Concentration of quantum equilibration and an estimate of the recurrence time

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(Dated: June 16, 2022)

We show that the dynamics of generic quantum systems concentrate around their equilibrium value when measuring at arbitrary times. This means that the probability of finding them away from equilibrium is exponentially suppressed, with a decay rate given by the effective dimension. Our result allows us to place a lower bound on the recurrence time of quantum systems, since recurrences corresponds to the rare events of finding a state away from equilibrium. In many-body systems, this bound is doubly exponential in system size. We also show corresponding results for free fermions, which display a weaker concentration and earlier recurrences.

I. INTRODUCTION

Closed quantum systems obey the Schrödinger equation, so that their dynamics are both unitary and reversible. Most large systems seem to quickly evolve towards a steady state for long times, with only very small out-of-equilibrium fluctuations around it. This process is usually called *equilibration*, and is associated with the emergence of statistical physics [1, 2]. The equilibrated or average expectation value of an observable A is

$$\langle A(\infty)\rangle = \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\langle A(t)\rangle, \tag{1}$$

where $\langle A(t)\rangle = \langle\Psi| e^{-iHt}Ae^{iHt}|\Psi\rangle$ for some initial state Ψ and Hamiltonian H.

If a system equilibrates, it is because the probability of finding $\langle A(t)\rangle$ very close to $\langle A(\infty)\rangle$ at any given time is overwhelmingly large. We show that this is indeed the case: the dynamics of quantum systems with a *generic* spectrum concentrate highly around the steady-state value $\langle A(\infty)\rangle$.

More specifically, we show that when sampling times at random $t \in [0,\infty)$ the probability of finding the system away from equilibrium is exponentially suppressed. The decay rate of that exponential is given by the *effective dimension* or *inverse participation ratio*. This is defined as $\mathrm{Tr}\left[\omega^2\right]^{-1}$, where ω is the diagonal ensemble

$$\omega = \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T} e^{-iHt}|\Psi\rangle\langle\Psi| e^{iHt}, \tag{2}$$

which is such that $\mathrm{Tr}\left[A\omega\right] = \langle A(\infty)\rangle$.

That this equilibration happens, leaving little or no memory from the initial conditions, seems to conflict with the unitarity and reversibility of the dynamics. This conflict can be seen by considering the Poincaré recurrence theorem in quantum mechanics [3–7], which states that any closed quantum evolution eventually returns arbitrarily close to its initial state.

The solution to this problem is that even if the initial state is eventually recovered to an arbitrarily good approximation, this only happens at extremely long times. These recurrences constitute large out-of-equilibrium fluctuations, that can be understood as the rare events of finding a system far from its equilibrated state.

Based on this idea, we show how a lower bound on the average spacing between recurrences follows from our concentration results, as the inverse of the tail bound. We find that recurrences occur at time intervals that are at least exponential in the effective dimension. This gives, for the first time, a mathematically rigorous scaling on the average recurrence time, that matches the scaling of previous estimates [8, 9]. See [10, 11] for other results.

We also show equivalent results for free fermion Hamiltonians with generic single-particle modes. We find that under the assumption of extensivity in the single particle eigenstates a similar concentration bound and recurrence time result hold, but with a slower exponential scaling on the lattice size. This shows a markedly different behaviour with respect to generic models. See Table I for a summary.

Our results constitute a qualitative improvement over previous bounds on out-of-equilibrium fluctuations [12–14] for systems with a generic spectrum. These only focused on the variance induced by the probability measure $\lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}$, while we are able to analyze arbitrarily high moments thereof. The improvement is exponential in the same sense in which the Chernoff-Hoeffding bound is exponentially better than Chebyshev's inequality.

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	$\langle A(t)\rangle$	$\lvert \langle\Psi\rvert\, e^{-itH}\,\lvert\Psi\rangle\,\rvert^2$
Generic	$e^{\Omega\left(\mathrm{Tr}[\omega^2]^{-1/2}\right)}$	$e^{\Omega\left(\mathrm{Tr}[\omega^2]^{-1}\right)}$
Free	$e^{\Omega(\sqrt{L})}$	$e^{\Omega(L)}$

TABLE I. Lower bounds on the recurrence time for different dynamical quantities. Tr $\left[\omega^2\right]^{-1}$ is the effective dimension of a generic system and L is the number of sites in a fermionic lattice. In the free case, the observable and initial state are restricted to specific forms. See Sec. IV for the precise statements.

II. THE CONCENTRATION BOUND

We consider functions of time $f(t)$ that track some physical property of interest. In the cases here, $f(t) = \langle A(t)\rangle$ is the expectation value of a time-evolved operator $A(t)$. This allows us to define the moments of a probability distribution

$$f(\infty) \equiv \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T} f(t), \tag{3}$$

$$\kappa_q \equiv \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\left(f(t)-f(\infty)\right)^q. \tag{4}$$

These moments are bounded for arbitrary q as $\kappa_q \leq (2||A||)^q$. This means that they uniquely determine a characteristic function with an infinite radius of convergence

$$\phi(\lambda) = \sum_q \frac{k_q\lambda}{q!}. \tag{5}$$

This function defines a probability distribution, which we can write formally as

$$P(x) = \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\delta(x - f(t)). \tag{6}$$

Here $P(x)$ should be understood as the probability that, if we pick a random time $t \in [0,\infty)$, the value of $f(t)$ is exactly x (see also [15] and [16] for an overview of previous results). Note that in order to compute $\int x^q P(x)dx$ the limit in T is swapped with the integral in x. For justification of this see Appendix A. An example of $P(x)$ forming for $f(t) = \langle A(t)\rangle$ is given in Fig. 1.

Below we prove that the moments κ_q are bounded by

$$\kappa_q \leq (qg)^q\,, \tag{7}$$

where g is some small quantity, decreasing quickly as the size of the system grows and such that $g \to 0$ in the thermodynamic limit. A bound of this form implies that the distribution concentrates highly around the average, as per the following elementary lemma.

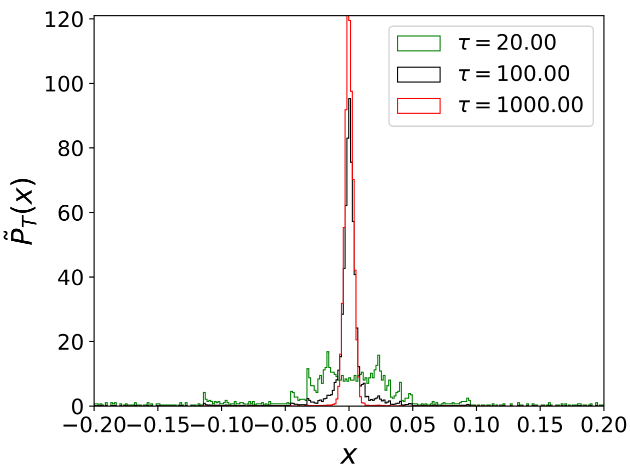

FIG. 1. Convergence of $P_T(x)$ to $P(x)$ where $P_T(x) = \int_0^T \frac{\mathrm{d}t}{T}\delta(x-\langle A(t)\rangle)$ and $P(x)$ is recovered as $T\to\infty$. $\tilde{P}_T(x)$ represents the approximation of $P_T(x)$ by binning samples and constructing a histogram. 1000 bins were used to create this histogram. Numerics were performed on a spin 1/2 chain with 22 sites. The data is normalized such that $\int_{-\infty}^{\infty}\tilde{P}_T(x)dx = 1$. The Hamiltonian is a Heisenberg type model with nearest and next nearest neighbour interactions. Further details can be found in Appendix D.

Lemma 1. *Let* $|\kappa_q| \leq (qg)^q$ *for* q *even. Then,*

$$\Pr\left[\left|f(t)-f(\infty)\right| \geq \delta\right] \leq 2e \times \exp\left(-\frac{\delta}{eg}\right). \tag{8}$$

Proof. Let us set $f(\infty) = 0$ for simplicity, and focus on the case $x \geq \delta$. We have that

$$\Pr\left[\langle A(t)\rangle \geq \delta\right] = \int_{x\geq\delta} P(x)\mathrm{d}x \tag{9}$$

$$\leq \frac{1}{\delta^q}\int_{x\geq\delta} x^q P(x)\mathrm{d}x \tag{10}$$

$$\leq \frac{\kappa_q}{\delta^q} \tag{11}$$

$$\leq \left(\frac{qg}{\delta}\right)^q. \tag{12}$$

A similar inequality holds for $x \leq -\delta$, so that $\Pr\left[\left|f(t)\right| \geq \delta\right] \leq 2\left(\frac{qg}{\delta}\right)^q$. The bound is obtained by choosing $q = \lfloor\frac{\delta}{eg}\rfloor$. □

We now simply need to find the corresponding g for the concentration bound to hold, which we do for various physical problems.

III. MODELS AND THEIR MOMENTS

A. Generic models

First we consider models governed by a Hamiltonian $H = \sum_{m=1}^{D} E_m|E_m\rangle\langle E_m|$, which we assume to have a discrete and *generic* spectrum.

Definition 1. *Let H be a Hamiltonian with spectrum $H = \sum_j E_j |E_j\rangle \langle E_j|$, and let Λ_q, Λ'_q be two arbitrary sets of q energy levels $\{E_j\}$. H is generic if for all $q \in \mathbb{N}$ and all Λ_q, Λ'_q, the equality*

$$\sum_{j\in\Lambda_q} E_j = \sum_{j\in\Lambda'_q} E_j \tag{13}$$

implies that $\Lambda_q = \Lambda'_q$.

This condition is expected to hold in non-integrable and chaotic models, such as those with Wigner-Dyson level statistics [17]. It is an extension of the well-known non-degenerate gaps condition, which is the $q = 2$ case [1]. Notably, the probability of uniformly choosing a non-generic Hamiltonian is zero, as seen in the following lemma.

Lemma 2. *For any positive integer $d \geq 2$, the set of $d \times d$ complex Hermitian matrices that are not generic has Lebesgue measure zero.*

The proof is a straightforward generalization of the $q = 2$ case in [18] and can be found in Appendix B.

Consider $f(t) = \langle\psi|A(t)|\psi\rangle$ to be the pure state time evolution of some observable A. The first concentration result is as follows.

Theorem 1. *Let H be a generic non-integrable Hamiltonian, ω the diagonal ensemble, and $||A||$ the largest singular value of A. The moments in Eq. 1 are such that*

$$\kappa_q \leq \left(q||A||\sqrt{\mathrm{Tr}[\omega^2]}\right)^q. \tag{14}$$

We thus have the bound

$$\begin{aligned} &\Pr\left[\left|\langle A(t)\rangle - \langle A(\infty)\rangle\right| \geq \delta\right] \\ &\leq 2e \times \exp\left(-\frac{\delta}{e||A||\sqrt{\mathrm{Tr}[\omega^2]}}\right). \end{aligned} \tag{15}$$

This states that the probability of finding $\langle A(t)\rangle$ away from $\langle A(\infty)\rangle$ even by a small amount is exponentially suppressed in $\sqrt{\mathrm{Tr}[\omega^2]}$. Previous results [12, 19] only yield the bound

$$\Pr\left[\left|\langle A(t)\rangle - \langle A(\infty)\rangle\right| \geq \delta\right] \leq \frac{||A||^2\mathrm{Tr}[\omega^2]}{\delta^2}. \tag{16}$$

A particular observable of interest is the initial state itself, $A = |\Psi\rangle\langle\Psi|$. In this case, the quantity at hand is the fidelity with the initial state

$$F(t) = \langle\Psi| e^{-itH} |\Psi\rangle \langle\Psi| e^{itH} |\Psi\rangle . \tag{17}$$

Theorem 2. *Let H be generic and let $A = |\Psi\rangle\langle\Psi|$, then*

$$\kappa_q \leq \left(q\,\mathrm{Tr}\left[\omega^2\right]\right)^q. \tag{18}$$

Notably, assuming a generic Hamiltonian, the average fidelity is $F(\infty) = \mathrm{Tr}\left[\omega^2\right]$, so that we have the concentration bound

$$\Pr\left[\left|F(t) - \mathrm{Tr}[\omega^2]\right| \geq \delta\right] \leq 2e \times \exp\left(-\frac{\delta}{e\mathrm{Tr}[\omega^2]}\right). \tag{19}$$

This improves on Eq. (15) by a factor of $\sqrt{\mathrm{Tr}\left[\omega^2\right]}$ when substituted into $A = |\Psi\rangle\langle\Psi|$ and $||A|| = 1$. Eq. (18) appeared previously in [20].

It is well known that $\mathrm{Tr}\left[\omega^2\right]$ is exponentially suppressed in system size for generic models for sufficiently well behaved initial conditions. As an example, see figure 2, which shows a clear exponential decay of $\mathrm{Tr}\left[\omega^2\right]$ with system size L.

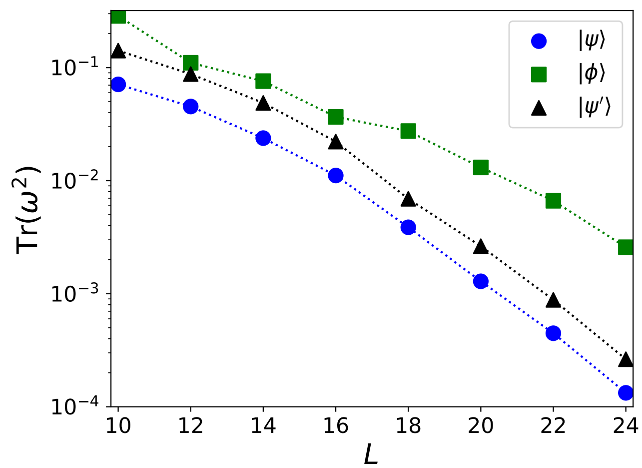

FIG. 2. $\mathrm{Tr}\left[\omega^2\right]$ for a variety of system sizes and states. Numerics were done with the same model as Fig. 1. More details on the states and the model can be found in App. D.

B. Generic extended free fermions

The second class that we consider are generic extended free fermionic models

$$H = \sum_{m,n=1}^{L} M_{m,n} f_m^\dagger f_n, \tag{20}$$

where f_n is a fermionic annihilation operator for the lattice site n. The fermionic operators obey the standard canonical anti-commutation relations $\{f_m, f_n\} = \{f_m^\dagger, f_n^\dagger\} = 0$, $\{f_m^\dagger, f_n\} = \delta_{m,n}$. We assume M is real symmetric, so it is diagonalized with a real orthogonal matrix O such that $M = ODO^T$. D is a diagonal matrix with entries $D_{k,k} = \epsilon_k$, which allows us to rewrite the Hamiltonian as

$$H = \sum_{k=1}^{L} \epsilon_k d_k^\dagger d_k, \tag{21}$$

where ϵ_k are the single particle energy eigenmodes, and we have new fermionic operators in eigenmode space defined in terms of the real space fermionic operators: $d_k = \sum_{j=1}^{L} O_{j,k} f_j$. This class of models notably does not obey Def. 1. However, we can instead give the following definition.

Definition 2. *Let $H = \sum_{k=1}^{L} \epsilon_k d_k^\dagger d_k$ be a free Hamiltonian. Let Λ_q, Λ'_q be two arbitrary sets of q eigenmodes $\{\epsilon_j\}$. Then H is a generic extended free fermionic model if, for all $q \in \mathbb{N}$ and all Λ_q, Λ'_q, the equality*

$$\sum_{j\in\Lambda_q} \epsilon_j = \sum_{j\in\Lambda'_q} \epsilon_j \tag{22}$$

implies that $\Lambda_q = \Lambda'_q$ and the entries of O are such that

$$O_{j,k} = \frac{c_{j,k}}{\sqrt{L}}, \tag{23}$$

with $c_{j,k} = \mathcal{O}(1)$.

The equivalent of Lemma 2 also holds here by applying it to the matrix M and the energy eigenmodes. This definition crucially excludes localized models, which have entries of the form, $O_{m,k} \sim e^{-|k-m|/\xi}$, with ξ the localization length. The bound on the moments is as follows.

Theorem 3. *Let H be a generic extended free fermionic Hamiltonian and let $A = f_m^\dagger f_n$. Then, for even q,*

$$\kappa_q \le \left(qc^2\sqrt{\frac{\nu}{L}}\right)^q, \tag{24}$$

where $\nu = \frac{N}{L}$ is the filling factor of the fermions on the lattice and $c = \sqrt{L}\max_{k_j}\{O_{m,k_j}, O_{n,k_j}\}$.

The corresponding concentration bound is

$$\begin{aligned} &\Pr\left[\left|\langle f_m^\dagger f_n(t)\rangle - \langle f_m^\dagger f_n(\infty)\rangle\right| \ge \delta\right] \\ &\le 2e \times \exp\left(-\frac{\delta}{ec^2}\sqrt{\frac{L}{\nu}}\right). \end{aligned} \tag{25}$$

Theorem 3 can be contrasted with the bound found in [21] for the second moment. The authors consider a potentially extensive observable and do not limit the analysis to extensive models, recovering $\kappa_2 \le ||a||^2 \nu L$ for an observable $A = \sum_{m,n} f_m^\dagger a_{m,n} f_n$.

The last quantity of interest is the single particle propagator

$$\{f_m^\dagger(t), f_n\} = a_{m,n}(t). \tag{26}$$

For example, if we initialize our state as $|\Psi\rangle = f_m^\dagger|0\rangle$, then the fidelity is

$$F(t) = |a_{m,m}(t)|^2. \tag{27}$$

The more general $|a_{m,n}(t)|^2$ is also studied in the context of out of time ordered correlators [22–25]. Consider

$$|a_{m,n}(t)|^2 = \sum_{k,l} O_{m,k}O_{n,k}O_{m,l}O_{n,l}e^{i(\epsilon_k-\epsilon_l)t}. \tag{28}$$

The infinite time average of this quantity is taken as

$$\omega_{m,n} = \sum_k O_{m,k}^2 O_{n,k}^2. \tag{29}$$

For extended models with non-degenerate frequencies this quantity decays to zero since $\omega_{m,n} \sim \frac{1}{L}$ and $\kappa_2 \le \frac{c}{L^2}$, where c is weakly dependent on system size and is $\mathcal{O}(1)$ in the thermodynamic limit [23]. We can bound the moments for the single particle propagator as follows.

Theorem 4. *Let H be a generic free fermionic Hamiltonian, and let our dynamical function $f(t) = |a_{m,n}(t)|^2$ be the squared single particle propagator, then we can bound the moments by*

$$\kappa_q \le \left(\frac{qc^4}{L}\right)^q, \tag{30}$$

where $c = \sqrt{L}\max_{k_j}\{O_{m,k_j}, O_{n,k_j}\}$.

Finally, the corresponding concentration bound is

$$\Pr\left[\left||a_{m,n}(t)|^2 - \omega_{m,n}\right| \ge \delta\right] \le 2e \times \exp\left(-\frac{\delta L}{ec^4}\right). \tag{31}$$

IV. RECURRENCE TIME

All the quantities analyzed above always come back arbitrarily close to their initial values at $t = 0$. For large systems, however, such recurrences only happen at astronomically large timescales, inaccessible to both experiments and numerical studies. We now put a lower bound on those timescales through a suitably defined notion of *average recurrence time*, both for observables and also the whole state.

Definition 3. *A (u, Δ, A)-recurrence occurs at a time interval $\mathcal{C}_\Delta = [t_\Delta, t_\Delta + \Delta]$ if, for all $t \in \mathcal{C}_\Delta$,*

$$|\langle A(t)\rangle - \langle A(0)\rangle| \le u||A||. \tag{32}$$

Similarly, a (u, Δ)-recurrence occurs if, for all $t \in \mathcal{C}_\Delta$,

$$1 - F(t) \le u. \tag{33}$$

Notice that an (u, Δ)-recurrence implies an (u, Δ, A)-recurrence for all A, and that conversely an (u, Δ, A)-recurrence for all A implies an (u, Δ)-recurrence. However, individual observables may have additional earlier recurrences.

Let us also define $t_\Delta^n(A)$ as the time for the n-th (u, Δ, A)-recurrence, so that $t_\Delta^n(A) < t_\Delta^{n+1}(A)$, and analogously, t_Δ^n for the fidelity. This motivates the following definition, inspired by that in [9].

Definition 4. *The average (u, Δ, A)-recurrence time is*

$$T(u, \Delta, A) \equiv \lim_{n\to\infty} \frac{t^n_\Delta(A)}{n}, \tag{34}$$

with $T(u, \Delta)$ analogously defined.

These quantities can be easily bounded with the concentration bounds above. First, for $T(u, \Delta, A)$.

Corollary 1. *Let H be generic, and let w.l.o.g. $\langle A(0)\rangle - \langle A(\infty)\rangle = c_A||A|| \geq 0$. Then, for $u \leq c_A$,*

$$\frac{\Delta}{2e}\exp\left(\frac{c_A - u}{e\sqrt{\mathrm{Tr}[\omega^2]}}\right) \leq T(u, \Delta, A). \tag{35}$$

Proof. From the definition of the distribution $P(x)$ in Eq. (6) and Eq. (15) we have that

$$\begin{aligned}\lim_{n\to\infty} \frac{\Delta n}{t^n_\Delta(A) + \Delta} &\leq \Pr\left[|\langle A(t)\rangle - \langle A(0)\rangle| \leq u||A||\right] \\ &\leq \Pr\left[|\langle A(t)\rangle - \langle A(\infty)\rangle| \geq (c_A - u)||A||\right] \\ &\leq 2e \times \exp\left(\frac{u - c_A}{e\sqrt{\mathrm{Tr}\left[\omega^2\right]}}\right).\end{aligned} \tag{36}$$

Notice that $\lim_{n\to\infty} \frac{\Delta n}{t^n_\Delta(A)+\Delta} = \Delta/T(u, \Delta, A)$. Solving for $T(u, \Delta, A)$ yields the result. □

With typical out-of-equilibrium initial conditions, we have that $c_A = \mathcal{O}(1)$. The larger recurrences should have a duration comparable to that of the initial equilibration time T^A_{eq}, which we can define as the time it takes for $\langle A(t)\rangle$ to settle around the steady value $\langle A(\infty)\rangle$. The recurrences with $u = \mathcal{O}(1)$ are then on average spaced by a time

$$T \gtrsim T^A_{\mathrm{eq}} e^{\Omega\left(\mathrm{Tr}[\omega^2]^{-1/2}\right)}. \tag{37}$$

Note that for local Hamiltonians and observables, T^A_{eq} is believed to generally scale as a low-degree polynomial in system size [26].

For the fidelity, the bound on the recurrence follows exactly the proof of Corollary 1 but using Eq. (19) instead.

Corollary 2. *Let H be generic. Then,*

$$\frac{\Delta}{2e^2}\exp\left(\frac{1-u}{e\mathrm{Tr}[\omega^2]}\right) \leq T(u, \Delta). \tag{38}$$

Proof. Again from Eq. (6) and Eq. (19) we have that

$$\begin{aligned}\lim_{n\to\infty} \frac{\Delta n}{t^n_\Delta + \Delta} &\leq \Pr\left[F(t) \geq 1-u\right] \\ &\leq \Pr\left[|F(t) - \mathrm{Tr}[\omega^2]| \geq 1 - u - \mathrm{Tr}[\omega^2]\right] \\ &\leq 2e \times \exp\left(-\frac{1 - u - \mathrm{Tr}[\omega^2]}{e\mathrm{Tr}[\omega^2]}\right) \\ &\leq 2e^2 \times \exp\left(-\frac{1-u}{e\mathrm{Tr}[\omega^2]}\right).\end{aligned} \tag{39}$$

Also, $\lim_{n\to\infty} \frac{\Delta n}{t^n_\Delta + \Delta} = \Delta/T(u, \Delta)$, and solving for $T(u, \Delta)$ yields the result. □

In many-body systems, the fidelity initially decays as $F(t) = e^{-\sigma^2 t^2/2}$ where $\sigma^2 = \langle\Phi| H^2 |\Phi\rangle - \langle\Phi| H |\Phi\rangle^2$ is the energy variance [27–29]. Recurrences with $u = \mathcal{O}(1)$ likely decay in a similar fashion, with $\Delta \sim \sigma^{-1}$, so that on average they are spaced by a time

$$T \gtrsim \sigma^{-1} e^{\Omega\left(\mathrm{Tr}[\omega^2]^{-1}\right)}. \tag{40}$$

This closely matches the behaviour found in [9], which gives an exact calculation of the average recurrence time assuming a Gaussian wavefunction. It also matches the scaling of other previous estimates [8], so Eq. (38) should be close to optimal.

Finally, we also have corresponding bounds for fermions.

Corollary 3. *Let H be a generic free fermionic Hamiltonian, and let w.l.o.g. $\langle f^\dagger_m f_n(0)\rangle - \langle f^\dagger_m f_n(\infty)\rangle = c_f \geq 0$. Then, for $u \leq c_f$,*

$$\frac{\Delta}{2e}\exp\left(\frac{c_f - u}{ec^2}\sqrt{\frac{L}{\nu}}\right) \leq T(u, \Delta, f^\dagger_m f_n), \tag{41}$$

as well as for the fidelity in Eq. (27).

Corollary 4. *Let H be a generic free fermionic Hamiltonian. Then,*

$$\frac{\Delta}{2e}\exp\left(\frac{(1-u)L}{ec^4}\right) \leq T(u, \Delta). \tag{42}$$

The analogue of Eq. (37) and Eq. (40) also holds following the same considerations. These bounds however scale as $e^{\Omega(\sqrt{L})}$ and $e^{\Omega(L)}$ respectively, which are exponential in the number of sites L. This is a fast scaling, but still exponentially slower than that from Corollaries 1 and 2. Even shorter recurrence times are also found in specific instances of Bose gases [30–32], which can even be experimentally tested [33] with cold atoms.

V. CONCLUSION

We have shown how in generic systems both observables and the fidelity with the initial state equilibrate around their time-averaged values, with out-of-equilibrium fluctuations suppressed exponentially in the effective dimension $\mathrm{Tr}\left[\omega^2\right]^{-1}$. This number scales exponentially under very general conditions on the state and the Hamiltonian [18, 34–36], so in generic systems fluctuations are most often doubly exponentially suppressed. Since partial or full recurrences are far from equilibrium fluctuations, our bounds yield an estimate of their occurrence, with a scaling that we believe is almost optimal. Equivalent results with a slower scaling also hold for free fermions.

Previous works [18, 34, 36–39] start with the bound on the second moment in [12, 13] to obtain results on equilibration, so the present findings naturally strengthen them. Also, Theorem 2 in [40] extends [12, 13] to two-point correlation functions, and the corresponding concentration bound is straightforward.

Our bounds on the recurrence time apply to individual states. A given Hamiltonian should also have other later state-independent recurrences. For instance, the recent result for random circuits [41] suggests that a recurrence in complexity of e^{-itH} might still doubly exponential, but with a larger exponent that Eq. (40).

ACKNOWLEDGMENTS

AMA acknowledges support from the Alexander von Humboldt foundation. J.R. and N.J.P. acknowledges support from the Natural Sciences and Engineering Research Council of Canada (NSERC).

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Appendix A: Defining moments

The average value of $\langle A(t)\rangle$ in Eq. (1) motivates the formal definition of the following probability distribution

$$P(x) = \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\delta(x - \langle A(t)\rangle).$$

We would like to compute the moments as

$$\kappa_q \equiv \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\left(\langle A(t)\rangle - \langle A(\infty)\rangle\right)^q.$$

This requires swapping the integral over x and limit in T and can be justified using the dominated convergence theorem. To use this famous result we must prove that the absolute value of the integrand is bounded by an integrable function. Consider

$$\kappa_q = \int x^q \lim_{T\to\infty} P_T(x)dx.$$

For book-keeping purposes let $g(t) = (\langle A(t)\rangle - \langle A(\infty)\rangle)$. We may bound the integrand inside the limit

$$|x^q P_T(x)| = |\frac{x^q}{T}\int_0^T \mathrm{d}t\delta(x - g(t))| \leq \frac{|x|^q}{T}\int_0^T \mathrm{d}t\delta(x - g(t)).$$

Integrating the absolute value of the right-hand side and applying Fubini's theorem we find

$$\int \left|\frac{|x|^q}{T}\int_0^T \mathrm{d}t\delta(x - g(t))\right| dx \leq \int \frac{|x|^q}{T}\left|\int_0^T \mathrm{d}t\delta(x - g(t))\right| dx \leq \int \frac{|x|^q}{T}\int_0^T \mathrm{d}t\delta(x - g(t))dx$$

$$\leq \frac{1}{T}\int_0^T \mathrm{d}t|g(t)|^q.$$

The function $g(t)$ is continuous and therefore bounded on the interval $[0, T]$, so the integral is finite, therefore its positive and negative components are as well, thus the q-th absolute moment is bounded. We may therefore conclude that the absolute value of the integrand is bounded by an integrable function, so by the dominated convergence theorem we may swap the integral and limit.

Note that equivalently, one may define the moments for finite T and then take the limit.

Appendix B: Generic spectra

Lemma 1. *The set of generic Hermitian matrices in $M_d(\mathbb{C})$ has full Lebesgue measure.*

Proof. Note that this proof is similar to the proof that the set of non-diagonalizable matrices has Lebesgue measure zero. Let H be some Hermitian $d \times d$ matrix. We start by defining the function

$$F(H) = \prod_{\substack{n_1,m_1,\ldots,n_q,m_q \\ n_i\neq m_i}} \left(\sum_{i=1}^{q} E_{n_i} - E_{m_i}\right).$$

This function is zero precisely when the spectrum is not generic. Clearly swapping eigenvalues does not change the function F, i.e. F is a symmetric polynomial of the eigenvalues. By the fundamental theorem of symmetric (real) polynomials F can be written uniquely as a polynomial in the elementary symmetric polynomials in E_i's, which are precisely trace powers of H.

Recall that H can be expressed in some basis. For example some generalized Pauli basis, or even the standard basis. This is conceptually equivalent to saying that the vector space of all possible H can be parameterized by the coefficients of the basis elements in the expansion. Thus, we may expand H in this basis and then take trace powers, showing us that F is a real polynomial in the space of these coefficients.

It is a well known fact from measure theory that the zero set of a multivariate polynomial has Lebesgue measure zero. □

Appendix C: Bounding the moments

1. Proof for generic models

Theorem 1. *Let H be a generic non-integrable Hamiltonian, ω the diagonal ensemble, and $||A||$ the largest singular value of A. The moments in Eq. 1 are such that*

$$\kappa_q \leq \left(q||A||\sqrt{\mathrm{Tr}[\omega^2]}\right)^q. \tag{C1}$$

Proof. First we prove the following inequality:

$$|\,\mathrm{Tr}\left[(A\omega)^q\right]| \leq \left(||A||\sqrt{\mathrm{Tr}\left[\omega^2\right]}\right)^q. \tag{C2}$$

To realize this, consider the matrix $A\omega$, which is not Hermitian and therefore may have complex eigenvalues and may potentially not be diagonalizable. This, however, does not prevent us from finding a complete set of eigenvalues such that their multiplicity summed is the dimension of $A\omega$. The matrix is always similar to its Jordan form, and we can in general always write

$$\mathrm{Tr}\left[A\omega\right] = \sum_i \lambda_i, \tag{C3}$$

where λ_i is the i-th (potentially complex) eigenvalue of $A\omega$. More generally, we can always write

$$\mathrm{Tr}\left[(A\omega)^q\right] = \sum_i \lambda_i^q. \tag{C4}$$

From here we can bound the following:

$$|\,\mathrm{Tr}\left[(A\omega)^q\right]| = |\sum_i \lambda_i^q| \tag{C5}$$

$$\leq \sum_i |\lambda_i^q| \tag{C6}$$

$$= ||\lambda||_q^q \tag{C7}$$

$$\leq ||\lambda||_2^q = \left(\sum_i |\lambda_i|^2\right)^{q/2} \tag{C8}$$

where we used the triangle inequality in (C6), and in (C8) we use the property $||x||_{p+a} \leq ||x||_p$ for any vector x and real numbers $p \geq 1$ and $a \geq 0$. Note that, using the Shur decomposition, we may write $A\omega = QUQ^\dagger$, where Q is a unitary matrix, and U is upper triangular with the same spectrum on the diagonal as $A\omega$. Using this, we have that

$$\mathrm{Tr}\left[(A\omega)(A\omega)^\dagger\right] = \mathrm{Tr}\left[QUQ^\dagger(QUQ^\dagger)^\dagger\right] = \mathrm{Tr}\left[UU^\dagger\right] = \sum_i |\lambda_i|^2 + \text{other non-negative terms.}$$

Thus,

$$\sum_i |\lambda_i|^2 \leq \mathrm{Tr}\left[A\omega\omega^\dagger A^\dagger\right] \leq ||AA^\dagger||\,\mathrm{Tr}\left[\omega\omega^\dagger\right] \leq ||A||^2\,\mathrm{Tr}\left[\omega\omega^\dagger\right] = ||A||^2\,\mathrm{Tr}\left[\omega^2\right].$$

This gives us our desired inequality.

Moving on, let us derive a general bound for the q-th moment of models satisfying Definition 1. For simplicity, and w.l.o.g., let us assume that $\langle A(\infty)\rangle = 0$. Expanding the definition of the moments we arrive at

$$\kappa_q = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt \sum_{m_1,n_1,\ldots,m_q,n_q}\prod_{i=1}^q \left(A_{m_i,n_i}\bar{c}_{m_i}c_{n_i}\right)e^{i(E_{m_i}-E_{n_i})t}. \tag{C9}$$

The assumption that the Hamiltonian is generic means that only certain terms in the sum survive after averaging over all time: those for which the sets of $\{m_i\}$ and $\{n_i\}$ coincide up to permutations, which we denote with $\{\sigma(i)\}$. Due to the equilibrium expectation value being zero, we can also eliminate all terms for which $i = \sigma(i)$ for $1 \leq i \leq q$. Thus, we want all permutations on q elements except those that have a fixed point. Such permutations are called derangements. The number of distinct derangements is denoted by $!q$, the subfactorial of q, and has no explicit formula. However, it may be computed recursively, the first few being $0, 1, 2, 9, 44, 265$. Let D_q denote the set of derangements on $\{1, 2, .., q\}$. Eq. (C9) becomes

$$\kappa_q = \sum_{m_1,...,m_q} \prod_{i=1}^{q} |c_{m_i}|^2 \sum_{\sigma \in D_q} \prod_{i=1}^{q} A_{m_i,\sigma(m_i)}. \tag{C10}$$

Given a derangement σ, it can be decomposed as the product of cycles $\sigma_1, \sigma_2, ..., \sigma_r$ with lengths $\ell_1, \ell_2, ..., \ell_r$, respectively, such that $\sum_{j=1}^{r} l_j = q$. In each term of the inner summation we can collect terms of the same cycle. For example for $q = 6$ and $\sigma = \sigma_1\sigma_2 = (m_1, m_2)(m_3, m_4, m_5, m_6)$ the term can be written as

$$(A_{m_1,m_2} A_{m2,m_1})(A_{m_3,m_4} A_{m_4,m_5} A_{m_5,m_6} A_{m_6,m_3}).$$

Summing over $m_1, m_2, m_3, m_4, m_5, m_6$ we have that this term is precisely

$$\mathrm{Tr}\left[(A\omega)^2\right] \mathrm{Tr}\left[(A\omega)^4\right].$$

In general, each cycle of a term will correspond to a product of trace powers i.e. $\sigma = \sigma_1, \sigma_2, ..., \sigma_r$ corresponds to

$$\mathrm{Tr}\left[(A\omega)^{\ell_1}\right] \mathrm{Tr}\left[(A\omega)^{\ell_2}\right] ... \mathrm{Tr}\left[(A\omega)^{\ell_r}\right].$$

We may apply Eq. (C2) term-wise to get

$$\mathrm{Tr}\left[(A\omega)^{\ell_1}\right] \mathrm{Tr}\left[(A\omega)^{\ell_2}\right] ... \mathrm{Tr}\left[(A\omega)^{\ell_r}\right] \leq \left(||A||\sqrt{\mathrm{Tr}\left[\omega^2\right]}\right)^{\ell_1+\ell_2+...+\ell_r} = \left(||A||\sqrt{\mathrm{Tr}\left[\omega^2\right]}\right)^{q}.$$

In each moment's inner summation there are precisely $!q$ terms of this form because there are $!q$ derangements, thus

$$\kappa_q \leq !q \left(||A||\sqrt{\mathrm{Tr}\left[\omega^2\right]}\right)^{q} \leq \left(q||A||\sqrt{\mathrm{Tr}\left[\omega^2\right]}\right)^{q}. \tag{C11}$$

□

The $q = 2$ case can be found in [19].

Theorem 2. *Let H be generic and let $A = |\Psi\rangle\langle\Psi|$, then*

$$\kappa_q \leq \left(q\,\mathrm{Tr}\left[\omega^2\right]\right)^q. \tag{C12}$$

Proof. The moments defined for the fidelity are defined as

$$\kappa_q = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} dt \prod_{i=1}^{q} \sum_{m_i \neq n_i} |c_{m_i}|^2 |c_{n_i}|^2 e^{i(E_{m_i}-E_{n_i})t}, \tag{C13}$$

$$= \sum_{m_1,...m_q} \prod_{i=1}^{q} |c_{m_i}|^2 \prod_{\sigma \in D_q} |c_{\sigma(m_i)}|^2. \tag{C14}$$

In the above expression we can note that there are $!q$ possible derangements using genericity given in Definition 1. Each m_i will have one pair given to us from $\sigma(m_i)$, implying each individual term in the sum is $\mathrm{Tr}\left[\omega^2\right]^q$, so our final expression is

$$\kappa_q = !q\,\mathrm{Tr}\left[\omega^2\right]^q \leq \left(q\,\mathrm{Tr}\left[\omega^2\right]\right)^q. \tag{C15}$$

□

2. Generic free models

This class of models conserves total particle number, which we will denote as

$$N = \sum_{j=1}^{L} \langle f_j^\dagger f_j \rangle = \sum_{k=1}^{L} \langle d_k^\dagger d_k \rangle. \tag{C16}$$

Theorem 3. *Let H be a generic extended free fermionic Hamiltonian and let $A = f_m^\dagger f_n$. Then, the even moments are bounded above by*

$$\kappa_q \leq \left(q c^2 \sqrt{\frac{\nu}{L}} \right)^q, \tag{C17}$$

where $\nu = \frac{N}{L}$ is the filling factor of the fermions on the lattice and $c = \sqrt{L} \max_{k_j} \{O_{m,k_j}, O_{n,k_j}\}$.

Proof. Consider

$$\kappa_{2n} = \lim_{T\to\infty} \frac{1}{T} \int_0^\infty dt \prod_{j=1}^{n} \sum_{k_j \neq l_j} O_{m,k_j} O_{n,l_j} \langle d_{k_j}^\dagger d_{l_j} \rangle e^{i(\epsilon_{k_j} - \epsilon_{l_j})t} \sum_{p_j \neq q_j} O_{m,p_j} O_{n,q_j} \langle d_{q_j}^\dagger d_{p_j} \rangle e^{i(\epsilon_{q_j} - \epsilon_{p_j})t}. \tag{C18}$$

Let us define the tensor (note the j dependence)

$$B_{k_j,l_j} = \begin{cases} O_{m,k_j} O_{n,l_j} \langle d_{k_j}^\dagger d_{l_j} \rangle & j \text{ odd} \\ O_{m,l_j} O_{n,k_j} \langle d_{k_j}^\dagger d_{l_j} \rangle & j \text{ even} \\ 0 & k_j = l_j \end{cases}. \tag{C19}$$

This allows us to rewrite our equation as

$$\kappa_{2n} = \lim_{T\to\infty} \frac{1}{T} \int_0^\infty dt \prod_{j=1}^{2n} \sum_{k_j,l_j} B_{k_j,l_j} e^{i(\epsilon_{k_j} - \epsilon_{l_j})t}. \tag{C20}$$

This can likewise be rewritten as

$$\kappa_{2n} = \lim_{T\to\infty} \frac{1}{T} \int_0^\infty dt \sum_{k_1,l_1,\ldots k_{2n},l_{2n}} \prod_{j=1}^{2n} B_{k_j,l_j} e^{i(\epsilon_{k_j} - \epsilon_{l_j})t}. \tag{C21}$$

Assuming a generic single-particle spectrum, this means we have the following surviving terms:

$$\kappa_{2n} = \sum_{k_1,\ldots k_{2n}} \sum_{\sigma \in S_{2n}} \prod_{j=1}^{2n} B_{k_j,\sigma(k_j)} \tag{C22}$$

where S_{2n} denotes the symmetric group on $1, 2 \ldots 2n$. We can then enforce the fact that these terms are zero if $k_j = \sigma(k_j)$ for $1 \leq j \leq 2n$. So denoting the derangements as D_{2n} as earlier, we arrive at

$$\kappa_{2n} = \sum_{k_1,\ldots k_{2n}} \sum_{\sigma \in D_{2n}} \prod_{j=1}^{2n} B_{k_j,\sigma(k_j)}. \tag{C23}$$

Next, recognizing that each definition of B contains two extensive terms multiplied, let $c = \sqrt{L} \max_{k_j} \{O_{m,k_j}, O_{n,k_j}\}$,

$$\kappa_{2n} \leq \frac{c^{4n}}{L^{2n}} \sum_{k_1,\dots k_{2n}} \sum_{\sigma \in D_{2n}} \prod_{j=1}^{2n} \langle d^\dagger_{k_j} d_{\sigma(k_j)} \rangle \tag{C24}$$

As in Theorem 1, each term will be a trace of powers of Λ, and there can be at most n products of traces of Λ. Since $0 \leq \Lambda \leq \mathbb{I}$, each trace of Λ can further be bounded by $\mathrm{Tr}\left[\Lambda^p\right] \leq \mathrm{Tr}\left[\Lambda\right] = N = \nu L$, which means we can bound κ_{2n} by

$$\kappa_{2n} \leq !(2n) \frac{c^{4n}\nu^n}{L^{2n-n}} \leq \left(\frac{4n^2 c^4 \nu}{L}\right)^n, \tag{C25}$$

where c' is weakly dependent on system size and $0 \leq \nu \leq 1$. Choosing $q = 2n$ and reorganizing gives the desired result. ☐

Theorem 4. *Let H be a generic free fermionic Hamiltonian, and let our dynamical function $f(t) = |a_{m,n}(t)|^2$ be the squared single particle propagator, then we can bound the moments by*

$$\kappa_q \leq \left(\frac{qc^4}{L}\right)^q, \tag{C26}$$

where $c = \sqrt{L}\max_{k_j}\{O_{m,k_j}, O_{n,k_j}\}$.

Proof. The q-th moment can be written as

$$\kappa_q = \lim_{T\to\infty} \frac{1}{T}\int_0^\infty dt \prod_{i=1}^{q} \sum_{k_i \neq l_i} O_{m,k_i} O_{n,k_i} O_{m,l_i} O_{n,l_i} e^{i(\epsilon_{k_i} - \epsilon_{l_i})t}, \tag{C27}$$

through the usual procedure and using definition 2 we recover

$$\kappa_q = \sum_{k_1,\dots k_q} \prod_{i=1}^{q} O_{m,k_i} O_{n,k_i} \prod_{\sigma \in D_q} O_{m,\sigma(k_i)} O_{n,\sigma(k_i)}, \tag{C28}$$

defining $c = \sqrt{L}\max_{k_j}\{O_{m,k_j}, O_{n,k_j}\}$, we factor out four of these, and sum up the indices, giving us

$$\kappa_q \leq \frac{!qc^{4q}}{L^q} \leq \left(\frac{qc^4}{L}\right)^q. \tag{C29}$$

☐

Appendix D: Numerics

The numerics for the figures in the main body were carried out on the spin 1/2 Hamiltonian,

$$H = \sum_{j=1}^{L} J_1 \left(S^+_j S^-_{j+1} + \mathrm{h.c}\right) + \gamma_1 S^Z_j S^Z_{j+1} + J_2 \left(S^+_j S^-_{j+2} + \mathrm{h.c}\right) + \gamma_2 S^Z_j S^Z_{j+2},$$

where $(J_1, \gamma_1, J_2, \gamma_2) = (-1, 1, -0.2, 0.5)$ giving us a non-integrable model. We perform exact diagonalization exploiting total spin conservation and translation invariance. We choose pure states that allow us to further exploit the Z_2 spin flip symmetry and the spatial reflection symmetry. In Fig. 1 we see the approximated probability distribution function $\tilde{P}_T(x)$ as a histogram. The observable is $A = \sigma^Z_1$, the Pauli-z matrix on the first lattice site. The initial state is a Néel type state:

$$|\psi\rangle = |\uparrow\downarrow\dots\rangle. \tag{D1}$$

In Fig. 2 we calculate the purity of the diagonal ensemble $\mathrm{Tr}\left[\omega^2\right]$ for three states. The states featured are

$$|\psi\rangle \coloneqq |\uparrow\downarrow\uparrow\downarrow \ldots..\rangle, \tag{D2}$$

$$|\psi'\rangle \coloneqq \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\uparrow\downarrow \ldots..\rangle + |\downarrow\uparrow\downarrow\uparrow \ldots..\rangle\right), \tag{D3}$$

$$|\phi\rangle \coloneqq \frac{1}{\sqrt{L}}\sum_{r=0}^{L-1}\hat{T}^r|\uparrow\uparrow \ldots \uparrow\downarrow \ldots \downarrow\downarrow\rangle. \tag{D4}$$