

Fundamental membranes and the string dilaton

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ABSTRACT: We study the quantization of the bosonic sector of supermembrane theory in double dimensional reduction, in order to extract the dependence of the resulting world-sheet action on the string dilaton (which cannot be obtained from a purely kinematic reduction). Our construction relies on a Polyakov-type approach with all six metric components on the world-volume as independent quantum fields, and shows that the correct and unique answer is only obtained if the target-space dimension of the theory is restricted to the critical value ($D = 11$ for the supermembrane). As a corollary, our analysis implies that there are no analogs of the non-critical string for (super-)membrane theory.

KEYWORDS: Bosonic Strings, M-Theory

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1 Introduction

Membranes are notoriously hard to quantize. This is because they are defined by world-volume actions that, unlike the world-sheet actions of (super-)string theory, involve seemingly intractable interactions on the world-volume. For this reason it is not even clear whether and under what circumstances a quantum (super-)membrane theory can be sensibly defined at all. Nevertheless, and intriguingly, the maximally extended $D = 11$ supermembrane theory [1, 2] stands out uniquely as a challenging candidate for the *non-perturbative* quantum unification of gravity and matter, with maximally extended $D = 11$ supergravity [3] as a ‘low energy limit’.

Early work on the quantization of the bosonic membrane in a Minkowski target-space background relied on a Hamiltonian formulation in the light-cone gauge and demonstrated its equivalence with the $N \rightarrow \infty$ limit of a certain $SU(N)$ matrix model [4, 5]. Building on these insights it was shown in [6] that the light-cone gauge formulation of the supermembrane (again in flat target-space) leads to a maximally supersymmetric matrix model, corresponding to the reduction of a maximally supersymmetric Yang-Mills theory to one (time) dimension. This result greatly improved prospects for making the supermembrane amenable to a quantum treatment which can accommodate $D = 11$ supergravity as a massless sector. Indeed, nine years later the very same model was proposed as a model of M theory [7]. A crucial role here is played by the fact that the spectrum of the supersymmetric Hamiltonian is continuous [8, 9]. Altogether, these developments clarified that the supermembrane, unlike (super-)string theory, does not admit a proper first quantized formulation, but must be regarded as a second quantized theory from the outset [10]. For a review of these developments and for different perspectives on them, see [10–14].

Nevertheless, despite these advances there has overall been only scant progress on the quantization of the supermembrane and the $N \rightarrow \infty$ limit of the matrix model, with major results mainly concerning the spectrum of the Hamiltonian and the existence (or non-existence) of a normalizable ground state wave function for fixed finite N (see [15–19]

and references therein), and the construction of (classical analogs of) superstring vertex operators for the massless supermembrane states [20]. Furthermore, very recent work [21] has provided evidence that supersymmetry is a necessary prerequisite for the $N \rightarrow \infty$ limit of the matrix model to exist, thus lending credence to old claims that the bosonic membrane is not renormalizable in any dimension. However, all this work pertains to the quantization of membrane theory in a *flat* target-space background only, and does not generalize in any obvious way to curved target-space geometries.

In this paper we return to the Lagrangian formulation of the membrane and supermembrane in an arbitrary curved target-space background, focusing on the links between membrane theory and the string dilaton. As is well known, the latter occupies a central place in string theory via its direct relation to the string coupling g_s . In a first step we here study the double dimensional reduction of the supermembrane and its quantization in a Polyakov-type formulation, with the world-volume metric as independent quantum variables. As shown in [22, 23] the kinematical reduction together with the embedding equations reproduces part of the type IIA superstring action in a Green-Schwarz formulation, in particular with the correct world-sheet couplings of the target-space metric $G_{\mu\nu}^{(10)}(X)$ and the two-form field $B_{\mu\nu}(X)$. However, apart from the (anticipated) disappearance of the Ramond-Ramond fields in this reduction, there remains no trace of the remaining missing piece of the type I subsector, namely the dilaton! More specifically, the origin of the crucial term [24]

$$\frac{1}{4\pi} \int d^2\sigma \sqrt{g} \phi R(g) \quad (1.1)$$

(where $R \equiv R^{(2)}$ is the world-sheet curvature) giving rise to the identification between the string coupling g_s and the dilaton vacuum expectation value remains unexplained. We note, however, that for constant ϕ a derivation of this term was already proposed in [25].

The main purpose of this work is to argue that the derivation of (1.1) and a proper understanding of the dilaton couplings on the world-sheet require a *quantum treatment* of the membrane, beyond the classical considerations of [22], and an approach where one integrates over *all* six world-volume degrees of freedom (see also [26] for an alternative approach). In addition there is the peculiar feature that the field ϕ in (1.1) requires a special wave-function renormalization from the membrane perspective, as we will explain in section 4. These conclusions are consistent with the fact that the term (1.1) does not come with a factor $(4\pi\alpha')^{-1}$, unlike the tree level couplings. Our construction furthermore reveals the necessity of restricting the target-space dimension to a critical value as a consistency condition (which, however, may not be sufficient), leading to the conclusion that the membrane and the supermembrane can be viable, if they are viable at all, only in target-space dimensions $D = 27$ and $D = 11$ (see also [27] for very early work where the same conclusion was reached by different arguments). The derivation also implies the relation

$$\frac{1}{2} T_3 R_{10} = \frac{1}{4\pi\alpha'} g_s^{2/3} \quad (1.2)$$

for the doubly dimensionally reduced membrane. This relation extends the known result $R_{10} \propto g_s^{2/3}$ [28] to membrane theory, and links the membrane tension T_3 and the compactification radius R_{10} to the two key parameters of string theory. Furthermore it shows how

the strong coupling limit of string theory gets intertwined with the large tension limit of the membrane [21].

Putting together the available evidence we thus conclude that, if at all, only the *maximally extended* supermembrane theory can give rise to a viable quantum theory, because (1) $D = 11$ is a necessary condition by the results of the present work, and (2) supersymmetry is required by the arguments of [21]. This implies that, unlike for string theory, there appears to be no such thing as ‘non-critical (super-)membrane theory’!

The structure of this paper, then, is as follows. In section 2 we review the kinematics of double dimensional reduction, following [22, 23]. In section 3 we analyze the quantization of the theory in a Polyakov-type approach. Finally in section 4 we show that the usual dilaton coupling is the only sensible, and in fact unique, outcome. Although our main concern is the supermembrane we restrict attention to its bosonic subsector, as the fermionic terms do not affect our main conclusions.

2 Double dimensional reduction

We start from the bosonic world-volume action in ‘Polyakov form’ [29, 30] with Euclidean signature

$$S = \frac{T_3}{2} \int d^3\sigma \sqrt{\gamma} \left(\gamma^{ij} \partial_i X^M \partial_j X^N G_{MN}(X) - 1 \right) + \frac{T_3}{6} \int d^3\sigma \varepsilon^{ijk} \partial_i X^M \partial_j X^N \partial_k X^P \mathcal{A}_{MNP}(X) \quad (2.1)$$

where $i, j = 0, 1, 2$ and $M, N = 0, \dots, 10$ with the world-volume coordinates $\sigma^i \equiv (\sigma^0, \sigma^1, \sigma^2)$; below we will occasionally write $\xi \equiv \sigma^2$ to distinguish the compactified coordinate. $G_{MN}(X)$ is the (in general curved) target space metric; $\mathcal{A}_{MNP}(X)$ is the 3-form field of $D = 11$ supergravity. For flat target (Minkowski) space we have $G_{MN} = \eta_{MN}$. The only dimensionful parameter is the membrane tension T_3 , of mass dimension three. An alternative way of writing S in terms of flat target-space indices is

$$S = \frac{T_3}{2} \int d^3\sigma \sqrt{\gamma} \left(\gamma^{ij} \Pi_i^A \Pi_j^B \eta_{AB} - 1 \right) + \frac{T_3}{6} \int d^3\sigma \varepsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C \mathcal{A}_{ABC} \quad (2.2)$$

where

$$\Pi_i^A \equiv \partial_i X^M E_M^A(X) \quad (2.3)$$

with the target-space elfbein E_M^A and flat (Lorentz) indices A, B, C, \dots . Both actions (2.1) and (2.2) are manifestly invariant under world-volume diffeomorphisms. Because $\partial_i X^M$ transforms as a vector in target-space, they are likewise invariant under target-space diffeomorphisms. Finally, the action is invariant under the target space gauge transformations $\delta \mathcal{A}_{MNP} = 3\partial_{[M} \Lambda_{NP]}$.

The solution of the (algebraic) equations of motion for γ_{ij} is

$$\gamma_{ij} = \partial_i X^M \partial_j X^N G_{MN}(X) \quad (2.4)$$

so γ_{ij} is just the induced metric on the world-volume; note that the second term in (2.1) does not depend on the world-volume metric, hence does not contribute to the embedding equations of motion. Varying the target-space coordinates gives

$$\begin{aligned} & \partial_i \left(\sqrt{\gamma} \gamma^{ij} \partial_j X^M \right) + \sqrt{\gamma} \gamma^{ij} \Gamma_{PQ}^M(X) \partial_i X^P \partial_j X^Q + \\ & + \frac{1}{6} \varepsilon^{ijk} \partial_i X^N \partial_j X^P \partial_k X^Q \mathcal{F}_{NPQ}^M(X) = 0 \end{aligned} \quad (2.5)$$

where $\mathcal{F}_{MNPQ} \equiv 4\partial_{[M} \mathcal{A}_{NPQ]}$, and Γ_{PQ}^M is the affine connection associated to the target space metric $G_{MN}(X)$. There is no other equation that relates \mathcal{A}_{MNP} to world-volume objects. As explained in [1, 2] the supersymmetry of the supermembrane action requires that both G_{MN} and \mathcal{A}_{MNP} satisfy their respective target superspace equations of motion.

Even before dimensionally reducing the theory, the world-volume metric can be decomposed via the standard Kaluza-Klein (KK) ansatz

$$\begin{aligned} \gamma_{ij} &= \begin{pmatrix} g_{\alpha\beta} + e^{2\tau} A_\alpha A_\beta & e^{2\tau} A_\alpha \\ e^{2\tau} A_\beta & e^{2\tau} \end{pmatrix}, \\ \gamma^{ij} &= \begin{pmatrix} g^{\alpha\beta} & -A^\alpha \\ -A^\beta & e^{-2\tau} + A^\gamma A_\gamma \end{pmatrix}, \end{aligned} \quad (2.6)$$

where the world-volume indices are split as $i = (\alpha, \dot{2})$ etc. into world-sheet indices α, β, \dots and the remaining third coordinate (we sometimes put a dot on the last index to indicate that it is a curved index). $g_{\alpha\beta}$ is the 2×2 world-sheet metric, A_α is a 2-vector and $\rho \equiv e^\tau$ is the world-volume dilaton. Then

$$\sqrt{\gamma} = e^\tau \sqrt{g} \quad (2.7)$$

The double dimensional reduction scheme, or ‘DDR’ for short, in part reproduces the string or superstring (Green-Schwarz) worldsheet actions by a simple kinematic reduction that makes partial use of the embedding equations (2.4), and is implemented by identifying the 10th target-space coordinate X^{10} with the third world-volume coordinate $\sigma^2 \equiv \xi$ [22]. Here we slightly generalize this ansatz by assuming the membrane world-volume to be of the following topological shape (see also [25])

$$\text{world-volume} \quad \sim \quad \Sigma_n \times S^1 \quad (2.8)$$

where Σ_n is a Riemann surface of genus n . Then we set

$$X^\mu = X^\mu(\sigma^0, \sigma^1) \quad \text{for } \mu = 0, \dots, 9 \quad (2.9)$$

and identify the 10th target-space coordinate with the third world-volume coordinate according to¹

$$\partial_{\dot{2}} X^{10} = 1, \quad \partial_\alpha X^{10} = 0 \quad (2.10)$$

¹In principle we could also use

$$\partial_\alpha X^{10} = v_\alpha$$

with v_α a harmonic vector field on Σ_n , which obeys $\partial_\alpha v_\beta - \partial_\beta v_\alpha = 0$ and $\partial_\alpha (\sqrt{g} g^{\alpha\beta} v_\beta) = 0$; there are $2n$ independent such vector fields on Σ_n . This means that in principle, the target space coordinate X^{10} can wrap around not only S^1 , but simultaneously around any non-trivial cycle on the Riemann surface.

For the target-space metric we proceed again from the standard KK ansatz

$$G_{MN}^{(11)} = \begin{pmatrix} e^{-\frac{2}{3}\phi} G_{\mu\nu}^{(10)} + e^{\frac{4}{3}\phi} \mathcal{A}_\mu \mathcal{A}_\nu & e^{\frac{4}{3}\phi} \mathcal{A}_\mu \\ e^{\frac{4}{3}\phi} \mathcal{A}_\nu & e^{\frac{4}{3}\phi} \end{pmatrix} \quad (2.11)$$

where $\mu, \nu = 0, \dots, 9$, and $G_{\mu\nu}^{(10)}(X) = e_\mu{}^a(X) e_{\nu a}(X)$ and $\mathcal{A}_\mu \equiv \mathcal{A}_\mu(X)$. The associated elfbein (in triangular gauge) is

$$E_M^A = \begin{pmatrix} e^{-\frac{1}{3}\phi} e_\mu{}^a & e^{\frac{2}{3}\phi} \mathcal{A}_\mu \\ 0 & e^{\frac{2}{3}\phi} \end{pmatrix} \Rightarrow E_A^M = \begin{pmatrix} e^{\frac{1}{3}\phi} e_a{}^\mu & -e^{\frac{1}{3}\phi} \mathcal{A}_a \\ 0 & e^{-\frac{2}{3}\phi} \end{pmatrix} \quad (2.12)$$

From these relations and the definition (2.3) we read off that

$$\begin{aligned} \Pi_\alpha^a &= e^{-\frac{1}{3}\phi} \partial_\alpha X^\mu e_\mu{}^a, & \Pi_2^a &= 0 \\ \Pi_\alpha^{10} &= e^{\frac{2}{3}\phi} \partial_\alpha X^\mu \mathcal{A}_\mu, & \Pi_2^{10} &= e^{\frac{2}{3}\phi} \end{aligned} \quad (2.13)$$

The dilatonic prefactors in (2.11) and (2.12) have been adjusted in order to end up with the standard bosonic action of $D = 10$ type IIA supergravity in string frame after dimensional reduction from $D = 11$ to $D = 10$ [28]

$$\begin{aligned} S_{11} &= \int d^{11}X \sqrt{-G^{(11)}} \left(R^{(11)} - \frac{1}{48} \mathcal{F}^{MNPQ} \mathcal{F}_{MNPQ} \right) \rightarrow \\ &\rightarrow \int d^{10}X \sqrt{-G^{(10)}} e^{-2\phi} \left(R^{(10)} - \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} + 4 G^{(10)\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) \\ &+ \int d^{10}X \sqrt{-G^{(10)}} \left(-\frac{1}{4} \mathcal{A}^{\mu\nu} \mathcal{A}_{\mu\nu} - \frac{1}{48} \mathcal{F}^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu\rho\sigma} \right) \equiv S_{10} \end{aligned} \quad (2.14)$$

With this normalization, the terms involving the Ramond-Ramond fields \mathcal{A}_μ and $\mathcal{A}_{\mu\nu\rho}$ carry no dilaton factors in the effective target-space Lagrangian (see also [31]).

The relation (2.4) implies in particular

$$\gamma_{\dot{2}\dot{2}} \equiv e^{2\tau} = \partial_2 X^M \partial_2 X^N G_{MN} = e^{\frac{4}{3}\phi} \quad (2.15)$$

from which we read off the on-shell relation between the world-volume dilaton τ and the target-space dilaton ϕ

$$\tau(\sigma) = \frac{2}{3} \phi(X(\sigma)) \quad (2.16)$$

The embedding formula (2.4) furthermore requires

$$\gamma_{\alpha\dot{2}} \equiv e^{2\tau} A_\alpha = \partial_\alpha X^M \partial_2 X^N G_{MN} \Rightarrow A_\alpha = \partial_\alpha X^\mu \mathcal{A}_\mu \quad (2.17)$$

whence on-shell the world-volume KK vector field A_α gets identified with the pull-back of the target-space KK vector field \mathcal{A}_μ .

Next we substitute (2.6) and (2.11) into (2.2); for the first term on the r.h.s. this gives

$$S_{\text{DDR}} = \frac{T_{3\text{f0}}}{2} \int d^2\sigma e^\tau \sqrt{g} \eta_{AB} \left(\gamma^{\alpha\beta} \Pi_\alpha^A \Pi_\beta^B + 2\gamma^{\alpha\dot{2}} \Pi_\alpha^A \Pi_{\dot{2}}^B + \gamma^{\dot{2}\dot{2}} \Pi_{\dot{2}}^A \Pi_{\dot{2}}^B - 1 \right) \quad (2.18)$$

where $r_0 = \int d\xi$ is a reference length; with the dilaton vacuum expectation value $\tau = \langle \tau \rangle_0$ the actual circumference of the compactified dimension is

$$R_{10} = \int d\xi e^{\langle \tau \rangle_0}, \quad (2.19)$$

which together with (2.16) implies the well-known relation $R_{10}/r_0 = g_s^{2/3}$ for constant ϕ [28]. Next we substitute (2.6) and (2.13) to obtain²

$$\begin{aligned} S_{\text{DDR}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \bigg[& e^{\tau - \frac{2}{3}\phi} \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}^{(10)}(X) \\ & + e^{\tau + \frac{4}{3}\phi} \sqrt{g} g^{\alpha\beta} \left(\partial_\alpha X^\mu \mathcal{A}_\mu - A_\alpha \right) \left(\partial_\beta X^\nu \mathcal{A}_\nu - A_\beta \right) \\ & + 2 e^{\frac{2}{3}\phi} \sqrt{g} \sinh \left(\frac{2}{3}\phi - \tau \right) \bigg] \end{aligned} \quad (2.20)$$

where we identify

$$\frac{1}{2} T_3 r_0 \equiv \frac{1}{4\pi\alpha'} \quad \Rightarrow \quad \frac{1}{2} T_3 R_{10} = \frac{1}{4\pi\alpha'} g_s^{2/3} \quad (2.21)$$

thus exhibiting the relation between the membrane tension T_3 and the compactification radius on the one hand, and α' and the string coupling g_s on the other. Now substituting the embedding equations (2.16) and (2.17) we see that the cosmological constant and the terms with KK vector fields cancel, and we are left with the canonical string σ -model world-sheet action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}^{(10)}(X) \quad (2.22)$$

The dependence on the dilaton field in (2.22) thus drops out on-shell, that is, upon partial use of the embedding equations (2.4). Similarly, the dilaton decouples from

$$\begin{aligned} & \frac{T_3}{6} \int d^2\sigma \int d\xi \varepsilon^{ijk} \partial_i X^M \partial_j X^N \partial_k X^P \mathcal{A}_{MNP}(X) \\ & = \frac{1}{4\pi\alpha'} \int d^2\sigma \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) \end{aligned} \quad (2.23)$$

but without the use of any embedding equations, since both ε^{ijk} and $\varepsilon^{\alpha\beta}$ are densities (that is, again no dilaton dependence) and $B_{\mu\nu} \equiv \mathcal{A}_{\mu\nu 10}$.

We therefore reach the conclusion that in this purely kinematical reduction of the membrane action (2.1) the dilaton disappears altogether from the string world-sheet action, even when restricting to the type I subsector. Likewise the Ramond-Ramond fields \mathcal{A}_μ and $\mathcal{A}_{\mu\nu\rho}$ drop out, but this was already clear from the well known fact that their inclusion requires an extension of the usual NSR formalism.

3 Quantization

This leaves the question how to generate the world-sheet dilaton coupling (1.1) from the membrane action (2.1). As is well known, in string theory this term is added in a somewhat

²An equivalent formula was already derived in [23].

ad hoc fashion in order to ensure conformal invariance (that is, vanishing β -functions for the target space fields) also for non-Ricci flat target space geometries [24, 32]. It is important that from the string theory perspective, this term should be interpreted as a σ -model one-loop term, because it comes without a factor of $(4\pi\alpha')^{-1}$ [24, 32]. The argument explains why (1.1) cannot be derived by a simple kinematical reduction from the membrane action, unlike (2.22) and (2.23). Here we wish to argue that in order to recover the correct dilaton dependence we must replace the above classical (on-shell) treatment by a *quantum mechanical (off-shell) treatment* where instead of using the embedding equations (2.16) and (2.17) one must keep the world volume fields τ and A_α as quantum fields with the action (2.20) and integrate over them. Such an approach is evidently much closer in spirit to a Polyakov-type treatment in the sense that all six components of the world-volume metric are to be integrated over. This is also a main difference with [25] where a more restricted form is assumed for the world-volume metric, cf. their eq. (11).

In order to implement the integration over all world-volume degrees of freedom in (2.20) we decompose the full functional measure of the world-volume theory according to

$$\begin{aligned} [\mathcal{D}\gamma_{ij}] &= [\mathcal{D}g_{\alpha\beta}] [\mathcal{D}A_\alpha] [\mathcal{D}e^\tau] \\ [\mathcal{D}X] &= [\mathcal{D}^{(10)}X] [\mathcal{D}X^{10}] \end{aligned} \quad (3.1)$$

where $[\mathcal{D}X^{(10)}] \equiv [\mathcal{D}X^0] \cdots [\mathcal{D}X^9]$. One new ingredient originating from the DDR of the membrane theory that is absent from the string world-sheet theory is the extra gauge invariance

$$\delta X^{10} = \Xi(X^0, \dots, X^9) \quad (3.2)$$

that follows from the invariance of the original membrane action (2.1) under target-space diffeomorphisms: indeed, such fluctuations drop out entirely from (2.20). Now, as is well known from general KK theory, the diffeomorphisms along the compactified direction descend to gauge transformations on the KK vector in the compactified theory according to

$$\delta \mathcal{A}_\mu(X) = \partial_\mu \Xi(X) \quad (3.3)$$

Likewise the world-volume diffeomorphisms split into world-sheet diffeomorphisms and the KK gauge transformations

$$\delta A_\alpha(\sigma) = \partial_\alpha \omega(\sigma) \quad (3.4)$$

On-shell, the transformations (3.3) and (3.4) are identified by (2.17)

$$\delta A_\alpha = \partial_\alpha \omega \equiv \partial_\alpha X^\mu \partial_\mu \Xi \quad (3.5)$$

where $\omega(\sigma) \equiv \Xi(X(\sigma))$ is the KK gauge transformation parameter induced on the world-volume. This argument also shows in what sense (2.20) is invariant under world-volume gauge transformations: any gauge transformation on the world-volume KK vector A_α can be absorbed into a target-space gauge transformation, where the function $\Xi(X)$ must coincide with ω on the world-volume for $X^\mu = X^\mu(\sigma)$, but is otherwise arbitrary. Because Ξ is thus a gauge transformation parameter in the effective target-space theory, it must not be

integrated over. Consequently, we can remove $[\mathcal{D}X^{10}]$ from (3.1), and thus replace the last line of (3.1) by

$$[\mathcal{D}X] \longrightarrow [\mathcal{D}^{(10)}X] \quad (3.6)$$

Next we observe that the first two lines on the r.h.s. of (2.20) depend only on the unimodular part $\sqrt{g}g^{\alpha\beta}$ of the world-sheet metric, and therefore the conformal factor $e^\lambda \equiv \sqrt{g}$ appears only in the last line of the action (2.20) as a linear factor. However, it is well known from string theory that a hidden dependence on the conformal factor may nevertheless arise via the functional measure in the form of a Liouville action [33–38]. This hidden dependence *disappears only in the critical dimension*. More precisely, only in the latter case we can exploit the invariance of the full functional measure under conformal rescalings

$$[\mathcal{D}g_{\alpha\beta}] [\mathcal{D}^{(10)}X] \Big|_{g=\hat{g}e^\lambda} = [\mathcal{D}\hat{g}_{\alpha\beta}] [\mathcal{D}^{(10)}X] \Big|_{\hat{g}} \quad (3.7)$$

To divide out world-sheet diffeomorphisms we follow the standard procedure (which is beautifully explained in the original papers [34–36]) by parametrizing the metric variations as

$$\delta g_{\alpha\beta} = \delta\lambda g_{\alpha\beta} + (Pv)_{\alpha\beta} + \delta\mu_r \Psi_{\alpha\beta}^{(r)} \quad (3.8)$$

with the traceless world-sheet diffeomorphisms

$$(Pv)_{\alpha\beta} := \nabla_\alpha v_\beta + \nabla_\beta v_\alpha - g_{\alpha\beta} \nabla_\gamma v^\gamma \quad (3.9)$$

Here μ coordinatize the moduli space of Σ_n , and the $\Psi_{\alpha\beta}^{(r)}$ form an orthonormal basis of $(\text{Im } P)^\perp = \ker(P^\dagger)$. Then it is shown [35, 36] that after dividing out the diffeomorphisms the measure can be represented in the form

$$[\mathcal{D}g_{\alpha\beta}] \rightarrow [\mathcal{D}e^\lambda] d\mu \left(\det' P_g^\dagger P_{\bar{g}} \right)^{1/2} \det^{1/2} H \left(P_{\bar{g}}^\dagger \right) \quad (3.10)$$

where $d\mu$ is a measure on the moduli space $\mathcal{M}(\Sigma_n)$, with $\bar{g}_{\alpha\beta} \equiv \bar{g}_{\alpha\beta}(\mu)$ a representative metric in the diffeomorphism and conformal equivalence class of metrics, and the finite-dimensional matrix $H_{rs}(P^\dagger) = \langle \Psi^{(r)} | \Psi^{(s)} \rangle$ [35, 36]. The determinant factors in (3.10) are finite functions of the modular parameters μ [35, 36] which we will disregard in the remainder. Importantly, $[\mathcal{D}e^\lambda]$ is still part of the measure, but crucially there is no other hidden dependence on the conformal factor in the full measure by (3.7), so we can take the operator $P_{\bar{g}}$ to depend on any representative metric $\bar{g}_{\alpha\beta}(\mu)$. Without any other dependence on the conformal factor — as in critical string theory — we can then divide the full measure by the (infinite) volume of the group of Weyl rescalings, and thus drop the integral over $[\mathcal{D}e^\lambda]$ altogether.

Here the situation is different because there still remains an explicit dependence on the conformal factor in (2.20). The necessity of restricting the target-space dimension to a critical value also for the (super-)membrane is now a consequence of the fact that only in this case the conformal factor can act as a Lagrange multiplier field and does not acquire any dynamics of its own (as in Liouville theory), so that the integration over $[\mathcal{D}e^\lambda]$ can be

explicitly performed to produce a δ -functional $\propto \delta(\tau - \frac{2}{3}\phi)$. Once this δ -functional is in place we can finally do the integral over $[De^\tau]$, which identifies the world-volume and the target-space dilaton fields also for the quantized theory, thus ensuring that the world-volume dilaton likewise does not develop any independent dynamics of its own. Altogether this leaves us with the world-sheet action (the coupling (2.23) emerges from the kinematical reduction as before)

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\bar{g}} \bar{g}^{\alpha\beta} \left[\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}^{(10)}(X) + e^{2\phi} \left(\partial_\alpha X^\mu \mathcal{A}_\mu - A_\alpha \right) \left(\partial_\beta X^\nu \mathcal{A}_\nu - A_\beta \right) \right] \quad (3.11)$$

which still depends on the target-space dilaton ϕ . From the membrane perspective, the conformal (Weyl) invariance of the world-sheet theory can thus be viewed as the result of integrating out the conformal factor, such that we are only left with the dependence on the representative metric $\bar{g}_{\alpha\beta}(\mu)$ and a finite-dimensional integral over the moduli space $\mathcal{M}(\Sigma_n)$ (see also [23, 26] for a somewhat different perspective on the emergence of conformal symmetry on the world-sheet).

We now recognize the necessity of restricting the target space dimension to a critical value also for the (super-)membrane, from a perspective that is quite different from string theory. Our arguments imply $D = 27$ and $D = 11$ as necessary (but not sufficient) consistency conditions for the bosonic membrane and the supermembrane to exist: only with this assumption, the extra world-volume degrees of freedom remain kinematical (Lagrange multiplier) degrees of freedom without dynamics of their own. With any other choice we would be left with a *de facto* intractable path integral! These arguments also imply that there is no such thing as non-critical membrane theory.

4 Dilaton coupling

However, we are not yet done since it remains to integrate over A_α , and thus to determine the dilaton dependence of the final result. For the normalization of the integral we choose

$$\int [\mathcal{D}A'_\alpha]_{\bar{g}} e^{-\|A'\|_{\bar{g}}^2} = 1 \quad (4.1)$$

with

$$\|A'\|_{\bar{g}}^2 \equiv \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\bar{g}} \bar{g}^{\alpha\beta} A'_\alpha A'_\beta \quad (4.2)$$

and the redefined (gauge invariant) field $A'_\alpha \equiv A_\alpha - \partial_\alpha X^\mu \mathcal{A}_\mu$. Our aim is to compute the (partial) renormalized effective action functional $W = W[\bar{g}_{\alpha\beta}, \phi]$

$$e^{-W[\bar{g}, \phi]} = \int [\mathcal{D}A_\alpha]_{\bar{g}} \exp \left(-\frac{1}{4\pi\alpha'} \int d^2\sigma e^{2\phi} \sqrt{\bar{g}} \bar{g}^{\alpha\beta} A_\alpha A_\beta \right) \quad (4.3)$$

where we drop primes from now on. It is not immediately obvious how to get a well-defined answer from this expression, but we now demonstrate, subject to the somewhat unusual

renormalization prescription (4.8) below, that if there is any sensible answer at all, it must be proportional to (1.1)!

The key observation is that (4.3) is an ultralocal Gaussian integral which can be done explicitly, apart from questions related to the continuum limit. In order to analyze it, let us discretize the r.h.s. of (4.3), with two-dimensional lattice points $a\mathbf{n}$ on (a local coordinate patch of) the discretized Riemann surface, and lattice spacing a and $\mathbf{n} \in \mathbb{Z}^2$. We wish to calculate the $a \rightarrow 0$ limit of the integral

$$e^{-f_a(\phi)} = \int \prod_{\mathbf{n}; \alpha=0,1} \left[\frac{a \cdot dA_\alpha(a\mathbf{n})}{2\pi\sqrt{\alpha'}} \right] \times \\ \times \exp \left(-\frac{a^2}{4\pi\alpha'} \sum_{\mathbf{n}} e^{2\phi(a\mathbf{n})} G^{\alpha\beta}(a\mathbf{n}) A_\alpha(a\mathbf{n}) A_\beta(a\mathbf{n}) \right) \quad (4.4)$$

where $G^{\alpha\beta}(a\mathbf{n}) \equiv \sqrt{\bar{g}} \bar{g}^{\alpha\beta}(a\mathbf{n})$ is unimodular; because of unimodularity no further normalization is required, and the integral is thus normalized in such a way that $f_a(0) = 0$. Now by rescaling integration variables it is easy to see that for non-vanishing $\phi(a\mathbf{n})$ we have, on the given patch,

$$f_a(\phi) = 2 \sum_{\mathbf{n}} \phi(a\mathbf{n}) \quad (4.5)$$

As desired, this is linear in ϕ , so the only question is how to perform the continuum limit $a \rightarrow 0$ in such a way as to get a sensible and well-defined result. However, as it stands, the limit $a \rightarrow 0$ does *not* exist because we are lacking a prefactor a^2 for this expression to be converted into a Riemann sum for an integral. The only way to remedy this situation is to insert a factor $a^2 \times (Z(a, \mathbf{n})/a^2)$ in such a way that the sum admits a finite limit. The relation, valid in two dimensions

$$\sqrt{\bar{g}} R(\bar{g}) = -\frac{1}{2} \square \ln \det \bar{g} \quad (4.6)$$

then suggests the introduction of a discretized Laplacian,³

$$Z(a, \mathbf{n}) = -\frac{1}{2} \ln \left[\frac{\bar{g}(a(\mathbf{n} + \mathbf{e}_0)) \bar{g}(a(\mathbf{n} - \mathbf{e}_0)) \bar{g}(a(\mathbf{n} + \mathbf{e}_1)) \bar{g}(a(\mathbf{n} - \mathbf{e}_1))}{\bar{g}(a\mathbf{n})^4} \right] \quad (4.7)$$

with lattice unit vectors \mathbf{e}_α , such that $\lim_{a \rightarrow 0} Z(a, \mathbf{n})/a^2 = \sqrt{\bar{g}} R(\bar{g})$. This procedure therefore amounts to a metric dependent ‘wave-function renormalization’

$$\phi(a\mathbf{n}) = CZ(a, \mathbf{n}) \phi_{\text{ren}}(a\mathbf{n}) \quad (4.8)$$

such that the sum (4.5) is replaced by

$$f_a(\phi) = Ca^2 \sum_{\mathbf{n}} \left(\frac{Z(a, \mathbf{n})}{a^2} \right) \phi_{\text{ren}}(a\mathbf{n}) \quad (4.9)$$

Therefore we have the renormalized result

$$\lim_{a \rightarrow 0} f_a(\phi) = C \int d^2\sigma \sqrt{\bar{g}} \phi_{\text{ren}} R(\bar{g}) \quad (4.10)$$

³For simplicity, we spell out this formula only for $G^{\alpha\beta} = \delta^{\alpha\beta}$.

This renormalization prescription is perhaps a bit unusual in view of the fact that, in flat space quantum field theory, the wave-function renormalization factor depends only on the cutoff, but not on the coordinates. However, for a non-trivial background one would expect the renormalization to also involve the background geometry.

We emphasize that, up to an overall factor C , this outcome is unique if we demand (i) the final result to be generally covariant, and (ii) the limit to be such that the a^2 factor is properly taken care of with a finite and well-defined limit. The latter requirement excludes also higher order derivative contributions such as R^2 , R^3 , \dots . Furthermore, by (4.5) there cannot appear any terms with derivatives acting on ϕ (which would not be accessible to arguments restricted to constant ϕ). Luckily, the above renormalization prescription does not affect any other terms in the world-sheet Lagrangian, precisely because the dilaton appears nowhere else in the final answer. From our derivation it is clear that if there is any sensible result at all for (4.3), it must be proportional to (1.1). This is indeed all there is to the issue of ‘renormalization’ for this particular sector of the theory! The overall prefactor $(4\pi)^{-1}$ in (1.1) is then fixed by adjusting the proportionality constant in (4.10), and is thus also part of the renormalization prescription (alternatively, its value can be pinned down by arguments along the lines of [25, 26]).

As a final comment, we remark that one could also try to apply more standard heat kernel techniques (see e.g. [39, 40]) to the evaluation of (4.3). More specifically, invoking the Hodge-de Rham decomposition

$$\delta A_\alpha = \partial_\alpha \omega + \epsilon_{\alpha\beta} \partial^\beta \varphi \quad (4.11)$$

we can change integration variables in (4.3) with the Jacobian (discarding zero modes)

$$\det \frac{\delta(A_0, A_1)}{\delta(\omega, \varphi)} = \det \square_{\tilde{g}} \quad (4.12)$$

to recast the exponent of (4.3) in a more familiar form with a scalar Laplacian. Inspection of the formulas in section 7.1 of [40] then shows that the desired term (1.1) does appear, as well as non-local terms that would be excluded by our arguments above, in addition to various renormalizations that must be taken into account. However, apart from the fact that the formulas given there cannot be directly applied to the determination of the relevant coefficients for the model at hand, the introduction of derivative terms in (4.3) by means of (4.11) appears rather artificial, in that it obscures the ultralocality of the original expression. For this reason we prefer the more direct argument given above.

5 Outlook

In summary, we have shown that the dilaton coupling which is missing in the double dimensional reduction of the (super-)membrane can be accounted for by properly quantizing the membrane, thus completing the derivation of the world-sheet action for the type I subsector of the full theory. We have also shown that the construction only works in the critical dimension, whence non-critical (super-)membrane theories are ruled out. As expected, the derivation does not extend to the Ramond-Ramond sector, although our

discussion in section 3 does clarify why the KK vector field \mathcal{A}_μ can only appear in a gauge invariant combination in the effective target-space Lagrangian, if it appears at all. The question of how to properly include these degrees of freedom in the world-sheet description of string theory has been under discussion for a long time. According to standard wisdom [41] this requires the extension to open strings and the incorporation of D -branes into the theory. By contrast, from the membrane perspective, the world-volume action (2.1) already includes them in a natural manner from the very outset. It would therefore be interesting to relate these two descriptions, but this is a task which we leave to future work.

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