) \quad \left\{ \frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^m)} \right\}_m \rightarrow \left\{ \frac{\widehat{K}_n^M(A, I)}{\widehat{K}_n^M(A, I^m)} \right\}_m$$



is bounded by an integer not depending on  $R$ .

We first assume that  $R$  is a field. If  $R$  is a finite field, then it is well known that  $\mathbb{W}_m \Omega_R^{n-1} = 0$ . It follows from Proposition 3.10 and Theorem 4.3 that both sides of (4.5) are bounded by zero. If  $R$  is an infinite field, then  $A$  contains an infinite field. Hence, the map  $\overline{K_n^M(A, I^m)} \rightarrow \widehat{K}_n^M(A, I^m)$  is an isomorphism (see [22, Proposition 10 (5)]). We can therefore replace  $\widehat{K}_n^M(A, I^m)$  by  $\overline{K_n^M(A, I^m)}$  in (4.5). We now use the maps

$$\left\{ \frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^m)} \right\}_m \rightarrow \left\{ \frac{\overline{K_n^M(A, I)}}{\overline{K_n^M(A, I^m)}} \right\}_m \rightarrow \left\{ \frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^m)} \right\}_m,$$

where the second arrow is induced by the inclusions  $\overline{K_n^M(A, I^m)} \hookrightarrow \widehat{K}_n^M(A|I^m)$ . We now note that the function  $\lambda: \mathbb{N} \rightarrow \mathbb{N}$  associated to the first arrow is  $\lambda(m) = m + 1$  and it is identity for the second arrow. More precisely, the composite arrow is induced by the canonical surjections  $\frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^{m+1})} \twoheadrightarrow \frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^m)}$ . It follows immediately that the kernel of the composite arrow is bounded by 1. We are therefore done.

In the general case, we let  $F$  be the fraction field of  $R$ . Let  $B = F[T]_{(T)} = A(F)$  and  $J = IB \subset B$ . It is clear that the diagram

$$(4.6) \quad \begin{array}{ccccc} \{\mathbb{W}_m \Omega_R^{n-1}\}_m & \xrightarrow{\cong} & \left\{ \frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^m)} \right\}_m & \rightarrow & \left\{ \frac{\widehat{K}_n^M(A, I)}{\widehat{K}_n^M(A, I^m)} \right\}_m \\ \downarrow & & \downarrow & & \downarrow \\ \{\mathbb{W}_m \Omega_F^{n-1}\}_m & \xrightarrow{\cong} & \left\{ \frac{\widehat{K}_n^M(B|J)}{\widehat{K}_n^M(B|J^m)} \right\}_m & \rightarrow & \left\{ \frac{\widehat{K}_n^M(B, J)}{\widehat{K}_n^M(B, J^m)} \right\}_m \end{array}$$

is commutative, where the vertical arrows are the canonical base change maps.

It follows from Lemma 4.1 that the left vertical arrow in (4.6) is level-wise injective. Since  $R$  is local, it follows from Theorem 4.3 that the left horizontal arrows on the top and the bottom are level-wise isomorphisms. It follows that the middle vertical arrow is level-wise injective. We showed above that the kernel of the second horizontal arrow on the bottom is bounded by 1. We deduce that the kernel of the second horizontal arrow on the top must also be bounded by 1. This proves the theorem.  $\square$

Combining Theorem 4.4 and (3.9), we get the following.

**Corollary 4.5.** *Let  $R$  be a regular semi-local ring containing a field  $k$  of characteristic  $p > 0$  and let  $X = \text{Spec}(R)$ . Let  $n \geq 0$  be an integer. Then there are isomorphisms of pro-abelian groups*

$$\{\mathbb{W}_m \Omega_R^{n-1}\}_m \xrightarrow[\cong]{\lambda_R} \left\{ H^0(X, \widehat{\mathcal{K}}_{n, X|I}^M / \widehat{\mathcal{K}}_{n, X|I^m}^M) \right\}_m \xrightarrow[\cong]{\theta_R} \{\widetilde{K}_n^M(R_m)\}_m,$$

which are natural in  $R$ .

*Proof.* Since  $U \mapsto \mathbb{W}_m \Omega_{\mathcal{O}(U)}^{n-1}$  is a Zariski sheaf on  $X$  (see [17, Proposition I.1.13.1]), the first isomorphism follows from Theorem 4.3. We now show the second isomorphism.

Theorem 4.4 yields a map  $\{\mathbb{W}_m \Omega_R^{n-1}\}_m \xrightarrow{\theta_R \circ \lambda_R} \{H^0(X, \widetilde{K}_{n, m, X}^M)\}_m$  whose kernel and cokernel are bounded by 1. It suffices therefore to show that  $\widetilde{K}_n^M(R_m) \cong H^0(X, \widetilde{K}_{n, m, X}^M)$ . Using the augmentation  $R_m \twoheadrightarrow R$ , it enough to show that  $\widehat{K}_n^M(R_m) \cong H^0(X, \widehat{\mathcal{K}}_{n, m, X}^M)$ . But this is clear from the definition of the improved Milnor  $K$ -theory given in § 3.3 once we observe that  $\widehat{\mathcal{K}}_{n, m, X}^M$  is nothing but the direct image sheaf  $\pi_* (\widehat{\mathcal{K}}_{n, X_m}^M)$ , where  $\pi: X_m \rightarrow X$  is the projection. Indeed, we have  $H^0(X, \widehat{\mathcal{K}}_{n, m, X}^M) = H^0(X, \pi_* (\widehat{\mathcal{K}}_{n, X_m}^M)) = H^0(X_m, \widehat{\mathcal{K}}_{n, X_m}^M) = \widehat{K}_n^M(R_m)$ .  $\square$

If  $R$  is local, the above result is equivalent to the following.

**Corollary 4.6.** *Let  $R$  be as in Corollary 4.5 and  $n \geq 0$  an integer. Assume  $R$  is local. Then there are isomorphisms of pro-abelian groups*

$$\{\mathbb{W}_m \Omega_R^{n-1}\}_m \xrightarrow[\cong]{\lambda_R} \left\{ \frac{\widehat{K}_n^M(A|I)}{\widehat{K}_n^M(A|I^m)} \right\}_m \xrightarrow[\cong]{\theta_R} \{\widetilde{K}_n^M(R_m)\}_m,$$

which are natural in  $R$ .

In particular, we have the following corollary stating that the improved Milnor  $K$ -groups of the truncated polynomials over finite fields are pro-zero.

**Corollary 4.7.** *Let  $k$  be a finite field and  $n \geq 2$ . Then the pro-group  $\{\widehat{K}_n^M(k_m)\}_m$  is zero.*

*Proof.* It follows from Corollary 4.6, using the fact that the Milnor  $K$ -groups  $K_n^M(k)$  are zero for  $n \geq 2$ .  $\square$

*Remark 4.8.* It should be noted that we actually showed in Corollaries 4.5 and 4.6 that  $\lambda_R$  is a level-wise isomorphism while the kernel and cokernel of  $\theta_R$  are bounded by 1.

*Remark 4.9.* Let  $R$  be as in Corollary 4.6, where we assume further that it is essentially of finite type over a perfect field and  $\dim(R) < p$ . In this special case, one can obtain a direct proof of Corollary 4.6 as follows.

We consider the maps

$$\{\mathbb{W}_m \Omega_R^{n-1}\}_m \xrightarrow{\theta_R \circ \lambda_R} \{\widetilde{K}_n^M(R_m)\}_m \twoheadrightarrow \{\widetilde{K}_n^{\text{sym}}(R_m)\}_m,$$

where  $\widetilde{K}_n^{\text{sym}}(R_m)$  is the reduced symbolic  $K$ -theory used by Bloch [3]. We observed in Proposition 3.10 that the first arrow is surjective. It must therefore be an isomorphism because the composite arrow is an isomorphism by [17, Théorème I.5.2]. We warn the reader however that Theorem 4.4 does not follow from the local case since we can not work with stalks in order to show that a given morphism between pro-sheaves is an isomorphism.

## 5. MILNOR VS QUILLEN RELATIVE $K$ -THEORIES

In this section, we shall use Theorem 4.4 to prove some relations between the relative Milnor and Quillen  $K$ -groups for regular semi-local rings (see Proposition 5.8). As an immediate consequence, we shall prove our main results sans Theorem 1.1(4).

**5.1. Milnor to Quillen  $K$ -theory.** Let  $R$  be a regular ring containing a field  $k$  and let  $X = \text{Spec}(R)$ . Let  $m, n \geq 0$  be two integers. Recall the sheaves  $\widetilde{\mathcal{K}}_{n,m,X}^M$  and  $\widetilde{\mathcal{K}}_{n,m,X}$  from § 4.2. There is a canonical map  $\widetilde{K}_n(R_m) \rightarrow H^0(X, \widetilde{\mathcal{K}}_{n,m,X})$  which is functorial in  $R$  and  $m \geq 0$ . We shall need to know that this map is close to being an isomorphism in order to construct a map from the relative Milnor to Quillen  $K$ -theory. To prove a precise statement, we need another result.

Suppose that  $\text{char}(k) = p > 0$ . By the main result of [12], there is a map of pro-abelian groups  $\{\mathbb{W}_m \Omega_R^{n-1}\}_m \rightarrow \{\widetilde{K}_n(R_m)\}_m$  which is natural in  $R$ . In particular, there is a map of sheaves of pro-abelian groups  $\{\mathbb{W}_m \Omega_X^{n-1}\}_m \rightarrow \{\widetilde{\mathcal{K}}_{n,m,X}\}_m$ . This map has the following property.

**Lemma 5.1.** *The maps  $\{\mathbb{W}_m \Omega_R^{n-1}\}_m \rightarrow \{\widetilde{K}_n(R_m)\}_m$  and  $\{\mathbb{W}_m \Omega_X^{n-1}\}_m \rightarrow \{\widetilde{\mathcal{K}}_{n,m,X}\}_m$  are isomorphisms.*

*Proof.* Let  $\mathcal{E}_m^n$  and  $\mathcal{F}_m^n$  denote the kernel and cokernel of the map  $\mathbb{W}_m \Omega_X^{n-1} \rightarrow \tilde{\mathcal{K}}_{n,m,X}$ , respectively. By Lemma 2.1, it suffices to show that for every  $m \geq 0$ , there are integers  $N(m)$  and  $N'(m)$  such that the maps of stalks  $\mathcal{E}_{m+N(m),x}^n \rightarrow \mathcal{E}_{m,x}^n$  and  $\mathcal{F}_{m+N'(m),x}^n \rightarrow \mathcal{F}_{m,x}^n$  are zero for all  $x \in X$ . But this follows directly from [12, Theorem A, Theorem 6.3 (iii)]. In fact, one can take  $N(m) = 1$  for all  $m$  and  $N'(m)$  depends only on  $m$  and  $p$ . The identical proof works for the map  $\{\mathbb{W}_m \Omega_R^{n-1}\}_m \rightarrow \{\tilde{K}_n(R_m)\}_m$  too because Hesselholt's result holds for  $R$  as well (using Néron-Popescu desingularization).  $\square$

Fix  $n \geq 0$ . We can now prove:

**Lemma 5.2.** *The map  $\tilde{K}_n(R_m) \rightarrow H^0(X, \tilde{\mathcal{K}}_{n,m,X})$  is an isomorphism if  $\text{char}(k) = 0$ . The map of pro-abelian groups  $\{\tilde{K}_n(R_m)\}_m \rightarrow \{H^0(X, \tilde{\mathcal{K}}_{n,m,X})\}_m$  is an isomorphism if  $\text{char}(k) > 0$ .*

*Proof.* Assume first that  $\text{char}(k) = 0$ . In this case, it follows from [11, Theorem 10] that  $\tilde{K}_n(A_m) \cong \bigoplus_{i \geq 1} (\Omega_A^{n+1-2i})^m$  for any regular ring  $A$  containing  $k$ . In particular,  $\tilde{\mathcal{K}}_{n,m,X}$  is a quasi-coherent sheaf on  $X$  defined by the  $R$ -module  $\tilde{K}_n(R_m)$ . This immediately implies the desired result.

Suppose now that  $\text{char}(k) > 0$  and consider the commutative diagram of pro-abelian groups

$$(5.1) \quad \begin{array}{ccc} \{\mathbb{W}_m \Omega_R^{n-1}\}_m & \longrightarrow & \{\tilde{K}_n(R_m)\}_m \\ \downarrow & & \downarrow \\ \{H^0(X, \mathbb{W}_m \Omega_X^{n-1})\}_m & \longrightarrow & \{H^0(X, \tilde{\mathcal{K}}_{n,m,X})\}_m. \end{array}$$

The left vertical arrow is an isomorphism by [17, Proposition I.1.13.1] and the usual  $p$ -typical decomposition argument. The top horizontal arrow is an isomorphism by Lemma 5.1. The bottom horizontal arrow is an isomorphism by Lemmas 2.1 and 5.1. We conclude that the right vertical arrow is an isomorphism too.  $\square$

Since the Quillen  $K$ -theory sheaf on the spectrum of a regular semi-local ring containing a field is acyclic by Quillen's Gersten resolution, Lemma 5.2 implies the following.

**Corollary 5.3.** *Let  $R$  be a regular semi-local ring containing a field  $k$ . Then the map  $K_n(R_m) \rightarrow H^0(X, \mathcal{K}_{n,m,X})$  is an isomorphism if  $\text{char}(k) = 0$ . The map of pro-abelian groups  $\{K_n(R_m)\}_m \rightarrow \{H^0(X, \mathcal{K}_{n,m,X})\}_m$  is an isomorphism if  $\text{char}(k) > 0$ .*

Recall that unless  $R$  is local, it is not known if there exists a canonical map from the Milnor  $K$ -theory defined by Gabber and Kerz to the Quillen  $K$ -theory. We can however now show using the previous results that such a map exists for the truncated polynomials in the pro-setting.

**Corollary 5.4.** *Let  $R$  be a regular semi-local ring containing a field  $k$  and let  $n \geq 0$  be an integer. Then there is a map of pro-abelian groups  $\{\widehat{K}_n^M(R_m)\}_m \rightarrow \{K_n(R_m)\}_m$  which is natural in  $R$ . The same holds for the relative  $K$ -groups.*

*Proof.* We let  $X = \text{Spec}(R)$ . At any rate, we have a natural map of sheaves  $\widehat{\mathcal{K}}_{n,m,X}^M \rightarrow \mathcal{K}_{n,m,X}$  by the main result of [22]. This gives rise to a commutative diagram

$$(5.2) \quad \begin{array}{ccc} \{\widehat{K}_n^M(R_m)\}_m & \cdots \cdots \cdots \longrightarrow & \{K_n(R_m)\}_m \\ \cong \downarrow & & \downarrow \cong \\ \{H^0(X, \widehat{\mathcal{K}}_{n,m,X}^M)\}_m & \longrightarrow & \{H^0(X, \mathcal{K}_{n,m,X})\}_m. \end{array}$$

The left vertical arrow is a level-wise isomorphism by definition and the right vertical arrow is an isomorphism by Corollary 5.3. The corollary follows.  $\square$

**5.2. Identification of relative Milnor and Quillen  $K$ -theory.** We shall now show that the canonical map from relative Milnor to Quillen  $K$ -theory that we constructed in § 5.1 is an isomorphism. We need the following to prove its injectivity.

**Lemma 5.5.** *Let  $X$  be a Noetherian regular scheme over a field of characteristic  $p > 0$ . Let  $q \geq 1$  be an integer. Then  $\{ {}_q\mathbb{W}_m\Omega_X^n \}_m = 0$  for all  $n \geq 0$ .*

*Proof.* It suffices to show that if  $R$  is a regular semi-local ring containing a field of characteristic  $p > 0$ , and  $m \geq 1$  is an integer, then the map  ${}_q\mathbb{W}_{m'}\Omega_R^n \rightarrow \mathbb{W}_m\Omega_R^n$  is zero for some integer  $m' \geq m$ , depending only on  $m$  and  $n$ .

Using Lemma 4.1, it suffices to prove this assertion for fields. So we let  $k$  be a field of characteristic  $p > 0$ . Write  $q = p^r s$ , where  $p \nmid s$ . It is then clear that  ${}_q\mathbb{W}_m\Omega_k^n = {}_{p^r}\mathbb{W}_m\Omega_k^n$ . We therefore need to show that given any integer  $m \geq 1$ , the map  ${}_{p^r}\mathbb{W}_{m'}\Omega_k^n \rightarrow \mathbb{W}_m\Omega_k^n$  is zero for all  $m' \gg m$ .

Using the  $p$ -typical decomposition of  $\mathbb{W}_m\Omega_k^n$  and the fact that this decomposition is finite and is compatible with the restriction maps  $\mathbb{W}_{m'}\Omega_k^n \rightarrow \mathbb{W}_m\Omega_k^n$ , it suffices to prove the last assertion for the  $p$ -typical de Rham-Witt forms  $W_m\Omega_k^n$ . We thus have to show that given  $m \geq 1$ , the map  ${}_{p^r}W_{m'}\Omega_k^n \rightarrow W_m\Omega_k^n$  is zero for all  $m' \gg m$ , depending only on  $m$  and  $n$ . However, this is an immediate consequence of a theorem of Illusie (see [17, Proposition I.3.4, p. 569], see also [35, Lemma 2.3]) that the canonical and  $p$ -filtrations (and also the  $V$ -filtration) of  $W_r\Omega_k^n$  coincide.  $\square$

**Lemma 5.6.** *Let  $X$  be a Noetherian regular scheme over a field  $k$  of characteristic  $p > 0$ . Let  $n \geq 0$  be an integer. Then the canonical map  $\{ \tilde{K}_{n,m,X}^M \}_m \rightarrow \{ \tilde{K}_{n,m,X} \}_m$  is injective.*

*Proof.* For  $n \leq 1$ , the lemma is obvious. So we assume  $n \geq 2$ . We fix an integer  $m \geq 1$ . It suffices to show that if  $R$  is regular local ring containing  $k$  and  $F_m^n$  is the kernel of the map  $\tilde{K}_n^M(R_m) \rightarrow \tilde{K}_n(R_m)$ , then there exists an integer  $m' \gg m$ , depending only on  $m$  and  $n$ , such that the map  $F_{m'}^n \rightarrow F_m^n$  is zero.

It follows from Lemma 4.1 and Corollary 4.5 that the kernel of the map  $\{ \tilde{K}_n^M(R_m) \}_m \rightarrow \{ \tilde{K}_n^M(F_m) \}_m$  is bounded by 1. Using the commutativity of this map with the similar map the between Quillen  $K$ -groups, it suffices therefore to prove our assertion for a field  $k$  with  $\text{char}(k) > 0$ .

If  $k$  is finite, then  $\{ \tilde{K}_n^M(k_m) \}_m$  is bounded by 1, again by Corollary 4.5. We can therefore assume that  $k$  is infinite. In this case, we know that  $F_m^n$  is a torsion group of exponent  $(n-1)!$  (see [33] or [22, Proposition 10 (6)]). On the other hand, it follows from Corollary 4.5 that the map  $\{ {}_q\mathbb{W}_m\Omega_k^{n-1} \}_m \rightarrow \{ {}_q\tilde{K}_n^M(k_m) \}_m$  has kernel and cokernel bounded by 1 for all  $q$ . It suffices therefore to show that for every pair of integers  $m, q \geq 1$ , there exists an integer  $m' \gg m$ , depending only on  $m$  and  $n$ , such that the map  ${}_q\mathbb{W}_{m'}\Omega_k^{n-1} \rightarrow {}_q\mathbb{W}_m\Omega_k^{n-1}$  is zero. But this is shown in the proof of Lemma 5.5.  $\square$

The above result implies the following (see the proof of Proposition 5.8).

**Corollary 5.7.** *Let  $R$  be a regular semi-local ring containing a field of characteristic  $p > 0$ . Then the canonical map  $\{ \tilde{K}_n^M(R_m) \}_m \rightarrow \{ \tilde{K}_n(R_m) \}_m$  (see Corollary 5.4) is injective for all  $n \geq 0$ .*

Recall that a ring  $R$  containing a field of characteristic  $p > 0$  is called  $F$ -finite, if it is a finitely generated algebra (equivalently, a finitely generated module) over  $R^p$ . One knows that  $R$  is  $F$ -finite if it is essentially of finite type over a perfect field. This is also true for the Henselization or completion of  $R$  along any ideal. In particular, any field

which is a finitely generated over a perfect field is  $F$ -finite. We say that a scheme is locally  $F$ -finite if all its local rings are so.

The main result of this section that we shall use later is the following.

**Proposition 5.8.** *Let  $R$  be an  $F$ -finite regular semi-local ring containing a field of characteristic  $p > 0$ . Then the canonical map  $\{\tilde{\mathcal{K}}_n^M(R_m)\}_m \rightarrow \{\tilde{K}_n(R_m)\}_m$  of Corollary 5.4 is an isomorphism for all  $n \geq 0$ .*

*Proof.* If  $R$  is local, this follows from Theorem 4.4 and [32, Theorem 6.1] (which implicitly uses Theorem 4.4 or Remark 4.9). To prove the general case, we write  $X = \text{Spec}(R)$  and  $X_m = \text{Spec}(R_m)$  as before. We then have the strict map of sheaves of pro-abelian groups  $\psi_X: \{\tilde{\mathcal{K}}_{n,m,X}^M\}_m \rightarrow \{\tilde{\mathcal{K}}_{n,m,X}\}_m$  on  $X$ . Using Lemmas 2.1, 5.2 and (the proof of) Corollary 4.5, it suffices to show that  $\psi_X$  is an isomorphism. Note that  $X$  is locally  $F$ -finite. In view of Lemma 5.6, we only have to show that  $\psi_X$  is surjective.

We consider the commutative diagram of exact sequences of sheaves

$$(5.3) \quad \begin{array}{ccccccc} 0 & \rightarrow & \tilde{\mathcal{K}}_{n,m,X}^M & \rightarrow & \widehat{\mathcal{K}}_{n,m,X}^M & \rightarrow & \widehat{\mathcal{K}}_{n,X}^M \rightarrow 0 \\ & & \psi_X \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \tilde{\mathcal{K}}_{n,m,X} & \rightarrow & \mathcal{K}_{n,m,X} & \rightarrow & \mathcal{K}_{n,X} \rightarrow 0. \end{array}$$

It follows by [8, Theorem 8.1] and the exactness of Gersten complexes for  $\widehat{\mathcal{K}}_{n,X}^M$  and  $\widehat{\mathcal{K}}_{n,X}$  (see [22, Proposition 10(8)]) that these sheaves have no  $p$ -torsion. In particular, the two rows of (5.3) remain exact with  $\mathbb{Z}/p^r$ -coefficients.

We first show that  $\psi_X$  is surjective with  $\mathbb{Z}/p^r$ -coefficients. Since the right vertical arrow in (5.3) is an isomorphism with  $\mathbb{Z}/p^r$ -coefficients by [8, Theorem 8.1], we need to show that the middle vertical arrow is surjective with  $\mathbb{Z}/p^r$ -coefficients. But this follows from [32, Corollary 5.5] (which uses the  $F$ -finiteness assumption). Morrow states this corollary in the case when  $X$  is the spectrum of a local ring. However, as he explains in [32, Remark 5.8], the result holds at the level of sheaves too and the proof is obtained by repeating the proof of the local ring case verbatim and observing that the bounds in the pro-systems are controlled while going from one point to another point of the underlying scheme.

Indeed, Morrow shows that there are maps

$$\{\widehat{\mathcal{K}}_{n,m,X}^M/p^r\}_m \rightarrow \{\mathcal{K}_{n,m,X}/p^r\}_m \xrightarrow{\text{dlog}} \{W_r \Omega_{(X_m, X), \log}^n\}_m$$

whose composition is level-wise surjective, where the last term is the relative  $p$ -typical logarithmic de Rham-Witt complex [17]. He then shows that the second map is an isomorphism. The main argument (one which involves the usages of pro-systems) in the proof of this isomorphism is the pro-HKR theorem of Dundas-Morrow [6]. And one checks that this pro-HKR theorem holds at the level of sheaves (see the footnote below § 5.3 in [32]).

To prove the surjectivity of  $\psi_X$ , we now claim that  $p^r \tilde{\mathcal{K}}_{n,m,X} = 0$  for some  $r$ , depending only on  $m$  and  $n$ . For this, we first note that  $\mathbb{W}_{m+n} \Omega_X^{n-1}$  is a sheaf of  $\mathbb{W}_{m+n}(\mathbb{F}_p)$ -modules. Hence,  $p^{m+n} \mathbb{W}_{m+n} \Omega_X^{n-1} = 0$ . We next use the Hesselholt-Madsen exact sequence [14]:

$$\bigoplus_{i \geq 0} \mathbb{W}_{(m+1)(i+1)} \Omega_X^{n-1-2i} \xrightarrow{\epsilon} \tilde{\mathcal{K}}_{n,m,X} \xrightarrow{\partial} \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_X^{n-2-2i}.$$

We have seen above that the two end terms are annihilated by some power of  $p$  (depending only on  $m$  and  $n$ ). It follows that the middle term has the same property. This proves the claim.

We now let  $\mathcal{E}_m^n = \text{Coker}(\tilde{\mathcal{K}}_{n,m,X}^M \rightarrow \tilde{\mathcal{K}}_{n,m,X})$ . We have shown previously that  $\{\mathcal{E}_m^n/p^r\}_m = 0$  for every  $r \geq 1$ . On the other hand, for a fixed integer  $m \geq 0$ , the claim implies that  $p^r \mathcal{E}_m^n = 0$  for some  $r \gg 0$ . It follows that the map  $\mathcal{E}_{m'}^n \rightarrow \mathcal{E}_m^n$  is zero for all  $m' \gg m$ . In particular,  $\{\mathcal{E}_m^n\}_m = 0$ . This shows that  $\psi_X$  is surjective and finishes the proof of the proposition.  $\square$

**5.3. The cycle class map to Milnor  $K$ -theory.** It was shown by Rülling [35] (for fields) and Krishna-Park [29] (for semi-local rings) that the additive higher Chow groups  $\text{TCH}^n(R, n; m)$  for  $m, n \geq 1$  together form the universal restricted Witt-complex (see [35, § 1] for definition) over  $R$ . In particular, there is an isomorphism of restricted Witt-complexes

$$(5.4) \quad \tau_R: \mathbb{W}_m \Omega_R^{n-1} \xrightarrow{\cong} \text{TCH}^n(R, n; m)$$

for every  $m, n \geq 1$ . This map is given by

$$(5.5) \quad \tau_R(\text{wdlog}[a_1] \cdots \text{dlog}[a_{n-1}]) = V(\gamma(w), y_1 - a_1, \dots, y_{n-1} - a_{n-1}),$$

where  $a_i \in R^\times$ ,  $\gamma(w) \in R[T]$  is the polynomial defined in (4.1) and  $V(I)$  denotes the closed subscheme of  $\text{Spec}(R) \times \square^{n-1} \cong \text{Spec}(R[T, y_1, \dots, y_{n-1}])$ , defined by the ideal  $I$ . We shall call  $\tau_R$  by the name ‘the de Rham-Witt-Chow isomorphism’. This is the additive analog of the Milnor-Chow isomorphism of [7], [33] and [38].

Using the Chow-Witt isomorphism and Corollary 4.5, we define our cycle class map from the additive higher Chow group of relative 0-cycles to relative Milnor  $K$ -theory as follows.

**Definition 5.9.** Let  $R$  be as above and  $n \geq 1$  an integer. We define the cycle class map to Milnor  $K$ -theory to be the composite map of pro-abelian groups

$$(5.6) \quad \text{cyc}_R^M = \theta_R \circ \lambda_R \circ \tau_R^{-1}: \{\text{TCH}^n(R, n; m)\}_m \rightarrow \{\tilde{K}_n^M(R_m)\}_m.$$

It follows from [29, Theorem 1.1] that  $\tau_R$  is functorial with respect to any  $k$ -algebra homomorphism between regular semi-local rings  $R \rightarrow R'$  essentially of finite type over  $k$ . The maps  $\lambda_R$  and  $\theta_R$  are clearly functorial in  $R$  by their construction. It follows that  $\text{cyc}_R^M$  is functorial in  $R$ . Notice also that  $\text{cyc}_R^M$  is an isomorphism.

**5.4. Proofs of Theorems 1.1((1)-(3)) and 1.3.** We let  $k$  and  $R$  be as in the these theorems. We define the cycle class map as the following composition

$$(5.7) \quad \text{cyc}'_R = \psi_R \circ \text{cyc}_R^M: \{\text{TCH}^n(R, n; m)\}_m \rightarrow \{\tilde{K}_n(R_m)\}_m,$$

where the map  $\psi_R$  is as in Corollary 5.4. The proofs of Theorem 1.3 and parts (1) to (3) of Theorem 1.1 follow immediately.  $\square$

## 6. THE CYCLE CLASS MAP FOR SEMI-LOCAL RINGS

The goal of the remaining two sections is to define  $\text{cyc}_R$  at the level of additive higher Chow groups and prove the final part of Theorem 1.1. In this section, we shall define  $\text{cyc}_R$  which generalizes the construction of (2.13) from fields to regular semi-local rings over a field. In the next section, we shall show the agreement between  $\text{cyc}_R$  and  $\text{cyc}'_R$  under our assumptions.

We fix a field  $k$  of characteristic  $p > 0$  and let  $R$  be a regular semi-local ring which is essentially of finite type over  $k$ . We let  $F$  denote the total ring of quotients of  $R$ . Let  $\Sigma$  denote the set of maximal ideals of  $R$ . Recall our function  $\lambda: \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  given by  $\lambda(m) = n(m+1) - 1$  in (2.13).

**6.1. A pro-Gersten for  $K$ -theory.** In [10, Theorem 10.2] (see its proof), it was shown that if  $R$  contains  $\mathbb{Q}$ , the base change map  $\tilde{K}_n(R_m) \rightarrow \tilde{K}_n(F_m)$  is injective for all  $m, n \geq 0$ , where  $F$  is the total ring of quotients of  $R$ . However, we do not know if this inclusion holds in positive characteristic. We shall use a result of Hesselholt-Madsen [14] to prove the following partial result which will imply the validity of this inclusion in the pro-setting. We shall need this result in order to construct our cycle class map.

**Lemma 6.1.** *Let  $n \geq 0$  be any integer and let  $e \geq 1$  be an integer not divisible by  $p$ . Then the base change map  $\eta_{R,e}: \tilde{K}_n(R_{e-1}) \rightarrow \tilde{K}_n(F_{e-1})$  is injective. In particular, for every  $m \geq 1$ , the canonical map  $\text{Ker}(\tilde{K}_n(R_{mp}) \rightarrow \tilde{K}_n(F_{mp})) \rightarrow \tilde{K}_n(R_m)$  is zero.*

*Proof.* We only have to show that  $\eta_{R,e}$  is injective as the second assertion of the lemma immediately follows from this. We can assume that  $R$  is an integral domain so that  $F$  is a field. We now fix an integer  $n \geq 0$ . It was shown by Hesselholt-Madsen [14] that there is a natural exact sequence

$$(6.1) \quad \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_R^{n-2i} \xrightarrow{V_e} \bigoplus_{i \geq 0} \mathbb{W}_{e(i+1)} \Omega_R^{n-2i} \xrightarrow{\epsilon} \tilde{K}_{n+1}(R_{e-1}) \xrightarrow{\partial} \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_R^{n-1-2i},$$

where  $V_e$  is the Verschiebung map. By comparing this exact sequence with the analogous exact sequence for  $F$  and using Lemma 4.1, the proof of the injectivity of  $\eta_{R,e}$  reduces to showing that for every  $n \geq 0$  and  $m \geq 1$ , the square

$$(6.2) \quad \begin{array}{ccc} \mathbb{W}_m \Omega_R^n & \xrightarrow{V_e} & \mathbb{W}_{me} \Omega_R^n \\ \downarrow & & \downarrow \\ \mathbb{W}_m \Omega_F^n & \xrightarrow{V_e} & \mathbb{W}_{me} \Omega_F^n \end{array}$$

is Cartesian.

To show this, let  $\alpha \in \mathbb{W}_{me} \Omega_R^n$  and  $\beta \in \mathbb{W}_m \Omega_F^n$  be such that  $\alpha = V_e(\beta) \in \mathbb{W}_{me} \Omega_R^n$ . We consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{W}_m \Omega_R^n & \xrightarrow{V_e} & \mathbb{W}_{me} \Omega_R^n & \xrightarrow{F_e} & \mathbb{W}_m \Omega_R^n \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{W}_m \Omega_F^n & \xrightarrow{V_e} & \mathbb{W}_{me} \Omega_F^n & \xrightarrow{F_e} & \mathbb{W}_m \Omega_F^n \end{array}$$

where  $F_e$  is the Frobenius map. Since  $F_e \circ V_e(\beta) = e\beta$  (see [35, Definition 1.4]), we get  $F_e(\alpha) = e\beta$ . Since  $p \nmid e$ , we have that  $e \in (\mathbb{W}_m(R))^\times$ . We thus get  $\beta = e^{-1} F_e(\alpha) \in \mathbb{W}_m \Omega_R^n$ . Since the all vertical arrows in the above diagram are inclusions, it follows that  $\beta \in \mathbb{W}_m \Omega_R^n$  and  $V_e(\beta) = \alpha$ . This finishes the proof.  $\square$

**6.2. The sfs cycles.** We need the notion of sfs-cycles in order to generalize the cycle class map of (2.13) from fields to semi-local rings. Let  $m \geq 0$  and  $n \geq 1$  be two integers. Recall (see § 2.4) that  $\text{TCH}^n(R, n; m)$  is defined as the middle homology of the complex  $\text{Tz}^n(R, n+1; m) \xrightarrow{\partial} \text{Tz}^n(R, n; m) \xrightarrow{\partial} \text{Tz}^{n-1}(R, n; m)$ . Note that a cycle in  $\text{Tz}^n(R, n; m)$  has relative dimension zero over  $R$ . We shall say that an extension of regular semi-local rings  $R \subset R'$  is simple if there is an irreducible monic polynomial  $f \in R[t]$  such that  $R' = R[t]/(f(t))$ .

Let  $X = \text{Spec}(R)$  and  $\Sigma$  the set of all maximal ideals of  $R$ . Let  $Z \subset X \times \mathbb{A}_k^1 \times \square^{n-1}$  be an irreducible admissible relative 0-cycle. Recall from [29, Definition 3.4] that  $Z$  is called an sfs-cycle if the following hold.

- (1)  $Z$  intersects  $\Sigma \times \mathbb{A}_k^1 \times F$  properly for all faces  $F \subset \square^{n-1}$ .
- (2) The projection  $Z \rightarrow X$  is finite and surjective.

- (3)  $Z$  meets no face of  $X \times \mathbb{A}_k^1 \times \square^{n-1}$ .  
(4)  $Z$  is closed in  $X \times \mathbb{A}_k^1 \times \mathbb{A}_k^{n-1} = \text{Spec}(R[t, y_1, \dots, y_{n-1}])$  (by (2) above) and there is a sequence of simple extensions of regular semi-local rings

$$R = R_{-1} \subset R_0 \subset \dots \subset R_{n-1} = k[Z]$$

such that  $R_0 = R[t]/(f_0(t))$  and  $R_i = R_{i-1}[y_i]/(f_i(y_i))$  for  $1 \leq i \leq n-1$ .

Note that an sfs-cycle has no boundary by (3) above. We let  $\text{Tz}_{\text{sfs}}^n(R, n; m) \subset \text{Tz}^n(R, n; m)$  be the subgroup of cycles whose irreducible components are sfs-cycles and define

$$(6.3) \quad \text{TCH}_{\text{sfs}}^n(R, n; m) = \frac{\text{Tz}_{\text{sfs}}^n(R, n; m)}{\partial(\text{Tz}^n(R, n+1; m)) \cap \text{Tz}_{\text{sfs}}^n(R, n; m)}.$$

It is clear that the canonical map  $\text{TCH}_{\text{sfs}}^n(R, n; m) \rightarrow \text{TCH}^n(R, n; m)$  is injective. The following result from [28, Theorem 1.1] says more.

**Theorem 6.2.** *The canonical map  $\text{TCH}_{\text{sfs}}^n(R, n; m) \rightarrow \text{TCH}^n(R, n; m)$  is an isomorphism if  $k$  is infinite and perfect.*

**6.3. The cycle class map to Quillen  $K$ -theory.** The construction of the map  $\text{cyc}_R$  for  $\text{TCH}_{\text{sfs}}^n(R, n; m)$  is obtained by word by word repetition of the construction of the cycle class map for fields described in § 2.5. So let  $Z \subset X \times \mathbb{A}_k^1 \times \square^{n-1}$  be an irreducible sfs-cycle and let  $R' = k[Z]$ . Let  $f: Z \rightarrow X \times \mathbb{A}_k^1$  be the projection map. Let  $g_i: Z \rightarrow \square$  denote the  $i$ -th projection. Then the sfs property implies that each  $g_i$  defines an element of  $R'^{\times}$  and this in turn gives a unique element  $\{g_1, \dots, g_{n-1}\} \in K_{n-1}^M(R')$ . We let  $\text{cyc}_{R'}([Z])$  be its image in  $K_{n-1}(R')$  under the map  $K_{n-1}^M(R') \rightarrow K_{n-1}(R')$ . Since  $Z$  does not meet  $X \times \{0\}$ , we see that the finite map  $f$  defines a push-forward map of spectra  $f_*: K(R') \rightarrow K(R[t], (t^{m+1}))$ . We let  $\text{cyc}_R([Z]) = f_*(\text{cyc}_{R'}([Z])) \in K_{n-1}(R[t], (t^{m+1}))$ . We extend this definition linearly to get a cycle map  $\text{cyc}_R: \text{Tz}_{\text{sfs}}^n(R, n; m) \rightarrow K_{n-1}(R[t], (t^{m+1}))$ .

**Lemma 6.3.** *The assignment  $[Z] \mapsto \text{cyc}_R([Z])$  defines a cycle class map*

$$\text{cyc}_R: \text{TCH}_{\text{sfs}}^n(R, n; \lambda(pm)) \rightarrow K_{n-1}(R[t], (t^{m+1}))$$

which is functorial for the inclusion  $R \hookrightarrow F$ .

*Proof.* Let  $\pi: \text{Spec}(F) \hookrightarrow \text{Spec}(R)$  be the inclusion. We consider the diagram

$$(6.4) \quad \begin{array}{ccccccc} \partial^{-1}(\text{Tz}_{\text{sfs}}^n(R, n; \lambda(pm))) & \xrightarrow{\partial} & \text{Tz}_{\text{sfs}}^n(R, n; \lambda(pm)) & \xrightarrow{\text{cyc}_R} & K_{n-1}(R[t], (t^{pm+1})) & \rightarrow & K_{n-1}(R[t], (t^{m+1})) \\ \pi^* \downarrow & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* \\ \text{Tz}^n(F, n+1; \lambda(pm)) & \xrightarrow{\partial} & \text{Tz}^n(F, n; \lambda(pm)) & \xrightarrow{\text{cyc}_F} & K_{n-1}(F[t], (t^{pm+1})) & \rightarrow & K_{n-1}(F[t], (t^{m+1})), \end{array}$$

where the horizontal arrows in the square on the right are the structure maps of the pro-abelian group  $\{K_{n-1}(R[t], (t^m))\}_{m \geq 1}$  (and for  $F$ ) because  $mp \geq m$ . In particular, this square is commutative. It was shown in [10, Theorem 10.2] that all the other squares are commutative. It follows from the case of fields (see (2.12)) that the composite map  $\text{cyc}_F \circ \partial \circ \pi^*$  is zero. We deduce from Lemma 6.1 that the composite  $\text{cyc}_R \circ \partial$  is zero. It follows that the composition of all horizontal arrows in the top row of (6.4) is zero. This proves the lemma.  $\square$

Since the map  $\text{cyc}_k$  is clearly functorial in  $m \geq 1$ , using the natural isomorphism  $\partial: \tilde{K}_n(R_m) \xrightarrow{\cong} K_{n-1}(R[t], (t^{m+1}))$ , we get

**Theorem 6.4.** *For every  $n \geq 1$ , there is a cycle class map between pro-abelian groups*

$$\text{cyc}_R: \{\text{TCH}_{\text{sfs}}^n(R, n; m)\}_m \rightarrow \{\tilde{K}_n(R_m)\}_m$$

which is functorial for the inclusion  $R \hookrightarrow F$  and coincides with (2.13) if  $R$  is a field.



## 7. END OF THE PROOF OF THEOREM 1.1

We shall now complete the proof of Theorem 1.1 by proving its remaining part (4). After § 5.4, the key lemma that remains to be proven for this purpose is Lemma 7.4. We shall prove this in few steps. We let our ring  $R$  and other notations be the same as in § 6. Since  $R$  is a product of integral domains and our proofs for the case of integral domains directly generalize to products of such rings, we shall assume that  $R$  is a regular semi-local integral domain.

Hence the standing assumption of this section is that  $R$  is a regular semi-local integral domain which is essentially of finite type over a field  $k$  of characteristic  $p > 0$ . We let  $F$  denote the fraction field of  $R$ . Recall from Corollary 5.4 that we have a well-defined map  $\psi_R : \{\widehat{K}_*^M(R_m)\}_m \rightarrow \{K_*(R_m)\}_m$ . If  $R$  is a field, then this map is induced by the canonical map  $\psi_{R_m} : \widehat{K}_*^M(R_m) \rightarrow K_*(R_m)$  from Milnor to Quillen  $K$ -theory.

**7.1. The case of fields.** We first consider the case when  $R$  is a field. So we let  $k$  be a field of characteristic  $p > 0$ . We fix an integer  $n \geq 1$  and consider the diagram

$$(7.1) \quad \begin{array}{ccc} \{\mathrm{TCH}^n(k, n; m)\}_m & \xrightarrow{\mathrm{cyc}_k} & \{K_{n-1}(\mathbb{A}_k^1, (m+1)\{0\})\}_m \\ \mathrm{cyc}_k^M \downarrow & & \uparrow \partial \\ \{\widetilde{K}_n^M(k_m)\}_m & \xrightarrow{\psi_{k_m}} & \{\widetilde{K}_n(k_m)\}_m \end{array}$$

where  $\mathrm{cyc}_k$  is the map of (2.13).

Our goal is to show that this diagram is commutative. We shall use the shortened notation  $\psi_k$  for  $\psi_{k_m}$  even if it is meant to be used for  $k_m$  for different values of  $m \geq 1$  in different parts of the proofs.

**Lemma 7.1.** *The diagram (7.1) is commutative for  $n = 1$ .*

*Proof.* It follows from (2.13) and (3.8) that all maps in (7.1) are strict maps of pro-abelian groups, i.e., the associated function  $\lambda : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  is identity (see § 2.2). Furthermore, it was shown in the initial part of the proof of [10, Proposition 5.1] that  $\mathrm{cyc}_k$  is a level-wise isomorphism. It follows from (4.1) and (5.4) that all other maps are also level-wise isomorphisms. Clearly, all these are functorial in  $k$ .

Finally, to show that (7.1) commutes level-wise for  $n = 1$ , let  $w \in \mathbb{W}_m(k)$  and let  $f(T) = 1 + Tp(T) \in k[T]$  be a polynomial such that  $\gamma(w) = f(T)$  modulo  $T^{m+1}$ . The construction of the cycle class map in § 2.5 then shows that  $\mathrm{cyc}_k(\gamma(w))$  is the class of the finitely generated  $k[T]$ -module  $k[T]/(f(T))$  in  $K_0(\mathbb{A}_k^1, (m+1)\{0\})$  (see [10, §2C]). Since  $\tau_k$  is an isomorphism, it suffices to show that this class coincides with  $\partial(\gamma(w))$ . But this follows from [10, Lemma 2.1]. This proves the lemma and also proves stronger versions of Theorems 1.1 and 1.3 when  $n = 1$ .  $\square$

Our next goal is to prove the commutativity of (7.1) when  $n \geq 2$ . We let  $n \geq 2$  and let  $\lambda_n : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  be given by  $\lambda_n(m) = n(m+1) - 1$ . It is then easy to see using (2.13) and (3.8) that all maps in (7.1) are morphisms of pro-abelian groups all of whose associated functions are same, namely, the function  $\lambda_n$  above (note that this requires  $n \geq 2$ ). Moreover, for  $m' \geq m$ , the diagram (2.2) already commutes when  $l = \lambda_n(m')$ . In the proofs below, we shall write  $\lambda_n(m)$  simply as  $\lambda(m)$  since  $n$  is fixed.

To prove that the diagram (7.1) is commutative for  $k$  and  $n \geq 2$ , it suffices therefore to show that for every  $m \geq 1$ , the square on the right in the diagram

$$(7.2) \quad \begin{array}{ccc} \mathbb{W}_{\lambda(m)} \Omega_k^{n-1} \xrightarrow{\tau_k} \mathrm{TCH}^n(k, n; \lambda(m)) & \xrightarrow{\mathrm{cyc}_k} & K_{n-1}(\mathbb{A}_k^1, (m+1)\{0\}) \\ \theta_k \circ \lambda_k \searrow & \mathrm{cyc}_k^M \downarrow & \uparrow \partial \\ & \widetilde{K}_n^M(k_m) & \xrightarrow{\psi_{k_m}} \widetilde{K}_n(k_m) \end{array}$$

is commutative. Since  $\tau_k$  is an isomorphism, this is equivalent to showing that the outer trapezium is commutative.

To show the commutativity of the trapezium, we shall use Proposition 4.2 for fields (due to Hyodo-Kato [16] and R ulling-Saito [36]). Using this, it suffices to show that the above diagram commutes for the generators of the two groups on the left hand side of (4.2).

We know from (4.1) that any element  $w \in \mathbb{W}_{\lambda(m)}(k)$  is of the form  $\gamma^{-1}(\overline{1 - Tp(T)})$  with  $p(T) \in k[T]$  and  $\overline{f(T)} = f(T)$  modulo  $T^{\lambda(m)+1}$ . Since we can write  $1 - Tp(T)$  as a product of irreducible polynomials of the form  $1 - Tq(T)$ , we see that  $w$  is a sum of elements of the form  $\gamma^{-1}(\overline{1 - Tq(T)})$  such that  $1 - Tq(T)$  is irreducible. We can therefore assume that  $w = \gamma^{-1}(\overline{f(T)})$ , where  $f(T) = 1 - Tp(T)$  is irreducible.

In what follows below, we write  $\phi_k = \theta_k \circ \lambda_k$  and  $\psi_{k_m} = \psi_k$  to simplify the notations, where the value of  $m \geq 1$  is allowed to vary. We also write  $\overline{f(T)} = f(t)$  in any  $k_m$ . We let  $A = k[T]_{(T)}$ .

**Lemma 7.2.** *For  $n \geq 2$ , we have*

$$\partial \circ \psi_k \circ \phi_k(\text{wdlog}[a_1] \cdots \text{dlog}[a_{n-1}]) = \text{cyc}'_k \circ \tau_k(\text{wdlog}[a_1] \cdots \text{dlog}[a_{n-1}]).$$

*Proof.* With the above notations, we have

$$\begin{aligned} (7.3) \quad \partial \circ \psi_k \circ \phi_k(\text{wdlog}[a_1] \cdots \text{dlog}[a_{n-1}]) &=^1 \partial \circ \psi_k(\{\gamma(w), a_1, \dots, a_{n-1}\}) \\ &= \partial \circ \psi_k(\{1 - tp(t), a_1, \dots, a_{n-1}\}) \\ &=^2 (\partial \circ \psi_k(\{1 - tp(t)\})) \cdot \psi_k(\{a_1, \dots, a_{n-1}\}) \\ &=^3 \pi_*(1) \cdot \psi_k(\{a_1, \dots, a_{n-1}\}) \\ &=^4 \pi_*(\{a_1, \dots, a_{n-1}\}) \\ &=^5 \text{cyc}'_k(V(f(T), y_1 - a_1, \dots, y_{n-1} - a_{n-1})) \\ &=^6 \text{cyc}'_k \circ \tau_k(\text{wdlog}[a_1] \cdots \text{dlog}[a_{n-1}]), \end{aligned}$$

where  $\pi: \text{Spec}(k[T]/(f(T))) \hookrightarrow \mathbb{A}_k^1$  is the closed immersion.

We explain various equalities. First,  $\theta_k$  being the restriction map  $\widehat{K}_*^M(A, (T)) \rightarrow \widehat{K}_*^M(k_m)$  (see (3.9)), it is clear that  $\theta_k(\{f(T), a_1, \dots, a_{n-1}\}) = \{\gamma(w), a_1, \dots, a_{n-1}\}$ , where  $f(T)$  is viewed as an element of  $(1 + (T)) \subset A^\times$ . The equality  $=^1$  therefore follows from the definition of the map  $\lambda_k$  in (4.4). The equality  $=^2$  follows because  $\partial$  is a  $K_*^M(k)$ -linear map. The equality  $=^3$  follows from the  $n = 1$  case shown in Lemma 7.1 and  $=^4$  follows because  $\pi_*$  is  $K_*^M(k)$ -linear (see [10, Lemma 2.2]). The equality  $=^5$  follows from the definition of the cycle class map in (2.10) and  $=^6$  follows from (5.5). This finishes the proof.  $\square$

The final step is the following.

**Lemma 7.3.** *The diagram (7.1) is commutative for  $n \geq 2$ .*

*Proof.* Using the above reductions and Lemma 7.2, we only have to show that

$$(7.4) \quad \partial \circ \psi_k \circ \phi_k(\text{dwdlog}[a_1] \cdots \text{dlog}[a_{n-2}]) = \text{cyc}_k \circ \tau_k(\text{dwdlog}[a_1] \cdots \text{dlog}[a_{n-2}]).$$

We shall continue to use the above simplified notations and make another simplification by setting  $\tilde{w} = \text{dwdlog}[a_1] \cdots \text{dlog}[a_{n-2}]$ . It is clear from the definition of the differential for the Witt-complex structure on the additive higher Chow groups (see [26, § 6.1]) that if we let  $\gamma(w) = f(t) = 1 - tp(t)$ , then  $\tau_k(dw) = d\tau_k(w)$  is the class of the cycle  $V(f(T), Ty_1 - 1) \subset \mathbb{A}_k^1 \times \square$  in  $\text{TCH}^2(k, 2; \lambda(m))$ . As  $f(T)$  is irreducible,  $V(f(T), y_1T - 1, y_2 - a_1, \dots, y_{n-1} - a_{n-2})$  is a closed point  $z \in \mathbb{A}_k^1 \times \square^{n-1}$  such that  $l = k(z) \cong k[T]/(f(T))$ . We therefore have an admissible  $l$ -rational point  $z_0 = V(1 -$

$\alpha^{-1}T, y_1 - \alpha^{-1}, y_2 - a_1, \dots, y_{n-1} - a_{n-2}$  of  $\mathbb{A}_l^1 \times_l \square_l^{n-1}$  such that  $[z] = \pi_*([z_0])$ , where we let  $\alpha = T$  modulo  $(f(T))$  and  $\pi: \text{Spec}(l) \rightarrow \text{Spec}(k)$  the projection.

We can now write

$$\begin{aligned} \text{cyc}_k \circ \tau_k(\tilde{w}) &= \text{cyc}_k([z]) \\ &\stackrel{=1}{=} \pi_* \circ \text{cyc}_l([z_0]) \\ &= \pi_* \circ \text{cyc}_l(V(1 - \alpha^{-1}T, y_1 - \alpha^{-1}, y_2 - a_1, \dots, y_{n-1} - a_{n-2})) \\ &\stackrel{=2}{=} \pi_* \circ \partial_l \circ \psi_l \circ \phi_l(w_l \text{dlog}[\alpha^{-1}] \text{dlog}[a_1] \cdots \text{dlog}[a_{n-2}]) \\ &\stackrel{=3}{=} \partial_k \circ \pi_* \circ \psi_l \circ \phi_l(w_l \text{dlog}[\alpha^{-1}] \text{dlog}[a_1] \cdots \text{dlog}[a_{n-2}]), \end{aligned}$$

where  $w_l = \gamma^{-1}(1 - \alpha^{-1}T) \in \mathbb{W}_{\lambda(m)}(l)$ .

The equality  $\stackrel{=1}{=}$  follows from the construction of the cycle class map (see [10, Lemma 4.4]),  $\stackrel{=2}{=}$  follows from Lemma 7.2 for  $\text{Spec}(l)$  and  $\stackrel{=3}{=}$  follows because the connecting homomorphism  $\partial$  commutes with the push-forward map  $\pi_*$ . Note that this push-forward map exists on the relative  $K$ -theory by (2.4). It suffices therefore to show that

$$(7.5) \quad \psi_k \circ \phi_k(\tilde{w}) = \pi_* \circ \psi_l \circ \phi_l(w_l \text{dlog}[\alpha^{-1}] \text{dlog}[a_1] \cdots \text{dlog}[a_{n-2}]).$$

However, we have

$$\begin{aligned} \psi_k \circ \phi_k(\tilde{w}) &= \psi_k \circ \theta_k \circ \lambda_k(\tilde{w}) \\ &= (-1)^{n-1} \psi_k \circ \theta_k(\{\gamma(w), a_1, \dots, a_{n-2}, T\}) \\ &\stackrel{=1}{=} -\psi_k \circ \theta_k(\{\gamma(w), T, a_1, \dots, a_{n-2}\}) \\ &= (-\psi_k \circ \theta_k(\{\gamma(w), T\})) \cdot \psi_k(\{a_1, \dots, a_{n-2}\}), \end{aligned}$$

where  $\stackrel{=1}{=}$  follows from [21, Lemma 2.2] as  $k(T)$  is infinite. On the other hand, letting  $\tilde{w}_l = w_l \text{dlog}[\alpha^{-1}] \text{dlog}[a_1] \cdots \text{dlog}[a_{n-2}]$ , we also have

$$\begin{aligned} \pi_* \circ \psi_l \circ \phi_l(\tilde{w}_l) &= \pi_* \circ \psi_l(\{\gamma(w_l), \alpha^{-1}, a_1, \dots, a_{n-2}\}) \\ &= (\pi_* \circ \psi_l(\{\gamma(w_l), \alpha^{-1}\})) \cdot \psi_k(\{a_1, \dots, a_{n-2}\}), \end{aligned}$$

where the last equality holds by the projection formula. Thus, (7.5) is reduced to showing that for every  $m \geq 1$ , we have

$$(7.6) \quad \psi_k \circ \theta_k(\{1 - Tp(T), T\}) = -\pi_* \circ \psi_l(\{\gamma(w), \alpha^{-1}\}) = -\pi_* \circ \psi_l(\{1 - \alpha^{-1}T, \alpha^{-1}\})$$

in  $\tilde{K}_2(k_m)$  under the composite map  $\frac{\tilde{K}_2^M(A, (T))}{\tilde{K}_2^M(A, (T^{m+1}))} \xrightarrow{\theta_k} \tilde{K}_2^M(k_m) \xrightarrow{\psi_k} \tilde{K}_2(k_m)$ . Here,  $\{1 - Tp(T), T\}$  is viewed as an element of  $\tilde{K}_2^M(A, (T))$  via the inclusion  $\tilde{K}_2^M(A|(T)) \subset \tilde{K}_2^M(A, (T))$  of Lemma 3.9.

The commutative diagram

$$(7.7) \quad \begin{array}{ccc} \frac{\tilde{K}_2^M(A, (T))}{\tilde{K}_2^M(A, (T^{m+1}))} & \xrightarrow{\theta_k} & \tilde{K}_2^M(k_m) \\ \psi_A \downarrow & & \downarrow \psi_k \\ \frac{K_2(A, (T))}{K_2(A, (T^{m+1}))} & \xrightarrow{\theta_k} & \tilde{K}_2(k_m) \end{array}$$

shows that verifying (7.6) is equivalent to showing that

$$(7.8) \quad \theta_k(\{1 - Tp(T), T\}) = -\pi_*(\{1 - \alpha^{-1}T, \alpha^{-1}\})$$

in  $\tilde{K}_2(k_m)$  under the map  $\frac{K_2(A, (T))}{K_2(A, (T^{m+1}))} \xrightarrow{\theta_k} \tilde{K}_2(k_m)$ , if we use the identical notation for  $\{1 - Tp(T), T\} \in \tilde{K}_2^M(A, (T))$  (resp.,  $\{1 - \alpha^{-1}T, \alpha^{-1}\} \in \tilde{K}_2^M(l_m)$ ) and its image in  $K_2(A, (T))$  via  $\psi_A$  (resp., in  $\tilde{K}_2(l_m)$  via  $\psi_l$ ). We shall use this convention in the rest of the proof.

To show (7.8), we let  $A' = l[T]_{(T)}$  as in the notations of Lemma 2.3. We showed in § 2.3 that  $A'$  is finite (and flat) over  $A$  and  $k_m \otimes_A A' \cong l_m$ . Using (2.4), we get push-forward maps  $\pi_*: K_2(A', (T^i)) \rightarrow K_2(A, (T^i))$  for all  $i \geq 0$  and a commutative diagram

$$(7.9) \quad \begin{array}{ccc} \frac{K_2(A', (T))}{K_2(A', (T^{m+1}))} & \xrightarrow{\theta_l} & \widetilde{K}_2(l_m) \\ \pi_* \downarrow & & \downarrow \pi_* \\ \frac{K_2(A, (T))}{K_2(A, (T^{m+1}))} & \xrightarrow{\theta_k} & \widetilde{K}_2(k_m). \end{array}$$

It suffices therefore to show that  $\pi_*({1 - \alpha^{-1}T, \alpha^{-1}}) = -{1 - Tp(T), T}$  holds under the left vertical arrow in (7.9).

Since  ${1 - Tp(T), T} \in K_2(A, (T)) \subset K_2(A)$  and  ${1 - \alpha^{-1}T, \alpha^{-1}} \in K_2(A', (T)) \subset K_2(A')$  (note that these inclusions use the splitting of  $A \rightarrow A/(T)$  and  $A' \rightarrow A'/(T)$ ), it suffices to show that  $\pi_*({1 - \alpha^{-1}T, \alpha^{-1}}) = -{1 - Tp(T), T}$  in  $K_2(A)$ . Using Lemma 2.3, we further reduce to showing that this equality holds in  $K_2(k(T))$  under the push-forward map  $\pi_*: K_2(l(T)) \rightarrow K_2(k(T))$ .

But in  $K_2(k(T))$ , we have

$$\begin{aligned} -\pi_*({1 - \alpha^{-1}T, \alpha^{-1}}) &= \pi_*({1 - \alpha^{-1}T, T}) \\ &\stackrel{=1}{=} \pi_*({1 - p(\alpha)T, T}) \\ &\stackrel{=2}{=} N_{l(T)/k(T)}({1 - p(\alpha)T, T}) \\ &\stackrel{=3}{=} \{N_{l(T)/k(T)}(1 - p(\alpha)T), T\} \\ &\stackrel{=4}{=} \{1 - p(T)T, T\}. \end{aligned}$$

Here,  $\stackrel{=1}{=}$  follows because  $1 - \alpha p(\alpha) = 0$  in  $l$ , the equality  $\stackrel{=2}{=}$  follows by the compatibility between the Norm in Milnor  $K$ -theory and push-forward in Quillen  $K$ -theory (see the proof of [10, Lemma 4.4]),  $\stackrel{=3}{=}$  follows from the projection formula for norm as  $T \in k(T)^\times$ , and  $\stackrel{=4}{=}$  is a straightforward calculation of the norm of  $1 - p(\alpha)T \in l(T)^\times$ . This finishes the proof of the lemma.  $\square$

**7.2. Back to the case of semi-local ring.** The following is our last key lemma before we prove the main results. Let  $cyc_R$  be the cycle class map of Theorem 6.4. Here,  $R$  is the semi-local integral domain satisfying the standing assumptions of this section.

**Lemma 7.4.** *The diagram*

$$(7.10) \quad \begin{array}{ccc} \{\mathrm{TCH}_{\mathrm{sfs}}^n(R, n; m)\}_m & & \\ \downarrow cyc_R^M & \searrow cyc_R & \\ \{\widetilde{K}_n^M(R_m)\}_m & \xrightarrow{\psi_R} & \{\widetilde{K}_n(R_m)\}_m \end{array}$$

is commutative. Equivalently,  $cyc_R = cyc'_R$ .

*Proof.* When  $R$  is a field, the lemma is equivalent to the commutativity of (7.1). We now prove the general case. Let  $\pi: \mathrm{Spec}(F) \hookrightarrow \mathrm{Spec}(R)$  be the inclusion of the generic

point. We consider the diagram

$$(7.11) \quad \begin{array}{ccc} \{\mathrm{TCH}_{\mathrm{sfs}}^n(R, n; m)\}_m & \xrightarrow{\pi^*} & \{\mathrm{TCH}^n(F, n; m)\}_m \\ \downarrow \mathrm{cyc}_R^M & \searrow \mathrm{cyc}_R & \downarrow \mathrm{cyc}_F \\ & \{\tilde{K}_n(R_m)\}_m & \xrightarrow{\pi^*} & \{\tilde{K}_n(F_m)\}_m \\ & \nearrow \psi_R & \downarrow \mathrm{cyc}_F^M & \nearrow \psi_F \\ \{\tilde{K}_n^M(R_m)\}_m & \xrightarrow{\pi^*} & \{\tilde{K}_n(F_m)\}_m \end{array}$$

We check the commutativity of various faces of (7.11). The front face clearly commutes and the back face commutes by Theorem 6.4. The right (triangular) face commutes because  $F$  is a field. The commutativity of the bottom face was shown in the construction of  $\mathrm{cyc}_R^M$  in § 5.3. A diagram chase shows that  $\pi^* \circ \psi_R \circ \mathrm{cyc}_R^M = \pi^* \circ \mathrm{cyc}_R$ . We can now apply Lemma 6.1 to conclude that (7.10) commutes. We use here an elementary fact that if a morphism between two pro-abelian groups factors through the zero pro-group, then this morphism itself is zero (see § 2.2).  $\square$

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