Unfolded Dynamics Approach and Quantum Field Theory

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Abstract

We study quantization of the self-interacting scalar field within the unfolded dynamics approach. To this end we present a classical unfolded system describing 4d off-shell scalar field with a general self-interaction potential. Then we systematically construct three different but related unfolded formulations of the corresponding quantum field theory, supporting them with illustrative calculations: unfolded functional Schwinger-Dyson system, unfolded system for correlation functions and an unfolded effective system which determines an effective action. The most curious feature we reveal is that an unfolded quantum commutator gets naturally regularized: standard delta-function is replaced with the heat kernel, parameterized by the unfolded proper time. We also identify an auxiliary 5d system, having this proper time as a physical time, which generates 4d scalar action as its on-shell action.

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1 Introduction

Quantum field theory (QFT) represents a powerful and elaborated theoretical framework, that unifies quantum mechanics and special relativity. It has been extremely successful in explaining and predicting various phenomena of the microworld and the properties of elementary particles. However, it is widely believed that solving the puzzle of quantum gravity will require a radical paradigm shift, since the direct application of QFT methods to general relativity does not lead to a meaningful theory.

One possible way out is to add new degrees of freedom while increasing the symmetry of the theory. A natural attempt is to consider theories with higher-spin fields. From the point of view of standard QFT, interactions with such fields seem problematic: if higher-spin (HS) fields are massive, corresponding interactions are in general non-renormalizable; if HS fields are massless, then related gauge symmetries turn out so restrictive that at first sight they forbid any interactions at all (for a review of HS no-go theorems see [1]). However, thanks to symmetries, in both cases we have remarkable examples of profound and highly nontrivial theories. String theory contains infinite sequences of massive HS fields, but infinite-dimensional superconformal symmetry organizes them in such a way that the whole theory is UV-finite. Vasiliev HS gravity describes massless fields of all spins and possesses an infinite-dimensional higher-spin gauge symmetry, but this symmetry requires a non-zero value of the cosmological constant, so the theory is formulated in anti-de Sitter space, where it eludes the taboos of the no-go theorems and generates non-zero HS vertices. For a partial review of the recent literature related to higher-spin problems see [2].

Available formulations of Vasiliev HS gravity represent generating systems for classical equations of motion, written within the framework of the unfolded dynamics approach [3–7]. This

includes Vasiliev theories in 4d [4, 5], in 3d [8], Vasiliev theory for symmetric bosonic fields in arbitrary dimensions [9], chiral HS gravity [10, 11] etc. All this models are purely classical.

One of the main problem of HS gravity is that its nonlinear action is unknown, although some alternatives for an action principle have been put forward, see e.g. [12–19]. This provides an obstacle to the systematic study of quantum HS gravity. In this paper we address the problem of quantization of the field theory within Vasiliev's unfolded dynamics approach, which does not directly use a classical action. Namely, we provide a systematic procedure for an unfolded quantization of a 4d self-interacting scalar field.

To this end we first present a classical unfolded formulation for this model. To quantize it, we use the method proposed in [20], where it has been shown that quantization can be performed via identifying a certain submodule of the off-shell unfolded system with an external source, conjugate to the unfolded field, and promoting classical unfolded equation to unfolded Schwinger-Dyson ones. For related proposal of the so-called Lagrange anchor see [21] (Let us note, that at the moment there are only few suitable classical off-shell unfolded systems, besides that presented in the paper; those include free integer-spin fields in 4d Minkowski space [20], in 4d anti-de Sitter space [22], and free chiral and gauge supermultiplets in 4d Minkowski space [23].) Developing the idea of [20], we construct three unfolded formulations of the corresponding quantum field theory: unfolded functional Schwinger-Dyson system, unfolded system for correlation functions and an unfolded system for a quantum effective action.

The paper is organized as follows. In Section 2 we give a brief overview of the unfolded dynamics approach and present and analyze a classical unfolded system for 4d self-interacting scalar field. In Section 3 we quantize this unfolded system by deducing unfolded functional Schwinger-Dyson equations that determine a partition function of the theory, and use them to calculate a free unfolded propagator. In Section 4 we present a closed system of equations on unfolded correlation functions and with its help evaluate a first perturbative correction to the unfolded propagator. In Section 5 we formulate a prescription for the system of unfolded effective equations that determines a quantum effective action and find the manifest form of this system in the one-loop approximation. In Section 6 we present a toy 5d model which leads to the classical 4d theory under consideration and reveal some curious quasi-holographic features. In Appendix A we present a general consistency analysis of the classical unfolded system for the self-interacting scalar field and comment on differences between its particular solutions.

2 Unfolded dynamics approach and classical scalar field

2.1 General construction

In the unfolded dynamics approach [3–7] a classical field theory is formulated through imposing unfolded equations

$$dW^{A}(x) + G^{A}(W) = 0, (2.1)$$

on unfolded fields $W^A(x)$, where A stands for all indices of the field. The theory is formulated on some spacetime manifold M^d with local coordinates x and de Rham differential d. Unfolded fields W are exterior forms on M^d , and $G^A(W)$ is built from exterior products of W (we omit the wedge symbol throughout the paper). Every unfolded field W^A is provided with one and only one own unfolded equation (2.1).

The nilpotency of the de Rham differential $d^2 \equiv 0$ entails a consistency condition for G

$$G^B \frac{\delta G^A}{\delta W^B} \equiv 0, \tag{2.2}$$

which is of central importance for the process of "unfolding" the field theory. Different solutions to (2.2) can, in general, provide different unfolded formulation for the same field theory.

If (2.2) is hold, unfolded equations (2.1) are manifestly invariant under infinitesimal gauge transformations

$$\delta W^A = d\varepsilon^A(x) - \varepsilon^B \frac{\delta G^A}{\delta W^B}.$$
 (2.3)

Here a gauge parameter $\varepsilon^A(x)$, representing a rank-(n-1) form, is generated by a rank-n unfolded field W^A . 0-form unfolded fields do not generate gauge symmetries and are transformed only by gauge symmetries of higher-rank fields through the second term in (2.3). At the linear level 0-forms get transformed only due to vacuum symmetries and hence correspond to gauge-invariant degrees of freedom. In a nutshell, unfolded field includes some physical field (we call it a primary field) and all its differential descendants, parameterized in a coordinate-independent way. In a nonlinear theory the basis of the differential descendants usually becomes nonlinear as well.

The two most important features of the unfolded dynamics approach are manifest gauge invariance, which allows one to efficiently control all gauge symmetries of the theory, and manifest coordinate-independence, achieved through exploiting the exterior form formalism.

2.2 Unfolded Minkowski vacuum

According to the ideology of the unfolded dynamics approach, the geometry of the spacetime manifold M^d must be encoded in some unfolded equations (2.1). This is achieved by using Cartan formalism.

One introduces a 1-form connection $\Omega = \mathrm{d} x^{\underline{a}} \Omega_{\underline{a}}^{A}(x) T_{A}$ that takes values in the Lie algebra of symmetries of M^{d} with generators T_{A} . Then maximally symmetric vacuum arises via imposing Maurer–Cartan equation on Ω

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0 \tag{2.4}$$

(square brackets stand for the Lie-algebra commutator). Fixing some particular solution Ω_0 to this equation breaks an associated gauge symmetry

$$\delta\Omega = d\varepsilon(x) + [\Omega, \varepsilon] \tag{2.5}$$

down to a residual global symmetry ε_{glob} that leaves the solution Ω_0 invariant and thus must satisfy

$$d\varepsilon_{glob} + [\Omega_0, \varepsilon_{glob}] = 0. (2.6)$$

In the paper we deal with 4d Minkowski space, so we consider a connection that takes values in Poincaré algebra iso(1,3)

$$\Omega^{AdS} = e^{\alpha\dot{\beta}} P_{\alpha\dot{\beta}} + \omega^{\alpha\beta} M_{\alpha\beta} + \bar{\omega}^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}, \tag{2.7}$$

where $P_{\alpha\dot{\alpha}}$, $M_{\alpha\beta}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}}$ represent generators of spacetime translations and (selfdual and antiselfdual part of) rotations. $e^{\alpha\dot{\beta}}$ and $\omega^{\alpha\beta}$ ($\bar{\omega}^{\dot{\alpha}\dot{\beta}}$) are 1-forms of vierbein and Lorentz connection, where two-valued indices α and $\dot{\beta}$ correspond to two spinor representations of the Lorentz algebra $so(3,1) \approx sp(2,\mathbb{C})$. The indices are raised and lowered by Lorentz-invariant spinor metric

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
 (2.8)

according to

$$v_{\alpha} = \epsilon_{\beta\alpha}v^{\beta}, \quad v^{\alpha} = \epsilon^{\alpha\beta}v_{\beta}, \quad \bar{v}_{\dot{\alpha}} = \epsilon_{\dot{\beta}\dot{\alpha}}\bar{v}^{\dot{\beta}}, \quad \bar{v}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{v}_{\dot{\beta}}.$$
 (2.9)

Expansion of (2.4) in generators gives

$$de^{\alpha\dot{\beta}} + \omega^{\alpha}{}_{\gamma}e^{\gamma\dot{\beta}} + \bar{\omega}^{\dot{\beta}}{}_{\dot{\gamma}}e^{\alpha\dot{\gamma}} = 0, \tag{2.10}$$

$$d\omega^{\alpha\beta} + \omega^{\alpha}{}_{\gamma}\omega^{\gamma\beta} = 0, \tag{2.11}$$

$$d\bar{\omega}^{\dot{\alpha}\dot{\beta}} + \bar{\omega}^{\dot{\alpha}}{}_{\dot{\gamma}}\bar{\omega}^{\dot{\gamma}\dot{\beta}} = 0. \tag{2.12}$$

The simplest solution to (2.10)-(2.12) (with a non-degenerate vierbein) is provided by Cartesian coordinates

$$e_{\underline{m}}{}^{\alpha\dot{\beta}} = (\sigma_{\underline{m}})^{\alpha\dot{\beta}}, \quad \omega_{\underline{m}}{}^{\alpha\beta} = 0, \quad \bar{\omega}_{\underline{m}}{}^{\dot{\alpha}\dot{\beta}} = 0.$$
 (2.13)

In Cartesian coordinates equation 2.6 comes down to a simple requirement from $\varepsilon_{glob}^{\alpha\dot{\beta}}$, $\varepsilon_{glob}^{\alpha\beta}$ and $\bar{\varepsilon}_{glob}^{\dot{\alpha}\dot{\beta}}$ to be x-independent, i.e. they are literally global. When we consider an unfolded scalar field in the next Subsection, 2.3 with global ε_{glob} will define a representation of Poincaré algebra on this unfolded scalar.

2.3 Unfolded self-interacting scalar field

Unfolded formulation of the scalar field $\phi(x)$ requires introduction of an infinite sequence of 0-forms, which as we will see, encode all its linearly independent differential descendants.

So we start with defining an unfolded scalar field as the following set of 0-forms

$$\Phi(Y,\tau|x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Phi_n^{(k)}(Y,\tau|x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(n!)^2} \Phi_{\alpha(n),\dot{\alpha}(n)}^{(k)}(x) y^{\alpha_1} \dots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1} \dots \bar{y}^{\dot{\alpha}_n} \frac{\tau^k}{k!}.$$
 (2.14)

Here we make use of condensed notations for symmetric spinor-tensors, so that

$$f_{\alpha(n)} := f_{\alpha_1 \dots \alpha_n}. \tag{2.15}$$

Contracting all spinor indices of $\Phi_{\alpha(n),\dot{\alpha}(n)}$ with σ -matrices one can see that it corresponds to the symmetric traceless rank-n Lorentz tensor

$$\Phi_{a_1 a_2 \dots a_n} = (\bar{\sigma}_{a_1})^{\dot{\alpha}_1 \alpha_1} \dots (\bar{\sigma}_{a_n})^{\dot{\alpha}_n \alpha_n} \Phi_{\alpha(n), \dot{\alpha}(n)}, \quad \eta^{a_1 a_2} \Phi_{a_1 a_2 \dots a_n} = 0. \tag{2.16}$$

Thus, (2.14) is equivalent to the set of symmetric traceless Lorentz tensors of all ranks, depending on spacetime coordinate x and an additional variable τ .

To conveniently operate with symmetric spinor-tensors, we introduce in 2.14 a pair of auxiliary commuting $sp(2,\mathbb{C})$ -spinors $Y=(y^{\alpha},\bar{y}^{\dot{\alpha}})$ which contract all spinor indices. Due to commutativity, they are null with respect to the antisymmetric spinor metric

$$y^{\alpha}y^{\beta}\epsilon_{\alpha\beta} = 0, \quad \bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}} = 0.$$
 (2.17)

We also define corresponding derivatives as

$$\partial_{\alpha}y^{\beta} = \delta_{\alpha}{}^{\beta}, \quad \bar{\partial}_{\dot{\alpha}}\bar{y}^{\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}$$
 (2.18)

and an Euler operator N

$$N = \frac{1}{2} y^{\alpha} \partial_{\alpha} + \frac{1}{2} \bar{y}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}}. \tag{2.19}$$

As it becomes clear below, higher powers in $y\bar{y}$ and τ in (2.14) correspond to differential descendants of a scalar field

$$\phi(x) = \Phi(Y = 0, \tau = 0|x), \tag{2.20}$$

which for this reason we call the primary field.

A consistent unfolded system, that describes a self-interacting primary scalar ϕ is

$$D\Phi + \frac{1}{N+1}e^{\alpha\dot{\beta}}\partial_{\alpha}\bar{\partial}_{\dot{\beta}}\Phi + \frac{1}{N+1}e^{\alpha\dot{\beta}}y_{\alpha}\bar{y}_{\dot{\beta}}\left(\frac{\partial}{\partial\tau}\Phi + m^{2}\Phi + gU'(\Phi)\right) = 0, \tag{2.21}$$

where U' corresponds to the first variation of the scalar potential, g is a coupling constant and D is the Lorentz-covariant derivative

$$Df(Y,\tau|x) := \left(d + \omega^{\alpha\beta}y_{\alpha}\partial_{\beta} + \bar{\omega}^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}}\right)f(Y,\tau|x), \tag{2.22}$$

which in Cartesian coordinates comes down to the de Rham differential. To simplify notations, throughout the paper we omit spinor indices contracted between a vierbein 1-form $e^{\alpha \dot{\beta}}$ and auxiliary spinors, so that

$$ey\bar{y} := e^{\alpha\dot{\beta}}y_{\alpha}\bar{y}_{\dot{\beta}}, \quad e\partial\bar{\partial} := e^{\alpha\dot{\beta}}\partial_{\alpha}\bar{\partial}_{\dot{\beta}}.$$
 (2.23)

Now let us analyze the content of (2.21). To this order we expand the Lorentz-covariant derivative in vierbeins as

$$D = e^{\alpha \dot{\alpha}} \nabla_{\alpha \dot{\alpha}} \tag{2.24}$$

and act on (2.21) with

$$y^{\alpha}\bar{y}^{\dot{\beta}}\frac{\delta}{\delta e^{\alpha\dot{\beta}}},\tag{2.25}$$

which yields a relation, that completely determines Y-dependence of Φ

$$\frac{1}{N}y^{\alpha}\bar{y}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\Phi + \Phi = 0. \tag{2.26}$$

We see that Y-dependence comes down to a simple shift of the coordinate x of the Y-independent component of Φ by $(-y^{\alpha}\bar{y}^{\dot{\alpha}})$,

$$\Phi(Y,\tau|x) = \exp\left(-y^{\alpha}\bar{y}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\right)\Phi(0,\tau|x),\tag{2.27}$$

or, treated another way, $y\bar{y}$ parameterize all traceless (because of (2.17)) derivatives of $\Phi(0, \tau|x)$. To determine τ -dependence of Φ we act on (2.21) with

$$\left(\nabla^{\alpha\dot{\alpha}} - \frac{1}{N+1} (\partial^{\alpha}\bar{\partial}^{\dot{\alpha}} + y^{\alpha}\bar{y}^{\dot{\alpha}}\frac{\partial}{\partial\tau} + y^{\alpha}\bar{y}^{\dot{\alpha}}m^{2})\right) \frac{\delta}{\delta e^{\alpha\dot{\alpha}}},$$
(2.28)

which leads to

$$\Box \Phi - m^2 \Phi - gU'(\Phi) = \dot{\Phi}, \tag{2.29}$$

where the dot stands for thes τ -derivative and d'Alembertian is defined as

$$\Box := \frac{1}{2} \nabla_{\alpha \dot{\alpha}} \nabla^{\alpha \dot{\alpha}}. \tag{2.30}$$

Making use of (2.27) we deduce hereof

$$(\Box - m^2)\Phi(0, \tau | x) - gU'(\Phi(0, \tau | x)) = \dot{\Phi}(0, \tau | x), \tag{2.31}$$

This equation fixes τ -dependence of Φ . But for a general potential this equation, of course, cannot be solved manifestly, as opposite to the Y-equation (2.26), so the dependence on τ is quite complicated.

But in the case of the free theory U'=0 it is easy to find a manifest solution to

$$\Box \Phi - m^2 \Phi = \dot{\Phi},\tag{2.32}$$

which is

$$\Phi^{free}(0,\tau|x) = \exp\left((\Box - m^2)\tau\right)\phi(x) \tag{2.33}$$

where ϕ is a primary scalar field (2.20). Combining (2.33) with (2.27), one can write down the full solution to the unfolded system (2.21) with U' = 0 as

$$\Phi^{free}(Y,\tau|x) = e^{\tau(\Box - m^2) - y^{\alpha}\bar{y}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}}\phi(x). \tag{2.34}$$

Now one can give a clear interpretation of Y- and τ -dependent components of the free unfolded field Φ : they provide a basis in the space of differential descendants of the primary scalar ϕ , with $y\bar{y}$ parameterizing traceless derivatives and τ parameterizing powers of the kinetic operator ($\Box - m^2$). These two sequences exhaust all possible types of descendants in the case of the scalar field.

Equation (2.31) with general U' cannot be resolved manifestly. Nevertheless, one can write down a formal implicit solution as

$$\Phi(Y,\tau|x) = e^{\tau(\Box - m^2)}\phi(x - y\bar{y}) - g\int_0^\tau d\tau' e^{(\tau - \tau')(\Box - m^2)}U'(\Phi(0,\tau'|x - y\bar{y})). \tag{2.35}$$

Here the first term coincides with the free unfolded field (2.34), so one can solve (2.35) perturbatively in g. Comparing (2.35) with (2.34), we see that τ -dependence of the self-interacting Φ plays the same role as in the free case: it encodes d'Alembertians of ϕ , but now sophisticatedly entangled with nonlinear corrections coming from the potential. Strictly speaking, all systems (2.21) with different U' are is some sense equivalent, as they simply provide different parameterizations for the space of differential descendants of ϕ . However, this equivalence is established, in general, by strongly non-local field redefinitions.

Unfolded system (2.21) is said to be off-shell, because the primary field $\phi(x)$ is not subjected to any differential constraints like e.g. equations of motion. To put the system on-shell, i.e. to subject ϕ to some differential constraints, one has to consistently remove some part of descendants inside of Φ . An advantage of systems (2.21) is that the on-shell reduction which

leads to standard e.o.m. for the self-interacting scalar with potential $U(\phi)$ is provided by a simple constraint

$$\dot{\Phi} = 0, \tag{2.36}$$

which eliminates all τ -descendants from Φ . Then an unfolded equation for $\Phi = \Phi(Y|x)$ becomes

$$d\Phi + \frac{1}{N+1}e\partial\bar{\partial}\Phi + \frac{1}{N+1}ey\bar{y}\left(m^2\Phi + gU'(\Phi)\right) = 0$$
 (2.37)

and imposes, as follows from (2.29), a differential constraint

$$\Box \Phi - m^2 \Phi - gU'(\Phi) = 0, \tag{2.38}$$

which includes, in Y = 0 sector, e.o.m. for the primary scalar

$$\Box \phi - m^2 \phi - qU'(\phi) = 0. \tag{2.39}$$

So in this case unfolded field Φ describes primary scalar ϕ subjected to nonlinear Klein–Gordon equation (2.39) and all its independent non-zero descendants encoded in Y-expansion (2.27) (note that imposing on-shell constraint (2.36) does not affect Y-sector of the problem)

$$\Phi^{on\text{-}shell}(Y|x) = \exp\left(-y^{\alpha}\bar{y}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\right)\phi(x). \tag{2.40}$$

Now let us return to the off-shell system (2.21). Keeping in mind the form of the on-shell reduction (2.36), we see that $\dot{\Phi}(Y=0,\tau=0|x)$ can be treated as an external source j(x) for the primary scalar $\phi(x)$, as follows from (2.29) projected to Y=0, $\tau=0$ sector

$$\Box \phi - m^2 \phi - gU'(\phi) = j(x), \quad j(x) := \dot{\Phi}(Y = 0, \tau = 0|x). \tag{2.41}$$

Thus an off-shell unfolded model can be as well treated as the on-shell one coupled to an external source [20]. This observation plays a decisive role when one turns to the problem of quantization of an unfolded model.

2.4 Relation to Vasiliev Higher-Spin Gravity

Let us take a cursory glance at how the pieces of the unfolding formalism we considered so far are built into Vasiliev's unfolded formulation of 4d on-shell HS gravity [4, 5].

First, the space of 0-forms of the higher-spin theory contains spinor-tensors of all possible ranks in dotted and undotted indices $C_{\alpha(m),\dot{\beta}(n)}$, not just a scalar unfolded module $\Phi_{\alpha(n),\dot{\alpha}(n)}$. These new 0-forms correspond to gauge-invariant strength tensors of all fields of the theory (Maxwell tensor and its descendants for s=1, Weyl tensor and its descendants for s=2 and so on for higher-spin fields). Analogously, 1-forms of the theory now include all possible $\omega_{\alpha(m),\dot{\beta}(n)}$ besides gravitational sector m+n=2 which we have used to describe Minkowski vacuum. They encode potentials of gauge fields and their gauge-noninvariant descendants (first (s-1) derivatives of the potential for a spin-s field). At the linear order, 1-forms get connected to corresponding 0-forms that makes them dynamical (in particular, gravitational gauge multiplet $e^{\alpha\dot{\beta}}$, $\omega^{\alpha\beta}$, $\bar{\omega}^{\dot{\alpha}\dot{\beta}}$ gets connected to 0-forms of Weyl tensor $C_{\alpha(4)}$, $\bar{C}_{\dot{\alpha}(4)}$, that allows metric to fluctuate). At the higher orders both 0- and 1-form equations receive nonlinear corrections that describe HS interactions.

All spinor indices are still contracted with Y-spinors, which now play a very important role, being generating elements of infinite-dimensional associative algebra of HS gauge symmetries.

Finally, Minkowski vacuum (2.10)-(2.12) is not a solution of Vasiliev theory. HS gauge symmetry requires a non-zero value of cosmological constant, so usually one considers expansion over AdS_4 background.

For a detailed review of Vasiliev theory see e.g. [24, 25].

3 Quantization of the unfolded scalar field

In this Section we show how one can quantize classical unfolded system presented in the previous Section. In a nutshell, we make use of the analogy with functional Schwinger-Dyson equations and promote off-shell unfolded equations

$$dW^A + G^A(W) = 0,$$

to operator equations

$$\left(\mathrm{d}\hat{W}^A + G^A(\hat{W})\right)Z = 0$$

for a partition function Z and determine unfolded operator algebra

$$[\hat{W}^A, \hat{W}^B] = F^{A,B}(\hat{W}).$$

3.1 Functional Schwinger-Dyson equation

In standard QFT with the classical action $S[\phi]$, the partition function

$$Z[j] := \int \mathcal{D}\phi \exp\{\frac{i}{\hbar}S[\phi] - \frac{i}{\hbar} \int d^4x \phi(x)j(x)\}$$
 (3.1)

satisfies functional Schwinger-Dyson equation

$$\frac{\delta S}{\delta \phi} [i\hbar \frac{\delta}{\delta j}] Z = jZ, \tag{3.2}$$

which can be deduced from the fact that a functional integral of a total derivative vanishes, so that

$$\int \mathcal{D}\phi \frac{\delta}{\delta \phi} e^{\frac{i}{\hbar}(S - \int d^4 x \phi j)} = 0. \tag{3.3}$$

One can obtain Schwinger-Dyson equation (3.2) as follows. One starts with the classical e.o.m. of the theory coupled to an external source j

$$\frac{\delta S}{\delta \phi}[\phi] = j \tag{3.4}$$

and quantize it by promoting a pair field-source to the operators acting on "wave function" Z and subjected to a canonical commutation relation

$$[\hat{\phi}(x_1), \hat{j}(x_2)] = i\hbar\delta^4(x_1 - x_2). \tag{3.5}$$

Then in j-representation one arrives at (3.2).

We are going to perform a similar procedure for the unfolded system (2.21). In this Subsection we deal with a free theory with $U'(\Phi) = 0$. The case of the non-vanishing potential will be considered in the next Subsection.

3.2 Free quantum scalar

We start with classical off-shell unfolded equation

$$d\Phi + \frac{1}{N+1}e\partial\bar{\partial}\Phi + \frac{1}{N+1}ey\bar{y}\left(m^2\Phi + \dot{\Phi}\right) = 0$$
(3.6)

and, accounting for (2.41), define an unfolded external source as

$$J(Y,\tau|x) = \dot{\Phi}(Y,\tau|x). \tag{3.7}$$

Note that this is to some extent analogous to the Legendre transform in classical mechanics: here we pass from "velocity" $\dot{\Phi}$ to "momentum" J.

Now we have a pair of unfolded equations

$$d\Phi + \frac{1}{N+1}e\partial\bar{\partial}\Phi + \frac{1}{N+1}ey\bar{y}\left(m^2\Phi + J\right) = 0,$$
(3.8)

$$dJ + \frac{1}{N+1}e\partial\bar{\partial}J + \frac{1}{N+1}ey\bar{y}\left(m^2J + \dot{J}\right) = 0.$$
(3.9)

Analogously to (2.34), a solution to (3.9) is

$$J(Y,\tau|x) = \exp\left((\Box - m^2)\tau - \nabla_{\alpha\dot{\alpha}}y^{\alpha}\bar{y}^{\dot{\alpha}}\right)j(x),\tag{3.10}$$

where j(x) is a primary source. Now for Φ one has, instead of (2.32), a wave equation

$$\Box \Phi - m^2 \Phi = J, \tag{3.11}$$

and in the primary sector, where Y = 0, $\tau = 0$,

$$\Box \phi - m^2 \phi = j. \tag{3.12}$$

Equation (3.12) should be treated as a classical e.o.m. with an external source j and then one can quantize the theory as described in Subsection 3.1. However, our aim is to get a closed formulation of the unfolded quantum theory in terms of the unfolded quantum fields $\hat{\Phi}$ and \hat{J} , without manifest appealing to primary fields, which are just particular components of $\hat{\Phi}$ and \hat{J} .

We want to promote system (3.8)-(3.9) to the quantum operator equations on the partition function Z

$$\left(d\hat{\Phi} + \frac{1}{N+1}e\partial\bar{\partial}\hat{\Phi} + \frac{1}{N+1}ey\bar{y}\left(m^2\hat{\Phi} + \hat{J}\right)\right)Z = 0,$$
(3.13)

$$\left(d\hat{J} + \frac{1}{N+1}e\partial\bar{\partial}\hat{J} + \frac{1}{N+1}ey\bar{y}\left(m^2\hat{J} + \frac{\partial}{\partial\tau}\hat{J}\right)\right)Z = 0.$$
(3.14)

This requires the definition of the commutator $[\hat{\Phi}_i, \hat{J}_k]$, satisfying "initial condition" (3.5) and consistent with (3.13)-(3.14). Resolving Y-dependence in (3.13)-(3.14) (which still reduces to a shift of space-time coordinates x by $(-y\bar{y})$, just as in the classical theory) one gets for Y = 0

$$\left(\Box\hat{\Phi} - m^2\hat{\Phi} - \hat{J}\right)Z = 0, \tag{3.15}$$

$$\left(\Box \hat{J} - m^2 \hat{J} - \frac{\partial}{\partial \tau} \hat{J}\right) Z = 0. \tag{3.16}$$

Assuming

$$[\hat{\Phi}_i, \hat{\Phi}_k] = [\hat{J}_i, \hat{J}_k] = 0,$$
 (3.17)

self-consistency of these equations requires

$$\left(\Box_i - m^2\right) \left[\hat{\Phi}_i, \hat{J}_k\right] = \left(\Box_k - m^2\right) \left[\hat{\Phi}_k, \hat{J}_i\right],\tag{3.18}$$

$$\left(\Box_i - m^2\right) \left(\Box_k - m^2 - \frac{\partial}{\partial \tau_k}\right) \left[\hat{\Phi}_i, \hat{J}_k\right] = 0. \tag{3.19}$$

Any solution to this system, respecting initial condition (3.5), defines some consistent quantization of the unfolded system (3.6).

We pick up a particular solution of the form

$$[\hat{\Phi}_i, \hat{J}_k] = i\hbar e^{-m^2(\tau_i + \tau_k)} K_{\tau_i + \tau_k} (x_i - y_i \bar{y}_i; x_k - y_k \bar{y}_k) \delta_{\tau_i, \tau_k}, \tag{3.20}$$

where the heat kernel $K_{\tau}(x_1; x_2)$ is defined in the usual way

$$K_{\tau}(x_1; x_2) := \frac{1}{(4\pi\tau)^2} e^{-\frac{(x_1 - x_2)^2}{2\tau}}$$
(3.21)

and possesses well-known properties

$$K_{\tau}(x_1; x_2) = e^{\Box \tau} \delta(x_1 - x_2), \tag{3.22}$$

$$\lim_{\tau \to 0} K_{\tau}(x_1; x_2) = \delta(x_1 - x_2). \tag{3.23}$$

$$\left(\Box_{i=\{1,2\}} - m^2 - \frac{\partial}{\partial \tau}\right) e^{-m^2 \tau} K_{\tau}(x_1; x_2) = 0, \tag{3.24}$$

$$\int d^4x_2 K_{\tau}(x_1; x_2) K_{\tau'}(x_2; x_3) = K_{\tau + \tau'}(x_1; x_3), \tag{3.25}$$

Relation (3.23) guarantees that (3.5) holds. We have also added an equal $-\tau$ -time factor

$$\delta_{\tau_i,\tau_k} = \begin{cases} 1, & \tau_i = \tau_k \\ 0, & \tau_i \neq \tau_k \end{cases}$$
 (3.26)

to (3.20). This is not required by quantum consistency (3.18)-(3.19), and the reason to have it will become clear when we consider a nonlinear problem.

Expression for the commutator of unfolded quantum fields (3.20) is one of the central in this paper. Let us pay attention to two of its most striking features:

(1) It naturally contains the heat kernel in the Schwinger proper-time parametrization. However, while proper time τ in Schwinger's method appears as a formal integration variable which allows for convenient representantions of Green's functions, one-loop determinants etc., in the unfolded dynamics approach it arises already at the classical level and possesses a clear interpretation – τ parametrizes off-shell descendants of the primary field, and, in addition, generates the Legendre transform (3.7) which defines an unfolded source conjugate to the unfolded field;

(2) Expression (3.20) does not immediately produces a singularity in coinciding spacetime points. Singularity appears only when $\tau_i = \tau_k = 0$, i.e. in the sector of primary fields and their on-shell descendants. Effectively, τ -dependent heat kernel replaces for unfolded fields the spacetime delta-function distribution in ordinary expressions of QFT. Thus, proper time τ serves as a natural regularizer, which potentially might manage and soften quantum divergences;

Finally, (3.20) is to some extent similar to the propagator of a non-relativistic quantum particle with τ playing the role of the time, as usual for the heat kernels. The important difference, however, is that (3.20) depends on the sum of the times, not on the difference, and contains an equal- τ -time factor.

Now let us use (3.20) to complete quantization of the free unfolded scalar. We go to J-representation, where operators are realized, according to (3.20), as

$$\hat{J}_i = J(Y_i, \tau_i | x_i) \tag{3.27}$$

$$\hat{\Phi}_i = i\hbar \int d^4 x_k \int d\tau_k \int d^4 Y_k K_{2\tau_i}(x_i - y_i \bar{y}_i; x_k - y_k \bar{y}_k) \delta_{\tau_i, \tau_k} \frac{\delta}{\delta J_k}.$$
 (3.28)

When dealing with a free theory, it is handy to introduce, instead of Z, a generator for connected correlation functions W as

$$W = -i\hbar \log Z. \tag{3.29}$$

Then instead of (3.15) one has

$$\left(\Box - m^2\right) \int d^4 x' K_{2\tau}(x; x') \frac{\delta W}{\delta J(\tau | x')} = J(\tau | x). \tag{3.30}$$

From here one finds, restoring Y-dependence, for W

$$W[J] = \frac{i}{2} \int_{0}^{+\infty} d\tau \int d^4x \int d^4y \int d^4x' \int d^4y' J(\tau, Y|x) K_{-2\tau}(x - y\bar{y}; x' - y'\bar{y}') J(\tau, Y'|x'). \quad (3.31)$$

One also has to take into account equation (3.14) which restricts an admissible form of \hat{J} . But in J-representation it literally coincides with classical equation (3.9), whose solution is (3.10). Substituting (3.10) into (3.31) one finds, after integrating by parts,

$$W[j] = -\frac{i}{2\hbar} \int d^4x j(x) \left(\Box - m^2\right)^{-1} j(x) \int_0^{+\infty} d\tau \int d^4Y \int d^4Y',$$
 (3.32)

which is a standard expression for W of the free scalar filed up to an infinite constant. resulting from integrals over space of auxiliary variables τ and Y.

From (3.31) one can recover an expression for the propagator of the unfolded free scalar field

$$\langle \Phi_i \Phi_k \rangle^0 = \frac{i\hbar}{\Box_i - m^2} K_{2\tau_i} (x_i - y_i \bar{y}_i; x_k - y_k \bar{y}_k) \delta_{\tau_i, \tau_k}. \tag{3.33}$$

Again, we see that the spacetime delta-function of the scalar propagator is replaced by the heat kernel.

In the primary sector one has, sending τ and Y to zero and using (3.23),

$$\langle \phi(x_i)\phi(x_k)\rangle^0 = \frac{i\hbar}{\Box_i - m^2}\delta(x_i - x_k),$$
 (3.34)

i.e. the standard expression for the propagator of the free scalar field.

3.3 Self-interacting quantum scalar

In the case of a non-zero potential one can proceed along the same lines. We start with classical unfolded equation

$$d\Phi + \frac{1}{N+1}e\partial\bar{\partial}\Phi + \frac{1}{N+1}ey\bar{y}\left(\dot{\Phi} + m^2\Phi + gU'(\Phi)\right) = 0$$
 (3.35)

and define a conjugate unfolded source as $J = \Phi$. But for J we leave the free equation, which does not spoil the consistency

$$dJ + \frac{1}{N+1}e\partial\bar{\partial}J + \frac{1}{N+1}ey\bar{y}\left(\dot{J} + m^2J\right) = 0. \tag{3.36}$$

We want to promote system (3.35)-(3.36) to the operator equations

$$\left[d\hat{\Phi} + \frac{1}{N+1}e\partial\bar{\partial}\hat{\Phi} + \frac{1}{N+1}ey\bar{y}\left(m^2\hat{\Phi} + gU'(\hat{\Phi}) + \hat{J}\right)\right]Z = 0. \tag{3.37}$$

$$\left[d\hat{J} + \frac{1}{N+1}e\partial\bar{\partial}\hat{J} + \frac{1}{N+1}ey\bar{y}\left(m^2\hat{J} + \frac{\partial}{\partial\tau}\hat{J}\right)\right]Z = 0.$$
 (3.38)

Then, assuming (3.17) again, self-consistency of (3.37)-(3.38) requires

$$\left(\Box_i - m^2 - gU''(\hat{\Phi}_i)\right) \left[\hat{\Phi}_i, \hat{J}_k\right] = \left(\Box_k - m^2 - gU''(\hat{\Phi}_k)\right) \left[\hat{\Phi}_k, \hat{J}_i\right]$$
(3.39)

$$\left(\Box_i - m^2 - gU''(\hat{\Phi}_i)\right) \left(\Box_k - m^2 - \frac{\partial}{\partial \tau_k}\right) [\hat{\Phi}_i, \hat{J}_k] = 0.$$
 (3.40)

To solve for (3.39) an analogy with non-relativistic quantum mechanics helps. One notices that $\left(\Box_i - m^2 - gU''(\hat{\Phi}_i)\right)$ can be considered, after performing Euclidean rotation that brings \Box_i to the spatial Laplacian and assigning τ to an imaginary non-relativistic time, as a Hamiltonian of a particle moving in a time-dependent potential $m^2 + gU''(\Phi(\tau|x))$. Then an evolution operator $\hat{\mathcal{U}}(\tau|x)$ for the corresponding imaginary-time Schrödinger equation satisfies

$$\left(\frac{\partial}{\partial \tau} - \Box + m^2 + gU''\left(\hat{\Phi}(\tau|x)\right)\right)\hat{\mathcal{U}}(\tau|x) = 0, \quad \hat{\mathcal{U}}(\tau = 0|x) = 1, \tag{3.41}$$

and has a form

$$\hat{\mathcal{U}}(\tau|x) = \hat{\mathcal{T}} \exp\left\{ \int_0^\tau d\tau' (\Box - m^2 - gU''(\hat{\Phi}(\tau'|x))) \right\},\tag{3.42}$$

where \hat{T} stands for the τ -chronological ordering. Using this one can write down a solution to (3.39)-(3.40) as

$$[\hat{\Phi}_i, \hat{J}_k] = i\hbar \hat{\mathcal{U}}(\tau_i | x_i - y_i \bar{y}_i) e^{-m^2 \tau_k} K_{\tau_k} (x_i - y_i \bar{y}_i; x_k - y_k \bar{y}_k) \delta_{\tau_i, \tau_k}.$$
(3.43)

(3.40) is solved due to the heat kernel property (3.24). (3.39) is satisfied because of (3.41) and an equal- τ -time factor δ_{τ_i,τ_k} . This is why we have introduced the same equal- τ -time factor to the commutator (3.20) of the free theory, because now (3.43) does have (3.20) as its free limit

g = 0. On the other hand, (3.43) still has the correct commutator (3.5) in the primary sector $\tau = 0, Y = 0$.

The commutator (3.43) has a quite complicated structure. In order to analyze it perturbatively, it is convenient to represent the evolution operator (3.42) in the interaction picture form as

$$\hat{\mathcal{U}}(\tau|x) = \exp\left\{\tau(\Box - m^2)\right\} \hat{\mathcal{T}} \exp\left\{-g \int_0^\tau d\tau' e^{-\tau'(\Box - m^2)} U''(\hat{\Phi}(\tau'|x)) e^{\tau'(\Box - m^2)}\right\}. \tag{3.44}$$

Using this one can evaluate (3.43) to the first order in g

$$[\hat{\Phi}_i, \hat{J}_k]|_{\tau_i = \tau_k = \tau} = i\hbar e^{-2m^2\tau} K_{2\tau}(x_i; x_k) - i\hbar g \int_0^\tau d\tau' \int d^4x' K_{\tau - \tau'}(x_i; x') U''(\hat{\Phi}(\tau'|x')) K_{\tau' + \tau}(x'; x_k) + O(g^2).$$
(3.45)

4 Unfolded correlation functions

Starting from the functional Schwinger–Dyson equation (3.2) one can deduce Schwinger–Dyson equations for correlation functions which are v.e.v. with vanishing sources. Let us rewrite (3.2) in the operator form

$$\frac{\delta S}{\delta \phi_i} [\hat{\phi}] Z = \hat{j}_i Z, \tag{4.1}$$

with (3.5) imposed. Acting on (4.1) with field operators $\hat{\phi}_{a_1}$, $\hat{\phi}_{a_2}$,... $\hat{\phi}_{a_n}$ and putting $\hat{j} = 0$ at the end, one finds

$$\hat{\phi}_{a_1}\hat{\phi}_{a_2}...\hat{\phi}_{a_n}\frac{\delta S}{\delta \phi_i}[\hat{\phi}]Z|_{\hat{j}=0} = \sum_{k=1}^n \hat{\phi}_{a_1}\hat{\phi}_{a_2}...[\hat{\phi}_{a_k}, \hat{j}_i]...\hat{\phi}_{a_n}Z|_{\hat{j}=0}.$$
(4.2)

Accounting that acting with field operators on Z with zero sources produces corresponding correlation functions

$$\hat{\phi}_{a_1}\hat{\phi}_{a_2}...\hat{\phi}_{a_n}Z|_{\hat{j}=0} = \langle \phi_{a_1}\phi_{a_2}...\phi_{a_n} \rangle$$
(4.3)

and using (3.5) one recovers from (4.2) Schwinger–Dyson equations for correlation functions

$$\left\langle \frac{\delta S}{\delta \phi_i} \phi_{a_1} \phi_{a_2} ... \phi_{a_n} \right\rangle = i\hbar \sum_{k=1}^n \left\langle \phi_{a_1} \phi_{a_2} ... \delta^4 (x_i - x_{a_k}) ... \phi_{a_n} \right\rangle. \tag{4.4}$$

Our goal is to provide an analogue of (4.4) for unfolded fields, so one can perturbatively solve for unfolded correlators. We naturally define unfolded n-point correlation functions as

$$\langle \Phi_{a_1} \Phi_{a_2} ... \Phi_{a_n} \rangle = \hat{\Phi}_{a_1} \hat{\Phi}_{a_2} ... \hat{\Phi}_{a_n} Z|_{\hat{J}=0},$$
 (4.5)

which in primary $\tau = 0$, Y = 0 sector coincides with (4.3). Acting on (3.37), which is an unfolded substitute for (4.1), with $\hat{\Phi}_{a_1}\hat{\Phi}_{a_2}...\hat{\Phi}_{a_n}$, we have

$$\left(d_i + \frac{1}{N_i + 1}(e\partial\bar{\partial})_i + \frac{1}{N_i + 1}(ey\bar{y})_i m^2\right) \left\langle \Phi_i \Phi_{a_1} \Phi_{a_2} \dots \Phi_{a_n} \right\rangle +
+ \frac{1}{N_i + 1}(ey\bar{y})_i \left(g \left\langle \mathcal{U}'(\Phi_i) \Phi_{a_1} \Phi_{a_2} \dots \Phi_{a_n} \right\rangle + \sum_{k=1}^n \left\langle \Phi_{a_1} \Phi_{a_2} \dots \left[\hat{\Phi}_{a_k}, \hat{J}_i\right] \dots \Phi_{a_n} \right\rangle \right) = 0. (4.6)$$

where $\left[\hat{\Phi}_{a_k}, \hat{J}_i\right]$ is defined in (3.43). (4.6) provides an unfolded form of a chain of Schwinger–Dyson equations and allows one to iteratively calculate unfolded correlation functions.

For a multi-point unfolded system of the form

$$d_i \Phi^{(n)} + G_i^{(n)}(\Phi) = 0 \tag{4.7}$$

one has a generalized consistency condition, arising from the identity $\{d_i, d_j\} \equiv 0$, which is

$$\sum_{k} \left(G_j^{(k)} \frac{\delta G_i^{(n)}}{\delta \Phi^{(k)}} + G_i^{(k)} \frac{\delta G_j^{(n)}}{\delta \Phi^{(k)}} \right) \equiv 0. \tag{4.8}$$

One can check that (4.6) satisfies it.

For the free theory with U' = 0 we get from (4.6)

$$\left(d_{i} + \frac{1}{N_{i} + 1}(e\partial\bar{\partial})_{i} + \frac{1}{N_{i} + 1}(ey\bar{y})_{i}m^{2}\right)\left\langle\Phi_{i}\Phi_{a_{1}}\Phi_{a_{2}}...\Phi_{a_{n}}\right\rangle + \\
+ i\hbar\frac{1}{N_{i} + 1}(ey\bar{y})_{i}\sum_{k=1}^{n}\left\langle\Phi_{a_{1}}\Phi_{a_{2}}...e^{-m^{2}(\tau_{i} + \tau_{a_{k}})}K_{\tau_{i} + \tau_{a_{k}}}(x_{i} - y_{i}\bar{y}_{i}; x_{a_{k}} - y_{a_{k}}\bar{y}_{a_{k}})\delta_{\tau_{i},\tau_{a_{k}}}...\Phi_{a_{n}}\right\rangle (4.9)$$

As expected, for unfolded fields contact terms get smoothed. In the nonlinear problem contact terms in additional get nonlinear field dressing.

Let us use (4.6) to calculate a first-order perturbative correction to the unfolded propagator. To this end we expand a full propagator in a coupling constant as

$$\langle \Phi_i \Phi_k \rangle = \langle \Phi_i \Phi_k \rangle^0 + \langle \Phi_i \Phi_k \rangle^g + O(g^2) \tag{4.10}$$

Equation for the full propagator is

$$\left(d_i + \frac{1}{N_i + 1}(e\partial\bar{\partial})_i + \frac{1}{N_i + 1}(ey\bar{y})_i m^2\right) \langle \Phi_i \Phi_k \rangle + g \frac{1}{N_i + 1}(ey\bar{y})_i \langle U'(\Phi_i)\Phi(\xi_k) \rangle + i\hbar \frac{1}{N_i + 1}(ey\bar{y})_i \left\langle \left[\hat{\Phi}_k, \hat{J}_i\right] \right\rangle = 0.$$
(4.11)

In zeroth order in q

$$\left(d_i + \frac{1}{N_i + 1}(e\partial\bar{\partial})_i + \frac{1}{N_i + 1}(ey\bar{y})_i m^2\right) \langle \Phi_i \Phi_k \rangle^0 + i\hbar \frac{1}{N_i + 1}(ey\bar{y})_i K_{\tau_i + \tau_k}(x_i - y_i\bar{y}_i; x_k - y_k\bar{y}_k) \delta_{\tau_i, \tau_k} = 0,$$
(4.12)

whose solution, of course, coincides with (3.33).

Then for the first-order correction we have, using (3.45),

$$\left(d_{i} + \frac{1}{N_{i} + 1}(e\partial\bar{\partial})_{i} + \frac{1}{N_{i} + 1}(ey\bar{y})_{i}m^{2}\right)\left\langle\Phi_{i}\Phi_{k}\right\rangle^{g} + g\frac{1}{N_{i} + 1}(ey\bar{y})_{i}\left\langle U'(\Phi_{i})\Phi(\xi_{k})\right\rangle^{0} - i\hbar g\frac{1}{N_{i} + 1}(ey\bar{y})_{i}\int_{0}^{\tau}d\tau'\int d^{4}x'K_{\tau-\tau'}(x_{i};x')\left\langle U''(\Phi(\tau'|x'))\right\rangle^{0}K_{\tau'+\tau}(x';x_{k})\delta_{\tau_{i},\tau_{k}} = 0, (4.13)$$

so that

$$\langle \Phi_i \Phi_k \rangle^g = \frac{ig\hbar}{\Box_i - m^2} \left(\langle U'(\Phi_i) \Phi_k \rangle^0 - \int_0^{\tau} d\tau' \int d^4 x' K_{\tau - \tau'}(x_i; x') \left\langle U''(\hat{\Phi}(\tau'|x')) \right\rangle^0 K_{\tau' + \tau}(x'; x_k) \delta_{\tau_i, \tau_k} \right). \tag{4.14}$$

5 Semiclassical quantization and unfolded effective equations

Quantum effective action $\Gamma[\bar{\phi}]$ is a generating functional for one-particle irreducible correlation functions. It is defined as the Legendre transform of a generator of connected correlation functions W (3.29)

$$\Gamma[\bar{\phi}] = \int d^4x \bar{\phi}j + W[j], \tag{5.1}$$

where $\bar{\phi}(x)$ is called mean (or classical) field and represents the expectation value of the corresponding quantum field

$$\bar{\phi}(x) = \langle \phi(x) \rangle_{J}. \tag{5.2}$$

A nice thing about Γ is that its expansion in powers of the Planck constant has a clear physical meaning: \hbar^n -order contribution corresponds to the n-loop correction to the classical action S.

One can deduce an equation that determines Γ from the functional Schwinger–Dyson equation (3.2). To this end one uses (3.29), (5.1) and

$$j(x) = \frac{\delta\Gamma}{\delta\bar{\phi}(x)} \tag{5.3}$$

following from (5.1) to represent Z as a functional of the mean field

$$Z[j(\bar{\phi})] = \exp\left(\frac{i}{\hbar}(1 - \int d^4x \bar{\phi}(x) \frac{\partial}{\partial \bar{\phi}(x)})\Gamma[\bar{\phi}]\right). \tag{5.4}$$

From (5.3) one can also express j-variational derivative through $\bar{\phi}$ -variation derivative as

$$\frac{\delta}{\delta j(\bar{\phi})} = \int d^4y \left(\frac{\delta^2 \Gamma}{\delta \bar{\phi}(x) \delta \bar{\phi}(y)} \right)^{-1} \frac{\delta}{\delta \bar{\phi}(y)}.$$
 (5.5)

Substituting (5.3), (5.4) and (5.5) to the functional Schwinger–Dyson equation (3.2) one finds following equation for Γ

$$\frac{\delta S}{\delta \phi} [\bar{\phi}(x) + i\hbar \int d^4 y (\frac{\delta^2 \Gamma}{\delta \bar{\phi}(x) \delta \bar{\phi}(y)})^{-1} \frac{\delta}{\delta \bar{\phi}(y)}] = \frac{\delta \Gamma}{\delta \bar{\phi}(x)}, \tag{5.6}$$

where ϕ -derivatives act to the right. This equation determines the effective action up to a field-independent contribution $\Gamma[0]$.

As is seen from (5.6), this equation can be easily obtained from classical e.o.m. coupled to an external source (3.4). One should simply replace

$$\phi(x) \to \bar{\phi}(x) + i\hbar \frac{\delta}{\delta j(x)}$$
 (5.7)

and then substitute

$$j(x) = \frac{\delta\Gamma}{\delta\bar{\phi}(x)}. (5.8)$$

Then one arrives at (5.6). Note, however, that while shift (5.7) can always be performed for arbitrary e.o.m., the substitute (5.8) requires from the e.o.m. to be Lagrangian. For the models

of self-interacting scalar we study in the paper this is always true, but in general one should check this for the consistent quantization. A systematic procedure to analyze Lagrangian properties of unfolded system is presented in [26].

It is a simple issue to repeat an analogous procedure for the unfolded theory. We want to formulate a system of effective unfolded equations determining Γ .

We start with (3.35), perform a Legendre transform $\Phi = J$ and then eliminate all τ -dependence (why we do this will be clear in the end of the Subsection). So we have an unfolded equation

$$d\Phi(Y|x) + \frac{1}{N+1}e\partial\bar{\partial}\Phi(Y|x) + \frac{1}{N+1}ey\bar{y}\left(J(Y|x) + m^2\Phi(Y|x) + gU'(\Phi(Y|x))\right) = 0 \quad (5.9)$$

Then we transform unfolded field Φ to the combination of the unfolded mean field $\bar{\Phi}(Y|x)$ and J-derivative, in analogy with (5.7)

$$\Phi(Y,\tau|x) \to \bar{\Phi}(Y|x) + i\hbar \frac{\partial}{\partial J(Y|x)},$$
(5.10)

and arrive at

$$d\bar{\Phi} + \frac{1}{N+1}e\partial\bar{\partial}\bar{\Phi} + \frac{1}{N+1}ey\bar{y}\left(J + m^2\bar{\Phi} + gU'(\bar{\Phi} + i\hbar\frac{\partial}{\partial J})\right) = 0.$$
 (5.11)

Here the source J is treated as a function of the mean field, expandable in the Planck constant,

$$J = J(\bar{\Phi}, \hbar), \tag{5.12}$$

and (5.11) should be considered as an equation that determines this function.

 \hbar -term in (5.11) generate functional derivatives $\frac{\partial^n \Phi}{\partial J...\partial J}$, for which one needs new unfolded equations, in order to have a closed system. These equations are generated from (5.11) by successively acting on it with $\frac{\delta}{\delta J}$. The higher orders in \hbar one studies, the higher functional derivatives appear in (5.11) and hence the more additional equations are required. Recently a similar idea has been put forward in [27]

We present the system to the first order in \hbar , which corresponds to the one-loop approximation. In this case $\frac{\partial}{\partial J}$ in (5.11) contributes only once, so one needs only one additional equation.

We rewrite (5.11) as

$$d_i\bar{\Phi}_i + \frac{1}{N_i + 1}(e\partial\bar{\partial})_i\bar{\Phi}_i + \frac{1}{N_i + 1}(ey\bar{y})_i\left(J_i + m^2\bar{\Phi}_i + gU'(\bar{\Phi}_i) + \frac{i\hbar}{2}gU'''(\bar{\Phi}_i)\frac{\partial\Phi_i}{\partial J_i}\right) + O(\hbar^2) = 0,$$
(5.13)

and acting on it with $\frac{\partial}{\partial J_k}$ obtain a missing equation

$$d_{i}\frac{\partial\bar{\Phi}_{i}}{\partial J_{k}} + \frac{1}{N_{i}+1}e\partial\bar{\partial}_{i}\frac{\partial\bar{\Phi}_{i}}{\partial J_{k}} + \frac{1}{N_{i}+1}(ey\bar{y})_{i}\left(\delta(x_{i}-x_{k}) + m^{2}\frac{\partial\bar{\Phi}_{i}}{\partial J_{k}} + gU''(\bar{\Phi}_{i})\frac{\partial\bar{\Phi}_{i}}{\partial J_{k}}\right) + \frac{1}{N_{i}+1}(ey\bar{y})_{i}\left(\frac{i\hbar}{2}gU^{(IV)}(\bar{\Phi}_{i})\frac{\partial\bar{\Phi}_{i}}{\partial J_{k}} + \frac{i\hbar}{2}gU'''(\bar{\Phi}_{i})\frac{\partial^{2}\bar{\Phi}_{i}}{\partial J_{i}\partial J_{k}}\right) + O(\hbar^{2}) = 0.$$
 (5.14)

Now we introduce an effective action $\Gamma[\bar{\Phi}]$ through

$$J(\bar{\Phi}) = \frac{\partial \Gamma}{\partial \bar{\Phi}} \tag{5.15}$$

and expand it to the first order in \hbar as

$$\Gamma = S + \hbar \Gamma^1 + O(\hbar^2). \tag{5.16}$$

As was said, replacing a source with the variation of the action requires some additional consistency conditions to be fulfilled. Namely, one has to check that

$$\frac{\partial J_i}{\partial \bar{\Phi}_k} = \frac{\partial J_k}{\partial \bar{\Phi}_i} \tag{5.17}$$

which arises the commutativity of variational derivatives, is in agreement with (5.11). For the theory we consider this is true.

Substitution of the expansion (5.16) to (5.13) and (5.14) gives, after simplification, a closed unfolded system

$$d\bar{\Phi}_{i} + \frac{1}{N_{i} + 1} e \partial \bar{\partial} \bar{\Phi}_{i} + \frac{1}{N_{i} + 1} e y \bar{y} \left(\frac{\partial S}{\partial \bar{\Phi}^{i}} + \hbar \frac{\partial \Gamma^{1}}{\partial \bar{\Phi}^{i}} + m^{2} \bar{\Phi}_{i} + g U'(\bar{\Phi}_{i}) + \frac{i\hbar}{2} g U'''(\bar{\Phi}_{i}) (\frac{\partial^{2} S}{\partial \bar{\Phi}^{i} \partial \bar{\Phi}^{i}})^{-1} \right) + O(\hbar^{2}) = 0,$$

$$(5.18)$$

$$d_{i}\left(\frac{\partial^{2} S}{\partial \bar{\Phi}^{i} \partial \bar{\Phi}^{k}}\right)^{-1} + \frac{1}{N_{i} + 1} e \partial \bar{\partial}_{i}\left(\frac{\partial^{2} S}{\partial \bar{\Phi}^{i} \partial \bar{\Phi}^{k}}\right)^{-1} + \frac{1}{N_{i} + 1} (ey\bar{y})_{i}\left(\delta(x_{i} - x_{k}) + m^{2}\left(\frac{\partial^{2} S}{\partial \bar{\Phi}^{i} \partial \bar{\Phi}^{k}}\right)^{-1} + gU''(\bar{\Phi}_{i})\left(\frac{\partial^{2} S}{\partial \bar{\Phi}^{i} \partial \bar{\Phi}^{k}}\right)^{-1}\right) + O(\hbar) = 0(5.19)$$

which determines the effective action of the theory in the semiclassical approximation.

Analysis of this system goes as follows. First one uses (5.19) to solve for $(\frac{\partial^2 S}{\partial \Phi^i \partial \Phi^k})^{-1}$. The one substitutes the answer to (5.18) and solves for $\frac{\partial \Gamma^1}{\partial \Phi}$ and $\frac{\partial S}{\partial \Phi}$ (the last one being purely classical is determined, of course, by (5.18) itself and does not need (5.19)). Then one can try to restore S and Γ^1 , provided that consistency condition (5.17) is fulfilled.

For the sake of formality, one may want to have unfolded equations for $\frac{\partial S}{\partial \overline{\Phi}}$ and $\frac{\partial \Gamma^1}{\partial \overline{\Phi}}$ as well. This is not a problem: one can always consistently write down unfolded stub equations. Consider, say, $\frac{\partial S}{\partial \overline{\Phi}}$. Let us introduce a new unfolded field Ψ having $\frac{\partial S}{\partial \overline{\Phi}}$ as the primary

$$\Psi = \Psi(Y, \tau | x), \quad \Psi(Y, \tau = 0 | x) = \frac{\partial S}{\partial \bar{\Phi}(Y | x)}.$$
 (5.20)

We impose the simplest unfolded off-shell equation on Ψ

$$d\Psi + \frac{1}{N+1}e\partial\bar{\partial}\Psi + \frac{1}{N+1}ey\bar{y}\dot{\Psi} = 0.$$
 (5.21)

This equation puts no restrictions on $\frac{\partial S}{\partial \Phi}$, while the value of $\frac{\partial S}{\partial \Phi}$ extracted from (5.18) will determine τ -dependence of Ψ according to

$$\Psi(Y,\tau|x) = e^{\tau \Box} \frac{\partial S}{\partial \bar{\Phi}(Y|x)}.$$
 (5.22)

The same can be done for the whole $\frac{\partial \Gamma}{\partial \Phi}$ as well. This also explains why we removed τ -dependence of the mean field $\bar{\Phi}$ and source J from the very beginning: as we saw, in order to determine

 Γ , their $\tau = 0$ components are enough, while all would-be τ -dependent components would be expressed in terms of τ -components of the stub-field Ψ .

Another question one can address by analyzing unfolded effective equations, is the dynamics of the mean field $\bar{\Phi}$. If one takes

$$\frac{\partial \Gamma}{\partial \bar{\Phi}} = 0, \tag{5.23}$$

then off-shell effective equations become on-shell and determine the possible v.e.v. of the unfolded quantum scalar field $\langle \Phi \rangle$. For instance, the system (5.18)-(5.19) with

$$\frac{\partial S}{\partial \bar{\Phi}^i} = \frac{\partial \Gamma^1}{\partial \bar{\Phi}^i} = 0 \tag{5.24}$$

determines possible values of $\langle \Phi \rangle$ in the one-loop approximation.

6 5d auxiliary model and τ as a physical time

Finally, let us briefly address the following issue: in a nutshell, quantization procedures we performed for the unfolded systems consisted in taking the equation

$$\Box \Phi - m^2 \Phi - gU'(\Phi) = \dot{\Phi} \tag{6.1}$$

and quantizing it by identifying $\dot{\Phi}$ with momentum conjugate to the field Φ ; one may wonder if there is any model which has τ as a physical time and somehow leads to the equation (6.1).

Such model does exist. In order to construct it, we first notice that upon identifying $\dot{\Phi}$ with the momentum, one cannot treat (6.1) as the classical equation of motion, because it is of first order in the time τ . Instead, one should consider (6.1) as the solution to the e.o.m, which in turn can be derived by τ -differentiating (6.1)

$$\ddot{\Phi} - (\Box - m^2 - gU'')(\Box \Phi - m^2 \Phi - gU') = 0.$$
(6.2)

This e.o.m. can be derived from the action

$$S = \int_{0}^{+\infty} d\tau \int d^{4}x \frac{1}{2} \left(\dot{\Phi}^{2} - (\Box \Phi - gU'(\Phi))^{2} \right), \tag{6.3}$$

which indeed leads to $\dot{\Phi}$ as a canonically conjugate momentum for Φ . This 5d model is non-relativistic and contains higher-derivatives, so its meaning is not immediately clear. But curiously, it mimics some holographical features. Namely, if one evaluates the action (6.3) on its minimal trajectory (6.1), then one gets 4d action of the underlying primary scalar $\phi(x) = \Phi(\tau = 0, x)$

$$S^{on\text{-}shell} = \int d^4x \left(-\phi \Box \phi + U(\phi) \right) + const, \tag{6.4}$$

assuming that asymptotics $\Phi(\tau \to \infty)$ is fixed. Thus, a classical 4d primary action arises as an on-shell 5d action (6.3) treated as a functional of the initial value ϕ .

However, it is not straightforward to continue this relation on the quantum level, because from the standpoint of 5d model the quantization procedure we perform is far from the standard one: we quantize a classical solution (6.1) by imposing equal-time commutation relation (3.43), which is canonical only at the initial moment $\tau = 0$.

7 Conclusion

In the paper we have studied the problem of quantization of the unfolded system of the 4d scalar field with a self-interaction potential of the general form. We have presented and analyzed corresponding classical unfolded system, which is of interest in itself, since the number of available unfolded models is quite limited.

We have proposed three different but related ways of formulating unfolded quantum field theory. All of them require classical off-shell unfolded system as the starting point. The first one consists in imposing functional Schwinger–Dyson equations as unfolded operator equations on the partition function of the theory. This requires finding a consistent commutation relation between unfolded field and unfolded source operators. This relation turns out to be quite remarkable: instead of delta-function presented in a standard QFT, an unfolded commutator represents a heat kernel, dependent on an auxiliary variable τ , which appears already in the classical unfolded system, where it parametrizes off-shell descendants of the primary scalar field. In the commutator of unfolded quantum fields τ plays the role of a natural regularizer and thus one may hope that unfolded dynamics approach will provide new instruments for dealing with the problem of divergences in QFT. Another curious feature is that the mentioned unfolded commutator becomes field-dependent in the nonlinear theory, that reflects the nonlinearity of relations between descendants and primaries in the unfolded module. We present this formulation and use it to solve the free model, while for nonlinear theories two other formulations seems more handy and promising.

The second way to formulate unfolded QFT is in terms of the infinite chain of unfolded Schwinger–Dyson equations for correlators, which allows one to perturbatively calculate unfolded correlation functions. However, to construct a corresponding unfolded system one has to have a unfolded functional Shwinger–Dyson system from the previous paragraph. We present corresponding unfolded system and use it to evaluate a first perturbative correction to the unfolded propagator.

The third way to quantize unfolded field theory is to write down an unfolded effective equations, which allows one to systematically restore an effective action from the semiclassical expansion. This, however, requires form the model in question to be Lagrangian, which is obviously true for the self-interacting scalar but may not be so clear for more complicated unfolded theories. We give a general prescription how to generate unfolded effective equations for the scalar theory to any order in \hbar and present a manifest form in the one-loop approximation.

Finally we present an auxiliary 5d model, which has τ -variable as the physical time, τ -equation of the 4d unfolded system as its classical solution and generates a correct 4d scalar action as an on-shell 5d action evaluated as a functional of initial values of the field. Although the status of this model is not entirely clear, its very existence indicates that the auxiliary variable τ of the unfolded dynamics approach may have some deeper meaning than a cursory glance suggests.

In this paper, we have focused mainly on the problem of formulating unfolded QFT, limiting ourselves to a few calculations for illustration purposes. Therefore it would be interesting to consider some concrete scalar field theory in order to develop a systematic technique of calculations in unfolded QFT, including Feynman diagrams, renormalizations etc.

On the other hand, the problem of formulating an unfolded quantum gauge field theory raises new questions related to gauge symmetry, ghosts etc. This requires additional thorough analysis, which is beyond the scope of this paper.

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Appendix A. General unfolded frame for the scalar field

Ansatz for unfolded equation

$$d\Phi + a_N e \partial \bar{\partial} \Phi + b_N e y \bar{y} (\dot{\Phi} + m^2 \Phi) + c_N e y \bar{y} g U'(f_N \Phi) = 0, \tag{7.1}$$

where \mathcal{U}' is understood as a formal series

$$U'(f_N\Phi) := \sum_{n=2}^{\infty} \frac{u_n}{n!} (f_N\Phi)^n$$
(7.2)

with every f_N acting only on the one following Φ .

Consistency condition from $d^2 \equiv 0$ then requires

$$b_N = \frac{b}{N(N+1)a_{N-1}},\tag{7.3}$$

$$c_N = \frac{c}{(N+1)!(a_0 \cdot a_1 \cdot \dots \cdot a_{N-1})},\tag{7.4}$$

$$f_N = f \cdot N!(a_0 \cdot a_1 \cdot \dots \cdot a_{N-1}). \tag{7.5}$$

Then Y-dependence of Φ is resolved as

$$\Phi(Y,\tau|x) = \sum_{n=0}^{\infty} \frac{(-y^{\alpha} \bar{y}^{\dot{\alpha}} \nabla_{\alpha \dot{\alpha}})^n}{(n!)^2 (a_0 \cdot \dots \cdot a_{n-1})} \Phi(0,\tau|x), \tag{7.6}$$

and τ -dependence has to be determined from

$$\Box \Phi - m^2 \Phi - c_N(N+1)U'(f_N \Phi) = \dot{\Phi}. \tag{7.7}$$

In the paper we pick up a particular solution

$$a_N = \frac{1}{N+1}, \ b_N = \frac{1}{N+1}, \ c_N = \frac{1}{N+1}, \ f_N = 1,$$
 (7.8)

But different choices are also possible. In the HS literature a standard choice for any unfolded equation is to demand $a_N = 1$. For the model under consideration this choice entails (up to overall scaling of variables)

$$a_N = 1, \ b_N = \frac{1}{N(N+1)}, \ c_N = \frac{1}{(N+1)!}, \ f_N = N!,$$
 (7.9)

so that Y-dependence is resolved as

$$\Phi(Y,\tau|x) = e^{\tau(\Box - m^2)} {}_{0}F_1(;1;y^{\alpha}\bar{y}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}})\phi(x), \tag{7.10}$$

where ${}_{0}F_{1}(;1;z)$ is a confluent hypergeometric limit function, which can be expressed through the modified Bessel function as ${}_{0}F_{1}(;1;z) = I_{0}(2\sqrt{z})$, and τ -equation takes the form

$$\Box \Phi - m^2 \Phi - \frac{1}{N!} gU'(N!\Phi) = \dot{\Phi}. \tag{7.11}$$

We see that the solution (7.8) has two important advantages over (7.9): first, Y-dependence comes down to a simple shift of x-coordinate by $(-y\bar{y})$ as seen from (2.27); second, Y- and τ -dependencies are completely separated and unfolded potential arises by trivially replacing the primary field ϕ with the unfolded field Φ .

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