

# Simulatability of locally-interacting open quantum spin systems

Rahul Trivedi<sup>1,2\*</sup> and J. Ignacio Cirac<sup>1,2</sup>

<sup>1</sup>Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, 85748 Garching, Germany.

<sup>2</sup>Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, D-80799 Munich, Germany.

(Dated: October 22, 2021)

We analyze the complexity of classically simulating the dynamics of locally interacting quantum spin systems with a constant rate of entanglement breaking noise. We show that for a system with  $n$  spins, a polynomial time classical algorithm can be used to sample within a  $O(1/\text{poly}(n))$  total-variation distance of the state of the spins when the rate of noise is sufficiently high. Furthermore, by encoding a 1D fault tolerant quantum computation into the dynamics of open quantum spin systems, we show that for certain physically relevant channels, the problem of sampling from the output state is BQP-complete in the low noise regime. Our results rigorously show the presence of phase transitions in the classical simulatability of the continuous time dynamics of open quantum spin systems.

*Introduction:* Simulating the dynamics of quantum spin systems with local interactions is of fundamental interest in many-body physics. Not only is this a model for many physically relevant systems, it also serves as a model for engineered quantum systems that underly quantum information processing technologies. There has been a long line of research aimed at developing classical algorithms for simulating dynamics of quantum spin systems using tensor-networks [1–5]. It is recognized that simulating quantum spin system dynamics on classical computers is generically hard due to the possibility of encoding a quantum computation in such dynamics [6–8]. However, for quantum systems interacting with an external environment, more opportunities open up for classically simulating their dynamics [9–14]. From a physical standpoint, a strongly interacting external environment prevents the spins from significantly entangling with each other and thus opens up the possibility of an efficient classical representation of the quantum state.

This question has received significant attention in the context of *circuit model* (or discrete-time model) of noisy quantum computation. It was shown very early on that a simulatability phase transition is expected for the circuit model of quantum computation on tuning the rate of noise. For sufficiently high rate of noise, there would exist a polynomial time classical algorithm to simulate a quantum computer [15, 16]. Furthermore, the threshold theorem for quantum computation [17, 18] implied that if the noise is small enough (below a threshold) and a fresh supply of auxiliary qubits is available, then a faultless quantum circuit can be encoded into a faulty quantum circuit. Through numerical simulations and analytical mappings to classical statistical mechanics, qualitatively similar behaviour has been found in the entanglement dynamics of random unitary circuits interspersed with measurements [19–24].

In this paper, we rigorously show the existence of simulatability phase transitions in the *continuous time dynamics* of locally interacting spin systems which have thus far only been treated numerically in specific models [25, 26]. Not only does this model underly the discrete-time circuit model, it also physically more relevant for analyzing near term quantum hardware

being used as analogue quantum simulators [27–34]. We first consider the high noise regime and show that for general Lindbladian (dissipative or non-dissipative) interactions between the spins, a classical polynomial time algorithm can sample within  $O(1/\text{poly}(n))$  total variation error of the spin states after  $\text{poly}(n)$  time (theorem 1). The key technical contribution here is to identify a map between the noisy continuous time dynamics and a correlated percolation problem. Next, in the low noise regime and even with non-dissipative local interactions, we show that spin models in two or higher dimensions are expected to be classically insimulatable if the noise channel is entanglement breaking and maps the spin to a thermal state on a known basis (theorem 2). The key technical contribution here is an encoding of 1D fault tolerant quantum computation [17, 18] into the dynamics of the spin system, which is enabled by implementing heat bath algorithmic cooling [35, 36] by leveraging the noise channel so as to refresh the bath qubits while maintaining the fault tolerance threshold theorem. A minor modification of this construction also shows that discrete-time models of unitary circuits interrupted with entanglement breaking noise (such as projective measurements) are also expected to be insimulatable at low noise rates (theorems 3 and 4), thus providing rigorous evidence for numerically and analytically observed entanglement phase transitions [21–24].

We first consider the problem of simulating the continuous-time dynamics of locally interacting quantum spin systems in the high noise regime, as made precise below.

**Definition 1** A superoperator  $\mathcal{L} : \mathfrak{L}((\mathbb{C}^2)^{\otimes n}) \rightarrow \mathfrak{L}((\mathbb{C}^2)^{\otimes n})$  over  $n$  spins arranged on  $\mathbb{Z}^d$  is a local Lindbladian with an interaction range  $R$  if  $\exists \mathcal{S} = \{\Lambda \subset \mathbb{Z}^d \mid \text{diam}(\Lambda) \leq R\}$  such that

$$\mathcal{L} = \sum_{\Lambda \in \mathcal{S}} \mathcal{L}^\Lambda,$$

where  $\forall \Lambda \in \mathcal{S}$ ,  $\mathcal{L}^\Lambda : \mathfrak{L}((\mathbb{C}^2)^{\otimes n}) \rightarrow \mathfrak{L}((\mathbb{C}^2)^{\otimes n})$  is a Lindbladian which is identity on spins outside  $\Lambda$ . The tuple  $(\mathcal{S}, \{\mathcal{L}^\Lambda \mid \Lambda \in \mathcal{S}\})$  is a local representation of  $\mathcal{L}$  with interaction range  $R$ .

Furthermore, we need to introduce a notion of the interaction strength of the local evolution, which would determine the threshold for the noise rate  $\kappa$  above which the dynamics of the

\* rahul.trivedi@mpq.mpg.de

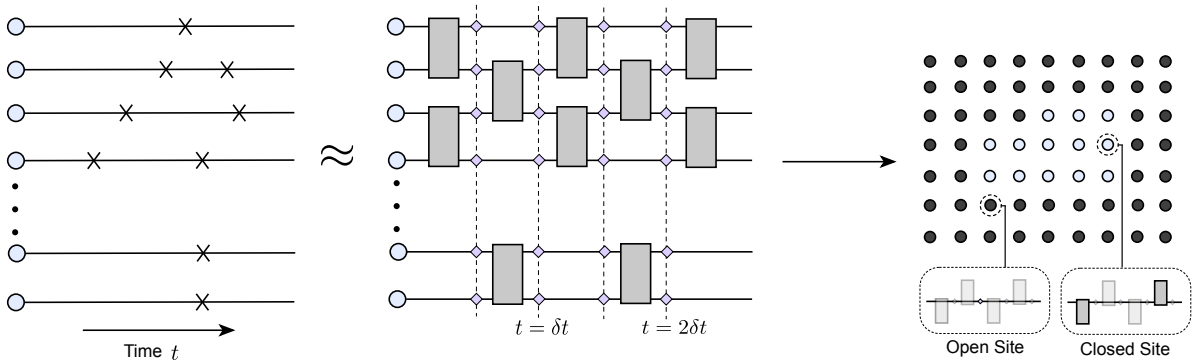


FIG. 1. Schematic depiction of the mapping of a continuous time model to an equivalent percolation problem in 1D (this mapping is easily generalized to higher dimensions, see the supplement). The continuous time evolution is first trotterized, followed by blocking  $\tau/\delta t$  time-steps together and identifying this block of time-steps per qubit with a site on the percolating lattice. The site is declared open if the qubit associated with the site experiences entanglement breaking noise at least once and does not couple to any of the neighbouring qubits via the channels obtained on trotterization of the Lindbladian  $\mathcal{L}(t)$ .

spin system is classically simulatable. The notion of interaction strength that will be of key importance in theorem 1 is related to the Choi-state of the local terms in the Lindbladian determining the interaction.

**Definition 2** Let  $\mathcal{L}$  be a local Lindbladian with an interaction range  $R$ , then the interaction strength of its local representation  $(\mathcal{S}, \{\mathcal{L}^\Lambda | \Lambda \in \mathcal{S}\})$  is given by

$$g = \max_{\Lambda \in \mathcal{S}} |\lambda_{\max}(\Phi_{\mathcal{L}^\Lambda})|,$$

where  $\Phi_{\mathcal{L}}$  is the Choi-state corresponding to the superoperator  $\mathcal{L}$ ,  $\lambda_{\max}(\Phi_{\mathcal{L}})$  is its maximum magnitude eigenvalue.

**Problem 1 (Continuous-time model)** For a fixed lattice dimension  $d \in \mathbb{N}$ , an interaction range  $R > 0$ , an interaction strength  $g$ , a noise rate  $\kappa > 0$  and a single spin entanglement-breaking channel  $\mathcal{N} : \mathfrak{L}(\mathbb{C}^2) \rightarrow \mathfrak{L}(\mathbb{C}^2)$ , sample in the computational basis from a family of states  $\{\rho_n \in \mathfrak{D}_1((\mathbb{C}^2)^{\otimes n}) : n \in \mathbb{N}\}$  where for  $n \in \mathbb{N}$ ,  $\rho_n$  is a state of  $n$ -spins arranged on  $\mathbb{Z}^d$  such that

$$\rho_n = \mathcal{T} \exp \left( \int_0^{t_n} \left( \mathcal{L}_n(s) + \kappa \sum_{i=1}^n (\mathcal{N}_i - \text{id}) \right) ds \right) \rho_n(0),$$

where

- $\forall n \in \mathbb{N}$ ,  $\rho_n(0)$  is a  $n$  spin product state that can be computed classically in  $O(\text{poly}(n))$  time,
- $t_n = O(\text{poly}(n))$  is the evolution time,
- $\forall n \in \mathbb{N}$  and  $s \in [0, t_n]$ ,  $\mathcal{L}_n(s) : \mathfrak{L}((\mathbb{C}^2)^{\otimes n}) \rightarrow \mathfrak{L}((\mathbb{C}^2)^{\otimes n})$  is a  $n$ -spin local Lindbladian superoperator which has an efficiently computable local representation of interaction range  $R$  and interaction strength  $g$ .
- $\forall i \in \{1, 2, \dots, n\}$ ,  $\mathcal{N}_i : \mathfrak{L}((\mathbb{C}^2)^{\otimes n}) \rightarrow \mathfrak{L}((\mathbb{C}^2)^{\otimes n})$  is the tensor product of the entanglement breaking channel  $\mathcal{N}$  acting on the  $i^{\text{th}}$  spin and identity on other spins.

**Theorem 1**  $\exists f_{\text{th}} > 0$  dependent on  $d$  and  $R$  such that for  $\kappa \geq f_{\text{th}}g$ , there is classical polynomial-time algorithm to sample within a  $O(1/\text{poly}(n))$  total variation distance of  $\rho_n$ , the  $n$  qubit state from the family of states specified in problem 1.

*Proof sketch:* The basic steps in the proof are schematically depicted in Fig. 1 — we first trotterize the continuous time evolution with time-step  $\delta t = O(1/\text{poly}(n))$  chosen to incur a total variation error  $\leq O(1/\text{poly}(n))$ . While in the trotterized picture the noise channels are applied at the same time, the probability of the noise actually being applied is proportional to  $\delta t$  and hence vanishingly small — consequently, the trotterized picture does not directly exhibit a percolation phase transition either. The key idea to resolve these issues is to note that the channels resulting from the trotterization of the local  $n$  qubit Lindbladian  $\mathcal{L}_n$  can be further approximated as a convex combination of an identity channel, applied with probability  $1 - \Omega(g\delta t)$  and another completely positive trace-preserving (CPTP) map, applied with probability  $O(g\delta t)$ . To sample from the trotterized circuit, we sample independently from the channels resulting from the trotterization of  $\mathcal{L}_n$  and the entanglement breaking channel  $\mathcal{N}$ . Each channel sampling is mapped to a percolation problem by associating a site in the equivalent percolation problem with a qubit and a block of  $m$  trotterized time steps, where  $m \approx \tau/\delta t$  for some  $\tau > 0$ . The site is declared open if the the associated qubit experiences the entanglement breaking channel at least once and does not couple to the neighbouring qubits through the channels corresponding to  $\mathcal{L}_n$ , else it is declared closed. The probability of the site being open is then lower bounded by  $\approx \exp(-\Omega(g\tau))(1 - \exp(-\kappa\tau))$  (see the supplement for more details). A simple analysis of this result shows that if  $\kappa$  is larger than a threshold determined by  $g$ ,  $\tau$  can be chosen such that this probability exceeds the site percolation threshold, making the percolation problem subcritical. Now, using the fact that the sizes of the clusters in the subcritically percolated lattice are almost surely  $O(\log n)$  [37], we then obtain that the resulting sequence of sampled channels can be contracted in  $\text{poly}(n)$  time on a classical computer. A subtlety in this analysis is

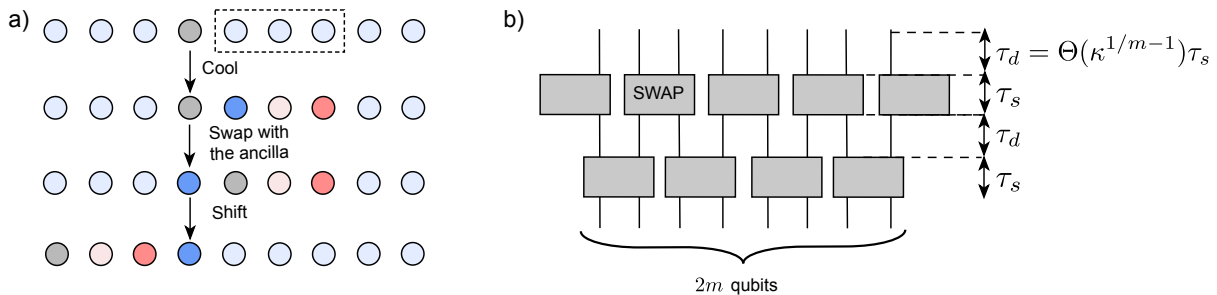


FIG. 2. Implementation of a RESTART gate on an a computational ancilla qubit by using the neighbouring auxillary qubits — (a) Schematic depiction of the basic steps involved — cooling the auxiliary qubits, swap with ancilla and shift for the next restart operation. (b) The shift operation implemented by layers of SWAP gates (which take time  $\tau_s$ ) followed by allowing the noise to act on the individual qubits for time  $\tau_d$ . Since the choice  $\tau_d = \Theta(\kappa^{1/m-1})\tau_s$  suffices to shift  $m$  qubits with a given fidelity, this construction does not break the fault-tolerant threshold theorem.

that the effective percolation problem obtained is not where each site is independent, but where the probability of a site being open or closed is dependent on the state of the site above or below it — however, by closely analyzing the conditional probability, we can show that the size of the clusters in this problem are also upper bounded by  $O(\log n)$ . A detailed formal proof is provided in the supplement.  $\square$ .

Intuitive reasoning might suggest that since the entanglement breaking noise, at any non-zero rate, disentangles an extensive number of qubits, it might be possible to classically simulate local spin systems at any rate of noise. For instance, if the entanglement breaking noise is depolarizing (i.e.  $\mathcal{N}(\cdot) = \text{tr}(\cdot)I/2$ ) and the local interactions are entirely non dissipative, then it follows simply by entropic considerations [38, 39] that the beyond  $t = \Theta(\log(n))$ , the joint state of the qubits is within a  $O(1/\text{poly}(n))$  total variation error of the maximally mixed state. In 1D locally interacting spin systems, this implies that the dynamics is always classically simulatable for all times  $t$ , and in two or higher dimensions, the dynamics is classically simulatable for  $t = \Omega(\log(n))$ . For depolarizing noise, therefore, we do not expect to have a simulatability phase transition on changing the rate of noise.

However, if the noise channel is not depolarizing, then the entropic argument presented in Refs. [38, 39] does not apply, and there is a possibility that for sufficiently small rate of noise  $\kappa$ , the dynamics of the spin system become hard to simulate. In the remainder of this paper, we investigate the relationship between the properties of the noise channel  $\mathcal{N}$  and the simulatability of dynamics in the low noise regime using the theory of fault-tolerant quantum computation. Our analysis relies on the threshold theorems proved by Aharonov [17] and Gottesman [18] where they showed that if the noise rate is below a particular threshold, a fault tolerant quantum computation can be encoded even on a 1D lattice of spins with nearest neighbour interaction provided that we have the ability to perform a RESTART gate (i.e. a channel which could replace a qubit with a  $|0\rangle$ ) in the middle of the computation.

We first show the insimulatability of the continuous time model introduced in Problem 1 if  $\mathcal{N}$  maps the spin to a thermal state on a known basis (e.g. amplitude damping channel).

**Theorem 2** *If  $\exists \varepsilon > 0$  and mutually orthogonal states*

$|\phi^+\rangle, |\phi^-\rangle \in \mathbb{C}^2$  such that  $\forall \sigma \in \mathcal{N}(\mathfrak{D}_1(\mathbb{C}^2))$  :

$$\sigma = \frac{1 + \varepsilon_\sigma}{2} |\phi^+\rangle \langle \phi^+| + \frac{1 - \varepsilon_\sigma}{2} |\phi^-\rangle \langle \phi^-|,$$

for some  $\varepsilon_\sigma > \varepsilon$ , then for  $d \geq 2$ ,  $\exists \kappa_{\text{th}} > 0$  such that for  $\kappa \leq \kappa_{\text{th}}$  problem 1 with purely non-dissipative Lindbladians is BQP complete.

*Proof sketch:* The key idea behind this proof is to encode a fault tolerant quantum computation within the continuous-time model. We first recall some preliminaries:

1. It has been shown in Refs. [6, 7] in the absence of noise, sampling from the quantum state generated by a sequence of local unitary gates even in a 1D lattice of qubits is BQP complete.
2. If allowed to controllably restart the qubits (i.e. replace them with a qubit in a pure state), fault tolerance in 1D circuits can be achieved with just nearest neighbour unitary gates [17], and thus the problem of sampling from the state of a 1D lattice of qubits generated by local unitaries and a RESTART gate is BQP complete if the rate of noise is below the error correction threshold.

However, we do not directly have access to a RESTART gate — we propose to implement this gate with unitaries and by exploiting the noise channel (Fig. 2a). Consider  $d = 2$  (BQP completeness for  $d = 2 \implies$  BQP completeness for  $d \geq 2$ ) — we pick one row of spins in the lattice as the computational qubits, comprising of data qubits (which hold the encoded state on which the BQP complete computation is being performed) and ancilla qubits (which are used to perform error correction on the encoded data qubits). The ancilla qubits need to be restarted during the computation — to implement the RESTART gate on an ancilla qubit, we utilize the qubits, henceforth called the auxillary qubits, in the column containing the ancilla. We initialize these qubits in a state  $\sigma \in \mathcal{N}(\mathfrak{D}(\mathbb{C}^2))$  — since these qubits are in a finite-temperature thermal state at the time the RESTART operation needs to be performed,  $\Theta(1)$  of them next to the ancilla qubit, cool them using a sorting unitary [35, 36], and swap them with the ancilla qubit.

While this restarts the ancilla qubit into the state  $|\phi^-\rangle$ , we need to replenish the auxillary qubits so as to prepare them for the next RESTART gate. This is done by shifting the qubits next to the used auxillary qubits in their place — a difficulty in performing this shift operation is that it could be faulty due to the noise. Since we need RESTART operations at  $\Theta(\text{poly}(n))$  time, the error in the shift operation is non negligible even at any constant, no matter how small, rate of noise. To resolve this issue, we propose to perform an imperfect shift operation, followed by allowing the noise to act on the shifted qubits for time  $\tau_d$  (Fig. 2b) — while an imperfect shift operation can possibly entangle the qubits and even put them into  $I/2$ , the action of the noise channel on the qubit disentangles the qubits and drives them to a finite temperature state. Clearly, if  $\tau_d$  is chosen to be large enough, then the qubits would be in a state which can be subsequently cooled. However, increasing  $\tau_d$  also increases the effective noise on the computational qubits since error correction is paused till the qubits are not restarted— however, a close analysis of this operation reveals that (see the supplement) that to replenish  $m$  auxillary qubits with the shift operation,  $\tau_d$  can be chosen to be  $\Theta(1/\kappa^{1-1/m})$  and hence the error sustained in the computational qubits while error correction is paused for this shifting, which is proportional to  $\kappa\tau_d$ , can be made smaller than the error correction threshold for sufficiently small  $\kappa$ .  $\square$

A simplified version of the construction of theorem 2 allows us to show the low-noise insimulatability of discrete-time models of unitaries interspersed with entanglement breaking channels. For dynamics of local random unitary circuits interspersed with projective measurements, it is found using numerical simulations and analytical mappings to statistical mechanics that for sufficiently low noise rate, a volume law entanglement develops in the states of spins [21, 22]. We make this observation rigorous by showing that whenever the noise channels have a fixed point different from  $I/2$ , a fault-tolerant quantum computation can again be encoded in noisy discrete time models — this condition is weaker than the one assumed in theorem 2 and additionally covers channels corresponding to projective measurements. We consider two unitary layers — a parallelized layer of unitary gates acting on disjoint sets of qubits, and matrix product unitary layers [40].

**Definition 3** A superoperator  $\mathcal{U} : \mathfrak{L}((\mathbb{C}^2)^{\otimes n}) \rightarrow \mathfrak{L}((\mathbb{C}^2)^{\otimes n})$  over  $n$  spins arranged on  $\mathbb{Z}^d$  is a local parallelized unitary layer if  $\forall \rho \in \mathfrak{L}((\mathbb{C}^2)^{\otimes n})$

$$\mathcal{U}\rho = \left( \prod_{i=1}^M U_i \right) \rho \left( \prod_{i=1}^M U_i^\dagger \right),$$

for some  $U_1, U_2 \dots U_M$  that are single or nearest-neighbour two-qubit unitaries acting on disjoint sets of spins.

**Definition 4** A superoperator  $\mathcal{U} : \mathfrak{L}((\mathbb{C}^2)^{\otimes n}) \rightarrow \mathfrak{L}((\mathbb{C}^2)^{\otimes n})$  over  $n$  spins is a matrix-product unitary layer of bond dimension  $D$  if  $\forall \rho \in \mathfrak{L}((\mathbb{C}^2)^{\otimes n})$ ,  $\mathcal{U}\rho = U\rho U^\dagger$ , where  $U$  is a  $n$  qubit

unitary such that for  $\forall i_1, i_2 \dots i_n, j_1, j_2 \dots j_n \in \{0, 1\}$

$$\langle i_1, i_2 \dots i_n | U | j_1, j_2 \dots j_n \rangle = \prod_{l=1}^n A_l^{i_l, j_l},$$

where  $\forall i_1, i_2 \dots i_n, j_1, j_2 \dots j_n \in \{0, 1\}$ ,  $A_1^{i_1, j_1} \in \mathbb{C}^{1 \times D}$ ,  $A_n^{i_n, j_n} \in \mathbb{C}^{D \times 1}$  and  $A_2^{i_2, j_2}, A_3^{i_3, j_3} \dots A_{n-1}^{i_{n-1}, j_{n-1}} \in \mathbb{C}^{D \times D}$ .

We point out that every local parallelized unitary layer in  $d = 1$  dimensions can be expressed as a matrix product unitary layer with bond-dimension 2, but there are matrix product unitary layers that cannot be expressed as a local parallelized unitary layers.

**Problem 2 (Discrete time model)** For a fixed noise rate  $p \in (0, 1)$  and a single spin entanglement-breaking channel  $\mathcal{N} : \mathfrak{L}(\mathbb{C}^2) \rightarrow \mathfrak{L}(\mathbb{C}^2)$ , sample in the computational basis from a family of states  $\{\rho_n \in \mathfrak{D}_1((\mathbb{C}^2)^{\otimes n}) : n \in \mathbb{N}\}$  where for  $n \in \mathbb{N}$ ,  $\rho_n$  is a state of  $n$  spins given by

$$\rho_n = \mathcal{E}_p^{\otimes n} \mathcal{U}_{t_n} \mathcal{E}_p^{\otimes n} \mathcal{U}_{t_n-1} \dots \mathcal{U}_1 \rho(0),$$

where

- $\forall n \in \mathbb{N}$ ,  $\rho_n(0)$  is a  $n$  spin product state that can be computed classically in  $O(\text{poly}(n))$  time,
- $t_n = O(\text{poly}(n))$  is the number of time steps,
- $\forall n \in \mathbb{N}, t \in \{1, 2 \dots t_n\}$ ,  $\mathcal{U}_t$  is a unitary layer,
- $\mathcal{E}_p = (1 - p)\text{id} + p\mathcal{N}$ .

**Theorem 3** If  $\exists \sigma \neq I/2$  such that  $\mathcal{N}(\sigma) = \sigma$ , then  $\exists p_{\text{th}} > 0$  such that for  $p \leq p_{\text{th}}$  the problem 2 with local parallelized unitary layers in two or higher dimensions (definition 3) is BQP complete.

*Proof sketch:* We follow the same construction as that of theorem 2, with the additional simplification that the shift operation can be done faultlessly within this model. This follows from the fact that the shift operation can be performed as two layers of two-qubit SWAP gates, and since in the model the errors are allowed only before or after the SWAP gate, a qubit initially in a fixed point of  $\mathcal{N}$  remains unchanged.  $\square$

**Theorem 4** If  $\exists \sigma \neq I/2$  such that  $\mathcal{N}(\sigma) = \sigma$ , then  $\exists p_{\text{th}} > 0$  such that for  $p \leq p_{\text{th}}$ , problem 2 with matrix-product unitary layers of bond dimension  $\geq 2$  (definition 4) is BQP complete.

*Proof sketch:* Note that a non-local SWAP operation can be implemented with a matrix product unitary (MPU) of bond dimension 2 by applying SWAP gates sequentially — consequently, all the ancilla qubits used to implement the RESTART gate can be maintained towards one end of the circuit and swapped into place using a matrix product unitary. As in theorem 3, since the model only allows the noise channel to act before or after the MPU, the SWAP gate can be done faultlessly  $\square$ .

In conclusion, we have considered the problem of simulating an open quantum system of  $n$  spins arranged on  $d$ -dimensional lattices and with local interactions. Our work provides rigorous evidence of existence of simulatability phase transitions in the continuous-time and discrete-time dynamics of locally interacting open quantum spin systems. We have also outlined how the nature of the noise channel can be instrumental in determining whether such a phase transition is possible — while for depolarizing channel, the final state of the quantum spin systems is very close to the maximally mixed state, for a number of other channels the existence of such phase transitions can be inferred simply by an application of the quantum computation threshold theorem.

Two specific technical problems left open in this work are extending the fault tolerant construction to 1D spin systems with local interactions and to extend theorem 2 to a larger class of noise channels. Another possible direction could be to extend theorems 2-4 to settings where the spins interact

purely dissipatively. Furthermore, we expect similar results to hold for cases where the interaction with the environment is non-Markovian — analyzing the impact of non-Markovianity on phase transitions in both worst-case and average-case problems could also be an interesting and experimentally relevant direction to pursue.

## ACKNOWLEDGMENTS

We thank Daniel Malz, Alavaro Alhambra and Georgios Styliaris for useful discussions. RT acknowledges Max Planck Harvard research center for quantum optics (MPHQ) post-doctoral fellowship. We acknowledge support from the Munich Quantum Valley, from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under the project number 414325145 in the framework of the Austrian Science Fund (FWF): SFB F7104 and within the D-ACH Lead-Agency Agreement through project No. 414325145 (BEYOND C).

- 
- [1] G. Vidal, Physical review letters **93**, 040502 (2004).
  - [2] G. Vidal, Physical review letters **91**, 147902 (2003).
  - [3] A. Müller-Hermes, J. I. Cirac, and M. C. Banuls, New Journal of Physics **14**, 075003 (2012).
  - [4] M. M. Rams and M. Zwolak, Physical review letters **124**, 137701 (2020).
  - [5] M. P. Zaletel, R. S. Mong, C. Karrasch, J. E. Moore, and F. Pollmann, Physical Review B **91**, 165112 (2015).
  - [6] J. Watrous, in *Proceedings of IEEE 36th Annual Foundations of Computer Science (IEEE, 1995)* pp. 528–537.
  - [7] K. G. Vollbrecht and J. I. Cirac, Physical Review A **73**, 012324 (2006).
  - [8] N. Schuch, M. M. Wolf, K. G. H. Vollbrecht, and J. I. Cirac, New Journal of Physics **10**, 033032 (2008).
  - [9] A. H. Werner, D. Jaschke, P. Silvi, M. Kliesch, T. Calarco, J. Eisert, and S. Montangero, Physical review letters **116**, 237201 (2016).
  - [10] J. Del Pino, F. A. Schröder, A. W. Chin, J. Feist, and F. J. Garcia-Vidal, Physical review letters **121**, 227401 (2018).
  - [11] D. Jaschke, S. Montangero, and L. D. Carr, Quantum Science and Technology **4**, 013001 (2018).
  - [12] I. Luchnikov, S. Vintskevich, H. Ouerdane, and S. Filippov, Physical review letters **122**, 160401 (2019).
  - [13] C. Mc Keever and M. Szymańska, Physical Review X **11**, 021035 (2021).
  - [14] H. Nakano, T. Shirai, and T. Mori, Physical Review E **103**, L040102 (2021).
  - [15] D. Aharonov, Physical Review A **62**, 062311 (2000).
  - [16] D. Aharonov and M. Ben-Or, in *Proceedings of 37th Conference on Foundations of Computer Science (IEEE, 1996)* pp. 46–55.
  - [17] D. Aharonov and M. Ben-Or, SIAM Journal on Computing (2008).
  - [18] D. Gottesman, Journal of Modern Optics **47**, 333 (2000).
  - [19] A. Nahum, S. Vijay, and J. Haah, Physical Review X **8**, 021014 (2018).
  - [20] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Physical Review X **7**, 031016 (2017).
  - [21] B. Skinner, J. Ruhman, and A. Nahum, Physical Review X **9**, 031009 (2019).
  - [22] Y. Bao, S. Choi, and E. Altman, Physical Review B **101**, 104301 (2020).
  - [23] M. Ippoliti, M. J. Gullans, S. Gopalakrishnan, D. A. Huse, and V. Khemani, Physical Review X **11**, 011030 (2021).
  - [24] A. Nahum, S. Roy, B. Skinner, and J. Ruhman, PRX Quantum **2**, 010352 (2021).
  - [25] S.-K. Jian, C. Liu, X. Chen, B. Swingle, and P. Zhang, Physical Review Letters **127**, 140601 (2021).
  - [26] X. Cao, A. Tilloy, and A. De Luca, arXiv preprint arXiv:1804.04638 (2018).
  - [27] R. Blatt and C. F. Roos, Nature Physics **8**, 277 (2012).
  - [28] P. Schindler, M. Müller, D. Nigg, J. T. Barreiro, E. A. Martinez, M. Hennrich, T. Monz, S. Diehl, P. Zoller, and R. Blatt, Nature Physics **9**, 361 (2013).
  - [29] K. Kim, M.-S. Chang, S. Korenblit, R. Islam, E. E. Edwards, J. K. Freericks, G.-D. Lin, L.-M. Duan, and C. Monroe, Nature **465**, 590 (2010).
  - [30] T. L. Nguyen, J.-M. Raimond, C. Sayrin, R. Cortinas, T. Cantat-Moltrecht, F. Assemat, I. Dotsenko, S. Gleyzes, S. Haroche, G. Roux, *et al.*, Physical Review X **8**, 011032 (2018).
  - [31] F. Schäfer, T. Fukuhara, S. Sugawa, Y. Takasu, and Y. Takahashi, Nature Reviews Physics **2**, 411 (2020).
  - [32] A. A. Houck, H. E. Türeci, and J. Koch, Nature Physics **8**, 292 (2012).
  - [33] J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck, Physical Review X **4**, 031043 (2014).
  - [34] M. Fitzpatrick, N. M. Sundaresan, A. C. Li, J. Koch, and A. A. Houck, Physical Review X **7**, 011016 (2017).
  - [35] L. J. Schulman and U. V. Vazirani, in *Proceedings of the thirty-first annual ACM symposium on Theory of computing* (1999) pp. 322–329.
  - [36] P. O. Boykin, T. Mor, V. Roychowdhury, F. Vatan, and R. Vrijen, Proceedings of the National Academy of Sciences **99**, 3388 (2002).
  - [37] M. Z. Bazant, Physical Review E **62**, 1660 (2000).
  - [38] D. Aharonov, M. Ben-Or, R. Impagliazzo, and N. Nisan, arXiv preprint quant-ph/9611028 (1996).
  - [39] F. Pastawski, A. Kay, N. Schuch, and I. Cirac, Physical review letters **103**, 080501 (2009).

- [40] J. I. Cirac, D. Perez-Garcia, N. Schuch, and F. Verstraete, *Journal of Statistical Mechanics: Theory and Experiment* **2017**, 083105 (2017).

# Supplementary to Simulatability phase transitions in the dynamics of open quantum spin systems

Rahul Trivedi<sup>1,2\*</sup> and J. Ignacio Cirac<sup>1,2</sup>

<sup>1</sup>Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, 85748 Garching, Germany.

<sup>2</sup>Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, D-80799 Munich, Germany.

(Dated: October 22, 2021)

## I. PROOF OF THEOREM 1

In this section, we outline a proof of theorem 1 of the main text. We begin the section by recalling a basic result from the theory of subcritical percolation [1, 2] which provides a bound on the likelihood of obtaining a very large connected cluster in a (site) percolating lattice. We also provide a simple extension of this result to a percolation problem where the state of each site is dependent on the states of (some finite number of) its neighbouring sites — this result is used for analyzing the run-time of the classical sampling algorithm used in the proof of theorem 1.

### A. Some results for subcritical percolation

**Definition 1 (Cluster)** Consider a percolation problem on a connected lattice  $L \subset \mathbb{Z}^d$  where each vertex is either open (set to 1) or closed (set to 0). A set of closed vertices  $S \subset L$  is called a cluster if  $\forall v_1, v_2 \in S$ , there exists a path (using only edges from  $\mathbb{Z}^d$ ) from  $v_1$  to  $v_2$ . We will denote by  $\mathcal{C}(L)$  as the set of all clusters on  $L$  i.e.  $\mathcal{C}(L) = \{S \subset L | S \text{ is a cluster}\}$ .

**Lemma 1 (Independent site-percolation from Ref. [2])** Consider a site-percolation problem on a connected lattice  $L \subset \mathbb{Z}^d$  with  $n$  sites where each vertex  $v$  is independently opened (set to 1) or closed (set to 0). Then,  $\exists p_c \in (0, 1)$ , referred to as the site percolation threshold, such that if  $\text{Prob}(X_v = 1) > p_c \forall v \in L$ , where  $X_v$  is the state of the vertex  $v \in L$  then

$$\text{Prob}\left(\max_{S \in \mathcal{C}(L)} |S| \geq s\right) \leq O(n^d s^d \exp(-s/s_p)),$$

for some  $s_p > 0$ .

Next, we consider a site percolation problem on  $\mathbb{Z}^d$  where each site is statistically dependent on its neighbours, and show that a similar percolation threshold can be derived for such a problem. The idea is to construct a channel from a percolation problem where all sites are independent to the percolation problem where each site is dependent on its neighbours in such a way that the maximum cluster size can only decrease.

**Lemma 2** Consider a site-percolation problem on a connected lattice  $L \subset \mathbb{Z}^d$  with  $n$  sites where each site is opened (set to 1) or closed (set to 0) with state of a vertex  $v \in L$  being conditionally dependent on state of the vertices in a deleted neighbourhood  $N_v \subset L$ , then if  $\inf_{v \in L} \inf_{x \in \{0,1\}^{|N_v|}} \text{Prob}(X_v = 1 | X_{N_v} = x) \geq p_c$ , where for  $v \in L$ ,  $X_v \in \{0, 1\}$  is the state of the vertex  $v$ , for  $S \subset L$ ,  $X_S \in \{0, 1\}^{|S|}$  is the state of the vertices contained in  $S$  and  $p_c$  is the site-percolation threshold defined in lemma 1, then

$$\text{Prob}\left(\max_{S \in \mathcal{C}(L)} |S| \geq s\right) \leq O(n^d s^d \exp(-s/s_p)),$$

for some  $s_p > 0$ .

**Proof:** We consider a site-percolation problem on  $L$  where each vertex  $v \in L$  is independently open or closed. For notational convenience,  $\forall v \in L$ , we define  $x_v \in \{0, 1\}^{|N_v|}$  via

$$x_v = \arg \min_{x \in \{0,1\}^{|N_v|}} \text{Prob}(X_v = 1 | X_{N_v} = x).$$

\* rahul.trivedi@mpq.mpg.de

Next,  $\forall v \in L$  and  $x \in \{0, 1\}^{|N_v|}$ , we define a Bernoulli random variable  $Z_{v,x}$  such that

$$\text{Prob}(Z_{v,x} = 1) = \begin{cases} \text{Prob}(X_v = 1 | X_{N_v} = x_v) & \text{if } x = x_v, \\ (\text{Prob}(X_v = 1 | X_{N_v} = x) - \text{Prob}(X_v = 1 | X_{N_v} = x_v)) / \text{Prob}(X_v = 0 | X_{N_v} = x_v) & \text{if } x \neq x_v. \end{cases}$$

The random variable  $X_v$  can then be generated from  $Z_{v,x}$  via

$$X_v = \begin{cases} 1 & \text{if } Z_{v,x_v} = 1, \text{ else} \\ Z_{v,x} & \text{where } x \in \{0, 1\}^{|N_v|} \text{ such that } X_{N_v} = x. \end{cases}$$

It is easy to verify that this definition of  $X_v$  gives the right conditional probability distributions. Furthermore, we can easily note that  $Z_{v,x_v} = 1 \implies X_v = 1$ . Therefore, the maximum cluster size in a lattice  $L$  where each vertex  $v$  has state  $Z_{v,x_v}$  is necessarily larger than the maximum cluster size in a lattice  $L$  where each vertex  $v$  has state  $X_v$ . Thus, from lemma 1, the bound on the likelihood of the maximum cluster size follows.  $\square$ .

## B. Sampling from the open-quantum spin system dynamics

The problem under consideration is precisely stated below,

**Problem:** We consider a noise rate  $\kappa$ , a single-qubit entanglement breaking channel  $\mathcal{N}(\cdot)$  and a family of Lindbladians  $(\mathcal{L}_n)_{n \in \mathbb{N}}$  over  $n$ -qubits on the  $d$ -dimensional lattice  $\mathbb{Z}^d$ . The local Lindbladian  $\mathcal{L}_n$  is expressible as a sum of  $M = O(\text{poly}(n))$  Lindbladians acting over qubits at members of the set  $\mathcal{S} = \{\Lambda_1, \Lambda_2 \dots \Lambda_M\}$ , where  $\Lambda_i \subset \mathbb{Z}^d \forall i \in [M]$  i.e.

$$\mathcal{L}_n = \sum_{\Lambda \in \mathcal{S}} \mathcal{L}_\Lambda,$$

where  $\forall \Lambda \in \mathcal{S}$ , (i)  $\mathcal{L}_\Lambda$  is identity when acting on qubits outside  $\Lambda$ , (ii) its diameter is bounded i.e.  $d(\Lambda) \leq O(1) \forall \Lambda \in \mathcal{S}$  and (iii) the Lindbladian has a bounded norm i.e.  $\|\mathcal{L}_\Lambda\|_{1 \rightarrow 1} \leq O(1)$ . We further assume that the set  $\mathcal{S}$  can be partitioned into  $L = \Theta(1)$  sets  $\mathcal{S}_1, \mathcal{S}_2 \dots \mathcal{S}_L$  such that  $\forall i \in [L]$  and  $\Lambda, \Lambda' \in \mathcal{S}$ ,  $\Lambda \cap \Lambda' = \emptyset$ . The master equation under consideration is then given by

$$\frac{d\rho(t)}{dt} = \mathcal{L}_n \rho(t) + \kappa \sum_{i=1}^n (\mathcal{N}_i - \text{Id}) \rho(t),$$

where  $\mathcal{N}_i$  is the entanglement breaking channel  $\mathcal{N}$  acting on the  $i^{\text{th}}$  qubit. The computational problem that we are interested in is to sample from the state  $\rho(t)$ , in the computational basis, for a time  $t = O(\text{poly}(n))$ .

We first trotterize the evolution and represent each channel in the trotterized evolution (be it the entanglement breaking channel due to trotterization of the noise, or the channels that are entangling neighbouring qubits) as a convex combination of the identity channel and a completely-positive trace preserving (CPTP) channel. This is made precise in the following lemma.

**Lemma 3** A parameter  $N = \Theta(\text{poly}(n))$  can be chosen such that  $\|\rho(t) - \hat{\sigma}_{t,N}\|_{\text{tr}} \leq O(1/\text{poly}(n))$ , where  $\hat{\sigma}_{t,N}$  is the trotterized state

$$\hat{\sigma}_{t,N} = \prod_{i=1}^L \left[ \prod_{\Lambda \in \mathcal{S}_i} \left( \left(1 - \frac{gt}{N}\right) \text{id} + \frac{gt}{N} \mathcal{E}_\Lambda \right) \prod_{i=1}^n \left( \left(1 - \frac{\kappa t}{NL}\right) \text{id} + \frac{\kappa t}{NL} \mathcal{N}_i \right) \right]^N \rho(0), \quad (1)$$

with  $g \geq \max_{\Lambda \in \mathcal{S}} \lambda_{\max}(\Phi_{\mathcal{L}_\Lambda})$  (where  $\lambda_{\max}(\Phi_{\mathcal{L}})$  is the maximum magnitude eigenvalue of the Choi state corresponding to  $\mathcal{L}_\Lambda$ ) and  $\mathcal{E}_\Lambda$  is a channel acting on the qubits contained in  $\Lambda$ . The channel  $(1 - gt/N)\text{id} + gt/N\mathcal{E}_\Lambda$  appearing in the trotterized channel, expressed as a convex combination of  $\text{id}$  and  $\mathcal{E}_\Lambda$ , will be referred to as a ‘horizontal’ channel.

**Proof:** We use the first order Trotter-Suzuki formula to approximate  $\rho(t)$  by  $\hat{\rho}_{t,N}$ , where for  $N \in \mathbb{N}$ ,

$$\hat{\rho}_{t,N} = \left[ \prod_{i=1}^L \left( \prod_{\Lambda \in \mathcal{S}_i} e^{\mathcal{L}_\Lambda t/N} \right) \left( \prod_{i=1}^n e^{\kappa(\mathcal{N}_i - \text{Id})t/NL} \right) \right]^N \rho(0). \quad (2)$$



A standard analysis of the trotterization error [3] allows us to bound the trace-norm error between  $\rho(t)$  and  $\hat{\rho}$ :

$$\|\rho(t) - \hat{\rho}_{t,N}\|_1 \leq O\left(\frac{t^2}{N} \left( \sum_{\Lambda} \|\mathcal{L}_{\Lambda}\|_{1 \rightarrow 1} + n\kappa \|\text{Id} - \mathcal{N}\|_{1 \rightarrow 1} \right)^2\right).$$

Since  $\kappa = \Theta(1)$ ,  $\|\mathcal{I} - \mathcal{N}\|_{1 \rightarrow 1} = O(1)$  and  $\forall \Lambda \in \mathcal{S} : \|\mathcal{L}_{\Lambda}\|_{1 \rightarrow 1} \leq O(1)$ , we obtain that  $\|\rho(t) - \hat{\rho}_{t,N}\|_1 \leq O(t^2 \text{poly}(n)/N)$  — a sufficiently large value of  $N$  which scales as  $\text{poly}(n)$  thus allows us to control the error incurred in this approximation.

Next, we consider each of the channels appearing in the trotterization (Eq. 2) and express them as the convex combination of the identity channel and a completely-positive trace-preserving (CPTP) map. For a Lindbladian  $\mathcal{L}$ , consider the channel  $e^{\mathcal{L}\tau}$  for some  $\tau > 0$ . A first order Taylor expansion of this map with respect to  $\tau$  yields the map  $\text{Id} + \tau\mathcal{L}$  — note that if  $\tau$  is sufficiently small (made precise below), this map is also CPTP. From Taylor's theorem it follows that

$$\|e^{\mathcal{L}\tau} - (\text{Id} + \mathcal{L}\tau)\|_{1 \rightarrow 1} \leq O(\|\mathcal{L}\|_{1 \rightarrow 1}^2 \tau^2).$$

Next, we note that for any  $g > 0$ ,  $\text{Id} + \mathcal{L}\tau = (1 - g\tau)\text{Id} + g\tau(\text{Id} + \mathcal{L}/g)$ . Choosing  $g > \lambda_{\max}(\Phi_{\mathcal{L}})$ , where  $\Phi_{\mathcal{L}}$  is the Choi-state corresponding to  $\mathcal{L}$ , it follows that  $\text{Id} + \mathcal{L}/g$  is completely positive. Furthermore, since  $\mathcal{L}$  is a Lindbladian, it follows that  $\text{Id} + \mathcal{L}/g$  is completely positive and trace preserving. For  $\tau < 1/g$ , we have thus approximated  $e^{\mathcal{L}\tau}$  by a convex combination of the identity channel (applied with probability  $1 - g\tau$ ) and a non-identity CPTP channel (applied with probability  $g\tau$ ). Now, using  $g > \sup_{\Lambda \in \mathcal{S}} \lambda_{\max}(\Phi_{\mathcal{L}_{\Lambda}})$ , we obtain that  $\forall \Lambda \in \mathcal{S}$ ,

$$e^{\mathcal{L}_{\Lambda}t/N} = \left(1 - \frac{gt}{N}\right)\text{Id} + \frac{gt}{N}\mathcal{E}_{\Lambda} + O\left(\frac{t^2}{N^2}\right)$$

for some channel  $\mathcal{E}_{\Lambda}$  that only acts on the qubits in  $\Lambda$ . Similarly,  $\forall i \in [n]$ :

$$e^{\kappa(\mathcal{N}_i - \text{Id})tN/L} = \left(1 - \frac{\kappa t}{NL}\right)\text{Id} + \frac{\kappa t}{NL}\mathcal{N}_i + O\left(\frac{t^2}{N^2}\right).$$

We can then construct a circuit whose output is  $\hat{\sigma}_{t,N}$ , where

$$\hat{\sigma}_{t,N} = \prod_{i=1}^L \left[ \prod_{\Lambda \in \mathcal{S}_i} \left( \left(1 - \frac{gt}{N}\right)\text{id} + \frac{gt}{N}\mathcal{E}_{\Lambda} \right) \prod_{i=1}^n \left( \left(1 - \frac{\kappa t}{NL}\right)\text{id} + \frac{\kappa t}{NL}\mathcal{N}_i \right) \right]^N \rho(0). \quad (3)$$

Furthermore, it easily follows from the abovementioned error estimates that  $\|\hat{\sigma}_{t,N} - \rho(t)\|_1 \leq O(t^2 \text{poly}(n)/N)$  — if  $t = O(\text{poly}(n))$ , then clearly there is a choice of  $N = \Theta(\text{poly}(n)/\varepsilon)$  that ensures that  $\|\hat{\sigma}_{t,N} - \rho(t)\|_1 \leq \varepsilon$ . Choosing  $\varepsilon = \Theta(1/\text{poly}(n))$  proves the lemma statement  $\square$ .

Next, we consider the problem of sampling from  $\hat{\sigma}_{t,N}$  — we first map it to a percolation problem on a  $(d+1)$ -dimensional lattice by choosing the identity channel, or the non-identity channel ( $\mathcal{E}_{\Lambda}$  or  $\mathcal{N}_i$ ). However, this sampling alone is not enough to ensure that the resulting problem percolates, since the probability of choosing the non-identity channel is very close to 0 — in order to map this to a problem that percolates, each site in the percolation problem needs to be mapped to an of block many time-steps per qubit. This is made precise in the definition below.

**Definition 2 (Equivalent percolation problem)** Choose  $\tau > 0$  and  $m = \lceil N\tau/t \rceil$  — a percolation problem on  $\mathbb{Z}^{d+1}$  corresponding to sampling from  $\hat{\sigma}_{t,N}$  in Eq. 3 is constructed by the first sampling from the convex combination of the channels ( $\mathcal{E}_{\Lambda}$ ,  $\mathcal{N}_i$  or the single- or two-qubit identity channels) in Eq. 3 — then a vertex  $v := (i, q) \in \mathbb{Z}^{d+1}$  (where  $i \in \mathbb{Z}^d$  and  $q \in \mathbb{N}$ ) is open if all the horizontal channels acting on the  $i^{\text{th}}$  qubit from time-step  $mq$  to  $m(q+1) - 1$  are sampled to be the identity channel, and the channel  $\mathcal{N}_i$  is applied at least once.

**Proof of theorem 1:** We next analyze the equivalent percolation problem and arrive at a proof of theorem 1 from the main text. Our goal is to show that for a sufficiently large  $\kappa$ ,  $\tau$  can be chosen to ensure that the equivalent percolation problem is subcritical which can then be efficiently sampled from. To do so we will use lemma 2 — we first provide a lower bound on the probability that a vertex on the lattice corresponding to the equivalent percolation problem is open, maximized over all possible configurations of the neighbouring vertices. For a vertex  $v = (i, q)$  in the equivalent percolation problem, we associate the neighbourhood  $N_v = \{(j, q), \text{ where } j \in \mathbb{Z}^d, s \in \mathbb{Z} \mid \exists \Lambda \in \mathcal{S} \text{ such that } i, j \in \Lambda\}$ . Note that the probability of a vertex  $v$ , conditioned on the state of the vertices in its neighbourhood  $N_v$ , being open is the smallest when the vertices in  $N_v$  are all closed — thus if this

probability can be made larger than the site-percolation threshold, the equivalent lattice will also percolate. Recall that

$$\text{Prob}(X_v = 1 | X_{N_v} = 0^{|N_v|}) = \frac{\text{Prob}(X_v = 1, X_{N_v} = 0^{|N_v|})}{\text{Prob}(X_{N_v} = 0^{|N_v|})}. \quad (4)$$

Consider the probability of the vertex  $v$  being open and all the vertices in its neighbourhood being closed. One configuration of samplings from the channels that achieve this is if all the horizontal channels acting on the qubit associated with  $v$  are open, the entanglement breaking channel is applied at least once on the qubit associated with  $v$  and the entanglement breaking channel is never applied with the qubits associated with the vertices in  $N_v$ . This provides us with the lower bound

$$\text{Prob}(X_v = 1, X_{N_v} = 0^{|N_v|}) \geq \left(1 - \frac{gt}{N}\right)^{ml_v} \left(1 - \left(1 - \frac{\kappa t}{LN}\right)^{Lm}\right) \left(1 - \frac{\kappa t}{LN}\right)^{Lm|N_v|},$$

where  $l_v$  is the the number of  $\Lambda \in \mathcal{S}$  such that  $i \in \Lambda$  (where  $v = (i, q)$ ). Consider now the probability of all the vertices in  $N_v$  being closed. This event can occur if either the entanglement breaking channel is not applied on any of the qubits associated with vertices in  $N_v$ , or at least one of the horizontal channels associated with the qubits in  $N_v$  for the time-steps in  $v$  is not identity. Thus, we obtain the upper bound

$$\text{Prob}(X_{N_v} = 0^{|N_v|}) \leq \left(1 - \frac{\kappa t}{LN}\right)^{Lm|N_v|} + \left(1 - \left(1 - \frac{gt}{N}\right)^{mr_{N_v}}\right),$$

where  $r_{N_v}$  is the number of  $\Lambda \in \mathcal{S}$  such that  $j \in \Lambda$  for some  $(j, v) \in N_v$ . Thus, we obtain a lower bound on the worst-case conditional probability defined in Eq. 4. Using  $m = \lceil N\tau/t \rceil$ , we obtain

$$\text{Prob}(X_v = 1 | X_{N_v} = 0^{|N_v|}) \geq e^{-g\tau l_v} (1 - e^{-\kappa\tau}) \left( \frac{e^{-\kappa|N_v|\tau}}{e^{-\kappa|N_v|\tau} + 1 - e^{-gr_{N_v}\tau}} \right) + O\left(\frac{1}{\text{poly}(N)}\right).$$

We can now show that for  $\kappa$  larger than a constant, the lower bound obtained above can be made larger than the site-percolation threshold  $p_c$ . Noting that since we only consider problems with local interactions (i.e. the diameter of the set of qubits interaction with each other is finite),  $l = \max_v l_v = O(1)$  and  $r = \max_v r_{N_v} = O(1)$ . Fixing  $\tau$  by  $e^{-\kappa\tau} = c < 1$  for some constant  $c$ , we obtain that

$$\forall v : \text{Prob}(X_v | X_{N_v} = 0^{|N_v|}) \geq 1 - c + O\left(\frac{g}{\kappa}\right) + O\left(\frac{1}{\text{poly}(N)}\right).$$

Recall from lemma 3 that  $N = \Omega(\text{poly}(n))$  — it thus follows that for large  $n$ , we can choose  $\kappa/g$  to be large enough for the lower bound to be very close to  $1 - c$  and thus an appropriate choice of  $c$  (which is equivalent to an appropriate choice of  $\kappa\tau$ ) will result in this probability being larger than the site-percolation threshold. It then follows from lemma 2 that with a probability  $1 - O(\text{polylog}(n)/\text{poly}(n))$ , the maximum size of the percolating clusters will be  $O(\log(n))$ . Using this fact, we can now provide an algorithm to sample approximately from the trotterized state  $\hat{\sigma}_{t,n}$ . We recall that an entanglement breaking single-qubit channel can be always be expressed as

$$\mathcal{N}(\rho) = \sum_i \sigma_i \text{Tr}(E_i \rho), \quad (5)$$

where  $\sigma_i \in \mathfrak{D}_1(\mathbb{C}^2)$ , and  $\{E_i\}$  form a POVM. Thus, an application of an entanglement breaking channel on a single-qubit state can be simulated by first measuring the qubit with respect to the POVM  $E_i$  and then replacing the qubit with  $\sigma_i$  if the outcome of the measurement is  $i$ . The sampling algorithm then proceeds in two steps:

1. First, we sample from the percolation problem and ignore any samples that have a cluster of size larger than  $c' \log(n)$  where  $c'$  is a sufficiently large constant. The error in total variation distance incurred due to this is  $O(\text{polylog}(n)/\text{poly}(n))$ .
2. We then replace the entanglement breaking channel by applying a POVM and replacing the qubit with an unentangled state as per Eq. 5. We note that this can be done efficiently and per each time-step. More specifically, suppose that upto the (discrete) time-step  $k$  all the entanglement breaking channels have been sampled in this way, and we wish to sample the entanglement breaking channels at time-step  $k + 1$  — we then need to sample from the measurement of the POVM corresponding to the qubits where the entanglement breaking channels are applied and trace over the remaining qubits. The sampling from the POVMs can be done sequentially — we pick any one qubit and measure it in the POVM corresponding to the entanglement breaking channel, and trace over all the other qubits (including the ones on which the other entanglement breaking channel is applied). We note that we can efficiently contract the circuit to calculate the probabilities of various

outcomes corresponding to the POVM since all the clusters of qubits can be contracted in  $\text{poly}(n)$  time. After having computed these probabilities, we produce a sample from the POVM and conditioned on this sample we sample the next qubit which has an entanglement breaking channel. Since there are at-most  $n$  such qubits, we need to do at most  $\text{poly}(n)$  such contractions.

The above two-steps outline a sampling algorithm that can be executed in  $\text{poly}(n)$  time. Furthermore, there are two sources of error — one is due to the trotterization, and the other is due to ignoring samples which have large clusters. Both of these errors are smaller than  $O(\text{polylog}(n)/\text{poly}(n))$  in the total-variation distance and this completes the proof of theorem 1.  $\square$ .

## II. IMPLICATION OF THRESHOLD THEOREM ON THE LOW NOISE REGIME

The threshold theorem in quantum computation [4] is a well known result which states that scalable quantum computation is possible if the noise rate in the quantum computer is low enough. These theorems are usually proved by performing encoded quantum computation together with error correction, and the reduction in the rate of noise needed to be able to implement scalable quantum computation depends on the error correcting code. While initially constructed and proved for general circuit model of computation, the threshold theorem has been proved for quantum circuits while constraining the interactions to be nearest neighbours on qubits arranged in 1D, 2D or 3D lattices [4, 5].

Here, we investigate various implication of fault tolerant constructions with nearest neighbour interaction on the existence of phase transitions in the simulation of open quantum spin systems. We begin by recalling a basic lemma proved by Aharonov et al, and Gottesman et al, which indicates that a fault tolerant quantum computation is possible with 1D nearest neighbour qubits provided that qubits can be restarted at any point during the time evolution. This lemma is stated within the circuit model of computation with independent errors (although several extensions are possible) at different locations within the circuit with an error probability  $\eta$ .

**Lemma 4 (Fault tolerance in 1D with nearest neighbour gates from Ref. [4])** *Let  $\varepsilon > 0$  and let  $\mathcal{G}$  a universal set of gates. Let  $Q$  be a quantum circuit on  $n$  qubits and depth  $\text{poly}(n)$  formed from gates from  $\mathcal{G}$  acting on either one or two neighbouring qubits. Then,  $\exists \eta_{\text{th}}$  such that for  $\eta \leq \eta_{\text{th}}$ , there exists a circuit  $C'$  formed from gates of  $\mathcal{G} \cup \{\text{SWAP}, \text{RESTART}\}$  acting over  $O(n \text{ polylog}(n/\varepsilon))$  qubits for depth  $O(\text{poly}(n)\text{polylog}(n/\varepsilon))$  such that in the presence of noise  $\eta \leq \eta_{\text{th}}$ , it computes a quantum state which is  $\varepsilon$ -close in total variation distance to the quantum state of the circuit  $C$ .*

We point out that in the fault-tolerant construction used in this lemma, both the RESTART and SWAP operations can be subjected to errors, and the concatenated error correction schemes subsequently applied on them will succeed if these errors are small enough. Building on this lemma and by providing implementations of the RESTART and SWAP gates, below we show that in many contexts, there we are able to encode a fault-tolerant quantum computation into the problem and hence we expect in the presence of low biased noise for the problem to be hard to simulate classically. We analyze three models in the following two subsections:

1. For  $d \geq 2$ , we consider a discrete-time model where the entanglement breaking channels applied with a probability  $p$  are interleaved with single and two-qubit unitaries applied locally on disjoint sets of qubits. Here, we show that as long as the entanglement breaking channel has a fixed point  $\neq I/2$ , we expect the model to be classically insimulatable at sufficiently low rate of noise.
2. For  $d = 1$ , we consider a discrete-time model where we apply a Matrix product unitary followed by the entanglement breaking channel applied with a probability  $p$  on all qubits. Here, we again show that if the entanglement breaking channel has a fixed point  $\neq I/2$ , we expect the model to be classically insimulatable at sufficiently low rate of noise.
3. For  $d \geq 2$ , we consider a continuous-time model where the entanglement breaking channels can act at any time, and thus might result in erroneous gate operations. Here, we show that if the image of space of single qubit density matrices under the entanglement breaking channel does not contain  $\neq I/2$ , then we expect this model to be classically insimulatable at sufficiently low rate of noise.

Another key ingredient that we will use in our analysis is algorithmic cooling [6, 7] i.e. having a supply of  $m$  qubits individually being in a mixed state  $\sigma_1 \otimes \sigma_2 \cdots \otimes \sigma_m$  where  $\sigma_i \neq I/2 \forall i \in [m]$ , we would like to be able to extract at least a single qubit in a pure state. This is a well known problem, and of technological relevance in NMR quantum computing where there is often a large supply of noisy qubits and it is of interest to concentrate the entropy on a few qubits. For our purposes, this will be a key ingredient in implementing the RESTART operations for noise-channels which do not necessarily drive the qubit state to identity (i.e. they are not depolarizing) and thus allow us to construct fault-tolerant circuits below a particular noise rate and hence point to a simulatability phase transition.

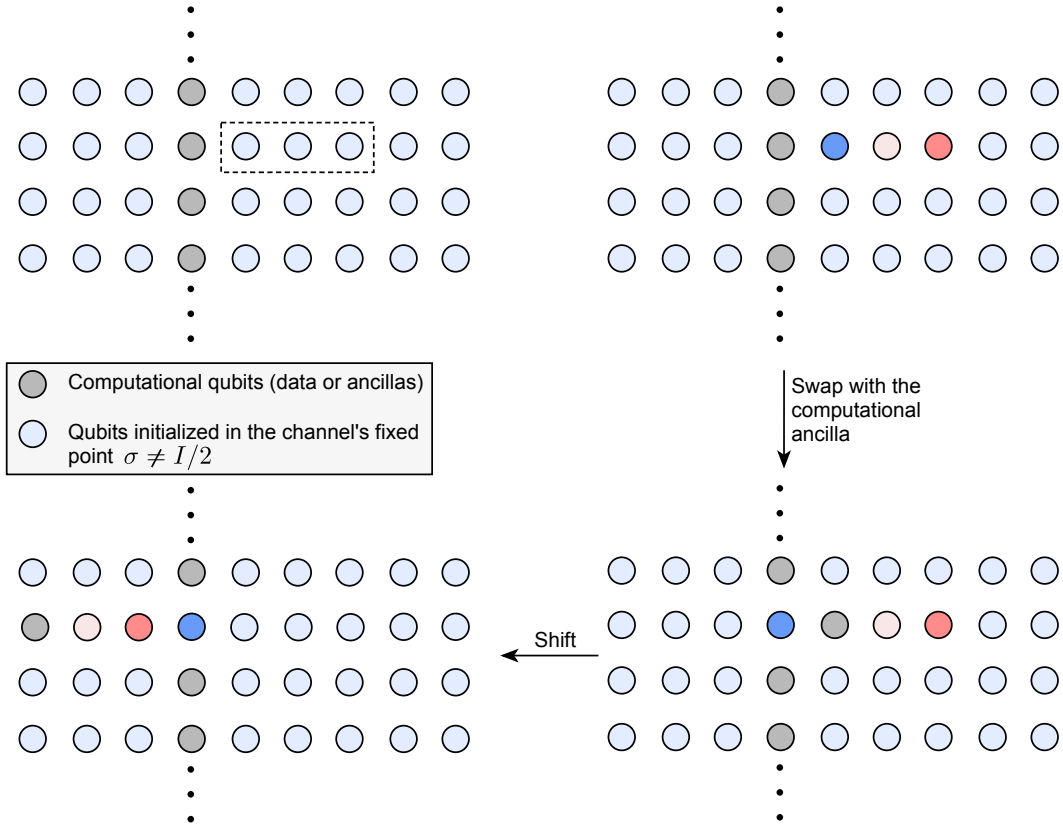


FIG. 1. A schematic depiction of the implementation of the RESTART operation on a 2D lattice of qubits. We note that the a shift operation can be implemented on the alternate qubits in any row (i.e. the qubits at the odd or even positions) by doing two consecutive swap operations — one layer of operations swapping the even with the odd qubits and then another layer of operations swapping the odd qubits with the even qubits. Taken together, these two operations will shift the odd qubits to one direction and the even qubits in the opposite direction. For simplicity, in this figure, we depict this shift operation as a single operation and hide one set of alternate qubits.

**Definition 3 (Polarization of a qubit state)** A qubit (mixed) state  $\sigma$  is said to have a polarization of  $\varepsilon$  if  $\lambda_{\max} - \lambda_{\min} = \varepsilon$ , where  $\lambda_{\max}$  and  $\lambda_{\min}$  are the maximum and minimum eigenvalues of  $\sigma$  respectively.

**Lemma 5 (Algorithmic cooling from Ref. [6])** Consider  $m$  qubits initially in the state  $\rho_{\text{init}} = \bigotimes_{i=1}^m \sigma_i$ , where for  $i \in [m]$ ,  $\sigma_i$  has polarization  $\varepsilon_i (> 0)$  and let  $\varepsilon_{\min} = \min_{i \in [m]} \varepsilon_i > 0$ , then for any  $\eta \in (0, 1)$ ,  $m$  can be chosen as a function of  $\eta, \varepsilon_{\min}$  such that  $\exists$  a unitary circuit  $U$  of depth dependent on  $\eta, \varepsilon_{\min}$  which yields (at least) one qubit whose reduced state has polarization  $1 - \eta$ .

**Proof of theorem 3:** We restrict ourselves to  $d = 2$  (i.e. a 2D lattice of spins), since if it is BQP hard to simulate the 2D problem in the presence of a constant, but small enough, rate of noise, it will be BQP hard to simulate higher dimensional problems as well. We will implement a 1D fault-tolerant quantum computation (lemma 4) in one of the columns of the 2D lattice, and call the qubits involved in this construction as the *computational qubits* — some of the computational qubits will be the data qubits (which encode the quantum computation) and some of these qubits will be ancilla qubits needed for the error correction operations. In order for the fault-tolerant construction to succeed, we need to implement a RESTART gate, with fidelity below the fault tolerance threshold, on the ancilla qubits. In order to restart a computational ancilla qubit, we use the non-computational qubits in its row. A schematic depiction of the restart operation is shown in Fig. 1 — we start off with all the qubits to the right of the qubit to be restarted in a fixed point  $\sigma \neq I/2$  of the noise channel. We then perform a sequence of three steps:

1. First, we perform algorithmic cooling on certain number of, say  $m$ , qubits on the right of the qubit to be restarted. At the end of this step, the qubits neighbouring the computational ancilla qubit will be in a state  $|0\rangle$ . We note that the algorithmic cooling operation can fail with some non-zero probability — however, this probability does not scale with the number of qubits involved and hence this construction still yields a threshold theorem. We analyze this point in more detail below.
2. Next, we swap the cooled qubit with the computational qubit. This effectively implements the RESTART operation.

3. Finally, we prepare the qubits for another RESTART operation - we note that the the previous two steps left the qubits immediately on the right of the computational qubit in a state that is no longer  $\sigma$  and consequently cannot be used to implement another RESTART operation. We therefore now perform a shift operation to shift these qubits to the left of the computational qubits and replace the  $m$  qubits on the right of the computational qubits with the state  $\sigma^{\otimes m}$ . This configuration can now be used to implement the next RESTART operation by using steps 1-3.

*Error analysis:* We need to ensure that a threshold theorem exists with the above outlined RESTART operation. Recall that at every time-step, we are allowed to apply either a single-qubit or a nearest neighbour two-qubit gate over disjoint set of qubits. Recall from lemma 5 that to achieve a cooled qubit in the state  $|0\rangle$  with specified probability  $p_{\text{cool}}$ , the number of qubits  $m$  required is a function of  $p_{\text{cool}}$  (which we denote by  $m(p_{\text{cool}})$ ), and the number of time-steps  $t$  required to accomplish this is also a function of  $p_{\text{cool}}$  (which we denote by  $t_{\text{cool}}(p_{\text{cool}})$ ). Furthermore, it follows from lemma 5 that to achieve  $p_{\text{cooling}} = 1 - O(1)$ , both  $m(p_{\text{cool}}) = O(1)$  and  $t_{\text{cool}}(p_{\text{cool}}) = O(1)$ . Consequently, both the following swap and shift operations (which needs to swap  $O(m(p_{\text{cool}}))$  qubits) can be done in time-steps  $t_{\text{swaps}}(p_{\text{cool}}) = O(1)$ . Importantly, we note that since in the model being considered in this theorem, we can perform the SWAP gate without an error *in the gate*, and since  $\sigma$  is a fixed-point of the noise channel, swapping a qubit in the state  $\sigma$  with another qubit will still generate an output qubit in the state  $\sigma$  (this is not necessarily true if an error occurs in the gate) and consequently the shift operation can always be performed to arrange the qubits properly for the next RESTART operation. Finally, since the number of time-steps and the number of qubits participating in the RESTART operation  $O(1)$ , we note that the RESTART operation can be made more efficient than required by the error correction threshold by making the probability of applying the entanglement breaking noise  $p$  small enough. This shows that below a particular noise threshold, we will be able to implement a BQP hard problem fault-tolerantly within the discrete-time model considered, which proves the theorem.  $\square$ .

**Proof of theorem 4:** This key difficulty with implementing a RESTART operation in the 1D nearest-neighbour setting is that the qubits that are cooled and swapped into the computational ancillas cannot be placed close to the computational ancillas, and  $\Theta(n)$  (where  $n$  is the number of computational qubits) swap operations are required in order to bring them close to the computational ancillas. Applying  $\Theta(n)$  swap gates without any error correction breaks the threshold theorem for quantum computation and hence does not allow us to fault tolerantly encode a BQP hard problem into the noisy quantum dynamics. However, if we allow ourselves to use constant bond dimension MPUs, then we can perform  $\Theta(n)$  swap operations with an MPU of bond dimension 2 in one time-step, and thus the construction used in the proof of theorem 2 works. In this case, we will just maintain the qubits used for restarting the computational ancilla on one end of the 1D lattice, and cool and swap them with the computational ancilla qubits in a single time-step.  $\square$ .

We now proceed to the proof of theorem 4 where we consider a continuous model of noise being applied on the spins. Here, the swap operations needed to shift the qubits to be cooled near the computational qubits cannot be done faultlessly. However, if we consider noise modelled with point channels with a fixed point different from identity, we show that a 1D fault tolerant quantum computation can still be implemented in  $d \geq 2$ .

**Proof of theorem 2:** For the two (or higher) dimensional models where the noise is applied continuously in time, one of the key difficulties in the construction of theorem 2 is that the SWAP operations can be faulty. Thus, even at a very low rate of noise, it is not guaranteed that the qubits that are being used to restart the computational ancillas at times  $\Theta(\text{poly}(n))$  are in a state that is sufficiently different from the identity. In particular, if we consider channels which have identity as a fixed point (in addition to fixed points which are not identity) then an error in the SWAP gate could result in one of the qubits being set into the identity and thus not be useful for the RESTART operation.

However, under the assumption  $\exists |\phi_+\rangle, |\phi_-\rangle \in \mathbb{C}^2$  such that the entanglement breaking channel  $\mathcal{N}$  maps every qubit state to a state of the form  $(1 - \varepsilon)/2 |\phi_-\rangle \langle \phi_-| + (1 + \varepsilon)/2 |\phi_+\rangle \langle \phi_+|$ , we can construct a shift operation that, despite being erroneous,

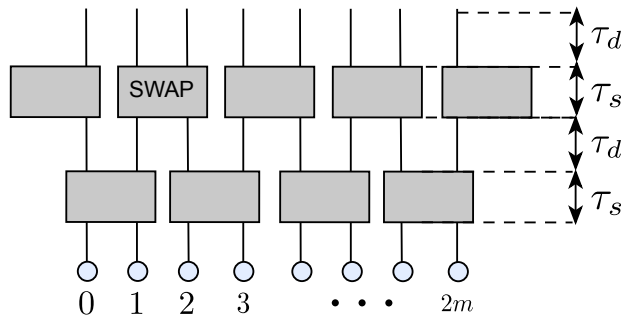


FIG. 2. Schematic of the shift operation used in the proof of theorem 4.

ensures with a sufficiently high success probability that the shifted qubits are unentangled and different from identity and hence can be used for the subsequent cooling algorithm. This shift operation, implemented with two layers of SWAP gates, is shown in Fig. 2 — we apply the Hamiltonian for the SWAP gate for a time period  $\tau_s$  (chosen such that in a noiseless setting, the SWAP gate would be perfectly executed) and then allow the qubits to evolve just under the influence of the entanglement breaking noise channel for a time period  $\tau_d$ . The latter evolution not only disentangles the output qubit due to the entanglement breaking nature of the channel, but also drives it away from  $I/2$ . We make this concrete by analyzing the operation of this shift operation on  $2m + 1$  qubits, out of which with probability  $1 - q$ , the qubits  $2, 4, 6 \dots 2m$  are in a separable state with the states of the individual qubits belonging to  $\mathcal{N}(\mathfrak{D}_1(\mathbb{C}^2))$  i.e.

$$\rho(0) = (1 - q)\rho_{0,1,3,5\dots 2m-1} \otimes \sigma_{2,4,6\dots 2m} + q\rho'.$$

where  $\sigma_{2,4,6\dots 2m}$  can be expressed as

$$\sigma_{2,4,6\dots 2m} = \sum_{\alpha} p_{\alpha} \sigma_2^{\alpha} \otimes \sigma_4^{\alpha} \cdots \otimes \sigma_{2m}^{\alpha}, \quad (6)$$

with  $\sum_{\alpha} p_{\alpha} = 1, p_{\alpha} \geq 0$  and  $\sigma_i^{\alpha} \in \mathcal{N}(\mathfrak{D}_1(\mathbb{C}^2)) \forall \alpha, i \in \{2, 4, 6 \dots 2m\}$ . Note that with probability  $\exp(-2m\kappa\tau_s)$ , the SWAP operation occurs perfectly and hence the output state can be expressed as

$$\rho(\tau_s) = \exp(-2m\kappa\tau_s)(1 - q)\sigma_{1,3,5\dots 2m-1} \otimes \rho'_{0,2,4\dots 2m} + (1 - \exp(-2\kappa\tau_s)(1 - q))\rho'',$$

where  $\sigma_{1,3,5\dots 2m-1}$  is the state  $\sigma_{2,4,6\dots 2m}$  but now shifted to the qubits  $1, 3, 5 \dots 2m - 1$  and  $\rho'_{0,2,4\dots 2m}$  ( $\rho''$ ) is some state over the qubits  $0, 2, 4 \dots 2m$  (all the qubits). Next, we allow the qubits to evolve under the entanglement breaking noise for a time  $\tau_d$  — we note that this is equivalent to applying the channel  $\mathcal{E}_{\tau_d} = e^{\kappa\tau_d(\mathcal{N} - \text{Id})}$  on each qubit. By a Taylor expansion of this exponential, it can easily be seen that

$$\mathcal{E}_{\tau_d} = e^{-\kappa\tau_d}\text{Id} + (1 - e^{-\kappa\tau_d})\mathcal{N}',$$

where  $\mathcal{N}' : \mathfrak{L}(\mathbb{C}^2) \rightarrow \mathfrak{L}(\mathbb{C}^2)$  is another entanglement breaking channel with the property that  $\mathcal{N}'(\mathfrak{D}_1(\mathbb{C}^2)) = \mathcal{N}(\mathfrak{D}_1(\mathbb{C}^2))$ . Therefore, with probability  $1 - e^{-\kappa\tau_d}$ , this channel disentangled the qubit and maps it to a state in  $\mathcal{N}(\mathfrak{D}_1(\mathbb{C}^2))$  and consequently,

$$\rho(\tau_s + \tau_d) = \mathcal{E}_{\tau_d}^{\otimes(2m+1)}\rho(\tau_s) = (1 - q')\sigma_{0,2,4\dots 2m-2} \otimes \rho''_{1,3,5\dots 2m-1,2m} + q'\rho''',$$

where  $1 - q' = \exp(-2m\kappa\tau_s)(1 - q) + (1 - \exp(-\kappa\tau_d))^m(1 - \exp(-2\kappa\tau_s)(1 - q))$ ,  $\sigma_{0,2,4\dots 2m-2}$  is a separable state of the form of Eq. 6 but over the qubits  $0, 2, 4 \dots 2m - 2$ . It immediately follows that  $q' \leq q$  if

$$\tau_d \geq -\frac{1}{\kappa} \log \left( 1 - \left( \frac{(1 - q)(1 - e^{-2m\kappa\tau_s})}{1 - (1 - q)e^{-2m\kappa\tau_s}} \right)^{1/m} \right) \quad (7)$$

A similar condition holds for the next layer of SWAP gates — if  $\tau_d$  is chosen as per Eq. 7, then the final state of the  $m$  qubits being shifted will be maintained in a separable state where the individual qubits are in  $\mathcal{N}(\mathfrak{D}_1(\mathbb{C}^2))$  with a probability at least  $1 - q$  — hence this property will also be maintained by the subsequent shift operation and we will have  $m$  qubits available for the cooling procedure. We note that even though the qubits now being purified are in a separable state as opposed to a product state, since by assumption the eigenstates of the individual qubit states are the same, the cooling unitary applied on the product state will also cool the separable state without any impact on the fidelity of the cooling algorithm. However, the cooling algorithm's fidelity will be lowered by a factor  $\geq 1 - q$ . Consequently, in order to implement the RESTART operation with fidelity large enough to ensure that the threshold theorem can be satisfied, we must be able to choose a sufficiently small  $q$ . Since the shift operation outlined above relies on the entanglement breaking channel to purify the state of the qubits, it is not clear if this condition can be satisfied (for small rates  $\kappa$ ) while at the same time ensuring that the noise in the computational qubits is smaller than the fault-tolerance threshold. However, this is indeed possible — to see this, we note that for small  $\kappa$ , we can use  $\tau_d$

$$\tau_d \approx \left( \frac{2m(1 - q)}{q} \right)^{1/m} \frac{(\kappa\tau_s)^{1/m}}{\kappa}.$$

However, note that for the error correction procedures to fault tolerantly encode a computation on the computational qubit, it is necessary to ensure that  $\kappa\tau_d \leq c_{\text{th}}$ , where  $c_{\text{th}}$  is a constant dependent on the spread of the error correcting code. It follows easily that for a given  $q$  and  $m$  (which are determined by the target fidelity of the cooling operation) if  $\kappa < qc_{\text{th}}^m/2m\tau_s(1 - q)$ , then for any  $\tau_d \in ((2m(1 - q)/q)^{1/m}\kappa^{1/m-1}\tau_s^{1/m}, c_{\text{th}}/\kappa)$ , the shift operation will successfully bring the auxiliary qubits near the computational qubits for the RESTART operation while at the same time ensuring that a quantum computation can be fault

tolerantly performed on the computational qubits.

---

- [1] G. Grimmett, in *Lectures on probability theory and statistics* (Springer, 1997) pp. 153–300.
- [2] M. Z. Bazant, *Physical Review E* **62**, 1660 (2000).
- [3] A. M. Childs, Y. Su, M. C. Tran, N. Wiebe, and S. Zhu, *Physical Review X* **11**, 011020 (2021).
- [4] D. Aharonov and M. Ben-Or, *SIAM Journal on Computing* (2008).
- [5] D. Gottesman, *Journal of Modern Optics* **47**, 333 (2000).
- [6] L. J. Schulman and U. V. Vazirani, in *Proceedings of the thirty-first annual ACM symposium on Theory of computing* (1999) pp. 322–329.
- [7] P. O. Boykin, T. Mor, V. Roychowdhury, F. Vatan, and R. Vrijen, *Proceedings of the National Academy of Sciences* **99**, 3388 (2002).